Propositional Language - Syntax

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Propositional language

- By using connectives, we can combine propositions, whether they are atomic themselves or compound.
- To prevent ambiguity, we introduce fully paranthesized expressions that can be parsed in a unique way.
- When possible, we will omit parantheses and use precedence rules.
- We construct the propositional language $\mathcal{L}^p$, which is the formal language of the propositional logic.

Propositional language $\mathcal{L}^p$: syntax

Propositional language $\mathcal{L}^p$ is the formal language for propositional logic.
It consists of:

- propositional variables: p, q, r.
- propositional constants: 1 and 0 (true and false).
- logical connectives: $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$.

Oral reading of logical connectives

$\neg$	not	negation
$\wedge$	and	conjunction
$\vee$	or (and/or)	(inclusive) disjunction
$\rightarrow$	if, then (imply)	implication
$\leftrightarrow$	iff (equivalent to)	equivalence

- left and right parantheses : (,).

Expressions of $\mathcal{L}^p$

- expressions $=$ finite strings of symbols
  – $p, pq, (r), p\wedge \rightarrow q$ and $\neg(p \vee q)$ are expressions in $\mathcal{L}^p$.
- length of expression: the number of occurrences of symbols in it
  – for expressions above, length is $1, 2, 3, 4$ and 6
- empty expression $=$ expression of length zero, $\emptyset$
- two expressions u and v are equal iff they are of the same length and have the same symbols in the same order.
- scanning of expressions proceeds from left to right

Expressions

- concatenating two expressions u, v in this order is denoted uv. Note that $\emptyset u = u\emptyset = u$
- if $u = w_1 v w_2$ where u, v, w_1, w_2 are expressions, then v is a segment of u.
- If $v \neq u$ then v is a proper segment of u.
- If $u = vw$, where u, v, w are expressions, then v is an initial segment of u. Similarly, w is a terminal segment of u.

Atoms and formulas

- An atom (atomic formula) is an expression consisting of a proposition variable or constant only.
- An expression in $\mathcal{L}^p$ is called a formula (well-formed formula) iff it can be constructed according to the following formation rules:
  1. Every atomic formula is formula;
  2. If A is a formula then so is $(\neg A)$;
  3. If A, B are formulas then so are $(A \vee B)$, $(A \wedge B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$.
  4. No other expression is a formula.

Definition. (Negation, Conjunction, Disjunction, Implication, Equivalence).

- $(\neg A)$ is called a negation (formula). It is the negation of A.
- $(A \wedge B)$ is called a conjunction (formula). It is the conjunction of A and B. A and B are called the conjuncts of $(A \wedge B)$.
- $(A \vee B)$ is called a disjunction (formula). It is the disjunction of A and B. A and B are called the disjuncts of $(A \vee B)$.
- $(A \rightarrow B)$ is called an implication (formula). It is the implication of A and B. A and B are called the antecedent and consequent of $(A \rightarrow B)$.
- $(A \leftrightarrow B)$ is called an equivalence (formula). It is the equivalence of A and B.

Formulas

- The set of atomic formulas in $\mathcal{L}^p$ is denoted by Atom($\mathcal{L}^p$).
- The set of all formulas (well-formed formulas, wff) in $\mathcal{L}^p$ is denoted by Form($\mathcal{L}^p$).
- Form($\mathcal{L}^p$) is the smallest class of expressions of $\mathcal{L}^p$ closed under the formation rules of formulas.

Comments

- Every formula of $\mathcal{L}^p$ has the same number of occurrences of left and right parantheses.
- Any proper initial segment of a formula in $\mathcal{L}^p$ has more occurrences of left than right parantheses. Any proper terminal segment of a formula in $\mathcal{L}^p$ has less occurrences of left than right parantheses.
  Thus neither a proper initial segment nor a proper terminal segment of a formula can itself be a formula of $\mathcal{L}^p$.
- Every formula of $\mathcal{L}^p$ is of exactly one of the six forms: an atom, $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, or $(A \leftrightarrow B)$, and in each case it is of the form in exactly one way.

Scope

- If $(\neg A)$ is a segment of a formula C, then A is called the scope of the negation $\neg$ in the formula C.
- If $(A \wedge B)$ is a segment of a formula C, then A is called the left scope of the conjunction and B is called the right scope of the conjunction in the formula C.
- Similar definitions hold for left/right scopes of disjunction, implication and equivalence.

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The scopes of $\rightarrow$ are $(\neg p)$ and r.

Example 1 (generating formulas)

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The expression

$$((p \vee q) \rightarrow ((\neg p) \leftrightarrow (q \wedge r)))$$

is a formula. How is it generated?

Example 2 (parsing formulas)

If Michelle wins at the Olympics, everyone will admire her, and she will get rich, but if she does not win, all her effort was in vain.

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q: Everyone admires Michelle.
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The compound proposition becomes

$$((p \rightarrow (q \wedge r)) \wedge ((\neg p) \rightarrow s))$$

One can use parse trees to analyze formulas.

Precedence rules

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Examples:

- $\neg p \vee q$ is to be understood as $(\neg p) \vee q$.
- $p \wedge q \vee r$ is to be understood as $((p \wedge q) \vee r)$.
- $p \rightarrow q \vee r$ is to be understood as $p \rightarrow (q \vee r)$
- $p \leftrightarrow p \rightarrow q$ must be understood as $p \leftrightarrow (p \rightarrow q)$
- proposition about Michelle: $(p \rightarrow q \wedge r) \wedge (\neg p \rightarrow s)$

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Translation into propositional language: $\neg p \rightarrow \neg q$.

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$$p \leftrightarrow q \wedge r$$

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$$\neg p \vee (q \wedge r)$$

Converse and contrapositive

Definition. Given the proposition $p \to q$,

the converse of $p \to q$ is the proposition $q \to p$

the contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$.

Induction on the complexity of formulas

Theorem. Suppose R is a property. If

1. For any atomic formula $p \in$Atom$(\mathcal{L}^p)$, p has the property R.
2. For any formula A in Form$(\mathcal{L}^p)$, if A has property R, then $(\neg A)$ has the property R.
3. For any formulas A and B in Form$(\mathcal{L}^p)$, if A and B have property R then $(A * B)$ has property R, where $* \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$.

then, any formula in Form$(\mathcal{L}^p)$ has property R.