Assignment #4

Student #:[redacted]

Name: Zaid Albirawi

UWO email: zalbiraw@uwo.ca

1. $\sum_{j=n}^{2n-1}(2j+1) \;=\; 3n^2$

Basis Step (1):

$\sum_{j=1}^{2-1}(2j+1)$ $\qquad\qquad$ $3(1)^2$

= $(2(1)+1)$ $\qquad\qquad$ $=\;3$

= 3

$$\sum_{j=n}^{2n-1}(2j+1)$$

= $\sum_{j=n}^{2n-1}(2j+1)$

= $(2(n)+1)+(2(n+1)+1)+\cdots+(2(2n-1)+1)$

= $(2n+1)+(2n+3)+\cdots+(4n-1)$

Induction Step: $\qquad \sum_{j=n+1}^{2(n+1)-1}(2j+1)$ $\qquad\qquad$ $3(n+1)^2$

= $\sum_{j=n+1}^{2n+2-1}(2j+1)$	=	$3(\mathrm{n}+1)(\mathrm{n}+1)$
= $\sum_{j=n+1}^{2n+1}(2j+1)$	=	$3(n^2+2\mathrm{n}+1)$
= $(2(n+1)+1)+\cdots+(2(2n-1)+1)+(2(2n+1)+1)$	=	$3n^2+6n+3$
= $(2n+3)+\cdots+(4n-1)+(4n+1)+(4n+3)$	=	$3n^2+6n+3$

Add and subtract the number$(2n+1)$.

= $(2n+1)+(2n+3)+\cdots+(4n+1)+(4n+3)-(2n+1)$= $\qquad$ $3n^2+6n+3$

By the induction hypothesis $(2n+1)+(2n+3)+\cdots+(4n-1)$ equals $3n^2$. Therefore,

= $3n^2+(4n+1)+(4n+3)-(2n+1)$	=	$3n^2+6n+3$
= $3n^2+(6n+3)$	=	$3n^2+6n+3$
= $3n^2+6n+6$	=	$3n^2+6n+3$

Therefore, RS equals LS. Proven by induction.

2. Using structural induction prove that $l(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T, #leaves = #vertices+1

   Basis Step: The root r is a leaf of the full binary tree with exactly one vertex r. This tree has no internal vertices.

   Recursive Step: The set of leaves of the tree $T = T_1 \cdot T_2$ is the union of the sets of leaves of T_1 and T_2. The internal vertices of T are the root r of T and the union of the set of internal vertices of T_1 and the set of internal vertices of T_2.

   Theorem: If T is a full binary tree, then $n(T) = i(T) + l(T)$

   Proof: Using structural induction.

   Basis Step: The result holds for a full binary tree consisting only of a root, $l(T) = 1$ and $i(T) = 0$. Hence, $i(T) + 1 = l(T), 0 + 1 = l(T)$

   Recursive Step: Assume (T_1) and (T_2) are full binary trees, $T = T_1 \cdot T_2$

   Since $n(T) = i(T) + l(T)$ and $T = T_1 \cdot T_2$ we can see that

$l(T) = l(T_1) + l(T_2)$	By definition of T
$l(T) = 1 + i(T_1) + l(T_2)$	By induction hypothesis of T_1
$l(T) = 1 + i(T_1) + 1 + i(T_2)$	By induction hypothesis of T_2
$l(T) = 1 + i(T_1 \cdot T_2)$	By definition of i

   This will cover all the recursive cases of the structural induction of binary tree until the base case is reached. Therefore, I have proven by structural induction that $l(T)$, the number of leaves of a full binary tree T, is 1 more than $i(T)$, the number of internal vertices of T in full binary search tree.

3. Consider all bit strings of length 12

   a. How many begin with 110?

      110 _ _ _ _ _ _ _ _ _
      X=9, hence we have 9 spaces to fill with 2 choices for each.
      $2^9 = 512$

      Therefore, 512 different bit strings of length 12 start with 110.

   b. How many begin with 11 and end with 10?

      11 _ _ _ _ _ _ _ _10
      X=8, hence we have 8 spaces to fill with 2 choices for each.
      $2^8 = 256$

      Therefore, 256 different bit strings of length 12 start with 11 and end with 10.

   c. How many begin with 11 or end with 10?

      11 _ _ _ _ _ _ _ _ _ _
      X=10, hence we have 10 spaces to fill with 2 choices for each.
      $2^{10} = 1024$

      _ _ _ _ _ _ _ _ _ _10
      X=10, hence we have 10 spaces to fill with 2 choices for each.
      $2^{10} = 1024$

      11 _ _ _ _ _ _ _ _ 10
      X=8, hence we have 8 spaces to fill with 2 choices for each.
      $2^8 = 256$

      1024 + 1024 – 256 = 1792

      Therefore, 1792 different bit strings of length 12 start with 11 or end with 10.

d. How many have exactly five 1’s?

Since the order does not matter then we want to choose exactly 5 out of 12 spots to have 1’s in them. Therefore, C (12, 5)

$$\frac{12!}{(12-5)!\,5!}$$

$$= \frac{12!}{7!\,5!}$$

$$= \frac{12\ x\ 11\ x\ 10\ x\ 9\ x\ 8\ x\ \cancel{7!}}{\cancel{7!}\,x\ 5\ x\ 4\ x\ 3\ x\ 2}$$

$$= \frac{\cancel{12}\ x\ 11\ x\ \cancel{10}\ x\ 9\ x\ 8}{\cancel{5}\ x\ 4\ x\ \cancel{3}\ x\ \cancel{2}}$$

$$= \frac{11\ x\ 9\ x\ 8}{1}$$

$$= 792$$

Therefore, 792 different bit strings of length 12 have exactly five 1’s

e. How many have exactly four 1's such as none of these 1's are adjacent to each other?

Since we have 12 bit strings to fill we need at 5 zeros to surround the ones therefore, we use n = 9 and r = 4.

$$\frac{9!}{(9-4)!\,4!}$$

$$= \frac{9!}{5!\,4!}$$

$$= \frac{9\ x\ 8\ x\ 7\ x\ \cancel{6\ x}\ \cancel{5!}}{\cancel{5!}\,4\ x\ \cancel{3\ x\ 2}}$$

$$= \frac{9\ x\ 8\ x\ 7}{4}$$

$$= 126$$

Therefore, there are 126 different bit strings of length 12 have exactly four 1's such as none of these 1's are adjacent to each other.

4. Solve the following counting problems:

   a. How many permutations of the seven letters A, B, C, D, E, F, G have E in the first position?

      E _ _ _ _ _ _

      X=6, hence we have to choose 6 letters from the set of the six letters, A, B, C, D, F and G, were order does matter.

      $6!$
      $= 6\ x\ 5\ x\ 4\ x\ 3\ x\ 2$
      $= 720$

      Therefore 720 permutations of the seven letters A, B, C, D, E, F, and G have E in the first position.

   b. How many permutations of the seven letters A, B, C, D, E, F, G have E in one of the first two positions?

      E _ _ _ _ _ _ or _ E _ _ _ _ _

      X=6, hence we have to choose 6 letters from the set of the six letters, A, B, C, D, F and G, were order does matter. Since both cases are similar, we double the result of the first case to obtain the answer of both cases.

      $2\ x\ 6!$
      $= 6\ x\ 5\ x\ 4\ x\ 3\ x\ 2\ x\ 2$
      $= 1440$

      Therefore 1440 permutations of the seven letters A, B, C, D, E, F, and G have E in one of the first two positions.

   c. How many permutations of the seven letters A, B, C, D, E, F, and G have the two vowels before the five consonants?

      EA _ _ _ _ _ or AE _ _ _ _ _

      X=5, hence we have to choose 5 letters from the set of the six letters, B, C, D, F and G, were order does matter. Since both cases are similar, we double the result of the first case to obtain the answer of both cases.

      $2\ x\ 5!$
      $= 5\ x\ 4\ x\ 3\ x\ 2\ x\ 2$
      $= 240$

Therefore, 240 permutations of the seven letters A, B, C, D, E, F, and G have the two vowels before the five consonants.

d. How many permutations of the seven letters A, B, C, D, E, F, and G neither begin nor end with A?

A _ _ _ _ _ _ or _ _ _ _ _ _ A

X=6, hence we have to choose 6 letters from the set of the six letters B, C, D, E, F and G, were order does matter. Since both cases are similar, we double the result of the first case to obtain the answer of both cases.

$$2\ x\ 6!$$
$$= 6\ x\ 5\ x\ 4\ x\ 3\ x\ 2\ x\ 2$$
$$= 1440$$

Therefore, the number of permutations of the seven letters A, B, C, D, E, F, and G begin or end with A is 1440. To find the number permutations of the seven letters A, B, C, D, E, F, and G neither begin nor end with A. We must subtract 1440 from the number of permutations of any order.

$$7! - (2\ x\ 6!)$$
$$= 7\ x\ 6! - 2\ x\ 6!$$
$$= 5\ x\ 6!$$
$$= 3600$$

Therefore 3600 permutations of the seven letters A, B, C, D, E, F, and G neither begin nor end with A.

e. How many permutations of the seven letters A, B, C, D, E, F, and G do not have the vowels next to each other?

Consider AE and EA as one letter therefore, we'll compute the permutations of these letters being selected out of 6 letters. Since both cases are similar, we double the result of the first case to obtain the answer of both cases.

$$2\ x\ 6!$$
$$= 6\ x\ 5\ x\ 4\ x\ 3\ x\ 2\ x\ 2$$
$$= 1440$$

Therefore, the number of permutations of the seven letters A, B, C, D, E, F, and G have the vowels next to each other is 1440. To find the number permutations of the seven letters A, B, C, D, E, F, and G that do not have the vowels next to each other. We must subtract 1440 from the number of permutations of any order.

$7! - (2\ x\ 6!)$
$= 7\ x\ 6! - 2\ x\ 6!$
$= 5\ x\ 6!$
$= 3600$

Therefore 3600 permutations of the seven letters A, B, C, D, E, F, and G do not have the vowels next to each other.

5. A student council consists of 15 students.

   a. In how many ways can the committee of six can be selected from the membership of the council?

   Since the order does not matter, the ways the committee of six can be selected from the membership of the council, we can use combinations equation.

$$\frac{15!}{(15-6)!\,6!}$$

$$= \frac{15\ x\ 14\ x\ 13\ x\ 12\ x\ 11\ x\ 10\ x\ \cancel{9!}}{\cancel{9!}\,6!}$$

$$= \frac{\cancel{15}\ x\ 14\ x\ 13\ x\ \cancel{12}\ x\ 11\ x\ 10}{\cancel{6}\ x\ \cancel{5}\ x\ 4\ x\ \cancel{3}\ x\ \cancel{2}}$$

$$= \frac{14\ x\ 13\ x\ 11\ x\ 10}{4}$$
$$= 5005$$

   Therefore, there are 5005 different ways to select the committee of six from the membership of the council.

   b. Two council members have the same major and are not permitted to serve together on a committee. How many ways can a committee of six be selected from the membership council?

   Since only one of these members can be on the committee, we only have to cases.

   First case: neither are on the committee

$$\frac{13!}{(13-6)!\,6!}$$

$$= \frac{13\ x\ 12\ x\ 11\ x\ 10\ x\ 9\ x\ 8\ x\ \cancel{7!}}{\cancel{7!}\,6!}$$

$$= \frac{13\ x\ \cancel{12}\ x\ 11\ x\ \cancel{10}\ x\ 9x\ 8}{6\ x\ \cancel{5}\ x\ \cancel{4}\ x\ \cancel{3}\ x\ \cancel{2}}$$

$$= \frac{13\ x\ 11\ x\ 9\ x\ 8}{6}$$
$$= 1716$$

Second case: one of them is in the committee

$$2\ x\ \frac{13!}{(13-5)!\,5!}$$

$$= \frac{2\ x\ 13\ x\ 12\ x\ 11\ x\ 10\ x\ 9\ x\ \cancel{8!}}{\cancel{8!}5!}$$

$$= \frac{2x\ 13\ x\ \cancel{12}\ x\ 11\ x\ \cancel{10}\ x\ 9}{\cancel{5}\ x\ \cancel{4}\ x\ \cancel{3}\ x\ \cancel{2}}$$

$$= 2\ x\ 13\ x\ 11\ x\ 9$$
$$= 2574$$

Therefore, there are $2574 + 1716 = 4290$ ways to select a committee of six from the membership council if two council members have the same major and are not permitted to serve together on a committee.

c. Two council members always insist on serving on committees together. If they cannot serve together, they won't serve at all. How many ways can a committee of six be selected from the council membership?

Since only both of these members can be on the committee, we only have to cases.

First case: neither are on the committee

$$\frac{13!}{(13-6)!\,6!}$$

$$= \frac{13\ x\ 12\ x\ 11\ x\ 10\ x\ 9\ x\ 8\ x\ \cancel{7!}}{\cancel{7!}6!}$$

$$= \frac{13\ x\ \cancel{12}\ x\ 11\ x\ \cancel{10}\ x\ 9x\ 8}{6\ x\ \cancel{5}\ x\ \cancel{4}\ x\ \cancel{3}\ x\ \cancel{2}}$$

$$= \frac{13\ x\ 11\ x\ 9\ x\ 8}{6}$$
$$= 1716$$

Second case: both of them are in the committee

$$\frac{13!}{(13-4)!\,4!}$$

$$= \frac{13\ x\ 12\ x\ 11\ x\ 10\ x\ \cancel{9!}}{\cancel{9}!\,4!}$$

$$= \frac{13\ x\ \cancel{12}\ x\ 11\ x\ \cancel{10}}{4\ x\ \cancel{3}\ x\ \cancel{2}}$$

$$= 13\ x\ 11\ x\ 5$$
$$= 715$$

Therefore, there are 1716 + 715 = 2431 ways to select a committee of six from the membership council if two council members always insist on serving on committees together.

d. Suppose the council contains eight men and seven women. How many committees of six contain three men and three women? How many committees of six contain at least one woman?

Part 1: 3 men and 3 women

$$\frac{8!}{(8-3)!\,3!}\ x\ \frac{7!}{(7-3)!\,3!}$$

$$= \frac{8!}{5!\,3!}\ x\ \frac{7!}{4!\,3!}$$

$$= \frac{8\ x\ 7\ x\ 6\ x\ \cancel{5}!}{\cancel{5}!\,3!}\ x\ \frac{7\ x\ 6\ x\ 5\ x\ \cancel{4}!}{\cancel{4}!\,3!}$$

$$= \frac{8\ x\ 7\ x\ \cancel{6}}{\cancel{3\ x\ 2}}\ x\ \frac{7\ x\ \cancel{6}\ x\ 5}{\cancel{3\ x\ 2}}$$

$$= 8\ x\ 7\ x\ 7\ x\ 5$$
$$= 1960$$

Therefore, there are 1960 ways to select a committee of six from the membership council if the committee contains at 3 men and 3 women.

Part 2: At least one woman

We calculate the ways that we can choose a committee with 0 women and then subtract that from the ways that we can choose a committee from the whole population.

$$\frac{15!}{(15-6)!\,6!} - \frac{8!}{(8-6)!\,6!}$$

$$= \frac{15\ x\ 14\ x\ 13\ x\ 12\ x\ 11\ x\ 10\ x\ 9!}{\cancel{9}!\,6!} - \frac{8\ x\ 7\ x\ \cancel{6}!}{2!\,\cancel{6}!}$$

$$= \frac{\cancel{15}\ x\ 14\ x\ 13\ x\ \cancel{12}\ x\ 11\ x\ 10}{\cancel{6\ x\ 5\ x}\ 4\ \cancel{x\ 3}\ x\ \cancel{2}} - \frac{8\ x\ 7}{2}$$

$$= \frac{14\ x\ 13\ x\ 11\ x\ 10}{4} - \frac{8\ x\ 7}{2}$$

$$= 5005 - 28$$
$$= 4977$$

Therefore, there are 4977 ways to select a committee of six from the membership council if the committee contains at least one woman.

e. Suppose the council consists of three freshmen, four sophomores, three juniors and five seniors. How many committees of eight contain two representatives from each class?

Therefore, we do each case separately then we multiply them all at the end.

$$\frac{3!}{(3-2)!\,2!}\ x\ \frac{4!}{(4-2)!\,2!}\ x\ \frac{3!}{(3-2)!\,2!}\ x\ \frac{5!}{(5-2)!\,2!}$$

$$= \frac{3!}{2!}\,x\,\frac{4!}{2!\,2!}\,x\,\frac{3!}{2!}\,x\,\frac{5!}{3!\,2!}$$

$$= \frac{3\ x\ \cancel{2}!}{\cancel{2!}} + \frac{3\ x\ 2}{\cancel{2!\,2!}} + \frac{3\ x\ \cancel{2}!}{\cancel{2!}} + \frac{5\ x\ 2\ x\ \cancel{2\ x\ 3!}}{\cancel{3!\,2!}}$$
$$= 3\ x\ 6\ x\ 3\ x\ 10$$
$$= 540$$

Therefore, there are 540 ways to select a committee of 8 that will contain two representatives from each class.