Eternal domination in dynamic graphs

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April 1, 2025

Abstract

We study eternal domination in the context of dynamic graphs.

1 Introduction

1.1 Eternal domination

Let $G = (V, E)$ be a graph. We denote by $N_G(u)$ the *(open) neighbourhood* of $u \in V(G)$, or the set of vertices which are adjacent to u in G (one may write $N(u)$ if G is clear from context). The *closed neighbourhood* of u is $N_G[u] = N_G(u) \cup \{u\}$. The closed neighbourhood of a set $X \subseteq V(G)$ is $N_G[X] = \cup_{u \in X} N_G[u]$ (the open neighbourhood of a set may be defined similarly). A set $X \subseteq V(G)$ is called a *dominating set* if $N_G[X] = V(G)$. The cardinality of a minimum dominating set in G is denoted $\gamma(G)$; this parameter is called the *domination number* of G.

Many recent papers have considered dynamic models of graph "protection", where agents move through a graph in a way that somehow responds to "attacks". Our work in this paper follows a line of research which originates from [2], where the "eternal domination" model was introduced (originally referred to as "eternal security"). We describe this model in terms of a two player game, played between *defender* and *attacker*. The defender controls a set of guards which occupy some subset of $V(G)$ (typically, only one guard is allowed to occupy any one vertex). The attacker will attack some vertex in the graph, which forces the defender to respond to that attack.

More precisely, the defender chooses some set $D_1 \subseteq V(G)$ as the starting positions for the guards, and will choose each subsequent set D_{i+1}, $i \geq 1$, in response to the game-play of the attacker in the i^{th} round (this is sometimes referred to as the *adaptive online model* of the game). For each $i \geq 1$, the attacker's move in the i^{th} round is to choose some vertex $v_i \notin D_i$. The defender must then choose some vertex $u_i \in D_i$ such that $v_i \in N(u_i)$, and set $D_{i+1} = D_i \cup \{v_i\} \setminus \{u_i\}$. The goal of the defender is to be able to respond to any infinite sequence of attacks. If the defender can win from some set D_1, then D_1 is called an eternal dominating set; note that such a set D_1 (and each subsequent D_i) must necessarily be a dominating set. The *eternal domination number* of G, denoted $\gamma^{\infty}(G)$, is the minimum cardinality of an eternal dominating set in G. Recall that a *clique* in G is a subset of $V(G)$ whose elements are pairwise adjancent, and an *independent set* (or *stable set*) is a subset of $V(G)$ whose elements are pairwise non-adjancent. The *clique cover number* of G, denoted $\theta(G)$, is the minimum cardinality of a collection of cliques of G whose union is $V(G)$. The *independence number* of G, denoted $\alpha(G)$, is the maximum cardinality of an independent set of G. It is easy to argue (see [2]), that

$$\alpha(G) \leq \gamma^{\infty}(G) \leq \theta(G).$$

One may consider a related model where, instead of only moving one guard to respond to an attack, one may reconfigure D_i to D_{i+1} by moving any number of guards between adjacent vertices so long as the attacked vertex receives one guard (that is, there is a matching between $D_i \setminus D_{i+1}$ and $D_{i+1} \setminus D_i$ and the attacked vertex v_i is in D_{i+1}). This is called the *m-eternal domination model*, and an initial set of vertices which can guard any sequence of attacks is called an *m-eternal dominating set*. The minimum cardinality of an m-eternal dominating set, denoted $\gamma^{\infty}_{\text{all}}(G)$, is the *$m$-eternal domination number* of G. It is clear that $\gamma(G) \leq \gamma^{\infty}_{\text{all}}(G)$. By a clever application of Hall's Theorem, given in [2], it has been shown that $\gamma^{\infty}_{\text{all}}(G) \leq \alpha(G)$; thus we have the following fundamental inequality chain:

$$\gamma(G) \leq \gamma^{\infty}_{\text{all}}(G) \leq \alpha(G) \leq \gamma^{\infty}(G) \leq \theta(G).$$

We refer the reader to [3] for a survey of models related to graph domination which includes both known results and many interesting conjectures.

In this paper, we initiate the study of the parameters $\gamma^{\infty}(G)$ and $\gamma^{\infty}_{\text{all}}(G)$ in graphs which change over time, often called *temporal graphs* or *dynamic graphs*.

1.2 Dynamic graph terminology

The following definition for dynamic graphs, taken from [1], attempts to unify the many approaches to graphs which change over time (these have also been called *temporal graphs* or *time-varying graphs*). We begin with the classic notation for vertices and edges, V and E respectively. The edges, representing relations, exist over an interval $\mathcal{T}$ called the lifetime of the graph, being a subset of $\mathbb{N}$ (discrete) or $\mathbb{R}^+$ (continuous), and more generally some time domain $\mathbb{T}$. In the general setting, a dynamic graph is a tuple $\mathcal{G} = (V, E, \mathcal{T}, \rho, \zeta)$ such that

- $\rho : E \times \mathcal{T} \to \{0, 1\}$, called the *presence function*, indicates if a given edge is available at a given time.
- $\zeta : E \times \mathcal{T} \to \mathbb{T}$, called the *latency function*, indicates the time it takes to cross a given edge at a given start time (the latency of an edge could itself vary in time).

The latency function is optional and other functions could equally be added, such as a node presence function $\psi : V \times \mathcal{T} \to \{0, 1\}$, a node latency function $\varphi : V \times \mathcal{T} \to \mathbb{T}$ (accounting e.g., for local processing times), etc. In the case where the lifetime is $\mathbb{N}$, which is the setting in which we are interested, it is equivalent to define $\mathcal{G}$ to be an infinite sequence of graphs on the same vertex set V, $\mathcal{G} = \{G_0, G_1, G_2, \ldots\}$. We call this a *discrete dynamic graph*, but use "dynamic graph" throughout for brevity's sake. Note that the usual definition of a (static) graph corresponds with that of a dynamic graph $\mathcal{G} = \{G_0, G_1, G_2, \ldots\}$ for which $G_i = G_j$ for all i and j.

Given a discrete dynamic graph $\mathcal{G} = \{G_0, G_1, G_2, \ldots\}$, the *footprint* of $\mathcal{G}$, also called the *underlying graph* of $\mathcal{G}$, is the graph with vertex set $\cup_{i \geq 0} V(G_i)$ and edge set $\cup_{i \geq 0} E(G_i)$; we denote this graph by $\cap \mathcal{G}$. The *eventual footprint* of $\mathcal{G}$ is the subgraph of $\cap \mathcal{G}$ containing precisely those edges which reappear infinitely often. We call G_t the *snapshot* of $\mathcal{G}$ at time t; the footprint can also be defined as the union of all snapshots, while the eventual footprint is the limsup of all snapshots. Conversely, we denote by $\cap \mathcal{G}$ the intersection of all snapshots of $\mathcal{G}$, which may be called *intersection graph* or *denominator* of $\mathcal{G}$. If G is a fixed graph and $\mathcal{G}$ is a dynamic graph with $\cup \mathcal{G} = G$, we call $\mathcal{G}$ a dynamic graph on G and write $\mathcal{G} \in [G]$. If $\mathcal{G}$ is a dynamic graph on G and for each snapshot G_i of $\mathcal{G}$ we have $|E(G) \setminus E(G_i)| \leq k$, then we say that $\mathcal{G}$ a dynamic graph on G with *edge deficiency* at most k and write $\mathcal{G} \in [G]_{\leq k}$.

1.3 Eternal domination in discrete dynamic graphs

How does one then define eternal domination on a dynamic graph? We adopt the following conventions for game play. Attacker and defender are given a static graph G and play eternal domination on some dynamic graph $\mathcal{G} \in [G]_{\leq}$. The defender will place some number of guards on the vertices of G. During the i^{th} round of play, the following steps occur:

1. the snapshot of $\mathcal{G}$ changes from G_{i-1} to G_i;
2. Attacker chooses a vertex;
3. Defender moves their guards along edges in $E(G_i)$ according to the usual eternal domination objectives outlined in Section 1.1

The minimum cardinality of an eternal dominating set (m-eternal dominating set) in this setting is denoted $\gamma^\infty(\mathcal{G})$ ($\gamma^\infty_{\text{all}}(\mathcal{G})$).

Now, we note that any setting involving dynamic graphs may be considered from an *online* perspective, where the presence and latency functions are not known to the user/viewer ahead of time, and an *offline* perspective, where these functions are public knowledge. This affects only Step 1 above – in online play, G_i is not known to attacker or defender ahead of time (though the full sequence is known in advance by some oracle), while in offline play both attacker and defender have perfect information. Note that in both cases the players play according to the adaptive online model – their moves are not predetermined and are made based on all current information available.

In this paper, we consider both the online and offline versions of play. In online play, we are able to translate the problem to an equivalent problem on static graphs where the attacker is given additional power when compared to usual play and thus prove bounds and exact values for $\gamma^\infty(\mathcal{G})$ and $\gamma^\infty_{\text{all}}(\mathcal{G})$. In offline play, we consider the computational complexity of the natural decision problems associated with the game.

2 Online dynamic graphs

In order to study the minimum number of guards needed to eternally dominate a dynamic graph $\mathcal{G}$, it can be immediately seen that some conditions on $\mathcal{G}$ will be needed for the problem to be meaningful. Indeed, if any presence function is allowed, then one needs a guard on each vertex. Assumptions on connectivity or temporal connectivity[1] in G do not seem to lead to meaningful problems. We will concern ourselves, in this work, with dynamic graphs having bounded edge deficiency.

At any given time step, the defender must be able to move their guards for *any* possible snapshot that may appear next. In other words, the defender must be able to respond to the worst possible scenario for each snapshot. Thus, it is equivalent to consider "game play" as though, at each time step, the attacker chooses the edges

[1] A dynamic graph $\mathcal{G}$ on a graph G is called *temporally connected* if, for any positive integer t, any initial vertex u, and any terminal vertex v, there exists a uv-walk in G having edges $e_1, e_2, e_3, \ldots$ and a sequence of times $t_1 < t_2 < t_3 < \cdots$ such that e_i is present in G_{t_i} for all i.

which are missing from the footprint of the graph.

2.1 Generalizing the eternal domination bounds

We begin by examining the role of degeneracy in the footprint of a dynamic graph. Recall that a graph G is called d-degenerate if every subgraph of G contains a vertex whose degree is at most d. Equivalently, an n-vertex graph G is d-degenerate if $V(G)$ can be ordered $v_1, v_2, \ldots, v_n$ such that for every i there are at most k values of $j \in \{1, 2, \ldots, i-1\}$ such that $v_j v_i \in E(G)$; we call such edges the *back edges* of v_i with respect to the ordering.

Lemma 1. *Let G be a k-degenerate graph. If $\mathcal{G} \in [G]_{\leq k}$, then $\gamma^\infty(\mathcal{G}) = n$.*

Proof. Order the vertices $v_1, v_2, \ldots, v_n$ such that each v_i has at most k back edges. Suppose that, at time step i, v_i is attacked and none of its back edges are present. In this case, a guard which does not occupy a vertex in $\{v_1, v_2, \ldots, v_{i-1}\}$ must move to occupy v_i. It immediately follows that there exists a sequence of attacks and snapshots for which a guard on every vertex is required. □

Given a graph G, let $\alpha_k(G)$ be the size of a maximum induced k-degenerate subgraph of G. Note that $\alpha_0(G)$ coincides with the usual definition of the independence number of G, $\alpha(G)$. We present our first main theorem, which generalizes the standard bounds for $\gamma^\infty(G)$ in static graphs.

Theorem 2. *Let G be a graph. If $\mathcal{G} \in [G]_{\leq k}$, then*

$$\alpha_k(G) \leq \gamma^\infty(\mathcal{G}) \leq (k+1)\theta(G).$$

Furthermore, there exist infinitely many graphs achieving $\gamma^\infty(\mathcal{G}) = \alpha_k(G)$ and $\gamma^\infty(\mathcal{G}) = (k+1)\theta(G)$.

Proof. The lower bound of $\alpha_k(G)$ immediately follows from Lemma 1, since the attacker could play solely on the largest induced k-degenerate subgraph in $\cap\mathcal{G}$; Lemma 1 also shows that any such graph with a k-degenerate footprint achieves this value for $\gamma^\infty(G)$. The upper bound can be obtained by observing that $k+1$ guards can eternally guard a clique with edge deficiency at most k. To see that an infinite family of graphs exists attaining this point, consider a collection of θ complete graphs of order at least $k+2$, $C_1, C_2, \ldots, C_\theta$, each with two distinct distinguished vertices $u_i, v_i \in V(C_i)$. Construct G by adding an edge between u_i and v_{i+1} for each $i \in \{1, \ldots, \theta\}$. We suppose that fewer than $(k+1)\theta(G)$ guards are used, and show that the attacker wins. The attacker iteratively attacks $k+1$ vertices in $V(C_i) \setminus \{u_i\}$, and before each attack suppresses k edges between the vertex to be attacked and the positions of at most k guards in C_i (if there are fewer guards, then arbitrary edges are suppressed). Note that after these attacks in C_i, the positions of the $k+1$ guards who responded to those attacks precludes them from responding to subsequent attacks in C_{i+1}. Once the attacker has attacked $C_1, C_2, \ldots, C_{\theta-1}$ in this way, either the attacker has won at a previous step or there at most k guards remaining in C_Θ and no guard occupying u_{i-1}. Thus, the attacker wins. $\square$

Note that one can achieve strict inequality in these bounds as well. Consider the graph $G = C_5[K_2]$ [note to self: introduce notation], in which one replaced each vertex of C_5 with a copy of K_2 and two copies of K_2 are complete to one another if and only if the corresponding vertices they have replaced in C_5 are adjacent in C_5. One can argue that $\alpha_1(G) = 4$ and that if $\mathcal{G} \in [G]_{\leq 1}$ then $\gamma^\infty(\mathcal{G}) \geq 5$. NOTE: can we show that it's $= 5$? Can we generalize the construction and get a theorem?

We now consider the bounds for m-eternal domination. Recall that $\gamma(G) \leq \gamma^\infty_{\text{all}}(G) \leq \alpha(G)$ for any (static) graph G. The lower bound is straightforward to generalize. We denote by $\gamma_k(G)$ the minimum size a so-called k-dominating set of vertices in G; that is, a set of vertices X such that every vertex not in X has at least k neighbours in X.

Proposition 3. *Let G be a graph. If $\mathcal{G} \in [G]_{\leq k}$, then $\gamma_{k+1}(G) \leq \gamma^\infty_{\text{all}}(\mathcal{G})$.*

Proof. Suppose that the defender has fewer than $\gamma_{k+1}(G)$ guards; let X denote their positions. Since X is not a $(k+1)$-dominating set of G, there is some vertex $v \in V(G) \setminus X$ with at most k neighbours in X. The attacker wins by suppressing all edges between X and v then attacking v. $\square$

We leave the completion of the full inequality chain as an open problem.

Problem 4. *For every graph G and every $\mathcal{G} \in [G]_{\leq k}$, is it true that $\gamma^\infty_{\text{all}}(\mathcal{G}) \leq \alpha_k(G)$?*

2.2 Realizable triples

We now turn our attention to realizable triples. For a positive integer k, we say that a triple of integers (a, g, t) *realizable for k* if there exists a graph G and dynamic graph $\mathcal{G}$ on G with edge failure rate k having $\alpha_k(G) = a$, $\gamma^\infty(\mathcal{G}) = g$, and $\theta(G) = t$.

PROBLEM: Determine which triples are realizable.

Here is one approach to constructing realizable triples. Given a set of disjoint (static) graphs $\{G_1, G_2, \ldots, G_s\}$, let $\mathcal{S}(G_1, G_2, \ldots, G_s)$ be the graph obtained by adding a universal vertex to $\{G_1, G_2, \ldots, G_s\}$ (Klostermeyer and MacGillivray refer to this as the star product of $G_1, G_2, \ldots, G_s$ in []). It is straightforward to check the following:

- $\alpha(\mathcal{S}(G_1, G_2, \ldots, G_s)) = \sum_{i=1}^{s} \alpha(G_i)$
- $\gamma^\infty(\mathcal{S}(G_1, G_2, \ldots, G_s)) = \sum_{i=1}^{s} \gamma^\infty(G_i)$
- $\theta(\mathcal{S}(G_1, G_2, \ldots, G_s)) = \sum_{i=1}^{s} \theta(G_i)$

It is much less clear what the effect the star product has on dynamic graphs. Indeed, we still do not know the effect of adding a universal vertex to a connected graph.

Question 1. *Let G be a graph, $H = G \vee K_1$, $\mathcal{G} \in [G]_{\leq k}$, and $\mathcal{H} \in [H]_{\leq k}$. Is true that $\gamma^\infty_{\text{all}}(\mathcal{H}) = \gamma^\infty_{\text{all}}(\mathcal{G})$?*

2.3 Graph classes

PROBLEM: Find exact values, or bounds, for as many graph classes as possible for BOTH $\gamma^\infty(\mathcal{G})$ and $\gamma^\infty_{\text{all}}(\mathcal{G})$.

- Note that the values below for grids are not truly proven yet, aside from the ones that come from the degeneracy result.

By applying the degeneracy bound from Lemma 1, we obtain the following trivial results.

Proposition 5. *Let G be a graph and $\mathcal{G} \in [G]_{\leq k}$. If*

1. *G is a tree;*
2. *$G = K_{m,n}$, $m \geq n$, and $k \geq n$;*
3. *G is outerplanar and $k \geq 2$; or*

4. G is planar and $k \geq 5$

then $\gamma^\infty(\mathcal{G}) = |V(G)|$.

Proposition 6. *If $\mathcal{G} \in [K_n]_{\leq k}$ and $1 \leq k < n$, then $\gamma^\infty(\mathcal{G}) = \gamma^\infty_{\text{all}}(\mathcal{G}) = k + 1$.*

Cartesian products of simple graphs, in particular paths and cycles, are often of interest in the field of domination. Lemma 1 implies that there are a limited number of cases to consider.

Proposition 7. *Let G be a graph and $\mathcal{G} \in [K_n]_{\leq k}$. If*

1. *$G = P_m \square P_n$ and $k \geq 2$;*
2. *$G = P_m \square C_n$ and $k \geq 3$;*
3. *$G = C_m \square C_n$ and $k \geq 4$;*

then $\gamma^\infty(\mathcal{G}) = nm$.

Detemine the (remaining) values of $\gamma^\infty(\mathcal{G})$ and $\gamma^\infty_{\text{all}}(\mathcal{G})$ when G is any of $P_m \square P_n, P_m \square C_n, C_m \square C_n$. Warmup propositions appear below.

Proposition 8. *If $\mathcal{G} \in [P_2 \square P_n]_{\leq 1}$, then $\gamma^\infty(\mathcal{G}) = \left\lceil \frac{3n}{2} \right\rceil$.*

SKETCH. Upper bound – partition into C_4s. Lower bound – Partition into C_4s and show we can force an "end" C_4 to have only two guards remaining and force a loss. □

Proposition 9. *If $\mathcal{G} \in [P_3 \square P_n]_{\leq 1}$, then $\gamma^\infty(\mathcal{G}) = \left\lceil \frac{5n}{2} \right\rceil$.*

Proof. Upper bound – 5 guards per 2 columns. Lower bound less clear; need to show can never "borrow" guards from neighbouring column. □

3 Offline dynamic graphs

Offline dynamic graphs present a very different challenge due to the sheer number of variations one can construct on a single footprint. As such, we focus our attention on the computational complexity of the decision problem "can k guards win eternal domination on some dynamic graph $\mathcal{G}$".

- Examine the known algorithms for deciding if k guards can win on static graphs and extend it to this setting.
- Prove complexity results.

4 Acknowledgements

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