Propositional Calculus

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In standard algebra, expressions in which the variables and constants represent numbers are manipulated.
Consider for instance the expression

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One sees at a glance that this expressions yields a. In fact, we are so accustomed to performing such algebraic manipulations that we are no longer aware of what is behind each step. Here we used the identities

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$$y - y = 0$$

$$x + 0 = x$$

Simplifications of logic formulas

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$$(A \wedge \neg A) \mathrel{\models\!\!\!\!\!\dashv} 0$$

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We can now apply these equivalences to conclude

$$(p \wedge q) \wedge \neg q \mathrel{\models\!\!\!\!\!\dashv} p \wedge (q \wedge \neg q) \mathrel{\models\!\!\!\!\!\dashv} p \wedge 0 \mathrel{\models\!\!\!\!\!\dashv} 0.$$

Removing conditionals and biconditionals

Since the symbolic treatment of conditionals and biconditionals is relatively cumbersome, one usually removes them before performing further formula manipulations.

To remove the conditional, one uses the following logical equivalence:

$$p \to q \mathrel{\models\!\!\!\!\dashv} \neg p \vee q.$$

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There are two ways to express the biconditional

$$p \leftrightarrow q \;\models\!\!\!\!\;\dashv\; (p \wedge q) \vee (\neg p \wedge \neg q)$$

$$p \leftrightarrow q \;\models\!\!\!\!\;\dashv\; (p \to q) \wedge (q \to p) \;\models\!\!\!\!\;\dashv\; (\neg p \vee q) \wedge (p \vee \neg q).$$

The first version expresses the fact that two formulas are equivalent if they have the same truth values.

The second version stresses the fact that two formulas are equivalent if the first implies the second and the second implies the first.

Example

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$$(p \rightarrow q \wedge r) \vee ((r \leftrightarrow s) \wedge (q \vee s)).$$

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$$(p \rightarrow q \wedge r) \vee ((r \leftrightarrow s) \wedge (q \vee s)).$$

Solution.

$$(\neg p \vee q \wedge r) \vee (((\neg r \vee s) \wedge (r \vee \neg s)) \wedge (q \vee s)).$$

Essential laws for propositional calculus

Law	Name
$p \vee \neg p \models\!\mid 1$	Excluded middle law

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Law	Name
$p \vee \neg p \models\!\!\!\!\!\!\dashv 1$	Excluded middle law
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$p \vee 0 \models\!\!\!\!\dashv p,\ p \wedge 1 \models\!\!\!\!\dashv p$	Identity laws

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$p \vee \neg p \models\!\mid 1$	Excluded middle law
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$p \vee 0 \models\!\mid p$, $p \wedge 1 \models\!\mid p$	Identity laws
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$\neg(\neg p) \models\!\!\!\mid p$	Double-negation law
$p \vee q \models\!\!\!\mid q \vee p$, $p \wedge q \models\!\!\!\mid q \wedge p$	Commutative laws
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$p \vee 0 \models\!\!\!\!= p$, $p \wedge 1 \models\!\!\!\!= p$	Identity laws
$p \vee 1 \models\!\!\!\!= 1$, $p \wedge 0 \models\!\!\!\!= 0$	Domination laws
$p \vee p \models\!\!\!\!= p$, $p \wedge p \models\!\!\!\!= p$	Idempotent laws
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$p \vee q \models\!\!\!\!= q \vee p$, $p \wedge q \models\!\!\!\!= q \wedge p$	Commutative laws
$(p \vee q) \vee r \models\!\!\!\!= p \vee (q \vee r)$ $(p \wedge q) \wedge r \models\!\!\!\!= p \wedge (q \wedge r)$	Associative laws
$(p \vee q) \wedge (p \vee r) \models\!\!\!\!= p \vee (q \wedge r)$ $(p \wedge q) \vee (p \wedge r) \models\!\!\!\!= p \wedge (q \vee r)$	Distributive laws

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$(p \vee q) \wedge (p \vee r) \models\!\!\!\dashv p \vee (q \wedge r)$ $(p \wedge q) \vee (p \wedge r) \models\!\!\!\dashv p \wedge (q \vee r)$	Distributive laws
$\neg(p \wedge q) \models\!\!\!\dashv \neg p \vee \neg q$ $\neg(p \vee q) \models\!\!\!\dashv \neg p \wedge \neg q$	De Morgan's laws

Essential laws for propositional calculus

• All laws can be proved by the truth table method.

• With the exception of the double-negation law all laws come in pairs, called dual pairs. For each formula depending only on the connectives $\neg, \wedge, \vee$, the dual is found by replacing all 1 by 0 and all 0 by 1 and by replacing all $\wedge$ by $\vee$ and all $\vee$ by $\wedge$.

• The laws allow one to simplify a formula and it is normally a good idea to apply them whenever it is possible. For instance, the formula $\neg\neg p \wedge (q \vee \neg q)$ is logically equivalent to p.

• The commutative, associative and distributive laws have their equivalents in standard algebra. In fact, the connective $\vee$ is often treated like $+$, and the connective $\wedge$ is often treated like $\times$. (The analogy sometimes breaks down.)

Further laws

- From these laws one can derive further laws, for example, the absorption laws

$$p \vee (p \wedge q) \;\mathrel{\models\!\!\!\!\dashv}\; p$$

$$p \wedge (p \vee q) \;\mathrel{\models\!\!\!\!\dashv}\; p.$$

(Hint: use identity law, distributive law, domination law and identity law again.)

- Another important law:

$$(p \wedge q) \vee (\neg p \wedge q) \;\mathrel{\models\!\!\!\!\dashv}\; q$$

$$(p \vee q) \wedge (\neg p \vee q) \;\mathrel{\models\!\!\!\!\dashv}\; q.$$

Shortcuts for manipulating formulas

Definition. A formula is called a literal if it is of the form p or $\neg p$, where p is a propositional variable. The two formulas p and $\neg p$ are called complementary literals.

The following rules are available to simplify conjunctions containing only literals.

- If a conjunction contains complementary literals or if it contains the propositional constant 0, it always yields 0; that is, it is a contradiction.
- All instances of the propositional constant 1 and all duplicate copies of any literal, may be dropped.

Shortcuts for manipulating formulas

To simplify disjunctions, the duals of these two rules can be used.

- If a disjunction contains complementary literals or if it contains the propositional constant 1, it always yields 1; that is, it is a tautology.
- All instances of the propositional constant 0 and all duplicate copies of any literal may be dropped.

Example. Simplify

$$(p_3 \wedge \neg p_2 \wedge p_3 \wedge \neg p_1) \vee (p_1 \wedge p_3 \wedge \neg p_1).$$

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To simplify disjunctions, the duals of these two rules can be used.

- If a disjunction contains complementary literals or if it contains the propositional constant 1, it always yields 1; that is, it is a tautology.
- All instances of the propositional constant 0 and all duplicate copies of any literal may be dropped.

Example. Simplify

$$(p_3 \wedge \neg p_2 \wedge p_3 \wedge \neg p_1) \vee (p_1 \wedge p_3 \wedge \neg p_1).$$

Solution: $\neg p_1 \wedge \neg p_2 \wedge p_3$.

Disjunctive normal form

Formulas can be transformed into standard forms so that they become more convenient for symbolic manipulations and make identification and comparison of two formulas easier. There are two types of normal forms in propositional calculus: the disjunctive and the conjunctive normal form.

Definition. A formula is said to be in disjunctive normal form if it is written as a disjunction, in which all the terms are conjunctions of literals.

Example: $(p \wedge q) \vee (p \wedge \neg q)$, $p \vee (q \wedge r)$, $\neg p \vee t$ are in disjunctive normal forms. The disjunction $\neg(p \wedge q) \vee r$ is not in normal form.

In general a formula in disjunctive normal form is

$$(A_{11} \wedge \ldots A_{1n_1}) \vee \ldots \vee (A_{k1} \wedge \ldots \wedge A_{kn_k}).$$

Conjunctive normal form

Definition. Disjunctions (conjunctions) with literals as disjuncts (conjuncts) are called disjunctive (conjunctive) clauses. Disjunctive and conjunctive clauses are simply called clauses.

Definition. A conjunction with disjunctive clauses as its conjuncts is said to be in conjunctive normal form.

Example: $p \wedge (q \vee r)$ and $p \wedge 0$ are in conjunctive normal form. However $p \wedge (r \vee (p \wedge q))$ is not in conjunctive normal form.

In general, a formula in conjunctive normal form is

$$(A_{11} \vee \ldots A_{1n_1}) \wedge \ldots \wedge (A_{k1} \vee \ldots \vee A_{kn_k}).$$

Normal form examples

Examples:

Observe the following formulas:

(1)p

(2)$\neg p \vee q$

(3)$\neg p \wedge q \wedge \neg r$

(4)$\neg p \vee (q \wedge \neg r)$

(5)$\neg p \wedge (q \vee \neg r) \wedge (\neg q \vee r)$

Example contd.

- (1) is an atom, and therefore a literal. It is a disjunction with only one disjunct. It is also a conjunction with only one conjunct. Hence it is a disjunctive or conjunctive clause with one literal. It is a formula in disjunctive normal form with one conjunctive clause p. It is also a formula in conjunctive normal form with one disjunctive clause p.
- (2) is a disjunction with two disjuncts, and a disjunctive normal form with two clauses, each with one literal. It is also a conjunction with one conjunct, and a formula in conjunctive normal form which consists of two literals.

Example contd.

- (3) is a conjunction and a formula in conjunctive normal form. It is also a disjunction and a formula in disjunctive normal form.
- (4) is a formula in disjunctive normal form, but not in conjunctive normal form.
- (5) is a formula in conjunctive normal form but not in disjunctive normal form.
- If $\vee$ is exchanged for $\wedge$ in (4) and (5), then (4) becomes a formula in conjunctive normal form and (5) a formula in disjunctive normal form.

Existence of normal forms

Theorem. Any formula $A \in \text{Form}(\mathcal{L}^p)$ is tautologically equivalent to some formula in disjunctive normal form.

Theorem. Any formula $A \in \text{Form}(\mathcal{L}^p)$ is tautologically equivalent to some formula in conjunctive normal form.

How to obtain normal forms?

Use the following tautological equivalences:

(1)$A \to B \models\mid \neg A \vee B$.

(2)$A \leftrightarrow B \models\mid (\neg A \vee B) \wedge (A \vee \neg B)$.

(3)$A \leftrightarrow B \models\mid (A \wedge B) \vee (\neg A \wedge \neg B)$.

(4)$\neg\neg A \models\mid A$.

(5)$\neg(A_1 \wedge \ldots \wedge A_n) \models\mid \neg A_1 \vee \ldots \vee \neg A_n$.

(6)$\neg(A_1 \vee \ldots \vee A_n) \models\mid \neg A_1 \wedge \ldots \wedge \neg A_n$.

(7)$A \wedge (B_1 \vee \ldots \vee B_n) \models\mid (A \wedge B_1) \vee \ldots \vee (A \wedge B_n)$.

$(B_1 \vee \ldots \vee B_n) \wedge A \models\mid (B_1 \wedge A) \vee \ldots \vee (B_n \wedge A)$.

(8)$A \vee (B_1 \wedge \ldots \wedge B_n) \models\mid (A \vee B_1) \wedge \ldots \wedge (A \vee B_n)$.

$(B_1 \wedge \ldots \wedge B_n) \vee A \models\mid (B_1 \vee A) \wedge \ldots \wedge (B_n \vee A)$.

How to obtain normal forms?

By the replaceability of tautological equivalences, we can replace the preceding formulas on the left with the corresponding ones on the right to yield a formula tautologically equivalent to the original one.

- By (1)–(3) we eliminate $\rightarrow$ and $\leftrightarrow$.
- By (4)–(6) we eliminate $\neg, \vee, \wedge$ from the scope of $\neg$ such that any $\neg$ has only an atom as its scope.
- By (7) we eliminate $\vee$ from the scope of $\wedge$.
- By (8) we eliminate $\wedge$ from the scope of $\vee$.

This method leads to obtaining the disjunctive or conjunctive normal forms.

Example

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The conjunctive normal form can be found by the following derivations:

$$\begin{array}{ll}
\neg((p \vee \neg q) \wedge \neg r) \models\!\!\!\!\dashv & \\
\models\!\!\!\!\dashv \neg(p \vee \neg q) \vee \neg\neg r & \text{De Morgan} \\
\models\!\!\!\!\dashv \neg(p \vee \neg q) \vee r & \text{Double negation} \\
\models\!\!\!\!\dashv (\neg p \wedge \neg\neg q) \vee r & \text{De Morgan} \\
\models\!\!\!\!\dashv (\neg p \wedge q) \vee r & \text{Double negation} \\
\models\!\!\!\!\dashv (\neg p \vee r) \wedge (q \vee r) & \text{Distributivity}
\end{array}$$

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Solution: $(p_1 \vee p_3) \wedge (p_1 \vee p_4 \vee p_5) \wedge (p_2 \vee p_3) \wedge (p_2 \vee p_4 \vee p_5)$.

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Once a conjunctive normal form is obtained, it pays to check if further simplifications are possible.

Example. Simplify the following conjunctive normal form:

$$(p \vee q) \wedge p \wedge (q \vee r) \wedge (p \vee \neg p \vee r) \wedge (\neg q \vee r).$$

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Convert the following formula into conjunctive normal form.

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Example. Simplify the following conjunctive normal form:

$$(p \vee q) \wedge p \wedge (q \vee r) \wedge (p \vee \neg p \vee r) \wedge (\neg q \vee r).$$

Solution: $(p \wedge r)$.

Disjunctive normal forms from truth tables

So far we have shown how to find the truth table of a logical formula. The reverse is also possible. One can convert any given truth table into a formula. The formula obtained in this way is already in disjunctive normal form.

In fact, the conceptually easiest method to find the normal form of a formula is by using truth tables. Unfortunately, truth tables grow exponentially with the number of variables, which makes this method unattractive for formulas with many variables.

Truth (Boolean) functions

Generally, a truth table gives truth values of some formula for all possible assignments. The table below gives an example of truth table for a certain formula f. The truth values of f depends on the three variables p, q, r.
This makes f a truth function, or a Boolean function with arguments p, q, r and range $\{0, 1\}$.

p	q	r	f
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

Minterms

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- Example: If a Boolean function has the variables p, q, r then $p \wedge \neg q \wedge r$ is a minterm but $p \wedge \neg q$ and $p \wedge p \wedge \neg q$ are not.

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• Each minterm is true for exactly one assignment. For example, $p \wedge \neg q \wedge r$ is true if p is 1, q is 0 and r is 1. Any deviation from this assignment would make this particular minterm false.
• A disjunction of minterms is true only if at least one of its constituents minterms is true. For example,

$$(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$$

is only true if at least one of $p \wedge q \wedge r$, $p \wedge \neg q \wedge r$ or $\neg p \wedge \neg q \wedge r$ is true.

DNF from truth table

If a function, such as f, is given by truth table, we know exactly for which assignments it is true. Consequently, we can select the minterms that make the function true and form the disjunction of these minterms.

The function f, for instance, is true for three assignments:

1. p, q, r are all true.
2. $p, \neg q, r$ are all true.
3. $\neg p, \neg q, r$ are all true.

The disjunction of the corresponding minterms is tautologically equivalent to f, which means that we have the following formula for f:

$$f \models\!\!\!\!\mid (p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r).$$

Full disjunctive normal form

Since the minterms are conjunctions, we have expressed the function in question in disjunctive normal form. Actually, we have a special type of normal form, the full disjunctive normal form.

Definition. If a Boolean function is expressed as a disjunction of minterms, it is said to be in full disjunctive normal form.

Conjunctive normal form and complementation

- Complementation can be used to obtain conjunctive normal forms from truth tables.
- If A is a formula containing only the connectives $\neg, \vee$ and $\wedge$, then its complement is formed by replacing all $\vee$ by $\wedge$, all $\wedge$ by $\vee$ and all atoms by their complements.
- Example: Find the complement of the formula $A = (p \wedge q) \vee \neg r$.
- Complementation can be used to find the conjunctive normal form from the truth table of some truth function (Boolean function) f.

CNF from truth tables

One first determines the disjunctive normal form for $\neg f$. If the resulting disjunctive normal form is A, then $A \models\!\!\mid \neg f$, and the complement of A must be logically equivalent to f.

Example: Find the full conjunctive normal form for f_1 given by the table

p	q	r	f_1
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	1

Example contd.

Solution: $\neg f_1$ is true for the following assignments:

$$p = 1, q = 0, r = 1$$

$$p = 1, q = 0, r = 0$$

$$p = 0, q = 0, r = 1$$

Example contd.

Solution: $\neg f_1$ is true for the following assignments:

$$p = 1, q = 0, r = 1$$

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The disjunctive normal form of $\neg f_1$ is therefore

$$(p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r).$$

Example contd.

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The disjunctive normal form of $\neg f_1$ is therefore

$$(p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r).$$

This formula has the complement

$$f_1 \models\!\!\!\dashv (\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r),$$

which is the desired conjunctive normal form.