Table of Contents

Selected Problems

Problem: Demonstrate that the 2D Gaussian function

$$f(x,y)=\frac{1}{2\pi\sigma^2}\exp\left(\frac{-x^2+y^2}{2\sigma^2}\right)$$

is a separable function.

Solution: Since

$$\exp\left(\frac{-x^2+y^2}{2\sigma^2}\right)=\exp\left(\frac{-x^2}{2\sigma^2}\right)\times\exp\left(\frac{y^2}{2\sigma^2}\right)$$

the function can be separated in terms of x and y . In general, the technique is to see if $f(x,y)$ can be expressed as a product $f(x)f(y)$.

Problem: Given a 1D Gaussian function

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-x^2}{2\sigma^2}\right)$$

compute a Gaussian kernel of size 5 for 1D convolution with $\sigma^2=1$.

Solution: Compute the values of the function for the kernel positions as if the function was centered on it and by assuming that pixel size is one:

$f(-2)$	$f(-1)$	$f(0)$	$f(1)$	$f(2)$
0.0764	0.3422	0.5642	0.3422	0.0764

Of course the coefficients do not sum to one, as we did not perform a numerical integral of the function. Hence we must divide the kernel entries by the sum of its entries.

Problem: Create a 2D Gaussian convolution kernel using the 1D kernel from the previous problem.

Solution: Use the kernel as a 1×5 matrix, consider its transpose, and matrix multiply the two, as in:

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$$

The result is a 5×5 2D convolution mask identical to what one would obtain by considering $f(x)f(y)=f(x,y)$.

Problem: Convolve a normalized 1D Gaussian kernel of size 5 with $\sigma^2=1$ with the following row of pixels:

0	0	0	0	0	10	20	30	40	50

and put the result in this table:

NULL	NULL							NULL	NULL

Problem: Let's say we have 1000 image points that we think fit a 2D line in the resolution of a given problem. Also suppose we know about 30 percent of that data is corrupt. No matter what, we decide we are going to use RanSaC and iterate ten times, no more. Hence, ten times we are going to select 2 image points and fit a line, and evaluate how good that fit is with the rest of the points. What is the probability that one of these ten random choices of two points yields the right solution (i.e. the two chosen points are inliers)?

Solution:

$$1-(1-\omega^n)^k=1-(1-0.7^2)^{10} \;=\; 0.9988$$

Problem: How many iterations would be needed to be sure at 90% that the best solution is correct?

Solution:

$$k = \frac{\log(1.0-0.9)}{\log(1.0-0.7^2)} = \frac{\log(0.1)}{\log(0.51)} = \frac{-1}{-0.2944} = 3.4$$

Problem: Given a 5-point central difference formula:

$$f'(x) = \frac{-f(x+2h)+8f(x+h)-8f(x-h)+f(x-2h)}{12h}$$

Find the corresponding derivative kernel.

Solution:

The solution is direct and given by $\frac{1}{12}\begin{bmatrix}1 & -8 & 0 & 8 & -1\end{bmatrix}$

Problem: Derive the following signal at $i=6$ using the kernel obtained in the previous question:

1	2	3	4	5	6	7	8	9	10	11	12	13

Solution:

Center the kernel at $i=6$ over the signal and proceed to convolve. One obtains

$$\frac{1}{12}(5-48+64-9) = 1$$

This derivative is exact.

Problem: Suppose we have these two 3D points $P_1=(X_{1,}Y_{1,}Z_1)=(0,0,50)$ and $P_2=(X_{2,}Y_{2,}Z_2)=(1,1,50)$ in a camera coordinate system with focal length $f=10$. What is the perspective projection of these two points onto the imaging plane of the camera?

Solution:

$$p_1 = f\frac{P_1}{Z_1} = (0,0,1.0)$$

and

$$p_2 = f\frac{P_2}{Z_2} = (0.2, 0.2, 1.0)$$

Problem: What is the image disparity between these two points?

Solution: Image disparity is defined as $p_2 - p_1 = (0.2, 0.2, 0.0)$.

Problem: Suppose a 3D point $P_1 = (10,10,100)$ in camera coordinates with $f=10$ as before. Let this point translate by $\vec{T} = (1.0, 0.0, 0.0)^T$ and rotate by $\vec{\omega} = (0.0, 0.0, 0.01)^T$. Compute the 3D velocity of this point.

Solution: We have:

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} -T_x - \omega_y Z + \omega_z Y \\ -T_y - \omega_z X + \omega_x Z \\ -T_z - \omega_x Y + \omega_y X \end{pmatrix} = \begin{pmatrix} -1.0 - 0.0 + 0.01 \times 10 \\ 0.0 - 0.01 \times 10 + 0.0 \\ 0.0 - 0.0 + 0.0 \end{pmatrix} = \begin{pmatrix} 0.9 \\ -0.1 \\ 0 \end{pmatrix}$$

Problem: Prove that:

$$f\frac{P}{Z}\frac{d}{dt} = \begin{bmatrix} \frac{T_z x - f T_x}{Z} - f\omega_y + \omega_z y + \frac{\omega_x x y}{f} - \frac{\omega_y x^2}{f} \\ \frac{T_z y - f T_y}{Z} + f\omega_x - \omega_z x - \frac{\omega_y x y}{f} + \frac{\omega_x y^2}{f} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

Solution: We have, by definition of derivative of quotients:

$$f\frac{P}{Z}\frac{d}{dt} = f\left(\frac{Z\vec{V} - V_z \vec{P}}{Z^2}\right)$$

Solving for the x component, by direct substitution, we obtain:

$$v_x = f\frac{[Z(-T_x - \omega_y Z + \omega_z Y) - (T_z - \omega_x Y + \omega_y X)X]}{Z^2}$$

Using perspective projection relations such as

$$\frac{X}{Z} = \frac{x}{f}$$

the equation becomes

$$v_x = \frac{T_z x - f T_x}{Z} - f\omega_y + \omega_z y + \frac{\omega_x xy}{f} - \frac{\omega_y x^2}{f}$$

Similar manipulations lead to the correct equation for v_y .

Problem: At an image point, and given the following values for image derivatives $I_x = 10,\ I_y = 5,\ I_t = 1$, compute the normal image velocity.

Solution: The normal velocity is a scalar in the direction of the gradient and it is given by the following equation:

$$v_n = \frac{-I_t}{\|\nabla I\|} = \frac{-1}{\sqrt{125}}$$

Problem: Give the vector for normal velocity.

Solution: Since this velocity is in the direction of the gradient, the normal velocity vector is simply given by:

$$\frac{\nabla I}{\|\nabla I\|} v_n$$

Problem: Suppose we have spatiotemporal derivatives I_{x1}, I_{y1}, I_{t1} and I_{x2}, I_{y2}, I_{t2} for two adjacent image points, all non-zero. Form a system of equations that yields the full velocity of this image neighborhood.

Solution: The motion constraint equation $I_x u + I_y v + I_t = 0$ is the basis for solving this problem. Using the derivatives for the two neighboring image points, we can form two equations based on it:

$$\begin{aligned} I_{x1}u+I_{y1}v+I_{t1} &= 0 \\ I_{x2}u+I_{y2}v+I_{t2} &= 0 \end{aligned}$$

We can then turn this simple system of equations into a matrix form such as:

$$\begin{bmatrix} I_{x1} & I_{y1} \\ I_{x2} & I_{y2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -I_{t1} \\ -I_{t2} \end{bmatrix}$$

This system of equations will have a solution only if the rank of its augmented matrix is equal to its number of rows.

Problem: In the development of the simple Fourier transform of 1D signal $f(x)=A\cos(k_0 x)$, indicate which properties are applied (in the order that they are) to obtain the Fourier spectrum.

Solution:

$\hat{f}(k)=A\int \cos(k_0 x)e^{-ikx}dx$	Definition of Fourier Transform
$\hat{f}(k)=\frac{A}{2}\int (e^{ik_0x}+e^{-ik_0x})e^{-ikx}dx$	Euler's formula
$\hat{f}(k)=\frac{A}{2}\int e^{-ix(k-k_0)}dx+\frac{A}{2}\int e^{-ix(k+k_0)}dx$	Law of exponents
$\hat{f}(k)=\frac{A}{2}\delta(k-k_0)+\frac{A}{2}\delta(k+k_0)$	Definition of Dirac delta functions

Problem: What happens to the 2D power spectrum of a 2D signal as it starts translating?

Solution: The motion adds a 3rd dimension to the 2D spectrum, and it is the temporal frequency dimension. The 2D spectrum of the signal gets tilted in this new dimension and forms a 2D plane within it. The vector normal to this plane yields the 2D velocity of the signal (this phenomenon was demonstrated in class).

Problem: Suppose we have a stereo system that is calibrated and rectified with both focal length $f=1$ and baseline $T=10.$ A stereo disparity is found as $(x_l, y_l) = (0.2, 0.2)$ and $(x_r, y_r) = (0.0, 0.2)$. What are the coordinates of the 3D point giving rise to this detected image disparity?

Solution: The reconstruction of the 3D point $P=(X, Y, Z)$ corresponding to the disparity can be obtained with very simple equations (why?) such as these:

$$Z=f\frac{T}{x_l-x_r} \quad X = x_l\frac{T}{x_l-x_r} \quad Y = y_l\frac{T}{x_l-x_r}$$

Direct substitution yields $P=(10,10,50)$. Conversely, given equations

$$x_l = f\frac{X}{Z} \quad x_r = f\frac{X-T}{Z} \quad y_l = y_r = f\frac{Y}{Z}$$

allow to verify that the stereo disparity generated by the 3D point effectively is

$$(x_l, y_l) \quad (x_r, y_r) = (0.2, 0.2) \quad (0.0, 0.2)$$

Problem: Given two points $p_1=(x_{1,} y_1)=(1,3)$ and $p_2=(x_{2,} y_2)=(3,5)$ that are on a 2D line, use the Hough transform to obtain the equation of this line.

Solution: These two points are on a line that can be described with the following equation:

$$y=ax+b$$

We parametrize this line in (a, b) space:

$$b=-ax+y$$

and use the two points to form the following equations:

$$b=-ax_1+y_1 \quad b=-ax_2+y_2$$

yielding:

$$b=-a+3 \quad b=-3\text{a}+5$$

Forming a system of equations to find the intersection gives us $(a_{1,} b_1)=(1,2)$ as the intersection point. Substituting these values in the original line equation $y=ax+b$ gives us line $y=x+2$. Verify this line contains points p_1 and p_2 .