Propositional Calculus

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Contents

- What is a logical argument?
- Some important logical arguments
- Propositions
- Logical connectives

Logical arguments, valid or invalid

- Logic is the analysis and appraisal of arguments.
- An argument is a set of statements consisting of premises and a conclusion.
- An argument here isn't a quarrel or fight. Rather it is the verbal expression of a reasoning process.
- Consider this argument about the Cuyahoga River:
  No pure water is burnable.
  Some Cuyahoga River water is burnable.

  Some Cuyahoga River water is not pure.
  (the horizontal line is short for *therefore*.)

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  Some Cuyahoga River water is not pure.
  (the horizontal line is short for *therefore*.)

The argument is valid. A valid (correct, sound) argument is one in which it would be contradictory for the premises to be true but the conclusion false.

Logical arguments, valid or invalid

- Logic studies forms of reasoning.
- The content might deal with anything - water purity, mathematics, cooking, nuclear physics, ethics, or whatever.
- When we learn logic, we are learning tools of reasoning that can be applied to any subject.
- Let us take another argument:
  No pure water is burnable.
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  Some Cuyahoga River water is pure water.

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- Let us take another argument:
  No pure water is burnable.
  Some Cuyahoga River water is not burnable.

  Some Cuyahoga River water is pure water.

This argument is invalid. (The whole Cuyahoga river could be polluted by non-burnables.)

Logical arguments

Example of a logical argument (A)

1. If the demand rises, then companies expand.
2. If companies expand, then companies hire workers.

---

3. If the demand rises, then companies hire workers.

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One can argue against the conclusions and claim that they are wrong. However, as soon as the premises are accepted, the conclusion must be accepted also.
The conclusion logically follows from the premises and, therefore, the argument is valid (sound).

Logical arguments

Example of logical argument (B)

1. This computer program has a bug, or the input is erroneous.
2. The input is not erroneous.

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3. This computer program has a bug.

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Compound statements consist of several parts, each of which is a statement in its own right.

• Example (A): “demand rises”, “companies expand”, connected by if, then
• Example (B): “this computer program has a bug”, “the input is erroneous” connected by or.

Some important logical arguments

To see which arguments are correct and which not, Aristotle abbreviated the essential statements by substituting letters p, q, r.

The letter p may express the statement that "demand rises",
The letter q may express the statement "companies expand",
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Then the logical argument (A) becomes:
1. If p then q.
2. If q then r.

---

3. If p then r.

This argument is called a hypothetical syllogism.

Some important logical arguments

1. p or q.
2. Not q.

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3. p.

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1. If p then q.
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This argument is called modus ponens.

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- p, q, r are called propositional variables.
- True and false (or 1 and 0) are propositional constants.
- Any propositional variable can be assigned the value 0 or 1.

Propositions

- Propositional variables and propositional constants are atomic propositions, that is, they cannot be further subdivided
- Compound propositions are obtained by combining several atomic propositions
- the function of the words “or”, “and”, “not” is to combine propositions, and they are therefore called logical connectives.

Propositions and truth tables

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- All nonatomic propositions are called compound propositions.
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How to calculate the truth value of a compound proposition?

Definition A truth table of a proposition gives the truth values of the proposition under all possible assignments.

Logical connectives

Statements formulated in natural languages are frequently ambiguous because the words can have more than one meaning. We want to avoid this. Therefore we introduce new mathematical symbols to take the role of connectives.

Convention: Stating a proposition in English implies that this proposition is true.

"it is true that cats eat fish" = "cats eat fish".

Similarly, if p is a proposition, then "p" means "p is true" or that "p holds".

Negation

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Truth table for negation

p	$\neg p$
1	0
0	1

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Truth table for conjunction

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
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Observations

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- Sometimes we use words other than "and" to denote a conjunction such as but, in addition to, and moreover.
- Not all instances of the word "and" denote conjunctions.
  Example: The word "and" in " Jack and Jill are cousins" is not conjunction at all!

Disjunction

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Truth table for disjunction

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Observations on disjunction

The English word "or" has two different meanings.

- Exclusive or
  "You can either have soup or salad" $=$
  soup or salad, but not both
- Inclusive or
  "The computer has a bug, or the input is erroneous"
- To avoid ambiguity one should translate $p \vee q$ as
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  "p or q or both"

Note: When performing the disjunction of two sentences, always make sure that the sentences are complete: each sentence must have its own subject and predicate.
"There was an error on line 15 or 16" must first be expanded to
"There was an error on line 15, or there was an error on line 16"

Conditional

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- $p \longrightarrow q$ is called the conditional of p and q.
- The conditional of p and q may be translated into English by using the "If...then" construct, as in "If p, then q", or to " It is not the case that p is true and q is false"
- $p \longrightarrow q$ means that, whenever p is correct, so is q.
- p is called antecedent, q is called consequent

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Definition. Let p and q be two propositions. Then $p \to q$ is false if p is true and q is false, and $p \to q$ is true otherwise.

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The truth table for conditional

p	q	$p \to q$
1	1	1
1	0	0
0	1	1
0	0	1

Conditional: observations

Generally, whenever p is false, "if p then q" is vacuously true, since in such case the verification of "if p then q" does not require doing anything to deduce q from p.

Although unusual, it yields no inconsistency with everyday speech.

Example: "If you climb Mount Everest, I will eat my hat."

My statement will never be contradicted (it is true) because I know that "You will climb Mount Everest" is false.

The meaning of conditional

" If a bottle contains acid, it carries a warning label"

p : "The bottle contains acid."
q : "The bottle carries a warning label."

What happens if a bottle does not contain acid, i.e., $p = 0$?

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It might not have a label because it contains orange juice ($q = 0$).

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It might not have a label because it contains orange juice ($q = 0$).

In either case, the statement $p \to q$ is not contradicted.

The meaning of conditional contd.

" If a bottle contains acid, it carries a warning label"

p : "The bottle contains acid."

q : "The bottle carries a warning label."

- One can say that $p \longrightarrow q$ translates into "p only if q", as in "The bottle contains acid only if it carries a warning label". (If the bottle does not have a warning label it could not have contained acid. Remember that we work under the assumption that $p \longrightarrow q$ is true.)

The meaning of conditional contd.

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- The warning label is a necessary condition for the bottle to contain acid. (The fact that the bottle has a warning label is necessary for the bottle to have contained acid because, if the bottle would not have a warning label it could not have contained acid.)

The meaning of conditional contd.

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- The warning label is a necessary condition for the bottle to contain acid. (The fact that the bottle has a warning label is necessary for the bottle to have contained acid because, if the bottle would not have a warning label it could not have contained acid.)
- The fact that the bottle contains acid is a sufficient condition for it to carry a warning label.

Conditional

- The following are logically equivalent:

1. $p \longrightarrow q$.

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7. q if p.

Conditional

- The following are logically equivalent:

1. $p \rightarrow q$.
2. If p then q.
3. Whenever p, then q.
4. p is sufficient for q.
5. p only if q.
6. p implies q.

7. q if p.
8. q whenever p.

Conditional

- The following are logically equivalent:

1. $p \to q$.
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7. q if p.
8. q whenever p.
9. q is necessary for p.
10. q is implied by p.

Conditional

Example: Try understanding the equivalence of the statements from the previous slide using the example wherein p means n is divisible by 6 and q means n is divisible by 3.

The word "only if" must be translated as "$\rightarrow$".

However, the word "if" corresponds to "$\leftarrow$".

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Truth table for biconditional

p	q	$p \leftrightarrow q$
1	1	1
1	0	0
0	1	0
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Sorting out equivalence and implication

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This sentence looks like an equivalence, but if we swap the sentence around, we can see that something is wrong.

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This sentence looks like an equivalence, but if we swap the sentence around, we can see that something is wrong.
Aiding the destruction of the rainforest is equivalent to eating hamburgers at a fast-food bar.

In fact, the intended meaning is implication:
If one eats hamburgers at a fast-food bar then one is aiding the destruction of the world's rainforest.

Ambiguity and imprecision

Logic helps to clarify the meanings of descriptions written, for example, in English. After all, one reason for our use of logic is to state precisely the requirements of computer systems.

Descriptions in natural languages can be imprecise and ambiguous.

- An ambiguous sentence can have more than one distinct meaning.
- In contrast, an imprecise or vague sentence has only one meaning, but, as a proposition, the distinction between the circumstances under which is true and the circumstances under which it is false is not clear-cut.

Ambiguous sentences: Examples

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  This may have two meanings: I know a much funnnier man than Bill does, or I know a much funnier man than Bill is.

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- I know a much funnier man than Bill.
  This may have two meanings: I know a much funnnier man than Bill does, or I know a much funnier man than Bill is.
- Don't leave animals in cars because they rapidly turns into ovens.(From *News Quiz*, BBC Radio 4, 10 October, 1994). The immediate reading is far from the intended meaning.

Imprecise sentences: Examples

- John is tall.
  We do not know exactly what tall means. A more precise description is John is over 2 meters tall.

Imprecise sentences: Examples

- John is tall.
  We do not know exactly what tall means. A more precise description is John is over 2 meters tall.
- This computer is fast.
  The meaning of "fast" is imprecise - fast compared to what? A more precise description would be This computer executes 2 million instructions per second.

Dealing with imprecision and ambiguity

- An ambiguous sentence usually has several interpretations. Ambiguity has to be eliminated by querying the author of the sentence or by examining the context.
- Imprecision or vagueness arises from the use of qualitative descriptions. Often we need to introduce some quantitative measure to remove vagueness.

Further remarks on connectives

- $\neg$ is the only unary connective, that is, $\neg p$ negates a single proposition.
- All other connectives are binary connectives (they require two propositions which are joined by the connective)
- The binary connectives $\vee, \wedge, \leftrightarrow$ are symmetric in the sense that the order of the two propositions joined by the connective does not affect the truth value of the resulting propositions.
  The truth value of $p \wedge q$ is the same as the truth value of $q \wedge p$.
- The connective $\rightarrow$ is not symmetric: $p \rightarrow q$ and $q \rightarrow p$ have different truth values.