The canonical HANK model

NBER Heterogeneous-Agent Macro Workshop

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This session

Just saw: How to solve steady states and PE transitional dynamics of neoclassical heterogeneous agent models.

Next: Introducing "HANK".

1. The canonical HANK model
2. Three instructive special cases
3. Solving the model using blocks and DAGs
4. Summary

The canonical HANK model

Introducing the canonical HANK model

- We now embed the standard incomplete markets model of consumption and saving into a New-Keynesian model

- Along the way, we will allow for a **government**: bonds, taxes, gov. spending

- Will mostly follow Auclert et al. (2018) (henceforth IKC), though allowing for monetary policy, too

- Will set up the model in the **sequence space**:
  - assume economy in steady state, feed in perfect foresight aggregate shocks at $t = 0$, study response thereafter ("MIT shocks")
  - keep in mind **certainty equivalence** !!

Household side

- Household i solves

$$\max_{c_{it}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(u(c_{it}) - v\left(n_{it}\right)\right)$$
$$c_{it} + a_{it} \leq (1 + r_t)a_{it-1} + z_{it}$$
$$a_{it} \geq \underline{a}$$

- This is the “sequence” analogue of the Bellman equation from before

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- This is the "sequence" analogue of the Bellman equation from before
- Real after tax income is

$$z_{it} = (1 - \tau_t)y_{it} \qquad y_{it} = \frac{W_t}{P_t} e_{it} n_{it}$$

- Here, e_{it} is idiosyncratic productivity, normalized so that $\mathbb{E}_i\left[e_{it}\right] = 1$
- Can capture progressive taxation as in Heathcote et al. (2017) (see IKC paper)

Unions and sticky wages

- For our canonical HANK model, we'll work with **sticky wages** (not prices)
  - with sticky prices, can get countercyclical profits

  ... redistribution from wage to profit earners in recession... matters in HANK!

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  - ... strange results can happen (examples: Bilbiie 2008, Broer et al. 2020)
- Microfound sticky wages extending Erceg et al. (2000) (see IKC)
  - + "labor allocation rule": which agent works what fraction of total labor N_t?
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  - today: assume all agents work same hours, $n_{it} = N_t$
- **Today:** will use simple ad-hoc wage Phillips curve (details won't matter)

$$\pi_t^w = \kappa \underbrace{\left(v'(N_t) - \frac{\epsilon - 1}{\epsilon} (1 - \tau_t) \frac{W_t}{P_t} u'(C_t) \right)}_{\text{wedge in labor FOC of "average" agent}} + \beta \pi_{t+1}^w$$

Production

- Representative firm with aggregate production function, linear in labor

$$Y_t = X_t N_t$$

  where X_t is TFP

- Assume **flexible prices** $\Rightarrow$

$$P_t = \frac{W_t}{X_t} \qquad \Leftrightarrow \qquad \frac{W_t}{P_t} = X_t$$

  Real wage is exogenous. No profits!

- Goods inflation π_t = wage inflation π_t^w minus TFP growth

Government: Fiscal policy

Government sets fiscal policy, consisting of paths

- G_t of gov spending
- T_t of total tax revenue, controlled via τ_t

$$T_t = \tau_t Y_t$$

- B_t of government bonds, uniformly bounded (no Ponzi schemes)

subject to government budget constraint

$$B_t = (1 + r_t) B_{t-1} + G_t - T_t$$

Can set any two of those and the third follows.

Government: How is after tax income distributed?

- Total after-tax income is

$$Z_t \equiv Y_t - T_t = (1 - \tau_t) Y_t$$

- Because of linear taxes, we find for individual after-tax income

$$y_{it} = e_{it} Y_t \qquad \Rightarrow \qquad z_{it} = e_{it} Z_t$$

- z_{it} simply a share of total after-tax income Z_t. Will be convenient.

Monetary authority follows an interest rate rule. Allow for two kinds of rules:

- **standard Taylor rule.** (linearized)

$$i_t = r + \phi_\pi \pi_t + \epsilon_t$$

  here: $r =$ steady state real rate, $\epsilon_t =$ monetary shock

- **real rate rule.**

$$r_{t+1} = r + \epsilon_t \qquad \Leftrightarrow \qquad i_t = r + \pi_{t+1} + \epsilon_t$$

  Equivalent to Taylor rule with coefficient 1 on expected inflation.
  Note: π_{t+1} vs π_t not key (same in cts time!), key is $\phi_\pi = 1$

Why allow for "real rate rule"? Huge gain in tractability! All monetary policy acts via changing real rate. Cost is small if Phillips curve is flat (π_t moves little).

Definition of equilibrium

- All agents optimize and markets clear

$$G_t + C_t = Y_t$$
$$A_t = B_t$$

where household aggregates are

$$C_t = \int c_t^* (a_-, e) \, dD_t (a_-, e)$$

$$A_t = \int a_t^* (a_-, e) dD_t (a_-, e)$$

Computing the steady state

How can we find the steady state of this model?

1. Normalize $Y = 1$, calibrate r and B, G. Set $T = G + rB$.
2. Can use **same code** as for standard incomplete markets model:
    - instead of $e_{it} Y$ now use $e_{it} \cdot (Y - T)$
    - choose β to match $A = B$.
3. $G + C = Y$ holds by Walras law! Done!

Three instructive special cases

Special cases

Will introduce three special cases that are helpful to analyze and compare the HA model to.

1. Representative-agent model (RA) — [Woodford 2003, Galí 2008]
2. Two-agent model (TA) — [Campbell and Mankiw 1989, Galí et al. 2007, Bilbiie 2008]
3. Zero-liquidity model (ZL) — [Werning 2015, Ravn and Sterk 2017, Bilbiie 2021]

Only difference across models: how C_t is determined given real rates r_t and after-tax incomes Z_t.

Steady state aggregates are identical across models.

Representative-agent model

- This is the standard NK model (with wage rigidities)
- Consumption solves

$$\max \sum_{t=0}^{\infty} \beta^t \frac{C_t^{1-\sigma}}{1-\sigma}$$
$$C_t + a_t \leq (1 + r_t)a_{t-1} + Z_t$$

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which has the solution

$$C_t = \frac{\beta^{t/\sigma} q_t^{-1/\sigma}}{\sum_{s \geq 0} \beta^{s/\sigma} q_s^{1-1/\sigma}} \left[\sum_{s \geq 0} q_s Z_s + (1 + r_0) a_{-1} \right]$$

where $q_t \equiv (1 + r_1)^{-1} \cdot \cdots \cdot (1 + r_t)^{-1}$.

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where $q_t \equiv (1+r_1)^{-1} \cdot \dots \cdot (1+r_t)^{-1}$. With $r_t = r = \beta^{-1} - 1$, this is just

$$C_t = \frac{r}{1+r} \sum_{s\geq 0} (1+r)^{-s} Z_s + r a_{-1}$$

Two-agent model

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- Jointly pin down aggregate consumption

$$C_t = (1-\mu)c_t^{PIH} + \mu c_t^{HTM}$$

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- Eventually, **all** agents must have zero assets, so $c_{it} = z_{it}$
- Does that mean all Euler equations fail? Consider:

$$z_{it}^{-\sigma} \geq \beta(1 + r_{t+1})\mathbb{E}_t\left[z_{it+1}^{-\sigma}\right] \qquad \Leftrightarrow \qquad Z_t^{-\sigma} \geq \beta(1 + r_{t+1})\underbrace{\mathbb{E}\left[\frac{(e')^{-\sigma}}{e^{-\sigma}}\middle| e\right]}_{\rho(e)} Z_{t+1}^{-\sigma}$$

- The last Euler equation to fail as we reduce β is that of $\overline{e} = \arg\max \rho(e)$
- Tractable! For instance, steady state r solves $\beta(1 + r)\rho(\overline{e}) = 1$

Solving the model using blocks and DAGs

Blocks

- Throughout this workshop, we will see that it is very useful to break models into "blocks"
- This language is often loosely used in practice, we will formally define them
  - reference is Auclert et al. (2021)
- We will write sequences of variables, e.g. $\{r_t\}$, as vectors $\mathbf{r} = (r_0, r_1, \ldots)'$.

Block: mapping from sequences of *inputs* to sequences of *outputs*.

Examples:

- Household block: $\mathbf{r}, \mathbf{Z} \to \mathbf{C}, \mathbf{A}$

$$V_t(e, a_-) = \max_{c,a} u(c) + \beta \mathbb{E}_t V_{t+1}(e', a)$$
$$\text{s.t. } c + a = (1 + r_t)a_- + e \cdot Z_t \qquad a \geq \underline{a}$$

1. backward iteration to get policies $a_t(e, a_-), c_t(e, a_-)$.
2. forward iteration to get distribution $D_t(e, a_-)$.
3. aggregation: $A_t = \int a_t(e, a_-) dD_t(e, a_-)$, $C_t = \int c_t(e, a_-) dD_t(e, a_-)$.

Defining blocks and models

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- Fiscal policy block: $\mathbf{r}, \mathbf{T}, \mathbf{G}, \mathbf{Y} \to \mathbf{B}, \mathbf{Z}$

$$B_t = G_t - T_t + (1 + r_t) B_{t-1}$$
$$Z_t = Y_t - T_t$$

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- Goods market clearing block: $\mathbf{Y}, \mathbf{C}, \mathbf{G} \to \mathbf{H} \equiv \mathbf{C} + \mathbf{G} - \mathbf{Y}$

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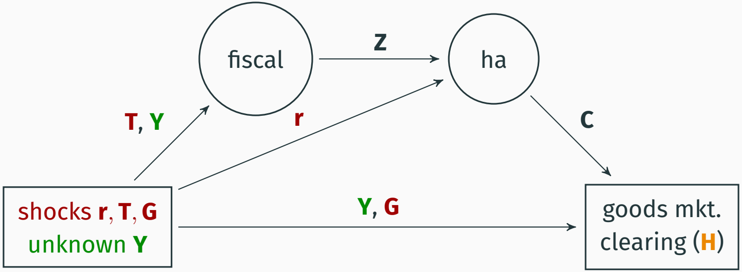
Model: combination of blocks

- some inputs are exogenous **shocks**, e.g. $\mathbf{r}, \mathbf{T}, \mathbf{G}$
- some inputs are endogenous **unknowns**, e.g. $\mathbf{Y}$
- some outputs are **targets** that must be zero in GE, e.g. $\mathbf{H}$ [#targets = #unknowns]

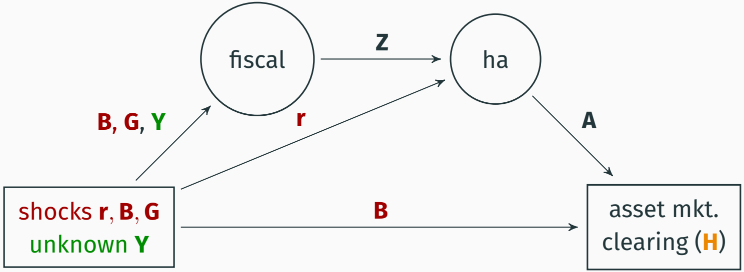
Most macro models can be written this way. Will help us solve them efficiently!

Require that models have no cycles → draw as directed acyclic graphs (DAGs).



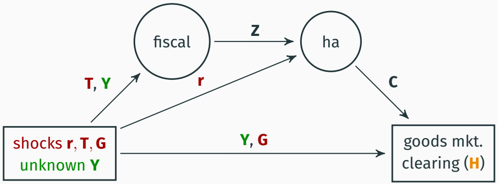
- Model is composite mapping: $(\mathbf{Y}, \mathbf{r}, \mathbf{T}, \mathbf{G}) \to \mathbf{H}$.
- GE response of $\mathbf{Y}$ to shocks satisfies $\mathbf{H} = 0$.

- E.g. instead of feeding in **G** and **T** shocks, could feed in **G** and **B** shocks
- Could use asset market rather than goods market clearing

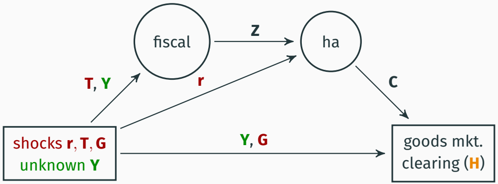


- We'll use this approach later in the tutorial.

Solving for output response to shocks



- Imagine we change the path of government spending **G**. How is **Y** affected?



- Imagine we change the path of government spending **G**. How is **Y** affected?
- We need to find **Y** such that

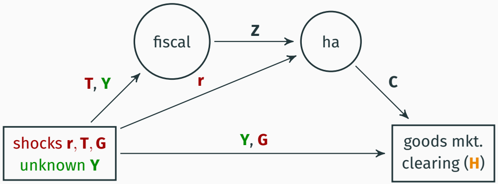
$$\mathbf{H}(\mathbf{Y}, \mathbf{G}) = 0$$

- First order shock $d\mathbf{G} \Rightarrow$ use implicit function theorem:

$$d\mathbf{Y} = -\left(\mathbf{H_Y}\right)^{-1} \cdot \mathbf{H_G} \cdot d\mathbf{G}$$

All we need is **H**'s Jacobians $\mathbf{H_Y}$ and $\mathbf{H_G}$ ...

How do we get the Jacobians?



First step is to compute **individual blocks' Jacobians**, e.g. $\mathcal{J}^{\mathbf{Z},\mathbf{Y}}$, $\mathcal{J}^{\mathbf{C},\mathbf{Z}}$, $\mathcal{J}^{\mathbf{H},\mathbf{Y}}$, $\mathcal{J}^{\mathbf{H},\mathbf{G}}$

- If block is analytical (`SimpleBlock`), its derivative is analytical too
  - e.g. $\mathcal{J}^{\mathbf{Z},\mathbf{Y}} = \mathbf{I}$ or $\mathcal{J}^{\mathbf{H},\mathbf{G}} = \mathbf{I}$
- If block has heterogeneous agents (`HetBlock`), solve Jacobian numerically
  - e.g. solve $\mathcal{J}^{\mathbf{C},\mathbf{Z}}$ using fake news algorithm

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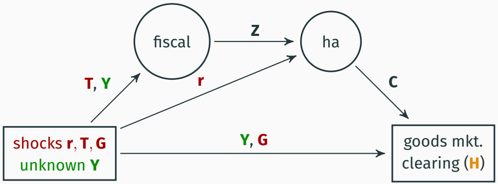

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Then "chain" the Jacobians together:

$$\mathbf{H}_{\mathbf{Y}} = \mathcal{J}^{\mathbf{H},\mathbf{Y}} + \mathcal{J}^{\mathbf{H},\mathbf{C}} \cdot \mathcal{J}^{\mathbf{C},\mathbf{Z}} \cdot \mathcal{J}^{\mathbf{Z},\mathbf{Y}} \qquad \mathbf{H}_{\mathbf{G}} = \mathcal{J}^{\mathbf{H},\mathbf{G}}$$

SSJ workflow (will use this many times!)

These ideas are at the heart of the workflow in our Sequence-Space Jacobian toolbox:

1. Define individual blocks: `SimpleBlock`, `HetBlock`, `SolvedBlock`
   - `SolvedBlock` allows to solve out recursions, e.g. solve an Euler equation
2. Combine the blocks into a model
3. Set steady state parameters and solve the model at the steady state.
4. Solve for the responses of the model directly, code handles all Jacobians.
   - e.g. `solve_impulse_linear` automatically computes $d\mathbf{Y} = -\left(\mathbf{H_Y}\right)^{-1} \cdot \mathbf{H_G} \cdot d\mathbf{G}$
   - but can also compute $\mathbf{H_Y}$, $\mathbf{H_G}$ individually, or even $\mathcal{J}^{\mathbf{C},\mathbf{Z}}$, $\mathcal{J}^{\mathbf{Z},\mathbf{Y}}$ etc
   - this will be helpful to inspect the model's mechanics!

Summary

Summary

We introduced a canonical HANK model:

- Standard incomplete markets households
- Standard New-Keynesian supply side, but sticky wages + flex prices
- Real rate rule for now (relax later)

Outlined how we can solve this model ...

- Set up as blocks. Many blocks $=$ a model
- SSJ toolbox solves out Jacobians, chains them, uses implicit function theorem to compute IRFs

Next: Analyze fiscal policy in this model. Tomorrow: Monetary policy.

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