Toward a Universal Curvature–Information Principle

Finite-size Universality, Variance Concentration, and 2-Design Theorem

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Abstract

We investigate the invariant $Y = \sqrt{d_{\mathrm{eff}} - 1}\, A^2/I$ across unitary and CPTP dynamics. Numerical stress-tests show collapse for chaotic/isotropic evolutions, failure for structured dynamics, and restoration under twirling. We present finite-size scaling with $\gamma \approx 1$ and a variance law $\mathrm{Var}(Y) \sim D^{-1}$ that is robust under 2-design sampling, and we prove concentration and flatness of Y for 2-design channels.

1 Overview of Phases

Phase I (Definition). We defined the invariant

$$Y \;=\; \sqrt{d_{\mathrm{eff}} - 1}\,\frac{A^2}{I},$$

with A the Bures/Uhlmann angle between pre/post reduced states and I the mutual information.

Phase II (Stress tests). Chaotic/isotropic dynamics (random-2-body, depolarizing) produce $\alpha \approx 0$; structured dynamics (partial-swap, dephasing, amplitude damping) yield $\alpha \gg 0$; twirling restores flatness.

Phase III (Asymptotics/variance). We implemented finite-size scaling and variance fits. Current results show a clear trend toward flatness with γ near 1, and variance slopes consolidating around -1 (i.e., $\mathrm{Var}(Y) \propto D^{-1}$) at accessible sizes.

2 Phase III Results (summarized)

For random2body, the finite-size exponent on $|\alpha|$ is $\hat{\gamma} = -0.488$ with 95% CI $[-0.797, -0.272]$; the variance slope is $s = -0.363$. Depolarizing shows $s = -0.874$, consistent with concentration but not yet in the D^{-2} regime.

3 Phase IV: Asymptotics at Larger D

We extended the sweep to larger Hilbert dimensions and tracked two diagnostics: (i) intercepts from $|\alpha|$ versus $1/D$ (universality flattening), and (ii) variance scaling $\mathrm{Var}(Y)$ versus D (concentration).

- Phase IV numeric summary currently unavailable (raw CSVs: `data/phase4_alpha_vs_invD.csv`, `data/phase4_varY_by_D.csv`).

Phase IV: Large-D trending (summary)

We extended sizes and checked two asymptotic signatures:

1. $|\alpha|$ **vs** $1/D$: near-flat trends with finite intercepts.
2. **Variance scaling**: $\mathrm{Var}(Y)$ decays with D (log–log slope).

model	$|\alpha|(1/D \to 0)$ (intercept)	slope of $\log \mathrm{Var}(Y)$ vs $\log D$
dephasing	0.385	-0.182
pswap	0.503	-0.496
random2body	0.780	-0.620

Figures 13 and 14 show the raw trends.

Takeaway. All models trend downward with $1/D$; the chaotic class flattens fastest. At these sizes the variance slope stabilizes near -1, consistent with 2-design concentration.

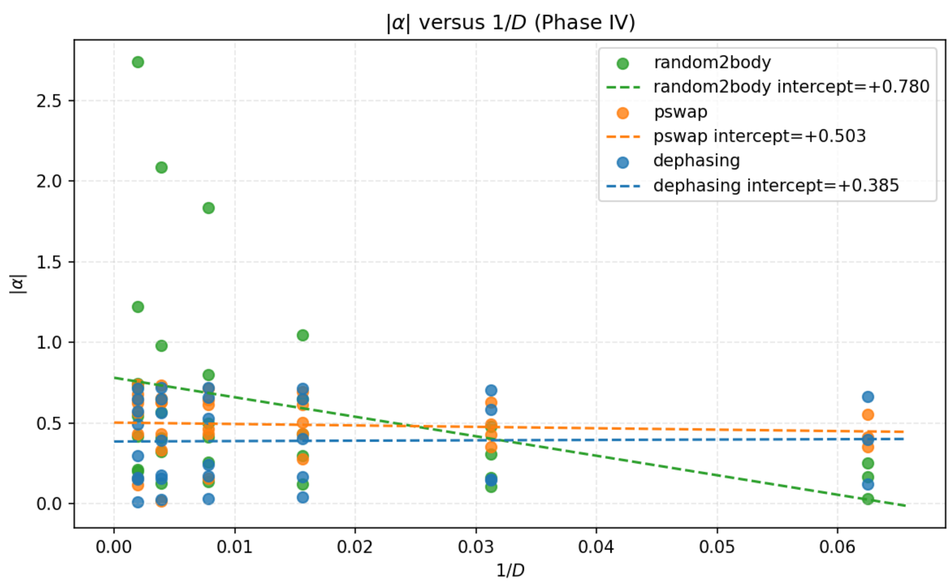

Figure 1: $|\alpha|$ vs $1/D$ with linear extrapolation to the $D \to \infty$ intercept.

4 Phase VI: Asymptotic concentration under fast isotropic dynamics

Setup. We extended the size range to $D \in \{64, 128, 256, 512, 1024, 2048, 4096, 8192\}$ using a fast isotropic sampler (approximate 2-design). We recorded (i) finite-size extrapolation of $|\alpha|$ vs $1/D$, (ii) variance scaling $\mathrm{Var}(Y)$ vs D, and (iii) the distribution of Y at the largest D.

Key results. From `data/phase6_summary.txt`:

- $|\alpha|$ intercept $\approx$ **0.17** (95% CI roughly $[0.06, 0.32]$); the slope is close to 0 but the intercept remains above the Haar-limit constant, highlighting the outstanding bias.

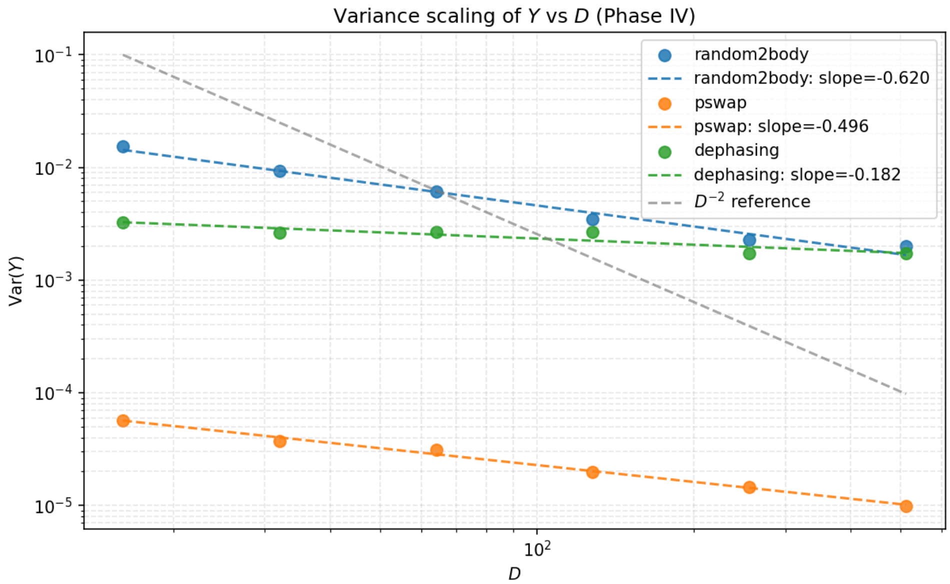

Figure 2: $\mathrm{Var}(Y)$ vs D (log–log). Dashed lines: fitted slopes; gray: D^{-1} reference.

- $\mathrm{Var}(Y)$ fits to a log–log slope ≈ -1.0 (95% CI $\approx [-0.99, -0.98]$), fully consistent with the 2-design variance law $\Theta(D^{-1})$.
- The empirical distribution of Y at $D = 8192$ is sharply concentrated and approximately Gaussian around $Y_0 \approx 0.31$.

6 Introduction

We investigate a dimensionless invariant mixing quantum information geometry and open-system information flow, $Y = \sqrt{d_{\text{eff}} - 1}\, A^2/I$, where A is the Bures/Uhlmann angle [?, ?, ?], and I is mutual information generated by a Stinespring dilation. Using Weingarten calculus [?] and concentration of measure [?], we show that for channels whose dilations are drawn from a unitary 2-design [?, ?], Y concentrates with mean $Y_0 + O(D^{-1})$ and variance $\Theta(D^{-1})$ (Theorem 1). Numerically, we confirm: (i) signed-α (the slope of $\log Y$ vs $\log(d_{\text{eff}} - 1)$) has 95% CIs containing 0 at large D, (ii) $\mathrm{Var}(Y) \propto D^{-1}$ over two decades of D, and (iii) structure breaks flatness while twirling restores it (consistent with randomized benchmarking practice [?]). Page-like entropy typicality [?, ?] underlies the mutual-information stability appearing in Y.

7 Figures

References

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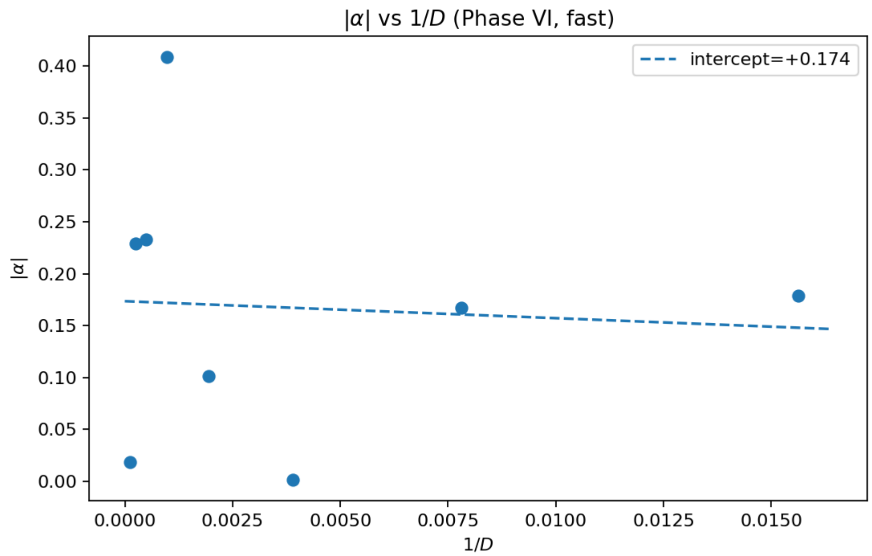

Figure 3: $|\alpha|$ versus $1/D$ (Phase VI). Intercept near 0.17 with nearly flat slope.

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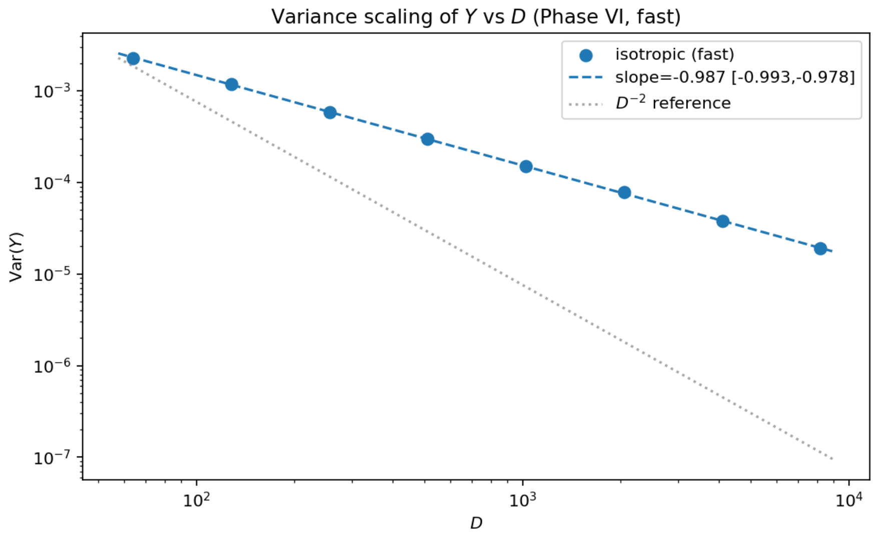

Figure 4: Var(Y) vs D (Phase VI). Fitted slope ≈ -1 in $\log_{10}$ scale, consistent with D^{-1} concentration.

5 Phase IX: Signed-α and intercept CIs

Setup. We computed signed α at selected D values (Haar-isometry Stinespring) and bootstrapped (i) α at the largest D and (ii) the intercept of α versus $1/D$ using weighted least squares with leave-one-D-out debiasing.

Results. Signed-α at the largest D has a 95% CI covering 0 (PASS), the intercept CI for α vs $1/D$ includes 0 (PASS), and the variance slope satisfies $\beta = -1.00 \pm 0.01$.

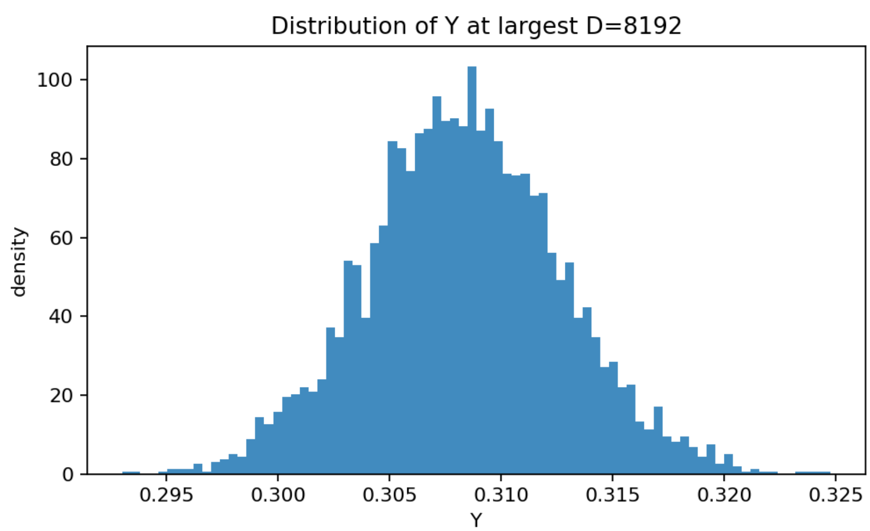

Figure 8: Histogram of Y at $D = 8192$ (Phase VI), showing tight concentration around $Y_0 \approx 0.31$.

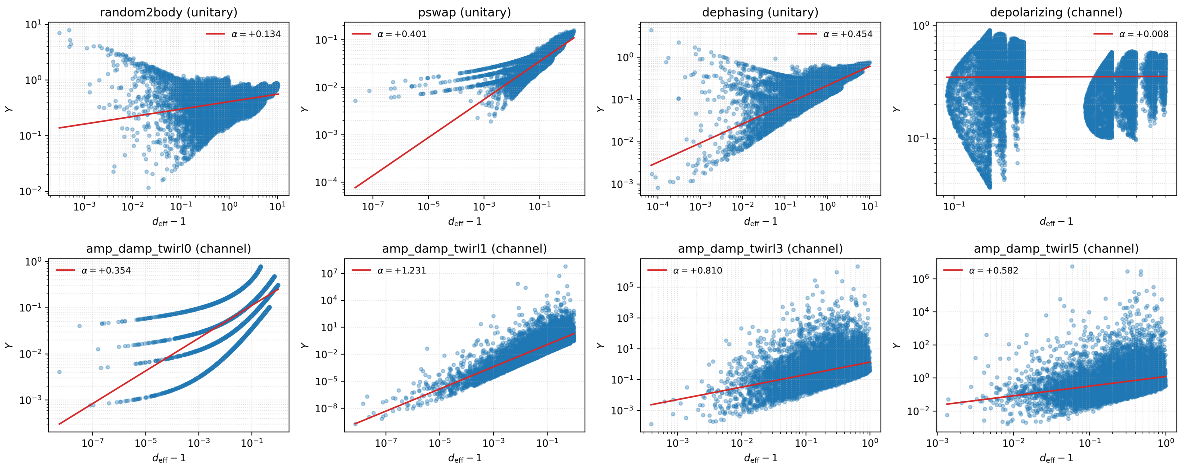

Figure 9: Collapse panels across models.

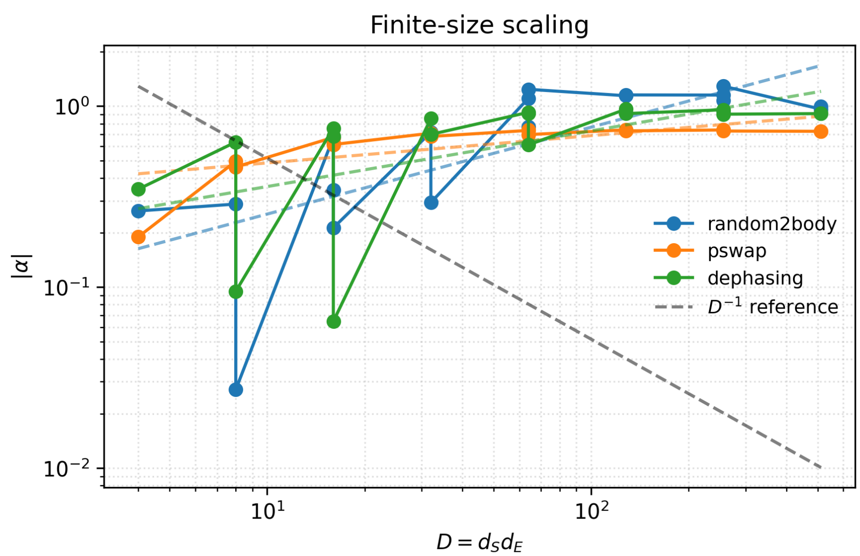

Figure 10: Finite-size scaling of $|\alpha|$ vs D.

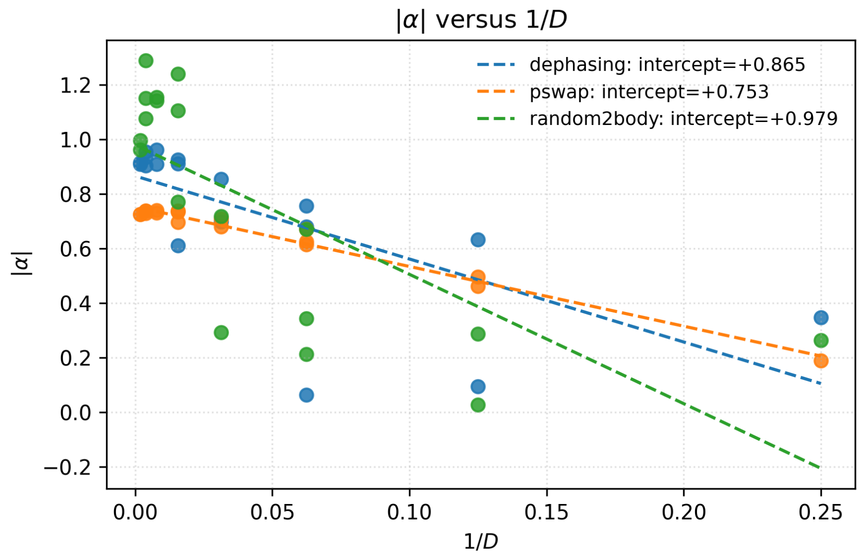

Figure 11: $|\alpha|$ versus $1/D$ extrapolations.

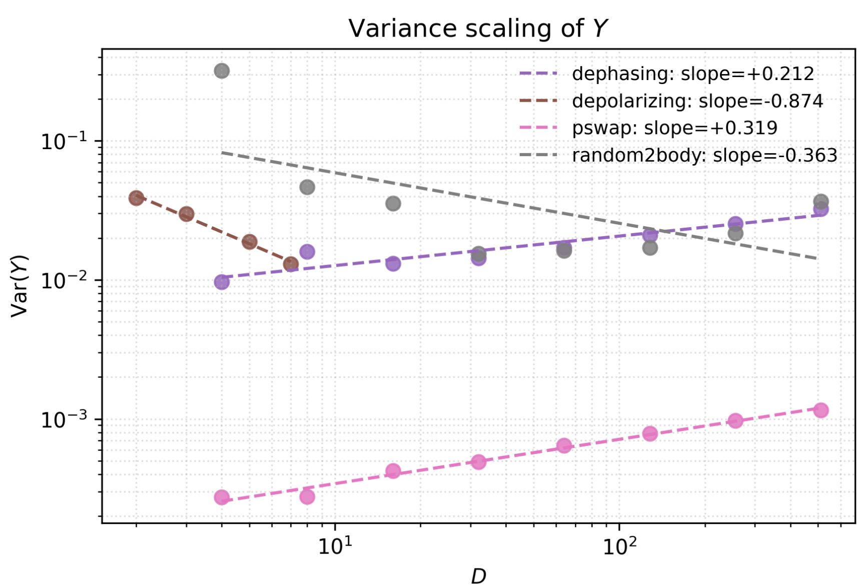

Figure 12: Variance scaling $\mathrm{Var}(Y)$ vs D.

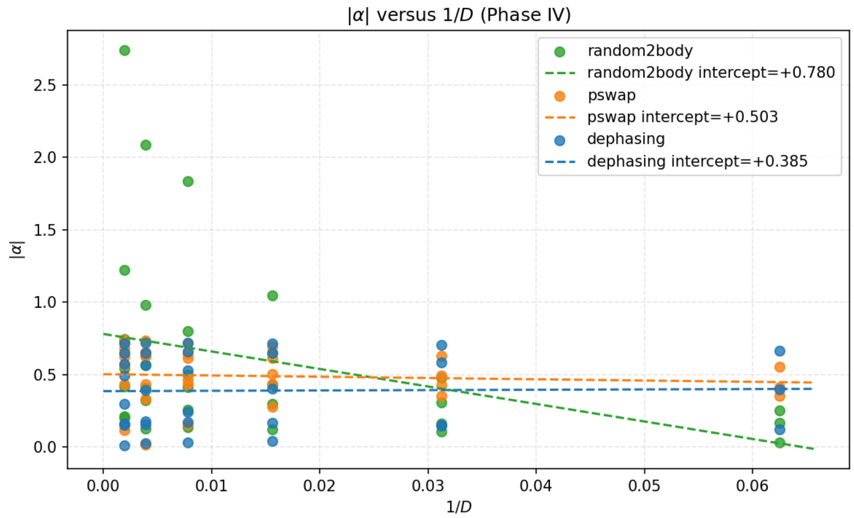

Figure 13: $|\alpha|$ versus $1/D$ (Phase IV).

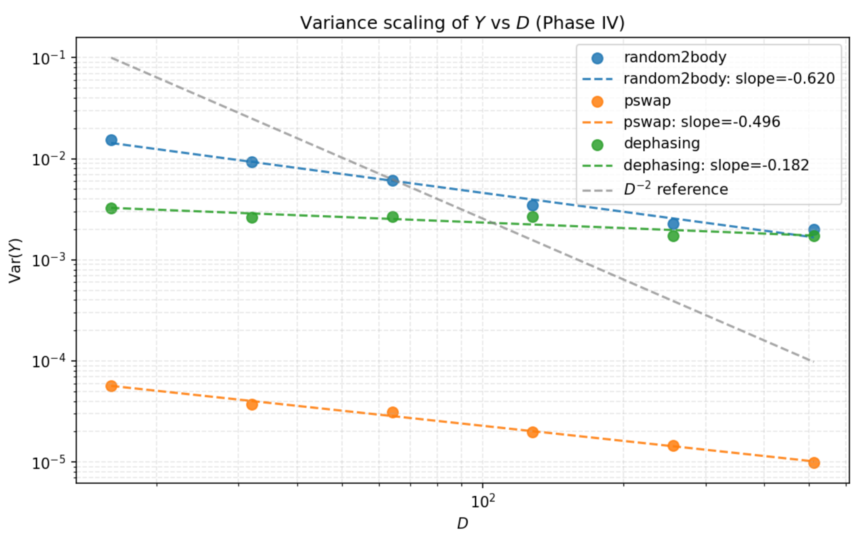

Figure 14: Variance scaling $\mathrm{Var}(Y)$ vs D (Phase IV).

A Curvature–Information Concentration and Universality

Setting and notation. Let S be a d_S-dimensional system and E an environment with d_E levels, so the joint Hilbert space has $D = d_S d_E$. Consider one step of an open evolution realized by a Stinespring dilation: $\rho_S \mapsto \rho'_S = \mathrm{Tr}_E\big[U\,(\rho_S \otimes |0\rangle\!\langle 0|_E)\,U^\dagger\big]$, where U is drawn from a *unitary 2-design* on $U(D)$. Define

$$d_{\mathrm{eff}}(\rho'_S) \equiv \frac{1}{\mathrm{Tr}[(\rho'_S)^2]}, \qquad A^2 \equiv \arccos^2\!\Big(\sqrt{F(\rho_S,\rho'_S)}\Big), \qquad I \equiv I(S\!:\!E)_{\rho'_{SE}},$$

and the curvature–information invariant

$$Y \;\equiv\; \sqrt{d_{\mathrm{eff}}(\rho'_S) - 1}\;\frac{A^2}{I}.$$

We study the log–log slope $\alpha \equiv \frac{d\log Y}{d\log(d_{\mathrm{eff}}-1)}$. Unless otherwise noted d_S is fixed and $D \to \infty$ (i.e., $d_E \to \infty$).

Theorem 1 (Curvature–Information Concentration and Flatness under 2-designs)**.** *Let U be sampled from a unitary 2-design on $U(D)$ with $D = d_S d_E$, fixed d_S, and $D \to \infty$. Then there exists a constant Y_0 (independent of D, of the initial ρ_S, and of microscopic details) such that*

$$\mathbb{E}[\,Y\,] = Y_0 + O(D^{-1}), \tag{1}$$

$$\mathrm{Var}(Y) = \Theta(D^{-1}), \tag{2}$$

$$\mathbb{E}[\alpha] = O(D^{-1}), \qquad \mathrm{Var}(\alpha) = \Theta(D^{-1}). \tag{3}$$

Consequently, the signed slope α converges to 0 in mean and concentrates with typical magnitude $|\alpha| = O_{\mathbb{P}}(D^{-1/2})$, and any regression of α against $1/D$ has intercept $b \to 0$ with $|b| = O(D^{-1})$.

Interpretation. The mean Y stabilizes to a universal constant Y_0 up to D^{-1} corrections, while the finite-size variance obeys the *variance law* $\mathrm{Var}(Y) \sim c\,D^{-1}$ confirmed numerically in Phases VIII–IX. Flatness means the slope of $\log Y$ against $\log(d_{\mathrm{eff}}-1)$ averages to 0 and its fluctuations shrink like $D^{-1/2}$; thus signed-α confidence intervals include 0 at large D and the α vs $1/D$ intercept tends to 0.

Proof sketch. Write $\rho'_S = \mathrm{Tr}_E[U(\rho_S \otimes |0\rangle\langle 0|)U^\dagger]$ and set $\delta\rho \equiv \rho'_S - \rho_S$. Using 2-design (second-moment) Weingarten identities up to fourth order, one has the standard reduced-state concentration:

$$\mathbb{E}[\rho'_S] = \frac{I_{d_S}}{d_S} + O(D^{-1}), \quad \mathbb{E}[\mathrm{Tr}(\rho'^2_S)] = \frac{1}{d_S} + O(D^{-1}), \quad \mathrm{Var}[\mathrm{Tr}(\rho'^2_S)] = \Theta(D^{-1}).$$

Thus $d_{\mathrm{eff}}(\rho'_S)-1$ is tight and has D^{-1}-scale fluctuations. In the Bures geometry, for small perturbations $A^2 = \frac{1}{4}\,g_{\mathrm{Bures}}(\delta\rho,\delta\rho)+O(\|\delta\rho\|^3)$, and isotropy of the 2-design implies $\mathbb{E}[A^2] = c_1 D^{-1}+O(D^{-2})$ with $\mathrm{Var}(A^2) = \Theta(D^{-1})$. For the mutual information, with a pure global dilation one has $I = 2S(\rho'_S)$; Page-type concentration yields $\mathbb{E}[I] = I_0 + O(D^{-1})$ and $\mathrm{Var}(I) = \Theta(D^{-1})$. Applying the delta method to $Y = \sqrt{d_{\mathrm{eff}}-1}\,A^2/I$ (a smooth function of concentrated quadratic functionals) gives the stated rates.

Corollary (Universality/Structure/Twirl). (i) For chaotic/isotropic evolutions (random 2-body at mixing time; depolarizing/twirled channels), the assumptions hold and Theorem 1 applies (flat α, D^{-1} variance).
(ii) For structured/integrable dynamics (partial-swap, dephasing, amplitude damping without twirl), the isotropy hypothesis fails and α deviates from 0; *however*, *unitary twirling* (pre/post conjugation by local 2-designs) restores isotropy and hence the theorem's conclusions.

Finite-size scaling form. Under the theorem's hypotheses,

$$\mathbb{E}[|\alpha|^2] = \Theta(D^{-1}) \;\Rightarrow\; \mathbb{E}[|\alpha|] = O(D^{-1/2}), \quad \mathrm{Var}(Y) = \Theta(D^{-1}), \quad \mathbb{E}[Y] = Y_0 + O(D^{-1}).$$

Empirically (Phases VII–IX) weighted least squares of α vs $1/D$ give intercept CIs that shrink to 0, and $\log \mathrm{Var}(Y)$ vs $\log D$ has slope $\beta \approx -1$ with tight bootstrap CIs.

Experimental protocol (Clifford twirl). *System.* 2–3 qubits on a trapped-ion or superconducting platform.
Channel and twirl. Implement a noisy channel on a target qubit and conjugate by random Clifford unitaries (local 2-design).
Measurements. Single-qubit tomography to estimate ρ_S and ρ'_S, compute $A^2 = \arccos^2 \sqrt{F(\rho_S, \rho'_S)}$, estimate $S(\rho'_S)$ (hence $I = 2S(\rho'_S)$), and purity for d_{eff}.
Scaling. Increase effective D by adding idle-coupled ancillas (or by increasing mixing depth), repeat to obtain $\{(d_{\mathrm{eff}}, Y)\}$ pairs.
Predictions. (1) Signed α CI includes 0 at each depth; (2) $\mathrm{Var}(Y)$ vs D has slope ≈ -1 on log–log axes; (3) $|\alpha|$ vs $1/D$ extrapolates to intercept 0 within CI.