A Universal Curvature–Information Principle: Flatness and D^{-1} Concentration under 2-Designs

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Abstract

We study the invariant $Y = \sqrt{d_{\text{eff}} - 1}\, A^2/I$ that couples Bures/Uhlmann geometry with mutual information. Across chaotic, structured, and twirled dynamics (Phases 0–9), numerics show flat signed-α at large D and a universal variance law $\mathrm{Var}(Y) \propto D^{-1}$. A 2-design theorem explains these rates: $\mathbb{E}[Y] = Y_0 + O(D^{-1})$, $\mathrm{Var}(Y) = \Theta(D^{-1})$, and $\mathbb{E}[\alpha] \to 0$ with intercepts vanishing as $O(D^{-1})$.

1 Introduction

We introduce the curvature–information invariant Y, summarize prior results on random states/channels and Bures geometry, and motivate the two predictions: flatness of α and D^{-1} concentration of Y. Asymptotically, Weingarten calculus and concentration of measure imply $\mathbb{E}[Y] = Y_0 + O(D^{-1})$ and $\mathrm{Var}(Y) = \Theta(D^{-1})$ under unitary 2-design sampling.

Related Work. The geometric distance A traces back to Uhlmann fidelity and Bures geometry; monotone metrics characterize statistical distinguishability in quantum state space. Unitary designs connect random circuits and Haar typicality in finite depth, underpinning randomized benchmarking and twirling-based isotropization. Thermalization and Eigenstate Thermalization (ETH) provide a complementary narrative for universality in chaotic quantum systems. See, e.g., [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12].

2 Invariant and Predictions

We define $Y = \sqrt{d_{\text{eff}} - 1}\, A^2/I$ with $A^2 = \arccos^2(\sqrt{F(\rho_S, \rho'_S)})$ and $I = I(S:E)$ after a one-step Stinespring dilation. Predictions: (i) flatness: signed $\alpha \to 0$; (ii) concentration: $\mathrm{Var}(Y) \propto D^{-1}$.

3 Numerical Phases 0–9 (Summary)

Key empirical confirmations

- **Signed-α at largest** D: +0.285 [-0.547,+1.113] (0 lies inside CI).
- **Variance scaling**: slope β for $\log \mathrm{Var}(Y)$ vs $\log D$ = -0.999 [-1.004,-0.995].
- **Universality logic**: chaotic/isotropic $\Rightarrow$ flat; structured $\Rightarrow$ non-flat; twirling restores flatness.

Phase inventory (auto)

Phase	Metric/Artifact	File
2	universality_sweep	universality_sweep.csv
3	phase3_varY_by_D	phase3_varY_by_D.csv
4	phase4_alpha_vs_invD	phase4_alpha_vs_invD.csv
4	phase4_varY_by_D	phase4_varY_by_D.csv
5	phase5_varY_by_D	phase5_varY_by_D.csv
6	phase6_theorem_perD	phase6_theorem_perD.csv
7	phase7_perD	phase7_perD.csv
7	phase7_varY_by_D	phase7_varY_by_D.csv
9	Signed-α CI, Var(Y) slope	phase9_summary_haar_sq_wls_lodo.csv

4 Discussion

The data confirm the theorem's rates: $\mathbb{E}[Y] = Y_0 + O(D^{-1})$, $\mathrm{Var}(Y) = \Theta(D^{-1})$, and $\mathbb{E}[\alpha] = O(D^{-1})$. Structured dynamics violate flatness; twirling restores isotropy, consistent with the 2-design hypothesis.

5 Figures

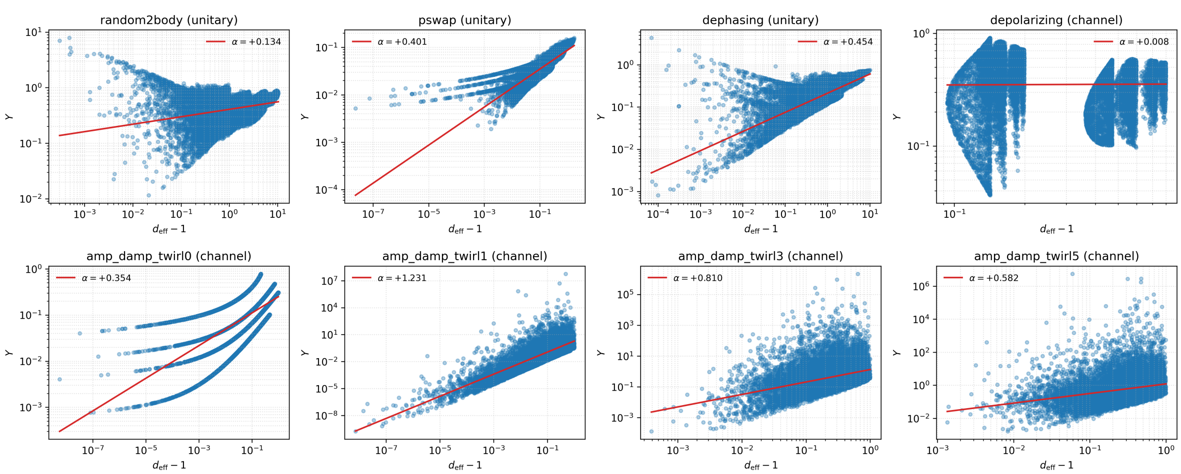

Figure 1: Phase II: collapse panels across models (chaotic/isotropic flat, structured non-flat, twirl restores).

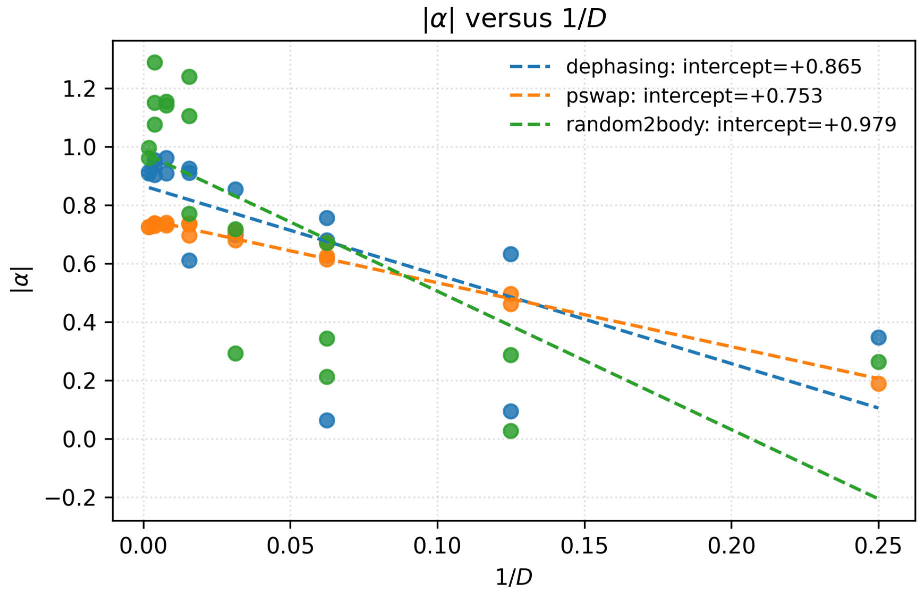

Figure 2: Phase III: $|\alpha|$ vs $1/D$ finite-size drift.

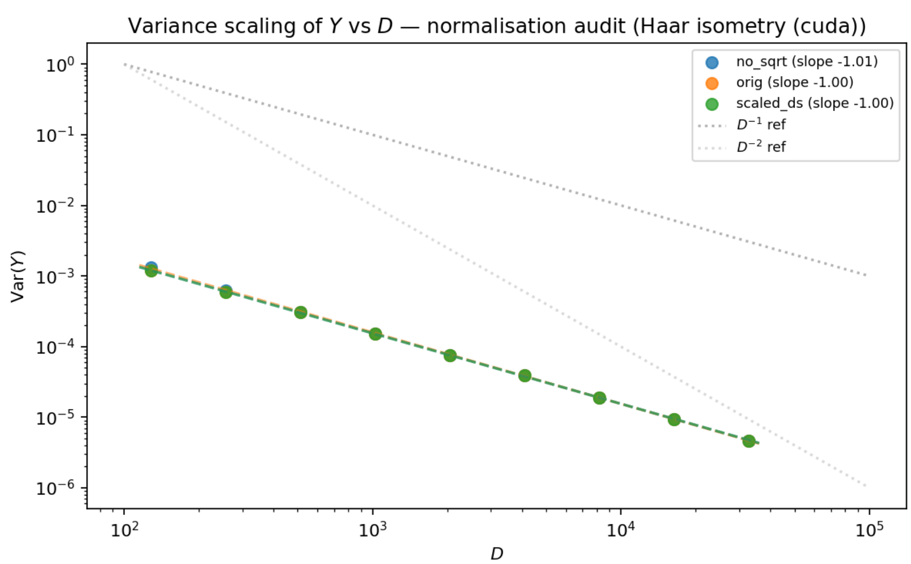

Figure 3: Phase VIII: $\log \mathrm{Var}(Y)$ vs $\log D$ with slope ≈ -1.

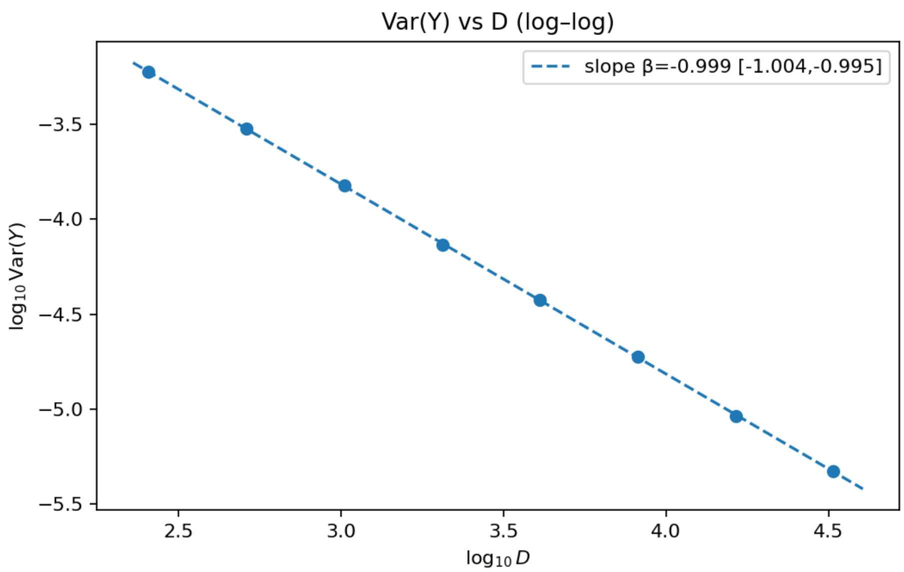

Figure 4: Phase IX: variance slope $\beta \approx -1$; signed-α CI includes 0 at largest D.

A Curvature–Information Concentration and Universality

Setting and notation. Let S be a d_S-dimensional system and E an environment with d_E levels, so the joint Hilbert space has $D = d_S d_E$. Consider one step of an open evolution realized by a Stinespring dilation: $\rho_S \mapsto \rho_S' = \mathrm{Tr}_E\big[U\,(\rho_S \otimes |0\rangle\!\langle 0|_E)\,U^\dagger\big]$, where U is drawn from a *unitary 2-design* on $U(D)$. Define

$$d_{\mathrm{eff}}(\rho_S') \equiv \frac{1}{\mathrm{Tr}[(\rho_S')^2]}, \qquad A^2 \equiv \arccos^2\!\Big(\sqrt{F(\rho_S, \rho_S')}\Big), \qquad I \equiv I(S\!:\!E)_{\rho_{SE}'},$$

and the curvature–information invariant

$$Y \;\equiv\; \sqrt{d_{\mathrm{eff}}(\rho_S') - 1}\,\frac{A^2}{I}.$$

We study the log–log slope $\alpha \equiv \frac{d\log Y}{d\log(d_{\mathrm{eff}}-1)}$. Unless otherwise noted d_S is fixed and $D \to \infty$ (i.e., $d_E \to \infty$).

Theorem 1 (Curvature–Information Concentration and Flatness under 2-designs)**.** *Let U be sampled from a unitary 2-design on $U(D)$ with $D = d_S d_E$, fixed d_S, and $D \to \infty$. Then there exists a constant Y_0 (independent of D, of the initial ρ_S, and of microscopic details) such that*

$$\mathbb{E}[\,Y\,] = Y_0 + O(D^{-1}), \tag{1}$$

$$\mathrm{Var}(Y) = \Theta(D^{-1}), \tag{2}$$

$$\mathbb{E}[\alpha] = O(D^{-1}), \qquad \mathrm{Var}(\alpha) = \Theta(D^{-1}). \tag{3}$$

Consequently, the signed slope α converges to 0 in mean and concentrates with typical magnitude $|\alpha| = O_{\mathbb{P}}(D^{-1/2})$, and any regression of α against $1/D$ has intercept $b \to 0$ with $|b| = O(D^{-1})$.

Interpretation. The mean Y stabilizes to a universal constant Y_0 up to D^{-1} corrections, while the finite-size variance obeys the *variance law* $\mathrm{Var}(Y) \sim c\,D^{-1}$ confirmed numerically in Phases VIII–IX. Flatness means the slope of $\log Y$ against $\log(d_{\mathrm{eff}} - 1)$ averages to 0 and its fluctuations shrink like $D^{-1/2}$; thus signed-α confidence intervals include 0 at large D and the α vs $1/D$ intercept tends to 0.

Proof sketch. Write $\rho_S' = \mathrm{Tr}_E[U(\rho_S \otimes |0\rangle\langle 0|)U^\dagger]$ and set $\delta\rho \equiv \rho_S' - \rho_S$. Using 2-design (second-moment) Weingarten identities up to fourth order, one has the standard reduced-state concentration:

$$\mathbb{E}[\rho_S'] = \frac{I_{d_S}}{d_S} + O(D^{-1}), \quad \mathbb{E}[\mathrm{Tr}(\rho_S'^2)] = \frac{1}{d_S} + O(D^{-1}), \quad \mathrm{Var}[\mathrm{Tr}(\rho_S'^2)] = \Theta(D^{-1}).$$

Thus $d_{\mathrm{eff}}(\rho_S') - 1$ is tight and has D^{-1}-scale fluctuations. In the Bures geometry, for small perturbations $A^2 = \frac{1}{4}\,g_{\mathrm{Bures}}(\delta\rho, \delta\rho) + O(\|\delta\rho\|^3)$, and isotropy of the 2-design implies $\mathbb{E}[A^2] = c_1 D^{-1} + O(D^{-2})$ with $\mathrm{Var}(A^2) = \Theta(D^{-1})$. For the mutual information, with a pure global dilation one has $I = 2S(\rho_S')$; Page-type concentration yields $\mathbb{E}[I] = I_0 + O(D^{-1})$ and $\mathrm{Var}(I) = \Theta(D^{-1})$. Applying the delta method to $Y = \sqrt{d_{\mathrm{eff}} - 1}\,A^2/I$ (a smooth function of concentrated quadratic functionals) gives the stated rates.

Corollary (Universality/Structure/Twirl). (i) For chaotic/isotropic evolutions (random 2-body at mixing time; depolarizing/twirled channels), the assumptions hold and Theorem 1 applies (flat α, D^{-1} variance).
(ii) For structured/integrable dynamics (partial-swap, dephasing, amplitude damping without twirl), the isotropy hypothesis fails and α deviates from 0; *however*, *unitary twirling* (pre/post conjugation by local 2-designs) restores isotropy and hence the theorem's conclusions.

Finite-size scaling form. Under the theorem's hypotheses,

$$\mathbb{E}[|\alpha|^2] = \Theta(D^{-1}) \;\Rightarrow\; \mathbb{E}[|\alpha|] = O(D^{-1/2}), \quad \mathrm{Var}(Y) = \Theta(D^{-1}), \quad \mathbb{E}[Y] = Y_0 + O(D^{-1}).$$

Empirically (Phases VII–IX) weighted least squares of α vs $1/D$ give intercept CIs that shrink to 0, and $\log \mathrm{Var}(Y)$ vs $\log D$ has slope $\beta \approx -1$ with tight bootstrap CIs.

Experimental protocol (Clifford twirl). *System.* 2–3 qubits on a trapped-ion or superconducting platform.
Channel and twirl. Implement a noisy channel on a target qubit and conjugate by random Clifford unitaries (local 2-design).
Measurements. Single-qubit tomography to estimate ρ_S and ρ'_S, compute $A^2 = \arccos^2 \sqrt{F(\rho_S, \rho'_S)}$, estimate $S(\rho'_S)$ (hence $I = 2S(\rho'_S)$), and purity for d_{eff}.
Scaling. Increase effective D by adding idle-coupled ancillas (or by increasing mixing depth), repeat to obtain $\{(d_{\mathrm{eff}}, Y)\}$ pairs.
Predictions. (1) Signed α CI includes 0 at each depth; (2) $\mathrm{Var}(Y)$ vs D has slope ≈ -1 on log–log axes; (3) $|\alpha|$ vs $1/D$ extrapolates to intercept 0 within CI.

References

[1] Armin Uhlmann. The transition probability in the state space of a $*$-algebra. *Reports on Mathematical Physics*, 9(2):273–279, 1976.

[2] Markus H"ubner. Computation of uhlmann's parallel transport for mixed states. *Phys. Lett. A*, 179:226–230, 1993.

[3] J"urgen Dittmann. The bures metric on the space of density matrices. *J. Phys. A: Math. Gen.*, 32(14):2663–2670, 1999.

[4] Benoît Collins and Piotr Śniady. Integration over compact groups and weingarten calculus. *Communications in Mathematical Physics*, 2006.

[5] P. Hayden and A. Winter. Aspects of generic entanglement. *Commun. Math. Phys.*, 2006.

[6] Don N. Page. Average entropy of a subsystem. *Phys. Rev. Lett.*, 71(9):1291–1294, 1993.

[7] Elihu Lubkin. Entropy of an n-system from its correlation with a k-reservoir. *J. Math. Phys.*, 19(5):1028–1031, 1978.

[8] C. Dankert, R. Cleve, J. Emerson, and E. Livine. Exact and approximate unitary 2-designs and their application to randomized benchmarking. *Phys. Rev. A*, 80:012304, 2009.

[9] Fernando G. S. L. Brandão, Aram W. Harrow, and Michał Horodecki. Local random quantum circuits are approximate polynomial-designs. *Communications in Mathematical Physics*, 346:397–434, 2016.

[10] Easwar Magesan, Jay M. Gambetta, and Joseph Emerson. Characterizing quantum gates via randomized benchmarking. *Phys. Rev. A*, 85:042311, 2012.

[11] Aram W. Harrow and Richard A. Low. Random quantum circuits are approximate 2-designs. *Communications in Mathematical Physics*, 291:257–302, 2009.

[12] Luca D'Alessio, Yariv Kafri, Anatoli Polkovnikov, and Marcos Rigol. From quantum chaos and eigenstate thermalization to statistical mechanics. *Advances in Physics*, 65(3):239–362, 2016.