Partial Differential Equations

Topic-1 (Unit-I)

Let z be a dependent Variables and x, y be dependent variables, that is, $z = z(x,y)$.

then first order derivative

$$z_x = p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} = z_y$$

2nd order partial derivative

$$r = \frac{\partial^2 z}{\partial x^2} = z_{xx}, \quad s = \frac{\partial^2 z}{\partial x \partial y} = z_{xx}, \quad t = \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

An eq^n contaning x, y, z, p, q defines a first order partial differential equation, that is

$$f(x, y, z, p, q) = 0$$

This eq^n is Linear, if it is linear in p, q.

An eq^n containing x, y, z, p, q, r, s, t defines a 2nd order partial differential eq^n, that is

$$g(x, y, z, p, q, r, s, t) = 0$$

This eq^n is linear if it is linear in p, q, r, s, t.

Partial differential Eq^n:- A differential eq^n containing partial derivatives of a function of two or more independent variables is called a partial differential eq^n, e.g. $f(x, y, z, p, q, r, s, t) = 0$

Order and degree of P.D.E

Order :- Order of a partial differential equation is the order of the highest ordered derivative

Degree :- Degree of a partial diff. eq^n is the greatest power of the highest ordered derivative present in the equation made free from radical sign and fractional powers.

e.g. $p + xq = x + y$ of order 1, degree 1

$r + s = x^2 + y$ of order 2, degree 1

Some Important P.D.E.

* One dimensional heat eq^n

$$u_t = u_{xx}$$

* ~~One dimensional~~ Laplace eq^n

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

* One-dimensional wave eq^n

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

Example of PDE & Solution

Ex. $u_t = u_{xx}$ Heat eq^n

Show that $u = \frac{1}{2}x^2 + t$ is sol^n.

Check $u_x = \frac{2x}{2}$, $u_{xx} = 1$

$u_t = 1$

$u_t = u_{xx}$ verify

from Ist given $u_t = u_{xx}$

Ex. $u = e^{ax+by}$

$u_t = e^{ax+by} \cdot b$

$u_x = a e^{ax+by}$

$u_{xx} = a^2 e^{ax+by} \Rightarrow \boxed{b = a^2}$

Method of formation of partial diff. eq^n

(i) Elimination of arbitrary constants

(ii) Elimination of arbitrary function

* If number of arbitrary constants to be eliminated is equal to the number of independent variables, we obtain a first-order P.D.E.

* If number of arbitrary constants to be eliminated is more than the number of independent variables, we get a PDE of higher-order.

(i) By eliminating arbitrary constants

(ii) By eliminating arbitrary function

(iii) By eliminating arbitrary function in the form $\phi(u,v)$

***Ex** $x^2+y^2+(z-a)^2=b^2$

Diff. partially w.r.t x

$$2x + 2(z-a)\frac{\partial z}{\partial x} = 0 \Rightarrow \cancel{2}x + \cancel{2}(z-a)p = 0$$

$$\Rightarrow (z-a) = -\frac{x}{p} \quad -(1)$$

Diff partially w.r.t y

$$2y + 2(z-a)\frac{\partial z}{\partial y} = 0 \Rightarrow y + (z-a)q = 0$$

$$z-a = -y/q \quad -(2)$$

equating (1) & (2)

$$\frac{x}{p} = \frac{y}{q} \Rightarrow \boxed{xq = yp} \text{ soln.}$$

2.) $ax^2+by^2+z^2=1 \quad -(1)$

w.r.t x

$2ax+0+2z\frac{\partial z}{\partial x}=0 \Rightarrow ax+zp=0$

$a=-\frac{zp}{x}$

w.r.t y

$2by+2z\frac{\partial z}{\partial y}=0 \Rightarrow by+zq=0 \Rightarrow$

$b=-\frac{zq}{y}$

Use in (1)

$-\frac{zp}{x}x^2-\frac{zq}{y}\cdot y^2+z^2=1$

$-z(px+qy)+z^2=1 \Rightarrow z^2-1=z(px+qy)$

3.) $2z=\frac{x^2}{a^2}+\frac{y^2}{b^2} \quad -(1)$

w.r.t x

$2\frac{\partial z}{\partial x}=\frac{2x}{a^2} \Rightarrow p=\frac{x}{a^2} \Rightarrow a^2=\frac{x}{p}$

w.r.t y

$2\frac{\partial z}{\partial y}=\frac{2y}{b^2} \Rightarrow q=y/b^2 \Rightarrow b^2=y/q$

Use in (1)

$2z=\frac{x^2}{x/p}+\frac{y^2}{y/q} \Rightarrow \boxed{2z=px+qy}$

4.) $z=(x+a)(y+b)$ Ex: $az+b=a^2x+y$

5.) $2z=(ax+y)^2+b$

6.) $x^2+y^2=(z-c)^2\tan^2\alpha$

w.r.t x

$2x=2(z-c)\tan^2\alpha\, p \Rightarrow (z-c)\tan^2\alpha=x/p$

w.r.t y

$2y=2(z-c)q\tan^2\alpha \Rightarrow y=q\cdot\frac{x}{p} \Rightarrow \boxed{yp=qx}$

Arbitrary Function (2)

$p = \frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x$

$q = \frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y$

$\frac{p}{q} = \frac{x}{y} \Rightarrow \boxed{py = xq}$ Ans

8.) $f(x+y+z,\ x^2+y^2+z^2) = 0$

$\cancel{f'(x+y+z,\ x^2+y^2+z^2)}$

$f(u, v) = 0$

$u = x+y+z \qquad v = x^2+y^2+z^2$

soln is

$Pp + Qq = R$

$P = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix},\ Q = -\begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}$

$R = +\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

$P = \begin{vmatrix} 1 & 1 \\ 2y & 2z \end{vmatrix} = 2z-2y \qquad Q = -\begin{vmatrix} 1 & 1 \\ 2x & 2z \end{vmatrix} = -(2z-2x)$

$R = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = (2y-2x)$

$2(z-y)p - 2(z-x)q = 2(y-z)$

$(z-y)p - (z-x)q = (y-z)$ A.

9.) $z = \phi(x)\psi(y)$

$p = \frac{\partial z}{\partial x} = \phi'(x)\psi(y)$

$q = \frac{\partial z}{\partial y} = \phi(x)\psi'(y)$

$s = \frac{\partial^2 z}{\partial y\partial x} = \phi'(x)\psi'(y)$

$p\cdot q = \phi'(x)\psi'(y)\phi(x)\psi(y)$

$\boxed{pq = s\cdot z}$

10.) $z = x+y+f(xy)$

$p = \frac{\partial z}{\partial x} = 1 + f'(xy)(y)$

$q = \frac{\partial z}{\partial y} = 1 + f'(xy)x$

$\frac{p-1}{y} = \frac{q-1}{x}$

$px - x = qy - y$

$\boxed{px - qy = x - y}$ A.

Ex. $z = f(x+it) + g(x-it)$

$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial t^2} = 0$

IV $F(xy+z^2,\ x+y+z) = 0$

V $F(x^2+y^2-z^2,\ z^2-2xy) = 0$

Ex(i) $z = f(xy/z)$

(ii) $xyz = \phi(x^2+y^2+z^2)$

(iii) $\phi(x^2+y^2,\ y^2+z^2) = 0$

Partial Differential Equations of first order:-

The general form of a first order partial differential eqⁿ is

$$F(x, y, z, p, q) = 0 \qquad (i)$$

where x, y are the two independent variables, z is the dependent variable

Complete Solution:-

Any function $f(x, y, z, a, b) = 0$ — (2) involving two arbitrary constant a, b and satisfying the PDE (1) is known as complete solution or complete integral or primitive.

General Solution:-

General solⁿ of P.D.E. (i) is any arbitrary function F of specific function u, v

$$F(u, v) = 0 \qquad — (3)$$

satisfying P.D.E. (1)

here $u = u(x, y, z)$ and $v = v(x, y, z)$ are known functions of x, y, z.

Linear: A partial differential Equation is said to be linear (after rationalization and cleared of fractions) if the dependent variable Z and its derivatives are of degree (power) one and product of Z and its derivatives do not appear in the equation.

Ex $x^2 p + y^2 q = z$

Quasi-Linear:- P.D.E is said to be quasi-linear if degree of highest ordered derivative is one and no products of partial derivatives of the highest order are present.

$$z \cdot z_{xx} + (z_y)^2 = 0$$

Non-Linear:- A P.D.E which is not linear is known as non-linear P.D.E.

$$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + u^2\left(\frac{\partial u}{\partial y}\right) = f(x, y)$$

Non-Linear in u & of 2nd order.

Linear Partial Differential Equations of First order:-

The general form of a quasi-Linear PDE of the first order is

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z) \quad -(1)$$

$$Pp + Qq = R$$

This eq^n (1) is known as Lagrange's Linear equation If P and Q are independent of z and R is linear in z then (1) is a linear equation.

The General soln of Lagrange's linear PDE

$$Pp + Qq = R$$

is given by $F(u,v) = 0$

Method of obtaining General solution

1. standard form $Pp + Qq = R$
2. Lagrange's auxilary equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

3. Nature of solution to the simultaneous eq^s of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$:-

$u(x,y,z) = c$, $u(x,y,z) = c_2$ are said to be the complete solution of the system of simultaneous equations (Provided u_1 + u_2 are L. Independent i.e $u_1/u_2 \neq$ constant)

Case 1: One of the variable is either absent or cancells out from the set of auxiliary eq^n.

Case 2: If $u = c_1$ is known but $v = c_2$ is not posible by case (1), use $u = c_1$ to get $v = c_2$

Case 3: Introducing Lagrange's Multipliers P_1, Q_1, R_1 which are function of x, y, z or constant, then each fraction in (3) =

$$\frac{P_1 dx + Q_1 dy + R_1 dz}{P_1 P + Q_1 Q + R_1 R}$$

If P_1, Q_1, R_1 are so choosen that $P_1 P + Q_1 Q + R_1 R = 0$

then $P_1 dx + Q_1 dy + R_1 dz = 0$ which can be integrated

Case 4 :- Multiplier may be choosen (More than one) s.t the numinator $P_1 dx + Q_1 dy + R_1 dz$ is an exact differential of the denominator $P_1 P + Q_1 Q + R_1 R$

4. General sol^n (1) is

$$F(u, v) = 0 \quad \text{or} \quad v = \phi(u)$$

Examples:-

1. $xp + yq = 3z$

$P = x,\quad Q = y,\quad R = 3z$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{3z}$$

$\frac{dx}{x} = \frac{dy}{y}$

Integrate

$\log x = \log y + \log c_1$

$x = c_1 y \Rightarrow c_1 = \frac{x}{y}$

$\frac{dy}{y} = \frac{dz}{3z}$

$3\log y = \log z + \log c_2$

$y^3 = c_2 z$

$c_2 = y^3/z$

$F(u, v) = 0$

$F(x/y,\ y^3/z)$ or $u = F(u)$

$y^3/z = F(x/y)$

$y^3 = z\,F(x/y)$

IIly $x^3 = z\,f(x/y)$ also be calculated

2. $yzp - xzq = xy$

$$\frac{dx}{yz} = \frac{dy}{-xz} = \frac{dz}{xy}$$

$\frac{dx}{yz} = \frac{dy}{-xz}$

$+x\,dx = -y\,dy$

Integrate

$\frac{x^2}{2} = -\frac{y^2}{2} + \frac{c_1}{2}$

$\boxed{x^2 + y^2 = c_1}$

$\frac{dy}{-xz} = \frac{dz}{xy}$

$y\,dy = -z\,dz$

$\frac{y^2}{2} = -\frac{z^2}{2} + \frac{c_2}{2}$

$\boxed{y^2 + z^2 = c_2}$

Thus the general solⁿ is

$$F(x^2+y^2,\ x^2-z^2) = 0$$

Ex. 3. $y^2p - xyq = x(z-2y)$

(i) $(x^2+y^2) = c_1$, (ii) $y^2 = yz + c_2$

Ex 4. $x^2p + y^2q = (x+y)z$

$\frac{dx}{x^2} = \frac{dy}{y^2}$, $\frac{dx-dy}{x^2-y^2} = \frac{dz}{(x+y)z}$

<u>Ex.</u> <u>dx</u> $p-q = \log(x+y)$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\log(x+y)}$$

$$\frac{dx}{1} = \frac{dy}{-1}$$

$x = -y + c$

$x+y = c_1$

Use this value in dz

$$\frac{dx}{1} = \frac{dz}{\log c_1}$$

$\log c_1 \, dx = dz$

$x \log c_1 = z + c_2$

$x \log(x+y) = z + c_2$

$F(x+y, \; x\log(x+y) - z) = 0$ A.

6\. <u>Ex.</u> $z(z^2+xy)(px-qy) = x^4$

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4}$$

$$\frac{dx}{x} = \frac{dy}{-y}$$

$\log x = -\log y + \log c_1$

$xy = c_1$

$$\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$$

Use $xy = c_1$

$x^3 \, dx = (z^3 + zc_1)\, dz$

$$\frac{x^4}{4} = \frac{z^4}{4} + \frac{z^2}{2} c_1 + c_2$$

$x^4 - z^4 - 2z^2xy = c_2$

$F(xy, \; x^4 - z^4 - 2z^2xy) = 0$

<u>Ex.</u> $y^2zp + x^2zq = xy^2$

<u>Ex.</u> $$\frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2+(x+y)^2}$$

<u>Ex.</u> $$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx-dy}{(x-y)(x+y+z)} \;\Big|\; \frac{dy-dz}{(y-z)(x+y+z)} \;\Big|\; \frac{dz-dx}{(z-x)(x+y+z)}$$

Equating (1) + (2) & (2+3)

$\log(x-y) = \log(y-z) + c_1$

$\log(y-z) = \log(z-x) + \log c_2$

$$F\left(\frac{x-y}{y-z}, \; \frac{y-z}{z-x}\right) = 0$$

$(mz-ny)p+(nx-lz)q = ly-mx$

$\frac{x\,dx+y\,dy+z\,dz}{0}$ | $\frac{l\,dx+m\,dy+n\,dz}{0}$

Ex. $x^2(y-z)p+y^2(z-x)q = z^2(x-y)$

$\frac{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}{0}$ | $\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{0}$

Ex. $p+3q = 5z+\tan(y-3x)$ Ans $\phi(y-3x, \log\{5z+\tan(y-3x)\})$

Ex. $(y^2+z^2-x^2)p-2xyq+2xz=0$ A $\phi\left(y/z, \frac{x^2+y^2+z^2}{z}\right)=0$

Ex. $(x^2-y^2-yz)p+(x^2-y^2-zx)q = z(x-y)$ An $\phi\left(x-y-z, \frac{x^2-y^2}{z^2}\right)=0$

Ex. $(y^2+z^2)p-xyq = -zx$ An $\phi(y/z, x^2+y^2+z^2)=0$

Charpit's Method

Charpit's method is a general method to find the complete solution of the first order non-linear P.D.E of the form

$$f(x, y, z, p, q) = 0 \qquad \text{— (1)}$$

We know

$$dz = p\,dx + q\,dy \qquad \text{— (2)}$$

Auxiliary Equations are:-

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

through these eq^n we get value of p & q

then $dz = p\,dx + q\,dy$

1 Ex.

$$px + qy = pq \quad \text{—(i)}$$

$$f = px + qy - pq$$

$$\frac{\partial f}{\partial x} = p, \quad \frac{\partial f}{\partial y} = q, \quad \frac{\partial f}{\partial z} = 0, \quad \frac{\partial f}{\partial p} = x - q, \quad \frac{\partial f}{\partial q} = y - p$$

$$\frac{dp}{-(p + p0)} = \frac{dq}{-(q + q(0))} = \frac{dz}{p(x-q) + q(y-p)} = \frac{dx}{p} = \frac{dy}{q}$$

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log c$$

$$\boxed{p = qc} \quad -(2)$$

Use in (1)

$$qcx + qy = pq$$

$$cx + y = p$$

from (2)

$$q = \frac{cx+y}{c}$$

$$dz = p\,dx + q\,dy$$

$$= (cx+y)\,dx + \left(\frac{cx+y}{c}\right)dy$$

$$= cx\,dx + \underline{y\,dx + x\,dy} + \frac{1}{c}y\,dy$$

$$dz = cx\,dx + d(xy) + \frac{1}{c}y\,dy$$

Integrate both side

$$z = \frac{cx^2}{2} + xy + \frac{y^2}{2c} + a$$

$$z = \frac{1}{2c}\left[c^2x^2 + 2cxy + y^2\right] + a$$

$$z = \frac{1}{2c}\left[(cx+y)^2\right] + a$$

$$cz = \frac{1}{2}\left[(cx+y)^2\right] + a \quad \text{A}$$

2. Ex. $f = 2(z + xp + yq) - yp^2 \quad -(1)$

Soln $\frac{dx}{2x - 2yp} = \frac{dy}{2y} = \frac{dz}{p(2x-2yp) + q(2y)} = \frac{dp}{-(2p+2p)} = \frac{dq}{-(2q+p^2+2q)}$

$$\frac{dx}{x - yp} = \frac{dy}{y} = \frac{dz}{xp - yp^2 + yq} = \frac{dp}{-2p} = \frac{dq}{-(2q - p^2/2)}$$

$$\frac{dy}{y} = \frac{dp}{-2p} \Rightarrow \log y = \frac{-1}{2}\log p + \log a$$

$$\log y^{-2} = \log pa \Rightarrow pa = y^{-2} \Rightarrow p = ay^{-2}$$

Use in (i)

$$x\left(z+\frac{xa}{y^2}+yq\right)-y\left(\frac{a^2}{y^4}\right)=0$$

$$2z+\frac{2xa}{y^2}-2yq-y\frac{a^2}{y^4}=0$$

$$q=\frac{a^2}{2y^4}-\frac{z}{y}-\frac{ax}{y^3}$$

Now $\quad dz = p\,dx + q\,dy$

$$=\frac{a}{y^2}dx+\left(\frac{a^2}{2y^4}-\frac{z}{y}-\frac{ax}{y^3}\right)dy$$

$$dz=\frac{a}{y^2}dx-\frac{ax}{y^3}dy+\frac{a^2}{2y^4}-\frac{z}{y}dy$$

$$\left(\frac{z}{y}dy+dz\right)=\left(\frac{ay\,dx-ax\,dy}{y^3}\right)+\frac{a^2}{2y^4}$$

$$\left(\frac{z\,dy+y\,dz}{y}\right)=\frac{ay\,dx-ax\,dy}{y^3}+\frac{a^2}{2y^4}$$

$$\int z\,dy+y\,dz=\int a\left[\frac{y\,dx-x\,dy}{y^2}\right]+\int\frac{a^2}{2y^3}$$

$$\int d(yz)=\int a\,d\left(\frac{x}{y}\right)+\int\frac{a^2}{2y^3}$$

$$yz=a\frac{x}{y}+\frac{a^2}{2}\left(\frac{-1}{2y^2}\right)+b$$

$$z=\frac{ax}{y^2}-\frac{a^2}{4y^3}+\frac{b}{y}\qquad A.$$

3. Ex. $\quad pxy+pq+qy=yz \quad -(1)$

$$f(x,y,z;p,q)=pxy-pq-qy-yz=0$$

$$\frac{\partial f}{\partial x}=py,\ \frac{\partial f}{\partial y}=px-q-z,\ \frac{\partial f}{\partial z}=-y,\ \frac{\partial f}{\partial p}=xy-q,\ \frac{\partial f}{\partial q}=-p-y$$

$$\frac{dp}{-(py-py)}=\frac{dq}{-(px-q-z-qy)}=\frac{dz}{+(p(xy-q)+q(-p-y))}=\frac{dx}{xy-q}=\frac{dy}{-p-y}$$

$$\frac{dp}{0}= \qquad =$$

$$\Rightarrow dp=0 \Rightarrow p=a$$

use in (i)

$$axy + aq + qy = yz$$

$$q(a+y) = yz - axy$$

$$q = \frac{y(z-ax)}{a+y}$$

Now $dz = p\,dx + q\,dy$

$$= a\,dx + \frac{y(z-ax)}{a+y}dy$$

$$\frac{dz - a\,dx}{z-ax} = \frac{y}{a+y}dy = \left(\frac{a+y-a}{a+y}\right)dy$$

$$\cancel{\frac{d(z-ax)}{z-ax}} =$$

$$\frac{d(z-ax)}{(z-ax)} = \left(1 - \frac{a}{a+y}\right)dy$$

$$\log(z-ax) = y - a\log(a+y) + c \quad \text{Ans}$$

4 **Ex.**

$$f = q + xp - p^2 = 0 \quad \text{—(i)}$$

$$\frac{\partial f}{\partial x} = p,\ \frac{\partial f}{\partial y} = 0,\ \frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial p} = x - 2p,\ \frac{\partial f}{\partial q} = 1$$

$$\frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + pf_z)} = \frac{dz}{pf_p + qf_q} = \frac{dx}{f_p} = \frac{dy}{f_q}$$

$$\frac{dp}{-(p)} = \frac{dq}{-(0+0)} = \frac{dz}{p(x-2p)+q(1)} = \frac{dx}{x-2p} = \frac{dy}{1}$$

$$\int \frac{dp}{p} = \int dy \Rightarrow \log p =$$

$$dq = 0 \Rightarrow q = c \quad \text{use in (i)}$$

$$c + xp - p^2 = 0$$

$$p^2 - xp - c = 0$$

$$\frac{x \pm \sqrt{x^2+4c}}{2},\quad \frac{1}{2}\left[x \pm \sqrt{x^2+4c}\right]$$

$$z = \int p\,dx + q\,dy$$
$$= \frac{1}{2}\left[x \pm \sqrt{x^2+4c}\right]dx + c\,dy$$
$$= \frac{1}{2}\left[\frac{x^2}{2} + \frac{x}{2}\sqrt{x^2+4c} + \frac{4c}{2}\log\left(x+\sqrt{x^2+4c}\right)\right] + cy + b$$

A

Ex. $f = 2xz - px^2 - 2qxy + pq = 0$

$f_x = 2z - 2px - 2py$, $f_y = -2qx$, $f_z = 2x$

$f_p = x^2 + q$, $f_q = -2xy + p$

$$\frac{dx}{x^2+q} = \frac{dy}{-2xy+p} = \frac{dz}{p(x^2+q)+q(-2xy+p)} = \frac{dp}{-(2z-2px-2py+p(2x))}$$

$$\frac{dq}{-(-2qx+q2x)} = 0$$

$dq = 0 \Rightarrow q = a$

$2xz - px^2 - 2axy + pa = 0$

$p(a - x^2) = 2axy - 2xz = 2x(ay - z)$

$$p = \frac{2x(ay-z)}{a-x^2} = \frac{2x(z-ay)}{x^2-a}$$

$dz = p\,dx + q\,dy$

$$= \frac{2x(ay-z)\,dx}{-a+x^2} + a\,dy$$

$$dz + \frac{2xz}{a-x^2}dx = \frac{2axy}{a-x^2}dx + a\,dy$$

$$\frac{2xz}{a-x^2}dx + dz = \frac{2xay}{a-x^2}dx + a\,dy$$

$-\log(a-x^2) + z = ay\log(a-x^2) + ay + c_2$

$\log(a-x^2) = -$

$$dz - a\,dy = \frac{2x(-ay+z)}{x^2-a}$$

$$\int\frac{dz-a\,dy}{z-ay} = \int\frac{2x}{x^2-a}$$

$\log(z-ay) = \log(x^2-a) + \log c_1$

$z - ay = c_1(x^2-a)$ A

Partial Differential Equations of Second order

An Equation is said to be a partial differential equation of the second order when it includes at least one of the partial derivatives r, s, t but none of the higher order.

Linear Partial Differential Equation with constant coeff.

A Partial diff. eqn in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together is called a linear partial diff. eqn.

$$A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + A_n \frac{\partial^n z}{\partial y^n} + B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \cdots + C_0 \frac{\partial z}{\partial x} + C_1 \frac{\partial z}{\partial y} + P_0 z = F(x, y) \quad (1)$$

i.e
$$\frac{\partial^3 z}{\partial x^3} + \frac{\partial^3 z}{\partial x^2 \partial y} + \frac{\partial^3 z}{\partial x \partial y^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

L.D.E with constant coeff.

* If all the derivative appearing in eqn (1) are of the same order then the diff. eqn is called linear Homogeneous p.d.e with constant coeff.

* If all the derivative are not of same order, then it is called a non-Homogenous.

Homogeneous Linear Partial differential eq^n with Constant coefficients :-

An equation of the form

$$a_0 \frac{\partial^n z}{\partial x^n} + a_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 \frac{\partial^2 z}{\partial x^{n-2}\partial y^2} + \text{——} + a_n \frac{\partial^n y}{\partial y^n} = F(x,y)$$

where $a_0, a_1, a_2 \text{ — } a_n$ are constants, is called a homogeneous L.P.D. Equation of the nth order with constant coefficients.

$$D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y}$$

$$(a_0 D^n + a_1 D^{n-1} D' + a_2 D^{n-2} D'^2 + \text{——} + a_n D'^n) z = F(x,y)$$

$$\phi(D, D') z = F(x,y)$$

(i) $C.S = C.F + P.I$

C.F $\quad \phi(D, D') z = 0$

It must contain n arbitrary functions, when n is the order of the dif. eq^n

(ii) P.I $\quad \phi(D, D') z = F(x,y)$

Complementary function :- (C.F) :-

Consider $\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$

Symbolic form is

$$(D^2 + a_1 DD' + a_2 D'^2) z = 0$$

A.E $\quad D^2 + a_1 DD' + a_2 D'^2 = 0$

Case I :- $(D - m_1 D')(D - m_2 D') z = 0$

$m_1 \neq m_2$

$$z = f_1(y + m_1 x) + f_2(y + m_2 x) +$$

Case II :- $m_1 = m_2$

$$z = \phi(y + mx) + x f(y + mx)$$

or $\quad z = \phi(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_3 x) + \text{——}$

Ex. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = 0$ $\qquad D' = m, \quad D' = 1$

$(D^2 - DD' - 6D'^2)z = 0$

A.E $\quad m^2 - m - 6 = 0$

$(m-3)(m+2) = 0$

$m = 3, -2$

$C.F = f_1(y+3x) + f_2(y-2x)$

$P.I = 0$

$z = C.F + P.I$

$= f_1(y+3x) + f_2(y-2x)$

Ex. $(D+2D')(D-3D')^2 z = 0$

$(m+2)(m-3)^2 = 0$

$m = -2, 3, 3$

C.F $\quad f_1(y-2x) + f_2(y+3x) + xf_3(y+3x)$

$P.I = 0$

Ex. $4r - 12s + 9t = 0$

$(4D^2 - 12DD' + 9D'^2)z = 0$

$4m^2 - 12m + 9 = 0$

$(2m-3)^2 = 0$

$m = 3/2, 3/2$

C.F $\quad f_1(y + \frac{3}{2}x) + xf_2(y + \frac{3}{2}x)$

$z = C.F + P.I = f_1(y + \frac{3}{2}x) + x f_2(y + \frac{3}{2}x)$

Ex. $(D^3 - 3D^2D' + 2DD'^2)z = 0$

$m^3 - 3m^2 + 2m = 0$

$m(m-1)(m-2) = 0$

$m = 0, 1, 2$

C.F $\quad f_1(y) + f_2(y+x) + f_3(y+2x)$

Ex. $(D^3 - 6D^2D' + 11DD'^2 + 6D'^3)z = 0$

$m^3 - 6m^2 + 11m - 6 = 0$

$m = 1, 2, 3.$

C.F $\quad f_1(y+x) + f_2(y+2x) + f_3(y+3x)$

Ex. $(D^3 - 6D^2D' + 12DD'^2 - 8D'^3)z = 0$

$m = 2, 2, 2$

Ex $\quad 25r - 40s + 16t = 0$

Ex. $\frac{\partial^4 z}{\partial x^4} + \frac{\partial^4 y}{\partial y^4} = 0$

$z_1 = \frac{-1+i}{\sqrt{2}}, \frac{+1+i}{\sqrt{2}}$

$= z_1, \bar{z}_1, z_2, \bar{z}_2$

$z = f_1(y + z_1 x) + f_2(y + \bar{z}_1 x)$
$+ f_3(y + z_2 x) + f_4(y + \bar{z}_2 x)$

Particular Integral:-

$F(D,D')z = \phi(x,y)$

$$P.I = \frac{1}{F(D,D')}\phi(x,y)$$

I:- $F(D,D')z = \phi(ax+by)$; $F(a,b) \neq 0$

where $F(D,D')$ is a homogeneous function of D & D' of degree n.

Step 1:- Replace D by a, D' by b in $F(a,b)$

Step 2:- Put $ax+by=u$ and Integrate $\phi(u)$, n times w.r.t u.

$$P.I = \frac{1}{F(a,b)} \underbrace{\iiint - - \int}_{n\text{ times}} \phi(u)\, \underbrace{du\,du\,du - du}_{n\text{ times}}$$

and replace u by $ax+by$

$F(a,b) \neq 0$

If $F(a,b) = 0$, the method fails

Step 3:- ~~Repla.~~ Differentiat $F(D,D')$ partially w.r.t D and Multiply by x

Step 4:- Check whether $F'(a,b) \neq 0$?

$$P.I \quad x \cdot \frac{1}{\frac{\partial}{\partial D}(F(D,D'))}\phi(ax+by)$$

fail if $F'(a,b) = 0$

Repeat the procedure whenever $F''(a,b) \neq 0$

Case (ii) If $F(D,D')$ Power of D' in Highest Degree Terms is greater than D, and case of failure Occur. then Diff w.r.t D' and Multiply the expression with y.

$$= y \cdot \frac{1}{\frac{\partial}{\partial D'}(F(D,D'))}\phi(ax+by)$$

Ex. $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$

Ex. $(D^2 - DD')z = \cos 2y(\sin x + \cos x)$

Ex. $r + 2s + t = 2(y-x) + \sin(x-y)$

$m^2 + 2m + 1 = 0 \Rightarrow m = -1, -1$

C.F $f_1(y-x) + x f_2(y-x)$

I P.I $\frac{2(y-x)}{D^2 + 2DD' + D'^2}$ = Case fail

$= x \cdot \frac{1}{2D + 2D'} 2(y-x)$ = case fail

$= x^2 \cdot \frac{1}{2} \cdot 2(y-x) = x^2(y-x)$

II P.I $\frac{\sin(x-y)}{(D+D')^2}$ = case fail

$= x \cdot \frac{1}{2(D+D')} \sin(x-y)$ = case fail

$= x^2 \cdot \frac{1}{2} \sin(x-y) = \frac{x^2}{2} \sin(x-y)$

$z = f_1(y-x) + x f_2(y-x) + x^2(y-x) + \frac{x^2}{2}\sin(x-y)$

Ex. $4r - 4s + t = 16 \log(x+2y)$

$(4D^2 - 4DD' + D'^2)z = 16(x+2y)$

$m = 1/2, 1/2$

C.F $f_1(y + \frac{x}{2}) + x f_2(y + \frac{x}{2})$

P.I $\frac{1}{(2D-D')^2} 16 \log(x+2y)$ Case fail

$= 16 \cdot x \cdot \frac{1}{2(2D-D') \cdot 2} \log(x+2y)$ Case fail

$= 4x^2 \cdot \frac{1}{2} \log(x+2y)$

$= 2x^2 \log(x+2y)$

Ex. $\frac{\partial^3 u}{\partial x^3} - 3\frac{\partial^3 u}{\partial x^2 \partial y} + 4\frac{\partial^3 u}{\partial y^3} = e^{x+2y}$

Sol^n $(D^3 - 3D^2D' + 4D'^3)z = e^{x+2y}$

A.E $(m^3 - 3m^2 + 4) = 0$

$(m-2)^2(m+1) = 0$

$m = 2, m = -1$

C.F $f_1(y+2x) + f_2(y-x) + xf_3(y+2x)$

P.I $\frac{1}{D^3 - 3D^2D' + 4D'^3} e^{x+2y}$

$= \frac{1}{1 - 3(1)(2) + 4(2)^3} \iiint e^u \, du = \frac{e^u}{27} = \frac{e^{x+2y}}{27}$

C.S $=$ C.F $+$ P.I

Ex. $\frac{\partial^3 z}{\partial x^3} + 3\frac{\partial^3 z}{\partial x^2 \partial y} + 2\frac{\partial^3 z}{\partial y^3} = x+y$

Sol^n $(D^3 + 3D^2D' + 2D'^3)z = x+y$

A.E $m^3 + 3m^2 + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1, -2$

C.F $f_1(y-x) + f_2(y-2x)$

P.I $\frac{x+y}{D^3 + 3D^2D' + 2D'^2} = \frac{1}{1+3+2} \iiint x \, dx$

$= \frac{1}{6}\left[\frac{x^3}{3\cdot 2}\right] = \frac{1}{36}(x+y)^3$

C.S $=$ C.F $+$ P.I

Ex. $(D^2 - 2DD' + D'^2)z = \sin(2x+3y)$

$m = 1, 1$

C.F $f_1(y+x) + x f_2(y+x)$

P.I $\frac{\sin(2x+3y)}{D^2 - 2DD' + D'^2} = \frac{1}{(D-D')^2} \sin(2x+3y)$

$= \frac{1}{(2-3)^2} \iint \sin u \, du = \frac{-\sin u}{1} = -\sin(2x+3y)$

Ex. $(2D^2 - 5DD' + 2D'^2)z = 24(y-x)$ $\quad a=-1, b=1 \quad m_1 = 1/2, 2$

Ex. $r + s - 2t = \sqrt{2x+y}$ $\quad a=2, b=1 \quad m = 1, -2$

P.I $= \frac{1}{15}(2x+y)^{5/2}$

Ex.

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2} = \cos mx \cos ny + 30(2x+y)$$

Solⁿ

$$(D^2+D'^2)z = \frac{1}{2}\left[\cos(mx+ny)+\cos(mx-ny)\right] + 30(2x+y)$$

$$m_1 = -i, i$$

C.F $f_1(y-ix) + f_2(y+ix)$

P.I $$\frac{\frac{1}{2}(\cos mx+ny) + \cos(mx-ny) + 30(2x+y)}{D^2+D'^2}$$

$$= \frac{1}{2}\left[\frac{1}{m^2+n^2}\iint \cos u\,du + \frac{1}{m^2+n^2}\iint \cos u\,du\right] + \frac{30}{4+1}\iint u\,du$$

$$= \frac{1}{2}\left[\frac{1}{m^2+n^2}(-\cos u) + \frac{1}{m^2+n^2}(-\cos u)\right] + \frac{30}{5}\frac{u^3}{3\cdot 2}$$

$$= \frac{-1}{2(m^2+n^2)}\left[\cos(mx+ny)+\cos(mx-ny)\right] + (2x+y)^3$$

$$= \frac{-1}{2(m^2+n^2)}\left[2\cos mx \cos ny\right] + (2x+y)^3$$

$$= \frac{-\cos mx \cos ny}{m^2+n^2} + (2x+y)^3$$

Ex. $(D^2 - 3DD' + 2D'^2)z = e^{2x+3y} + \sin(x-2y)$

Ex. $(D^2 - 2DD' + D'^2) = \sin x$

Ex. $[D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y)$

Ex. $(D^2 - DD') = \sin x \cos y$

$$m^2 - m = 0$$

$$m = 0, 1$$

C.F $f_1(y) + f_2(y+x)$

P.I $$\frac{1}{D^2-DD'}(\sin x \cos y) = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$$

$$P_1 = \frac{1}{D^2-DD'}\sin(x+y) = x\cdot\frac{1}{2D-D'}\sin(x+y) = x\cdot\frac{1}{2-1}\int \sin u\,du = x(-\cos u)$$
$$= -x\cos(x+y)$$

Case fail

$$P_2 = \frac{1}{D^2-DD'}\sin(x-y) = \frac{1}{1^2-1(-1)}\iint \sin u\,du = \frac{-1}{2}\sin(x-y) =$$

$$P.I = \frac{1}{2}\left[-x\cos(x+y) - \frac{1}{2}\sin(x-y)\right]$$

Ex. $(D-D')^2 z = x + \phi(x+y)$

$m = 1, 1$

C.F $f_1(y+x) + x f_2(y+x)$

$P_1 =$ P.I $\frac{1}{(D-D')^2}(x+0y) = \frac{1}{(1-0)^2}\iint x\,dx\,dx = \frac{x^3}{3\cdot 2} = \frac{x^3}{6}$

$P_2 = \frac{1}{(D-D')^2}\phi(x+y) = x \cdot \frac{1}{2(D-D')}\phi(x+y) = \frac{x^2}{2}\cdot\phi(x+y)$

P.I $= \frac{x^3}{6} + \frac{x^2}{2}\cdot\phi(x+y)$

$z = f_1(y+x) + x f_2(y+x) + \frac{x^3}{6} + \frac{x^2}{2}\cdot\phi(x+y)$

Ex. $D'(D^2 - 2DD' + D'^2)z = \frac{1}{x^2}$

$D' = 0 \quad , 1, 1$

$f_1(x) + f_2(y+x) + x f_3(y+x)$

P.I $\left[\frac{1}{D'^3 + D^2D' - 2DD'^2}\right]\left(\frac{1}{x^2}\right)$

$= y \cdot \frac{1}{3D'^2 + D^2 - 2DD'}\left(\frac{1}{x^2}\right)$

$= y \cdot \frac{1}{0+1-2}\iint \frac{1}{x^2}\,dx\,dx$

$= y\int\left(\frac{-1}{x}\right)dx = y(-\log x) = -y\log x$

$z =$ C.F $+$ P.I

$= f_1(x) + f_2(y+x) + x f_3(y+x) - y\log x$

Ex. $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$

Ex. $(D^2 - 8DD' + 7D'^2)z = \sin(7x+y)$

Ex. $(D^2 - 5DD' + 4D'^2)z = \sin(4x+y)$

Ex. $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x+2y)$

Case II P.I

when $F(D,D')z = \phi(ax+by) = \phi(x,y)$

Now $\phi(x,y) = x^m y^n$

In this case P.I is obtained by expanding $F(D,D')$ in an infinite series of ascending power of D & D' or $\frac{D'}{D}$.

Ex. $\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$

sol^n $(D^3 - D'^3) z = x^3 y^3$

$m^3 - 1 = 0$

$m = 1, \omega, \omega^2$

C.F $f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$

P.I $\left[\frac{1}{D^3 - D'^3}\right](x^3y^3)$

$= \frac{1}{D^3\left(1 - \frac{D'^3}{D^3}\right)}(x^3y^3)$

$= \frac{1}{D^3}\left[1 - \frac{D'^3}{D^3}\right]^{-1} x^3y^3$

$= \frac{1}{D^3}\left[1 + \frac{D'^3}{D^3} + \frac{D'^6}{D^6} - -\right] x^3y^3$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3} D'^3(x^3y^3)\right]$

$= \frac{1}{D^3}\left[x^3y^3 + \frac{1}{D^3}(6x^3)\right] = \frac{1}{D^3}\left[x^3y^3 + \frac{6x^6}{6 \cdot 5 \cdot 4}\right]$

$= \frac{x^6}{6 \cdot 5 \cdot 4} \cdot y^3 + \frac{x^9}{9 \cdot 8 \cdot 7 \cdot 5 \cdot 4}$

$= \frac{x^6y^3}{120} + \frac{x^9}{10080}$

C.S = C.F + P.I

2. $(D^2 - 6DD' + 9D'^2)z = 12x^2 + 36xy$

A.E $\quad m^2 - 6m + 9 = 0$

$m = 3, 3$

C.F $\quad f_1(y+3x) + x f_2(y+3x)$

P.I $\quad \dfrac{12x^2}{D^2 - 6DD' + 9D'^2} + \dfrac{36xy}{D^2 - 6DD' + 9D'^2}$

$\dfrac{12}{(1) - 6(1)(0) + 9(0)} \iint x^2\,dx\,dx + \dfrac{36}{D^2\left(1 - 6\frac{D'}{D} + 9\frac{D'^2}{D^2}\right)} [xy]$

$= \dfrac{12}{1} \dfrac{x^4}{4.3} + \dfrac{36}{D^2}\left(1 - 6\frac{D'}{D} + 9\frac{D'^2}{D^2}\right)^{-1} (xy)$

$= x^4 + \dfrac{36}{D^2}\left(1 + \left(\frac{6D'}{D} - 9\frac{D'^2}{D^2}\right) - - \right) xy$

$= x^4 + \dfrac{36}{D^2}\left(xy + \frac{6}{D}(x) - 0\right)$

$= x^4 + \dfrac{36}{D^2}\left(xy + 6\frac{x^2}{2}\right)$

$= x^4 + 36\left[\left(\frac{x^3}{3.2}y + 6\frac{x^4}{4.3.2}\right)\right]$

$= x^4 + 6x^3y + 9x^4$

$= 10x^4 + 6x^3y$

C.S $= f_1(y+3x) + x f_2(y+3x) + 10x^4 + 6x^3y$

Ex. $\quad D^3 - 2D^2D' = 2e^{2x} + 3x^2y = \dfrac{\partial^3 z}{\partial x^3} - 2\dfrac{\partial^3 z}{\partial x^2 \partial y}$

A.E $\quad m^3 - 2m^2 = 0$

$m^2(m-2) = 0$

$m = 0, 0, 2$

C.F $\quad f_1(y) + f_2(y) + f_3(y+2x)$

P.I $\quad \dfrac{2e^{2x} + 3x^2y}{D^3 - 2D^2D'}$

$= 2\dfrac{1}{2^3 - 2(2)^2 0} \iiint e^{2x}\,dx\,dx\,dx + \dfrac{3}{D^3\left(1 - \frac{2D'}{D}\right)} x^2y$

put $x = 2x$

$= \dfrac{1}{4}\dfrac{e^{2x}}{8} + \dfrac{3}{D^3}\left(1 + \frac{2D'}{D} + \left(\frac{2D'}{D}\right)^2 - - \right) x^2y$

$$= \frac{1}{4}e^{4} + \frac{3}{D^3}\left[x^2y + \frac{2}{D}x^2 + 0\right]$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left[x^2y + 2\cdot\frac{x^3}{4}\right]$$

$$= \left[\frac{1}{4}e^{2x} + 3\,\frac{x^5}{5\cdot4\cdot3}y + 2\cdot\frac{x^6}{6\cdot5\cdot4}\right]$$

$$= \frac{1}{4}e^{2x} + \frac{3}{60}x^5y + \frac{1}{60}x^6$$

$$= \frac{1}{4}e^{2x} + \frac{1}{60}\left[3x^5y + x^6\right]$$

C.S = C.F + P.I

Ex. $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3$

Ex. $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Ex. $(D^2 + (a+b)DD' + abD'^2)z = xy$

General Method of P.I :-

$F(D, D')$ can be factorised, in general, into n linear factors.

$$P.I = \frac{1}{F(D,D')}\phi(x,y)$$

$$= \frac{1}{(D-m_1D')(D-m_2D')(\cdots(D-m_nD')}\phi(x,y)$$

$$= \frac{1}{D-m_1D'}\cdot\frac{1}{(D-m_2D')}\cdots\left(\frac{1}{D-m_nD'}\right)\phi(x,y)$$

Evaluate

$$\left(\frac{1}{D-mD'}\right)\phi(x,y) = \int \phi(x, c-mx)\,dx$$

put $y = c-mx$
where c is replaced by $y+mx$

Example $(D^2+DD'-6D'^2)z = y\cos x$

A.E $m^2+m-6=0$

$(m-2)(m+3)=0$

$m=2,-3$

C.F $f_1(y+2x)+f_2(y-3x)$

P.I $\frac{1}{(D^2+DD'-6D'^2)}y\cos x$

$$= \frac{1}{(D-2D')(D+3D')}y\cos x$$

$y-3x = c$

$y = c+3x$

$$= \frac{1}{(D-2D')}\int (c+3x)\cos x\,dx$$

$$= \frac{1}{D-2D'}\left\{(c+3x)\sin x - 3\frac{\cos x}{-1}\right\}$$

$$= \frac{1}{D-2D'}\left\{y\sin x + 3\cos x\right\}$$

~~$y = c+2x$~~

$y+2x = c$

$y = c-2x$

$$= \int (c-2x)\sin x + 3\cos x$$

$$= (c-2x)(-\cos x) - (-2)(-\sin x) + 3\sin x$$

$$P.I = -y\cos x + \sin x$$

$$z = f_1(y+2x) + f_2(y-3x) - y\cos x + \sin x$$

Ex. $(D^2z + DD'z - 2D'^2z) = (y-1)e^x$

$D^2 + DD' - 2D'^2 = 0$

$m^2 + m - 2 = 0$

$(m-1)(m+2) = 0$

$m = 1 \quad m = -2$

C.F $f_1(y+x) + f_2(y-2x)$

P.I $\frac{1}{(D-D')(D+2D')}(y-1)e^x$ $\qquad y - 2x = c$

$= \left(\frac{1}{D-D'}\right)\int\{(c+2x)-1\}e^x\,dx$

$= \frac{1}{(D-D')}\left[\int(c-1)e^x + 2xe^x\,dx\right]$

$= \frac{1}{D-D'}\left[(c-1)e^x + 2(xe^x - e^x)\right]$

$= \frac{1}{D-D'}\left[(c+2x)e^x - \cancel{3}e^x\right] = \frac{1}{D-D'}\left[ye^x - 3e^x\right]$

$y + x = c$

$y = c - x$

$= \int(c-x)e^x - 3e^x\,dx$

$= (c-x)e^x - (-1)e^x - 3e^x$

P.I $= ye^x - 2e^x$

$z = f_1(y+x) + f_2(y-2x) + (y-2)e^x$

Ex. $r - s - 2t = (2x^2 + xy - y^2)(\sin xy) - \cos xy$

Soln $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$

$(m+1)(m-2) = 0$

$m = -1, 2$

C.F $f_1(y-x) + f_2(y+2x)$ $\qquad y + 2x = c$

$y = c - 2x$

P.I $\frac{1}{(D+D')(D-2D')}\left[(2x-y)(x+y)\sin xy - \cos xy\right]$

$= \frac{1}{D+D'}\int\{2x-(c-2x)\}\{x+(c-2x)\}\sin x(c-2x) - \cos x(c-2x)\}\,dx$

$= \frac{1}{D+D'}\int(4x-c)(c-x)\sin(cx-2x^2) - \cos(cx-2x^2)\,dx$

$= \frac{1}{D+D'}\left[\underbrace{(x-c)}_{II}\ (c-4x)\underbrace{\sin(cx-2x^2)}_{I} - \cos(cx-2x^2)\right]dx$

$= \frac{1}{D+D'}\left[(x-c)\{-\cos(cx-2x^2)\} - \int (1)(-\cos(cx-\cancel{2}x^2))\,dx - \int \cos(\cancel{c}x-2x^2)\,dx \right.$

$= \frac{1}{D+D'} \cancel{(\ldots)}\{(c-x)\cos x(c-2x)\}$

$= \frac{1}{D+D'} (y+x)\cos xy$ $\qquad c = y+2x$

$y - x = c$

$= \int (b+2x)\cos x(b+x)\,dx$ $\qquad y = b+x$

$= \sin(x(b+x))$

$= \sin(xy)$

$z = C.F + P.I = f_1(y-x) + f_2(y+2x) + \sin xy$

Ex. $(D^2 + DD' - 6D'^2)z = y\sin x$

Ex. $(D^3 + 2D^2D' - DD'^2 - 2D'^3)\cancel{z} = (y+2)e^x$

Ex. ~~$(D^2 + DD' - 6D'^2)z = y\sin x$~~

Ex. $r - 4t = \frac{4x}{y^2} - \frac{y}{x^2}$

If $f(D)$ can't be resolved into linear factor

$Z = Ae^{hx+ky}$

$D Z = Ah\, e^{hx+ky}$, $D^2 Z = h^2 A e^{hx+ky}$

Ed.

$(D^2 + D'^2 - p^2)Z = 0$

Let $Z = Ae^{hx+ky}$

$D'^2 = k^2 A e^{hx+ky}$

$(D^2 + D'^2 - p^2)Z = A(h^2 + k^2 - p^2)\, e^{hx+ky} = 0$

$A(h^2 + k^2 - p^2)\, e^{hx+ky} = 0$

$h^2 + k^2 - p^2 = 0 \Rightarrow h^2 + k^2 = p^2$

$C{\cdot}F = \Sigma A e^{hx+ky}$ where $h^2 + k^2 - p^2 = 0$

$Z = \Sigma A e^{hx+ky}$ where $h^2 + k^2 - p^2 = 0$

h maybe taken as $p\cos\alpha$ k may be $p\sin\alpha$

C.S $Z = \Sigma A e^{p(x\cos\alpha + y\sin\alpha)}$

Non-homogeneous C.F

In the form $(\alpha D + \beta D' + \gamma) Z = 0$

* $Z = e^{-\gamma/\alpha\, x} f_1(\alpha y - \beta x)$

* If D, αD absent means in the absence of first term this result not applicable

* If $(\alpha D' + b)$ the C.f $e^{-b/\alpha\, y} f_1(\alpha x)$

* If $(\alpha D' + b)^n$ the C.f $e^{-b/\alpha\, y}[f_1(\alpha x) + y f_2(\alpha x) + y^2 f_3(\alpha x) \text{ ---- } y^{n-1} f_n(\alpha x)]$

* If $b = 0$, $\alpha = 1$ means $D'Z = 0$

C.f $f_1(x)$

If $b = 0$, $\alpha = 1$ $D'^n Z = 0$

C.f $f_1(x) + y f_2(x) + y^2 f_3(x)$ ——

Non-Homogeneous Linear Partial Differential Eqⁿ with Constant coefficients :-

Complementary function Rule :-

$$F(D,D') = \phi(x,y)$$

if the polynomial $\phi(D,D')$ in D, D' is not homogeneous then (1) is ~~not~~ called a non-homogeneous L.P.D. Equation

$$Z = C.F + P.I$$

(a) $\phi(D,D')$ into linear factors of the form $D - mD' - a$

$$(D - m_1D' - a_1)(D - m_2D' - a_2) \cdots (D - m_nD' - a_n) z = 0$$

$$Z = e^{a_1x} f_1(y + m_1x) + e^{a_2x} f_2(y + m_2x) + \cdots e^{a_nx} f_n(y + m_nx)$$

(b) $(D - mD' - a)^3 z = 0$

$$z = e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx)$$

(c) If first term absent $(aD' + b)$ C.F is $e^{-b/a\, y}$

& $(aD' + b)^n = e^{-b/a\, y}[\phi_1(ax) + y\phi_2(ax) + \cdots y^{n-1}\phi_n(ax)]$

Ex. $(D + D' - 1)(D + 2D' - 2) z = 0$

C.F $e^{x}(f_1(y - x)) + e^{2x} f_2(y - 2x)$

$z = C.F + P.I = e^{x} f_1(y - x) + e^{2x} f_2(y - 2x)$

Ex. $DD'(D + 2D' + 1) z = 0$

$D = 0$, $D' = 0$, $D + 2D' + 1 = 0$

Part of C.F $= f_1(y)$

C.F of $(0D + D') = f_2(x)$

C.F of $(D + 2D' + 1) = e^{-x} f_3(y - 2x)$

$C.S = f_1(y) + f_2(x) + e^{-x} f_3(y - 2x)$

Ex. $(D + 1)(D + D' - p) z = 0$

C.F $e^{-x} f_1(y) + e^{x} f_2(y - x)$

C.S = C.F

Ex. $(D + 4D' + 5)^2 z = 0$

C.F. $e^{-5x} f_1(y - 4x) + x e^{-5x} f_2(y - 4x)$.

Ex. $(D^2 - D'^2 + D + 3D' - 2) z = 0$

$[D^2 - D'^2 + 2D - D + 2D' + D' - 2 + DD' - DD'] Z = 0$

$[D^2 - DD' + 2D + DD' - D'^2 + 2D' - D + D' - 2] z = 0$

$D[D - D' + 2] + D'(D - D' + 2) - (D - D' + 2] z = 0$

$(D - D' + 2)(D + D' - 1) = 0$ C.F $e^{-2x} f_1(y + x) + e^{x} f_2(y - x)$ A

P.I for Non-homogeneous L.P.D.E with constant coeff:-

$$F(D,D')z = \phi(x,y)$$

$$P.I = \frac{1}{F(D,D')}\phi(x,y)$$

Case I

$\phi(x,y) = e^{ax+by}$ and $F(a,b) \neq 0$

$P.I = \frac{1}{F(a,b)} e^{ax+by}$ replace D by a, D' by b

Case II

$\phi(x,y) =$ sin(ax+by) or cos(ax+by)

$P.I = \frac{1}{F(D^2, DD', D'^2)} \{\sin ax+by \text{ or } \cos ax+by\}$

Replace $D^2 = -a^2$, $DD' = -ab$, $D'^2 = -b^2$

$P.I = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by)$ or $\cos(ax+by)$

Case III

$\phi(x,y) = x^m y^m$

$P.I = \frac{1}{F(D,D')}(x^m y^n) = [F(D,D')]^{-1}(x^m y^m)$

* This can be evaluating after expanding $(F(D,D'))^{-1}$ in ascending power of $\frac{D'}{D}$ (when $m>n$) or $\frac{D}{D'}$ (when $m<n$)

* If a seperate constant is present in $\phi(D,D')$ then it should be given preferences in taking the terms outside the bracket.

Case IV

When $\phi(x,y) = e^{ax+by}.V$

$e^{ax+by}\frac{1}{F(D+a, D+b)}.V$

Ex. $s + ap + bq + abz = e^{mx+ny}$

$$[DD' + aD + bD' + ab]z = e^{mx+ny}$$

$$[D(D'+a) + b(D'+a)]z = e^{mx+ny}$$

$$(D'+a)(D+b)z = e^{mx+ny}$$

C.F $\quad e^{-ay} f_1(x) + e^{-bx} f_2(y)$

P.I $\quad \dfrac{1}{(D'+a)(D+b)} e^{mx+ny}$

replace $D' = n \quad D = m$

$$= \frac{1}{(m+a)(m+b)} \cdot e^{mx+ny}$$

$$C.S = e^{-bx} f_2(y) + e^{-ay} f_1(x) + \frac{e^{mx+ny}}{(m+a)(m+b)}$$

Ex. $D(D - 2D' - 3)z = e^{x+2y}$

C.f $\quad f_1(y) + e^{3x} f_2(y+2x)$

P.I $\quad \dfrac{1}{D(D-2D'-3)} \cdot e^{x+2y}$

$$= \frac{1}{1-4-3} e^{x+2y} = \frac{-1}{6} e^{x+2y}$$

$$C.S = f_1(y) + e^{3x} f_2(y+2x) - \frac{1}{6} e^{x+2y}$$

Ex. $(D^2 - D'^2 + D - D')z = e^{2x+3y}$

Ex. $(D^2 - 4DD' + D - 1)z = e^{3x-2y}$

Here $D^2 - 4DD' + D - 1$ can't be resolved into factor

$z = Ae^{hx+ky}$, $D^2z = h^2 Ae^{hx+ky}$, $Dz = hAe^{hx+ky}$

$DD'z = khAe^{hx+ky}$

$(D^2-4DD'+D-1)z = A[h^2-4kh+h-1]e^{hx+ky} = 0$

iff $h^2-4kh+h-1=0$

$\therefore$ ~~cof~~ $(D^2-4DD'+D-1)z = 0$

then c.f is $z = \sum Ae^{hx+ky}$, where $h^2-4hk+h-1=0$

P.I $\dfrac{e^{3x-2y}}{D^2-4DD'+D-1} = \dfrac{1}{35}e^{3x-2y}$ $\quad$ $9-4\cdot3\cdot(-2)+3-1$ $\quad$ $9+24+3-1=35$

C.S $= \sum Ae^{hx+ky} + \dfrac{1}{35}e^{3x-2y}$ when $h^2-4hk+h-1=0$

Ex. $(D^2-DD'+D'-1)z = \sin(x+2y)$

Soln $D(D-D')+(D'-1)z = \sin(x+2y)$

$[(D^2-1)-D'(D-1)]z = \sin(x+2y)$

$(D-1)(D+1-D')z = \sin(x+2y)$

$(D-1)(D-D'+1)z = 0$

C.f $e^{x}f_1(y) + e^{-x}f_2(y+x)$

P.I $\dfrac{1}{(D^2-DD'+D'-1)}\sin(x+2y)$ $\quad$ Put $D^2=-1^2$, $DD'=-2$, $D'^2=-4$

$= \dfrac{1}{-1+2+D'-1}\sin(x+2y)$

$= \dfrac{1}{D'}\sin(x+2y)$

$= \dfrac{\cos(x+2y)}{-2}$

C.S

$z = e^{x}f_1(y) + e^{-x}f_2(y+x) - \frac{1}{2}\cos(x+2y)$

Ex. $2s+t-3q = 5\cos(3x-2y)$ $\quad$ Ans P.I $= \frac{1}{10}[4\cos(3x-2y) + 3\sin(3x-2y)]$

Ex. $(D^2-DD'-2D'^2+2D+2D')z = \sin(2x+y)$

Ans $z = f_1(y-x) + e^{-2x}f_2(2x+y) - \frac{1}{6}\cos(2x+y)$

Ex. $(D^2+DD'+D'-1)z = \sin(x+2y)$

Ans $z = e^{-x} f_1(y) + e^{x} f_2(y-x) - \frac{1}{10}\{2\sin(x+2y) + \cos(x+2y)\}$

Ex. $(D-D'^2)z = \cos(x-2y)$

Ans $z = \sum A e^{k^2 x + ky} + \frac{1}{82}[\sin(x-3y) + 9\cos(x-3y)]$

Ex. $(D-D'-1)(D-D'-2)z = e^{3x-y} + x$

C.F $e^{x} f_1(y+x) + e^{2x} f_2(y+x)$

P.I $\underbrace{\frac{1}{(D-D'-1)(D-D'-2)} e^{3x-y}}_{P_1} + \underbrace{\frac{1}{(D-D'-1)(D-D'-2)} \cdot x}_{P_2}$

$P_1 = \frac{1}{(3+1-1)(3+1-2)} e^{3x-y} = \frac{1}{6} e^{3x-y}$

$P_2 = \frac{1}{-(1-D+D')(-2)(1-\frac{D}{2}+\frac{D'}{2})} x$

$= \left[\frac{1}{2(1-(D-D'))(1-(\frac{D}{2}-\frac{D'}{2}))}\right] x$

$= \frac{1}{2}\left[[1-(D-D')]^{-1}(1-(\frac{D}{2}-\frac{D'}{2}))^{-1} x\right.$

$= \frac{1}{2}\left[(1+D-D')(1+\frac{D}{2}-\frac{D'}{2}) x\right.$

$= \frac{1}{2}\left[1+\frac{D}{2}-\frac{D'}{2}+D-D'\right] x$

$= \frac{1}{2}\left[x+\frac{1}{2}-0+1-0\right] = \frac{1}{2}(x+\frac{3}{2})$

C.S $z = e^{x} f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{2}(x+\frac{3}{2})$

Ex. $(D^2-D'^2-3D+3D')z = xy + e^{x+2y}$

Soln $z = f_1(y+x) + e^{3x} f_2(y-x) - \frac{1}{3}\left[\frac{x^2 y}{2} + \frac{xy}{3} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{2x}{9}\right] - x e^{x+y}$

Ex. $(s+p+q) = z + xy$

A $e^{x} f_1(y) + e^{-y} f_2(x) + x + 1 - xy - y$

Ex. $r-4s+4t+p-2q = e^{x+y}$

$D^2-4DD'+4D'^2+D-2D' = e^{x+2y}$

$[(D-2D')^2+(D-2D')]z = e^{x+2y}$

$(D-2D')(D-2D'+1)z = e^{x+2y}$

C.f $\quad f_1(y+2x)+e^{-x}f_2(y+2x)$

P.I $\quad \dfrac{e^{x+2y}}{(D-2D')(D-2D'+1)}$

$= \dfrac{1}{(D-2D'+1)}\left[\dfrac{1}{D-2D'}\cdot e^{x+2y}\right]$

$= \dfrac{1}{(D-2D'+1)}\dfrac{1}{1-\cancel{2}}\int e^{x}dx$

$= -$

$= \dfrac{-1}{D-2D'+1}\; e^{(x+y)}$

$= -e^{x+y}\dfrac{1}{(D+1)-2(D'+1)+1}\cdot 1$

$= -e^{x+y}\left(\dfrac{1}{D-2D'}\right)1\cdot e^{0x+0y}$ $\quad$ case fail

$= -e^{x+y}\dfrac{x}{1} = -xe^{x+y}$

C.S = C.f + P.I

Ex. $(D^2-D')z = xe^{ax+a^2y}$ $\quad$ Find P.I

Ans $\quad e^{ax+a^2y}\cdot\left(\dfrac{x^2}{4a}-\dfrac{x}{4a^2}\right)$

Ex $(D^2-DD'+D'-1)z = \cos(x+2y)+e^{x}$

Ex. $(D^2+DD'+D+D'-1)z = e^{-2x}(x^2+y^2)$

Ans $z = \sum Ae^{hx+ky} + \frac{1}{27}e^{-2x}(9x^2+9y^2+18x+6y+14)$

where $h^2+hk+h+k+1=0$

Ex $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

C.F $\quad e^{2x}[f_1(y+3x) + x f_2(y+3x)]$

P.I $\quad \dfrac{2e^{2x}\tan(y+3x)}{(D-3D'-2)^2}$

$= 2e^{2x}\dfrac{1}{[(D+2)-3(D'+0)-2]^2}\tan(y+3x)$

$= 2e^{2x}\dfrac{1}{(D-3D')^2}\tan(y+3x)$ [Non homogeneous]

$= 2e^{2x}\dfrac{x^2}{2}\tan(y+3x)$

$= e^{2x}.x^2\tan(y+3x)$

$z =$ C.F + P.I

Equation Reducible to Partial Differential equations with constant coefficients :-

This process is applicable if coeff of derivatives are variable.

The purpose of the problem is to covert into constant coeff. by using

$$x = e^{X}, \quad y = e^{Y}$$

$$X = \log x, \quad Y = \log y$$

and $x\frac{\partial}{\partial x} = D$, $x^2\frac{\partial^2}{\partial x^2} = D(D-1)$, $x^3\frac{\partial^3}{\partial x^3} = D(D-1)(D-2)$

$xy\frac{\partial^2}{\partial x \partial y} = DD'$

$y\frac{\partial}{\partial y} = D'$, $y^2\frac{\partial^2}{\partial y^2} = D'(D'-1)$ -

Ex. $x^2\frac{\partial^2 z}{\partial x^2} - 4xy\frac{\partial^2 z}{\partial x \partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + 6y\frac{\partial z}{\partial y} = x^3y^4$

sol^n $x = e^{X}$, $y = e^{Y}$, $X = \log x$, $Y = \log y$

$D = \frac{\partial}{\partial X}$, $D' = \frac{\partial}{\partial Y}$

$$(D(D-1) - 4DD' + 4(D'(D'-1)) + 6D')z = e^{3X+4Y}$$

$$(D^2 - 4DD' + 4D'^2 - D - 4D' + 6D')z = e^{3X+4Y}$$

$$(D-2D')^2 - (D-2D')z = e^{3X+4Y}$$

$$(D-2D')(D-2D'-1)z = e^{3X+4Y}$$

C.F $\cancel{f_1(y+2x) + e^{x} f_2(y+2x)}$

C.F $f_1(Y+2X) + e^{X} f_2(Y+2X)$

$= f_1(\log y + 2\log x) + x\, f_2(\log y + 2\log x)$

$= f_1(\log yx^2) + x\, f_2(\log yx^2) =$

P.I $\dfrac{1}{(D-2D'-1)(D-2D')}\left[e^{3x+4y}\right]$

$f(D,D')$
$=(3-8-1)(\cancel{3-8})$
$=(-6)(-5)$
$((D-3D')$ homogen$)$

$= \cancel{\dfrac{1}{(D-2D'-1)}\left[\dfrac{1}{3}\right.} = \dfrac{1}{(3-8-1)(3-8)} e^{3x+4y}$

$= \dfrac{1}{30} e^{3x+4y}$

$= \dfrac{1}{30} x^3 y^4$

$Z = C.S = C.f + P.I$

$= f_1(\log y x^2) + x f_2 \log(y x^2) + \dfrac{1}{30} x^3 y^4$

$= g_1(y x^2) + x g_2(y x^2) + \dfrac{1}{30} x^3 y^4$

Ex. $x^2 r - y^2 t + p x - q y = \log x$

$D(D-1) - D'(D'-1) + Dx - D'y = X$

$(D^2 - D'^2) z = X$

$(D-D')(D+D') z = X$

C.f $f_1(Y+X) + f_2(Y-X)$

$= f_1(\log y + \log x) + f_2(\log y - \log x)$

$= f_1(\log(yx)) + f_2(\log(y/x))$

P.I $\dfrac{1}{D^2 - D'^2} X$

$ax+by$
$a=1, b=0$

$= \dfrac{1}{1-0} \iint u\,du\,du$

$= \dfrac{u^3}{6} = \dfrac{X^3}{6} = \dfrac{(\log x)^3}{6}$

$C.S = f_1(\log(yx)) + f_2 \log(y/x) + \dfrac{(\log x)^3}{6}$ A

Ex.

$$x^2\frac{\partial^2 z}{\partial x^2} - y^2\frac{\partial^2 z}{\partial y^2} = xy$$

soln

$x = e^X$, $y = e^Y$ $\quad X = \log x$, $Y = \log y$

$$[D(D-1) - D'(D'-1)]z = e^{X+Y}$$

$$(D^2 - D'^2 - D + D')z = e^{X+Y}$$

$$(D-D')(D+D'-1)z = e^{X+Y}$$

C.F $\quad f_1(Y+X) + e^X f_2(Y-X)$

$$= f_1(\log y + \log x) + x f_2(\log(y) - \log x)$$

P.I $\quad \frac{1}{(D-D')(D+D'-1)} e^{X+Y}$

$$= \frac{1}{(D-D')}\left[\frac{1}{1+1-1} e^{X+Y}\right] = \frac{1}{(D-D')} e^{X+Y} \quad \text{case fail}$$

$$= X e^{X+Y}$$

$$= xy \log x$$

C.S $\quad z = f_1(\log(yx)) + x f_2 \log(y/x) + xy\log x.$

Ex. $\quad (x^2D^2 + 2xyDD' + y^2D'^2)z = x^m y^n$

Cauchy's Method of characterstics

Consider first order P.D.E

$$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = f(x,y) + hu \quad (1) \qquad ; \quad u(0,y) = h(y)$$

here a, b, f depend on x, y and u

But not on the derivatives of u.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \quad - (2)$$

frm (i) & (ii)

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y)+hu}$$

from $\frac{dx}{a} = \frac{dy}{b}$ we get a sol^n $bx - ay = c$

Now use in another

$$\frac{dx}{a} = \frac{du}{f\left(x, \frac{bx-c}{a}\right) + hu}$$

$$\frac{du}{dx} - \frac{h}{a}u = \frac{1}{a} f\left(x, \frac{bx-c}{a}\right)$$

$$I.F = e^{-h/a\, x}$$

$$u = G(x,c) + c_1$$

where $c_1 = g(c)$

$$u = G(x,c) + g(c) \qquad A$$

Ex. $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x+y$; $u(x,0) = 0$

Sol^n

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{x+y}$$

$$\frac{dx}{1} = \frac{dy}{1}$$

$$x = y + c$$

$$\boxed{x - y = c}$$

$$\frac{dy}{1} = \frac{du}{x+y}$$

$$\frac{dy}{1} = \frac{du}{2y+c}$$

$$(2y+c)dy \; du$$

$$u(x,y) = y^2 + cy + c_1$$

Let $c_1 = g(c)$

$$u(x,y) = y^2 + cy + g(c)$$

$$= y^2 + (x-y)y + g(x-y)$$

where $g(x-y)$ is an arbitrary function

$$u(x,0) = 0$$

$$u(x,0) = g(x) = 0$$

$$g(x-y) = 0$$

$$u(x,y) = y^2 + (x-y)y = xy \quad \text{Ans}$$

Ex.

$u_x - u_y = 0$; $u(x,0) = x$

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{du}{0}$$

$$x = -y + c$$

$$x + y = c$$

$$du = 0$$

$$u(x,y) = c_1 = g(c) = g(x+y) \quad \text{—(i)}$$

$$u(x,0) = x = g(x)$$

$$g(x+y) = x+y$$

Use in (i)

$$u(x,y) = x+y$$

Ex.

$u_x - yu = 0 \quad ; \quad u(0,y) = 1$

$$\frac{dx}{1} = \frac{dy}{0} = \frac{du}{uy}$$

$$\int dy = \int 0 \Rightarrow y = c$$

$$\frac{dx}{1} = \frac{du}{uy} = \frac{du}{uc}$$

$$g(c) + cx = \log u$$

$$u(0,y) = 1$$

$$\log u = 0 + g(c) \cancel{\text{or } 1}$$

$$\log u = cx + g(c)$$

$$\log u = cx + g(y)$$

$$x = 0$$

$$\log 1 = 0 + g(y) \Rightarrow g(y) = 0$$

$$\log u = cx$$

$$\log u = yx$$

$$\boxed{u = e^{xy}}$$

Ex. $u_x + u_y = u$ $\quad u(x,0) = 1+e^x$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{u}$$

$$\frac{dx}{1} = \frac{dy}{1}$$

$$\boxed{x-y=c} \quad -(1)$$

$$\frac{dy}{1} = \frac{du}{u}$$

$$\log u = y + \log c_1$$

$$u = c_1 e^y = g(c) e^y \quad -(2)$$

Now use (1) in (2)

$$u(x,y) = g(x-y) e^y \quad — (3)$$

Condition is $u(x,0) = 1+e^x$

$\Rightarrow$ $u(x,0) = g(x-0)e^0 \Rightarrow g(x) = 1+e^x$

$\Rightarrow$ $g(x) = 1+e^x$

$\Rightarrow$ $g(x-y) = 1+e^{x-y}$

g is arbitrary function

using (3)

$$u(x,y) = \left(1+e^{x-y}\right) e^y$$

$$u(x,y) = e^y + e^x$$

Ex. $x u_x + u_y = x\sinh y + u$, $u(0,y) = 0$

$$\frac{dx}{x} = \frac{dy}{1} = \frac{du}{x\sinh y + u}$$

$$\log x = y + \log c$$

$$x = e^y c$$

$$c = xe^{-y}$$

$$\frac{dy}{1} = \frac{du}{e^y c\left(\frac{e^y - e^{-y}}{2}\right) + u}$$

$$\frac{dy}{1} = \frac{du}{\frac{c}{2}(e^{2y}-1) + u}$$

$$\frac{dx}{x} = \frac{du}{x\sinh y + u}$$

$$x\sinh y \, dx + u\,dx = x\,du$$

Ex.

$u(x,0) = x^2$

$$u_x + u_y = 2x + 2y$$

$$\frac{dx}{1} = \frac{dy}{1} = \frac{du}{2x+2y}$$

$$x = y + c$$

$$x - y = c$$

$$\frac{dy}{1} = \frac{du}{2c}$$

$$c_1 + y = \frac{1}{2c}(u)$$

$$u = 2cy + c_1$$

$$u = 2cy + g(c)$$

$$u = 2cy + g(x-y) \quad -(1)$$

Use in (i)

$$u = 2cy + g(x-y) \quad -(2)$$

$$u(x,0) = 0 + g(x) = x^2$$

$$g(x) = x^2$$

$$g(x-y) = (x-y)^2$$

Eq^n (i), we get

$$u(x,y) = 2(x-y)y + (x-y)^2 = 2(xy - y^2) + x^2 + y^2 - 2xy$$

$$u(x,y) = x^2 - y^2$$