Propositional Language - Semantics

Lila Kari

The University of Western Ontario

Syntax and semantics

Syntax	**Semantics**
analyzes Form	analyzes Meaning
$(p \wedge q) \rightarrow r$ - syntactically correct $p \wedge \wedge\wedge)))))$ - syntactically incorrect	when is a formula true, when it is false

Propositional language: Semantics

The "meaning" of a nonatomic formula, that is, its truth values (true or false) must be derived from the truth values of its constituent atomic formulas. Before this can be done, the formula must be parsed; that is, all subformulas of the formula must be found.

Propositional language: Semantics

The "meaning" of a nonatomic formula, that is, its truth values (true or false) must be derived from the truth values of its constituent atomic formulas. Before this can be done, the formula must be parsed; that is, all subformulas of the formula must be found.

If you take a class in computers and if you do not understand recursion, you will not pass.

We want to know exactly when this statement is true and when it is false.

Propositional language: Semantics

The "meaning" of a nonatomic formula, that is, its truth values (true or false) must be derived from the truth values of its constituent atomic formulas. Before this can be done, the formula must be parsed; that is, all subformulas of the formula must be found.

If you take a class in computers and if you do not understand recursion, you will not pass.

We want to know exactly when this statement is true and when it is false.

Define:

p: "You take a class in computers."

q: "You understand recursion."

r: "You pass."

The stament becomes $(p \wedge \neg q) \rightarrow \neg r$.

“Meaning” of a formula - truth table

Truth table for $(p \wedge \neg q) \rightarrow \neg r$

p	q	r	$\neg q$	$p \wedge \neg q$	$\neg r$	$(p \wedge \neg q) \rightarrow \neg r$
1	1	1	0	0	0	1
1	1	0	0	0	1	1
1	0	1	1	1	0	0
1	0	0	1	1	1	1
0	1	1	0	0	0	1
0	1	0	0	0	1	1
0	0	1	1	0	0	1
0	0	0	1	0	1	1

Value (truth) assignments

- Fix a set $\{0, 1\}$ of truth values. We interpret 0 as false and 1 as true.
- Definition. A (Boolean) value assignment (truth assignment) is a function v

$$v : \mathsf{Atom}(\mathcal{L}^p) \longrightarrow \{0, 1\},$$

with the set of all proposition variables and constants as domain and $\{0, 1\}$ as range.

Truth value of a formula under a value assignment

Definition. The value of a formula A in $\mathcal{L}^p$ with respect to the value assignment v is defined recursively as follows:

(1) If A is an atomic formula in Atom$(\mathcal{L}^p)$ then $v(f)$ is in $\{0, 1\}$ as defined above.

(2) $v(\neg A) = 1$ if $v(A) = 0$, and 0 otherwise.

(3) $v(A \wedge B) = 1$ if $v(A) = v(B) = 1$,
and 0 otherwise.

(4) $v(A \vee B) = 1$ if $v(A) = 1$ or $v(B) = 1$ (or both),
and 0 otherwise.

(5) $v(A \rightarrow B) = 1$ if $v(A) = 0$ or $v(B) = 1$,
and 0 otherwise.

(6) $v(A \leftrightarrow B) = 1$ if $v(A) = v(B)$,
and 0 otherwise.

Example

Suppose $A = p \vee q \to q \wedge r$, and v is a value assignment such that

$$v(p) = v(q) = v(r) = 1.$$

Example

Suppose $A = p \vee q \to q \wedge r$, and v is a value assignment such that

$$v(p) = v(q) = v(r) = 1.$$

Then we have $v(p \vee q) = 1$, $v(q \wedge r) = 1$ and therefore $v(A) = 1$.

Example

Suppose $A = p \vee q \to q \wedge r$, and v is a value assignment such that

$$v(p) = v(q) = v(r) = 1.$$

Then we have $v(p \vee q) = 1$, $v(q \wedge r) = 1$ and therefore $v(A) = 1$.

Suppose v_1 is another value assignment,

$$v_1(p) = v_1(q) = v_1(r) = 0.$$

Example

Suppose $A = p \vee q \rightarrow q \wedge r$, and v is a value assignment such that

$$v(p) = v(q) = v(r) = 1.$$

Then we have $v(p \vee q) = 1$, $v(q \wedge r) = 1$ and therefore $v(A) = 1$.

Suppose v_1 is another value assignment,

$$v_1(p) = v_1(q) = v_1(r) = 0.$$

Then we have $v_1(p \vee q) = 0$, $v_1(q \wedge r) = 0$ and therefore $v_1(A) = 1$.

Example

Suppose $A = p \vee q \to q \wedge r$, and v is a value assignment such that

$$v(p) = v(q) = v(r) = 1.$$

Then we have $v(p \vee q) = 1$, $v(q \wedge r) = 1$ and therefore $v(A) = 1$.

Suppose v_1 is another value assignment,

$$v_1(p) = v_1(q) = v_1(r) = 0.$$

Then we have $v_1(p \vee q) = 0$, $v_1(q \wedge r) = 0$ and therefore $v_1(A) = 1$.

If v_2 is a third value assignment such that $v_2(p) = 1$ and $v_2(r) = v_2(q) = 0$, then $v_2(A) = 0$.

The above example illustrates that the values which various value assignments assign to a formula may or may not be different.

Satisfiability

Definition. We say that a value assignment v satisfies a formula $A \in \mathcal{L}^p$ iff $v(A) = 1$.

We use the Greek letter Σ to denote any set of formulas.

Define

$$v(\Sigma) = \begin{cases} 1 & \text{if for each formula } B \in \Sigma, v(B) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Satisfiability

Definition. We say that a value assignment v satisfies a formula $A \in \mathcal{L}^p$ iff $v(A) = 1$.

We use the Greek letter Σ to denote any set of formulas.

Define

$$v(\Sigma) = \begin{cases} 1 & \text{if for each formula } B \in \Sigma, v(B) = 1, \\ 0 & \text{otherwise} \end{cases}$$

Definition. A set of formulas Σ is satisfiable iff there exists a value assignment v such that $v(\Sigma) = 1$.

When $v(\Sigma) = 1$, then v is said to satisfy Σ, and Σ is said to be true under v.

Observations

Note that $v(\Sigma) = 1$ means that under the value assignment v, all the formulas of Σ are true.

Note that $v(\Sigma) = 0$ means that for at least one formula $B \in \Sigma$, $v(B) = 0$.

(This does not necessarily mean that $v(C) = 0$ for every formula C in Σ.)

Tautologies and contradictions

- Definition A formula A is a tautology iff it is true under all possible value assignments, i.e. iff for any value assignment v, $v(A) = 1$.
- Definition. A formula A is a contradiction iff it is false under all possible value assignments, i.e., iff for any value assignment v, $v(A) = 0$.
- Definition A formula that is neither a tautology nor a contradiction is called contingent.

Determining that a formula is a tautology

- The most direct way to determine whether a formula is a tautology, a contradiction, or a contingent is by means of truth tables.
- If $p_1, p_2, \ldots, p_n$ are the n different propositional variables occurring in A, there are 2^n possible different value assignments for $p_1, p_2 \ldots p_n$. In each of the cases, that values of A can be obtained effectively.

Tautology example

Consider the formula $\neg(p \wedge q) \vee q$. Is this formula a tautology?

Tautology example

Consider the formula $\neg(p \wedge q) \vee q$. Is this formula a tautology?
Truth table for a tautology

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg(p \wedge q) \vee q$
1	1	1	0	1
1	0	0	1	1
0	1	0	1	1
0	0	0	1	1

Important tautologies

Law of the excluded middle states that $p \vee \neg p$ is a tautology. In other words, p is either true or false, everything else is excluded.

Important tautologies

Law of the excluded middle states that $p \vee \neg p$ is a tautology. In other words, p is either true or false, everything else is excluded.

The table below proves this law by showing that it is true for all value assignments:

p	$\neg p$	$p \vee \neg p$
1	0	1
0	1	1

Tautology: Observations

If A is a tautology that contains the variable p, one can determine a new expression by replacing all instances of p by an arbitrary formula. The resulting formula A' is a tautology.

For example, $p \vee \neg p$ is a tautology.
Replace all instances of p by any formula we like, say by $p \wedge q$.
The resulting formula $A' = (p \wedge q) \vee \neg(p \wedge q)$ is again a tautology.

Theorem. Let A be a tautology and let $p_1, p_2, \ldots p_n$ be the propositional variables of A. Suppose that $B_1, B_2, \ldots B_n$ are arbitrary formulas. Then, the formula obtained by replacing p_1 by B_1, p_2 by B_2... p_n by B_n is a tautology.
Example. Use the fact that $\neg(p \wedge q) \vee q$ is a tautology to prove that $\neg((p \vee q) \wedge r) \vee r$ is a tautology.

Contradiction example

An expression is a contradiction if it yields false under all assignments.
Prove that $p \wedge \neg p$ is a contradiction.

Contradiction example

An expression is a contradiction if it yields false under all assignments.
Prove that $p \wedge \neg p$ is a contradiction.

Truth table for a contradiction

p	$\neg p$	$p \wedge \neg p$
1	0	0
0	1	0

Contradiction example

An expression is a contradiction if it yields false under all assignments.
Prove that $p \land \neg p$ is a contradiction.

Truth table for a contradiction

p	$\neg p$	$p \land \neg p$
1	0	0
0	1	0

Contradictions are related to tautologies: A is a tautology if and only if $\neg A$ is a contradiction.