INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY

HYDERABAD

# International Institute of Information Technology

## Real analysis - MA4.101

Report On

# SERIES AND HISTORICAL PARADOXES RELATED TO SERIES

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In particular. We feel obliged to thank our teacher as well as our teaching assistant for allowing us to develop a passion in the field of real analysis and delve into the realm of pure mathematics.

# ABSTRACT

Through this report, we would like to shed light on the impact and importance of series, particularly infinite series and the paradoxes related to series. We would also try to link the connection of series to the field of mathematics as a whole.

We would start by formally defining what a series is, followed by the various types of series and the different tests for finding whether a series is convergent or not. In the next section, we start by defining what a paradox is and then talk about seven different paradoxes related to convergence, oscillating, harmonic series, etc. While defining each paradox, We will explain the problem, its solution, and its significance in the real world.

By the end, We would get a clear insight into how vital series are in pure mathematics and how much we can learn from the various paradoxes related to them.

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# INTRODUCTION

Series are a very indispensable part of mathematics, giving us precise ways to express many numbers and functions.

We will discuss various types of series like Geometric series, Harmonic series, P-series, Oscillating series and find conditions for which they would be converging and diverging. This will further help us to find solutions to the paradoxes which we will discuss in the second section.

We then try to answer the fundamental questions like if a task involves an infinite number of steps then does it necessarily mean that it would take infinite time as well and that the task would never be completed?

We will also see whether the sum of infinite terms can be finite or not. We also try to see the relation between the rate of growth of a series with its nature(of convergence or divergence).

# SERIES

To define series, we must first understand what a sequence is. We informally define a sequence as a list of numbers. List of numbers $X_1$ ,$X_2$ , $X_3$... represents a sequence with $X_n$ being a term of the sequence. Strictly speaking, a sequence is defined as a function f defined on the set 'N' of all positive integers such that $f_n = x_n$ where n∈N.

In a general sense, series and sequence are nearly synonymous; however, in mathematics, the distinction between the two is that a series is the sum of the terms of a sequence.

Let $\{a_n\}$ be a sequence and we define a new sequence $\{S_n\}$ by the recursion relation $S_1 = a_1$, and $S_{n+1} = S_n + a_{n+1}$. The sequence $\{S_n\}$ is called the sequence of partial sums of $\{a_n\}$.

We can also write $S_n$ as

$$S_n = a_1 + a_2 + \ldots + a_n .$$

For a finite series, a shorthand form of writing this sum is by using the sigma notation:

$$S_n = \sum_{j=1}^{n} a_j$$

**Sum of series-**

Let $\{a_n\}$ be a sequence and let $\{S_n\}$ be the sequence of partial sums of $\{a_n\}$.

If $\{S_n\}$ converges, we say that $\{a_n\}$ is summable. In this case, we denote the $\lim_{n\to\infty} S_n$ by

$$\sum_{j=1}^{\infty} a_j .$$

Example of a series-

$S = 2 + 4 + 6 + 8 + 10 + 12 \ldots.$

# TYPES OF SERIES

**Geometric Series-**

A geometric series is a type of series for which every two consecutive terms' ratio is a constant number or commonly known as the ratio of geometric series.

The series is represented mathematically as-

$$\sum_{n=0}^{\infty} ar^n$$

$a$ = first term, and $r$ = common ratio

The sum of a finite series of $n$ terms is

$$\frac{a(1-r^n)}{1-r}$$

The series will be convergent if and only if $|r| < 1$ to value-

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} - 0 = \frac{a}{1-r}$$

Example of geometric series-

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = \sum_{n=0}^{\infty} \frac{1}{2^n} = 2.$$

## Alternating Series-

An alternating or oscillating series is an infinite series which is of the form

$$\sum_{n=0}^{\infty} (-1)^n a_n$$

Where $a_n$ is greater than 0 for all n.

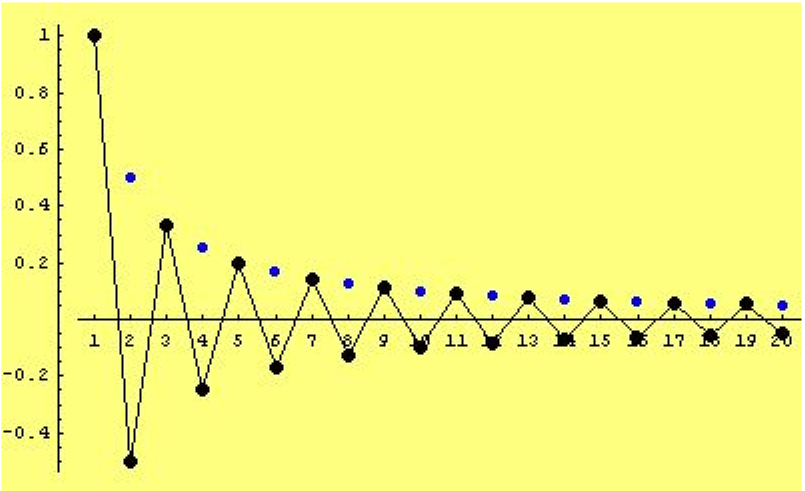
The signs of the general terms alternate between negative and positive. An alternating or oscillatory series converges only if the sequence of partial sum converges.

Example of oscillating series-

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$$

The sum ln(2) is calculated with the help of the Maclaurin series from where we know that the expansion of ln(1+x) for $|x| < 1$ is given by

$$ln(1+x) = x - x^2/2 + x^3/3 \ ........$$




## P- Series-

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is called a p-series.
A p-series converges if p > 1 and diverges for p ≤ 1.

Example of P-series-

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

For p=1, the resultant series is **harmonic series-**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots.$$

## Telescoping Series-

A telescoping series is a type of series whose partial sums eventually have a finite number of terms after the cancellation of terms.

Telescoping sums are finite; in these sums, pairs of consecutive terms cancel each other, leaving only the initial and final terms in the series.

Let $a_n$ be a sequence of numbers. Then,

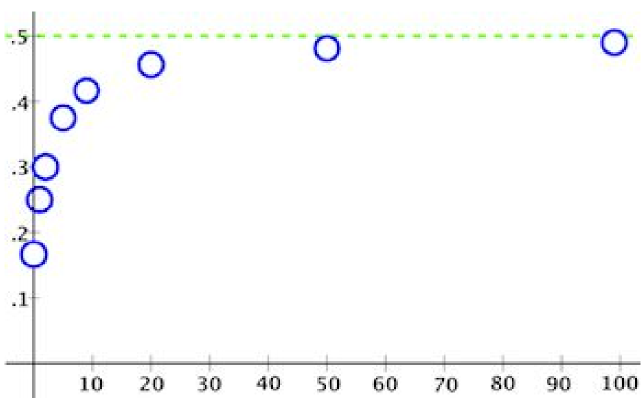
$$\sum_{n=1}^{N}(a_n - a_{n-1}) = a_N - a_0$$

Where $a_0$ is the first term and $a_N$ is the $N^{th}$ term.

Example of telescoping series- Let S be

$$\sum_{n=0}^{\infty}\frac{1}{n^2+5n+6}$$

For this series,




As $n \to \infty$, the value of series tends to $1/2$.

$S = 1/n^2 + 5n + 6 = 1/(n+3)(n+2) = 1/n+2 - 1/n+3$

So we can see that pair of consecutive terms will cancel each other and we will be left with only

½ as the answer as to when n tends to infinity, $1/n+3$ will tend to zero.

# The convergence of series

## Convergent series-

A series is convergent, if there exists a number $l$ such that for every arbitrarily small positive number $\varepsilon$, there is a (sufficiently large) inter $N$ such that for all $n \geq N$,

$$|S_n - l| < \varepsilon.$$

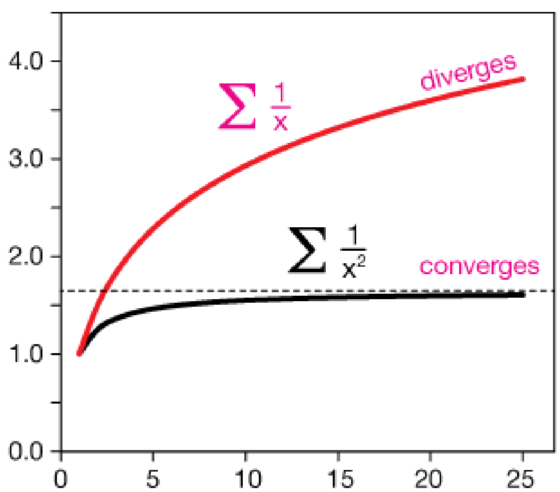
Where $S_n$ is the $n^{th}$ partial sum of the first $n$ terms of the sequence.

An example of converging series can be a geometric series with a common ratio 1/2.

## Divergent series-

A divergent series is a type of an infinite series that is not convergent, meaning that the infinite sequence of the series's partial sums does not have a finite limit.

For eg-



# Tests for convergence of series

## $n^{th}$ term test-

The *n*th-term test for divergence is a simple test for the divergence of an infinite series:

If $\lim_{n\to\infty} a_n \neq 0$ or if the limit does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges. The term test cannot prove by itself that a series converges.

If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ may or may not converge, the test is inconclusive.

## Ratio test-

The test makes use of the limit-

$$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| .$$

The ratio test states:

- if $L < 1$, the series converges absolutely;
- if $L > 1$, the series is divergent;
- if $L = 1$ or if the limit fails to exist, then the test is inconclusive, as there exists convergent and divergent series that satisfy this case.

## $n^{th}$ root test-

For the series $\sum a_n$,

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} (|a_n|)^{1/n}$$

Then,

1. if L<1, the series is absolutely convergent (and hence convergent).

2. if L>1, the series is divergent.

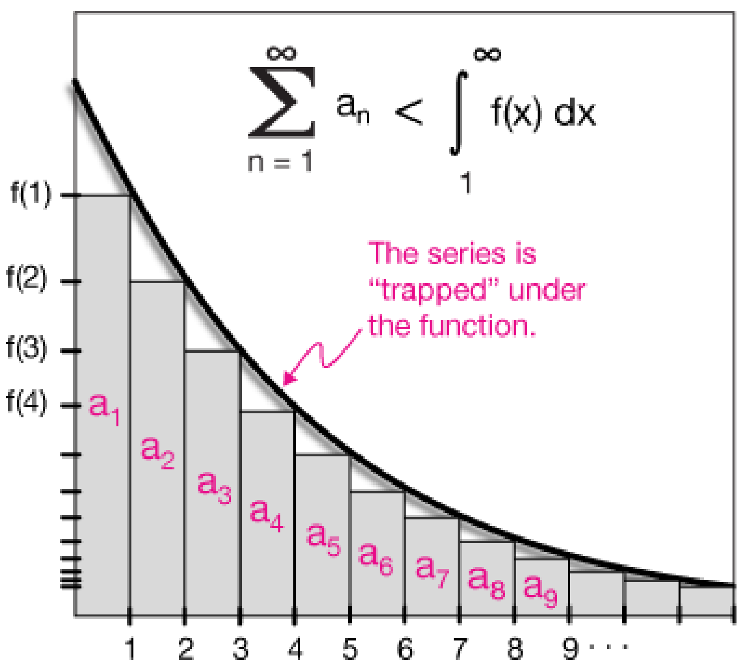
3. If L=1, the series may be divergent, conditionally convergent, or absolutely convergent.

## Integral test-

For an integer *N* and a non-negative(positive and null) function *f* defined on an interval which is unbounded [*N*, ∞), on which it monotonously decreases. Then such an infinite series

$$\sum_{n=N}^{\infty} f(n)$$

converges to $\Re$, a real number if and only if the integral is finite.



## Direct comparison test-

The comparison test is for series made up of non-negative terms.

If the infinite series $\sum b_n$ converges and $0 \le a_n \le b_n$ for all sufficiently large *n* (that is, for all $n > N$ for some fixed value $N$), then the infinite series $\sum a_n$ also converges.

If the infinite series $\sum b_n$ diverges and $0 \le b_n \le a_n$ for all sufficiently large *n*, then the infinite series $\sum a_n$ also diverges.

## Limit comparison test-

If $\sum a_n$ and $\sum b_n$ are two positive series,

$a_n, b_n > 0$ for all $n$

$$\lim_{n\to\infty} \frac{a_n}{b_n} > 0 \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges}\right)$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ converges}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \infty \text{ and } \sum_{n=1}^{\infty} b_n \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} a_n \text{ diverges}$$

## Absolute convergence Test-

A series $\sum_{n=0}^{\infty} a_n$ converges absolutely if the series of absolute values converge.

# HISTORICAL PARADOXES ON SERIES

Having discussed series, in this section we will discuss a few paradoxes related to series, explain how each paradox is resolved and give their significance. Before we begin let's first define what a paradox is.

The term *paradox* comes from the Greek words *para* ("contrary to") and *doxa* ("opinion"). The paradox is a seemingly absurd or contradictory statement or proposition which when investigated may prove to be well-founded or true.

They allow us to think in a different manner and that allows us to find several new results or theorems. It also allows us to improve our understanding of the topic.

A very simple example of a paradox can be this line from the famous TV series ‘Dark’ -“The beginning is the end”. This is kind of contradictory as to how can the beginning of something result in its end. These are the kind of seemingly illogical statements that we work on while studying paradoxes.

Let’s talk about perhaps the most famous paradox of all time “The Zeno’s Paradox”

# ZENO’S PARADOX

## INTRODUCTION

Before looking at the paradoxes themselves, let's sketch some of their historical significance.

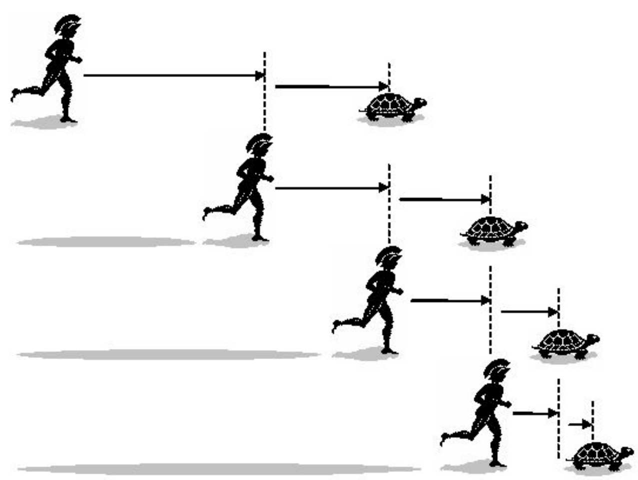
Zeno's Paradoxes are a set of philosophical problems devised by Greek philosopher Zeno of Elea to support Parmenides' doctrine that in particular emphasizes that motion is nothing but an illusion.

Zeno's arguments might be the first examples of proof by contradiction or *reductio ad absurdum.*

Here we will discuss three of his important paradoxes namely Achilles and the tortoise paradox, dichotomy paradox, and the arrow paradox. All of these intend to show that if we assume motion exists then it would lead us into some contradictions.

## ACHILLES AND THE TORTOISE PARADOX

Let's assume Achilles and tortoise in a race and Achilles is 10 times faster than the tortoise. They do not start from the same point. Instead, the tortoise is given a 100-meter head start, then they start to race. Now Achilles sprint 100 meters and reaches the position where the tortoise was. By this time the tortoise would have moved 10 meters from that position. Now Achilles sprints 10 meters and reaches the position where the tortoise was. In that time tortoise has moved another 1 meter. Now Achilles has to sprint to catch where the tortoise reaches. And by the time the tortoise would have traveled some more distance. So every time Achilles reaches the point, the tortoise would have moved some more distance.

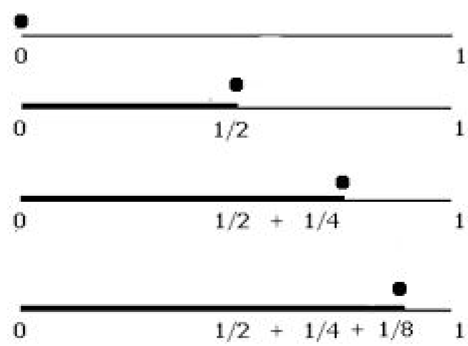


So it seems like Achilles will never be able to catch the tortoise. But our common sense tells us that Achilles being a fast runner should be able to catch the tortoise which is a slow runner at some point.

So will the Achilles be able to catch the tortoise or not?

## THE DICHOTOMY PARADOX

Suppose that a runner has to run from point 0 to point 1 and cover a distance of 1 unit. Now, to reach any destination we have to first reach its halfway point. So the runner will first reach the halfway point between 0 and 1,i.e., ½. Now for the remaining distance, he has to reach another halfway point between this halfway point and 1. Now there will be a third halfway point between this point and 1which has to be reached first and then there will be 4th, 5th, 6th,.... points. This way there would be infinite halfway points.

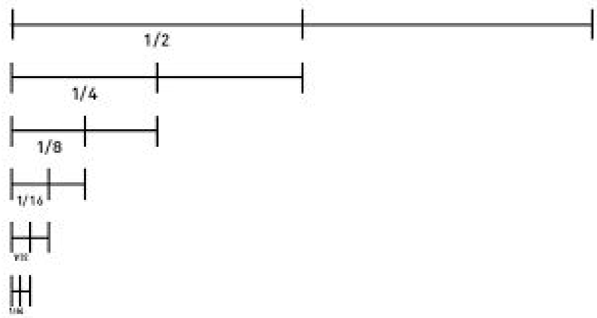



If the runner has to reach infinite halfway points in the way, will he/she ever be able to reach his/her destination(in a limited amount of time)?

Now let's look at the other side of the story;

There would be a halfway point between the starting point and first halfway point (½) and another one between the starting point and this one and similarly 3rd,4th,5th, and so on. Now that means that the runner can't even move as there would be an infinite no. of halfway points for any distance he/she decides to move.

So the distance is infinitely divisible and seems impossible to cross.




How will any movement take place now?

## SOLUTION OF ACHILLES AND DICHOTOMY PARADOXES

In both these paradoxes the problem we have is how can we complete an infinite number of steps or tasks? This issue can be solved by taking into account the fact that the time taken to cover the distances decreases proportionally to the distances.

### In the Achilles and Tortoise Paradox:

Let the distance from the starting point of Achilles where the tortoise and Achilles meet(if ever) be x and the speed of tortoise be v. So the speed of Achilles will be 10v.
So for finding x we have two algebraic equations:

$$x = (10v)t$$
$$x = vt + 100$$

This gives us x = 100/0.9 or 111.11 metres.

Now the total distance traveled by Achilles till catching the tortoise,i.e., x can also be calculated by adding distance for each interval where Achilles reaches the point where the tortoise was before as described above.

$x = 100 + 10 + 1 + 0.1 + 0.01 + \ldots$

We can see that it is a geometric series with a=first term=100 and r=common ratio=0.1. Since r<1 so the sum up to infinity can be given by the formula below:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} - 0 = \frac{a}{1-r}$$

This gives us sum= 100/0.9 or approx 111.11 meters.

So by both methods, we see we get the same value of x. So the Achilles will eventually catch the tortoise after running for 1000/9 meters though this task can be divided into infinite steps.

Since the distance is directly proportional to time so the time will also be limited to complete this task.This resolves everything.

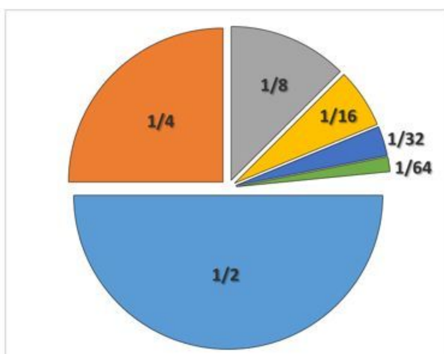
## In the dichotomy paradox:

This paradox can be easily solved by modern mathematics. Here we see that the geometric series is formed by ½,¼,⅛,...So the nth term = 1/ $2^n$ . Hence for n tending to infinity nth term will tend to zero.

Also, we see that the GP formed is converging as common ratio r=½. Its sum is given by

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} - 0 = \frac{a}{1-r}$$

Putting a=½ and r=½ we get sum=1.



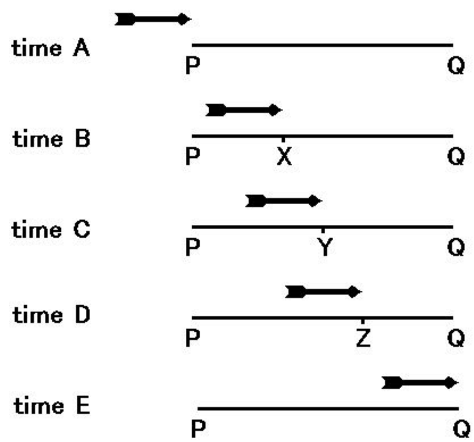

So the infinite seeming distance isn't actually infinite and the increasingly short amount of time needed to traverse these distances balances the infinite number of steps needed. Hence the motion is possible. Again the same logic that the distance is directly proportional to time so the time will also be limited to complete this task resolves everything.

We realize that Zenos left the task of finding the sum and just made a conclusion since the steps required are infinite, the tasks can’t be completed.

# THE ARROW PARADOX

Zenos states that if an object is in motion, it should change its position.

Imagine an arrow. It is moving from one place to another. Now consider a number of different points or moments or nows or (duration-less) instants as it is flying.




We know that anything that occupies a place just its own size is at rest.

At any given point or moment when we think about it, the arrow occupies the same space which is equal to its size. But that is what it means to be at rest. At the moment, the arrow can't move anywhere: Neither can the arrow move to where it is as it is already there, nor to where it is not because time elapsed is zero for it to move there. So at every single point in its motion, the arrow is at rest. So while it is in motion, it's also at rest. But aren't rest and motion supposed to be opposite to each other? Where are we going wrong?

## SOLUTION

So if time is entirely composed of instants then every moving object will be motionless at every such instant so the motion will not be possible.

The problem here lies in dividing time into points and not into segments.

The conclusion depends on dividing time into nows or moments that could be isolated apart from each other. And if we take that away then the argument fails.

## SIGNIFICANCE

Zeno's paradoxes show us that the concept of infinity can be problematic and care must be taken while dealing with tasks involving infinite steps. We cannot simply think that if some task requires infinite steps then it cannot be completed in finite time as we have seen in Zeno's paradoxes. While solving Zeno's paradoxes, mathematicians came across the idea of infinite summations which encouraged the formalization of these concepts

Zeno's paradoxes might be simply considered mathematical problems which now can be easily solved by using geometric series but Zeno's paradoxes are still relevant metaphysical problems.

Also, Zeno's arguments might be the first examples of proof by contradiction or *reductio ad absurdum.*

# GRANDI’S SERIES

In 1703, an Italian mathematician, philosopher, and priest Guido Grandi gave a series: 1-1+1-1+1-1+1-1+1……….

which can also be written as $\sum_{n=0}^{\infty} (-1)^n$ and known as Grandi’s series after Guido Grandi.

Inserting parentheses in the Grandi’s series in two different ways produced two different results:

1) (1-1) + (1-1) + (1-1)+...=0
2) 1 + (-1+1) + (-1+1) +...=1

You might think that the sum is this series is either 1 or 0

Now consider the following method:

Let S be the sum of series then

$S = 1-1+1-1+ \ldots$

Now, 1-S

$= 1-(1-1+1-1+\ldots)$

$= 1-1+1-1 + \ldots = S$

$1-S=S$

$1=2S$

$\frac{1}{2} = S$

Now using this third method,we can say that the sum of series is ½.

This is an infinite series. The sum of an infinite series is said to be the limit of the sequence of its partial sums if it exists. In the case of Grandi's series, the sequence of partial sums is 1,0,1,0,1,0….. which does not approach any number. Therefore, Grandi's series is divergent.

Also Grandi's series is a geometric series with first term(a) = 1 and common ratio(r) = -1.We know that the geometric series is divergent for $|r| >= 1$.

We see that the above methods do not consider what the sum of a series actually means. We have not considered that we are applying algebraic methods to a divergent geometric series.

## SIGNIFICANCE

If we consider $1 - 1 + 1 - 1 + \cdot \cdot \cdot = \frac{1}{2}$ as offered by Grandi in 1710, we can offer the following explanation. Let two brothers inherit a priceless gem that they can't sell according to their father's will. Now their father wants them to have an equal share in that gem so he decides that it will reside in both their museums in alternating years. Now if this goes on for all eternity between both the brothers' descendants, then by Grandi's series explanation we can say that the two families will each have equal possession of the gem, as it will alternate its position infinite number of times.

# THOMSON’S LAMP PARADOX

This paradox is an application of Grandi’s paradox. It was put forth by James Thomson, a British philosopher. He used the paradox to analyze the probability of a supertask, which means the execution of an infinite number of tasks but in some sense is completed in only a finite amount of time.
In this paradox, We consider a lamp with a toggle switch that can be turned on or off by flicking its switch. Suppose we assign a person to perform the following jobs-

1. Turns the lamp on and simultaneously starts a timer.
2. After one minute, he turns it off.
3. At the end of another half minute, turns it on.
4. At the end of another quarter minute, he turns it off.
5. He repeats this process (that is flicking the switch each time after waiting exactly half the time he waited before flicking it previously) an infinite number of times.

Let $S$ represent time , 1 represent switch on and -1 represent switch off. So

$$S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} \ldots..$$

(switch status) 1 -1 1 -1 1 .....

The series S is a geometric series and its sum can be easily computed as the common ratio is less than 1. So we know the sum of geometric series is given by

$$S = \frac{a}{1-r} \qquad \text{Here, } a=1 \text{ and } r=½$$

Hence on substituting the values we get that the total interval is 2 minutes.

So the paradox is whether the lamp is on or off at the 2-minute mark?
This is difficult to answer because whatsoever our answer may be , if we consider the switch to be on then we can give a counter-argument that we

turned it off halfway between the time we switched it on and the 2-minute mark. The same argument would work for switching off as well.
This results in a paradox.

**SOLUTION:**

The solution again takes us back to the behavior of Grandi's series. We saw that mathematicians now say that the sum of Grandi's series is 1/2. However, this has no meaning here as there cannot be a situation where the switch is half-open and half-closed. So it is impossible to answer whether the switch will be open or closed.

**SIGNIFICANCE:**

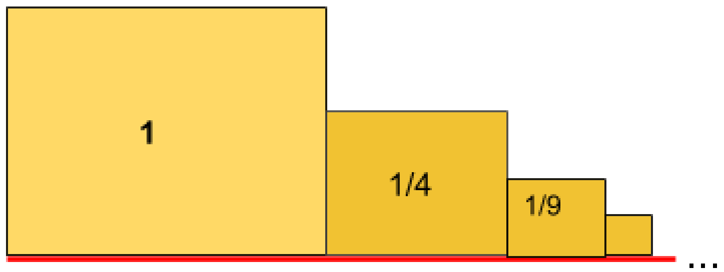
The observations from this paradox tells us that the divergence of a series does not provide information about its supertask. So even though mathematicians have told us that the sum of Grandi's series is 1/2, this does not mean we can find the output of the supertask.

Hence, the impossibility of a super task does not depend at all on the convergence or divergence of a series.

# PAINTER'S PARADOX

This is a simpler version of the painter's paradox (also known as Gabriel's Horn paradox) put forth by Evangelista Torricelli in the 17th century.

Consider a square of edge length 1.We draw another square of edge length $1/2$ adjacent to the first square. We repeat this process of drawing adjacent squares halving the edge length each time until we get an infinitesimally small value of edge length(implying we are drawing infinite squares).




If we try to paint the area inside the square, then that would be equal to

$$S = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots..$$

Now let the edge of the squares have a thickness ɑ, if we try to paint the bottom edge length of all the squares(the red line), then that would be equal to

$$S' = \alpha \left( 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots.. \right)$$

We know by p-series(on page- 8) that sum S is converging (p=2) and has a finite value while the sum S′ is diverging (p=1) and is equal to infinity.

The apparent paradox here is that the squares could be filled with a limited amount of paint, even then that paint would be insufficient to paint the edge of the squares!

**SOLUTION:**

The solution to the paradox lies in the fact that the rate of usage of the paint is decreasing more quickly in sum S as compared to sum S′.We need to realize that a finite amount of paint is able to paint the inside of the squares because it is getting thinner at a fast enough rate (much like the series gets smaller fast enough such that its sum is finite) while drawing along the line would require infinite motion of the pen so essentially the rate of usage of paint does not change. Hence the paint required to fill the inside of squares is finite while that required to paint the edges is infinite.

**SIGNIFICANCE:**

This paradox has allowed us to solve this contradictory problem: How can the sum of edge length of a figure be larger than the sum of area of that figure? It also allows us to appreciate the P-series and how a small variation in the rate of change can make the sum go from finite to infinite.

# ANT ON A RUBBER ROPE PARADOX

The ant on a rubber rope is a mathematical riddle whose result appears illogical but can be solved using the divergence of harmonic series. The puzzle says that consider a taut infinitely stretchable rubber rope whose initial length is 1 km and its left end is at x=0. Say at T=0, the rope starts to stretch uniformly at a constant rate of 1 km per sec such that the left end of the rope remains where it was (x=0) and only the right end of the rope moves. Also at T=0, an ant starts crawling along the rope at a speed of 1 cm per sec relative to the rope. The problem says will the ant ever be able to reach the end of the rope.

At first glance, it looks impossible that the ant will reach the end as the rate of stretching of rope is a lot more than the speed of the ant but surprisingly the answer is yes!

No matter what the speed of the ant is and what the rate of stretching might be, the ant will always reach the end as long as both these values are constant. This is the paradox!

**SOLUTION:**

Let $c$ represent the initial rope length, $a$ represents the distance ant moves each second and v represents the amount rope stretches each second.

In our example $c = 1km$, $a = 1cm$, $v = 1km$

So the movement of the ant on the rope will look like this

| | The ant covers | Total length of rope |
|---|---|---|
| First second | $a = 1cm$ | $c + v$ |
| Second second | $a = 1cm$ | $c + 2v$ |
| Third second | $a = 1cm$ | $c + 3v$ |
| And so on | ........... | ........... |

Instead of playing with the distance covered by the ant, let us consider the fraction of the rope the ant covers in any second, so the fraction covered in each second will be

| First second | Second second | Third second | Fourth second |
|---|---|---|---|
| $\frac{a}{c+v}$ | $\frac{a}{c+2v}$ | $\frac{a}{c+3v}$ | ......... |

Let $S$ represent the sum of the fractions covered by the ant, so

$$S = \frac{a}{c+v} + \frac{a}{c+2v} + \frac{a}{c+3v} + .....$$

So in the k'th second, the ant covers $a/(c+kv)$ fraction of rope. Using comparison test, We can say that

$$\frac{a}{c+kv} > \frac{a}{kc+kv}$$

So we can say that the sum S will be greater than

$$S' = \frac{a}{c+v}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ...\right)$$

As we know that the harmonic series

$$S'' = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} .....$$

diverges from p series (p=1); hence S would also diverge as S' diverges since S' is constant times S'' which we know is not convergent.

Since S is diverging, it can reach any value including 1 . Hence the ant will reach the end of the rope although it will take a very long time. In this particular example, the time taken by an ant is of the order $10^{43421}$ which is more than the number of atoms in the known universe. How fascinating!

## SIGNIFICANCE:

The solution to this problem allows us to answer similar problems like we know that the universe is expanding which is leading to an increase in the distance between galaxies, so we want to find if the light coming from a galaxy will ever reach us. So from the solution of this paradox, we can infer that no matter how fast the universe is expanding, the light will eventually reach us.

Hence this paradox along with the unusual nature of the harmonic series allows us to answer many questions that may seem misleading at first.

# CONCLUSION

This report involved a study of series and the historical paradoxes related to series.

We explored various properties of series, various types of series, tests for the convergence or divergence of series, paradoxes related to infinite series, and their significance.

Initially, we talked about the relation between series and sequences and then discussed Geometric series and convergent series. This allowed us to solve Zeno's paradox which implied that a task involving infinite steps can be completed in a limited amount of time.

Next, we talked about Alternating and Oscillating series and backed it with Grandi’s series and the Thomson Lamp paradox. Here we proved that the impossibility of a super task does not depend at all on the convergence or divergence of a series.

We also talked about P-series and saw the Painter’s paradox from which we inferred that a small change in the rate of growth can heavily determine the nature of the series.

Next, we discussed the ant on a rubber rope paradox and solved it using the Harmonic series and again saw that the fractional change of rate can determine the divergence and convergence of series.

We also explored the relation of these paradoxes with real-life problems like whether the light from a distant galaxy would ever reach, etc.

At last, we feel that we experienced the real beauty of math and developed a deep interest in it through the means of this project!

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