Adequate set of connectives

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- A remarkable property of the standard set of connectives $(\neg, \wedge, \vee, \to, \leftrightarrow)$ is the fact that for every truth table

p	q	$\dots$	
1	1	$\dots$	*
			*
			*
0	0	$\dots$	*

  there is a formula (depending on the variables $p, q, \dots$ and using only the standard connectives) that has exactly this truth table.

- For every truth table there is Boolean function $f(p, q, \dots)$ with exactly this truth table.

Post's observation

- Any set of connectives with the capability to express all truth tables is said to be adequate.
- As Post (1921) observed that the standard connectives are adequate.

Emil Post, 1897-1954

Proving adequacy of a connective set

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- Formulas $A \to B$ and $\neg A \vee B$ are tautologically equivalent.
- Then $\to$ is definable in terms of (or is reducible to or can be expressed in terms of) $\neg$ and $\vee$.
- Similarly, $\vee$ is definable in terms of $\neg$ and $\to$ because $A \vee B$ is tautologically equivalent to $\neg A \to B$.

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Proof.

For any formulas A, B,

$$\neg A \models\!\!\!\dashv \neg A$$

$$A \wedge B \models\!\!\!\dashv A \wedge B$$

$$A \vee B \models\!\!\!\dashv A \vee B$$

$$A \to B \models\!\!\!\dashv \neg A \vee B$$

$$A \leftrightarrow B \models\!\!\!\dashv (A \to B) \wedge (B \to A).$$

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Corollary $\{\neg, \wedge\}$, $\{\neg, \vee\}$, and $\{\neg, \rightarrow\}$ are adequate.

One connective is enough

Schroder showed in 1880 that each of the standard connectives is definable in terms of a single binary connective $\downarrow$, where the truth table associated with $\downarrow$ is

p	q	$p \downarrow q$
1	1	0
1	0	0
0	1	0
0	0	1

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$$\begin{aligned} \neg p &\mathrel{\models\!\!\!\dashv} p \downarrow p \\ p \wedge q &\mathrel{\models\!\!\!\dashv} (p \downarrow p) \downarrow (q \downarrow q) \end{aligned}$$

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$$\begin{array}{rcl} \neg p & \models\mid & p \downarrow p \\ p \wedge q & \models\mid & (p \downarrow p) \downarrow (q \downarrow q) \\ p \vee q & \models\mid & (p \downarrow q) \downarrow (p \downarrow q) \end{array}$$

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Thus it follows that a single connective $\downarrow$ is adequate.

Consequently, to test a given $\mathcal{S}$ for being adequate it suffices to test if $\downarrow$ can be expressed by $\mathcal{S}$.

Sheffer stroke

In 1913 Sheffer showed that the Sheffer stroke $|$ with associated truth table

p	q	$p \mid q$
1	1	0
1	0	1
0	1	1
0	0	1

is also a single binary connective in terms of which the standard connectives can be expressed.

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Proof. To see this, note that a formula depending on only one variable and which uses only the connective $\wedge$ has the property that its truth value for a value assignment that makes $p = 0$ is always 0.

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In order to define the negation $\neg p$ in terms of $\wedge$, there should exist a formula f depending on the variable p and using only the connective $\wedge$ such that $\neg p \models\!\!\!\!\dashv f$.

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In order to define the negation $\neg p$ in terms of $\wedge$, there should exist a formula f depending on the variable p and using only the connective $\wedge$ such that $\neg p \models\!\!\!\!\!\dashv f$.

However, for a value assignment v such that $v(\neg p) = 1$, we have $v(p) = 0$ and therefore $v(f) = 0$, which shows that $\neg p$ and f cannot be tautologically equivalent.

A ternary connective

Let us use the symbol τ for the ternary connective whose truth table is given by

p	q	r	τ
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

A ternary connective

It is easy to see that for any value assignment v we have

$$v(\tau(p,q,r)) = v(q) \text{ if } v(p) = 1 \text{ and } v(r) \text{ if } v(p) = 0.$$

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This is the familiar if-then-else connective from computer science, namely

if p then q else r

Comments

We can consider now propositional logic based not upon the five common connectives, but upon any adequate set of connectives, for instance $\{\neg, \wedge\}$.

Let $\mathcal{L}_0^p$ be a sublanguage of $\mathcal{L}^p$ obtained by deleting from $\mathcal{L}^p$ the three connectives $\vee, \rightarrow, \leftrightarrow$, and let $\mathsf{Form}(\mathcal{L}_0^p)$ be the set of formulas of $\mathcal{L}_0^p$.

Theorem. $\mathsf{Form}(\mathcal{L}_0^p) = \mathsf{Form}(\mathcal{L}^p)$.

Proof

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Obviously, Form($\mathcal{L}_0^p$) $\subseteq$ Form($\mathcal{L}^p$).
Conversely, for $A \in$ Form($\mathcal{L}^p$) we define (by recursion) its translation A_0 into $\mathcal{L}_0^p$ as follows:
$A_0 = A$ for atomic variable A,
$(\neg A)_0 = \neg A_0$,
$(A \wedge B)_0 = A_0 \wedge B_0$,
$(A \vee B)_0 = \neg(\neg A_0 \wedge \neg B_0)$,
$(A \rightarrow B)_0 = \neg(A_0 \wedge \neg B_0)$,
$(A \leftrightarrow B)_0 = (A \rightarrow B)_0 \wedge (B \rightarrow A)_0 =$
$= \neg(A_0 \wedge \neg B_0) \wedge \neg(\neg A_0 \wedge B_0)$