On shortest weak Hamiltonian path problem of the cactus graph

TOPIC : Graph Theory, Combinatorial Optimization

강원과학고등학교 김준혁

quickn@kangwon-sh.gwe.hs.kr

TABLE OF CONTENTS

Shortest weak Hamiltonian path problem

Shortest weak Hamiltonian path problem

1 Shortest weak Hamiltonian path problem

1. An undirected connected simple graph $G = (V, E_G, w_G)$ with non-negative weight function $w_G: V \times V \to \mathbb{R}$
2. Output: A path $p = v_1 \dots v_n$ through all vertices **at least once**
3. Optimization: $\min(\sum_{i=1}^{n-1} w_G(v_i, v_{i+1}))$

Shortest weak Hamiltonian path problem

2 Frequency Misunderstanding(s)

Q It is equivalent to TSP?

A Currently, we haven’t any **known proof**. I think it’s **not true**. Because reduction to the TSP or reduction from the TSP is **not easy**. This problem guarantees that existence of solution for all undirected simple non-negative weighted graph. However Hamiltonian path or Hamiltonian is not.

Shortest weak Hamiltonian path problem

2 Frequency Misunderstanding(s)

Q We have an easy reduction from the Hamiltonian path problem.

A There exists a countercase.

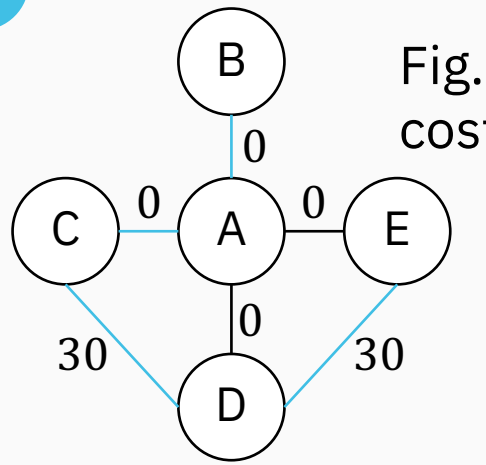


Fig. 1. Optimal cost: 60

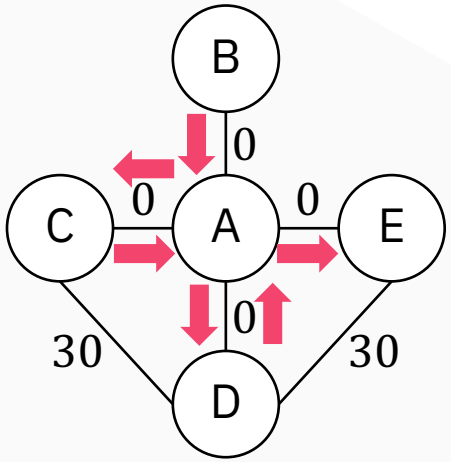


Fig. 2. Optimal cost: 0

Shortest weak Hamiltonian path problem

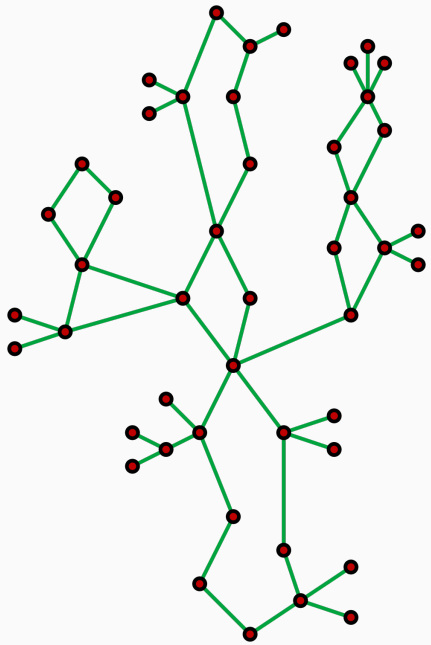
3 Hypothesis

Hypothesis - NP-hard?

- Unknown deterministic-polynomial time algorithm
- It can be considered possible reduction from the Hamiltonian path. (We can use Rahman–Kaykobad's theorem [1] and may select another shortest path of vertex by vertex from the solution of the shortest weak Hamiltonian path)

Shortest weak Hamiltonian path problem

4 Case – Cactus graph



- **Cactus graph** is a finite, connected, undirected, simple graph in which any two simple cycles have at most one vertex in common. [2]
- This study gives a deterministic-polynomial time solution of the Shortest weak Hamiltonian path problem on the cactus graph.

Fig. 3. An example of the cactus graph [3]

Algorithm and correctness

Algorithm and correctness

1 Cycle component transform of the cactus graph

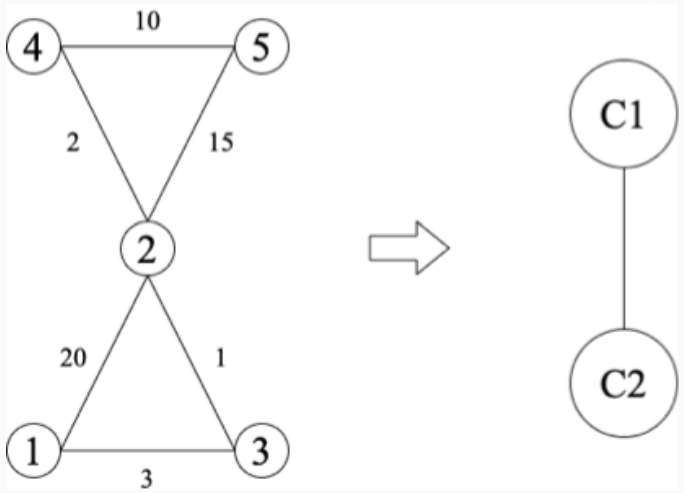


Fig. 5. Graphic Representation of the cycle component transform

- **Cycle component transform of the cactus graph** is a representation of the cycles of the cactus graph.
- Let C be a finite set of the cycles of the cactus graph G.
- INPUT: A cactus graph G
- OUTPUT: A tree $T := (C, E_C)$ where $E_C := \{(C_i, C_j) | C_i, C_j \in C \wedge \exists u \in C_i \cap C_j\}$

Algorithm and correctness

1 Cycle component transform of the cactus graph

- Cycle component transform of the cactus graph can be perform in the $O(|V_G| + |E_G|)$ time.
- Run DFS(Depth-first-search) in the graph and push each vertex to a stack by pre-order and pop vertices belong to a cycle if it detects back edge.

Algorithm and correctness

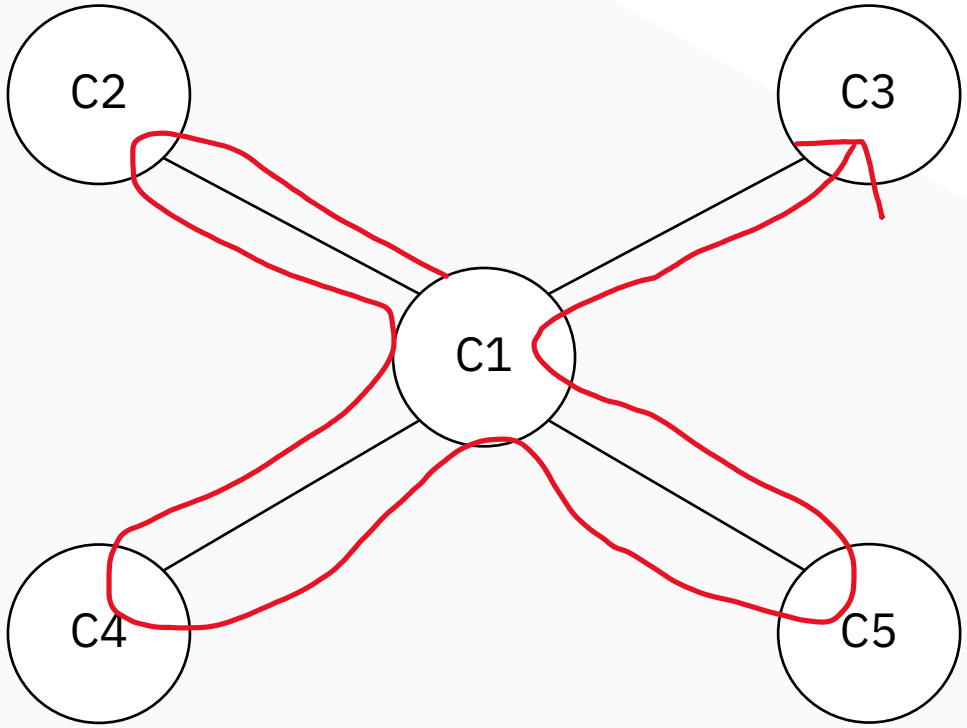


Fig. 6. An example of the optimal path

- We have rooted tree $T = (C, E_C)$ with an arbitrary root $C_r \in C$.
- Firstly, We can find a path through all vertices and visit all edges twice.
- We can remove a path with maximum cost it leads to leaf from root.

Algorithm and correctness

2 STEP 1 – Find optimal in the cycle

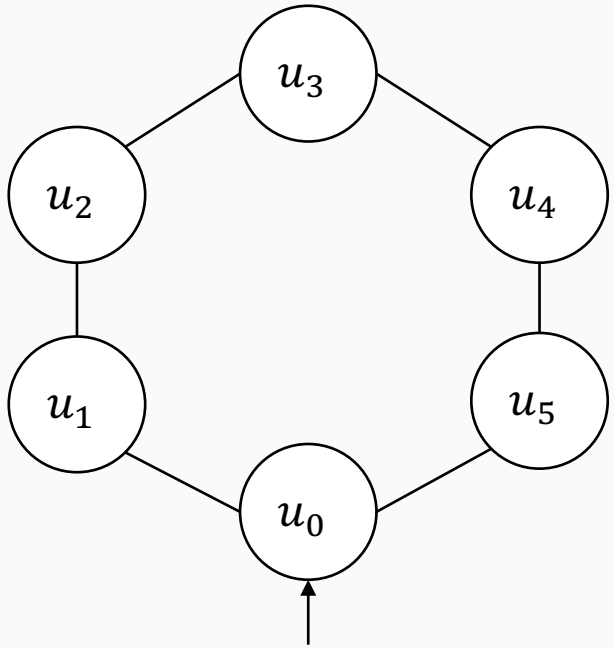


Fig. 7. A cycle with special order.

- For a cycle $C_i \in C$, Let $u_0 \in C_i$ be a vertex lead from the parent of the C_i. (If C_i is a root, u_0 is an arbitrary vertex)
- We have an order left to right $u_0 < \cdots < u_m$. (clockwise order)

Algorithm and correctness

2 STEP 1 – Find optimal in the cycle

- We define

$A_{0k}[i] = (Minimum\ cost\ with\ visit\ i-th\ vertex\ by\ clockwise\ order)$

$A_{1k}[i] = (Minimum\ cost\ with\ visit\ i-th\ vertex\ by\ counterclockwise\ order)$

$B_k[i] = (Minimum\ cost\ of\ a\ path\ through\ all\ vertices\ and\ ends\ at\ u_i)$

$\pi_u^1[v] = (Previous\ vertex\ of\ the\ vertex\ v\ in\ the\ optimal\ path)$

- The time complexity is $O(|V_G| + |E_G|)$

Algorithm and correctness

2

STEP 1 – Find optimal in the cycle

Algorithm 1 위 부분 문제를 해결하는 알고리즘

INPUT: Cycle component $C_k \in C$
OUTPUT: Arrays A_{0k}, A_{1k}, B_k
Let A_{0k}, A_{1k} *be* $0-$*indexed arrays with size* $m+1$
$A_{0k}[0] \leftarrow 0,\ A_{1k}[0] \leftarrow 0$
Let π_k^1 *be arrays with size* $m+1$
for $i \leftarrow 1$ to $|C_i|$, **do**
  $A_{0k}[i] \leftarrow A_{0k}[i-1] + w(u_{i-1}, u_i)$
  $A_{1k}[i] \leftarrow A_{1k}[i-1] + w(u_{(m-i+2)\ mod\ (m+1)}, u_{m-i+1})$
Let B_k *be an array with size* $m+1$
$\forall i \in \{0, \dots, m\}: B_k[i] \leftarrow \infty$
for $i \leftarrow 1$ to $m-1$, **do**
  if $B_k[i+1] > 2A_{0k}[i] + A_{1k}[m-i]$, **then**
    $B_k[i+1] \leftarrow 2A_{0k}[i] + A_{1k}[m-i]$)
    $\pi_k^1[i+1] \leftarrow (i, 1)$

Algorithm and correctness

2

STEP 1 – Find optimal in the cycle

```
16  if B_k[m − i] > 2A_1k[i] + A_0k[m − i], then
17    B_k[m − i] ← 2A_1k[i] + A_0k[m − i]
18    π_k^1[m − i] ← (m − i + 1,0)
19  if B_k[1] > A_1k[m], then
20    B_k[1] ← A_1k[m]
21    π_k^1[1] ← (0,0)
22  if B_k[m] > A_0k[m], then
23    π_k^1[m] ← (0,1)
24    B_k[m] ← A_0k[m]
25  for i ← 1 to m, do
26    if B_k[0] > B_k[i] + min(A_0k[i], A_1k[m − i + 1]), then
27      if A_0k[i] ≤ A_1k[m − i + 1], then
28        B_k[0] ← B_k[i] + A_0k[i]
29        π_k^1[0] ← (i, 1)
30      else
31        B_k[0] ← B_k[i] + A_1k[m − i + 1]
32    π_k^1[0] ← (i, 0)
```

Algorithm and correctness

2 Correctness – Find optimal in the cycle

Lemma 1. 위와 같은 cycle의 최단 경로에서
한 방향(clockwise or counterclockwise)으로
진행하다가 다른 방향으로 전환되는 것을 ‘틀어짐’이라고 할때,
이런 ’틀어짐’은 최대 한번만 발생한다.

Algorithm and correctness

2 Correctness – Find optimal in the cycle

Proof. Let define $S_i := \sum_{e \in E_i} w_G(e)$ and assume that
It occurs at u_i. Then, we have $w_G\left(u_i, u_{(i+1) mod (m+1)}\right) > S_i/2$.
And assume that it occurs at another u_j. And it implies that
$w_G\left(u_j, u_{(m+1+j-1) mod (m+1)}\right) > S_i/2$. Then, which satisfies
$w_G\left(u_i, u_{(i+1) mod (m+1)}\right) + w_G\left(u_j, u_{(m+1+j-1) mod (m+1)}\right) > S_i$.
Which is a contradiction. Thus, It occurs at most once. ■

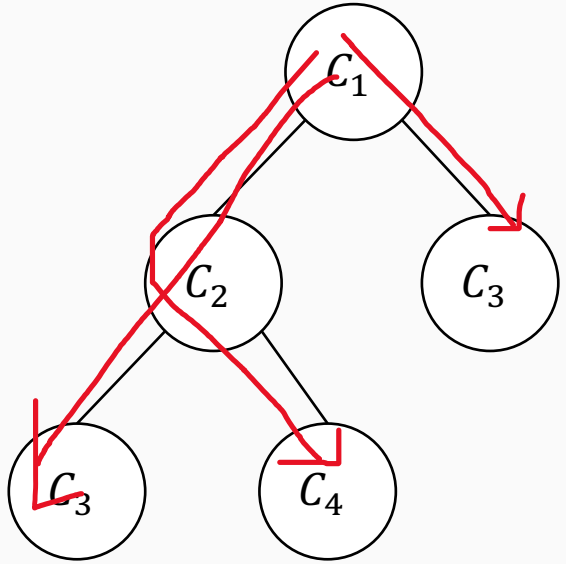
Algorithm and correctness

2 Correctness – Calculation in the cycle

Theorem 1. 위 알고리즘을 사이클 C_k에 대하여 호출한 뒤, $B_k[0]$은 사이클 C_k의 모든 정점을 순회하는 회로의 최소 비용이다.

Algorithm and correctness

2 STEP 2 – Lead to leaf



- We define a dynamic programming array $D[i] \coloneqq \left(\max_{(C_j,C_i)\in E_C,\pi[i]\neq j} D[j]\right) + B_i[0] - \min(A_{0i}[m], A_{1i}[m])$.
- By optimizing this, We have an optimal longest path root to leaf.
- The time complexity is $O(|C| + |E_C|)$.

Fig. 8. An example of the path root to leaf.

Algorithm and correctness

2

STEP 2 – Lead to leaf

Algorithm 2 배열 D, π^2을 채우는 알고리즘

INPUT: Tree $T = (C, E')$ **and arrays** A_{0k}, A_{1k}, B_k **for all** $C_k \in C$
OUTPUT: An array π^2
Let E, π^2 *be an array with size* $|C|$
$\forall C_i \in C: D[i] = 0$
procedure $DFS(u)$
  for $(C_j, C_u) \in E' \wedge \pi[u] \neq j$, **do**
    call DFS(j)
    if $D[u] < D[j]$, **then**
      $D[u] \leftarrow D[j]$
      $\pi^2[u] \leftarrow j$
  $D[u] \leftarrow D[u] + B_i[0] - \min(A_{0k}[m], A_{1k}[m])$

Algorithm and correctness

2 STEP 3 – Generate a path

- Primary Idea: Construct a shortest path through all vertices in C. And subtract longest path leads to leaf.
- Let $Component(u)$ be a function k which satisfies $u \in C_k$
- Let $IsBelongOptimal(u)$ is a boolean function that decides belong to the optimal path.

Algorithm and correctness

2

STEP 3 – Generate a path

Algorithm 3 최적의 경로 p를 생성하는 알고리즘이다.

INPUT: Tree $T = (C, E')$ and cactus graph $G = (V, E)$, an array π^2 and for all $C_k \in C$, $A_{0k}, A_{1k}, B_k, \pi^1_k$
OUTPUT: Optimal solution path p
Let p be a dynamic array
procedure $Move(S, m, u, v, b)$
  if $b == 1$, **then**
    $i \leftarrow u$
    while $i \neq v$, **do**
      call $Push(S, i)$
      $i \leftarrow (i + 1)\ mod\ (m + 1)$
    call $Push(S, i)$
  else
    $i \leftarrow u$
    while $i \neq v$, **do**
      call $Push(S, i)$
      $i \leftarrow (m + i)\ mod\ (m + 1)$
    call $Push(S, i)$
procedure $DFS2(u)$
  $\{u_0, \dots, u_m\} \leftarrow AllVertices(u)$
  Let V be an array with size $m + 1$
  Let S be an stack
  $\forall i \in \{0, \dots, m\}: V[i] = false$
  $t \leftarrow \pi^1_u[0]$
  $s \leftarrow t.first$
  if $IsBelongOptimal(u) \neq true$, **then**
    call $Move(S, m, 0, t.first, t.second)$
    $t \leftarrow \pi^1_u[s]$
    call $Move(S, m, s, t.first, t.second)$
    $s \leftarrow t.first$
    if $A_{0u}[s] < A_{1u}[s]$, **then**
      call $Move(S, m, s, 0, 0)$
    else
      call $Move(S, m, s, 0, 1)$
  else
    $u_{min} \leftarrow 0, val_{min} \leftarrow \infty$

Algorithm and correctness

2

STEP 3 – Generate a path

```
36   for i ← 1 to m, do
37    if val_min > B_u[i], then
38     val_min ← B_u[i]
39     u_min ← i
40   t ← π_u^1[u_min]
41   call Move(S, m, u_min, t.first, t.second)
42   s ← t.first
43   if A_0k[s] < A_1k[s], then
44    call Move(S, m, s, 0,0)
45   else
46    call Move(S, m, s, 0,1)
47  while IsEmpty(S) = false, do
48   v ← Pop(S)
49   call Pushback(p, u_v)
50   if V[v] = false, then
51    V[v] ← true
52    for ∀(q, u_v) ∈ E, do
53     if Component(v_v) ≠ Component(q), then
54      call DFS2(q)
55
```

Conclusion and its applications

Conclusion and its applications

1 Conclusion

1. NP-hard 문제로 예상되는 “Shortest weak Hamiltonian path problem”에 대해서 소개하고, 다항 시간에 결정론적으로 해결되는 인스턴스인 Cactus 그래프의 경우를 소개하고, 해법을 제시함.

2. Cactus 그래프를 부분 그래프로 가지는 그래프에 대한 효율적인 휴리스틱의 제시 가능성.

References

1. Rahman, M. S.; Kaykobad, M. (April 2005). "On Hamiltonian cycles and Hamiltonian paths". Information Processing Letters. 94: 37–41. doi:10.1016/j.ipl.2004.12.002.
2. K. Das, “Cactus Graphs and Some Algorithms”, arXiv:1408.4005, 2014.
3. Anon. 2020. Cactus graph. (November 2020). Retrieved December 8, 2020 from https://en.wikipedia.org/wiki/Cactus_graph (Image)

본 발표 자료 (*.pptx)는 다음 링크에서 확인하실 수 있습니다.

https://github.com/ANEP-Research/KSC2020