A numerical example of Stackelberg's model

Assume that in a duopoly market the demand function is

$$P = 100 - 0{\cdot}5(X_1 + X_2)$$

and the duopolists' costs are

$$C_1 = 5X_1 \qquad \text{and} \qquad C_2 = 0{\cdot}5X_2^2$$

The reaction functions are found by taking the partial derivatives of the duopolists' profit functions and equating them to zero:

$$\Pi_1 = PX_1 - C_1 = 95X_1 - 0{\cdot}5X_1^2 - 0{\cdot}5X_1X_2$$
$$\Pi_2 = PX_2 - C_2 = 100X_2 - X_2^2 - 0{\cdot}5X_1X_2$$

The partial derivatives are

$$\frac{\partial \Pi_1}{\partial X_1} = 95 - X_1 - 0{\cdot}5X_2 = 0$$

$$\frac{\partial \Pi_2}{\partial X_2} = 100 - 2X_2 - 0{\cdot}5X_1 = 0$$

The reaction functions are

$$X_1 = 95 - 0{\cdot}5X_2 \rightarrow A\text{'s reaction curve}$$
$$X_2 = 50 - 0{\cdot}25X_1 \rightarrow B\text{'s reaction curve}$$

(1) *Stackelberg's solution with* A *being the sophisticated leader*

Firm A will substitute B's reaction function in its own profit equation, which it will then maximise as if it were a monopolist:

$$\Pi_1 = PX_1 - C_1 = 95X_1 - 0{\cdot}5X_1^2 - 0{\cdot}5X_1X_2$$

Substitute $\quad X_2 = 50 - 0{\cdot}25X_1$

Maximise $\quad \Pi_1 = 70X_1 - 0{\cdot}375X_1^2$

(*a*) First-order condition: $\quad \dfrac{\partial \Pi_1}{\partial X_1} = 70 - 0{\cdot}75X_1 = 0$

This yields output: $\quad X_1 = 93\tfrac{1}{3}$

and profit: $\quad \Pi_1 = 70X_1 - 0{\cdot}375X_1 = 3267$

(*b*) The second-order condition for profit maximisation is fulfilled.

Firm B would be the follower. It would assume that A would produce $93\frac{1}{3}$ units; thus B substitutes this amount in its reaction function

$$X_2 = 50 - 0{\cdot}25X_1 = 26\tfrac{2}{3}$$

and its profit would be

$$\Pi_2 = 100X_2 - X_2^2 - 0{\cdot}5X_1X_2 = 155{\cdot}5$$

(2) *Stackelberg's solution if firm* B *is the sophisticated duopolist*

Firm B will substitute A's reaction function in its own profit function, and it will proceed to maximise this profit as a monopolist

$$\Pi_2 = PX_2 - C_2 = 100X_2 - X_2^2 - 0{\cdot}5X_1X_2$$

Substitute $X_1 = 95 - 0{\cdot}5X_2$ (i.e., A's reaction function)

$$\Pi_2 = 52{\cdot}5X_2 - 0{\cdot}75X_2^2$$

(*a*) The first-order condition for the maximisation of Π_2 requires

$$\frac{\partial \Pi_2}{\partial X_2} = 52{\cdot}5 - 1{\cdot}5X_2 = 0$$

which yields output: $X_2 = 35$

and profit: $\Pi_2 = 52{\cdot}5X_2 - 0{\cdot}75X_2^2 = 918{\cdot}75$

(*b*) The second-order condition for the maximisation of Π_2 is fulfilled.

The follower is now firm A which will act on the Cournot assumption; it will assume that the rival will keep his quantity at $X_2 = 35$, and will find its own output by substituting this quantity in its reaction function.

$$X_1 = 95 - 0{\cdot}5X_2 = 77{\cdot}5$$

and its profit is

$$\Pi_1 = 95X_1 - 0{\cdot}5X_1^2 - 0{\cdot}5X_1X_2 = 3003$$

(3) *Stackelberg's disequilibrium*

If both entrepreneurs adopt Stackelberg's sophisticated pattern of behaviour, each will examine his profits if he acts as a leader and if he acts as a follower, and will adopt the action that will yield him the greatest profit.

Firm A calculates its profits both as a leader and as a follower:
If A is the leader his profits are 3267
If A is the follower his profits are 3003
Clearly firm A will prefer to act as the leader.

Firm B similarly, calculates its profits as a leader and as a follower:
If B is the leader his profits are 918·75
If B acts as the follower his profits are 155·50
Thus firm B will also choose to act as the leader.[1]

With both firms acting in the sophisticated way implied by Stackelberg's behavioural hypothesis both will want to act as leaders. As they attempt to do so they find that their expectations about the rival are not fulfilled and 'warfare' will start, unless they decide to come to a collusive agreement.

We may now summarise Stackelberg's model. Each duopolist estimates the maximum profit that he would earn (*a*) if he acted as leader, (*b*) if he acted as follower, and chooses the behaviour which yields the largest maximum. Four situations may arise: (1) Duopolist A wants to be leader and B wants to be follower. (2) Duopolist B wants to be leader and A wants to be follower. (3) Both firms want to be followers. (4) Both firms desire to be leaders.

In situations (1) and (2) the result is a determinate equilibrium (provided that the first- and second-order conditions for maxima are fulfilled).

If both firms desire to be followers, their expectations do not materialise (since each assumes that the rival will act as a leader), and they must revise them. Two behavioural patterns are possible. If each duopolist recognises that his rival wants also to be a follower, the Cournot equilibrium is reached. Otherwise, one of the rivals must alter his behaviour and act as a leader before equilibrium is attained.

Finally, if both duopolists want to be leaders a disequilibrium arises, whose outcome, according to Stackelberg, is economic warfare. Equilibrium will be reached either by collusion, or after the 'weaker' firm is eliminated or succumbs to the leadership of the other.

[1] The numerical example is taken from J. M. Henderson and R. E. Quandt, *Microeconomic Theory* (McGraw-Hill 1958) p. 181.