Asset Pricing Model

1. Introduction:

Overcoming the drawbacks of CAPM and APT, in asset pricing model, we estimate the price of options for an underlying stock at different time steps beyond one period. Considering the asset price dynamics such as drift and volatility, various strategies are discussed in this assignment which are explained below and the results from various methods are compared.

2. Model Formulation:

European option is considered for the purpose of this study. In this option, execution happens only at its maturity. The underlying stock price at a future time period is assumed to follow Geometric Brownian motion and the stock price can be given by **Black-Scholes** partial differential equation as:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Where V is the price of the option as a function of stock price 'S' and time 't', R is risk free rate. Call and Put values corresponding to the stocks at time 't' are given by C(S, t) and P(S, t):

$$\begin{aligned} P(S,t) &= Ke^{-r(T-t)} - S + C(S,t) \\ &= \mathcal{N}(-d_2)Ke^{-r(T-t)} - \mathcal{N}(-d_1)S \end{aligned}$$

$$C(S,t) = \mathcal{N}(d_1)S - \mathcal{N}(d_2)Ke^{-r(T-t)}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right]$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

The results thus obtained are compared against results obtained from generating simulations using Monte Carlo method. Using finite number of time steps and finite number of paths, the stock price is computed for call and put options. The given '*mu*' and '*sigma*' were assumed to be annual drift and volatility respectively. The appropriate boundary conditions for a European option are

$$f_{call} = max\ (S - K;\ 0)$$

$$f_{put} = max\ (K - S;\ 0)$$

where S is the stock price and K is the strike price.

One-Step Monte Carlo:

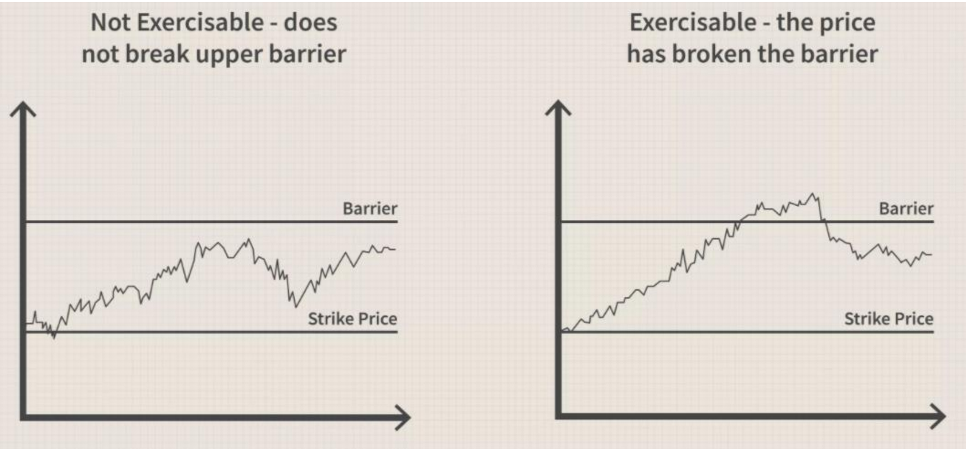
In one step Monte Carlo, the time step dT is considered as 1 i.e., dT = T (Time to Maturity), and price 'S' is computed at each step. If the number of paths chosen is small, the discretization error will be high, and we cannot get better approximation of the results from Black Scholes's whereas, increase in the number of paths increases computational time and cost. Hence, for various values from the range [10, 100, 1000, 5000, 7500] are chosen and the error is computed, then an optimal value for number of paths is chosen.

Multi-Step Monte Carlo:

In multi-step simulation, the time T is divided into small time steps and the price 'S' is computed at each time step. A set of significant time steps such as [4, 52, 252, 500] are chosen. Using 1 year to maturity, 4-time steps represent one movement per quarter, 52-time steps represent weekly movements and 252-time steps are used to account for the 252 trading days per year. An additional value of 500-time steps roughly accounts for two movements per day, and optimal value is selected by analyzing the error of call and put options at maturity to that of Black-Scholes equation's.

Barrier one-step/multi-step knock in Monte Carlo:

A barrier option is a type of option whose payoff depends on whether the underlying asset has reached or exceeded a predetermined price - barrier. There are two general types of barrier options, in and out options. A knock-out option only pays off if the stock does not hit the barrier level throughout the life of the option. If the stock hits a specified barrier, then it has knocked out and expires worthless. A knock-in option on the other hand only pays out if the barrier is crossed during the life of the option. The one-step and multi-step processes explained above are performed using this barrier fundamental too and the results are explained in subsequent sections.

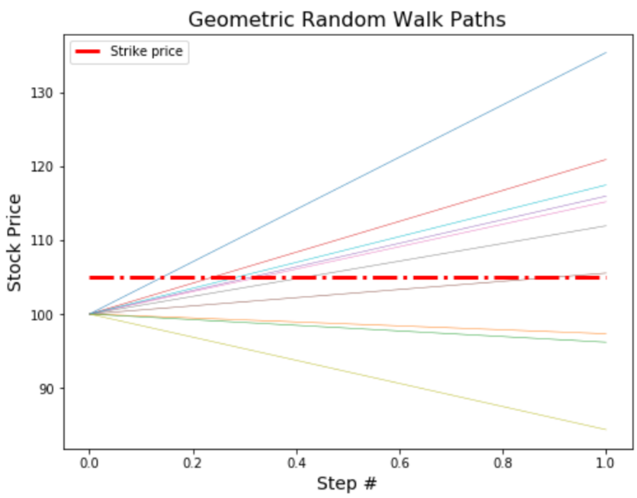


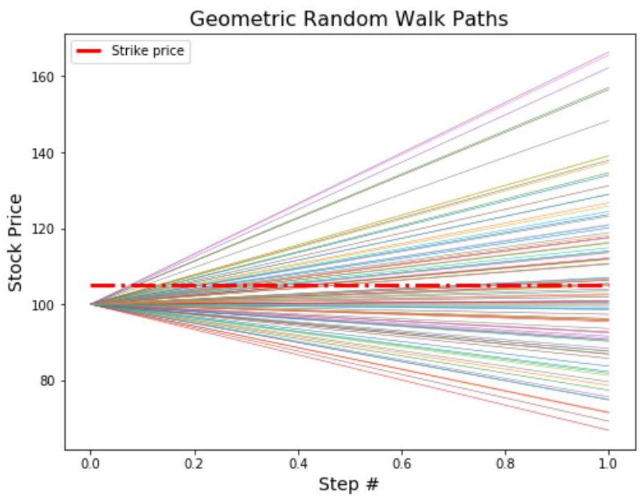
3. Results and Analysis:

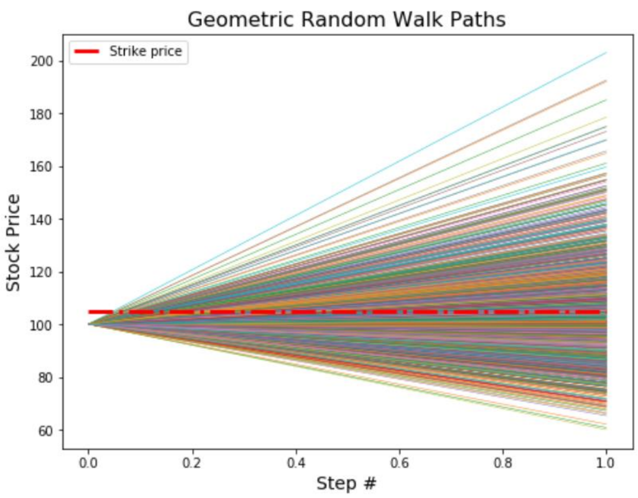
For Number of paths - One step MC				
# of paths	Put	Call	Difference	
			p	c
Black-Scholes	7.9004	8.0213	0.00%	0.00%
10	3.5312	8.3080	-55.30%	3.57%
100	6.9048	8.7871	-12.60%	9.55%
1000	7.9861	8.3059	1.08%	3.55%
2500	7.7252	8.3624	-2.22%	4.25%
5000	7.8230	7.9218	-0.98%	-1.24%
7500	7.8309	8.0734	-0.88%	0.65%

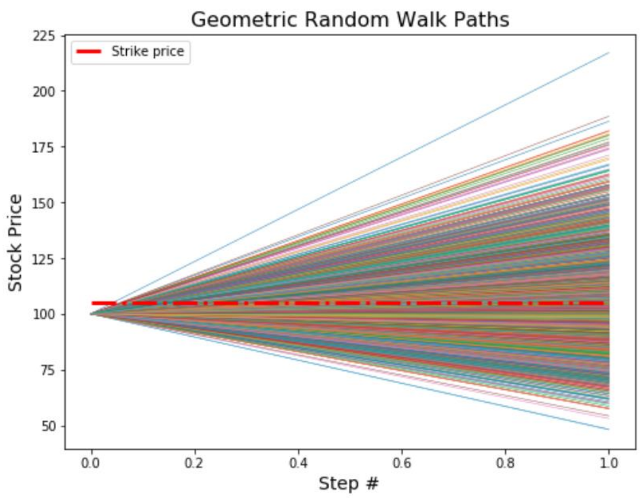
Call and put values for stock prices using Black-Scholes are tabulated above, and the results from MC simulations for various number of paths for single step are obtained and the error from each scenario is calculated. It is observed that, as the number of paths is increased for a given time step, the values of simulation are converging towards the benchmark value and at the same time computation time is increased linearly. Also, error post the number of paths as 7500 is approximately same. **Hence, number**

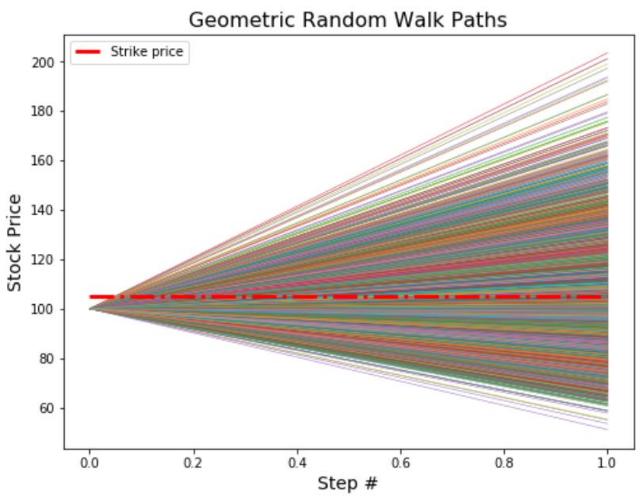
of paths are chosen to be 7500 for all methods in this study. Stock prices for various number of paths are shown below for paths [10, 100, 1000, 2500, 5000, 7500] respectively.

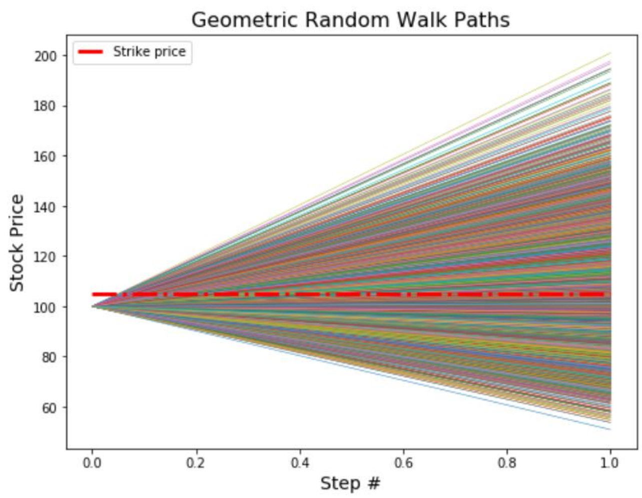








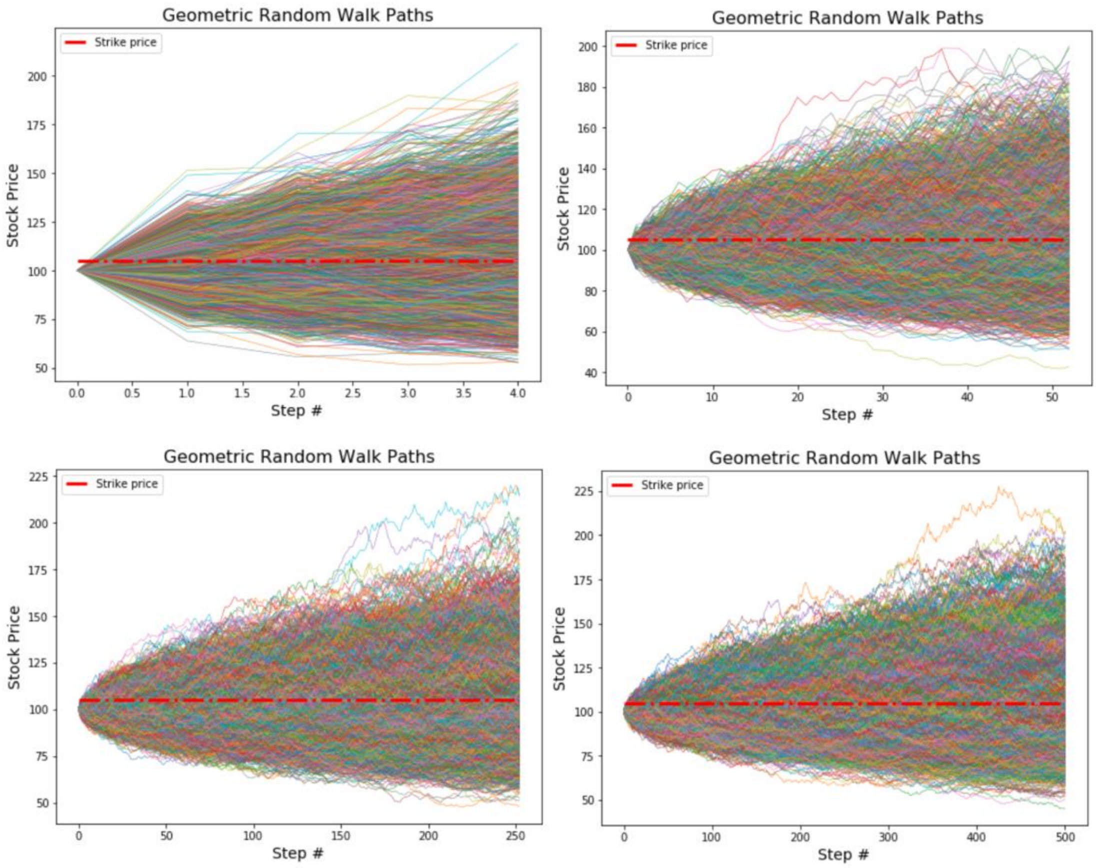




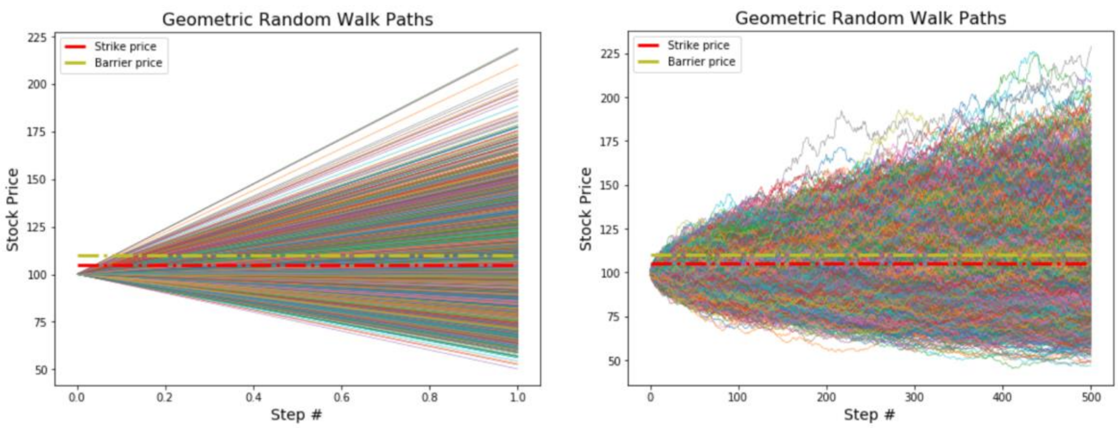
As the number of paths increases, we get a greater number of samples for estimating stock price as shown in the above graphs. Keeping the number of paths constant, now we perform similar analysis by varying number of timesteps by diving the time to maturity into smaller parts. Even in this analysis, as we increase number of steps, a greater number of samples re obtained for estimation of stock price. The call and put values of the stock at last time step (at Maturity) for different step values are tabulated below. Also, it is evident that the error of greater number of steps is less in both call and put options. **Hence, optimal number of steps is chosen as 500** since the discretization error is less than 1% and any further increase in number of steps makes the analysis cost inefficient.

For Number of Steps - Multi step MC				
# of Steps	Put	Call	Difference	
			p	c
Black-Scholes	7.9004	8.0213	0.00%	0.00%
4	7.9973	7.8001	1.23%	-2.76%
52	8.0463	7.8693	1.85%	-1.89%
252	8.0690	7.8116	2.13%	-2.61%
500	7.9688	7.9903	0.87%	-0.39%

Below graphs shows the GRW paths of stock price is waive with increase in the number of steps. The step values are considered [4, 52, 252, 500] respectively for below figures.



Using the same optimal number of steps and paths as that of MC method, the below results are obtained for barrier knock-in single and multi-steps. For barrier knock-in method, of many paths considered, only few satisfies either put or call condition in multistep. Hence a greater number of paths are needed to obtain sufficient valid samples for estimation of stock price, but for a fair comparison between different methods, the optimal values for number of steps and number of paths in all methods are kept the same. Below graphs shows GRW paths for 7500 paths and single step and 500 steps



Comparision of various strategies				
Strategies	Put	Call	Difference	
			p	c
Black-Scholes	7.9004	8.0213	0.00%	0.00%
One Step MC	7.8309	8.0734	-0.88%	0.65%
Multi Step MC	7.9688	7.9903	0.87%	-0.39%
Single Step Knock-in	-	7.9244	-100.00%	-1.21%
Multi Step Knock-in	2.0595	8.2086	-73.93%	2.34%

Comparison of strategies:

For 7500 paths and 500 steps, the above results show the comparison between different strategies. Since Black Scholes formula is calculated by solving the differential equation for an asset with infinitesimal time intervals, thus it gives the most accurate approximation of the options pricing. The barrier option pricing was not as close to approximating the price of the option since barrier option adds a constraint in the payoff which leads to its deviation from the estimate of the Black Scholes formula.

Comparison of call and put in Barrier and European option:

Above results show a different value for call and put in European and Barrier options because, as explained in methodology, Barrier option can be visualized as a subset of European option. Due to imposing a additional constraint, many paths which still gives a payoff are not considered in barrier option because they do not cross the barrier mark at any time step before the. Maturity date. The put price for the one step barrier option is 0 because the path that crosses the barrier never retraces back below the barrier in one step method.

Also, it can be noted that the discretization error for both call and put are moving inversely as the number of steps are increased contrary to the trend that has been followed in European option where the error is decreasing with increase in number of steps for both call and put options.

For Number of Steps - Multi step MC Barrier Knock-in				
# of Steps	Put	Call	Difference	
			p	c
Black-Scholes	7.9004	8.0213	0.00%	0.00%
4	0.919	8.4774	-88.37%	5.69%
52	2.1866	8.8456	-72.32%	10.28%
252	2.5149	8.5183	-68.17%	6.20%
500	2.7186	8.6975	-65.59%	8.43%

Effect of variance on Barrier option:

To analyze the effect of variance on call and put payoffs, we vary the volatility upwards and downwards by 10% and calculate the call and put option values at the time of maturity and tabulated as below. It can be seen that both the call and put payoffs are proportional to the volatility. With increase in volatility, we can observe an increase in both single and multi-step payoffs.

Effect of Variance on call and put payoffs					
Strategies	Variance	Put	Call	Difference	
				p	c
Single Step Knock-in	0.18	-	7.2482	-	-8.53%
	0.20	-	7.9244	-	-
	0.22	-	8.7728	-	10.71%
Multi Step Knock-in	0.18	1.6569	7.0748	-19.55%	-13.81%
	0.20	2.0595	8.2086	-	-
	0.22	2.4688	8.8701	19.87%	8.06%

Graphs corresponding to 0.18% and 0.22% volatility are plotted below for single and multi-step methods with 500 steps and 7500 paths.