Propositional Language - Semantics

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How to prove arguments are valid?

Logical argument:

$$\frac{\begin{array}{c}\text{Premise 1}\\ \text{Premise 2}\\ \dots\\ \text{Premise } n\end{array}}{\text{Conclusion}}$$

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Logical argument:

Premise 1

Premise 2

...

Premise n

Conclusion

Logical arguments can be

- Correct (valid, sound)
- Incorrect (invalid, unsound)

Examples of arguments

Example 1

If the sun is shining I will go to the beach.

The sun is shining.

I do not go to the beach.

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It is not night

It is day
This is a valid argument.

Proving validity using tautologies

- A logical argument is sound if the conclusion logically follows from the premises.
- If all premises are true (i.e. the conjuction of all the premises yields true), then the conclusions must also be true.
- Hence, if the conjunction of the premises is A and if the conclusion is C, then $A \rightarrow C$ must be true under all assignments: it must be a tautology.

Example

This idea is used in the table below to prove that disjunctive syllogism is sound. For the proof, one only has to establish that $((p \vee q) \wedge \neg p) \to q)$ is always true, which can be seen from the last column of the truth table.

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p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$((p \vee q) \wedge \neg p) \rightarrow q)$
1	1	1	0	0	1
1	0	1	0	0	1
0	1	1	1	1	1
0	0	0	1	0	1

Example: Show that from $p \wedge q$ and $p \wedge q \rightarrow \neg r$ one may conclude $\neg r$.

Proving validity using contradictions

- Contradictions are related to tautologies. If A is a tautology the $\neg A$ is a contradiction.
- Contradictions may be used to prove that logical arguments are sound. To do this, note that an argument cannot be sound if all premises are true, yet the conclusion is false.
- In other words, it is impossible that the negation of the conclusion and the premises are all true simultaneously.
- The conjunction of all premises with the negation of the conclusion must therefore always be false, that is, it must be a contradiction.

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The table below shows that this is indeed the case, demonstrating once more that the disjunctive syllogism is sound.

p	q	$p \vee q$	$\neg p$	$(p \vee q) \wedge \neg p$	$\neg q$	$(p \vee q) \wedge \neg p \wedge \neg q$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	1	1	0	0
0	0	0	1	0	1	0

Argument validity and deducibility

Suppose $A_1, A_2, \ldots, A_n$ and A are propositions. Deductive logic studies whether

$$A \text{ is deducible from } A_1, \ldots A_n$$

(that is, the truth of $A_1, \ldots A_n$ implies that of A.).

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Supposing the propositions are expressed by formulas, the question raised is what kind of relation between $\{A_1, \ldots A_n\}$ and A corresponds to the deducibility relation?

Validity and deducibility

Formulas are assigned values by value assignments. This makes us consider the following relation between $\{A_1, \ldots A_n\}$ and A, to correspond to the notion of deducibility:

If a value assignment assigns truth to each of $A_1, \ldots A_n$ then it assigns truth to A.

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However, since different value assignments may assign different truth values to the same formula, this relation should be stated more precisely by requiring the value assignment to be arbitrary.

Tautological consequence

Definition Suppose $\Sigma \subseteq$ Form($\mathcal{L}^p$) and $A \in$Form($\mathcal{L}^p$).
A is a tautological consequence of Σ (that is, of the formulas in Σ), written as $\Sigma \models A$, iff for any value assignment v, we have that $v(\Sigma) = 1$ implies $v(A) = 1$.

Notes

- $\models$ is not a symbol of the formal propositional language and $\Sigma \models A$ is not a formula.
- $\Sigma \models A$ is a statement (in the metalanguage) about Σ and A.
- We write $\Sigma \not\models A$ for "not $\Sigma \models A$".

Special case of tautological consequence

When Σ is the empty set, we obtain the important special case $\emptyset \models A$ of tautological consequences.

By definition, $\emptyset \models A$ means

(1) For any value assignment v if $v(\emptyset) = 1$ then $v(A) = 1$.

where $v(\emptyset) = 1$ means

(2) For any B if $B \in \emptyset$ then $v(B) = 1$.

Because $B \in \emptyset$ is false, (2) is vacuously true.

Hence, "if $v(\emptyset) = 1$ then $v(A) = 1$" in (1) is equivalent to $v(A) = 1$.

Thus, $\emptyset \models A$ means that A is a tautology.

Intuitively speaking, $\Sigma \models A$ means that the truth of the formulas in Σ is the sufficient condition of the truth of A. Since $\emptyset$ consists of no formula, $\emptyset \models A$ means that the truth of A is unconditional, hence A is a tautology.

Tautological equivalence

For two formulas we write

$$A \models\mid B$$

to denote "$A \models B$ and $B \models A$."

A and B are said to be tautologically equivalent (or simply equivalent) iff $A \models\mid B$ holds.

Tautologically equivalent formulas are assigned the same truth values by any value assignment.

Tautological consequence and valid arguments

Let $\Sigma = \{A_1, A_2, \ldots, A_n\}$ be a set of formulas (premises) and A be a formula (conclusion). The following are equivalent:

- The argument with premises $A_1, A_2, \ldots, A_n$ and conclusion A is valid.
- $(A_1 \wedge A_2 \wedge \ldots \wedge A_n) \to A$ is a tautology.
- $(A_1 \wedge A_2 \ldots \wedge A_n \wedge \neg A)$ is a contradiction.
- A is a tautological consequence of Σ, i.e.

$$\{A_1, A_2, \ldots, A_n\} \models A.$$

If A and B are formulas and $A \models B$ we say that A (tauto)logically implies B.

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If A and B are two formulas and if A and B have the same truth value they are said to be “(tauto)logically equivalent”. In other words, $A \models\!\mid B$ if and only if $A \leftrightarrow B$ is a tautology.

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Note that "(tauto)logically equivalent" is different from "equivalent".

Connections

- A is a tautology if $A \models\mid 1$.
- A is a contradiction if $A \models\mid 0$.
- If A is a tautology then $1 \models A$.
- If A is a contradiction then $A \models 0$.
- If $A \models\mid B$ then $A \models B$ and $B \models A$. Hence, each equivalence can be used to derive two logical implications.
  Conversely, if $A \models B$ and $B \models A$ one can always conclude that $A \models\mid B$.

Proving tautological consequence by truth tables

To prove a tautological consequence $\Sigma \models A$ we must show that any value assignment v satisfying Σ also satisfies A. One way to show this is by using truth tables.

Example

Show that $\{A \to B, B \to C\} \models (A \to C)$

Proof by using truth tables

A	B	C	$A \to B$	$B \to C$	conj.	$A \to C$
0	0	0	1	1	1	1
0	0	1	1	1	1	1
0	1	0	1	0	0	1
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	0	1	0	1
1	1	0	1	0	0	0
1	1	1	1	1	1	1

The value assignments in rows 1, 2, 4, 8 are the only ones that satisfy $\Sigma = \{A \to B, B \to C\}$. For these value assignments also $A \to C$ is satisfied. Hence $\Sigma \models (A \to C)$

Tautological consequence and argument validity

The proof on the previous slide shows that
$\{A \to B, B \to C\} \models (A \to C)$,
which means that the argument

$$\begin{array}{l} A \to B \\ B \to C \\ \hline A \to C \end{array}$$

is a valid argument.

General proof method for tautological consequence

Proof by contradiction.

Caution - "Proof by contradiction" is different from proof using contradictions, wherein the word "contradiction" is used with the meaning of a logical formula which is always false.

Suppose $\{A \to B, B \to C\} \not\models (A \to C)$. That is, there is a value assignment v such that
(1) $v(A \to B) = 1$,
(2) $v(B \to C) = 1$,
(3) $v(A \to C) = 0$.

General proof method, contd.

By (3) we have
(4) $v(A) = 1$.
(5) $v(C) = 0$.
By (1) and (4) we have $v(B) = 1$.
From $v(B) = 1$ and (2) we have $v(C) = 1$ which contradicts (5). As we reached a contradiction, our assumption must have been false, hence the tautological consequence is proved.

Comment on example

We began with (3) because we can deduce (4) and (5) from it. Thus the tautological consequence is proved.
If we would have begun with (1), and deduce from it that "$v(A) = 1$ and $v(B) = 1$" or "$v(A) = 0$ and $v(B) = 1$" or "$v(A) = 0$ and $v(B) = 0$", then the proof would be more complicated.

Proving an argument invalid - general method

To refute $\Sigma \models A$ (that is, to prove $\Sigma \not\models A$) we must construct a value assignment v satisfying Σ but not satisfying A.

Prove that $\{(A \to \neg B) \vee C, B \wedge \neg C, A \leftrightarrow C\} \not\models (\neg A \wedge (B \to C))$.

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Prove that $\{(A \to \neg B) \vee C, B \wedge \neg C, A \leftrightarrow C\} \not\models (\neg A \wedge (B \to C))$.

Proof.

Let v a value assignment such that $v(A) = 0$, $v(B) = 1$ and $v(C) = 0$.

Then we have

$v((A \to \neg B) \vee C) = 1$

$v(B \wedge \neg C) = 1.$

$v(A \leftrightarrow C) = 1$

$v(\neg A \wedge (B \to C)) = 0.$

which proves the statement.

Comment on example

It is convenient to first make v satisfy $v(B \wedge \neg C) = 1$, from which we get $v(B) = 1$ and $v(C) = 0$, then $v(A) = 0$ is obtained from $v(A \leftrightarrow C) = 1$.
Alternatively, we can use other empirical methods to find that value assignment which satisfies the set of premises and does not satisfy the conclusion.

(Tauto)logical equivalences

Consider the following two statements

He is either not informed, or he is not honest.

It is not true that he is informed and honest.

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It is not true that he is informed and honest.

Intuitively, these two statements are logically equivalent. We prove this now. Define p and q to be the statements that "he is honest" and that "he is well informed" respectively.

The first statement translates into $\neg p \vee \neg q$, whereas the second into $\neg(p \wedge q)$.

Tautological equivalences contd.

Claim. The two formulas are logically equivalent, that is, $\neg(p \wedge q) \models\!\!\!\!= (\neg p \vee \neg q)$ which means that $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ is a tautology.

Tautological equivalences contd.

Claim. The two formulas are logically equivalent, that is, $\neg(p \wedge q) \models\!\!\!\!\!\models (\neg p \vee \neg q)$ which means that $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ is a tautology.
(De Morgan's law)

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$	$\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$
1	1	1	0	0	1
1	0	0	1	1	1
0	1	0	1	1	1
0	0	0	1	1	1

How to negate a conjunction: take the disjunction of the negations of the conjuncts.

(Tauto)logical equivalences - contrapositives

Consider the following pair of statements:

If the goods were not delivered, the customer cannot have paid.

If the customer has paid, the goods must have been delivered.

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If q and p stand for "goods were delivered" and "customer paid" respectively, then these two statements translate into $\neg q \rightarrow \neg p$ and $p \rightarrow q$.

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If the customer has paid, the goods must have been delivered.

If q and p stand for "goods were delivered" and "customer paid" respectively, then these two statements translate into $\neg q \rightarrow \neg p$ and $p \rightarrow q$.

$\neg q \rightarrow \neg p$ and $p \rightarrow q$ are contrapositives of each other.

(Tauto)logical equivalences

The table below shows that contrapositives are equivalent, that is

$$\neg q \to \neg p \models\mathrel{\mkern-6mu}\dashv p \to q.$$

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$$\neg q \to \neg p \models\!\mid p \to q.$$

p	q	$\neg p$	$\neg q$	$\neg q \to \neg p$	$p \to q$
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	1	1
0	0	1	1	1	1

Sometimes in a proof it is easier to show that $\neg q \to \neg p$ instead of showing that $p \to q$.

(Tauto)logical equivalences - biconditional

The following is a very important logical equivalence.
(1) p and q have the same truth value
(2) If p, then q, and if q then p.
The second statement can be reworded as "p if and only if q.".
The first statement becomes $p \leftrightarrow q$ and the second one $(p \to q) \land (q \to p)$. The table below shows that these two formulas are equivalent.

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The first statement becomes $p \leftrightarrow q$ and the second one $(p \to q) \wedge (q \to p)$. The table below shows that these two formulas are equivalent.

p	q	$p \leftrightarrow q$	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
1	1	1	1	1	1
1	0	0	0	1	0
0	1	0	1	0	0
0	0	1	1	1	1

Proving a logical equivalence amounts to proving two logical implications

Tautological equivalences

If $A \models\!\!\!\!\dashv A'$ and $B \models\!\!\!\!\dashv B'$, then

1. $\neg A \models\!\!\!\!\dashv \neg A'$.
2. $A \wedge B \models\!\!\!\!\dashv A' \wedge B'$.
3. $A \vee B \models\!\!\!\!\dashv A' \vee B'$.
4. $A \rightarrow B \models\!\!\!\!\dashv A' \rightarrow B'$.
5. $A \leftrightarrow B \models\!\!\!\!\dashv A' \leftrightarrow B'$.

Replaceability of equivalent formula

Theorem. Let $B \models\!\!\!\dashv C$. If A' results from A by replacing some (not necessarily all) occurrences of B in A by C, then $A \models\!\!\!\dashv A'$.

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Proof. By induction on the complexity of A.

If $B = A$, then $C = A'$. We have thus $A = B \models\!\dashv C = A'$ and the theorem holds.

Basis. A is an atom. Then $B = A$; the theorem holds.

Induction step. A is of one of the five forms: $\neg A_1$, $A_1 \wedge A_2$, $A_1 \vee A_2$, $A_1 \to A_2$, or $A_1 \leftrightarrow A_2$.

Suppose $A = \neg A_1$. If $B = A$, the theorem holds as stated above. If $B \neq A$, then B is a segment of A_1.

Let A_1' be the formula resulting from A_1 by the replacement stated in the theorem. Then, $A' = \neg A_1'$. We have

$A_1 \models\!\dashv A_1'$ (by ind. hyp.),

$\neg A_1 \models\!\dashv \neg A_1'$ (by preceding lemma)

which implies $A \models\!\dashv A'$.

Proof contd.

Suppose $A = A_1 * A_2$. ($*$ denotes any one of $\wedge, \vee, \rightarrow, \leftrightarrow$).
If $B = A$, the theorem holds true as in the above case.
If $B \neq A$, then B is a segment of A_1 or A_2. Let A_1' and A_2' result respectively from A_1 and A_2 by the replacement stated in the theorem. Then $A' = A_1' * A_2'$. We have
$A_1 \models\!\!\!\dashv A_1'$, $A_2 \models\!\!\!\dashv A_2'$ (by ind. hyp.)
$A_1 * A_2 \models\!\!\!\dashv A_1' * A_2'$ (by preceding Lemma).
Consequently, $A \models\!\!\!\dashv A'$.
By the basis and induction step, the theorem is proved.

How to negate a formula

Theorem. Suppose A is a formula composed by atoms and the connectives $\neg, \vee, \wedge$ only, by the formation rules concerned. Suppose that $\Delta(A)$ results by replacing in A $\wedge$ with $\vee$, $\vee$ with $\wedge$, and each propositional variable with its negation. Then $A \models\!\mid \neg\Delta(A)$.

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Proof. By induction on the complexity of A.

Note: The above theorem provides a formal method to find the negation of a formula.

Example. Let $A = (p \wedge \neg q) \wedge (\neg r \wedge s)$. Find the negation of A, i.e. find $\neg A$.