for vectors $u = e_i$ and $v = e_j$ being elementary vectors (vectors with a single 1 in them) we consider the matrix A and modifications $B = A + vu^T$ and $C = B + uv^T = A + vu^T + uv^T$.

Matrix determinant lemma for $det(C)$ is:

$$det(C) = det(B + uv^T) = (1 + v^T B^{-1} u) det(B)$$

Matrix determinant lemma for $det(B)$ is:

$$det(B) = det(A + vu^T) = (1 + u^T A^{-1} v) det(A)$$

Thus:

$$det(C) = (1 + v^T B^{-1} u)(1 + u^T A^{-1} v) det(A)$$

The Sherman–Morrison formula for B^{-1} is:

$$B^{-1} = (A + vu^T)^{-1} = A^{-1} - \frac{A^{-1} vu^T A^{-1}}{1 + u^T Av}$$

Thus:

$$\begin{aligned} det(C) &= \left(1 + v^T \left(A^{-1} - \frac{A^{-1} vu^T A^{-1}}{1 + u^T Av}\right) u\right)(1 + u^T A^{-1} v) det(A) \\ &= \left(1 + v^T A^{-1} u - \left(\frac{v^T A^{-1} vu^T A^{-1} u}{1 + u^T Av}\right)\right)(1 + u^T A^{-1} v) det(A) \\ &= ((1 + v^T A^{-1} u)(1 + u^T A^{-1} v) - (v^T A^{-1} vu^T A^{-1} u)) det(A) \\ &= (1 + 2v^T A^{-1} u + v^T A^{-1} uu^T A^{-1} v - v^T A^{-1} vu^T A^{-1} u) det(A) \end{aligned}$$

Thus if $det(A)$ is positive then $det(C)$ will be greater than $det(A)$ if:

$$2v^T A^{-1} u + v^T A^{-1} uu^T A^{-1} v - v^T A^{-1} vu^T A^{-1} u > 0$$

Since A is symmetrical, so too is A^{-1} thus $u^T A^{-1} v = v^T A^{-1} u$, so:

$$2v^T A^{-1} u + v^T A^{-1} uv^T A^{-1} u - v^T A^{-1} vu^T A^{-1} u > 0$$

Oh, shit! thoes two terms dont actually cancel.

if $v = \sqrt{\epsilon} e_i$ and $u = \sqrt{\epsilon} e_j$ then this term would be:
$2\epsilon v^T A^{-1} u + \epsilon^2 (v^T A^{-1} uv^T A^{-1} u - v^T A^{-1} vu^T A^{-1} u)$
Thus we have linear and a quadratic term, the question then is whether we could make adjacency matricies with do have maximum positive determinant and where this quadratic term is pivotal in its being maximal; hence errantly excluded them by our approach. I feel that it would be unlikely, I will investigate.