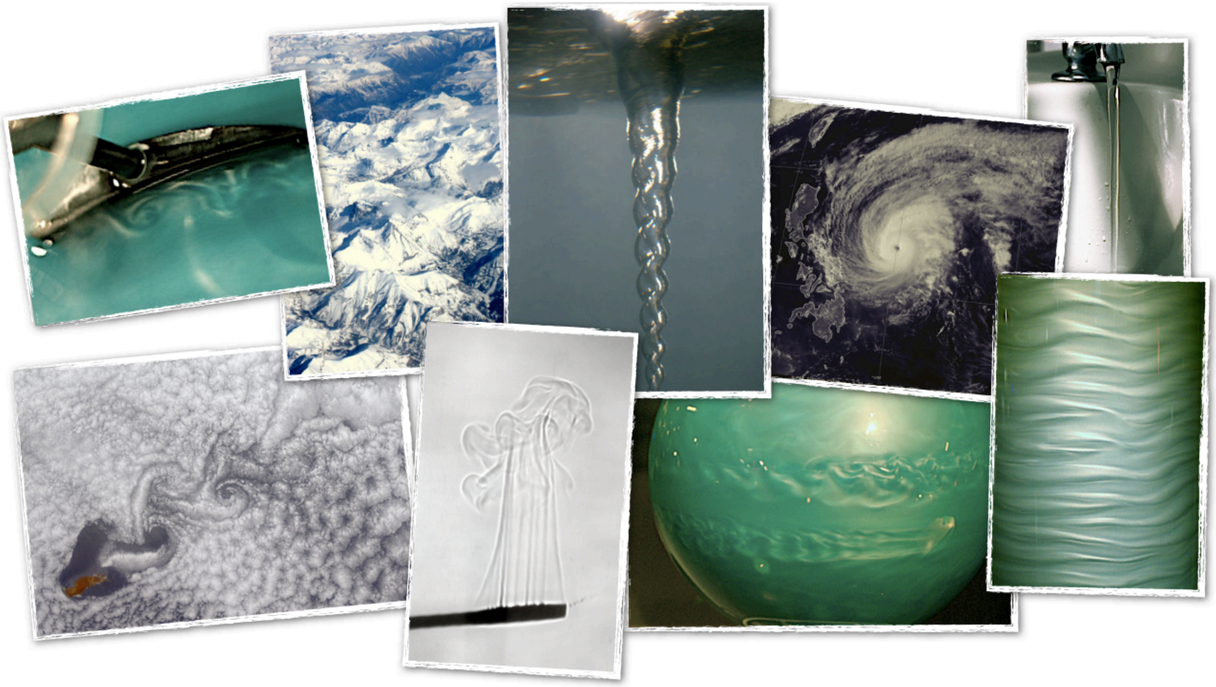
MONASH University

MTH3360 — Fluid Mechanics

Part 2: Incompressible Fluids



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2010; 2011

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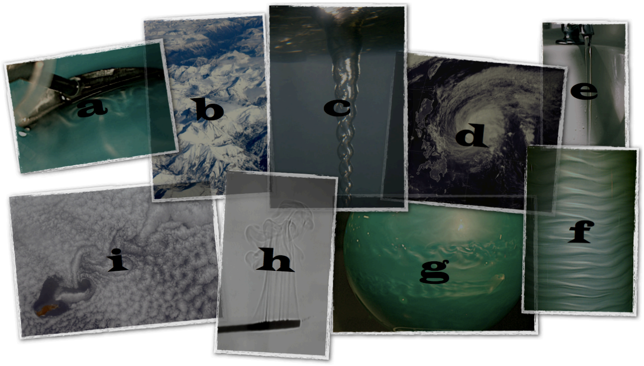
Acknowledgements

These notes draw heavily on the lectures given by Professor Bruce Morton, Professor Michael Reeder, Dr Simon Clarke at Monash University and Professor Roger Smith at the University of Munich.

Cover images



a — Flow past a wing-cross-section visualised by suspended particles in a thin layer of slowly flowing water.

b — The French alps seen from the air in the Spring of 2010. Viscous flow is an important component of mountain building (orogeny) on geological timescales.

c — The free-surface of a rapidly rotating vortex in a cylindrical column of water seen from beneath.

d — A Geostationary Meteorological Satellite (GMS) image of super typhoon Mitag, 5 March 2002. The image comes from the University of Madison, Wisconsin

e — Water from a running tap showing how the cross sectional area of the flow reduces as the fluid gains speed

f — Taylor-Couette flow at a rotation rate above the point where secondary vortices appear, but just below the onset of instability.

g — Zonal flows which develop spontaneously in a rapidly-rotating, fluid-filled sphere

h — Buoyant plume of warm water created by switching on a heating element in a tank of water and visualised by its shadow

i — MISR images of a von Karman vortex street. The alternating double row of vortices form in the wake of an obstacle, in this instance the eastern Pacific island of Guadalupe. This volcanic Mexican island reaches a maximum elevation of 1.3 kilometers. Theisland is about 35 kilometers long and is located 260 kilometers west of Baja California. The vortex pattern is made visible by the marine stratocumulus clouds around Guadalupe Island.

Images in *(a,e,f,g,h)* taken at the Swiss Science Center Technorama Winterthur, Switzerland, 2010, by LM – (http://www.technorama.ch). Photograph of the alps *(b)* by LM of a view provided by Swiss International Airlines. Image *(d)* comes from the University of Madison, Wisconsin. *(i)* NASA visible earth – http://visibleearth.nasa.gov/view_rec.php?vev1id=6586

Contents

Chapter 1

Fluid Flow and Conservation of Mass

1.1 Description of fluid flow

Fluids, generally liquids or gases, but, under the right conditions, also some solids, flow when subjected to stresses. In describing fluid flow, we are usually not at all interested in describing the relative displacement of points in the fluid as we must do when we describe the deformation of elastic solids, for example. Instead, the description of an ideal fluid flow requires a specification or determination of the **velocity field**, i.e. a specification of the fluid velocity at every point in the region. In general, this will define a **vector field** of position and time, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$.

Steady flow occurs when $\mathbf{u}$ is independent of time (i.e., $\partial \mathbf{u}/\partial t = 0$). Otherwise the flow is **unsteady**.

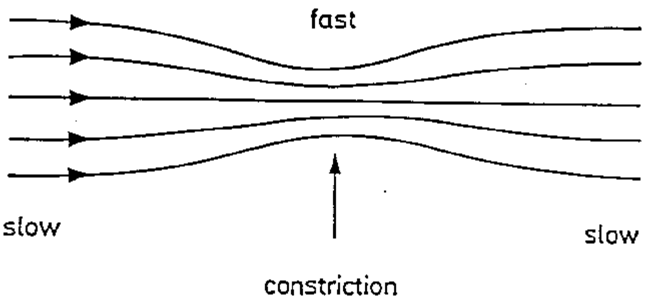


Figure 1.1: Streamlines of flow through a constriction

Streamlines are lines which at a given instant are everywhere in the direction of the velocity (analogous to electric or magnetic field lines). In steady flow the streamlines are independent of time, but the velocity can vary in magnitude along a streamline (as in flow through a constriction in a pipe).

Particle paths are lines traced out by "marked" particles as time evolves. In steady flow particle paths are identical to streamlines; in unsteady flow they are different, and sometimes very different. Particle paths are visualized in the laboratory using small floating particles of the same density as the fluid. Sometimes they are referred to as **trajectories**.

Filament lines or **streaklines** are traced out over time by all particles passing through a given point; they may be visualized, for example, using a hypodermic needle and releasing a slow stream of dye. In steady flow these are streamlines; in unsteady flow they are neither streamlines nor particle paths.

It should be emphasised that streamlines represent the velocity field at a specific instant of time, whereas particle paths and streak lines provide a representation of the velocity field over a finite period of time. In the laboratory one can obtain a record of streamlines photographically by seeding the fluid with small neutrally buoyant particles that move with the flow and taking a short exposure (e.g. 0.1 sec), long enough for each particle to trace out a short segment of line; the eye readily links these segments into continuous streamlines. Particle paths and streak lines are obtained from a time exposure long enough for the particle or dye trace to traverse the region of observation.

1.2 Equations for streamlines

The streamline through the point P, say (x, y, z), has the direction of $\mathbf{u} = (u, v, w)$. Let Q be the neighbouring point $(x+\delta x, y+\delta y, z+\delta z)$ on the streamline. Then $\delta x \approx u\delta t, \delta y \approx v\delta t, \delta z \approx w\delta t$ and as $\delta t \to 0$, we obtain the differential relationship

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \tag{1.1}$$

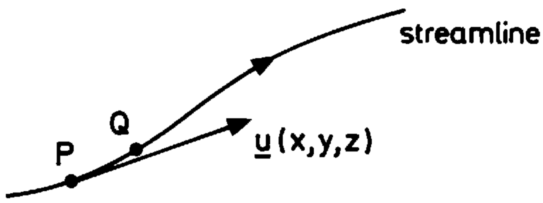


Figure 1.2: Fluid motion at a point is tangential to the streamline at a point

between the displacement $d\mathbf{x}$ along a streamline and the velocity components. Equation (1.1) gives two differential equations (why?). Alternatively, we can represent the streamline parametrically (with time as parameter) as

$$\int \frac{dx}{u} = \int dt, \quad \int \frac{dy}{v} = \int dt, \quad \int \frac{dz}{w} = \int dt. \tag{1.2}$$

Example 1.1

Find the streamlines for the velocity field $u = (-\Omega y, \Omega x, 0)$, where Ω is a constant.

Trajectories can be calculated from the definition of the velocity field,

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w,$$

where $u = u(x, y, z, t)$, $v = v(x, y, z, t)$ and $w = w(x, y, z, t)$. Generally this is very difficult as one has to know the velocity field for all time, and the velocity field is one of the principle unknowns in the problem !

1.3 Distinctive properties of fluids

Although fluids are molecular in nature, they can be treated as *continuous media* for most practical purposes, exceptions being rarefied gases, granular materials in a 'fluidized' state, and some suspensions such as concrete where the suspended particles are large compared to other 'important' length scales in the flow such as boundary layer thicknesses.

Real fluids generally show some **compressibility** defined as

$$\frac{1}{\rho}\frac{d\rho}{dp} = \frac{\text{changes in density per unit change in pressure}}{\text{density}},$$

but at normal atmospheric flow speed, the compressibility of air is a relatively small effect and for liquids it is generally negligible. Note that sound waves owe their existence to compressibility effects as do "supersonic bangs", produced by aircraft flying faster than sound. For many purposes it is accurate to assume our fluids are **incompressible**, i.e. they suffer no change in density with pressure. For the present we shall assume also that they are **homogeneous**, i.e., density ρ = constant.

When one solid body slides over another, **frictional forces** act between them to reduce the relative motion. Friction acts also when layers of fluid flow over one another. When two solid bodies are in contact (more precisely when there is a normal force acting between them) at rest, there is a threshold tangential force *below which* relative motion will not occur. It is called the **limiting friction**. An example is a solid body resting on a flat surface under the action of gravity (see figure below).

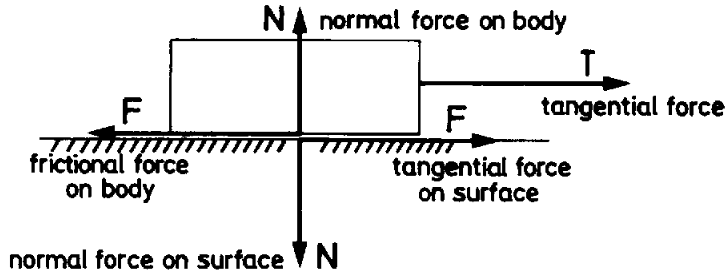


Figure 1.3: *Force balance for a block sliding on a surface and resisted by friction*

As T is increased from zero, $F = T$ until $T = \mu N$, where μ is the so-called coefficient of limiting friction and depends on the degree of roughness between the surface. For $T > \mu N$, the body will overcome the frictional force and accelerate. A distinguishing characteristic of most fluids in their inability to support tangential stresses between layers without motion occurring; i.e. there is no analogue of limiting friction. Exceptions are certain types of so-called visco-elastic fluids such as paints, slurries and foods.

Fluid friction is characterized by **viscosity**, which is a measure of the magnitude of tangential frictional forces in flows with velocity gradients. **Viscous forces** are important in many flows, but least important in flow past "streamlined" bodies. We shall be concerned mainly with **inviscid** flows where friction is not important, but it is essential to acquire some idea of the sort of flow in which friction may be neglected without completely misrepresenting the behaviour. As we shall see, neglecting friction is risky!

To begin with we shall be concerned mainly with **homogeneous, incompressible inviscid flows**.

1.4 Incompressible flows

We generalise the idea of a streamline and consider an element of fluid bounded by a "tube of streamlines", known as a **stream tube.** No fluid can cross the walls of the stream tube as they are everywhere in the direction of flow.

Hence for incompressible fluids the mass flux (= mass flow per unit time) across section 1 (= $\rho v_1 S_1$) is equal to that across section 2 (= $\rho v_2 S_2$), as there can be no accumulation of fluid between these sections. Hence vS = constant and in the limit, for stream tubes of small cross-section,

$vS =$ **constant along an elementary stream tube**.

This result is true for both steady and unsteady flows. The statement is obvious for steady flow as the stream tube does not change shape – think about a the analogy with a pipe.

It follows that, where streamlines contract the velocity increases, where they expand it decreases. Clearly, the streamline pattern contains a great deal of information about the velocity distribution.

All vector fields with the property that

(vector magnitude) $\times$ (area of tube)

remains constant along a tube are called **solenoidal**. The velocity field for an incompressible fluid is solenoidal.

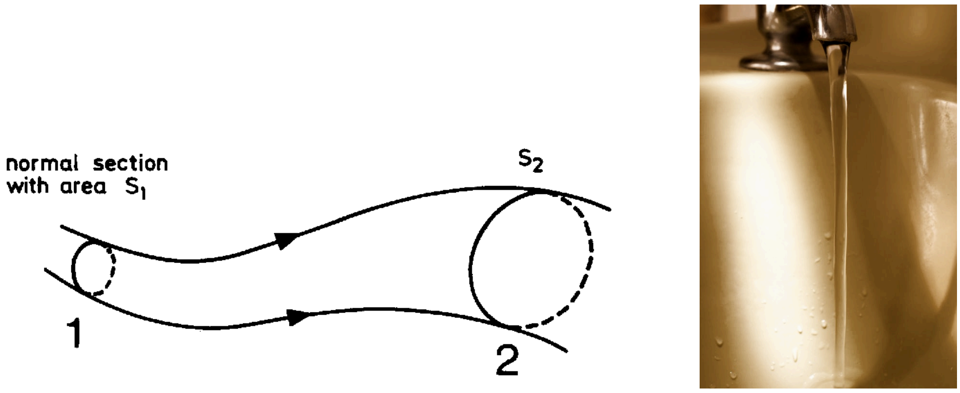


Figure 1.4: A conceptual diagram of a stream tube and (right) something not dissimilar from real life

1.5 Conservation of mass: the continuity equation

Apply the divergence theorem

$$\int_V \nabla.\mathbf{u}dV = \int_S \mathbf{u}.\mathbf{n}dS$$

to an arbitrarily chosen volume V with **closed** surface S. The mass flux across the elemental area dS is $\rho\mathbf{u}.\mathbf{n}dS$.

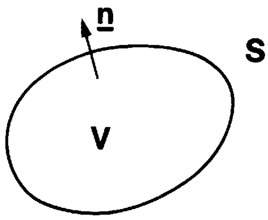


Figure 1.5: Volume element, V bounded by surface S

If the fluid is incompressible and there are no mass sources or sinks within S, then there can be neither continuing accumulation of fluid within V nor continuing loss. It follows that the net flux of fluid across the surface S must be zero, i.e.,

$$\int_S \mathbf{u}.\mathbf{n}dS = 0,$$

whereupon. $\int_V \nabla.\mathbf{u}dV = 0$. This holds for an arbitrary volume V, and therefore $\nabla.\mathbf{u} = 0$ throughout an incompressible flow without mass sources or sinks. This is the continuity equation for a **homogeneous, incompressible** fluid. It corresponds with mass conservation.

The stream tube in figure 1.4 is a special choice of the surface S in which the flow is perpendicular to the ends of the tube (S_1,S_2), and parallel to the sides. Thus the fluxes across the ends must be equal (and opposite – since we consider the sign)

1.6 Strain rate, Velocity gradients, and Vorticity

In elasticity theory, the constitutive behaviour of materials is described through the relationship between the applied forces and the resulting deformation. In fluids, the appropriate variable to relate to the applied forces or stresses is the *strain-rate*.

In Cartesian coordinates, the strain rate tensor, **D**, is given by:

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.3}$$

It is the symmetric part of the velocity gradient tensor, **L**,

$$L_{ij} = \frac{\partial u_i}{\partial x_j} \tag{1.4}$$

and non-symmetric part of the tensor is a rotation rate, $\mathbf{W}$

$$L_{ij} = D_{ij} + W_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{1.5}$$

$\mathbf{W}$ is half the usual fluid vorticity, $\nabla \times \mathbf{u}$, which we will encounter in chapter 4.

The strain-rate tensor can be decomposed into a (scalar) *volumetric* part D_v which describes the rate of change of volume, and a *deviatoric* part, $\mathbf{D}'$ which describes shearing motions:

$$D_v = D_{kk} \tag{1.6}$$

$$D'_{ij} = D_{ij} - \frac{1}{3}D_v\delta_{ij} \tag{1.7}$$

The deviatoric part is a traceless tensor.

1.7 Exercises

Question 1.1*

Find streamlines for the velocity field $\mathbf{u} = (\alpha x, -\alpha y, 0)$, where α is constant, and sketch them for the case $\alpha > 0$.

Question 1.2*

Calculate the parcel trajectories for the velocity field defined by $\mathbf{u} = (e^{-t}z, 0, 0)$.

Question 1.3*

Consider the unsteady flow $\mathbf{u} = (u_0, kt, 0)$, where u_0 and k are positive constants.

- Show that the streamlines are straight lines, and sketch them at two different times.
- Show that the fluid follows a parabolic path as time proceeds.

Question 1.4

Show that the expression $u = \Omega \cdot (\mathbf{k} \times \mathbf{x})$ describes solid body rotation about the $\mathbf{k}$ axis with angular velocity Ω — you should show that the velocity is consistent with rotation at a constant rate, Ω about the origin *and* you should show that the flow does not produce any change in shape. *Hint: which means the flow must have a zero strain rate.*

Question 1.5

A stream is broad and shallow with width 8 m, mean depth 0.5 m and mean speed 1 m s^{-1}.

- What is its volume flux (rate of flow per second) in m^3 s^{-1} ?
- It enters a pool of mean depth 3 m and width 6 m: what then is its mean speed ?
- It continues over a waterfall in a single column with mean speed 10 ms^{-1} at its base: what is the mean diameter of this column at the base of the waterfall?
- Will the diameter of the water column at the top of the waterfall be greater, equal to, or less at its base? Why?

Chapter 2

Equation of Motion

2.1 Newton's Second Law

The equation of motion is an expression of Newton's second law:

mass $\times$ acceleration = force.

However, in contrast to the case of rigid-body dynamics, we need to apply an additional constraint, a continuity equation, or mass-conservation equation. Because of this constraint, it is not possible, in general, to specify the *force*, or more precisely *force field* independently, as in rigid body problems.

To apply Newton's second law we must focus our attention on a particular element of fluid, say the small rectangular element, which at time t has vertex at $P[= (x, y, z)]$ and edges of length δx, δy, δz. The mass of this element is $\rho\ \delta x \delta y \delta z$, where ρ is the fluid **density** (or mass per unit volume), which we shall assume constant.

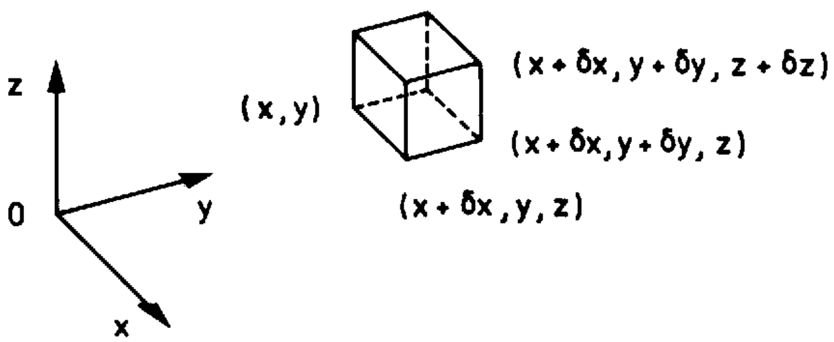


Figure 2.1: A rectangular fluid volume element

The velocity in the fluid, $\mathbf{u} = \mathbf{u}(x, y, z)$ is a function both of position (x, y, z) and time t, and from this we must derive a formula for the acceleration of the element of fluid which is changing its position with time. Consider, for example, steady flow through a constriction in a pipe (see the first diagram in Chapter 1). Elements of fluid must accelerate into the constriction as the streamlines close in and decelerate beyond as they open out again. Thus, in general, the acceleration of an element (i.e., the rate of change of $\mathbf{u}$ with time for that element) includes a rate of change at a fixed position $\partial \mathbf{u}/\partial t$ plus a change associated with its change of position with time. We derive an expression for the latter below.

The forces acting on the elements $\delta x\ \delta y\ \delta z$ comprise:

(i) **body forces**, which are forces per unit mass acting throughout the fluid because of external causes, such as the gravitational **weight**, and

(ii) **contact forces** acting across the surface of the element from adjacent elements.

These are discussed further below.

2.2 Rate-of-change moving with the fluid

We consider first the rate of change of a scalar property, for example the temperature of a fluid, following a fluid element. The temperature of a fluid, $T = T(x, y, z, t)$, comprises a scalar field in which T will vary, in general, both with the position and with time (as in the water in a kettle which is on the boil). Suppose that an element of fluid moves from $P[= (x, y, z)]$ at time t to $Q[= (x + \delta x, y + \delta y, z + \delta z)]$at time $t + \delta t$. Note that if we stay at a particular point (x_0, y_0, z_0), then $T(x_0, y_0, z_0, t)$ is effectively a function of t only, but that if we move with the fluid, $T(x, y, z)$ is a function both of position (x, y, z) and time t. It follows that the total change in T between P and Q in time δt is

$$\delta T = T_Q - T_P = T(x + \delta x, y + \delta y, z + \delta z, t + \delta t) - T(x, y, z, t) \quad ,$$

and hence the total rate of change of T moving with the fluid is

$$\lim_{\delta t \to 0} \frac{\delta T}{\delta t} = \lim_{\delta t \to 0} \frac{T(x + \delta x, y + \delta y, z + \delta z, t + \delta t) - T(x, y, z, t)}{\delta t}.$$

For small increments δx, δy, δz, δt we may use a Taylor expansion

$$\begin{aligned} &T(x + \delta x, y + \delta y, z + \delta z, t + \delta t) \\ &= T(x, y, z, t) + \left[\frac{\partial T}{\partial x}\right]_P \delta x + \left[\frac{\partial T}{\partial y}\right]_P \delta y + \left[\frac{\partial T}{\partial z}\right]_P \delta z + \left[\frac{\partial T}{\partial t}\right]_P \delta t \\ &\quad + \text{higher order terms in } \delta x, \delta y, \delta z, \delta t. \end{aligned}$$

Hence the rate of change moving with the fluid element

$$\begin{aligned} \lim_{\delta t \to 0} \frac{\delta T}{\delta t} &= \lim_{\delta t \to 0} \left[\frac{\partial T}{\partial t}\delta t + \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z\right] / \delta t \\ &= \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \end{aligned}$$

since higher order terms $\to 0$ and $u = dx/dt$, $v = dy/dt$, $w = dz/dt$, where $\mathbf{x} = \mathbf{x}(t)$ is the coordinate vector of the moving fluid element. To emphasize that we mean the **total rate of change moving with the fluid** we write

$$\frac{DT}{Dt} = \frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \quad . \tag{2.1}$$

Here, $\partial T/\partial t$ is the **local rate of change** with time at a fixed position (x, y, z), while

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \mathbf{u}.\nabla T$$

is the **advective rate of change** associated with the movement of the fluid element.

Example 2.1

Show that

$$\frac{D\mathbf{F}}{Dt} = \frac{\partial \mathbf{F}}{\partial t} + (\mathbf{u}.\nabla)\,\mathbf{F}$$

represents the total rate-of-change of any vector field **F** moving with the fluid velocity (velocity field **u**), and in particular that the acceleration (or total change in **u** moving with the fluid) is

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\,\mathbf{u}.$$

Example 2.2

Show that

$$\frac{D\mathbf{x}}{Dt} = \mathbf{u}$$

2.3 Internal forces in a fluid (see Acheson p. 203)

An element of fluid experiences *contact* or internal forces across its surface due to the action of adjacent elements. These are in many respects similar to the normal reaction and tangential friction forces exerted between two rigid bodies, except, as noted earlier, friction in fluids acts only when the fluid is in non-uniform motion.

Consider a region of fluid divided into two parts by the (imaginary) surface S, and let δS be a small element of S containing the point P and with region 1 below and region 2 above S. Let $(\delta X, \delta Y, \delta Z)$ denote the force exerted **on** fluid in region 1 **by** fluid region 2 across δS.

This elementary force is the resultant (vector sum) of a set of contact forces acting across δS, in general it will not act through P; alternatively, resolution of the forces will yield a force (δX, δY, δZ) acting through P together with an elementary couple with moment of magnitude on the order of $(\delta S)^{1/2}(\delta X^2 + \delta Y^2 + \delta Z^2)^1/2$, which $\to 0$ as $\delta S \to 0$.

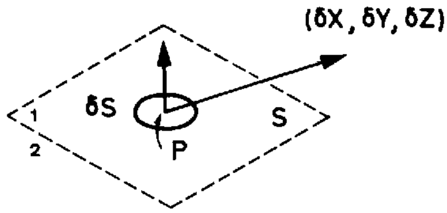


Figure 2.2: A surface element separating two regions of a fluid

The main force per unit area exerted by fluid 2 on fluid 1 across δS,

$$\left[\frac{\delta X}{\delta S}, \frac{\delta Y}{\delta S}, \frac{\delta Z}{\delta S}\right]$$

is called the **mean stress**. The limit as $\delta S \to 0$ in such a way that it always contains P, if it exists, is the **stress** at P across S. Stress is a force per unit area. The stress **F** is generally inclined to the normal **n** to S at P, and varies both in magnitude and direction as the orientation **n** of S is varied about the fixed point P.

The stress **F** may be resolved into a **normal reaction** N, or **tension**, acting normal to S and **shearing stress** T, tangential to S, each per unit area.

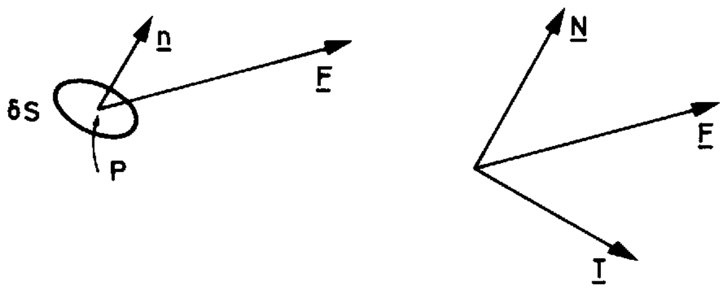


Figure 2.3: Surface traction on a surface element

The stress and its reaction (exerted by fluid in region 1 on fluid in region 2) are equal and opposite. [This follows by considering the equilibrium of an infinitesimal slice at P].

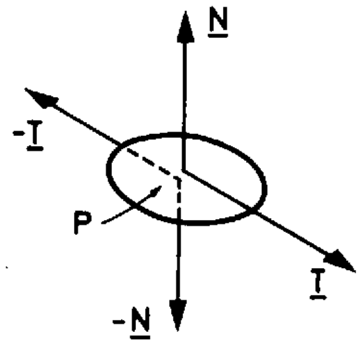


Figure 2.4: Force balance on a surface element

2.4 Fluid and solids: pressure

If the stress in a material **at rest** (or in uniform motion) is always normal to the measuring surface for all points P and surfaces S, the material is termed a **fluid**; otherwise it is a **solid**. Solids at rest sustain tangential stresses because of their elasticity (for example, a drawn bow). By assuming the material to be at rest we eliminate the shearing stress due to internal friction. Many real fluids conform closely to this definition including air and water (although there are more complex fluids possessing both viscosity and elasticity). A fluid can be defined also as a material offering no **initial** resistance to shear stress, although it is important to realize that frictional shearing stresses appear as soon as motion begins, and even the smallest force will initiate motion in a fluid in time. The property of internal friction in a fluid is known as **viscosity**.

Although the term tension is usual in the theory of elasticity, in fluid dynamics the term **pressure** is used to denote the hydrostatic stress, reversed in sign. In a fluid at rest the stress acts normally outwards from a surface, whereas the pressure thrust acts normally **inwards** from the fluid towards the surface.

Physically, pressure is the transfer of momentum per unit area per unit time (the momentum flux) across any surface δS.

2.4.1 Isotropy of pressure

The pressure at a point P in a continuous fluid is **isotropic**; i.e., it is the same for all directions $\mathbf{n}$. This is proved by considering the equilibrium of a small tetrahedral element of fluid with three faces normal to the coordinate axes and one slant face. The proof may be found in any text on fluid mechanics.

In an ideal fluid the force exerted on the fluid separated by a small imaginary surface $\mathbf{n}\delta S$ is $p\mathbf{n}\delta S$. The **pressure**

thrust on a surface is the gross force on the surface due to pressure acting normally inwards, i.e. $\int_S -p\mathbf{n}dS$ evaluated over the surface S. The pressure is independent of the surface element chosen.

2.4.2 Pressure gradient forces in a fluid in macroscopic equilibrium

Pressure is independent of direction at a point, but may vary from point to point in a fluid.

Consider the equilibrium of a thin cylindrical element of fluid PQ of length δs and cross-section A, and with its ends normal to PQ. In particular, consider the forces in the direction P for the fluid at rest. Then pressure acts normally inwards on the curved cylindrical surface and has no component in the direction of PQ. Thus the only contributions are from the plane ends.

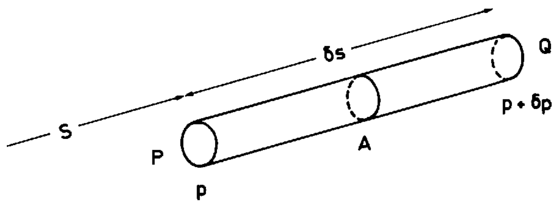


Figure 2.5: Forces due to pressure gradients in a fluid

The net force in the direction in the direction s (i.e. the direction PQ) due to the pressure thrusts on the surface of the element is

$$pA - (p + \delta p)A = -\frac{\partial p}{\partial s}A\delta s = -\frac{\partial p}{\partial s}\delta V,$$

where δV is the volume of the cylinder. In the limit $\delta s \to 0$ and $A \to 0$, the net pressure thrust $\to -(\partial p/\partial s)dV$, or $-\partial p/\partial s = -\hat{\mathbf{s}}.\nabla p$ per unit volume of fluid ($-\hat{\mathbf{s}}$ being a unit vector in the direction PQ). It follows that -∇p is the pressure gradient force per unit volume of fluid, and $-\hat{\mathbf{n}}.\nabla p$ is the component of pressure gradient force per unit volume in a given direction $\hat{\mathbf{n}}$.

The cylindrical element shown right is in equilibrium under the action of the pressure over its surface and its weight. The horizontal component of pressure gradient force per unit volume is $-\mathbf{i}.\nabla p = -\partial p/\partial x = 0$.

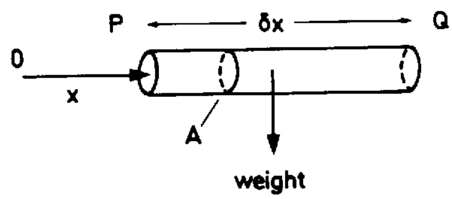


Figure 2.6

Thus p is independent of horizontal distance x, and is similarly independent of horizontal distance y. It follows that $p = p(z)$ and surfaces of equal pressure (**isobaric surfaces**) are horizontal in a fluid at rest.

2.4.3 Equilibrium of a vertical element

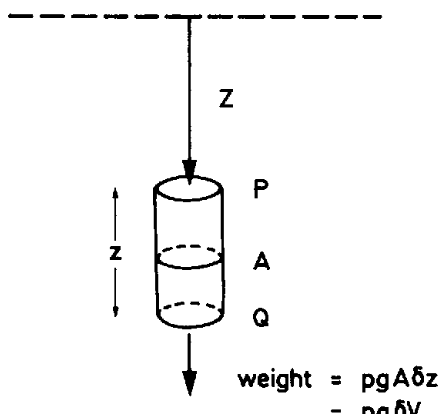
For a vertical cylindrical element at rest in equilibrium under the action of pressure thrusts and the weight of fluid

$$-\mathbf{k}.\nabla p\delta V + \rho g\delta V = 0,$$

where $\mathbf{k} = (0, 0, 1)$.

Thus $dp/dz = \rho g$, per unit volume, since $p = p(z)$ only [otherwise we would write $\partial p/\partial z$]. Hence $\rho = (dp/dz)/g$ is a function of z at most, i.e., $\rho = \rho(z)$. Here z measures downwards so that $\text{sgn}(\delta z) = \text{sgn}(\delta p)$. Normally we take z upwards whereupon

$$\frac{dp}{dz} = -\rho g.$$



(Show this.) The equation above is generally called the **hydrostatic equation**.

2.4.4 Liquids and gases

Liquids undergo little change in volume with pressure over a very large range in pressure and it is frequently a good assumption to assume that ρ = constant (which is the definition of a homogeneous fluid). In that case, the foregoing equation (with z pointing downwards) integrates to give

$$p = p_0 + \rho g z,$$

where $p = p_0$ at the level $z = 0$.

Ideal gases are such that pressure, density and temperature are related through the ideal gas equation, $p = \rho RT$, where T is the absolute temperature and R is the specific gas constant. If a certain volume of gas is isothermal (i.e., has constant temperature), then pressure and density vary exponentially with depth with a so-called **e-folding** scale $H_s = RT/g$ (see exercises).

2.4.5 Archimedes Principle

In a fluid at rest the net pressure gradient force per unit volume acts vertically upwards and is equal to $-dp/dz$ (when z points upwards) and the gravitational force per unit volume is ρg. Hence, for equilibrium, $dp/dz = -\rho g$.

Consider the vertically-oriented cylindrical element P_1P_2 of an immersed body in equilibrium, which intersects the surface of the body to form surface elements δS_1 and δS_2 which have normals $\mathbf{n}_1$, $\mathbf{n}_2$ inclined at angles θ_1, θ_2 to the vertical. The net upward thrust on these small surfaces is:

$$\text{Thrust} = p_2 \cos\theta_2 \delta S_2 - p_1 \cos\theta_1 \delta S_1 = (p_2 - p_1)\delta S \quad ,$$

where $\delta S_1 \cos\theta_1 = \delta S_2 \cos\theta_2 = \delta S$ is the horizontal cross-sectional area of the cylinder.

Since, $p_2 - p_1 = -\int_{z_1}^{z_2} \rho g dz$, the net upward thrust is:

$$\text{Thrust} = \left(\int_{z_2}^{z_1} \rho g dz \right) \delta S = \text{weight of liquid displaced.}$$

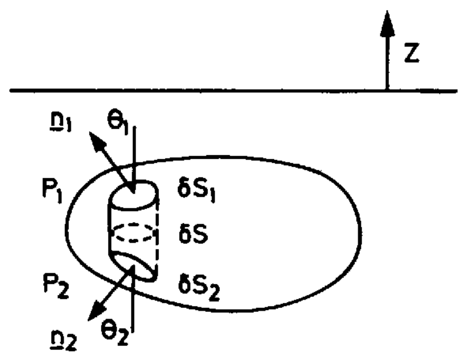


Figure 2.8

If this integration is now continued over the whole body we obtain one version of **Archimedes Principle** which states that the resultant thrust on an immersed body in equilibrium has magnitude equal to the weight of fluid displaced and acts upward through the centre of mass of the displaced fluid (provided that the gravitational field is uniform).

2.5 Equation of Motion for an inviscid fluid

If we apply Newton's second law to a unit volume of fluid:

(i) the mass of the element is ρ kg m^{-3} ;

(ii) the acceleration must be that **following the fluid element** to take account both of the change in velocity with time at a fixed point and of the change with position of the velocity field at a fixed time,

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = \frac{\partial \mathbf{u}}{\partial t} + u\frac{\partial \mathbf{u}}{\partial x} + v\frac{\partial \mathbf{u}}{\partial y} + w\frac{\partial \mathbf{u}}{\partial z};$$

(iii) the total force acting on the element (neglecting viscosity or fluid friction) comprises the contact force acting across the surface of the element -∇p per unit volume, which is a **pressure gradient force** arising from the difference in pressure across the element, and any body forces **F**, acting throughout the fluid including especially the gravitational weight per unit volume, $-g\rho\mathbf{k}$.

The resulting **equation of motion** or **momentum equation** for inviscid fluid flow, known as **Euler's equation**, is

$$\underset{\text{(A)}}{\rho\frac{\partial u}{\partial t}} + \underset{\text{(B)}}{\rho\left(\mathbf{u}.\nabla\right)\mathbf{u}} = \underset{\text{(C)}}{-\nabla p} + \underset{\text{(D)}}{\mathbf{F}},$$

where (A) — local acceleration, (B) — acceleration due to advection of momentum, (C) — pressure gradient force and (D) — body force.

In rectangular cartesian coordinates (x, y, z) with velocity components (u, v, w) the component equations are

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + X, \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + Y, \\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + Z
\end{aligned}$$

where $\mathbf{F} = (X, Y, Z)$ is the external force per unit mass (or body force). These are **three** partial differential equations in the **four** dependent variables u, v, w, p and four independent variables x, y, z, t. For a complete system we require **four** equations in the four variables, and the extra equation is the conservation of mass or **continuity equation,** which for an incompressible fluid has the form

$$\nabla.\mathbf{u} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad .$$

There is no equivalent to the continuity equation in either particle or rigid body mechanics, because in general mass is permanently associated with bodies. In fluids, however, we must ensure that holes do not appear or that fluid does not double up, and we do this by requiring that $\nabla.\mathbf{u} = 0$ which implies that in the absence of sources or sinks there can be no net flow either into or out of any closed surface. We may regard this as a geometric condition on the flow of an incompressible fluid. It is not, of course, satisfied by a compressible fluid (c.f. a bicycle pump). We say that any incompressible flow satisfying the continuity equation $\nabla.\mathbf{u} = 0$ is a **kinematically possible motion**.

The Euler equation plus continuity equation are extremely important but extremely difficult to solve. With possible further force terms on the right, they represent the behaviour of gaseous stars, the flow of oceans and atmosphere, the motion of the earth's mantle, blood flow, air flow in the lungs, many processes of chemistry and chemical engineering, the flow of water in rivers and in the permeable earth, aerodynamics of aeroplanes, and so forth.

The difficulty of solution, and there are probably no more than a dozen or so solutions known for very simple geometries, arises from the **non-linear** term $(\mathbf{u}.\nabla)\mathbf{u}$ as a result of which if $\mathbf{u}_1$ and $\mathbf{u}_2$ are two solutions of the equation $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$ (where c_1 and c_2 are constants) is in general *not* a solution, so that we lose one of our main methods of solution.

Example 2.3

Suppose that the velocity field is $\mathbf{u} = (-\Omega y, \Omega x, 0)$ for Ω constant. Show that it is a kinematically possible flow for an incompressible liquid in a uniform gravitational field $F \equiv g = (0, 0, -g)$. Determine the corresponding pressure field.

2.6 Equations of motion in cylindrical polars

Take the cylindrical polars (r, θ, z) and velocity (v_r, v_θ, v_z). This is more complicated than rectangular cartesians as v_r, and v_θ change in direction with P (in fact $0P$ rotates about $0z$ with angular velocity v_θ/r). Suppose that $\hat{\mathbf{r}}, \hat{\mathbf{n}}, \hat{\mathbf{k}}$ are the unit vectors at P in the radial, azimuthal and axial directions, as sketched. Then $\hat{\mathbf{k}}$ is fixed in direction (and, of course, magnitude) but $\hat{\mathbf{r}}$ and $\hat{\mathbf{n}}$ rotate in the plane $z = 0$ as P moves, and it follows that $d\hat{\mathbf{k}}/dt = \mathbf{0}$, but that

$$\frac{d}{dt}\hat{\mathbf{r}} = \dot{\hat{\mathbf{r}}} = \hat{\mathbf{n}}\dot{\theta}, \dot{\hat{\mathbf{n}}} = (-\hat{\mathbf{r}})\dot{\theta} = -\hat{\mathbf{r}}\dot{\theta}$$

Hence, as $\dot{\theta} = v_\theta/r$,

$$\begin{aligned}
\mathbf{v} &= (v_r\hat{\mathbf{r}} + v_\theta\hat{\mathbf{n}} + v_z\hat{\mathbf{k}}), \\
\dot{\mathbf{v}} &= \dot{v}_r\hat{\mathbf{r}} + v_r\dot{\hat{\mathbf{r}}} + \dot{v}_\theta\hat{\mathbf{n}} + v_\theta\dot{\hat{\mathbf{n}}} + \dot{v}_z\hat{\mathbf{k}} \\
&= (\dot{v}_r - v_\theta^2/r)\hat{\mathbf{r}} + (\dot{v}_\theta + v_r v_\theta/r)\hat{\mathbf{n}} + \dot{v}_z\hat{\mathbf{k}}.
\end{aligned}$$

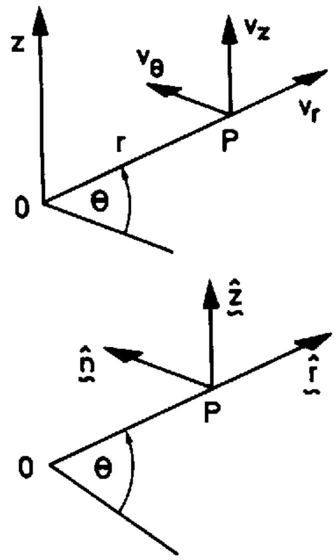


Figure 2.9: Cylindrical polar coordinates

Recalling also that d/dt must be interpreted here as D/Dt, the acceleration is

$$\left[\frac{Dv_r}{Dt} - \frac{v_\theta^2}{r}, \frac{Dv_\theta}{Dt} + \frac{v_r v_\theta}{r}, \frac{Dv_z}{Dt}\right] \quad .$$

If we now write (u, v, w) in place of (v_r, v_θ, v_z), Euler's equations in cylindrical polar coordinates take the form

$$\begin{aligned}
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r, \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \theta} + w\frac{\partial v}{\partial z} + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + F_\theta, \\
&\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho r}\frac{\partial p}{\partial z} + F_z, \\
&\frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0
\end{aligned}$$

2.7 Dynamic or perturbation pressure

If in the Euler equation for an incompressible fluid,

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \mathbf{g} \quad , \tag{2.2}$$

we put $\mathbf{u} = 0$ to represent the equilibrium or rest state,

$$\mathbf{0} = -\frac{1}{\rho}\nabla p_0 + \mathbf{g} \tag{2.3}$$

This is merely the hydrostatic equation

$$\nabla p_0 = \rho \mathbf{g} \quad \text{or} \quad \frac{\partial p_0}{\partial x} = 0, \frac{\partial p_0}{\partial y} = 0, \frac{\partial p_0}{\partial z} = \rho g$$

where p_0 is the hydrostatic pressure. Subtracting (2.2) - (2.3) we obtain

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla(p - p_0) = -\frac{1}{\rho}\nabla p_d$$

where $p_d = p - p_0$ = (total pressure) - (hydrostatic pressure) is known as the **dynamic pressure** (or sometimes, especially in dynamical meteorology, the perturbation pressure). The dynamic pressure is the excess of total pressure over hydrostatic pressure, and is the only part of the pressure field associated with motion.

We shall usually omit the suffix d since it is fairly clear that if **g** is included we are using **total** pressure, and if no **g** appears we are using the dynamic pressure,

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p \quad .$$

2.8 Incompressible viscous fluids

It can be shown that the viscous (frictional) forces in a fluid may be expressed as $\mu\nabla^2\mathbf{u} = \rho\nu\nabla^2\mathbf{u}$ where μ the **coefficient of viscosity** and $\nu = \mu/\rho$ the **kinematic viscosity** provide a measure of the magnitude of the frictional forces in particular fluid, i.e., μ and ν are properties of the fluid and are relatively small in air or water and large in glycerine or heavy oil.

In a viscous fluid the equation of motion for unit mass,

$$\underset{(A)}{\frac{\partial \mathbf{u}}{\partial t}} + \underset{(B)}{(\mathbf{u}.\nabla)\,\mathbf{u}} = \underset{(C)}{-\frac{1}{\rho}\nabla p} + \underset{(D)}{\mathbf{F}} + \underset{(E)}{\nu\nabla^2\mathbf{u}}, \tag{2.4}$$

is known as the **Navier-Stokes' equation**. The various terms are: (A) — local acceleration, (B) — advective acceleration, (C) — pressure gradient force, (D) — body force, and (E) — viscous force.

The viscous term is small relative to other terms except close to boundaries, yet it contains the highest order derivatives $(\partial_x^2\mathbf{u}, \partial_y^2\mathbf{u}, \partial_z^2\mathbf{u})$, and hence determines the number of spatial boundary conditions that must be imposed to determine a solution.

We require also the continuity equation,

$$\nabla.\mathbf{u} = 0,$$

to close the system of four differential equations in four dependent variables.

The Navier-Stokes equation is too difficult for us to handle at present and we shall concentrate on Euler's equation from which we can learn much about fluid flow. Euler's equation is still non-linear, but there are clever methods to bypass this difficulty.

2.9 Boundary conditions for fluid flow

(i) **Solid boundaries**: there can be no normal component of velocity (through the boundary). If friction is neglected there may be free slip along the boundary, but friction has the effect of slowing down fluid near the boundary and it is observed experimentally that there is no relative motion at the boundary, either normal or tangential to the boundary. In fluids with low viscosity this tangential slowing down occurs in a thin **boundary layer**, and in a number of important applications this boundary layer is so thin that it can be neglected and we can say approximately that the fluid slips at the surface; in many other cases the entire boundary layer separates from the boundary and the inviscid model is a very poor approximation.

Thus, in an inviscid flow (also called an ideal fluid) the fluid velocity must be tangential at a rigid body, and:

- for a surface at rest $\mathbf{n}.\mathbf{u} = \mathbf{0}$;
- for a surface with velocity $\mathbf{u}_s$ then $\mathbf{n}.(\mathbf{u} - \mathbf{u}_s) = \mathbf{0}$.

(ii) Free boundaries: at an interface between two fluids (of which one might be water and one air) the pressure must be continuous, or else there would be a finite force on an infinitesimally small element of fluid causing unbounded acceleration; and the component of velocity **normal** to the interface must be continuous. If viscosity is neglected the two fluids may slip over each other. If there is liquid under air, we may take $p = p_0$ =atmospheric pressure at the interface, where p_0 is taken as constant. If surface tension is important there may be a pressure difference across the curved interface.

2.9.1 An alternative boundary condition

As the velocity at a boundary of an **inviscid fluid** must be wholly tangential, it follows that a fluid particle once at the surface must always remain at the surface. Hence for a surface or boundary with equation

$$F(x, y, z, t) = 0 \quad ,$$

if the coordinates of a fluid particle satisfy this equation at one instant, they must satisfy it always. Hence, moving with the fluid at the boundary,

$$\frac{DF}{Dt} = 0$$

or

$$\frac{\partial F}{\partial t} + \mathbf{u}.\nabla F = 0,$$

as F must remain zero for all time for each particle at the surface.

2.10 Exercises

Question 2.1*

Under what condition is the advective rate-of-change equal to the total rate-of-change?

Question 2.2*

Show that, in hydrostatic equilibrium, the pressure and density in an isothermal (i.e. constant temperature) atmosphere vary with height according to the formulae

$$p(z) = p(0)\exp(-z/H_s) \quad \text{and} \quad \rho(z) = \rho(0)\exp(-z/H_s).$$

where $H_s = RT/g$ and z points vertically upwards. Show that for realistic values of T in the troposphere and $R = 287$ J K^{-1}kg^{-1}, the e-folding height scale H_s is on the order of 8 km.

Question 2.3*

The vector differential operator del (or nabla) is defined as

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

in rectangular cartesian coordinates. Express in full cartesian form the quantities: $\nabla.\mathbf{u}$, $\nabla \times \mathbf{u}$, $\mathbf{u}.\nabla$,$\nabla.\nabla$, and identify each.

Question 2.4*

Consider a homogeneous, incompressible fluid with velocity $\mathbf{u} = (\alpha x, -\alpha y, 0)$ where α is constant. Further suppose that the concentration of a certain pollutant in the fluid is found to be

$$c(x, y, z, t) = c_0 + \gamma t + \beta x^2 y e^{-\alpha t},$$

where c_0, γ and β are constants.

What is the (time) rate of change of the concentration of the pollutant at a particular point? How does the concentration change following fluid parcels? Consider a parcel at the point $(1, 1, 0)$ at $t = 0$. What is its concentration at $t = 1$?

Question 2.5

Show that

$$\frac{\partial}{\partial x}\left(\frac{Df}{Dt}\right) = \frac{D}{Dt}\left(\frac{\partial f}{\partial x}\right) + \frac{\partial \mathbf{u}}{\partial x}\cdot\nabla f.$$

Clarify the meaning of the final term using index notation.

Question 2.6

You are on a small raft floating in a pool, and you throw a heavy anchor overboard. Does the water level of the pool rise, fall or remain the same? *Carefully* justify your answer.

Question 2.7

Is the velocity field $\mathbf{u} = \left(\cos x, xz^2, z \sin x\right)$ kinematically admissible for an incompressible fluid ? Explain your answer ?

Question 2.8

Find the pressure field in the inviscid, incompressible flow with velocity field $\mathbf{u} = (kx, -ky, 0)$, where k is a constant.

Question 2.9

Show that $\dot{\hat{\mathbf{r}}} = \hat{\mathbf{n}}\dot{\theta}$ and $\dot{\hat{\mathbf{n}}} = -\hat{\mathbf{r}}\dot{\theta}$.

Question 2.10

Show that the continuity equation in cylindrical polar coordinates is

$$\frac{1}{r}\frac{\partial\left(ru\right)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0.$$

Chapter 3

Steady Inviscid Flows

3.1 Bernoulli's equation

For steady inviscid flow under external forces which have a potential Ω such that **F**=-$\nabla\Omega$ the Euler equation reduces to

$$(\mathbf{u}.\nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Omega \quad ,$$

and for incompressible fluids

$$(\mathbf{u}.\nabla)\,\mathbf{u} + \frac{1}{\rho}\nabla(p+\rho\Omega) = \mathbf{0} \quad .$$

We may regard $p+\rho\Omega$ as a more general **dynamic pressure**; but for the particular case of gravitation potential, $\Omega = gz$, and $\mathbf{F} = -\nabla\Omega = -(0,0,g) = -g\mathbf{k}$.

We note that

$$\begin{aligned}\mathbf{u}.(\mathbf{u}.\nabla)\mathbf{u} &= u(\mathbf{u}.\nabla)u + v(\mathbf{u}.\nabla)v + w(\mathbf{u}.\nabla)w \\ &= \mathbf{u}.\nabla\tfrac{1}{2}(u^2+v^2+w^2) \\ &= (\mathbf{u}.\nabla)\tfrac{1}{2}\mathbf{u}^2\end{aligned}$$

using the fact that $\mathbf{u}.\nabla$ is a scalar differential operator. Hence,

$$\mathbf{u}.\left[(\mathbf{u}.\nabla)\,\mathbf{u} + \frac{1}{\rho}\nabla\,(p+\rho\Omega)\right] = \mathbf{u}.\nabla\left(\frac{1}{2}\mathbf{u}^2 + \frac{p}{\rho} + \Omega\right) = 0 \quad ,$$

and it follows that $(\frac{1}{2}\mathbf{u}^2 + p/\rho + \Omega)$ is constant along each streamline (as $\mathbf{u}.\nabla$ is the derivative in the direction $\mathbf{u}$, and hence proportional to the rate of change in the direction $\mathbf{u}$ of streamlines).

Thus for steady, incompressible, inviscid flow, Bernoulli's equation states

$$\frac{1}{2}\mathbf{u}^2 + \frac{p}{\rho} + \Omega \quad = \text{constant on a streamline,} \tag{3.1}$$

although the constant will generally be different on each different streamline.

Typically, Ω might represent the gravitational potential gz, or, in a rotating reference frame, $\Omega = gz - \omega^2 r^2/2$

Example 3.1

In a 'static' fluid in a rotating reference frame, verify that this definition of Ω predicts that the free surface take the form of a parabola

3.2 Applications of Bernoulli's equation

3.2.1 Draining a reservoir through a small hole

If the draining opening is of much smaller cross-section than the reservoir, the water surface in the tank will fall very slowly and the flow may be regarded as approximately steady. We may take the outflow speed u_A as approximately uniform across the jet and the pressure p_A uniform across the jet and equal to the atmospheric pressure p_0 outside the jet (for, if this were not so, there would be a difference in pressure across the surface of the jet, and this would accelerate the jet surface radially, which is not observed, although the jet is accelerated downwards by its weight). Hence, on the streamline AB,

$$\frac{1}{2}u_A^2 + \frac{p_0}{\rho} = \tfrac{1}{2}u_B^2 + \tfrac{p_0}{\rho} + gh \quad ,$$

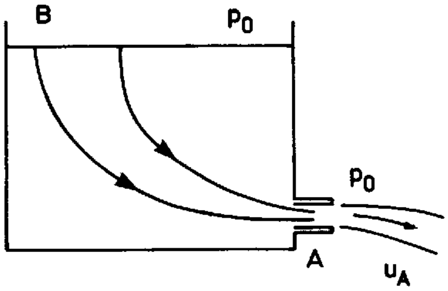


Figure 3.1: Flow from a draining tank

and as $u_B \ll u_A$ then $u_A = \sqrt{2gh}$.

This is known as Toricelli's theorem. Note that the outflow speed is that of free fall from B under gravity; this clearly neglects any viscous dissipation of energy.

3.2.2 Bluff body in a stream; Pitot tube

Suppose that a stream has uniform speed U_0 and pressure p_0 far from any obstacle, and that it then flows round a bluff body. The flow must be slowed down in front of the body and there must be one **dividing streamline** separating fluid which follows past one side of the body or the other. This dividing streamline must end on the body at a **stagnation point** at which the velocity is zero and the pressure

$$p = p_0 + \frac{1}{2}\rho U_0^2.$$

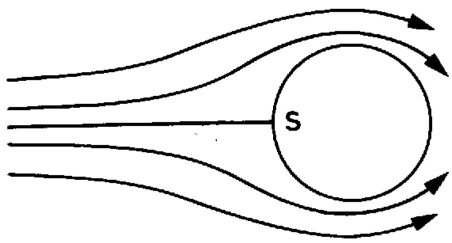


Figure 3.2: Flow around an obstruction

This provides the basis for the Pitot tube in which a pressure measurement is used to obtain the free stream velocity U_0. The pressure $p = p_0 + \frac{1}{2}\rho U_0^2$ is the **total** or **Pitot pressure** (also known as the **total head**) of the free stream, and differs from the static pressure p_0 by the dynamic pressure $\frac{1}{2}\rho U_0^2$. The Pitot tube consists of a tube directed into the stream with a small central hole connected to a **manometer** for measuring pressure difference $p - p_0$. At equilibrium there is no flow through the tube, and hence the left hand pressure on the manometer is the total pressure $p_0 + \frac{1}{2}\rho U_0^2$. The static pressure p_0 can be obtained from a **static tube** which is normal to the flow.

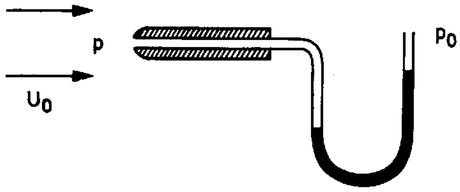


Figure 3.3: Pressure measurement with Pitot tube

3.3 Exercises

Question 3.1

Prove the vector identity

$$\nabla(\mathbf{a}.\mathbf{b}) = (\mathbf{b}.\nabla)\,\mathbf{a} + (\mathbf{a}.\nabla)\,\mathbf{b} + \mathbf{b}\times(\nabla\times\mathbf{a}) + \mathbf{a}\times(\nabla\times\mathbf{b})\,.$$

Hence show that

$$(\mathbf{u}.\nabla)\,\mathbf{u} = \nabla\left(\tfrac{1}{2}\mathbf{u}^2\right) - \mathbf{u}\times(\nabla\times\mathbf{u}) = \nabla\left(\tfrac{1}{2}\mathbf{u}^2\right) - \mathbf{u}\times\boldsymbol{\omega},$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is known as the **vorticity**.

Question 3.2

A uniform straight open rectangular channel carries a water flow of mean speed U and depth h. The channel has a constriction, which reduces its width by half and it is observed that the depth of water in the constriction is only $\frac{1}{2}h$. By applying Bernoulli's theorem to a surface streamline find U in terms of g and h.

Question 3.3

Explain why there is an increase in pressure on the side of a building facing the wind.

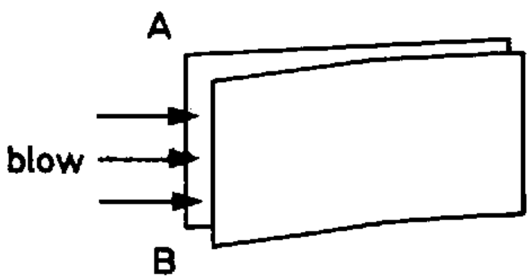
Question 3.4

Calculate the curl and divergence of the Cartesian velocity

$$\mathbf{u} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r}\right],$$

where $r = |(x, y, z)|$. Is this flow incompressible or irrotational?

Question 3.5



Hold two sheets of paper at A and B with a finger between the two at top and bottom, and blow between the sheets as illustrated in the diagram. The trailing edges of the sheets will not move apart as you might have anticipated, but together. Explain this in terms of Bernoulli's equation, assuming the flow to be steady.

Question 3.6

Using Bernoulli's equation (often referred to as Bernoulli's theorem):

1. show that air from a balloon at excess pressure p_1 above atmospheric will emerge with approximate speed $\sqrt{2p_1/\rho}$;
2. find the depth of water in the steady state in which a vessel, with a waste pipe of length 0.01 m and cross-sectional area $2 \times 10^{-5}\mathrm{m}^2$ protruding vertically below its base, is filled at the constant rate $3 \times 10^{-5}\mathrm{m}^3\mathrm{s}^{-1}$.

Question 3.7

A vertical round post stands in a river, and it is observed that the water level at the upstream face of the post is slightly higher than the level at some distance to either side. Explain why this is so, and find the increase in the height in terms of the surface stream speed U and acceleration of gravity g. Estimate the increase in height for a stream with undisturbed surface speed 1 ms^{-1}

Chapter 4

Vortex Motion

Vortex:*noun* (pl. vortexes or vortices)

A mass of whirling fluid or air esp. a whirlpool or whirlwind; cyclone, gyre, maelstrom, eddy, swirl, spiral; *figurative* something regarded as a whirling mass: *the vortex of existence*.

4.1 The vorticity field

The vector $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ is twice the local angular velocity in the flow, and $\boldsymbol{\omega}$ is termed the **vorticity** of the flow (from Latin for a whirlpool).

Vortex lines are everywhere in the direction of the vorticity field (c.f. streamlines); bundles of vortex lines make up **vortex tubes**; and thin vortex tubes, such that their constituent vortex lines are approximately parallel to the tube axis are **vortex filaments** (see below).

The vorticity field is **solenoidal**, i.e.

$$\nabla.\boldsymbol{\omega} = 0,$$

since

$$\begin{aligned}
\nabla.\boldsymbol{\omega} &= \nabla.\left(\nabla \times \mathbf{u}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \\
&= 0.
\end{aligned}$$

From the divergence theorem, for any volume V with boundary surface S,

$$\int_S \boldsymbol{\omega}.\mathbf{n} dS = \int_V \nabla.\boldsymbol{\omega} dV = 0,$$

and there is zero net flux of vorticity (or vortex tubes) out of any volume: hence there can be no sources of vorticity in the interior of a fluid.

Consider a length $P_1 P_2$ of vortex tube. From the divergence theorem

$$\int_S \boldsymbol{\omega}.\mathbf{n} dS = \int_V \nabla.\boldsymbol{\omega} dV = 0.$$

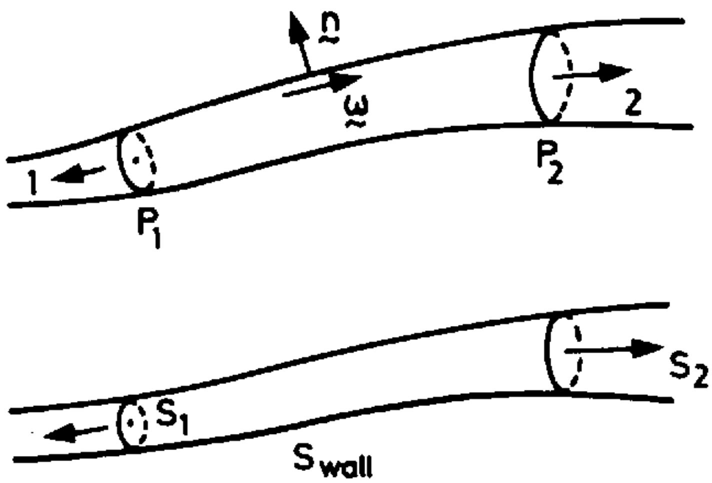


Figure 4.1

We can divide the surface of the length P_1P_2 into cross-sections and the tube wall, $S = S_1 + S_2 + S_{wall}$, then

$$\int_S \boldsymbol{\omega}.\mathbf{n}dS = \int_{S_1} \boldsymbol{\omega}.\mathbf{n}dS + \int_{S_2} \boldsymbol{\omega}.\mathbf{n}dS + \int_{S_{wall}} \boldsymbol{\omega}.\mathbf{n}dS = 0$$

However, the contribution from the wall (where $\boldsymbol{\omega}\perp\mathbf{n}$) is zero, and hence

$$\int_{S_1} \boldsymbol{\omega}.\mathbf{n}dS = \int_{S_2} \boldsymbol{\omega}.\left(-\mathbf{n}\right)dS$$

where the positive sense for normals is that of increasing distance along the tube from the origin. Hence $\int_S \boldsymbol{\omega}.\mathbf{n}dS$ measured over a cross-section of the vortex tube with **n** taken in the same sense is constant, and defined as the **strength of the vortex tube**.

In a **thin** vortex tube, we have approximately:

$$\int_S \boldsymbol{\omega}.\mathbf{n}dS \approx (\boldsymbol{\omega}.\mathbf{n}) \int_S dS = \omega S,$$

and $\omega \times$ area = constant along the tube (a property of all solenoidal fields), where $\omega = |\boldsymbol{\omega}|$.

Example 4.1

Show that $(\frac{1}{2}\mathbf{u}^2 + p/\rho + \Omega)$ is constant along a vortex line for steady, incompressible, inviscid flow under conservative external forces.

4.2 Circulation

From Stokes' theorem

$$\int_S (\nabla \times \mathbf{u}).\mathbf{n}dS = \oint_C \mathbf{u}.\mathbf{dx},$$

where $\mathbf{n}$ is the unit normal to S, with the orientation of C and the direction of $\mathbf{n}$ being given by the right hand rule. Hence the line integral of the velocity field in any circuit C that passes once round a vortex tube is equal to the total vorticity cutting any cap S on C, and is therefore equal to the strength of the vortex tube. We measure the strength of a vortex tube by calculating $\oint_C \mathbf{u}.\mathbf{dx}$ around **any circuit** C enclosing the tube once only. The quantity

$$\Gamma = \oint_C \mathbf{u}.\mathbf{dx}$$

is termed the **circulation**. Vorticity may be regarded as **circulation per unit area**, and the component in any direction of $\boldsymbol{\omega}$ is

$$\lim_{S\to 0} \frac{1}{S} \oint_C \mathbf{u}.\mathbf{dx}$$

where C is a loop of area S perpendicular to the direction specified.

4.3 The Helmholtz equation for vorticity

From Euler's equation for an incompressible fluid in a conservative force field

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p - \nabla\Omega$$

or

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla\left(\frac{1}{2}\mathbf{u}^2\right) - \mathbf{u}\times\boldsymbol{\omega} = -\frac{1}{\rho}\nabla p - \nabla\Omega;$$

taking the curl,

$$\nabla\times\left(\frac{\partial \mathbf{u}}{\partial t}\right) + \nabla\times\nabla\left(\frac{1}{2}\mathbf{u}^2\right) - \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = -\nabla\times\nabla\left(\frac{p}{\rho}+\Omega\right).$$

But $\nabla\times\nabla\chi = 0$ for all χ, and

$$\nabla\times(\mathbf{u}\times\boldsymbol{\omega}) = \mathbf{u}\,(\nabla.\boldsymbol{\omega}) - \boldsymbol{\omega}\,(\nabla.\mathbf{u}) + (\boldsymbol{\omega}.\nabla)\,\mathbf{u} - (\mathbf{u}.\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}.\nabla)\,\mathbf{u} - (\mathbf{u}.\nabla)\,\boldsymbol{\omega}$$

as $\boldsymbol{\omega}$ is always solenoidal and $\mathbf{u}$ is solenoidal in an incompressible fluid. Hence we obtain

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}.\nabla)\,\boldsymbol{\omega} = (\boldsymbol{\omega}.\nabla)\,\mathbf{u},$$

which is the **Helmholtz vorticity equation.**

4.3.1 Physical significance of $(\boldsymbol{\omega}.\nabla)\,\mathbf{u}$

We can understand the significance of the term $(\boldsymbol{\omega}.\nabla)\,\mathbf{u}$ in the Helmholtz equation by recalling that ∇ is a directional derivative and $(\boldsymbol{\omega}.\nabla)$ is proportional to the derivative in the direction of $\boldsymbol{\omega}$ along the vortex line:

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega}.\nabla)\,\mathbf{u} = \omega\hat{\boldsymbol{\omega}}.\nabla\mathbf{u} = \omega\frac{\partial\mathbf{u}}{\partial s},$$

where $\delta\mathbf{s}$ is the length of an element of vortex tube. We now resolve $\mathbf{u}$ into components $\mathbf{u}_\omega$ parallel to $\boldsymbol{\omega}$ and $\mathbf{u}_\perp$ at right

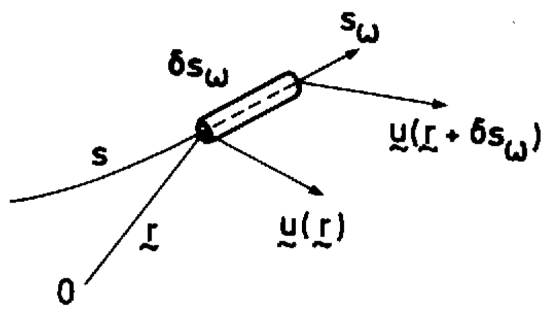


Figure 4.2

angles to $\boldsymbol{\omega}$ and hence to $\delta\mathbf{s}$. Then

$$\begin{aligned}\frac{\delta s}{\omega}\frac{Dw}{Dt} &= \frac{\partial}{\partial s}\left(\mathbf{u}_\omega + \mathbf{u}_\perp\right)\delta s \\ &= \frac{\partial \mathbf{u}_\omega}{\partial s}\delta s + \frac{\partial \mathbf{u}_\perp}{\partial s}\delta s \\ &= \left[\mathbf{u}_\omega\left(\mathbf{x}+d\mathbf{s}\right) - \mathbf{u}_\omega\left(\mathbf{x}\right)\right] + \left[\mathbf{u}_\perp\left(\mathbf{x}+d\mathbf{s}\right) - \mathbf{u}_\perp\left(\mathbf{x}\right)\right] \\ & \qquad\qquad (\mathbf{A}) \qquad\qquad\qquad\qquad (\mathbf{B})\end{aligned}$$

(A) Rate of stretching of an element: **stretching** along the length of the filament causes relative amplification of the vorticity field;

(B) Rate of turning of an element: **turning** away from the line of the filament causes a reduction of the vorticity in that direction, but an increase in the new direction. It also can include the twisting of the element by the flow which directly changes the vorticity.

4.3.2 In the presence of viscosity

The addition of viscosity to the equation of motion gives the Navier-Stokes equation (2.4) which is simply the Euler equation with the addition of $\nu\nabla^2\boldsymbol{u}$. Taking the curl of Navier-Stokes equation gives

$$\frac{D\boldsymbol{\omega}}{Dt} = \left(\boldsymbol{\omega}.\nabla\right)\mathbf{u} + \nabla\times\nu\nabla^2\boldsymbol{u} = \left(\boldsymbol{\omega}.\nabla\right)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$

(viscosity is constant in this form of the Navier-Stokes equation).

Where the first term on the right hand side distorts a vortex tube, the second term represents diffusion of the vorticity. This is most clearly demonstrated by considering a 2d flow.

In 2d, $\boldsymbol{u}.\boldsymbol{\omega} = 0$, because the vorticity vector is directed out of the plane containing the flow. Hence

$$\frac{D\omega}{Dt} = \nu\nabla^2\omega,$$

a scalar diffusion equation for the magnitude of the vorticity.

4.4 Kelvin's Theorem

The ideas of vorticity and circulation are important because of the permanence of circulation under deformation of the flow due to pressure forces. We next look at the rate of change of circulation round a circuit moving with an incompressible, inviscid fluid:

$$\begin{aligned}\frac{D}{Dt}\oint \mathbf{u}.\mathbf{dx} &= \oint \frac{D}{Dt}\left(\mathbf{u}.\mathbf{dx}\right) \\ &= \oint \frac{D\mathbf{u}}{Dt}.\mathbf{dx} + \oint \mathbf{u}.\frac{D\mathbf{dx}}{Dt}\end{aligned}$$

But $\oint \frac{D\mathbf{u}}{Dt}.\mathbf{dx} = \oint -\nabla\left(p/\rho + \Omega\right).\mathbf{dx}$ and $\oint \mathbf{u}.\frac{D\mathbf{dx}}{Dt} = \oint \mathbf{u}.\mathbf{du}$ (see Example below).

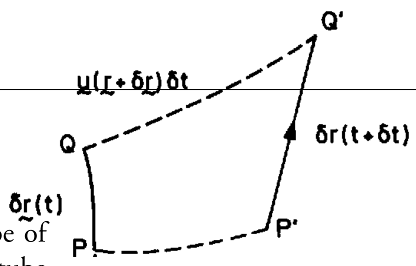
Hence

$$\begin{aligned}\frac{D\Gamma}{Dt} &= -\oint \nabla\left(p/\rho + \Omega\right).\mathbf{dx} + \oint \mathbf{u}.\mathbf{du}\\ &= \oint d\left(-p\rho - \Omega + \tfrac{1}{2}\mathbf{u}^2\right)\\ &= 0\end{aligned}$$

as $-p/\rho - \Omega + \frac{1}{2}\mathbf{u}^2$ returns to its initial value after one circuit since it is a single valued function.

Example 4.2

Show, with the help of the diagram in Figure 4.2 that $\oint \mathbf{u}.\frac{D\mathbf{dx}}{Dt} = \oint \mathbf{u}.\mathbf{du}$.



4.4.1 Helmholtz theorem: vortex lines move with the fluid

Consider a tube of particles T, which at the instant t forms a vortex tube of strength κ. At that time the circulation round any circuit C' lying in the tube wall, but *not* linking (i.e. embracing) the tube is zero, while that in an circuit C linking the tube *once* is κ. These circulations suffer no change moving with the fluid: hence the circulation in C' remains zero and that in C remains κ, i.e. the fluid comprising the vortex tube at T continues to comprise a vortex tube (as the vorticity component normal to the tube wall — measured in C' — is always zero), *and* the strength of the vortex remains constant. A vortex line is the limiting case of a small vortex tube: hence vortex lines move with (are frozen into) inviscid fluids.

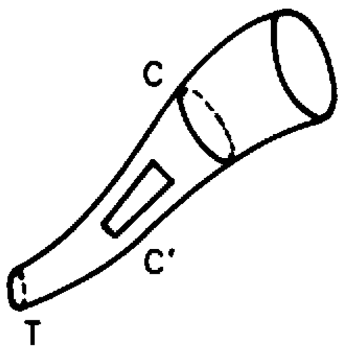


Figure 4.4

4.4.2 A flow which is initially irrotational remains irrotational

Circulation is advected with the fluid in inviscid flows, and vorticity is "circulation per unit area". If initially $\Gamma = 0$ for all closed circuits in some region of flow, it must remain so for all subsequent times. Motion started from rest is initially irrotational (free from vorticity) and will therefore remain irrotational provided that it is inviscid.

The direction of the vorticity turns as the vortex line turns, and its magnitude increases as the vortex line is stretched.

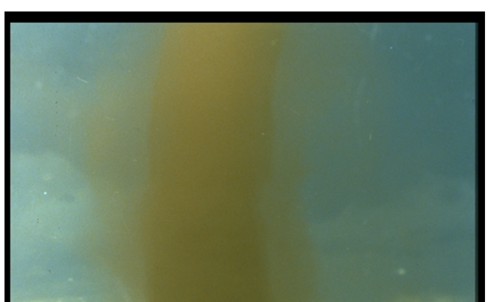
The circulation round a thin vortex tube remains the same; as it stretches the area of section decreases and

Circulation / Area = Vorticity

increases in proportion to the stretch.

4.5 Rotational and irrotational flow

Flow in which the vorticity is everywhere zero $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ is called **irrotational**. Other terms in use are **vortex free, ideal, and perfect**. Much of fluid dynamics used to be concerned with analysing irrotational flows



and deciding where these give a good representation of real flows, and where they are quite wrong.

Example 4.3

Consider a small circular element of fluid in solid body rotation with angular velocity Ω. Using cylindrical polar coordinates, show that the circulation around the circular element is

$$\Gamma = 2\Omega\pi r^2,$$

and that the vorticity is

$$\boldsymbol{\omega} = 2\Omega\mathbf{k}.$$

Example 4.4

Consider the shear flow **u** = (Λz,0,0). Show that although the streamlines are straight and parallel to the x-axis, the flow is not irrotational.

We have neglected **compressibility** and **viscosity**. It can be shown that the neglect of compressibility is not very serious even at moderately high speeds, but the effect of neglecting viscosity can be disastrous. Viscosity **diffuses** the vorticity (much as conductivity diffuses heat) and progressively blurs the results derived above, the errors increasing with time.

There is no term in the Helmholtz equation $D\boldsymbol{\omega}/Dt = (\boldsymbol{\omega}.\nabla)\,\mathbf{u}$ corresponding to the generation of vorticity (the term $\boldsymbol{\omega}.\nabla\mathbf{u}$ represents processing by stretching and turning of vorticity already present). It follows, therefore, that in homogeneous fluids all **vorticity must be generated at boundaries** (more on this later). In real (viscous) fluids, this vorticity is carried away from the boundary by diffusion and is then advected into the body of the flow. But in inviscid flow vorticity cannot leave the surface by diffusion, nor can it leave by advection with the fluid as no fluid particles can leave the surface. It is this inability of inviscid flows to model the diffusion/advection of vorticity generated at boundaries out into the body of the flow that causes most of the failures of the model.

In inviscid flows we are left with a free slip velocity at the boundaries, which we may interpret as a thin vortex sheet wrapped around the boundary.

4.6 Vortex sheets

Consider a thin layer of thickness δ in which the vorticity is large and is directed along the layer (parallel to $0y$), as sketched. The vorticity is

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \quad ,$$

where $\partial u/\partial z$ is large (but not $\partial w/\partial x$, which would lead to very large w).

We can suppose that within the vortex layer $u = u_0 + \omega z$ changing from u_0 to $u_0 + \omega\delta$ between $z = 0$ and δ, with mean vorticity

$$\frac{(u_0 + \omega\delta) - u_0}{\delta} = \omega$$

This vortex layer provides a sort of roller action, though it is not of course rigid, and it also suffers high rate-of-strain.

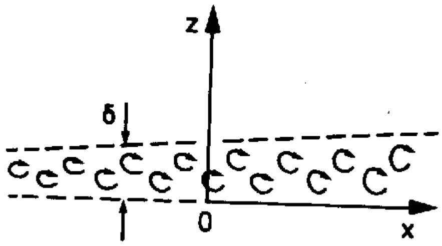


Figure 4.6

If we idealize this vortex layer by taking the limit $\delta \to 0$, $\omega \to \infty$, with $\omega\delta$ remaining finite, we obtain a **vortex sheet**, which is manifest only through the free slip velocity. Such vortex sheets follow the contours of the boundary and clearly may be curved. They are infinitely thin sheets of vorticity with infinite magnitude across which there is finite difference in tangential velocity, $\omega\delta$.

4.7 Line vortices

We can represent approximately also strong thin vortex tubes (e.g. tornadoes, waterspouts, draining vortices) by **vortex lines** without thickness. The circulation in a circuit round the tube tends to a definite non-zero limit as the circuit area $S \to 0$. If the flow outside the vortex is irrotational then all circuits round the vortex have the same circulation (by Stokes' theorem), the strength κ of the vortex being

$$\oint \mathbf{u}.\mathbf{dx} \to \kappa \quad \text{as} \quad C \to 0.$$

As the flow outside the vortex is irrotational, the circulation is zero around any circuit that does not enclose the vortex.

Example 4.5

Consider an isolated line vortex aligned with the z-axis. Show that a radially symmetric solution is

$$u = 0, \quad v = \frac{\kappa}{2\pi r},$$

where κ is the circulation around a circuit enclosing the z-axis.

The effect of viscosity is to thicken vortex sheets and line vortices by diffusion; however, the effect of diffusion is often slow relative to that of advection by the flow, and as a result large regions of flow will often remain free from vorticity. Vortex sheets at surfaces diffuse to form **boundary layers** in contact with the surfaces; or if free they often break up into line vortices. Boundary layers on bluff bodies often *separate* or break away from the body, forming a **wake** of rotational, retarded flow behind the body, and it is these wakes that are associated with the **drag** on the body. (More on separation later.)

4.8 Motion started from rest impulsively

Viscosity (which is really just distributed internal fluid friction) is responsible for retarding or damping forces which cannot begin to act until the motion has started; i.e. **take time to act**. Hence any flow will be **initially** irrotational everywhere except at actual boundaries. Within increasing time, vorticity will be diffused from boundaries and advected and diffused out into the flow.

Motion started from rest by an **instantaneous impulse** must be irrotational. If we integrate the Euler equation over the time interval $(0, \delta t)$:

$$\int_0^{\delta t} \frac{D\mathbf{u}}{Dt} dt = \int_0^{\delta t} \mathbf{F} dt - \int_0^{\delta t} \frac{\nabla p}{\rho} dt$$

or

$$[\mathbf{u}]_0^\delta = \int_0^{\delta t} \mathbf{F} dt - \frac{1}{\rho} \nabla \int_0^{\delta t} p dt.$$

In the limit $\delta t \to 0$ for start-up by an instantaneous impulse, the impulse of the body force $\to 0$ (as the body force is unaffected by the impulsive nature of the start) and

$$\mathbf{u} - \mathbf{u}_0 = -\frac{1}{\rho} \nabla P,$$

where the fluid responds instantaneously with the impulsive pressure field $P = \int_0^{\delta t} p dt$, and the impulse on a fluid element is $-\nabla P$ per unit volume, producing a velocity from rest of

$$\mathbf{u} = -\frac{1}{\rho} \nabla P. \tag{4.1}$$

This is irrotational as $\nabla \times \mathbf{u} = -\frac{1}{\rho} \nabla \times \nabla P = 0$.

4.9 Generation of circulation at boundaries

Consider now how circulation (or vorticity) is generated. An infinite plate moves to the right with speed U. Consider the circulation around a small circuit $ABCD$ centred about the point O on the surface of the plate, where $\overline{BC}/\overline{AB} \ll 1$. The circulation is

$$\begin{aligned} \int_{ABCDA} \mathbf{u}.\mathbf{dx} &= \int_{A'A} w \, dz + \int_{AB} (u - U) \, dx + \int_{BB'} w dz \\ &= \int_{-\delta x/2}^{\delta x/2} (u - U) \, dx \\ &= (u - U) \, \delta x \end{aligned}$$

Hence, the circulation **per unit length** (in the x direction) is $L = (u - U)$. Had the boundary been positioned on top of the fluid instead of below, we would have found the circulation per unit length to be $-(u - U)$.

Taking the material derivative of this expression and using the x-component of the Navier-Stokes equation gives,

$$\frac{DL}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u - \frac{dU}{dt}, \tag{4.2}$$

at the lower boundary. We see that the generation of circulation per unit length (and hence vorticity) at a boundary is due solely to the tangential pressure gradient and the acceleration of the boundary. The viscous diffusion simply redistributes the vorticity.

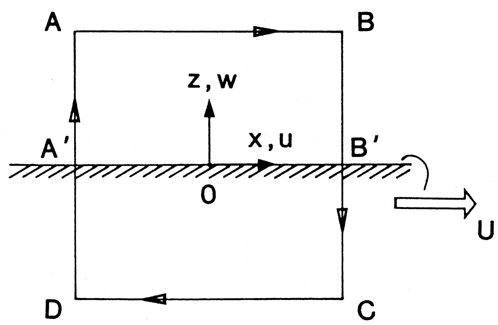


Figure 4.7

4.10 Exercises

Question 4.1*

Consider the shear flow $\mathbf{u} = (\Lambda z, 0, 0)$. Choose any initially rectangular or circular contour and calculate its position at arbitrary time t.

Calculate the circulation at $t = 0$ and at subsequent times.

Show that the circulation per unit area is Λ and is conserved following the flow.

Question 4.2*

Assume that a tornado can be approximated as a line vortex in which the velocity decays as (distance)$^{-1}$. Suppose that the tangential wind is 20 ms^{-1} at a radius of 2 km. What is the circulation? What is the wind speed at a radius of 500 m?

Question 4.3*

Does fluid with velocity

$$\mathbf{u} = \left[z - \frac{2x}{r}, 2y - 3z - \frac{2y}{r}, x - 3y - \frac{2z}{r}\right]$$

possess vorticity ?

(Here, $\mathbf{u} = (u, v, w)$ is the velocity in the Cartesian frame, $\mathbf{r} = (x, y, z)$ and $r^2 = x^2 + y^2 + z^2$)

What is the circulation in the circle $x^2 + y^2 = 9, z = 0$?

Is this flow incompressible?

Question 4.4

Given a velocity field $\mathbf{u} = (xy, 0, 0)$, calculate the vorticity.

Now calculate the circulation around the rectangular circuit $(1, 1, 0)$, $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$.

Is the flow irrotational? Why?

Question 4.5*

Consider the closed circuit $C(t)$ which at $t = 0$ is given by

$$\mathbf{x} = (a \cos s, a \sin s, 0), \quad 0 \leq s \leq 2\pi,$$

and is advected by the velocity

$$\mathbf{u} = (\alpha y, 0, 0).$$

Determine the position of C in terms of the initial label s and show that the circulation

$$\Gamma = \oint_{C(t)} \mathbf{u}.\mathbf{dx} = \int_0^{2\pi} \mathbf{u}.\frac{\partial \mathbf{x}}{\partial s} ds,$$

is conserved.

Question 4.6*

Prove that for two-dimensional motion in cylindrical polar coordinates given by $\mathbf{u} = (u, v, 0)$, the vorticity is:

$$\boldsymbol{\omega} = \left(0, 0, \frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right)\right)$$

Question 4.7

Inviscid fluid occupies the region $x \geq 0$, $y \geq 0$ bounded by two rigid boundaries $x = 0$, $y = 0$. Its motion results wholly from the presence of a line vortex, which itself moves according to the Helmholtz vortex theorem. Show that the condition of zero transverse velocity on the boundaries can be satisfied by introducing vortices of strength $-\kappa$ at $(-x, y)$ and $(x, -y)$ and a vortex of strength κ at $(-x, -y)$. Determine the velocity due to these "image vortices" on the original vortex and show that the path taken by the vortex is

$$\frac{1}{x^2} + \frac{1}{y^2} = \text{const.}$$

Chapter 5

One Dimensional Viscous Flows

A viscous flow is one in which the fluid immediately above some level exerts a stress on the fluid below it and visa versa. An inviscid fluid is one in which this stress has no tangential component.

Newtonian fluids are those for which the shear stress τ is proportional to the velocity gradient, i.e.,

$$\tau = \mu \frac{du}{dz},$$

where μ is the coefficient of viscosity. The kinematic viscosity is $\nu = \mu/\rho$.

Roughly speaking, viscosity is more important close to the boundary than in the overlying fluid because it is close to the boundary that the velocity gradients are largest. In general the velocity gradients are largest near the boundary because the fluid must satisfy a no slip boundary condition.

We examine now a class of viscous flow for which exact solutions can be found.

5.1 One-dimensional flows

The central difficulty of solving the Navier-Stokes equations lies in the non-linear term $\mathbf{u}.\nabla\mathbf{u}$. However, in one-dimensional flows only a single component of velocity is non-zero, say $\mathbf{u} = (0, 0, w)$. Then from the continuity equation,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

whence w is independent of z. It follows that

$$\begin{aligned}
\mathbf{u}.\nabla\mathbf{u} &= \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(u, v, w) \\
&= \left(0, 0, w\frac{\partial w}{\partial z}\right) \\
&= \mathbf{0}
\end{aligned}$$

and in this case many of the problems of solution disappear.

5.2 Steady plane Couette flow

Consider a steady ($\partial/\partial t = 0$) viscous flow confined between two rigid plates, one at $z = 0$ and the other at $z = h$. Let the lower boundary $z = 0$ be fixed; while driving the upper boundary $z = h$ in its own plane with velocity $(U, 0, 0)$. Suppose that the flow independent of the y coordinate ($\partial/\partial y = 0$ and $v = 0$). Assume that all x-positions are identical so that there can be no change in the x-direction ($\partial/\partial x = 0$). The solution to this problem is called *Couette* flow and is one of the classical problems in fluid dynamics.

From continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

giving

$$\frac{\partial w}{\partial z} = 0.$$

Hence w is independent of z as well as x, y and t. Since $w = 0$ on the bottom boundary, it must vanish everywhere. Consequently $\mathbf{u} = [u(z), 0, 0]$.

The Navier-Stokes equations reduce to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dz^2}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z},$$

where p, the dynamic pressure, is independent of x, y and z (although there is, of course, a hydrostatic pressure satisfying $\partial p/\partial z = -\rho g$. It follows that

$$\frac{d^2u}{dz^2} = 0,$$

from which $u = Az + B$.

But at $z = 0, u = 0 \Rightarrow B = 0$ and at $z = h, u = U \Rightarrow A = U/h$. Hence

$$u = U\frac{z}{h}.$$

The velocity profile is linear and the vorticity is therefore constant throughout the fluid. The vorticity is generated as the upper boundary is set in motion and subsequently diffuses across the channel.

The shearing stress is $\tau = \mu(du/dz) = \mu U/h$. This result implies that in steady motion the tangential stress exerted *by* the upper plate is transmitted across the channel without change in value to the lower plate (c.f. fluid transmission).

5.3 Steady plane Poiseuille flow

Steady plane Poiseuille flow is driven between parallel fixed walls by an applied pressure gradient, and was studied initially in connection with blood flow. At the boundaries $\mathbf{u} = \mathbf{0}$ at $z = 0$ and $z = h$. As in Couette flow the velocity field is assumed to be independent of x and y, which from continuity and the lower boundary condition implies that $w = 0$. Consequently, $\mathbf{u} = [u(z), 0, 0]$. However, if the flow is driven left to right there must be a corresponding drop in (dynamic) pressure (i.e. $\partial p/\partial x < 0$).

The Navier-Stokes equations reduce exactly as for Couette flow to

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{d^2u}{dz^2}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Thus, p is independent of t, y and z so that $p = p(x)$, whence

$$\frac{d^2u}{dz^2} = \frac{1}{\rho\nu}\frac{dp}{dx}.$$

But d^2u/dz^2 is a function of z only, and dp/dx a function of x only. Consequently, the only possibility is that both are equal to a constant, $-\gamma$ (say). Thus

$$\frac{d^2u}{dz^2} = -\frac{\gamma}{\mu},$$

from which

$$u = -\frac{\gamma}{2\mu}z^2 + Az + B,$$

where

$$0 = -\frac{\gamma}{2\mu}.0 + A.0 + B \quad \Rightarrow \quad B = 0,$$

and

$$0 = -\frac{\gamma}{2\mu}h^2 + Ah \quad \Rightarrow \quad A = \frac{\gamma h}{2\mu}.$$

Therefore,

$$u = \frac{\gamma}{2\mu}z(h - z),$$

which defines a quadratic profile.

The shearing stress $\tau = \gamma(h - 2z)/2$ is zero at the centre plane of the channel and takes its maximum values at the wall. In this case shearing stress is transmitted out to both walls, acting forwards along each (in this sign convention), and serving to transmit the pressure gradient force out to the walls, allowing steady flow (without acceleration).

The vorticity is $\eta = \gamma(h - 2z)/(2\mu)$. Physically, positive (negative) vorticity is continuously generated at the lower (upper) boundary and diffuses upward (downward). These positive and negative vorticities mutually annihilate at $z = 0$.

5.4 Exercises

Question 5.1

Mixed Couette–Poiseuille flow is driven in a layer of uniform incompressible fluid between two parallel plates at $z = 0$ and $z = h$ by moving the upper plate steadily in its own plane in the x-direction with velocity U and applying an *opposing* pressure gradient $dp/dx = \gamma > 0$.

Find the velocity profile in the flow and the volume flux and find the pressure gradient required to produce zero net volume flux per unit y-width.

Question 5.2

Consider the time dependent Couette problem. A viscous fluid at rest is confined between two rigid plates, one at $z = 0$ and the other at $z = h$. The lower boundary is fixed, but at $t = 0$ the upper boundary is impulsively set in motion in its own plane with velocity $(U, 0, 0)$.

Calculate the velocity and vorticity profiles as a function of time.

Question 5.3

Find the velocity field for a steady viscous flow through an axisymmetric pipe of radius a under a constant applied pressure gradient (i.e. Poiseuille flow in cylindrical polar coordinates).

In cylindrical coordinates for axi-symmetric problems:

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{\partial^2 f}{\partial z^2}.$$

Question 5.4

Show that in the (slab symmetric) version of the Poiseuille flow problem, negative/positive vorticity is being continuously generated at the upper/lower boundary. Consider the time dependent problem where the flow is impulsively accelerated by a uniform pressure gradient.

Show that the unsteady component of velocity satisfies the heat equation and determine the initial condition for this equation.

Question 5.5

An incompressible fluid occupies the space $0 < z < \infty$ above a plane rigid boundary $z = 0$, which oscillates in the x-direction with velocity $U \cos(\alpha t)$. (Assume no applied pressure gradient.)

Show the velocity field has the form $\mathbf{u} = [u(z,t), 0, 0]$ and satisfies

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2}.$$

Seek a solution of the form $u = Re\left\{f(z)\, e^{i\alpha t}\right\}$, where $Re\{\bullet\}$ means "the real part of".

Show that

$$u(z,t) = Ue^{-kz}\cos(kz - \alpha t),$$

where $k = \sqrt{\alpha/2\nu}$.

Chapter 6

Boundary Layers in Nonrotating Fluids

The Navier-Stokes' equation is the statement of Newton's second law of motion for a viscous fluid. It reads

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}. \tag{6.1}$$

The quantity ν is called the **kinematic viscosity**. For air, $\nu = 1.5 \times 10^{-5}$ m^{2}s^{-1}; for water $\nu = 1.0 \times 10^{-6}$ m^{2}s^{-1}.

The relative importance of viscous effects is characterized by the Reynolds number Re, a nondimensional number defined by

$$Re = \frac{UL}{\nu},$$

where U and L are typical velocity and length scales respectively. The Reynold's number is a measure of the ratio of the acceleration term to the viscous term in (6.1). For example, the diameter of a cricket ball is about 75 mm. If the balled is bowled at 100 km hr^{-1} (= 28 ms^{-1}), the Reynolds number is 1.4×10^5.

For many flows of interest, $Re \gg 1$ and viscous effects are relatively unimportant. However, these effects are **always** important near boundaries, even if only in a thin "boundary-layer" adjacent to the boundary.

6.1 Flow over a flat plate

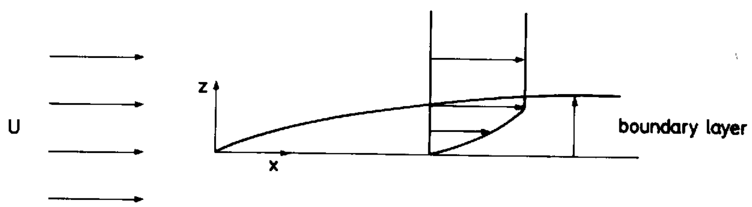


Figure 6.1

We consider the boundary layer on a flat plate at normal incidence to a uniform stream U as shown.

The Navier Stokes' equations for steady two-dimensional flow with typical scales written below each component are:

$$
\begin{array}{ccccccc}
u\dfrac{\partial u}{\partial x} & + w\dfrac{\partial u}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x} & + \nu\Bigg(& \dfrac{\partial^2 u}{\partial x^2} & + \dfrac{\partial^2 u}{\partial z^2} & \Bigg), \\
\dfrac{U^2}{L} & \dfrac{UW}{H} & \dfrac{\Delta p}{\rho L} & & \nu\dfrac{U}{L^2} & \nu\dfrac{U}{H^2} &
\end{array}
\tag{6.2}
$$

$$
\begin{array}{ccccccc}
u\dfrac{\partial w}{\partial x} & + w\dfrac{\partial w}{\partial z} & = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} & + \nu\Bigg(& \dfrac{\partial^2 w}{\partial x^2} & + \dfrac{\partial^2 w}{\partial z^2} & \Bigg), \\
\dfrac{UW}{L} & \dfrac{W^2}{H} & \dfrac{\Delta p}{\rho H} & & \nu\dfrac{W}{L^2} & \nu\dfrac{W}{H^2} &
\end{array}
\tag{6.3}
$$

and the continuity equation is

$$
\begin{array}{ccc}
\dfrac{\partial u}{\partial x} & + \dfrac{\partial w}{\partial z} & = 0. \\
\dfrac{U}{L} & \dfrac{W}{H} &
\end{array}
\tag{6.4}
$$

From the continuity equation we conclude that since $|\partial u/\partial x| = |\partial w/\partial z|$, $W \sim UH/L$ and hence the two advection terms on the left hand sides of (6.2) and (6.3) are the same order of magnitude: U^2/L in (6.2) and $(U^2/L)(H/L)$ in (6.3). Now, for a thin boundary layer, $H/L \ll 1$ so that the derivatives $\partial^2/\partial x^2$ in (6.2) and (6.3) can be neglected compared with $\partial^2/\partial z^2$. Then in (6.2), assuming that the pressure gradient term is not larger than both inertial or friction terms[1], we have

$$\frac{U^2}{L} \sim \nu\frac{U}{H^2} \geq \frac{\Delta p}{\rho L}.$$

The first two terms imply that $H \sim L\ Re^{-1/2}$. Alternatively, this expression implies that the boundary thickness increases downstream like $x^{1/2}$ [i.e., $H \sim L^{1/2}(\nu/U)^{1/2}$]. Now from (6.3) we find that

$$
\begin{aligned}
\frac{\Delta p}{\rho H}/\frac{UW}{L} &\sim \frac{\rho U^2}{\rho H}/\frac{U^2 H}{L^2} \sim \frac{L^2}{H^2} >> 1 \\
\frac{\Delta p}{\rho H}/\frac{\nu W}{H^2} &\sim \frac{\rho U^2}{\rho H}/\frac{\nu U}{HL} \sim \frac{UL}{\nu} = Re >> 1.
\end{aligned}
$$

But if both the inertia terms and friction terms in (6.3) are much less than the pressure gradient term, the equation must be accurately approximated by

$$\frac{\partial p}{\partial z} = 0.$$

This implies that the perturbation pressure is constant across the boundary layer. It follows that the horizontal pressure gradient in the boundary layer is equal to that in free stream.

Collecting these results together we find that an approximate form of the Navier-Stokes' equations for the boundary layer to be

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial z^2} \quad , \tag{6.5}$$

[1] Note that if this were not true, steady flow would not be possible as the large pressure gradient would accelerate the flow further.

with

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{6.6}$$

and $U = U(x)$ being the (possible variable) free stream velocity above the boundary layer. Equations (6.5) and (6.6) are called the **boundary layer equations**.

6.2 Blasius' solution (U = constant)

Equation (6.5) reduces to

$$u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = \nu\frac{\partial^2 u}{\partial z^2} \quad , \tag{6.7}$$

and we look for a solution satisfying the boundary conditions $u = 0$, $w = 0$ at $z = 0$, $u \to U$ as $z \to \infty$ and $u = U$ at $x = 0$. Equation (6.6) suggests that we introduce a streamfunction ψ such that

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x},$$

whereupon ψ must satisfy the conditions ψ = constant, $\partial\psi/\partial z = 0$ at $z = 0$, $\psi \sim Uz$ as $z \to \infty$ and $\psi = Uz$ at $x = 0$. It is easy to verify that a solution satisfying these conditions is

$$\psi = (2\nu U x)^{\frac{1}{2}} f(\chi), \tag{6.8}$$

where

$$\chi = (U/2\nu x)^{\frac{1}{2}} z, \tag{6.9}$$

and $f(\chi)$ satisfies the ordinary differential equation

$$f''' + ff'' = 0, \tag{6.10}$$

subject to the boundary conditions

$$f(0) = f'(0) = 0; \quad f'(\infty) = 1. \tag{6.11}$$

Here, a prime denotes differentiation with respect to χ. It is easy to solve Eq. (6.10) subject to (6.11) numerically (see e.g. Rosenhead, 1966, *Laminar Boundary Layers*, p. 222-224). The profile of f' which characterizes the variation of u across the boundary layer thickness is proportional to χ and we might take $\chi = 4$ as corresponding with the edge of the boundary layer. Then (6.9) shows that the dimensional boundary thickness $\delta(x) = 4(2\nu x/U)^{\frac{1}{2}}$, i.e., increases like the square root of the distance from the leading edge of the plate. We can understand the thickening of the boundary layers as due to the progressive retardation of more and more fluid as the fictional force acts over a progressively longer distance downstream.

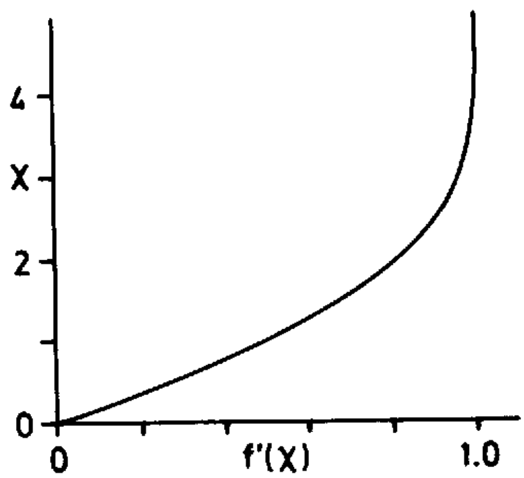


Figure 6.2

Note that the boundary layer is **rotational** since $\boldsymbol{\omega} = (0, \eta, 0)$, where $\eta = \partial u/\partial z - \partial w/\partial x$, or approximately just $\partial u/\partial z$.

Often the boundary layer is relatively thin. Consider for example the boundary layer in an aeroplane wing. Assuming the wing to have a span of 3 m and that the aeroplane flies at 200 ms^{-1}, the boundary layer at the trailing edge of the wing (assuming the wing to be a flat plate) would have thickness of $4(2 \times 1.5 \times 10^{-5} \times 3/200)^{1/2} = 2.7 \times 10^{-3}$ m using the value $\nu = 1.5 \times 10^{-5}$ m^2s^{-1} for the viscosity of air. The calculation assumes that the boundary layer remains laminar; if it becomes turbulent, the random eddies in the turbulence have a much larger effect on the lateral momentum transfer than do random molecular motions, thereby increasing the effective value of ν, possibly by an order of magnitude or more, and hence the boundary layer thickness.

6.3 Exercises

Question 6.1*

Define the Reynolds number and explain its physical significance. Estimate the Reynolds number for a ball of radius 10 cm thrown in air at a speed of 15 ms^{-1}. What speed gives an equivalent Reynolds number in water. The kinematic viscosity for air is 1.5×10^{-5} $\mathrm{m}^2\mathrm{s}^{-1}$ and for water is 10^{-6} $\mathrm{m}^2\mathrm{s}^{-1}$.

Question 6.2

Show that the boundary layers equations for $U_x = 0$ are satisfied by the similarity solution

$$\psi = (2\nu U x)^{\frac{1}{2}}\, f(\chi), \quad \chi = (U/2\nu x)^{\frac{1}{2}}\, z,$$

where

$$f''' + f f'' = 0,$$
$$f(0) = f'(0) = 0; \quad f'(\infty) = 1,$$

and

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}.$$

Chapter 7

Two Dimensional Flow Past a Cylinder

In two dimensions (x, z), the Euler equations of motion are

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad , \tag{7.1}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad , \tag{7.2}$$

and the continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0. \tag{7.3}$$

The vorticity $\boldsymbol{\omega}$ has only one non-zero component, the y-component, i.e., $\boldsymbol{\omega} = (0, \eta, 0)$, where

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}. \tag{7.4}$$

Taking $(\partial/\partial z)$(7.1) - $(\partial/\partial x)$(7.2) and using the continuity equation we can show that

$$\frac{D\eta}{Dt} = 0. \tag{7.5}$$

This equation states that fluid particles conserve their vorticity as they move around. This is a powerful and useful constraint. In some problems, $\eta = 0$ for all particles. Such flows are called **irrotational**.

7.1 Flow past a cylinder without circulation

Consider, for example, the steady flow around a cylinder. All fluid particles originate from far upstream ($x \to -\infty$) where $u = 0$, $w = 0$, and therefore $\eta = 0$. It follows that fluid particles have zero vorticity for all time.

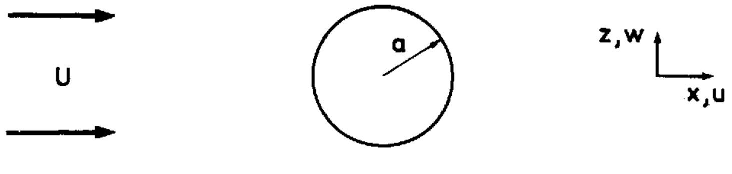


Figure 7.1

The inviscid flow problem can be solved as follows. Note that the continuity equation (7.3) suggests that we introduce a **streamfunction** ψ, defined by the equations

$$u = \frac{\partial \psi}{\partial z}, \quad w = -\frac{\partial \psi}{\partial x}. \tag{7.6}$$

Then (7.5) is automatically satisfied and it follows from (7.4) that

$$\eta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} \quad . \tag{7.7}$$

In the case of **irrotational flow**, η = 0 and ψ satisfies Laplace's equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \quad . \tag{7.8}$$

Appropriate boundary conditions are found using (7.6). For example, on a solid boundary, the normal velocity must be zero, i.e., $\mathbf{u}.\mathbf{n} = 0$ on the boundary. If $\mathbf{n} = (n_1, 0, n_3)$, it follows using (7.6) that $n_1 \partial \psi / \partial z - n_3 / \partial x = 0$, or $\mathbf{n} \times \nabla \psi$ = **0** on the boundary. We deduce that $\nabla \psi$ is in the direction of **n**, whereupon ψ is a constant on the boundary itself.

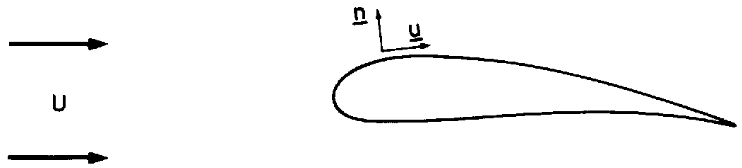


Figure 7.2

Let us return to the example of uniform flow past a cylinder of radius a:

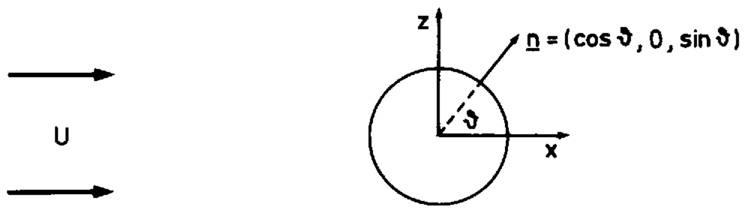


Figure 7.3

The problem is to solve (7.8) in the region outside the cylinder (i.e. $r > a$) subject to the boundary condition that

$$\mathbf{u} = \left(\frac{\partial \psi}{\partial z}, 0, -\frac{\partial \psi}{\partial x} \right) \to (U, 0, 0) \quad \text{as} \quad r \to \infty \quad , \tag{7.9}$$

and

$$\mathbf{u}.\mathbf{n} = 0 \quad \text{on} \quad r = a, \tag{7.10}$$

where $r = \sqrt{x^2 + z^2}$. For this problem it turns out to be easier to work in cylindrical polar coordinates centred on the cylinder. Since $x = r\cos\theta$ and $z = r\sin\theta$, we can show that $\partial r/\partial x = \cos\theta$, $\partial r/\partial z = \sin\theta$, $\partial\theta/\partial x = -(\sin\theta)/r$ and $\partial\theta/\partial z = (\cos\theta)/r$. Then

$$\begin{aligned} \frac{\partial \psi}{\partial z} &= \frac{\partial \psi}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial \psi}{\partial \theta}\frac{\partial \theta}{\partial z}, \\ &= \sin\theta \frac{\partial \psi}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \psi}{\partial \theta}. \end{aligned} \tag{7.11}$$

Similarly,

$$\begin{aligned}\frac{\partial\psi}{\partial x} &= \frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x},\\ &= \cos\theta\frac{\partial\psi}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\psi}{\partial\theta}.\end{aligned} \tag{7.12}$$

One can use (7.11) and (7.12) to transform (7.8) to cylindrical polar coordinates giving,

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2\psi}{\partial\theta^2}\right] = 0 \quad . \tag{7.13}$$

The boundary condition on the cylinder expressed by (7.10) requires that

$$\cos\theta\frac{\partial\psi}{\partial z} - \sin\theta\frac{\partial\psi}{\partial x} = 0$$

at $r = a$ and for all θ. Using (7.11) and (7.12), this reduces to

$$\frac{\partial\psi}{\partial\theta} = 0 \quad \text{at} \quad r = a.$$

This equation implies that ψ is a constant on the cylinder; i.e. the surface of the cylinder must be a streamline. Note also that for large r, $u = \partial\psi/\partial z \sim U$ and hence

$$\psi \sim Uz = Ur\sin\theta. \tag{7.14}$$

The far field solution (7.14) suggests seeking separable solutions to (7.13) of the form $\psi = f(r)\sin\theta$. Upon substituting this expression into (7.13), we get

$$r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} - f = 0. \tag{7.15}$$

Equation (7.15) is an Euler–Cauchy equation and can be solved by seeking solutions of the form $f = r^k$, whereupon we find that $k = \pm 1$. Hence,

$$f(r) = c_1 r + \frac{c_2}{r}.$$

Apply the boundary conditions (7.9) and (7.10) shows that $c_1 = U$ and $c_2 = -Ua^2$, and consequently

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta. \tag{7.16}$$

Note that $\psi = 0$ on the cylinder. However, the solution for ψ is unique only to within a constant value; if we add any constant to it, it will satisfy equation (7.8) or (7.13), but the velocity field would be unchanged.

In cylindrical polar coordinates the radial and tangential components of velocity, v_r and v_θ, are related to the stream-function by

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} \quad \text{and} \quad v_\theta = -\frac{\partial\psi}{\partial r},$$

Hence, from (1.1),

$$v_r = Ur\left(1 - \frac{a^2}{r^2}\right)\cos\theta \quad \text{and} \quad v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta.$$

On the boundary of the cylinder ($r = a$) $v_r = 0$ and $v_\theta = -2U \sin\theta$.

Recall that v_θ is positive when the flow is anticlockwise. For example, on the top of the cylinder ($\theta = \pi/2$) the tangential velocity is $v_\theta = -2U$, which is directed from left to right.

It is important to note that we have obtained a solution without reference to the pressure field, but the pressure distribution determines the force field that drives the flow! We seem, therefore, to have by-passed Newton's second law, and have obviously avoided dealing with the nonlinear nature of the momentum equations (7.1) and (7.2). Looking back we will see that the trick was to use the vorticity equation, a derivative of the momentum equations. For a homogeneous fluid, the vorticity equation does not involve the pressure since $\nabla \times \nabla p \equiv 0$. We infer from the vorticity constraint, (7.5), that the flow must be irrotational everywhere and use this, together with the continuity constraint (which is automatically satisfied when we introduce the streamfunction) to infer the flow field. If desired, the pressure field can be determined, for example, by integrating (7.1) and (7.2), or by using Bernoulli's equation along streamlines.

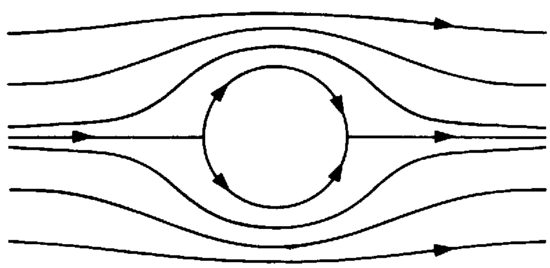


Figure 7.4

Now the solution itself. The streamline corresponding with (7.16) are sketched. Note that they are symmetrical around the cylinder. Applying Bernoulli's equation to the streamline around the cylinder we find that the pressure distribution is symmetrical also so that the total pressure force on the up-stream side of the cylinder is exactly equal to the pressure on the downwind side. In other words, the net pressure force on the cylinder is zero! This result, which in fact is a general one for irrotational inviscid flow past a body of any shape, is known as **d'Alembert's Paradox**. It is not in accord with our experience as you know full well when you try to cycle against a strong wind. What then is wrong with the theory? Indeed, what does the flow round a cylinder look like in reality? The reasons for the breakdown of the theory help us to understand the limitations of inviscid flow theory in general and help us to see the circumstances under which it may be applied with confidence. First, let us return to the viscous theory.

7.2 Flow past a cylinder with friction

The dynamics of the boundary layer plays a crucial role in flow past a circular cylinder. In particular it has important consequences for the solution downstream. The observed streamline pattern in this case at large Reynolds numbers is sketched in the figure below. The figure below shows how the flow past a cylinder changes with changing Reynolds number. Upstream of the cylinder the flow is similar to that predicted by the inviscid theory, except in a thin viscous boundary-layer adjacent to the cylinder. At points on the downstream side of the cylinder the flow separates and there is an unsteady turbulent wake behind it. For very small Reynolds number ($Re < 1$) viscosity is important, yet the flow is symmetrical and similar to the inviscid solution. As the Reynolds number increases ($1 < Re < 30$) the flow behind the cylinder stretches out and two symmetrically-placed eddies form. For higher Reynolds number ($40 < Re < 4,000$), a time-dependent but ordered wake forms behind the cylinder (Karman vortex streets). This wake becomes turbulent as the Reynolds number increases further ($10^3 < Re < 10^5$). At very large Reynolds number ($Re > 10^5$) the turbulent boundary layer reaches around the cylinder.

The existence of the wake destroys the symmetry in the pressure field predicted by the inviscid theory and there is net pressure force or **form drag** acting on the cylinder. Viscous stresses at the boundary itself cause additional drag on the body. However, as the Reynolds number increases from below about $Re > 10^5$, the drag drops sharply (which is critical for swing bowling). This is because the boundary layer becomes turbulent ahead of the separation point - more on this later.

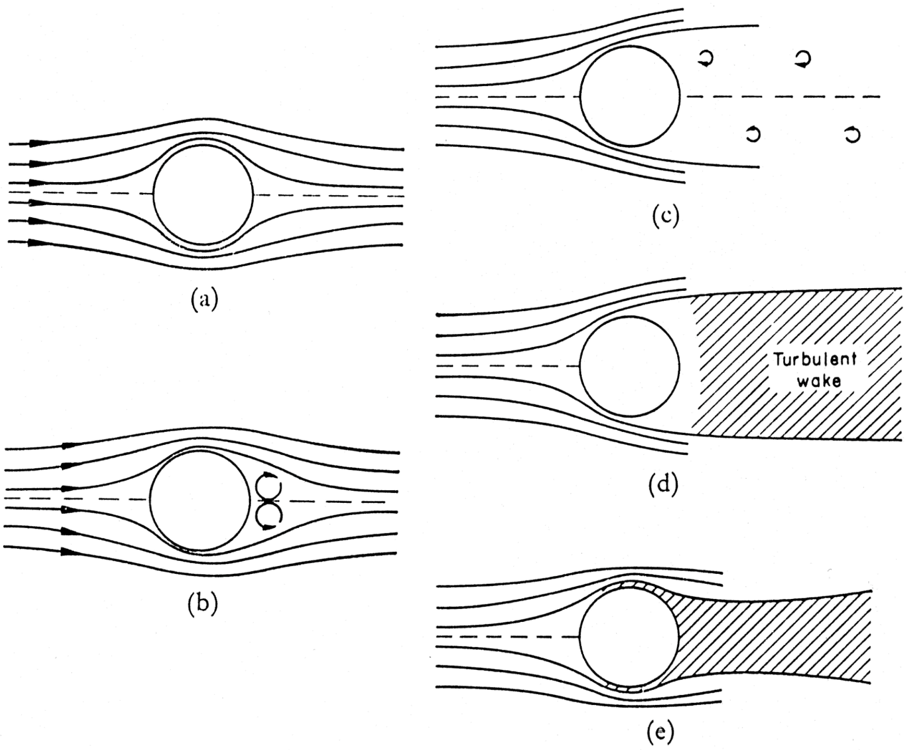


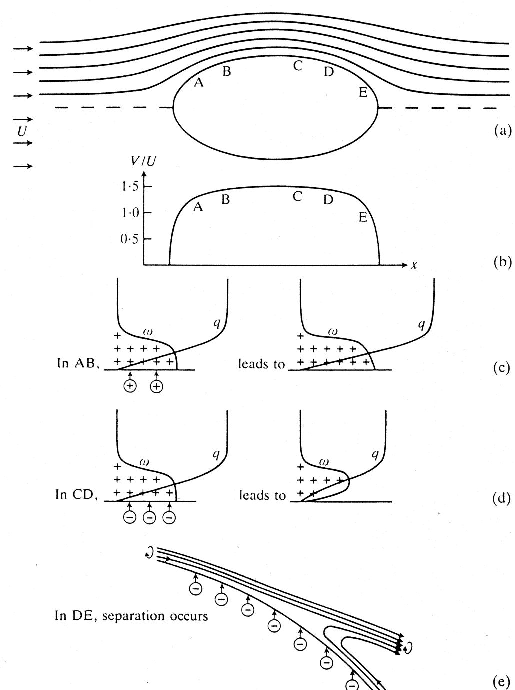
Figure 7.5: Flow past a cylinder. (a) $Re < 1$. *(b)* $1 < Re < 30$. *(c)* $40 < Re < 4,000$. *(d)* $10^3 < Re < 10^5$. *(e)* $Re > 10^5$. (From Modern Fluid Dynamics. Volume 1: Incompressible Flow, N. Curle and H.J. Davies, 1967.)

7.3 Boundary Layer Separation

If the Reynolds number is large, then the boundary layer may remain very thin and behave essentially like a vortex sheet. In this case the flow above the boundary layer may be well approximated by inviscid flow. However, under some circumstances the boundary layer separates from the solid boundary, in which case the interior flow may be radically different from that predicted by inviscid theory. For example, this is what happens when irrotational fluid flows around a cylinder.

We investigate now the circumstances under which a boundary layer separates from the surface. The problem is mathematically very complex, so we will confine ourselves to a qualitative discussion of the key physical principles.

The figure to the right shows schematically the separation of a boundary layer around an elliptic cylinder at rest in an oncoming flow with velocity U. Assuming that the flow is irrotational outside the boundary layer, we can calculate the streamlines and velocity field using a method similar to that used in Section 7.1. The streamlines are shown in (a) and velocity just outside the boundary layer in shown in panel (b). Where the streamlines are compressed the stream speed is high and pressure is low (by Bernoulli's theorem). To the extent that the boundary layer can be treated as a vortex sheet, the strength $(\omega\delta)$ is largest where the velocity is largest.



The flow accelerates, and the strength of the vortex sheet increases, along a streamline between points A and B (panel c). Consequently, vorticity is removed from B faster than it is

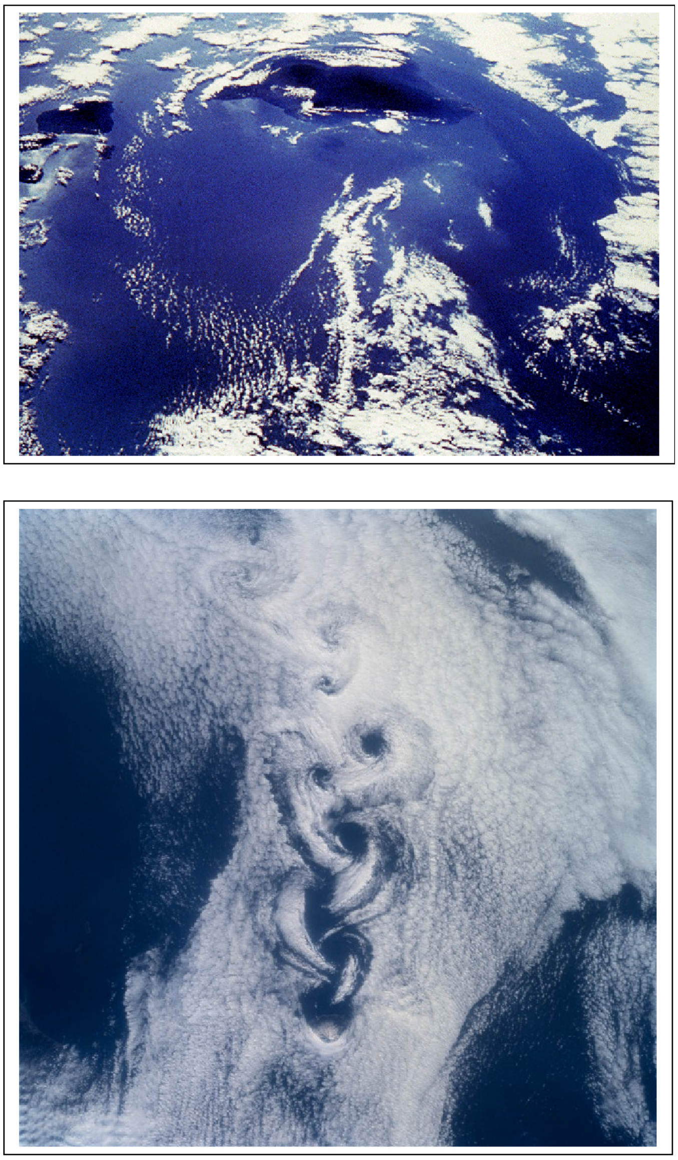


Figure 7.6: *Examples of wakes in geophysical flows. Upper panel: a wake formed by airflow around Hawaii. Lower panel: a Karmen vortex street formed in the lee of volcanic island near Japan.* (From NASA.)

replaced from A. On the other hand vorticity of the **same sense** as the vortex sheet is generated at the boundary by the (negative) pressure gradient. In addition, the newly generated vorticity diffuses **slowly** away from the boundary. The relative importance of the tangential pressure gradient and diffusion to the circulation budget at the surface can be assessed by comparing $-\rho^{-1}\partial p/\partial x$ to $\nu\partial^2 u/\partial z^2$.

Between C and D a similar argument can be made with the necessary changes (panel d). Vorticity is advected to D faster than it is advected away, but vorticity of the **opposite sign** to that in the vortex sheet is generated by the adverse (positive) pressure gradient between C and D. Diffusion slowly transports the newly generated (negative) vorticity away from the surface into the boundary layer and transports (positive) boundary layer vorticity to the surface. Provided that negative vorticity is not generated too quickly by the boundary pressure gradient, diffusion will ensure that a reversed circulation does not develop at the solid surface.

The flow is strongly retarded by the pressure gradient at E. Here the rate of diffusion is much smaller than the rate of generation by the pressure gradient, and circulation in the sense opposite to that in the boundary layer is generated. If the adverse pressure gradient is large enough the generation of negative vorticity may produce a local region wherein the vorticity changes sign. Consequently, the flow lifts off the surface (or **separates**) as shown in panel (e). Note that Δp is **large** and positive at E; separation occurs only in regions of very strong adverse pressure gradient, and for this reason aircraft wings a highly tapered.

As the Reynolds number increases beyond about 10^5 flows often become turbulent. Turbulence acts to prevent boundary layer separation by increasing the mixing in the boundary layer, effectively increasing the value of ν.

7.4 Inviscid flow past a cylinder with circulation

Consider, now the problem of a steady, inviscid, uniform flow U past a cylinder of radius a with circulation. We ignore the question of how the circulation was generated, which must involve the acceleration of the boundary relative to the interior flow and the subsequent diffusion of vorticity into the interior.

The circulation is specified and we calculate the flow past the cylinder. As the problem is linear, the solution is just that for flow past a non-rotating cylinder and that for the line vortex,

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{\kappa}{2\pi}\ln r \quad . \tag{7.17}$$

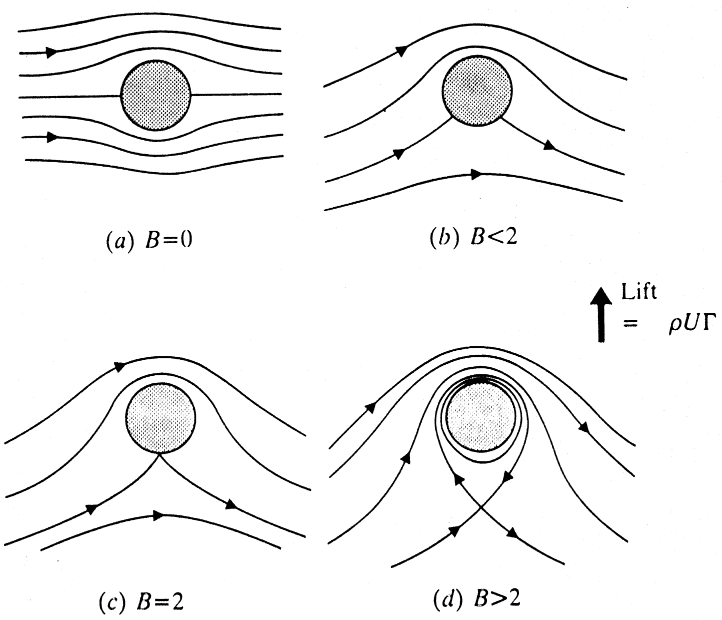
Consequently,

$$v_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta - \frac{\kappa}{2\pi r},$$

so that on the boundary of the cylinder ($r = a$),

$$v_\theta = -U\sin\theta - \frac{\kappa}{2\pi a}. \tag{7.18}$$

Equation (7.18) shows two stagnation points occur on the surface of the cylinder when $B = |\kappa/(4\pi Ua)| < 1$. These two points coalesce when $B = 1$. When $B > 1$, there are no stagnation points on the cylinder, although one occurs off the cylinder. The solution for various values of B is shown below.



(From Elementary Fluid Dynamics. D.J. Acheson, 1990.)

Figure 7.8

As the surface of the cylinder is a streamline, Bernoulli's theorem requires that

$$\frac{p}{\rho} + \frac{\mathbf{u}^2}{2} = \text{constant along } r = a.$$

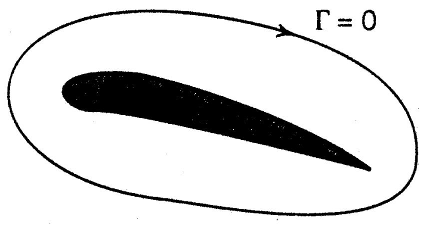
Therefore,

$$\begin{aligned}
\frac{p}{\rho} &= \text{constant} - \frac{v_\theta^2}{2}, \\
&= \text{constant} - \frac{1}{2}\left(4U^2\sin^2\theta + \frac{2U\kappa}{\pi a} + \frac{\kappa^2}{4\pi^2 a^2}\right), \\
&= \text{constant} - 2U^2\sin^2\theta - \frac{U\kappa}{\pi a}\sin\theta.
\end{aligned}$$

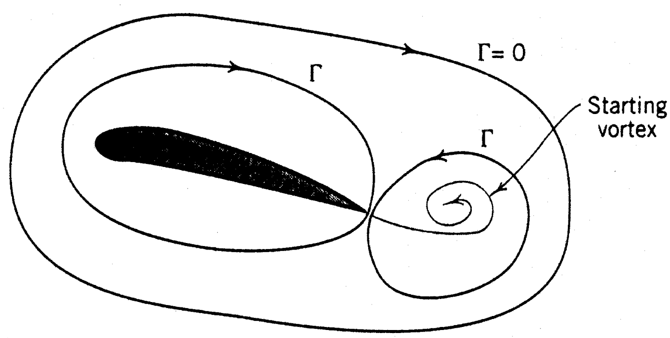
The pressure thrust (per unit length in the y direction) on a small element of the cylinder is $-(pa)\,d\theta$, from which the vertical component is $-(pa\sin\theta)\,d\theta$. It follows that the vertical component of the net thrust on the cylinder is

$$\begin{aligned}
\text{Lift} &= \int_0^{2\pi} \rho\left(2U^2\sin^2\theta + \frac{U\kappa}{\pi a}\sin\theta - \text{constant}\right) a\sin\theta d\theta, \\
&= \frac{\rho U\kappa}{\pi}\int_0^{2\pi}\sin^2\theta d\theta \\
&= \rho U\kappa.
\end{aligned}$$

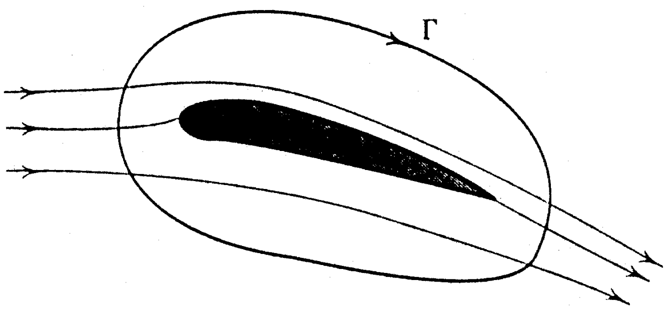
This is an example of the celebrated **Kutta-Joukowski lift theorem**. A similar calculation shows that the force in the direction of the stream, the **drag**, is zero. If the flow is irrotational, there is no drag on the body as the drag depends on the formation of a wake. This is, of course, an unrealistic feature of the solution. On the other hand, the body experiences **lift**, which is a force normal to the stream. The magnitude of the lift is $\rho U\kappa$ and is independent of the shape of the body (although the circulation depends on its shape). Physically, the addition of a clockwise (irrotational) swirling stream to that produced by uniform flow past the cylinder increases the velocity on the top of the cylinder



(a)



(b)



(c)

Figure 7.9

and reduces it on the bottom. Bernoulli's theorem requires the pressure to decrease where the velocity increases and the pressure to increase where the velocity decreases. Consequently, the pressure field produces a net transverse force on the cylinder toward increasing z. Circulation is the mechanism by which aircraft wings (as well as many other objects including golf and cricket balls) produce *lift*.

The circulation is zero around an aerofoil prior to the take-off of an aircraft (panel a). As the aircraft accelerates during take-off the aerofoil generates circulation in a thin boundary layer, which is subsequently advected into a thin wake. The thin wake *rolls up*, producing what is known as a **starting vortex**. (You can generate similar vortices on each side of a spoon by drawing along through the surface of a cup of coffee.) As the circulation is zero at the initial time, it remains zero for all time by Kelvin's theorem, provided we take a contour large enough to enclose both the aerofoil and the starting vortex (panel b). Consequently, there must be circulation around the aerofoil which is equal and opposite of that in the wake (panel c). The circulation generated around the aerofoil generated this way provides the lift.

7.5 Exercises

Question 7.1*

For a two-dimensional flow, $\boldsymbol{u} = (u, 0, w)$, and all terms $\partial/\partial y = 0$

$$u = \frac{\partial \psi}{\partial z}; \quad w = -\frac{\partial \psi}{\partial x}$$

Show that

1. $(\mathbf{u}.\nabla)\,\psi = 0$, and hence the streamfunction is constant along a streamline.
2. $\mathbf{u} = -\nabla \times (\psi \mathbf{j})$.
3. In cylindrical polar coordinates the radial and tangential components of velocity, v_r and v_θ, are related to the streamfunction by:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad v_\theta = -\frac{\partial \psi}{\partial r}.$$

Question 7.2

In rectangular Cartesian coordinates, the two-dimensional form of Laplace's equation is

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

Show that in cylindrical polar coordinates Laplace's equation can be written

$$\frac{1}{r}\left[\frac{\partial}{\partial x}\left(r\frac{\partial \psi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta^2}\right] = 0.$$

Question 7.3

For uniform flow of speed U with circulation $-\kappa$ (i.e. rotation is clockwise for κ positive) past a cylinder of radius a the streamfunction is

$$\psi = Ur\left(1 - \frac{a^2}{r^2}\right)\sin\theta + \frac{\kappa}{2\pi}\ln r.$$

Show that the drag force on the cylinder is zero.