Formal deduction in propositional logic

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'I know what you're thinking about,' said Tweedledum; 'but it isn't so, nohow.'

'Contrariwise,' continued Tweedledee, 'if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic.'

(Lewis Caroll, "Alice in Wonderland")

Formal deducibility

- We have seen how to prove arguments valid by using truth tables and other semantic methods.
- We now want to replace this approach by a purely syntactic one, that is, we give formal rules for deduction which are purely syntactical.
- We want to define a relation called formal deducibility to allow us to mechanically check the validity of a proof.
- The significance of the word formal will be explained later. The important point is that formal deducibility is concerned with the syntactic structure of formulas and proofs can be checked mechanically.

Notational conventions

- Suppose $\Sigma = \{A_1, A_2, A_3, \dots\}$.
  For convenience, Σ may be written as a sequence $A_1, A_2, A_3, \dots$.
- The sets $\Sigma \cup \{A\}$ and $\Sigma \cup \Sigma'$ may be written as Σ, A and Σ, Σ' respectively.
- We use the symbol $\vdash$ to denote the relation of formal deducibility and write

$$\Sigma \ \vdash \ A$$

  to mean that A is formally deducible (or provable) from Σ.
- Formal deducibility is a relation between Σ (a set of formulas which are the premises) and A (a formula which is the conclusion).

The 11 rules of formal deduction

(1)	(Ref)	$A \vdash A$	Reflexivity
(2)	$(+)$	If $\Sigma \vdash A$, then $\Sigma, \Sigma' \vdash A$.	Addition of premises
(3)	$(\neg -)$	If $\Sigma, \neg A \vdash B$, $\Sigma, \neg A \vdash \neg B$, then $\Sigma \vdash A$.	$\neg$ elimination
(4)	$(\rightarrow -)$	If $\Sigma \vdash A \rightarrow B$, $\Sigma \vdash A$, then $\Sigma \vdash B$.	$\rightarrow$ elimination
(5)	$(\rightarrow +)$	If $\Sigma, A \vdash B$, then $\Sigma \vdash A \rightarrow B$.	$\rightarrow$ introduction

The 11 rules of formal deduction (contd.)

(6)	$(\wedge -)$	If $\Sigma \vdash A \wedge B$, then $\Sigma \vdash A$, $\Sigma \vdash B$.	$\wedge$ elimination
(7)	$(\wedge +)$	If $\Sigma \vdash A$, $\Sigma \vdash B$, then $\Sigma \vdash A \wedge B$.	$\wedge$ introduction
(8)	$(\vee -)$	If $\Sigma, A \vdash C$, $\Sigma, B \vdash C$, then $\Sigma, A \vee B \vdash C$.	$\vee$ elimination
(9)	$(\vee +)$	If $\Sigma \vdash A$, then $\Sigma \vdash A \vee B$, $\Sigma \vdash B \vee A$.	$\vee$ introduction

The 11 rules of formal deduction (contd.)

(10)	$(\leftrightarrow -)$	If $\Sigma \vdash A \leftrightarrow B$, $\Sigma \vdash A$, then $\Sigma \vdash B$.	
		If $\Sigma \vdash A \leftrightarrow B$, $\Sigma \vdash B$, then $\Sigma \vdash A$.	$\leftrightarrow$ elimination
(11)	$(\leftrightarrow +)$	If $\Sigma, A \vdash B$, $\Sigma, B \vdash A$, then $\Sigma \vdash A \leftrightarrow B$.	$\leftrightarrow$ introduction

Observation: Each of these rules is not a single rule, but a scheme of rules, because Σ is any set of formulas, and A, B, C are any formulas.

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$(\in)$ If $A \in \Sigma$
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(2) $A, \Sigma' \vdash A \quad (by(+), (1))$
(That is, $\Sigma \vdash A$.)

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- At each of the steps, the rule appplied and the preceding steps concerned (if any) form a justification for this step, and are written on the right.

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- At each of the steps, the rule appplied and the preceding steps concerned (if any) form a justification for this step, and are written on the right.
- These steps are said to form a formal proof of the last line, $\Sigma \vdash A$.

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The following sequence
(1)$A \to B, B \to C, A \vdash A \to B \quad (by(\in))$
(2)$A \to B, B \to C, A \to A \quad (by(\in))$
(3)$A \to B, B \to C, A \vdash B \quad (by(\to -), (1), (2))$
(4)$A \to B, B \to C, A \vdash B \to C \quad (by(\in))$
(5)$A \to B, B \to C, A \vdash C \quad (by(\to -), (4), (3))$
(6)$A \to B, B \to C \vdash A \to C \quad (by(\to +), (5))$
consists of six steps.

At each step, one of the eleven rules or $(\in)$, which has just been proved, is applied.

On the right are written justifications for the steps.

These steps form a proof of $A \to B, B \to C \vdash A \to C$ which is generated at the last step.

Comments

Rules of formal deduction are only concerned with the syntactic structure of formulas.

For instance, from

$$(*) \quad \Sigma, \neg A \vdash B$$
$$(**) \quad \Sigma, \neg A \vdash \neg B$$

we can generate

$$(***) \quad \Sigma \vdash A$$

by applying $(\neg -)$.

- The premise Σ from $(***)$ is the Σ in the premises from $(*)$ and $(**)$.
- The conclusion A of $(***)$ results by deleting the leftmost $\neg$ of $\neg A$ in the premises of (*) and (**).
- The B of $(*)$ is an arbitrary formula.
- Therefore it can be checked mechanically whether the rules are used correctly.

Intuitive meaning of rules

- The elimination (introduction) of a connective means that one occurrence of this connective is eliminated (introduced) in the conclusion of the scheme of formal deducibility generated by the rule.
- Remark: In $(\vee -)$ it is the $\vee$ between A and B in $A \vee B$ that is eliminated in the conclusion C.
- $(\neg\ -)$ expresses the method of indirect proof or proof by contradiction: if a contradiction (denoted by B and $\neg B$) follows from certain premises (denoted by Σ) with an additional supposition that a certain proposition does not hold (denoted by $\neg A$), then this proposition is deducible from the premises (denoted by $\Sigma \vdash A$.)

Intuitive meaning of rules

- $(\vee -)$ expresses the method of proof by cases. If C follows from A and B separately, then C follows from "A or B".
- $(\to +)$ expresses that to prove "If A then B" from certain premises (denoted by $\Sigma \vdash A \to B$), it is sufficient to prove B from the premises together with A (denoted by $\Sigma, A \vdash B$).

Formal deducibility: Finding a proof

Definition. A is formally deducible from Σ, written as $\Sigma \vdash A$, iff $\Sigma \vdash A$ is generated by a (finite number of applications of) the rules of formal deduction. The sequence of rules generating $\Sigma \vdash A$ is called a formal proof.

• A scheme of formal deducibility may have various formal proofs. Perhaps one may not know how to construct a formal proof for it.

Formal deducibility: Checking a proof

• It is significant however that any proposed formal proof can be checked mechanically to decide whether it is indeed a formal proof of this scheme.

• This is done by checking 1) whether the rules of formal deduction are correctly applied, and 2) whether the last term of the formal proof is identical with this scheme.

• In this sense, rules of formal deduction and formal proofs serve to clarify the concepts of inference and proofs in informal reasoning.

Formal deducibility and formal proof

By definition, the terms in a formal proof should be generated by the rules of formal deduction. But in writing formal proofs we can use the demonstrated schemes of formal deducibility because they can be reduced to rules. Therefore the 11 rules are the 11 axioms of formal deduction, while the schemes we have proved are theorems.

Observations

- Tautological consequence ($\Sigma \models A$) and formal deducibility ($\Sigma \vdash A$) are different matters. The former belongs to semantics while the latter belongs to syntax.
- Both tautological consequence and formal deducibility are studied in the metalanguage by means of reasoning which is informal.
- $\models$ and $\vdash$ are not symbols in $\mathcal{L}^p$. They should not be confused with $\to$ which is a symbol in $\mathcal{L}^p$, a connective used for forming formulas..
- The connection between $\models$ and $\to$ is that $A \models B$ iff $A \to B$ is a tautology.
- The connection between $\vdash$ and $\to$ is that $A \vdash B$ iff $\emptyset \vdash A \to B$.

Formal deducibility: Complete definition

Definition (Formal deducibility). A formula A is formally deducible from Σ, written as $\Sigma \vdash A$, iff $\Sigma \vdash A$ is generated by (a finite number of applications of) the rules of formal deduction.

By the above definition, $\Sigma \vdash A$ holds iff there is a finite sequence

$$\Sigma_1 \vdash A_1$$

$$\dots$$

$$\Sigma_n \vdash A_n$$

such that each term $\Sigma_k \vdash A_k$ ($k = 1, \dots, n$) is generated by one rule of formal deduction, and $\Sigma_n \vdash A_n$ is $\Sigma \vdash A$ (that is, $\Sigma_n = \Sigma$ and $A_n = A$).

Comments

To say that $\Sigma_k \vdash A_k$ is generated by a rule of formal deduction, say $(\neg -)$, means that in the subsequence

$$\Sigma_1 \vdash A_1$$

$$\ldots$$

$$\Sigma_{k-1} \vdash A_{k-1}$$

which precedes $\Sigma_k \vdash A_k$, there are two terms

$$\Sigma_k, \neg A_k \vdash B,$$

$$\Sigma_k, \neg A_k \vdash \neg B$$

where B is an arbitrary formula.

Comments

In another example, if $\Sigma_k \vdash A_k$ is generated by $(\vee -)$, then there are in the subsequence preceding $\Sigma_k \vdash A_k$ two terms

$$\Sigma', B \vdash A_k,$$

$$\Sigma', C \vdash A_k$$

where B and C are arbitrary formulas such that $\Sigma', B \vee C = \Sigma_k$.

Formal proof: Complete definition

- The sequence

$$\Sigma_1 \vdash A_1$$
$$\dots$$
$$\Sigma_n \vdash A_n$$

  is called a formal proof. It is a formal proof of its last term $\Sigma_n \to A_n$.
- Now the significance of the word "formal" has been explained in full.
- The definition of formal deducibility is an inductive one. We may compare this definition with the definition of Form$(\mathcal{L}^p)$ to see that schemes of deducibility correspond to formulas, rules of formal deduction to formation rules.

Proving statements about formal deducibility

Statements concerning formal deducibility can be proved by induction on its complexity.

The basis of induction is to prove that $A \vdash A$, which is generated directly by rule (Ref), has a certain property.

The induction step is to prove that the other ten rules preserve this property

For instance, in the case of $(\vee -)$, we suppose

$$\Sigma, A \vdash C$$

$$\Sigma, B \vdash C$$

have the required property and show that

$$\Sigma, A \vee B \vdash C$$

also has this property.

Finiteness of premise set

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Proof. By induction on the complexity of $\Sigma \vdash A$.

Basis: The premise A of $A \vdash A$ generated by (Ref) is itself finite.

Induction Step: We distinguish ten cases. For each case, assume that the premises have the property and show that the corresponding conclusion has the property.

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Induction Step: We distinguish ten cases. For each case, assume that the premises have the property and show that the corresponding conclusion has the property.

Example: Case of $(\rightarrow -)$: "If $\Sigma \vdash A \rightarrow B$, $\Sigma \vdash A$, then $\Sigma \vdash B$."
By induction hypothesis, there exist finite subsets Σ_1 and Σ_2 of Σ such that $\Sigma_1 \vdash A \rightarrow B$ and $\Sigma_2 \vdash A$. By $(+)$ we have
$\Sigma_1, \Sigma_2 \vdash A \rightarrow B$ and $\Sigma_1, \Sigma_2 \rightarrow A$.
Then, by $(\rightarrow -)$, we have $\Sigma_1, \Sigma_2 \vdash B$, where Σ_1, Σ_2 is a finite subset of Σ.
The proof of the other cases is left as exercise.

Finitness of premise set

This theorem captures the intuition that, in a proof involving only finitely many steps, we can only use finitely many formulas in Σ.

Transitivity of deducibility

Theorem.
Let $\Sigma \subseteq \text{Form}(\mathcal{L}^p)$ and $A_1, A_2, \ldots, A_n$ be formulas in $\mathcal{L}^p$. If $\Sigma \vdash A_i$ for all $i = 1, \ldots n$ and $A_1, A_2, \ldots, A_n \vdash A$, then $\Sigma \vdash A$.

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Proof.

(1)	$A_1, \ldots A_n \vdash A$	by supposition
(2)	$A_1, \ldots, A_{n-1} \vdash A_n \to A$	by $(\to +), (1)$
(3)	$\emptyset \vdash A_1 \to (\ldots (A_n \to A) \ldots)$	analogous to (2)
(4)	$\Sigma \vdash A_1 \to (\ldots (A_n \to A) \ldots)$	by $(+), (3)$
(5)	$\Sigma \vdash A_1$	by supposition
(6)	$\Sigma \vdash A_2 \to (\ldots (A_n \to A) \ldots)$	by $(\to -), (4), (5)$
(7)	$\Sigma \vdash A_n \to A$	analogous to (6)
(8)	$\Sigma \vdash A_n$	by supposition
(9)	$\Sigma \vdash A$	by$(\to -), (7), (8)$

Remarks

- The theorem of transitivity of deducibility is denoted by (Tr).
- The sequence $A_1, \dots A_n$ in (Tr) is finite. Otherwise (3) in the above proof would be obtained by applying $(\rightarrow\ +)$ infinitely many times, contradicting the finiteness of a formal proof.
- The conclusion of a scheme of formal deducibility consists of one formula. When a number of schemes of formal deducibility have the same premises, we may write $\Sigma\ \vdash\ A_1, \dots, A_n$ for $\Sigma\ \vdash\ A_1, \dots, \Sigma\ \vdash\ A_n$.
- Thus, (Tr) may be written as:
  If $\Sigma\ \vdash\ A_1, \dots, A_n$,
  and $A_1, \dots, A_n\ \vdash\ A$,
  then $\Sigma\ \vdash\ A$.

Natural deduction

Since the rules of formal deduction (for propositional logic) express naturally and intuitively the rules of informal reasoning, the formal deduction based upon these rules is called natural deduction. There are other types of formal deduction, one of which will be introduced later.

Some useful theorems

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Proof.

$$\begin{array}{lll} (1) & \neg\neg A, \neg A \ \vdash \ \neg A & \text{by } (\in) \\ (2) & \neg\neg A, \neg A \ \vdash \ \neg\neg A & \text{by } (\in) \\ (3) & \neg\neg A \ \vdash \ A & \text{by } (\neg -), (1), (2). \end{array}$$

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Theorem If $\Sigma, A \vdash B$ and $\Sigma, A \vdash \neg B$, then $\Sigma \vdash \neg A$. (Reductio ad absurdum, $(\neg+)$).

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Proof

$$
\begin{array}{lll}
(1) & \Sigma, A \vdash B & \text{by supposition} \\
(2) & \Sigma^0, A \vdash B & \text{take finite } \Sigma^0 \subseteq \Sigma \\
(3) & \Sigma, \neg\neg A \vdash \Sigma^0 & \text{by } (\in) \\
(4) & \neg\neg A \vdash A & \text{by } (a) \\
(5) & \Sigma, \neg\neg A \vdash A & \text{by } (+), (4) \\
(6) & \Sigma, \neg\neg A \vdash B & \text{by (Tr) }, (3), (5), (2) \\
(7) & \Sigma, \neg\neg A \vdash \neg B & \text{analogous to}(6) \\
(8) & \Sigma \vdash \neg A & \text{by } (\neg-), (6), (7).
\end{array}
$$

Comments

- In (2) and (3) of the preceding proof, we used a finite subset Σ^0 to replace Σ because Σ may be infinite and accordingly not available in (Tr).
- Suppose $\Sigma^0 = C_1, \ldots C_n$. Then (3) consists of n steps

$$\Sigma, \neg\neg A \vdash C_1,$$

$$\ldots$$

$$\Sigma, \neg\neg A \vdash C_n.$$

  These can be written in one step because they are generated by the same rule ($\in$).
- The theorem of reductio ad absurdum is denoted by $(\neg+)$ and sometimes called $\neg$ introduction.

$(\neg-)$ and $(\neg+)$

- $(\neg+)$ and $(\neg-)$ are similar in shape but different in strength.
- $(\neg-)$ is stronger than $(\neg+)$.
- $(\neg+)$ has just been proved. But, if $(\neg-)$ is replaced by $(\neg+)$ in the rules, $(\neg-)$ cannot be proved.

Syntactical equivalence

For two formulas A and B we write

$$A \vdash\dashv B$$

for $A \vdash B$ and $B \vdash A$.

A and B are said to be syntactically equivalent iff $A \vdash\dashv B$ holds.

We write $\dashv$ to denote the converse of $\vdash$.

Lemma If $A \vdash\dashv A'$ and $B \vdash\dashv B'$ then

(1)$\neg A \vdash\dashv \neg A'$.
(2)$A \wedge B \vdash\dashv A' \wedge B'$.
(3)$A \vee B \vdash\dashv A' \vee B'$.
(4)$A \to B \vdash\dashv A' \to B'$.
(5)$A \leftrightarrow B \vdash\dashv A' \leftrightarrow B'$.

Note the resemblance to analogous results about tautological equivalences.

Replaceability of syntactically equivalent formulas in formal deduction, and other theorems

Theorem (Replacement of syntactically equivalent formulas)
If $B \vdash\dashv C$ and A' results from A by replacing some (not necessarily all) occurrences of B in A by C, then $A \vdash\dashv A'$.

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$A_1, A_2, \ldots A_n \vdash A$ iff $\emptyset \vdash A_1 \wedge \ldots \wedge A_n \to A$.

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Special cases

- When the premise is empty we have the special case $\emptyset \vdash A$ of formal deducibility.
- Obviously, $\emptyset \vdash A$ iff $\Sigma \vdash A$ for any Σ.
- It has been mentioned before that A is said to be formally provable from Σ when $\Sigma \vdash A$ holds.
- Now A is said to be formally provable when $\emptyset \vdash A$ holds.
- The laws of non-contradiction $\neg(A \wedge \neg A)$ and excluded middle $A \vee \neg A$ are instances of formally provable formulas.

Soundness and Completeness

- Mathematical logic is the study of reasoning; The (informal) deducibility relations between the premises and conclusions are established by their truth values.
- (Tauto)logical consequence, which is defined in terms of value assignments and truth values corresponds to (informal) deducibility and involves semantics.
- Formal deducibility, which is defined by a finite number of rules of formal deduction, is concerned with the syntactical structures of formulas and involves syntax.

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The Soundness Theorem.
The system of natural deduction based on the eleven given rules of formal deduction is sound.

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- The Soundness and Completeness Theorem says that with natural deduction (as defined by the 11 rules) we can prove
  The truth,
  the whole truth (completeness),
  and nothing but the truth. (soundness)

Another system of formal deduction

(1)	$A, B \vdash A \wedge B$	Law of combination
(2)	$A \wedge B \vdash B$	Law of simplification
(3)	$A \wedge B \vdash A$	Var. of law of simplification
(4)	$A \vdash A \vee B$	Law of addition
(5)	$B \vdash A \vee B$	Var. of law of addition
(6)	$A, A \to B \vdash B$	Modus ponens
(7)	$\neg B, A \to B \vdash \neg A$	Modus tollens
(8)	$A \to B, B \to C \vdash A \to C$	Hypothetical syllogism
(9)	$A \vee B, \neg A \vdash B$	Disjunctive syllogism
(10)	$A \vee B, \neg B \vdash A$	Var. of disjunctive syllogism
(11)	$A \to B, \neg A \to B \vdash B$	Law of cases
(12)	$A \leftrightarrow B \vdash A \to B$	Equivalence elimination
(13)	$A \leftrightarrow B \vdash B \to A$	Var. of equivalence elimination
(14)	$A \to B, B \to A \vdash A \leftrightarrow B$	Equivalence introduction
(15)	$A, \neg A \vdash B$	Inconsistency law

Exercise

Exercise

Assuming that we are in the system of natural deduction, prove the laws (1) - (15) on the preceding slide.

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Consequently, $p \vee \neg p$ cannot be derived if the rules of deduction are restricted those listed to rules (1) - (15) from the new system.

Deduction Law

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If

$$B, A_1, A_2, A_3, \ldots \vdash C$$

then

$$A_1, A_2, A_3, \ldots \vdash B \to C.$$

Together with this law, the new system is complete.

The deduction law

To prove $A \to B$ in mathematics one often uses the following informal argument.

1. Assume A, and add A to the premises.
2. Prove B, using A if necessary.
3. Discharge A, which means that A is no longer necessarily true, and write $A \to B$.

Example

Example. A couple has a boy, and they are expecting a second child. Prove that if the second child is a girl, then the couple has a girl and a boy.

Solution: Let p be "the first child is a boy" and q be "the second child is a girl".

We want to prove $q \to p \wedge q$, given that the premise is p. According to the method under discussion, this can be done as follows:
1. p is true: the couple has a boy.
2. Assume q: that is, assume that the second child is a girl.
3. From p and q, conclude $p \wedge q$ by the deduction rule of $\wedge$ introduction.
4. At this stage, one is allowed to conclude that $q \to p \wedge q$.

q can be now discharged; that is, $q \to p \wedge q$ is true even if q turns out to be false: in this case, $q \to p \wedge q$ is trivially true.

Example contd.

The reason is clear why this proof pattern holds. When proving $A \to B$, one only needs to consider the case where A is true: if A is false, $A \to B$ is trivially false.

If A is true, then it may be added to the premises.

This shows the soundness of the procedure.

Essentially, the argument states that an assumption may be converted into an antecedent of a conditional.