Case Study: Stability of a 5 DOF Coupled Spring–Mass System Through Eigenvalue Analysis

1. Introduction

A multi-degree-of-freedom (MDOF) mechanical system consists of multiple masses interconnected through springs (and possibly dampers). Such systems are commonly found in structural engineering, vehicle suspension systems, machinery, and vibration analysis of buildings.

The stability and dynamic behavior of these systems can be analyzed using **eigenvalue analysis**, a fundamental concept of linear algebra. The eigenvalues of the system matrix determine:

- Natural frequencies
- Oscillation characteristics
- Stability of the system

If all eigenvalues satisfy the stability condition, the system remains dynamically stable.

This case study applies matrix formulation and eigenvalue computation using MATLAB to analyze the stability of a **5 Degree of Freedom (5 DOF) coupled spring–mass system**.

2. Problem Description

Consider a mechanical system consisting of:

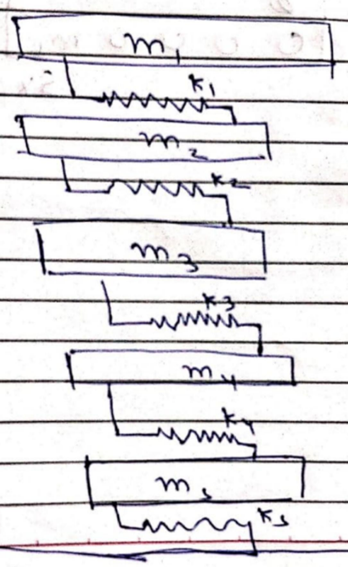
- Five masses:
  m1, m2, m3, m4, m5
- Five spring elements:
  k1 ,k2 ,k3 ,k4 ,k5

Each mass is connected to its neighboring mass through springs, forming a coupled system.

The objective of this case study is to:

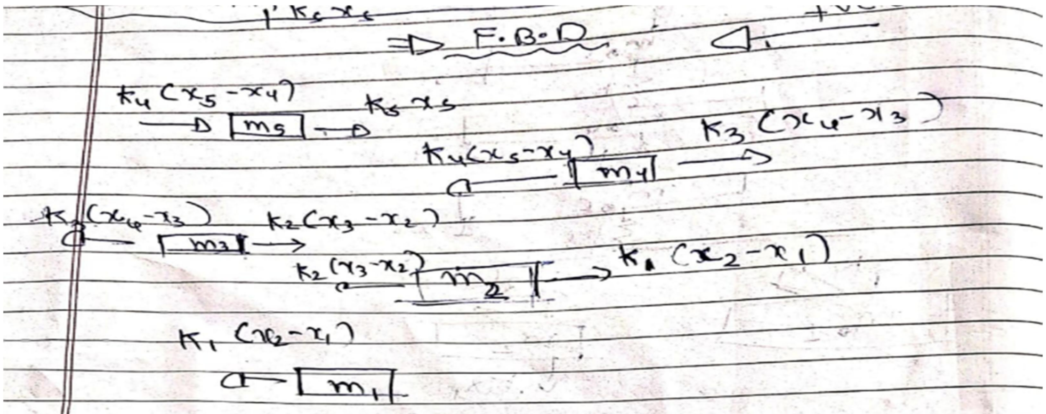
1. Formulate the system matrix using mass and stiffness relationships
2. Compute eigenvalues of the system
3. Determine natural frequencies
4. Analyze system stability based on eigenvalues

3. Mathematical Formulation



It's displaced in such a way

$x_5 > x_4 > x_3 > x_2 > x_1$



$$-(k_4+k_5)x_5 + k_4x_4 = m_5a_5 \quad \text{- (eq}$$

$$\Rightarrow +k_4x_5 - (k_4+k_3)x_4 + k_3x_3 = m_4a_4 \quad \text{- (eq}$$

$$\Rightarrow +k_3x_4 - (k_3+k_2)x_3 + k_2x_2 = m_3a_3 \quad \text{- (e}$$

$$\Rightarrow +k_2x_3 - (k_2+k_1)x_2 + k_1x_1 = m_2a_2 \quad \text{- (}$$

$$\Rightarrow +k_1x_2 - k_1x_1 = m_1a_1 \quad \text{- (eq-1)}$$

In Matrix form

$$\begin{bmatrix} -k_1 & k_1 & 0 & 0 & 0 \\ k_1 & -(k_1+k_2) & k_2 & 0 & 0 \\ 0 & k_2 & -(k_2+k_3) & k_3 & 0 \\ 0 & 0 & k_3 & -(k_3+k_4) & k_4 \\ 0 & 0 & 0 & k_4 & -(k_4+k_3) \end{bmatrix}_{5\times 5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times 1}$$

$$= \begin{bmatrix} m_1 & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 \\ 0 & 0 & m_3 & 0 & 0 \\ 0 & 0 & 0 & m_4 & 0 \\ 0 & 0 & 0 & 0 & m_5 \end{bmatrix}_{5\times 5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}_{5\times 1}$$

In eq (2)

Subtituting $x_i = X_i \sin\omega t$

$x_1 = X_1 \sin\omega t$

$x_2 = X_2 \sin\omega t$

$\therefore \quad m_1 a_1 = -k_1 x_2 + k_1 x_1$

$\Rightarrow m_1 \frac{\partial^2 x_1}{dt^2} = +k_1 X_2 \sin\omega t - k_1 X_1 \sin\omega t$

$\Rightarrow m_1 \frac{\partial^2 (X_1 \sin\omega t)}{dt^2} = +k_1 X_2 \sin\omega t - k_1 X_1 \sin\omega t$

$\Rightarrow m_1 (-\cancel{\sin\omega t})\, \omega^2 X_1 = k_1 X_2 \cancel{\sin\omega t} - k_1 X_1 \cancel{\sin\omega t}$

$-m_1 \omega^2 X_1 = k_1 X_2 - k_1 X_1$

$m_1 \omega^2 X_1 = +k_1 X_1 - k_1 X_2$

$$\boxed{\omega_1^2 X_1 = \frac{k_1 X_1}{m_1} - \frac{k_1 X_2}{m_1}} \quad - eq\ \boxed{a}$$

Same way

$\Rightarrow \omega_2^2 x_2 = \left(\frac{-k_1}{m_2}\right) x_1 + \left(\frac{k_1+k_2}{m_2}\right) x_2 + \left(\frac{-k_2 x_3}{m_2}\right)$

$\Rightarrow \omega_3^3 x_3 = \left(\frac{-k_2}{m_3}\right) x_2 + \left(\frac{k_2+k_3}{m_3}\right) x_4 + \left(\frac{-k_3 x_5}{m_3}\right)$

$\Rightarrow \omega_4^3 x_4 = \left(\frac{-k_3}{m_4}\right) x_4 + \left(\frac{k_3+k_4}{m_4}\right) x_5 + \left(\frac{-k_4 x_6}{m_4}\right)$

$\Rightarrow \omega_5^3 x_5 = -\left(\frac{k_4}{m_1}\right) x_5 + \left(\frac{k_4+k_5}{m_5}\right) x_6$

In Matrix Form $(Ax = \lambda x)$

$$\Rightarrow \underbrace{\begin{bmatrix} \frac{k_1}{m_1} & \frac{-k_1}{m_1} & 0 & 0 & 0 \\ \frac{-k_1}{m_2} & \frac{(k_1+k_2)}{m_2} & \frac{-k_2}{m_2} & 0 & 0 \\ 0 & \frac{-k_2}{m_3} & \frac{k_2+k_3}{m_3} & \frac{-k_3}{m_3} & 0 \\ 0 & 0 & \frac{-k_3}{m_4} & \frac{(k_3+k_4)}{m_4} & -k_4 \\ 0 & 0 & 0 & \frac{-k_4}{m_1} & \frac{(k_4+k_5)}{m_5} \end{bmatrix}}_{5\times5} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{5\times1} = \begin{bmatrix} \omega_1^2 x_1 \\ \omega_2^2 x_2 \\ \omega_3^2 x_3 \\ \omega_4^2 x_4 \\ \omega_5^2 x_5 \end{bmatrix}$$

Solving this system manually would require solving a 5×5 characteristic polynomial, which is computationally complex.

Therefore, MATLAB is used to:

- Compute eigenvalues
- Calculate natural frequencies
- Visualize eigenvalues on the complex plane
- Determine stability automatically

4. Stability Condition

The stability of the mechanical system depends on eigenvalues λi\lambda_iλi.

- If all $\lambda_i > 0$ → Natural frequencies are real → System is **stable**
- If any $\lambda_i < 0$→ Imaginary frequency → System is **unstable**
- If any $\lambda_i = 0$→ System is **marginally stable**

Thus, eigenvalue analysis directly determines the system's dynamic stability.