Matrix Theory

Definition: Matrix

A set of $m \times n$ numbers arranged in a rectangular form of m rows & n columns enclosed between a pair of square brackets is called a matrix of order $m \times n$ (read as m by n).

Matrices are generally denoted by capital alphabets & its elements are denoted by small alphabets.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}_{3\times 3}$$

In short, $A = [a_{ij}]_{m\times n}$ where $\mathrm{i} = \mathrm{No.\ of\ rows} = 1,2,3, \ldots, \mathrm{m}$

& $\mathrm{j} = \mathrm{No.\ of\ columns} = 1,2,3, \ldots, \mathrm{n}.$

In general, an $m \times n$ matrix has the following rectangular array:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & a_{i3} \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

Definition: Order of a matrix

The order of a matrix is defined as $m \times n$ if it contains m rows & n columns.

Examples:1. $A = [2 \quad 3 \quad -1]$ Order of A is 1×3

2. $B = \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix}$ Order of B is 3×2

Type of Matrix

1)

Row matrix: Matrix having only one row is called row matrix.

For e.g. : $A = \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}$.

2)

Column matrix: Matrix having only one column is called column matrix.

For e.g. : $D = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$.

3)

Rectangular Matrix:

If number of rows and column of matrix are different, such a matrix is called as rectangular matrix.

For Example: $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 4 \end{bmatrix}$

Note: In rectangular matrix, $m \neq n$

4)

Square matrix : Matrix having equal number of rows & columns is called square matrix.

For e.g. $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix}$

Note: In matrix A, elements 2,3,4 are diagonal elements and remaining are non-diagonal elements.

5)

Zero matrix: A matrix having all elements equal to zero is called zero matrix.

For e.g. : $A = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$; $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

6)

Diagonal matrix: A square matrix where all non diagonal elements are zero is called a diagonal matrix.

For e.g. : $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

7)

Scalar matrix: A diagonal matrix where all diagonal elements are equal is called a scalar matrix.

For e.g. : $K = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

8)

Identity matrix OR Unit matrix: A scalar matrix where all diagonal elements are one (unit) is called an identity matrix or unit matrix denoted by I.

For e.g. : $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

9)

Upper Triangular Matrix:

A Square matrix in which, all elements below the principal diagonal are zero, is called lower triangular matrix.

For Example: $\begin{bmatrix} 1 & -5 & 4 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \end{bmatrix}$

10)

Lower Triangular Matrix:

A Square matrix in which, all elements above the principal diagonal are zero, is called lower triangular matrix.

For Example: $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

Determinant of a matrix***:***

Let ‘A’ be a square matrix. The determinant formed by the elements of

A is said to be the determinant of matrix A. This is denoted by | A |.

For Example: If $A = \begin{bmatrix} 1 & 2 & -2 \\ 3 & 1 & 0 \\ 4 & 1 & 5 \end{bmatrix}$ then $|A| = \begin{vmatrix} 1 & 2 & -2 \\ 3 & 1 & 0 \\ 4 & 1 & 5 \end{vmatrix}$

11)

Singular and Non-Singular matrix*:*

A square matrix 'A' is called a singular matrix if |A| = 0.

If | A | ≠ 0, then the matrix A is called as Non-singular matrix.

Ex.:- Check whether following matrix are singular or Non-singular?

a) $A = \begin{bmatrix} 2 & 3 \\ 1 & 7 \end{bmatrix}$ b) $B = \begin{bmatrix} -5 & 2 \\ 10 & -4 \end{bmatrix}$

Solution:- a) Let $|A| = \begin{vmatrix} 2 & 3 \\ 1 & 7 \end{vmatrix} = (2 \times 7) - (1 \times 3) = 14 - 3 = 11 \neq 0$

Hence matrix 'A' is Non-singular matrix.

b) Let $|B| = \begin{vmatrix} -5 & 2 \\ 10 & -4 \end{vmatrix} = (-5 \times -4) - (10 \times 2) = 20 - 20 = 0$

Hence matrix 'B' is Singular matrix.

Algebra of Matrix

1. Addition of matrices: If two matrices A, B are of same order then the addition matrix 'A+B' can be obtained by adding the corresponding elements. Order of matrix $A+B$ is same as that of A and B.

For e.g. if $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 3 \\ -3 & 1 & -2 \end{bmatrix}$

then $A+B = \begin{bmatrix} 5+4 & 6+2 & 1+3 \\ 0-3 & 2+1 & 9-2 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 4 \\ -3 & 3 & 7 \end{bmatrix}$

Note

1. We emphasise that if A and B are not of the same order, then A + B is not defined. For example if $A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$, then A + B is not defined.
2. We may observe that addition of matrices is an example of binary operation on the set of matrices of the same order.

Properties of Addition of matrix

The addition of matrices satisfy the following properties:

(i) **Commutative Law** If $A = [a_{ij}]$, $B = [b_{ij}]$ are matrices of the same order, say $m \times n$, then $A + B = B + A$.

Now $A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$

$= [b_{ij} + a_{ij}]$ (addition of numbers is commutative)

$= ([b_{ij}] + [a_{ij}]) = B + A$

(ii) **Associative Law** For any three matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $C = [c_{ij}]$ of the same order, say $m \times n$, $(A + B) + C = A + (B + C)$.

Now $(A + B) + C = ([a_{ij}] + [b_{ij}]) + [c_{ij}]$

$$= [a_{ij} + b_{ij}] + [c_{ij}] = [(a_{ij} + b_{ij}) + c_{ij}]$$

$$= [a_{ij} + (b_{ij} + c_{ij})] \qquad \text{(Why?)}$$

$$= [a_{ij}] + [(b_{ij} + c_{ij})] = [a_{ij}] + ([b_{ij}] + [c_{ij}]) = A + (B + C)$$

(iii) **Existence of additive identity** Let $A = [a_{ij}]$ be an $m \times n$ matrix and O be an $m \times n$ zero matrix, then $A + O = O + A = A$. In other words, O is the additive identity for matrix addition.

(iv) **The existence of additive inverse** Let $A = [a_{ij}]_{m \times n}$ be any matrix, then we have another matrix as $-A = [-a_{ij}]_{m \times n}$ such that $A + (-A) = (-A) + A = O$. So $-A$ is the additive inverse of A or negative of A.

2. Subtraction of matrices: If two matrices A, B are of same order then matrix '$A - B$' can be obtained by subtracting the corresponding elements. Order of matrix $A - B$ is same as that of A and B.

For e.g. if $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 2 & 3 \\ -3 & 1 & -2 \end{bmatrix}$

then $A - B = \begin{bmatrix} 5-4 & 6-2 & 1-3 \\ 0+3 & 2-1 & 9+2 \end{bmatrix} = \begin{bmatrix} 1 & 4 & -2 \\ 3 & 1 & 11 \end{bmatrix}$

Matrix introduction link: https://youtu.be/xyAuNHPsq-g

3. Scalar Multiplication: If A is a matrix and 'k' is a scalar then the matrix 'kA' is obtained by multiplying every element of the matrix A by 'k'.

For e.g. if $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 9 \end{bmatrix}$ $then\ 5A = \begin{bmatrix} 25 & 30 & 5 \\ 0 & 10 & 45 \end{bmatrix}$ where k=5

Properties of Scalar Multiplication

(i) $k(\mathrm{A}+\mathrm{B}) = k\,\mathrm{A} + k\mathrm{B}$, (ii) $(k + l)\mathrm{A} = k\,\mathrm{A} + l\,\mathrm{A}$

For example, if $\mathrm{A} = \begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix}$, then

$$3\mathrm{A} = 3\begin{bmatrix} 3 & 1 & 1.5 \\ \sqrt{5} & 7 & -3 \\ 2 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 & 4.5 \\ 3\sqrt{5} & 21 & -9 \\ 6 & 0 & 15 \end{bmatrix}$$

Negative of a matrix The negative of a matrix is denoted by $-\mathrm{A}$. We define $-\mathrm{A} = (-1)\,\mathrm{A}$.

For example, let $A = \begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix}$, then $-A$ is given by

$$-A = (-1)A = (-1)\begin{bmatrix} 3 & 1 \\ -5 & x \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 5 & -x \end{bmatrix}$$

1. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 6 & -1 \\ 0 & 3 \end{bmatrix}$ find $2A + 3B$.

Solution: $2A + 3B = 2\begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 2 \\ 6 & -1 \\ 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -6 \\ 8 & 0 \\ -2 & -4 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 18 & -3 \\ 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3 & -6+6 \\ 8+18 & 0-3 \\ -2+0 & -4+9 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 0 \\ 26 & -3 \\ -2 & 5 \end{bmatrix}$$

2. Find the value of x and y satisfying the equation

$$\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

Solution : $\begin{bmatrix} 1 & x & 0 \\ y & 2 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 1 & 2 \\ 4 & 3 & -2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$

$$\therefore \begin{bmatrix} 1+3 & x+1 & 0+2 \\ y+4 & 2+3 & 4-2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4 & x+1 & 2 \\ y+4 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 2 \\ 6 & 5 & 2 \end{bmatrix}$$

By using equality of matrices, $x+1=2$ and $y+4=6$

$\therefore x=1$ & $y=2$

3. If $A = \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$

then prove that $(A+B)+C = A+(B+C)$

Solution: $L.H.S. = (A+B)+C$

$$= \left(\begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} \right) + \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 7 & 3 \end{bmatrix}$$

$R.H.S. = A+(B+C)$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \left(\begin{bmatrix} 4 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 7 \\ 5 & 2 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 2 & 5 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 11 \\ 7 & 3 \end{bmatrix}$$

$\therefore L.H.S. = R.H.S.$

If $A = \begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix}$, find the matrix 'X' such that $2A + X = 3B$

Solution : $\mathbf{2A + X = 3B}$

$$\therefore \quad X = 3B - 2A$$

$$\therefore \quad X = 3\begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} - 2\begin{bmatrix} 2 & -1 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & -6 \\ -3 & 12 \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ 8 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4 & -6+2 \\ -3-8 & 12-6 \end{bmatrix}$$

$$X = \begin{bmatrix} 5 & -4 \\ -11 & 6 \end{bmatrix}$$

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 0 & 2 \end{bmatrix}$, then find $2A - B$.

We have

$$2A - B = 2 \begin{matrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{matrix} \qquad \begin{matrix} 3 & 1 & 3 \\ 1 & 0 & 2 \end{matrix}$$

$$= \begin{bmatrix} 2 & 4 & 6 \\ 4 & 6 & 2 \end{bmatrix} + \begin{bmatrix} -3 & 1 & -3 \\ 1 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2-3 & 4+1 & 6-3 \\ 4+1 & 6+0 & 2-2 \end{bmatrix} = \begin{bmatrix} -1 & 5 & 3 \\ 5 & 6 & 0 \end{bmatrix}$$

Find X and Y, if $X + Y = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix}$ and $X - Y = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

We have $(X+Y)+(X-Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$.

$$(X + X) + (Y - Y) = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} \Rightarrow 2X = \begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix}$$

$$X = \frac{1}{2}\begin{bmatrix} 8 & 8 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$$

$$(X + Y) - (X - Y) = \begin{bmatrix} 5 & 2 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix}$$

$$(X - X) + (Y + Y) = \begin{bmatrix} 5-3 & 2-6 \\ 0 & 9+1 \end{bmatrix} \Rightarrow 2Y = \begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix}$$

$$Y = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 5 \end{bmatrix}$$

Exercise

A. If $A = \begin{bmatrix} 3 & 2 \\ 1 & -1 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 3 & 2 \\ 4 & -2 \end{bmatrix}$

verify that $A + B = B + A$

B. Find the values of x & y such that

$$\begin{bmatrix} 2x+1 & -1 \\ 3 & 4y \end{bmatrix} + \begin{bmatrix} -1 & 6 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 3 & 12 \end{bmatrix}$$

C. If $X = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$, $Y = \begin{bmatrix} 4 & 5 \\ 1 & -3 \end{bmatrix}$, $Z = \begin{bmatrix} 7 & 11 \\ -8 & 9 \end{bmatrix}$ then show that $3X + Y = Z$

D. If $A = \begin{bmatrix} x & 2 & -5 \\ 3 & 1 & 2y \end{bmatrix}$, $B = \begin{bmatrix} 2y+5 & 6 & -15 \\ 9 & 3 & -6 \end{bmatrix}$ and if $3A = B$ then find x and y

E. If $A=\begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix}, B=\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}$

then find $2A-4B+3I$

F. If $A=\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}, B=\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}, C=\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

Verify that $(A+B)+C=A+(B+C)$

G. If $A=\begin{bmatrix} 2 & -3 \\ 3 & 4 \end{bmatrix}, B=\begin{bmatrix} 4 & 5 \\ 3 & -2 \end{bmatrix}, C=\begin{bmatrix} 3 & -1 \\ 0 & 6 \end{bmatrix}$

then find $3A+4B-2C$

H. Decide whether the matrix $\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -1 & -2 & 3 \end{bmatrix}$

is non-singular?

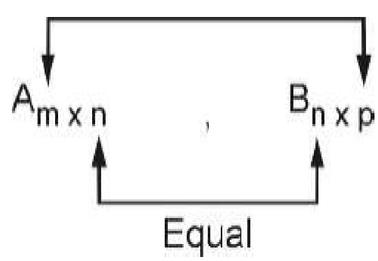
Multiplication of Two Matrix

Condition for multiplication of two matrices

The product of two matrices A and B is possible only if the **number of columns in A is equal** to the **number of rows in B**.

Let $A = [a_{ij}]$ be a '$m \times n$' matrix and $B = [b_{ij}]$ be a '$n \times p$' matrix.

then Order of $A \times B$ is $m \times p$



Matrix multiplication link: https://youtu.be/aKhhYguY0DQ

Method of Multiplication of two matrices:

Let A = $\begin{array}{c} R_1 \longrightarrow \\ R_2 \longrightarrow \end{array} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{array}{c} \begin{bmatrix} p & q & r \\ x & y & z \end{bmatrix} \\ \begin{array}{ccc} \downarrow & \downarrow & \downarrow \\ C_1 & C_2 & C_3 \end{array} \end{array}$

then $AB = \begin{bmatrix} R_1C_1 & R_1C_2 & R_1C_3 \\ R_2C_1 & R_2C_2 & R_2C_3 \end{bmatrix} = \begin{bmatrix} \text{ap + bx} & \text{aq + by} & \text{ar + bz} \\ \text{cp + dx} & \text{cq + dy} & \text{cr + dz} \end{bmatrix}$

Note: R_1C_1 means multiplying the elements of first row of A with corresponding elements of first column of B.

For example, if $C = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$, then the product CD is defined

and is given by $CD = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix}$. This is a 2×2 matrix in which each entry is the sum of the products across some row of C with the corresponding entries down some column of D. These four computations are

Entry in first row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} (1)(2) + (-1)(-1) + (2)(5) & ? \\ ? & ? \end{bmatrix}$

Entry in first row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & (1)(7) + (-1)(1) + 2(-4) \\ ? & ? \end{bmatrix}$

Entry in second row first column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 0(2) + 3(-1) + 4(5) & ? \end{bmatrix}$

Entry in second row second column $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 7 \\ -1 & 1 \\ 5 & -4 \end{bmatrix} = \begin{bmatrix} 13 & -2 \\ 17 & 0(7) + 3(1) + 4(-4) \end{bmatrix}$

Thus $CD = \begin{bmatrix} 13 & -2 \\ 17 & -13 \end{bmatrix}$

Note: In matrices, matrix multiplication is not commutative.

i.e. $A \times B \neq B \times A$ in general

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $AB = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Clearly $AB \neq BA$.

Thus matrix multiplication is not commutative.

Note This does not mean that $AB \neq BA$ for every pair of matrices A, B for which AB and BA, are defined. For instance,

If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}$, then $AB = BA = \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix}$

Observe that multiplication of diagonal matrices of same order will be commutative.

Zero matrix as the product of two non zero matrices

We know that, for real numbers a, b if $ab = 0$, then either $a = 0$ or $b = 0$. This need not be true for matrices, we will observe this through an example.

Find AB, if $A = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix}$.

We have $AB = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Properties of matrix multiplication

1. **The associative law** For any three matrices A, B and C. We have (AB) C = A (BC), whenever both sides of the equality are defined.
2. **The distributive law** For three matrices A, B and C.
   (i) A (B+C) = AB + AC
   (ii) (A+B) C = AC + BC, whenever both sides of equality are defined.
3. **The existence of multiplicative identity** For every square matrix A, there exist an identity matrix of same order such that IA = AI = A.

For Example:

If $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}, C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$

verify that $(A + B)C = AC + BC$

Solution Now, $A+B = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix}$

So $$(A + B)\, C = \begin{bmatrix} 0 & 7 & 8 \\ -5 & 0 & 10 \\ 8 & -6 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-14+24 \\ -10+0+30 \\ 16+12+0 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

Further $$AC = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-12+21 \\ -12+0+24 \\ 14+16+0 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix}$$

and $$BC = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0-2+3 \\ 2+0+6 \\ 2-4+0 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix}$$

So $$AC + BC = \begin{bmatrix} 9 \\ 12 \\ 30 \end{bmatrix} + \begin{bmatrix} 1 \\ 8 \\ -2 \end{bmatrix} = \begin{bmatrix} 10 \\ 20 \\ 28 \end{bmatrix}$$

Clearly, $(A + B)\, C = AC + BC$

<u>**Example**</u>

1. If $A = \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & -3 \end{bmatrix}$ find i). $A \times B$

ii). $B \times A$

Solution: $A \times B = \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix} \times \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 6-6 & 8-0 & -2+9 \\ 12+0 & 16+0 & -4+0 \\ -3-4 & -4+0 & 1+6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 8 & 7 \\ 12 & 16 & -4 \\ -7 & -4 & 7 \end{bmatrix}$$

$$B \times A = \begin{bmatrix} 3 & 4 & -1 \\ 2 & 0 & -3 \end{bmatrix} \times \begin{bmatrix} 2 & -3 \\ 4 & 0 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6+16+1 & -9+0+2 \\ 4+0+3 & -6+0+6 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & -7 \\ 7 & 0 \end{bmatrix}$$

2. If $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix}$ $then\ find\ \ A^2 - 3\mathrm{I}.$

Solution: $A^2 = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 0 \\ 1 & 3 & -4 \\ 5 & -3 & 4 \end{bmatrix}$

$$= \begin{bmatrix} 4-1+0 & -2-3+0 & 0+4+0 \\ 2+3-20 & -1+9+12 & 0-12-16 \\ 10-3+20 & -5-9-12 & 0+12+16 \end{bmatrix} = \begin{bmatrix} \mathbf{3} & \mathbf{-5} & \mathbf{4} \\ \mathbf{-15} & \mathbf{20} & \mathbf{-28} \\ \mathbf{27} & \mathbf{-26} & \mathbf{28} \end{bmatrix}$$

$$3I = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- $\therefore A^2 - 3\mathrm{I} = \begin{bmatrix} 3 & -5 & 4 \\ -15 & 20 & -28 \\ 27 & -26 & 28 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 4 \\ -15 & 17 & -28 \\ 27 & -26 & 25 \end{bmatrix}$

3. Find x & y if $\left\{4\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & 4 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

Solution: $Given\ \ \left\{4\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 3 \end{bmatrix} - 2\begin{bmatrix} 1 & 3 & -1 \\ 2 & -3 & 4 \end{bmatrix}\right\}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 2 & 2 & 2 \\ 4 & 2 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 4+0-2 \\ 8+0-4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

x = 2 and y = 4

12)

Transpose of matrix:

If ‘A’ is any matrix of order “m x n” then matrix obtain by interchanging row and column of matrix ‘A’ is called as “Transpose of matrix A”.
It is denoted by A^T or A'

For Example, If $A = \begin{bmatrix} 1 & -2 & 3 \\ 4 & 1 & 6 \end{bmatrix}_{2\times 3}$ then $A^T = \begin{bmatrix} 1 & 4 \\ -2 & 1 \\ 3 & 6 \end{bmatrix}_{3\times 2}$

Note: If ‘A’ is matrix of order “m x n” then order of matrix A^T is “m x n”

Properties of Transpose of matrix:

1) $\left(A^T\right)^T = A$

2) $(kA)^T = k.A^T$, Where ‘k’ is scalar multiple.

3) $(A \pm B)^T = A^T \pm B^T$

4) $(A.B)^T = B^T.A^T$

5) If $A.A^T = I$ then matrix ‘A’ is called as “Orthogonal matrix”.

1. If $A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$ then verify that $(AB)' = B'A'$

Solution :

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A' = \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} \quad \text{and} \quad B' = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6-3 & -2+0 & 4-3 \\ 3+5 & -1+0 & 2+5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -2 & 1 \\ 8 & -1 & 7 \end{bmatrix}$$

$$(AB)' = \begin{bmatrix} 3 & 8 \\ -2 & -1 \\ 1 & 7 \end{bmatrix} \qquad \ldots \text{(i)}$$

$$B'\ A' = \begin{bmatrix} 3 & 1 \\ -1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -3 & 5 \end{bmatrix} = \begin{bmatrix} 6-3 & 3+5 \\ -2+0 & -1+0 \\ 4-3 & 2+5 \end{bmatrix}$$

$$B'\ A' = \begin{bmatrix} 3 & 8 \\ -2 & -1 \\ 1 & 7 \end{bmatrix} \qquad \ldots \text{(ii)}$$

From (i) and (ii) $\quad (AB)' = \quad B' \cdot A'$

13)

Symmetric Matrix :

Definition: In a matrix A, if $a_{ij} = a_{ji}$ for all i and j then matrix is known as symmetric matrix i.e. if $A = A'$ then matrix is known as symmetric matrix.

For e.g. $A = \begin{bmatrix} 1 & 2 & -4 \\ 2 & 5 & 3 \\ -4 & 3 & 9 \end{bmatrix}$

Note:

1) In symmetric matrix, $a_{ij} = a_{ji}, i \neq j$

14)

Skew Symmetric Matrix:

Definition: In a matrix A, if $a_{ij} = -a_{ji}$ for all i and j then matrix is known as skew symmetric matrix i.e. if $A = -A'$ then matrix is skew symmetric matrix.

For e.g. $A = \begin{bmatrix} 0 & 2 & -3 \\ -2 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}$

Note:

1) In skew symmetric matrix, $a_{ij} = -a_{ji}, i \neq j$

2) In skew symmetric matrix, $a_{ii} = 0$ i.e. all diagonal numbers are always '0'

Theorem: Any Square matrix 'A' can be expressed as sum of symmetric and skew symmetric matrix.

Note:

Let A be a square matrix, then we can write

$$A = \underbrace{\frac{1}{2}(A + A')}_{\text{Symmetric}} + \underbrace{\frac{1}{2}(A - A')}_{\text{Skew symmetric}}$$

Express the matrix $B = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ as the sum of a symmetric and a skew symmetric matrix.

Solution Here

$$B' = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 3 & -2 \\ -4 & 4 & -3 \end{bmatrix}$$

Let $$P = \frac{1}{2}(B + B') = \frac{1}{2}\begin{bmatrix} 4 & -3 & -3 \\ -3 & 6 & 2 \\ -3 & 2 & -6 \end{bmatrix} = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix},$$

Now $$P' = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} = P$$

Thus $P = \frac{1}{2}(B + B')$ is a symmetric matrix.

Also, let $$Q = \frac{1}{2}(B - B') = \frac{1}{2}\begin{bmatrix} 0 & -1 & -5 \\ 1 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix}$$

Then $$Q' = \begin{bmatrix} 0 & \frac{1}{2} & \frac{5}{3} \\ \frac{-1}{2} & 0 & -3 \\ \frac{-5}{2} & 3 & 0 \end{bmatrix} = -Q$$

Thus $\quad Q = \frac{1}{2}(B - B')$ is a skew symmetric matrix.

Now $$P + Q = \begin{bmatrix} 2 & \frac{-3}{2} & \frac{-3}{2} \\ \frac{-3}{2} & 3 & 1 \\ \frac{-3}{2} & 1 & -3 \end{bmatrix} + \begin{bmatrix} 0 & \frac{-1}{2} & \frac{-5}{2} \\ \frac{1}{2} & 0 & 3 \\ \frac{5}{2} & -3 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = B$$

Express the following matrices as the sum of a symmetric and a skew symmetric matrix:

(i) $\begin{bmatrix} 3 & 5 \\ 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Minor of Element in matrix

It is a value of determinant obtain by eliminating row and column in which element is present. It is denoted by "M_{ij}".

M_{ij} = Minor of element a_{ij} present in i^{th} row and j^{th} column.

For Example: If A = $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 1 & 0 \\ 4 & 1 & 5 \end{bmatrix}$

M_{21} = Minor of number 3 = $\begin{vmatrix} 1 & 2 & -2 \\ 3 & 1 & 0 \\ 4 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 2 & -2 \\ 1 & 5 \end{vmatrix} = (2\times5)-(1\times-2) = 10+2 = 12$

$M_{21} = 12$

Note: 1) Element has a minor value only if matrix is a square matrix.

2) For matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ minor of 'a' = d, minor of 'b' = c, minor of 'c' = b, minor of 'd' = a

Co-factor of Element in matrix

It is a minor of element with proper sign. It is denoted as "C_{ij}".

It is given as, $C_{ij} = (-1)^{i+j} . M_{ij}$

Sign convention

For 2×2 matrix, $\begin{bmatrix} + & - \\ - & + \end{bmatrix}$ and for 3×3 matrix, $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

For Example: If A = $\begin{bmatrix} 1 & 2 & -2 \\ 3 & 1 & 0 \\ 4 & 1 & 5 \end{bmatrix}$

$M_{21} = 12$ then $C_{21} = -M_{21} - (12) = -12$

$C_{21} =$-12

Adjoint matrix of a matrix

If 'A' is any square matrix then adjoint matrix of matrix 'A' is transpose of co-factor matrix of matrix 'A'.
It is given as,

$$\text{Adjoint of matrix 'A'} = (\text{Co-factor matrix of A})^T$$

For Example,

If A= $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, find Adj A

Solution.: Given A= $\begin{bmatrix} -1 & 1 & 1 \\ 2 & 4 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$$c_{11} = (-1)^{1+1} \times \begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = +(4-4) = 0$$

$$c_{12} = (-1)^{1+2} \times \begin{vmatrix} 2 & 2 \\ 3 & 1 \end{vmatrix} = -(2-6) = 4$$

$$c_{13} = (-1)^{1+3} \times \begin{vmatrix} 2 & 4 \\ 3 & 2 \end{vmatrix} = +(4-12) = -8$$

$$c_{21} = (-1)^{2+1} \times \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -(1-2) = 1$$

$$c_{22} = (-1)^{2+2} \times \begin{vmatrix} -1 & 1 \\ 3 & 1 \end{vmatrix} = +(-1-3) = -4$$

$$c_{23} = (-1)^{2+3} \times \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -(-2-3) = 5$$

$$c_{31} = (-1)^{3+1} \times \begin{vmatrix} 1 & 1 \\ 4 & 2 \end{vmatrix} = +(2-4) = -2$$

$$c_{32} = (-1)^{3+2} \times \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = -(-2-2) = 4$$

$$c_{33} = (-1)^{3+3} \times \begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix} = +(-4-2) = -6$$

$$\therefore Matrix\ of\ cofactors = C = \begin{bmatrix} 0 & 4 & -8 \\ 1 & -4 & 5 \\ -2 & 4 & -6 \end{bmatrix}$$

$$\therefore Adj\ A = C^t = \begin{bmatrix} 0 & 1 & -2 \\ 4 & -4 & 4 \\ -8 & 5 & -6 \end{bmatrix}$$

Exercise

Ex. Calculate adjoint of a matrix $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

Note: Adjoint of a 2×2 Matrix:

The adjoint of matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is denoted by adjA is defined as

$$\text{adj}A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Inverse of a matrix

Definition If A is a square matrix of order m, and if there exists another square matrix B of the same order m, such that AB = BA = I, then B is called the *inverse* matrix of A and it is denoted by A^{-1}. In that case A is said to be invertible.

For example, let $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$ be two matrices.

Now
$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & -6+6 \\ 2-2 & -3+4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Also $BA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$. Thus B is the inverse of A, in other words $B = A^{-1}$ and A is inverse of B, i.e., $A = B^{-1}$

Note

1. A rectangular matrix does not possess inverse matrix, since for products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.
2. If B is the inverse of A, then A is also the inverse of B.

- Inverse of a square matrix, if it exists, is unique.

Matrix Inverse Method (By Adjoint matrix)

In this method inverse of matrix is obtain by,

Formula: $\mathbf{A^{-1}} = \frac{\mathbf{1}}{\mathbf{det\ A}} \times \mathbf{Adj\ A}$

For example if matrix $A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$

Then $\text{adj } A = \begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 6 - 4 = 2$$

Hence $A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2}\begin{bmatrix} 2 & -4 \\ -1 & 3 \end{bmatrix}$

Example

Find the inverse of the matrix $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution : Given $A = \begin{bmatrix} 3 & 1 & 2 \\ 4 & 1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

$$|A| = 3(3+1) - 1(12-2) + 2(-4-2)$$

$$= 12 - 10 - 12 = -10 \neq 0$$

$\therefore A^{-1}$ exists

$$c_{11} = (-1)^{1+1} \times \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = +(3+1) = 4$$

$$c_{12} = (-1)^{1+2} \times \begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix} = -(12-2) = -10$$

$$c_{13} = (-1)^{1+3} \times \begin{vmatrix} 4 & 1 \\ 2 & -1 \end{vmatrix} = +(-4-2) = -6$$

$$c_{21} = (-1)^{2+1} \times \begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = -(3+2) = -5$$

$$c_{22} = (-1)^{2+2} \times \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} = +(9-4) = 5$$

$$c_{23} = (-1)^{2+3} \times \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -(-3-2) = 5$$

$$c_{31} = (-1)^{3+1} \times \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = +(1-2) = -1$$

$$c_{32} = (-1)^{3+2} \times \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} = -(3-8) = 5$$

$$c_{33} = (-1)^{3+3} \times \begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} = +(3-4) = -1$$

$$\therefore C = \begin{bmatrix} 4 & -10 & -6 \\ -5 & 5 & 5 \\ -1 & 5 & -1 \end{bmatrix} \qquad \therefore Adj\ A = C^t = \begin{bmatrix} 4 & -5 & -1 \\ -10 & 5 & 5 \\ -6 & 5 & -1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{-10} \times \begin{bmatrix} 4 & -5 & -1 \\ -10 & 5 & 5 \\ -6 & 5 & -1 \end{bmatrix}$$

Inverse of matrix link: https://youtu.be/xfhzwNkMNg4

Exercise

Ex:

Find the inverse, if it exists, of the matrix.

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$$

Application of Inverse of matrix

Solution of simultaneous equations:

Suppose $$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

are the given simultaneous equations.

These equations can be represented in matrix form as follows:

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad \text{i.e. } A \times X = B \text{ where}$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix};\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix};\ B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

The solution of the system of equations is given by $X = A^{-1} \times B$ where $A^{-1} = \frac{1}{\det A} \times Adj\ A$

Inverse matrix application link: https://youtu.be/MS-BFkhXSQw

Remark:

If A is a singular matrix, then of course it has no inverse, and either the system has no solution or the solution is not unique.

Example : Use matrices to find the solution set of

$$x + y - 2z = 3$$
$$3x - y + z = 5$$
$$3x + 3y - 6z = 9$$

Solution:

Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Since $|A| = 3 + 21 - 24 = 0$

Hence the solution of the given linear equations does not exists.

Example

Solve by matrix inverse method

1. $x + y + z = 3;\ x + 2y + 3z = 4;\ x + 4y + 9z = 6$

Solution: $x + y + z = 3;\ x + 2y + 3z;\ \ x + 4y + 9z = 6$

Matrix Equation: $A \times X = B$

Where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}; B = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$

$\therefore |A| = 1(18 - 12) - 1(9 - 3) + 1(4 - 2) = 6 - 6 + 2 = 2 \neq 0$

$\therefore A^{-1}$ exists

$$c_{11} = (-1)^{1+1} \times \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = +(18 - 12) = 6$$

$$c_{12} = (-1)^{1+2} \times \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -(9 - 3) = -6$$

$$c_{13} = (-1)^{1+3} \times \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = +(4 - 2) = 2$$

$$c_{21} = (-1)^{2+1} \times \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -(9 - 4) = -5$$

$$c_{22} = (-1)^{2+2} \times \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = +(9 - 1) = 8$$

$$c_{23} = (-1)^{2+3} \times \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -(4-1) = -3$$

$$c_{31} = (-1)^{3+1} \times \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = +(3-2) = 1$$

$$c_{32} = (-1)^{3+2} \times \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -(3-1) = -2$$

$$c_{33} = (-1)^{3+3} \times \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = +(2-1) = 1$$

$$\therefore C = \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore Adj\ A = C^t = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det A} \times AdjA = \frac{1}{2} \times \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\therefore X = A^{-1} \times \mathrm{B} = \frac{1}{2} \times \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 18-20+6 \\ -18+32-12 \\ 6-12+6 \end{bmatrix} = \frac{1}{2} \times \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

The solution is $x = 2$; $y = 1$; $z = 0$

Example : Use matrices to find the solution set of

$$4x + 8y + z = -6$$
$$2x - 3y + 2z = 0$$
$$x + 7y - 3z = -8$$

Solution:

Let $A = \begin{bmatrix} 4 & 8 & 1 \\ 2 & -3 & 2 \\ 1 & 7 & -3 \end{bmatrix}$

Since $|A| = -32 + 48 + 17 = 61$

So A^{-1} exists.

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$= \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}$$

Now since,

$X = A^{-1} B$, we have

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{61}\begin{bmatrix} -5 & 31 & 19 \\ 8 & -13 & -16 \\ 17 & -20 & -28 \end{bmatrix}\begin{bmatrix} -6 \\ 0 \\ -8 \end{bmatrix}$$

$$= \frac{1}{61}\begin{bmatrix} 30+152 \\ -48+48 \\ -102+224 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$$

Hence Solution set: $\{(x, y, z)\} = \{(-2, 0, 2)\}$

Inverse matrix application link: https://youtu.be/MS-BFkhXSQw

Exercise

Q.1 Which of the following matrices are singular or non-singular.

(i) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ 3 & 3 & -6 \end{bmatrix}$

Q.2 Which of the following matrices are symmetric and skew-symmetric

(i) $\begin{bmatrix} 2 & 6 & 7 \\ 6 & -2 & 3 \\ 7 & 3 & 0 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 3 & -5 \\ -3 & 0 & 6 \\ 5 & -6 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Q.3 Find K such that the following matrices are singular

(i) $\begin{vmatrix} K & 6 \\ 4 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 2 & -1 \\ -3 & 4 & K \\ -4 & 2 & 6 \end{vmatrix}$ (iii) $\begin{vmatrix} 1 & 1 & -2 \\ 3 & -1 & 1 \\ k & 3 & -6 \end{vmatrix}$

Q.4 Find the inverse if it exists, of the following matrices

(i) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 0 & 2 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{bmatrix}$

Q.5 Find the solution set of the following system by means of matrices:

(i) $2x - 3y = -1$
$x + 4y = 5$

(ii) $x + y = 2$
$2x - z = 1$
$2y - 3z = -1$

(iii) $x - 2y + z = -1$
$3x + y - 2z = 4$
$y - z = 1$

(iv) $-4x + 2y - 9z = 2$
$3x + 4y + z = 5$
$x - 3y + 2z = 8$

(v) $x + y - 2z = 3$
$3x - y + z = 0$
$3x + 3y - 6z = 8$

Answer

Q.1 (i) Non-singular (ii) Singular
(iii) Singular

Q.2 (i) Symmetric (ii) Skew-symmetric
(iii) Symmetric

Q.3 (i) 8 (ii) 5 (iii) 3

Q.4 (i) $\begin{bmatrix} \frac{1}{7} & \frac{3}{7} \\ \frac{2}{7} & \frac{1}{-7} \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 2 & 3 \\ -1 & -3 & -3 \\ 1 & 2 & 2 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{4}{5} & -\frac{1}{5} & -\frac{4}{5} \\ -\frac{1}{5} & -\frac{1}{5} & \frac{7}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$

(iv) A^{-1} does not exist.

Q.5 (i) {(1, 1)} (ii) {(1, 1, 1)} (iii) {(1, 1, 0)}
(iv) {(7, -3, -4)} (v) no solution