The Foundations: Logic and Proofs

Chapter 1, Part III: Proofs

Summary

- Proof Methods
- Proof Strategies

Introduction to Proofs

Section 1.7

Section Summary

- Mathematical Proofs
- Forms of Theorems
- Direct Proofs
- Indirect Proofs
  - Proof of the Contrapositive
  - Proof by Contradiction

Proofs of Mathematical Statements

- A *proof* is a valid argument that establishes the truth of a statement.
- Proofs have many practical applications:
  - verification that computer programs are correct
  - establishing that operating systems are secure
  - enabling programs to make inferences in artificial intelligence
  - showing that system specifications are consistent

Definitions

- A *theorem* is a statement that can be shown to be true using:
  - definitions
  - other theorems
  - *axioms* (statements which are given as true)
  - rules of inference
- A *lemma* is a 'helping theorem' or a result which is needed to prove a theorem.
- A *corollary* is a result which follows directly from a theorem.
- Less important theorems are sometimes called *propositions*.
- A *conjecture* is a statement that is being proposed to be true. Once a proof of a conjecture is found, it becomes a theorem. However, a conjecture may turn out to be false.

Forms of Theorems

- Many theorems assert that a property holds for all elements in a domain, such as the integers, the real numbers, or some of the discrete structures that we will study in this class.
- Often the universal quantifier (needed for a precise statement of a theorem) is omitted by standard mathematical convention.

For example, the statement:

"If $x > y$, where x and y are positive real numbers, then $x^2 > y^2$ "

really means

"For all positive real numbers x and y, if $x > y$, then $x^2 > y^2$."

Proving Theorems

- Many theorems have the form:

$$\forall x (P(x) \rightarrow Q(x))$$

- To prove them, we show that where *c* is an arbitrary element of the domain, $P(c) \rightarrow Q(c)$
- So, we must prove something of the form: $p \rightarrow q$

Proving Conditional Statements: $p \to q$
(0) The easy cases

- *Trivial Proof*: If we know q is true, then
  $p \to q$ is true as well.

"If it is raining then 1=1."

- *Vacuous Proof*: If we know p is false then
  $p \to q$ is true as well.

"If I am both rich and poor then $2 + 2 = 5$."

[Even though these examples seem silly, both trivial and vacuous proofs are often used in mathematical induction, as we will see in Chapter 5)]

Even and Odd Integers

Definition: The integer n is even if there exists an integer k such that $n = 2k$, and n is odd if there exists an integer k, such that $n = 2k + 1$. Note that every integer is either even or odd and no integer is both even and odd.

We will need this basic fact about the integers in some of the example proofs to follow.

Proving Conditional Statements: $p \rightarrow q$
(1) Direct proof

- *Direct Proof*: Assume that p is true. Use rules of inference, axioms, and logical equivalences to show that q must also be true.

Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Proving Conditional Statements: $p \rightarrow q$

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Example: Give a direct proof of the theorem "If n is an odd integer, then n^2 is odd."

Solution: Assume that n is odd. Then $n = 2k + 1$ for an integer k. Squaring both sides of the equation, we get:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2r + 1,$$

where $r = 2k^2 + 2k$, an integer.

We have proved that if n is an odd integer, then n^2 is an odd integer. ◀

(◀ marks the end of the proof. Sometimes **QED** is used instead.)

Proving Conditional Statements: $p \rightarrow q$

Definition: The real number r is *rational* if there exist integers p and q where $q \neq 0$ such that $r = p/q$

Example: Prove that the sum of two rational numbers is rational.

Proving Conditional Statements: $p \rightarrow q$

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Example: Prove that the sum of two rational numbers is rational.

Solution*:* Assume *r* and *s* are two rational numbers. Then there must be integers *p, q* and also *t, u* such that

$$r = p/q, \quad s = t/u, \quad u \neq 0, \; q \neq 0$$

$$r + s = \frac{p}{q} + \frac{t}{u} = \frac{pu + qt}{qu} = \frac{v}{w}$$ where $v = pu + qt$, $w = qu \neq 0$

Thus the sum is rational. ◀

Proving Conditional Statements: $p \rightarrow q$
(2) Proof by Contraposition

- *Proof by Contraposition*: Assume $\neg q$ and show $\neg p$ is true also. This is sometimes called an *indirect proof* method. If we give a direct proof of $\neg q \rightarrow \neg p$ then we have a proof of $p \rightarrow$ q.

  Why does this work?

Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

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Example: Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Solution: Assume n is even. So, $n = 2k$ for some integer k. Thus

$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1) = 2j$ for $j = 3k + 1$

Therefore $3n + 2$ is even. Since we have shown $\neg q \rightarrow \neg p$, $p \rightarrow q$ must hold as well. If n is an integer and $3n + 2$ is odd (not even), then n is odd (not even). ◀

Proving Conditional Statements: $p \to q$

Example: Prove that for an integer n, if n^2 is odd, then n is odd.

Proving Conditional Statements: $p \rightarrow q$

Example: Prove that for an integer n, if n^2 is odd, then n is odd.

Solution: Use proof by contraposition. Assume n is even (i.e., not odd). Therefore, there exists an integer k such that $n = 2k$. Hence,

$$n^2 = 4k^2 = 2\,(2k^2)$$

and n^2 is even(i.e., not odd).

We have shown that if n is an even integer, then n^2 is even. Therefore by contraposition, for an integer n, if n^2 is odd, then n is odd. ◀

Proving Conditional Statements: $p \rightarrow q$
(3) Proof by Contradiction

- *Proof by Contradiction*: (AKA *reductio ad absurdum*).

  To prove p, assume $\neg p$ and derive a contradiction such as $p \wedge \neg p$. (an indirect form of proof). Since we have shown that $\neg p \rightarrow \mathbf{F}$ is true , it follows that the contrapositive $\mathbf{T} \rightarrow p$ also holds.

Proof by Contradiction

- A preview of Chapter 4.

  Example: Use a proof by contradiction to give a proof that $\sqrt{2}$ is irrational.

Proof by Contradiction

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Example: Use a proof by contradiction to give a proof that $\sqrt{2}$ is irrational.

Solution: Suppose $\sqrt{2}$ is rational. Then there exists integers a and b with $\sqrt{2} = a/b$, where $b \neq 0$ and a and b have no common factors (see Chapter 4). Then

$$2 = \frac{a^2}{b^2} \qquad\qquad 2b^2 = a^2$$

Therefore a^2 must be even. If a^2 is even then a must be even (an exercise). Since a is even, $a = 2c$ for some integer c. Thus,

$$2b^2 = 4c^2 \qquad\qquad b^2 = 2c^2$$

Therefore b^2 is even. Again then b must be even as well.

But then 2 must divide both a and b. This contradicts our assumption that a and b have no common factors. We have proved by contradiction that our initial assumption must be false and therefore $\sqrt{2}$ is irrational .

Proof by Contradiction

- A preview of Chapter 4.

Example: Prove that there is no largest prime number.

Proof by Contradiction

- A preview of Chapter 4.

Example: Prove that there is no largest prime number.

Solution: Assume that there is a largest prime number. Call it p_n. Hence, we can list all the primes 2,3,.., p_n. Form

$$r = p_1 \times p_2 \times \ldots \times p_n + 1$$

None of the prime numbers on the list divides *r*. Therefore, by a theorem in Chapter 4, either *r* is prime or there is a smaller prime that divides *r*. This contradicts the assumption that there is a largest prime. Therefore, there is no largest prime.

◀

Theorems that are Biconditional Statements

- To prove a theorem that is a biconditional statement, that is, a statement of the form $p \leftrightarrow q$, we show that $p \rightarrow q$ and $q \rightarrow p$ are both true.

Example: Prove the theorem: "If n is an integer, then n is odd if and only if n^2 is odd."

Solution: We have already shown (previous slides) that both $p \rightarrow q$ and $q \rightarrow p$. Therefore we can conclude $p \leftrightarrow q$.

Sometimes *iff* is used as an abbreviation for "if an only if," as in

"If n is an integer, then n is odd iff n^2 is odd."

What is wrong with this?

“Proof” that *1 = 2. We use these steps, where a and b are two equal positive integers.*

Step	**Reason**
1. $a = b$	Premise
2. $a^2 = a \times b$	Multiply both sides of (1) by a
3. $a^2 - b^2 = a \times b - b^2$	Subtract b^2 from both sides of (2)
4. $(a-b)(a+b) = b(a-b)$	Algebra on (3)
5. $a + b = b$	Divide both sides by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$
7. $2 = 1$	Divide both sides of (6) by b

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Solution: Step 5. a - b = 0 by the premise and division by 0 is undefined.

Looking Ahead

- If direct methods of proof do not work:
  - We may need a clever use of a proof by contraposition.
  - Or a proof by contradiction.
  - In the next section, we will see strategies that can be used when straightforward approaches do not work.
  - In Chapter 5, we will see mathematical induction and related techniques.
  - In Chapter 6, we will see combinatorial proofs

Proof Methods and Strategy

Section 1.8

Section Summary

- Proof by Cases
- Without Loss of Generality
- Existence Proofs
  - Constructive
  - Nonconstructive
- Disproof by Counterexample
- Uniqueness Proofs
- Proof and Disproof
- Open Problems

Proof by Cases

- To prove a conditional statement of the form:

$$(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q$$

- Use the tautology

$$\begin{aligned}[(p_1 \vee p_2 \vee \ldots \vee p_n) \rightarrow q] \leftrightarrow \\ [(p_1 \rightarrow q) \wedge (p_2 \rightarrow q) \wedge \ldots \wedge (p_n \rightarrow q)]\end{aligned}$$

- Each of the implications $p_i \rightarrow q$ is a *case*.

Proof by Cases

Example: Let $a \,@\, b = \max\{a, b\} = a$ if $a \geq b$, and

$$a \,@\, b = \max\{a, b\} = b, \; otherwise.$$

Show that for all real numbers a, b, c

$$(a \,@ b) \,@\, c = a \,@\, (b \,@\, c)$$

(This means the operation @ is associative.)

Proof by Cases

Example: Let a @ b = max{a, b} = a if $a \geq b$, otherwise a @ b = max{a, b} = b.

Show that for all real numbers a, b, c

(a @b) @ c = a @ (b @ c)

(This means the operation @ is associative.)

Proof: Let a, b, and c be arbitrary real numbers.

Then one of the following 6 cases must hold.

1. $a \geq b \geq c$
2. $a \geq c \geq b$
3. $b \geq a \geq c$
4. $b \geq c \geq a$
5. $c \geq a \geq b$
6. $c \geq b \geq a$

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Proof by Cases

Case 1: $a \geq b \geq c$

$(a @ b) = a$, $a @ c = a$, $b @ c = b$

Hence $(a @ b) @ c = a = a @ (b @ c)$

Therefore the equality holds for the first case.

A complete proof requires that the equality be shown to hold for all 6 cases. But the proofs of the remaining cases are similar. Try them.

Without Loss of Generality

Example: Show that if x and y are integers and both $x \cdot y$ *and* $x+y$ are even, then both x and y are even.

Without Loss of Generality

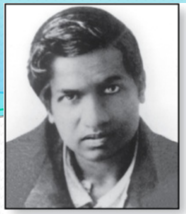
Example: Show that if x and y are integers and both $x \cdot y$ *and* $x+y$ are even, then both x and y are even.

Proof: Use a proof by contraposition. Suppose x and y are not both even. Then, one or both are odd. Without loss of generality, assume that x is odd. Then $x = 2m + 1$ for some integer k.

Case 1: y is even. Then $y = 2n$ for some integer n, so $x + y = (2m + 1) + 2n = 2(m + n) + 1$ is odd.

Case 2: y is odd. Then $y = 2n + 1$ for some integer n, so $x \cdot y = (2m + 1)\,(2n + 1) = 2(2m \cdot n + m + n) + 1$ is odd. ◀

We only cover the case where x is odd because the case where y is odd is similar. The use phrase *without loss of generality* (WLOG) indicates this.



Srinivasa Ramanujan
(1887-1920)

Existence Proofs

- Proof of theorems of the form $\exists x P(x)$.
- **Constructive** existence proof:
  - Find an explicit value of *c*, for which *P(c)* is true.

  - **Example**: Show that there is a positive integer that can be written as the sum of cubes of positive integers in two different ways:

Proof: 1729 is such a number since

$$1729 = 10^3 + 9^3 = 12^3 + 1^3 \quad \blacktriangleleft$$

Godfrey Harold Hardy
(1877-1947)

Nonconstructive Existence Proofs

Example: Show that there exist irrational numbers x and y such that x^y is rational.

Nonconstructive Existence Proofs

Example: Show that there exist irrational numbers x and y such that x^y is rational.

Proof: We know that $\sqrt{2}$ is irrational. Consider the number $\sqrt{2}^{\sqrt{2}}$.

* If it is *rational*, we have two irrational numbers x and y with x^y rational, namely $x = \sqrt{2}$ and $y = \sqrt{2}$.

* But, if $\sqrt{2}^{\sqrt{2}}$ is *irrational*, then we can let $x = \sqrt{2}^{\sqrt{2}}$ and $y = \sqrt{2}$ so that $x^y = (\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^{(\sqrt{2}\sqrt{2})} = \sqrt{2}^{2} = 2$.

◀

Counterexamples

- Recall $\exists x \neg P(x) \equiv \neg \forall x P(x)$.
- To establish that $\neg \forall x P(x)$ is true (or $\forall x P(x)$ is false) find a c such that $\neg P(c)$ is true or $P(c)$ is false.
- In this case c is called a *counterexample* to the assertion $\forall x P(x)$.

Example: "Every positive integer is the sum of the squares of 3 integers." The integer 7 is a counterexample. So the claim is false.

Uniqueness Proofs

- Some theorems asset the existence of a unique element with a particular property, $\exists! x\, P(x)$. The two parts of a *uniqueness proof* are
  - *Existence*: We show that an element x with the property exists.
  - *Uniqueness*: We show that if $y \neq x$, then y does not have the property.

Example: Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

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Example: Show that if a and b are real numbers and $a \neq 0$, then there is a unique real number r such that $ar + b = 0$.

Solution:

- Existence: The real number $r = -b/a$ is a solution of $ar + b = 0$ because $a(-b/a) + b = -b + b = 0$.
- Uniqueness: Suppose that s is a real number such that $as + b = 0$. Then $ar + b = as + b$, where $r = -b/a$. Subtracting b from both sides and dividing by a shows that $r = s$.

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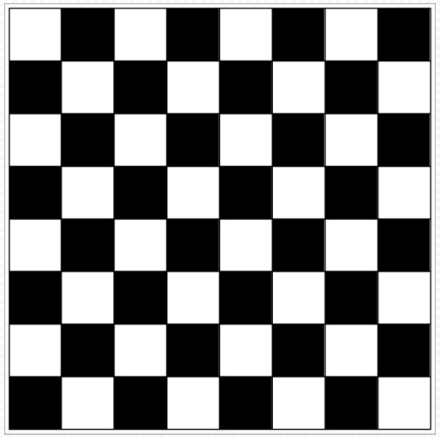
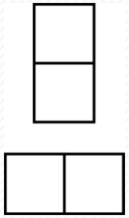
Proof and Disproof: Tilings

Example 1: Can we tile the standard checkerboard using dominos?

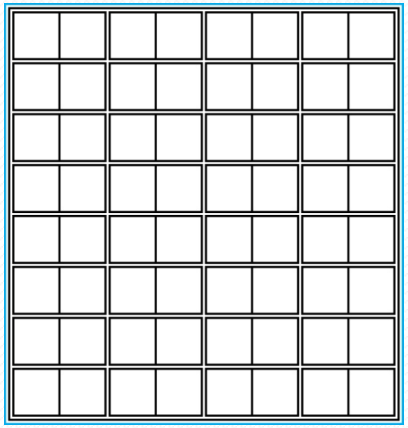
Solution: Yes! One example provides a constructive existence proof.



The Standard Checkerboard



Two Dominoes



One Possible Solution

Tilings

Example 2: Can we tile a checkerboard obtained by removing one of the four corner squares of a standard checkerboard?

Tilings

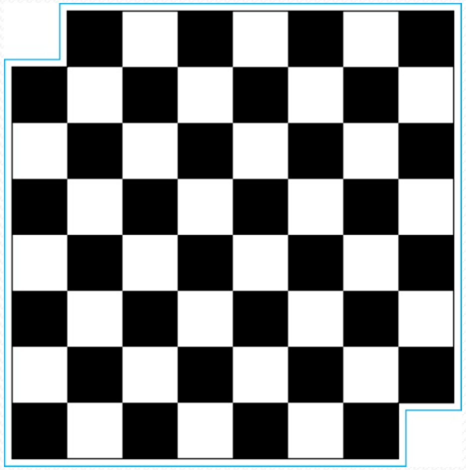
Example 2: Can we tile a checkerboard obtained by removing one of the four corner squares of a standard checkerboard?

Solution:

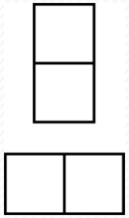
- Our checkerboard has $64 - 1 = 63$ squares.
- Since each domino has two squares, a board with a tiling must have an even number of squares.
- The number 63 is not even.
- We have a contradiction. ◀

Tilings

Example 3: Can we tile a board obtained by removing both the upper left and the lower right squares of a standard checkerboard?



Nonstandard Checkerboard



Dominoes

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Tilings

Solution:

- There are 62 squares in this board.
- To tile it we need 31 dominos.
- *Key fact*: Each domino covers one black and one white square.
- Therefore the tiling covers 31 black squares and 31 white squares.
- Our board has either 30 black squares and 32 white squares or 32 black squares and 30 white squares.
- Contradiction!

The Role of Open Problems

- Unsolved problems have motivated much work in mathematics. Fermat's Last Theorem was conjectured more than 300 years ago. It has only recently been finally solved.

Fermat's Last Theorem: The equation $x^n + y^n = z^n$

has no solutions in integers x, y, and z, with $xyz \neq 0$ whenever n is an integer with $n > 2$.

A proof was found by Andrew Wiles in the 1990s.

An Open Problem

- **The 3*x* + 1 Conjecture**: Let T be the transformation that sends an even integer x to $x/2$ and an odd integer x to $3x + 1$. For all positive integers x, when we repeatedly apply the transformation T, we will eventually reach the integer 1.

  For example, starting with $x = 13$:

  $T(13) = 3 \cdot 13 + 1 = 40$, $T(40) = 40/2 = 20$, $T(20) = 20/2 = 10$,
  $T(10) = 10/2 = 5$, $T(5) = 3 \cdot 5 + 1 = 16$, $T(16) = 16/2 = 8$,
  $T(8) = 8/2 = 4$, $T(4) = 4/2 = 2$, $T(2) = 2/2 = 1$

  The conjecture has been verified using computers up to $5.6 \cdot 10^{13}$.

Additional Proof Methods

- Later we will see many other proof methods:
  - Mathematical induction, which is a useful method for proving statements of the form $\forall n\, P(n)$, where the domain consists of all positive integers.
  - Structural induction, which can be used to prove such results about recursively defined sets.
  - Combinatorial proofs use counting arguments.