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REGULARIZATION USING QR FACTORIZATION AND THE ESTIMATION OF THE OPTIMAL PARAMETER *

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Abstract.

In this paper we propose a direct regularization method using QR factorization for solving linear discrete ill-posed problems. The decomposition of the coefficient matrix requires less computational cost than the singular value decomposition which is usually used for Tikhonov regularization. This method requires a parameter which is similar to the regularization parameter of Tikhonov's method. In order to estimate the optimal parameter, we apply three well-known parameter choice methods for Tikhonov regularization.

AMS subject classification: 65R30, 25F20.

Key words: Ill-posed problems, regularization, QR factorization, parameter choice.

1 Introduction.

In this paper we consider a linear discrete ill-posed problem

$$A\boldsymbol{x} = \boldsymbol{b}, \quad A \in \mathbb{R}^{m\times n}, \quad m \geq n, \tag{1.1}$$

where A is an ill-conditioned matrix and $\boldsymbol{b}$ is a vector which consists of data measured discretely with errors. Typically, these problems arise from continuous ill-posed problems such as Fredholm integral equations of the first kind

$$\int_0^1 K(s,t)f(t)dt = g(s), \quad 0 \leq s \leq 1. \tag{1.2}$$

Due to the ill-posedness, it is necessary to regularize equation (1.1). Tikhonov regularization [1, 3] using Singular Value Decomposition (SVD) is a well-known method for equation (1.1). The solution of Tikhonov regularization is defined as the unique minimizer of the functional

$$F_\lambda^{(\mathrm{S})}(\boldsymbol{x}) = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 + \lambda^2\|\boldsymbol{x}\|^2, \quad \lambda > 0, \tag{1.3}$$

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where λ is a regularization parameter. The symbol (S) of $F_\lambda^{(\mathrm{S})}$ in (1.3) simply represents the formulation for the regularization using SVD, while we use the symbol (Q) for our formulation based on QR factorization for clear distinction (see e.g., (2.4)–(2.6)). Throughout this paper let $\|\cdot\|$ be the Euclidean norm. Let the SVD of A be

$$A = U\Sigma V^{\mathrm{T}} = \sum_{i=1}^{n} \sigma_i \boldsymbol{u}_i \boldsymbol{v}_i^{\mathrm{T}}, \quad \Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n), \tag{1.4}$$

where $\boldsymbol{u}_i$ and $\boldsymbol{v}_i$ are the left and right singular vectors respectively and σ_i are the singular values. The minimizer $\boldsymbol{x}_\lambda^{(\mathrm{S})}$ of the functional (1.3) is given using the SVD (1.4) by

$$\boldsymbol{x}_\lambda^{(\mathrm{S})} = V(\Sigma^2 + \lambda^2 I)^{-1}\Sigma U^{\mathrm{T}}\boldsymbol{b} = \sum_{i=1}^{n} f_i^{(\mathrm{S})} \frac{\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b}}{\sigma_i}\boldsymbol{v}_i, \tag{1.5}$$

where $f_i^{(\mathrm{S})}$ are filter factors [6, 7] defined by

$$f_i^{(\mathrm{S})} = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}, \quad i = 1, \ldots, n. \tag{1.6}$$

The filter factors f_i have the property that the error in $\boldsymbol{b}$ are effectively filtered out as σ_i decreases.

In this paper we introduce a direct regularization method using QR factorization. The decomposition of A is obtained by applying QR factorization two times and the cost for this decomposition is less than that for SVD. In this method, the solution is regularized using filter factors determined by a regularization parameter which have the same form as (1.6).

In regularization methods, it is necessary to choose a good value of regularization parameter which gives the level of filtering. Three well-known parameter choice methods for the Tikhonov regularization, Generalized Cross Validation (GCV) [4], L-curve [6, 7] and Quasi-Optimal [8, 9], are applied to estimate the optimal parameter λ.

2 Regularization using QR factorization.

In this section we present a regularization method using QR factorization for discrete ill-posed problems (1.1). Here, we consider the decomposition of A as follows:

$$\begin{aligned} &A = UDRV^{\mathrm{T}}, \\ &U = [\boldsymbol{u}_1, \ldots, \boldsymbol{u}_k] \in \mathbb{R}^{m\times k}, \qquad V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k] \in \mathbb{R}^{n\times k}, \\ &D = \mathrm{diag}(d_1, \ldots, d_k), \quad d_1 \geq \cdots \geq d_k > 0, \end{aligned} \tag{2.1}$$

where k is the numerical rank of A determined by the QR factorization with column pivoting for some tolerance $\varepsilon > 0$ [5], U and V are orthonormal matrices and R is a well-conditioned upper triangular matrix. This can be computed cheaply using QR factorization (see Algorithm 2.1).

Since the Moore–Penrose inverse of A is written as

$$A^{\dagger} = VR^{-1}D^{-1}U^{\mathrm{T}}, \tag{2.2}$$

the least square solution of minimal norm for (1.1) is given by

$$\boldsymbol{x}_0 = VR^{-1}D^{-1}U^{\mathrm{T}}\boldsymbol{b} = \sum_{i=1}^{k} \frac{\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b}}{d_i}\boldsymbol{w}_i, \tag{2.3}$$

where $W \equiv VR^{-1} = [\boldsymbol{w}_1, \dots, \boldsymbol{w}_k]$. However, if the data $\boldsymbol{b}$ includes noise, the solution computed by (2.3) includes large oscillation due to the ill-posedness of (1.1).

We now consider a regularized solution given by

$$\boldsymbol{x}_\lambda^{(\mathrm{Q})} \equiv \sum_{i=1}^{k} f_i^{(\mathrm{Q})} \frac{\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b}}{d_i}\boldsymbol{w}_i, \tag{2.4}$$

where $f_i^{(\mathrm{Q})}$ are filter factors defined by

$$f_i^{(\mathrm{Q})} \equiv \frac{d_i^2}{d_i^2 + \lambda^2}, \quad i = 1, \dots, k. \tag{2.5}$$

The solution (2.4) is also written as

$$\boldsymbol{x}_\lambda^{(\mathrm{Q})} = VR^{-1}(D^2 + \lambda^2 I)^{-1}DU^{\mathrm{T}}\boldsymbol{b}. \tag{2.6}$$

The filter factors (2.5) play a role to reduce the influence of the noise in the vector $\boldsymbol{b}$ corresponding to the small diagonal elements d_i and we have the following results.

THEOREM 2.1. *Let $\boldsymbol{x}_\lambda^{(\mathrm{Q})}$ be as in* (2.4). *Then*

$$\boldsymbol{x}_\lambda^{(\mathrm{Q})} \longrightarrow A^{\dagger}\boldsymbol{b} \quad \text{as} \quad \lambda \longrightarrow 0,$$
$$\boldsymbol{x}_\lambda^{(\mathrm{Q})} \longrightarrow \mathbf{0} \quad \text{as} \quad \lambda \longrightarrow \infty.$$

PROOF. The proof follows immediately from (2.3), (2.4) and (2.5). □

From Theorem 2.1 it follows that the solution (2.4) is considered as an approximation of the least square solution of minimal norm.

This regularization has the following characterization:

THEOREM 2.2. *Let $\boldsymbol{x}_\lambda^{(\mathrm{Q})}$ be as in* (2.4). *Then $\boldsymbol{x}_\lambda^{(\mathrm{Q})}$ is the unique minimizer of the minimization problem*

$$\min_{X} F_\lambda^{(\mathrm{Q})}(\boldsymbol{x}),$$

where

$$F_\lambda^{(\mathrm{Q})}(\boldsymbol{x}) = \|A\boldsymbol{x} - \boldsymbol{b}\|^2 + \lambda^2\|RV^T\boldsymbol{x}\|^2, \quad X = \mathrm{span}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}. \tag{2.7}$$

Proof. From (2.7) we have

$$F_\lambda^{(\mathrm{Q})\prime}(\boldsymbol{x}) = 2(A^{\mathrm{T}}A\boldsymbol{x} - A^{\mathrm{T}}\boldsymbol{b} + \lambda^2 V R^{\mathrm{T}} R V^{\mathrm{T}}\boldsymbol{x}), \tag{2.8}$$

and

$$\delta^2 F_\lambda^{(\mathrm{Q})}(\boldsymbol{x},\boldsymbol{h}) = 2\|A\boldsymbol{h}\|^2 + 2\lambda^2\|RV^{\mathrm{T}}\boldsymbol{h}\|^2, \quad \boldsymbol{h} \in X,$$

where

$$\delta^2 F_\lambda^{(\mathrm{Q})}(\boldsymbol{x},\boldsymbol{h}) \equiv \left.\frac{d^2 F_\lambda^{(\mathrm{Q})}(\boldsymbol{x}+t\boldsymbol{h})}{dt^2}\right|_{t=0}.$$

If $\boldsymbol{x},\boldsymbol{h} \in X$ and $\boldsymbol{h} \neq 0$, then we have $\delta^2 F_\lambda^{(\mathrm{Q})}(\boldsymbol{x},\boldsymbol{h}) > 0$. Therefore the functional (2.7) is strictly convex and has a unique minimizer in $\boldsymbol{x} \in X$ [10]. The unique minimizer is characterized by the condition

$$F_\lambda^{(\mathrm{Q})\prime}(\boldsymbol{x})\boldsymbol{h} = 0, \quad \forall \boldsymbol{h} \in X.$$

Now, let $\boldsymbol{x}_\lambda^{(\mathrm{Q})}$ be as in (2.4), then $F_\lambda^{(\mathrm{Q})\prime}(\boldsymbol{x}_\lambda^{(\mathrm{Q})}) = \mathbf{0}$ from (2.8). Hence, (2.4) is the unique minimizer of the functional (2.7). ☐

The regularized solution (2.4) of the linear equation (1.1) can be computed by the following algorithm.

Algorithm 2.1.

1. Decomposition of the coefficient matrix A:
   (a) *Compute the QR factorization of the matrix A^{T} with column pivoting:*
   $$A^{\mathrm{T}}\Pi = VDL^{\mathrm{T}}, \quad V \in \mathbb{R}^{n\times k}, \quad L \in \mathbb{R}^{m\times k}, \quad D = \mathrm{diag}(d_1,\dots,d_k),$$
   where Π is the permutation matrix, L is the lower triangular matrix whose diagonal elements are 1 and k is the numerical rank of A [5].
   (b) *Compute the QR factorization of the matrix ΠL without pivoting:*
   $$\Pi L = U\hat{R}, \quad U \in \mathbb{R}^{m\times k}, \quad \hat{R} \in \mathbb{R}^{k\times k},$$
   where $\hat{R}$ is an upper triangular matrix.
   (c) $R := D^{-1}\hat{R}D$.
2. Choice of the regularization parameter λ (see Section 3).
3. Computation of the regularized solution:
   (a) $y_i := f_i^{(\mathrm{Q})}\dfrac{\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b}}{d_i}\boldsymbol{u}_i$ for $i = 1,\dots,k$, $\boldsymbol{y}_\lambda := [y_1,\dots,y_k]^{\mathrm{T}}$.
   (b) *Compute the solution of $RV\boldsymbol{x}_\lambda^{(\mathrm{Q})} = \boldsymbol{y}_\lambda$ by back-substitution.*

The decomposition of A is obtained by applying Householder QR two times and it requires $2kmn - k^2n + k^3/3$ flops. By comparing its cost with that for SVD (Golub–Reinsch SVD [5]: $2mn^2 - 2n^3/3$ flops) we see that it requires fewer computations, especially in the case where the numerical rank k is comparatively small.

3 Estimation of the optimal regularization parameter.

In regularized solution (2.4), noise in vector $\boldsymbol{b}$ is filtered out by the filter factors (2.5) determined by a regularization parameter λ and we have to select an appropriate regularization parameter λ to obtain a reasonable solution.

In order to estimate the optimal parameter, we apply well-known parameter choice methods for Tikhonov regularization, Generalized Cross-Validation (GCV) [4], the L-curve criterion [6, 7] and Quasi-Optimal [8], to the regularization using QR factorization.

3.1 GCV.

The parameter chosen by the method GCV [4] is defined as the minimizer of the function

$$G(\lambda) \equiv \frac{\|(I - A^I(\lambda))\boldsymbol{b}\|^2}{\mathrm{trace}(I - A^I(\lambda))}, \tag{3.1}$$

where $A^I(\lambda)$ is a matrix which satisfies

$$A\boldsymbol{x}_\lambda = A^I(\lambda)\boldsymbol{b}.$$

This is based on the idea of cross-validation [1].

Now, we consider the function (3.1) for the regularization using QR factorization. From (2.1) and (2.6) the matrix $A^I(\lambda)$ is written as

$$A^I(\lambda) = U(D^2 + \lambda^2 I)^{-1} D^2 U^{\mathrm{T}}$$

and from (2.5) the GCV function (3.1) is given by

$$G^{(\mathrm{Q})}(\lambda) = \frac{\{\sum_{i=1}^k (1 - f_i^{(\mathrm{Q})})\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b}\}^2 + \|\boldsymbol{b} - UU^{\mathrm{T}}\boldsymbol{b}\|^2}{\left(m - \sum_{i=1}^k f_i^{(\mathrm{Q})}\right)^2}. \tag{3.2}$$

The minimizer of this function is the estimation of the optimal regularization parameter for the regularization using QR factorization.

3.2 The L-curve method.

From the Tikhonov functional (1.3) we see that the regularization parameter λ play a role to control the weight between the two values, $\|A\boldsymbol{x} - \boldsymbol{b}\|$ and $\|\boldsymbol{x}\|$. The L-curve for the Tikhonov regularization is defined as a parametric plot of these two values [6, 7], i.e., $(\|A\boldsymbol{x}_\lambda - \boldsymbol{b}\|, \|\boldsymbol{x}_\lambda\|)$ and this curve has a characteristic that it tends to have a corner at the point corresponding to the optimal parameter.

The criterion of this method is the curvature of the L-curve and the parameter which maximize the curvature is the estimation of the optimal regularization parameter.

In regularization using QR factorization, we define the L-curve as

$$(y_1(\lambda), y_2(\lambda)) \equiv (\|A\boldsymbol{x}_\lambda - \boldsymbol{b}\|, \|RV^{\mathrm{T}}\boldsymbol{x}_\lambda\|), \tag{3.3}$$

using two values in the functional (2.7). The curvature $\kappa(\lambda)$ of the L-curve is given as follows:

$$\kappa^{(Q)}(\lambda) = \frac{|y_1'(\lambda)y_2''(\lambda) - y_1''(\lambda)y_2'(\lambda)|}{(y_1'(\lambda)^2 + y_2'(\lambda)^2)^{3/2}}, \tag{3.4}$$

where

$$y'(\lambda) \equiv \frac{dy}{d\lambda} \quad \text{and} \quad y''(\lambda) \equiv \frac{d^2y}{d\lambda^2}.$$

From (2.4), the plot of the L-curve (3.3) is given by

$$y_1(\lambda) = \sum_{i=1}^{k}((1 - f_i^{(Q)})\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b})^2 + \|\Delta\boldsymbol{b}\|^2,$$

$$y_2(\lambda) = \sum_{i=1}^{k}(f_i^{(Q)}\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b})^2,$$

and the curvature (3.4) for each λ can be computed as follows:

$$\kappa^{(Q)}(\lambda) = \frac{|y_1(\lambda)y_2(\lambda)/S - \lambda^2(y_1(\lambda) + \lambda^2 y_2(\lambda))|}{(y_1(\lambda) + \lambda^4 y_2(\lambda))^{3/2}}, \tag{3.5}$$

where

$$S = \sum_{i=1}^{k}\frac{(d_i\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b})^2}{(d_i^2 + \lambda^2)^3}.$$

The formula (3.5) has the same form as that for Tikhonov regularization and we see that the computational cost is the same order.

3.3 Quasi-optimal method.

The Quasi-Optimal method [8, 9] estimates the optimal regularization parameter using the behavior of the differentiation of the error function. In Tikhonov regularization, we use the error function of the regularized solution defined by

$$e^{(S)}(\lambda) \equiv \|\boldsymbol{x}_\lambda^{(S)} - \boldsymbol{x}_0\|, \tag{3.6}$$

where $\boldsymbol{x}_0$ is the least square solution of minimal norm for (1.1). The error function satisfies

$$\left|\frac{d}{d\xi}\|e^{(S)}(\lambda)\|\right| \leq \left\|\frac{d}{d\xi}\boldsymbol{x}_\lambda^{(S)}\right\|, \quad \xi \equiv \log_\beta \lambda, \quad \beta > 1, \tag{3.7}$$

and let

$$\zeta^{(S)}(\lambda) \equiv \left\|\frac{d}{d\xi}\boldsymbol{x}_\lambda^{(S)}\right\|^2.$$

If λ_0 is the minimizer of $e(\lambda)$, then

$$\left|\frac{d}{d\xi}\|e^{(S)}(\lambda_0)\|\right| = 0,$$

and we see from (3.7) that $\zeta^{(\mathrm{S})}(\lambda)^{1/2}$ is one of the upper bound the differentiation of $e^{(\mathrm{S})}(\lambda)$ with respect to ξ. The main idea of Quasi-Optimal method is that the minimizer of $\zeta(\lambda)$ can be an estimation of the optimal regularization parameter from that point of view.

In regularization using QR factorization, we define the error function by

$$e^{(\mathrm{Q})}(\lambda) \equiv \|RV^{\mathrm{T}}(\boldsymbol{x}_\lambda^{(\mathrm{Q})} - \boldsymbol{x}_0)\|, \tag{3.8}$$

where R and V are the matrices in (2.1) and it satisfies the following theorem:

THEOREM 3.1. *Let* $e^{(\mathrm{Q})}(\lambda)$ *be as in* (3.8). *Then*

$$\left|\frac{d}{d\xi}\|e^{(\mathrm{Q})}(\lambda)\|\right| \leq \left\|\frac{d}{d\xi}\boldsymbol{x}_\lambda^{(\mathrm{Q})}\right\| \equiv \zeta^{(\mathrm{Q})}(\lambda)^{\frac{1}{2}}, \quad \xi \equiv \log_\beta \lambda, \tag{3.9}$$

for some $\beta > 1$.

PROOF. From (3.8),

$$\begin{aligned}|e^{(\mathrm{Q})}(\lambda+\Delta\lambda) - e^{(\mathrm{Q})}(\lambda)| &= \left|\|RV^{\mathrm{T}}(\boldsymbol{x}_{(\lambda+\Delta\lambda)}^{(\mathrm{Q})} - \boldsymbol{x}_0)\| - \|RV^{\mathrm{T}}(\boldsymbol{x}_\lambda^{(\mathrm{Q})} - \boldsymbol{x}_0)\|\right| \\ &\leq \|RV^{\mathrm{T}}(\boldsymbol{x}_{(\lambda+\Delta\lambda)}^{(\mathrm{Q})} - \boldsymbol{x}_0) - RV^{\mathrm{T}}(\boldsymbol{x}_\lambda^{(\mathrm{Q})} - \boldsymbol{x}_0)\| \\ &= \|RV^{\mathrm{T}}(\boldsymbol{x}_{(\lambda+\Delta\lambda)}^{(\mathrm{Q})} - \boldsymbol{x}_\lambda^{(\mathrm{Q})})\|,\end{aligned}$$

and we have

$$\frac{1}{\lambda}|e^{(\mathrm{Q})}(\lambda+\Delta\lambda) - e^{(\mathrm{Q})}(\lambda)| \leq \frac{1}{\lambda}\|RV^{\mathrm{T}}(\boldsymbol{x}_{(\lambda+\Delta\lambda)}^{(\mathrm{Q})} - \boldsymbol{x}_\lambda^{(\mathrm{Q})})\|,$$

for any $\lambda > 0$. Let $\lambda \to 0$, then we have

$$\left|\frac{d}{d\lambda}e^{(\mathrm{Q})}(\lambda)\right| \leq \left\|\frac{d}{d\lambda}RV^{\mathrm{T}}\boldsymbol{x}_\lambda^{(\mathrm{Q})}\right\|, \qquad \left|\frac{d\lambda}{d\xi}\cdot\frac{d}{d\lambda}e^{(\mathrm{Q})}(\lambda)\right| \leq \left\|\frac{d\lambda}{d\xi}\cdot\frac{d}{d\lambda}RV^{\mathrm{T}}\boldsymbol{x}_\lambda^{(\mathrm{Q})}\right\|,$$

from which the assertion follows. □

This method uses the parameter which minimizes the function $\zeta^{(\mathrm{Q})}(\lambda)$. From (2.4), this function is given by

$$\zeta^{(\mathrm{Q})}(\lambda) = \sum_{i=1}^{k}\left(\frac{d_i\lambda^2\log\beta}{(d_i^2+\lambda^2)^2}\boldsymbol{u}_i^{\mathrm{T}}\boldsymbol{b}\right)^2. \tag{3.10}$$

The computational cost required for this function is the same order as that of the Tikhonov regularization. Therefore the regularization method proposed here is compatible with these three optimization processes.

4 Numerical results.

In this section, we consider an example of the discrete linear equation arising from the Fredholm integral equation of the first kind (1.2) with kernel K and right-hand side g given by

$$K(s,t) = (s^2+t^2)^{1/2}, \quad g(s) = \left((1+s^2)^{3/2} - s^3\right)/3. \tag{4.1}$$

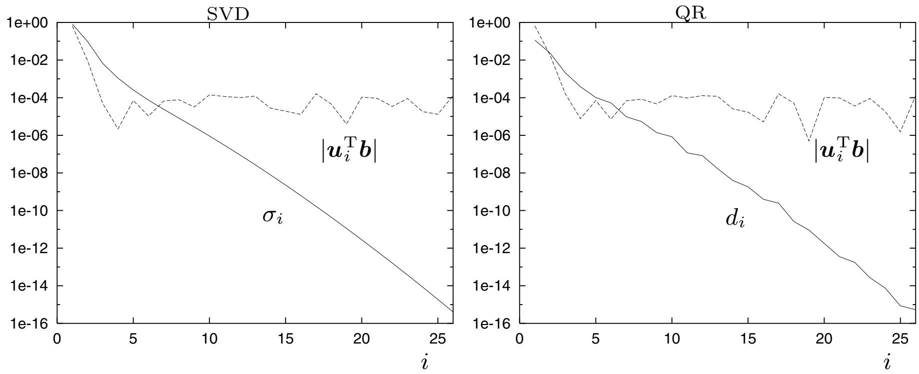

Figure 4.1: Diagonal elements and coefficients for the right-hand side.

Here the exact solution is $f(t) = t$. We assume that data are given at 100 points and the integral equation is discretized by the midpoint rule at 100 points. Here, the size of A is 100×100.

In order to test the influence of the noise, we added random numbers distributed normally with standard deviation 10^{-4} to $\boldsymbol{b}$ and let the right-hand side vector with noise be $\boldsymbol{b}^\delta$. Here $\|\boldsymbol{b}^\delta - \boldsymbol{b}\|/\|\boldsymbol{b}\| \approx 2.10 \times 10^{-4}$.

Table 4.1: CPU-time for both decompositions.

	CPU-time (sec)
$A = U\Sigma V^{\mathrm{T}}$	0.30
$A = UDRV^{\mathrm{T}}$	0.03

We applied two regularization methods to the discrete linear equation. Table 4.1 shows the comparison of the CPU-time for SVD (1.4) and that for the decomposition (2.1). Here, we used $\varepsilon = 1.0 \times 10^{-16}$ as the tolerance for the numerical rank of A and then the numerical rank k is 26. In this method, we see that the cost for (2.1) is about 10 times less than that for the SVD.

The diagonal elements of the diagonal matrices for both methods and the coefficients of $\boldsymbol{b}$ corresponding to the orthogonal basis $\boldsymbol{u}_i$ are shown in Figure 4.1. From Figure 4.1, we see that the diagonal elements d_i in (2.1) are almost the same as the singular values σ_i and that the coefficients behave like those of the Tikhonov regularization. Hence it is expected that the noise in $\boldsymbol{b}$ is effectively filtered out by the filter factors (2.5).

Table 4.2: Regularization parameters and errors.

		Optimal	GCV	L-curve	Quasi-Optimal
SVD	λ	3.98×10^{-3}	5.01×10^{-3}	1.00×10^{-2}	3.98×10^{-3}
	$e(\lambda)$	2.54×10^{-3}	3.00×10^{-3}	5.86×10^{-3}	2.54×10^{-3}
QR	λ	1.00×10^{-3}	7.94×10^{-4}	1.00×10^{-3}	1.00×10^{-3}
	$e(\lambda)$	5.71×10^{-3}	6.37×10^{-3}	5.71×10^{-3}	5.71×10^{-3}

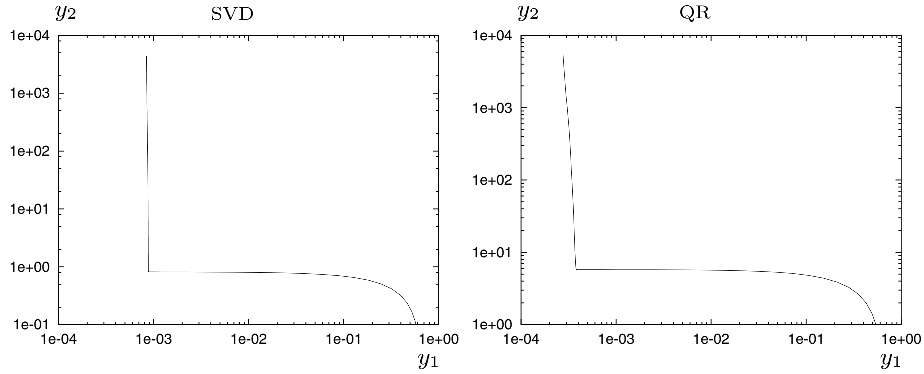

Figure 4.2: L-curves.

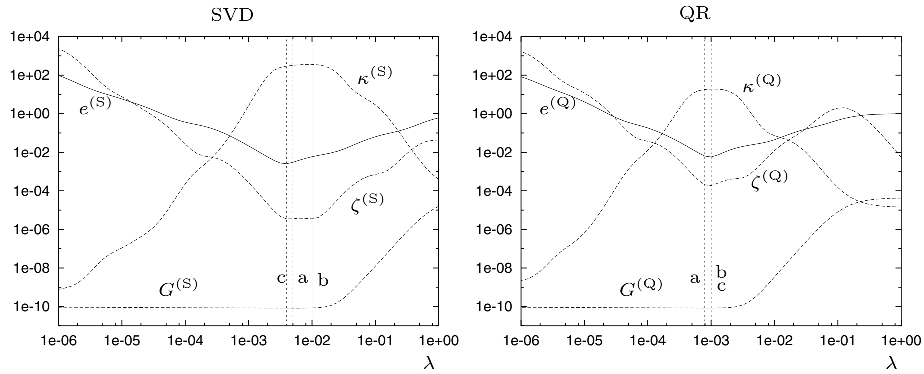

Figure 4.3: Criteria for the choice of regularization parameter and the parameters chosen by (a) GCV, (b) L-curve and (c) Quasi-Optimal.

The L-curves for both methods are shown in Figure 4.2. The L-curve for the regularization using QR factorization also has a distinct corner.

The criteria for the choice of regularization parameters and errors of the regularized solutions defined by $e(\lambda) = \|\boldsymbol{x}_\lambda - \boldsymbol{x}_0\| / \|\boldsymbol{x}_0\|$ are shown in Figure 4.3. Table 4.2 shows the parameters chosen by those criteria and the errors corresponding to these parameters.

We see that the parameters chosen by the criteria for the regularization using QR factorization give solutions which are almost optimal and those precisions are almost same as those given by Tikhonov regularization with SVD.

5 Conclusions.

The regularization using QR factorization is defined by applying the filter factors which have the same form as those of Tikhonov regularization based on the SVD. The cost for the decomposition of the coefficient matrix depends on the numerical rank of the coefficient matrix and is less than that for the SVD especially in cases when the numerical rank is relatively small.

The methods to estimate the optimal regularization parameter, GCV, L-curve and Quasi-Optimal, can be applied in the same way as those for Tikhonov regularization. Numerical results in this paper illustrate that the regularization using QR factorization gives a reasonable solution with less computational cost.

The computation of the regularized solution includes back-substitution of the upper triangular matrix and the upper triangular matrix is assumed to be well-conditioned in (2.1). However, the QR factorization with column pivoting does not always give a well-conditioned upper triangular matrix (see [2, 5]). In order to avoid such situation, it can be considered to use Rank Revealing QR factorization [2] but it requires further discussion.

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