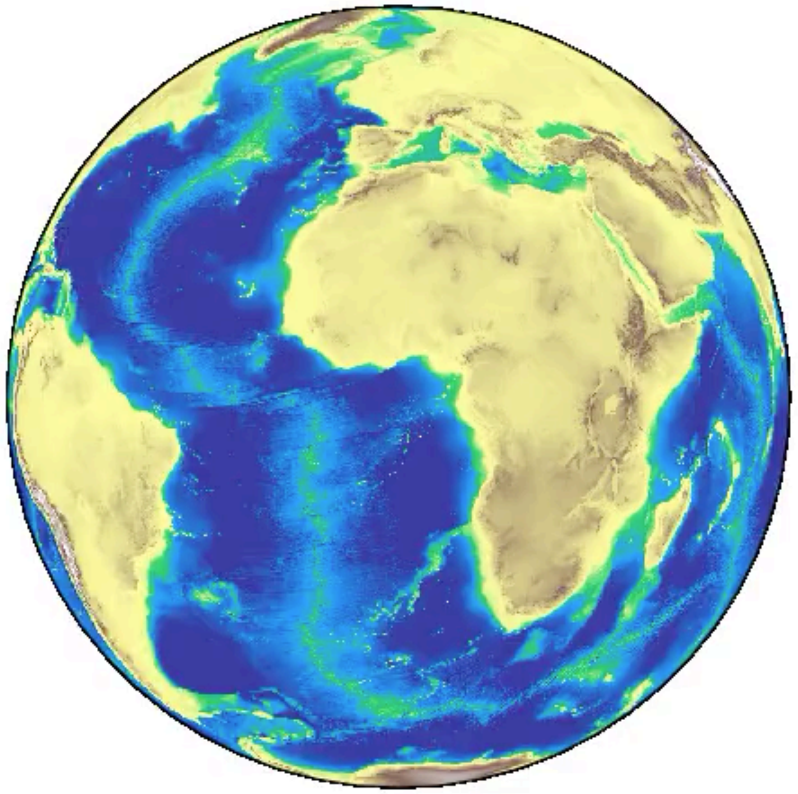
PHYS 3070 Section 2:

Convection and Observables — the oceanic lithosphere



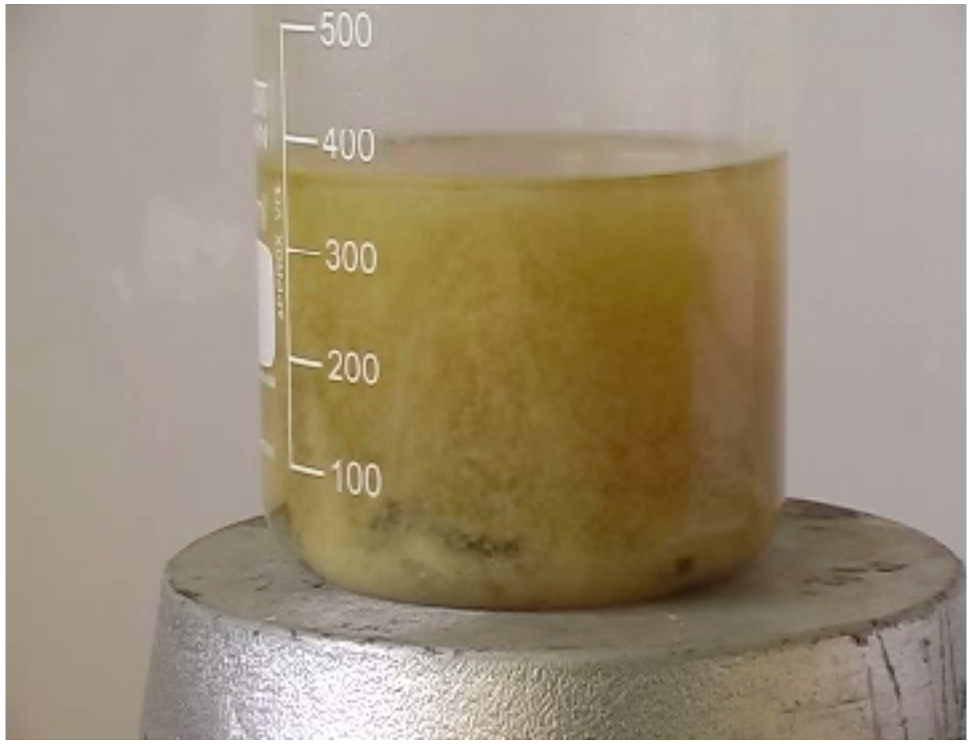
In which we try to understand the oceanic lithosphere from first principles and see how the Earth's heat engine reveals itself

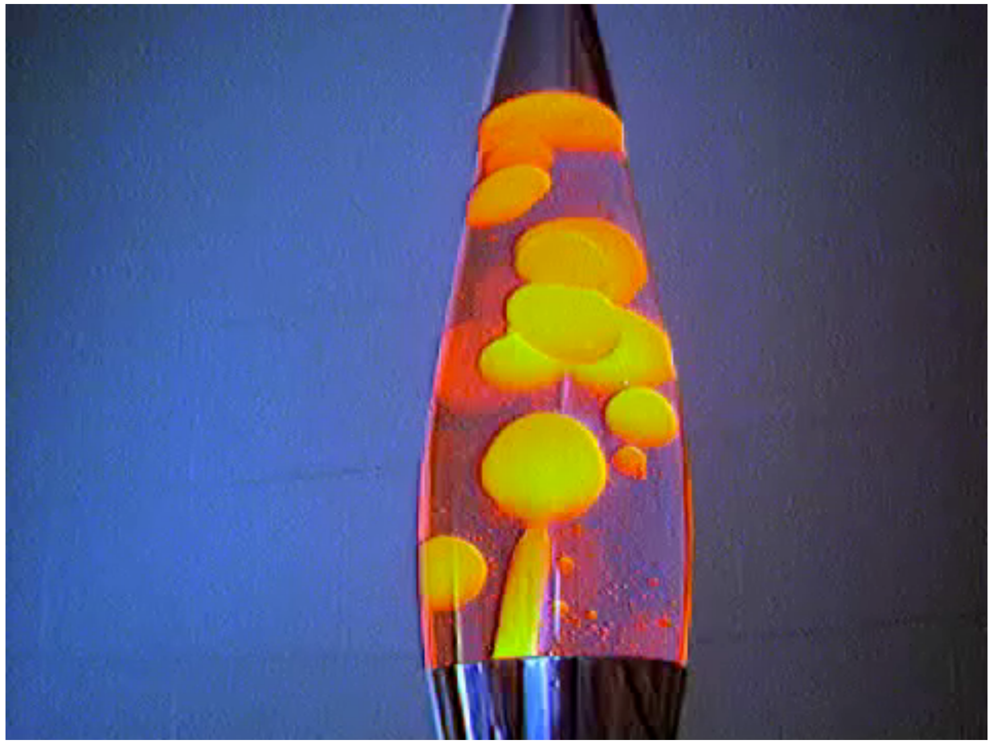
How does the Earth lose heat ?

The Earth loses about 4.4 x 10^{13}W through the surface.

This is about an equal mix of radioactive heating and secular cooling. The exact balance is not known (and the best way to tell is to measure the neutrino flux ... a work in progress).

The process at work is convection of the **solid** mantle — convection is a common phenomenon in which heat escapes by mechanical stirring of the fluid (either self-stirring or forced). Convection is a form of heat engine, turning heat into motion and making energy available.





Energy Sources

Geological change requires an energy source.

What energy sources might be available to be turned into mechanical motion at depth ?

Solar energy ?

- Drives the weather & life — therefore influences sedimentation / erosion.
- 1.75×10^{17} W available at the surface

Tidal dissipation ?

- Total power 4×10^{12} W - 60% dissipated within the planet
- Io & early Mercury

Impacts ?

- Kinetic energy of KT boundary impactor $\sim 10^{23}$ J — one per 100Myr corresponds to 3×10^{7} W
- Early Earth history

Energy Sources ...

Heating / Cooling of the Core and Mantle

- Total heat loss from within the Earth is 4.4 x 10^{13} W.
- Secular cooling, and / or radioactive decay.
- Shallow sources / deep ?

Chemical differentiation

- Gravitational settling of core ~10^{32} J
- 10^{14} W averaged over age of Earth
- Early Earth process

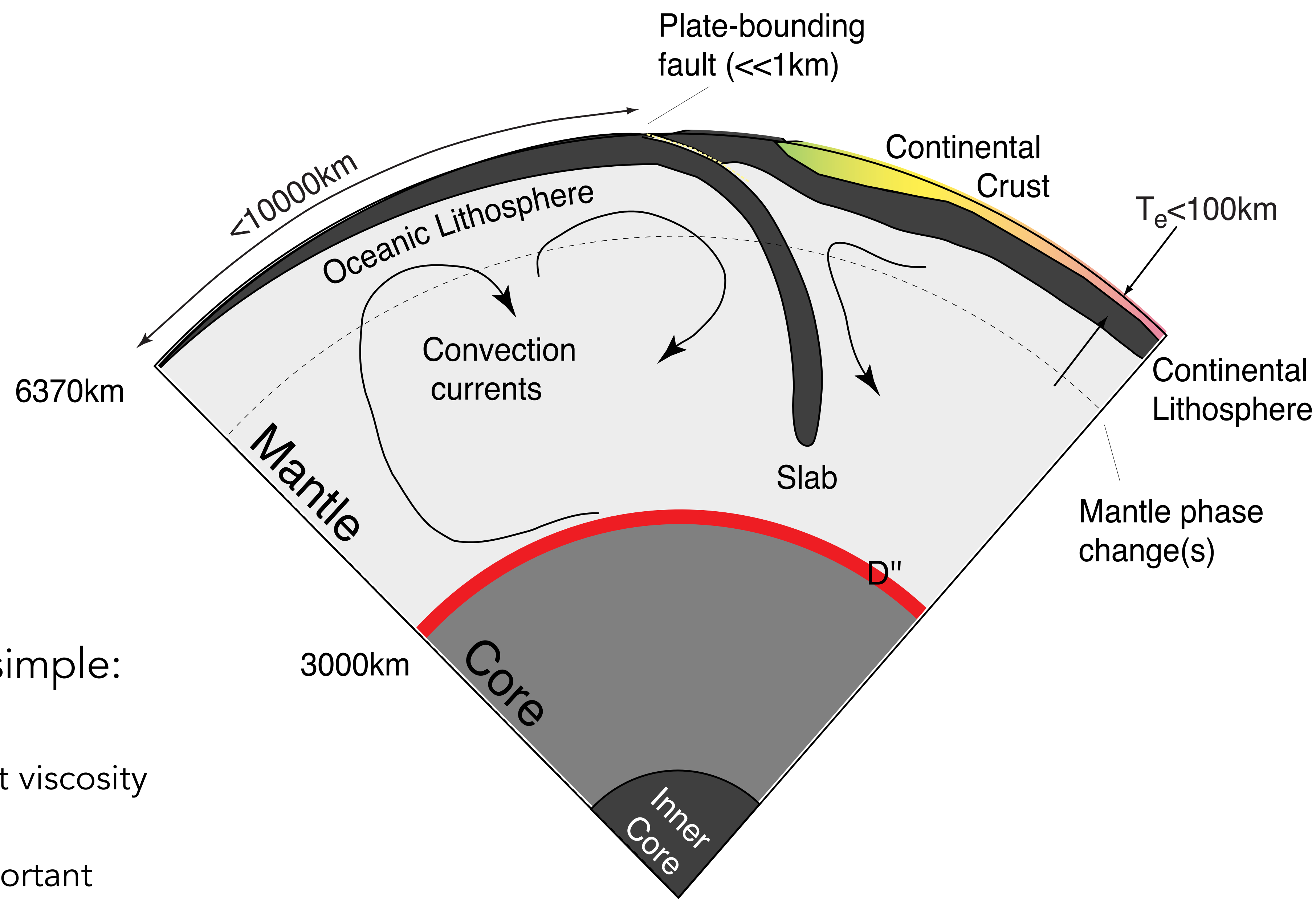
Chemical reactions / phase changes

- Solidification of inner core (Geodynamo ?)

But how does this Energy produce *Geological Change* ?

Convection in solid planets

The Earth's mantle is solid but over geological time it deforms in precisely the same way as a very viscous fluid. This occurs by reorganization of atoms in the crystal lattice. The mantle remains solid the entire time.



The lava-lamp picture is over simple:

- Cool / warm have a very different viscosity
- Elastic and brittle effects are important at low temperature
- This produces cold plates instead of blobs

Preliminaries

Before we can understand geological deformation produced by heat trying to escape from the Earth we have to know more about deformation of solids in general when they are stressed.

- Elastic deformation
- Viscous / ductile deformation
- Brittle deformation

We also need an unambiguous definition of the forces / stresses acting on a sample of material including

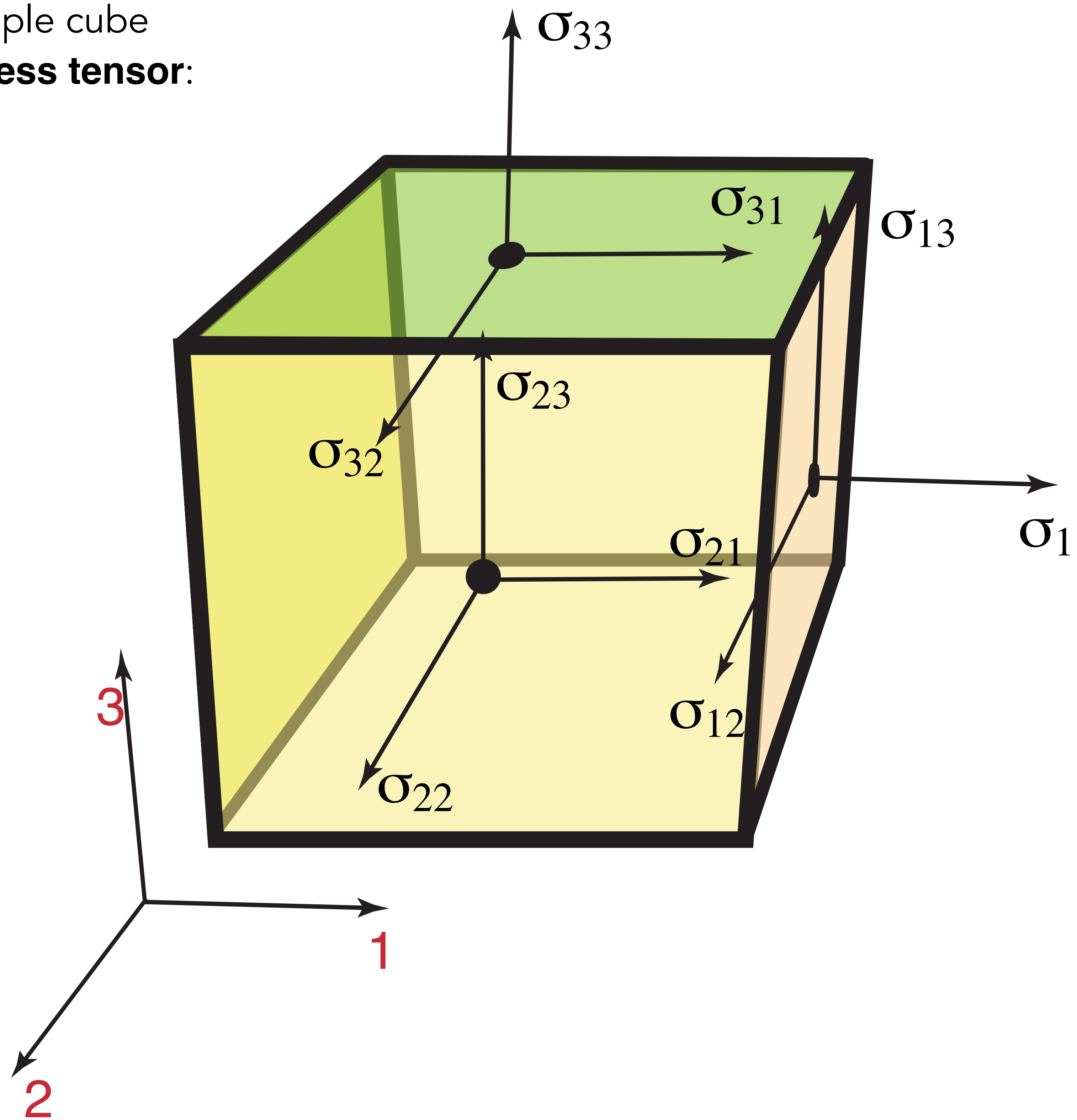
- Pressure (compression / dilation)
- Deviatoric stresses (shear)

Stress and strain

All possible forces and shears acting on a sample cube (assuming symmetry) can be written into a **stress tensor**:

$$\sigma = \begin{pmatrix} \sigma_{11} \sigma_{12} \sigma_{13} \\ \sigma_{21} \sigma_{22} \sigma_{23} \\ \sigma_{31} \sigma_{32} \sigma_{33} \end{pmatrix}$$

Symmetry: $\sigma_{ij} = \sigma_{ji}$



In general, stress/strain relations are tensor equations

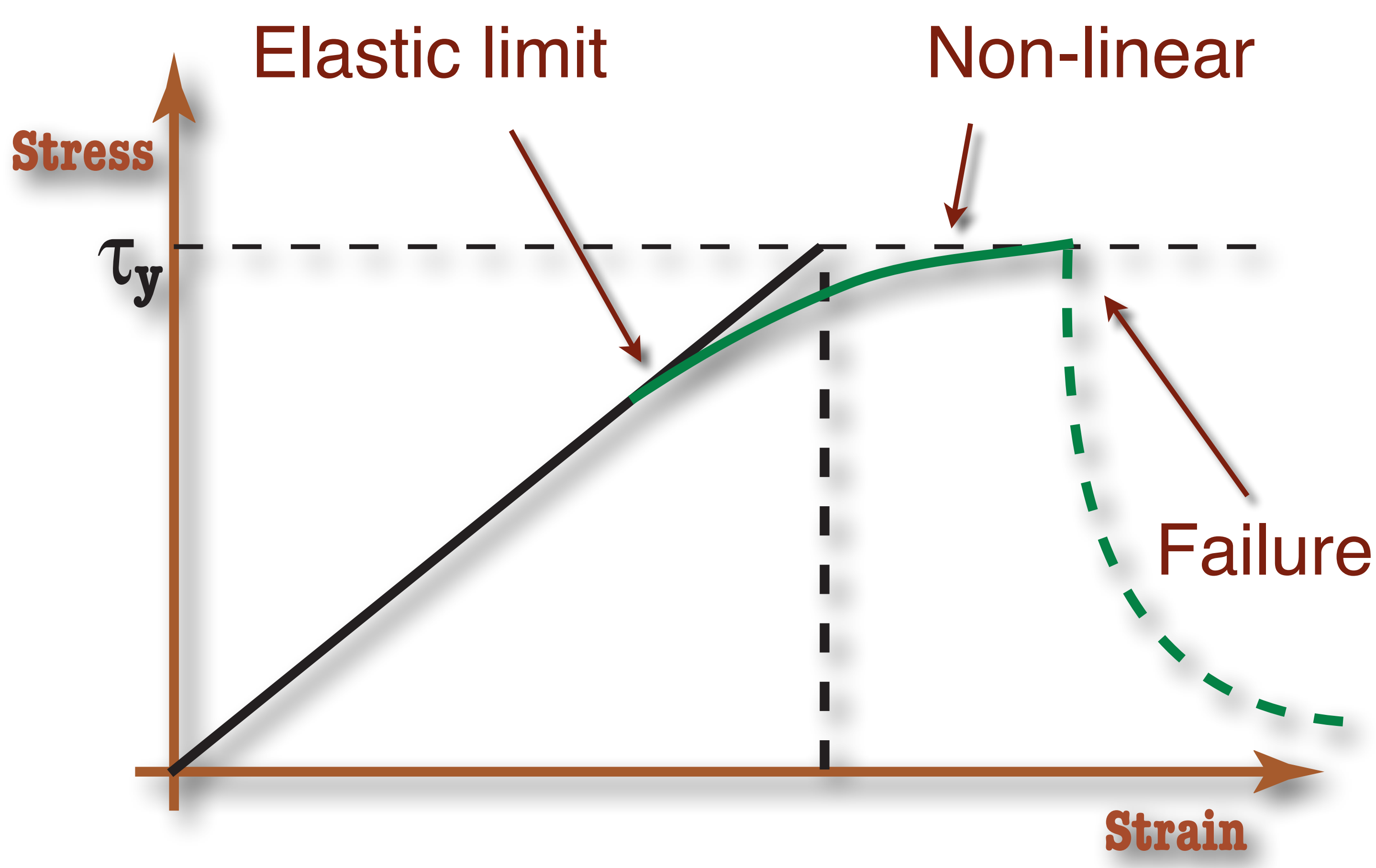
In 1D, it's considerably easier

What is elasticity ?

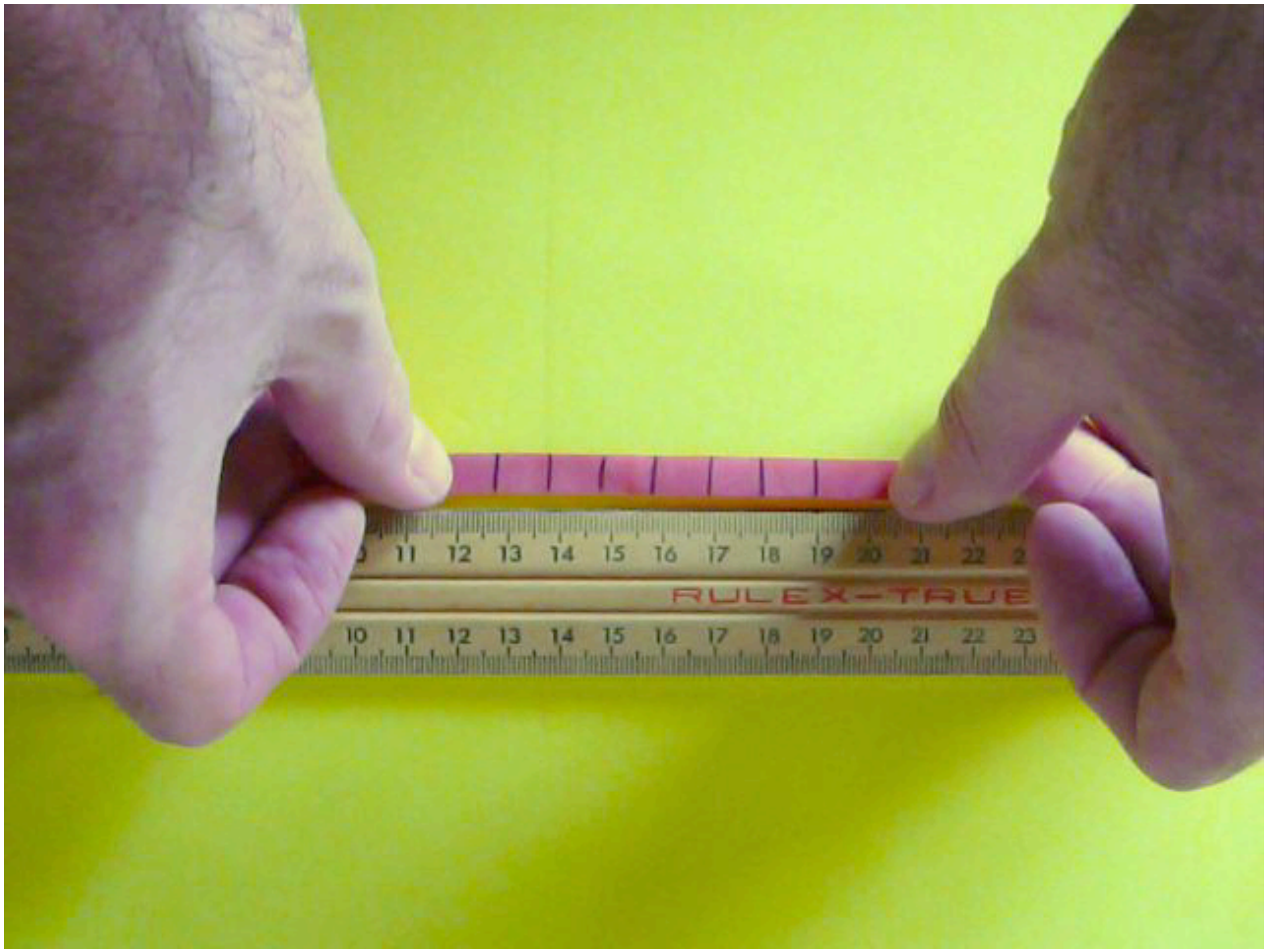
Pure elastic deformation is reversible - it doesn't matter how fast the deformation occurs, the behaviour is almost the same. However, there is an elastic limit — if the strain is larger than this, deformation is irreversible

1D response is quite simple to follow: stress is proportional to strain ...

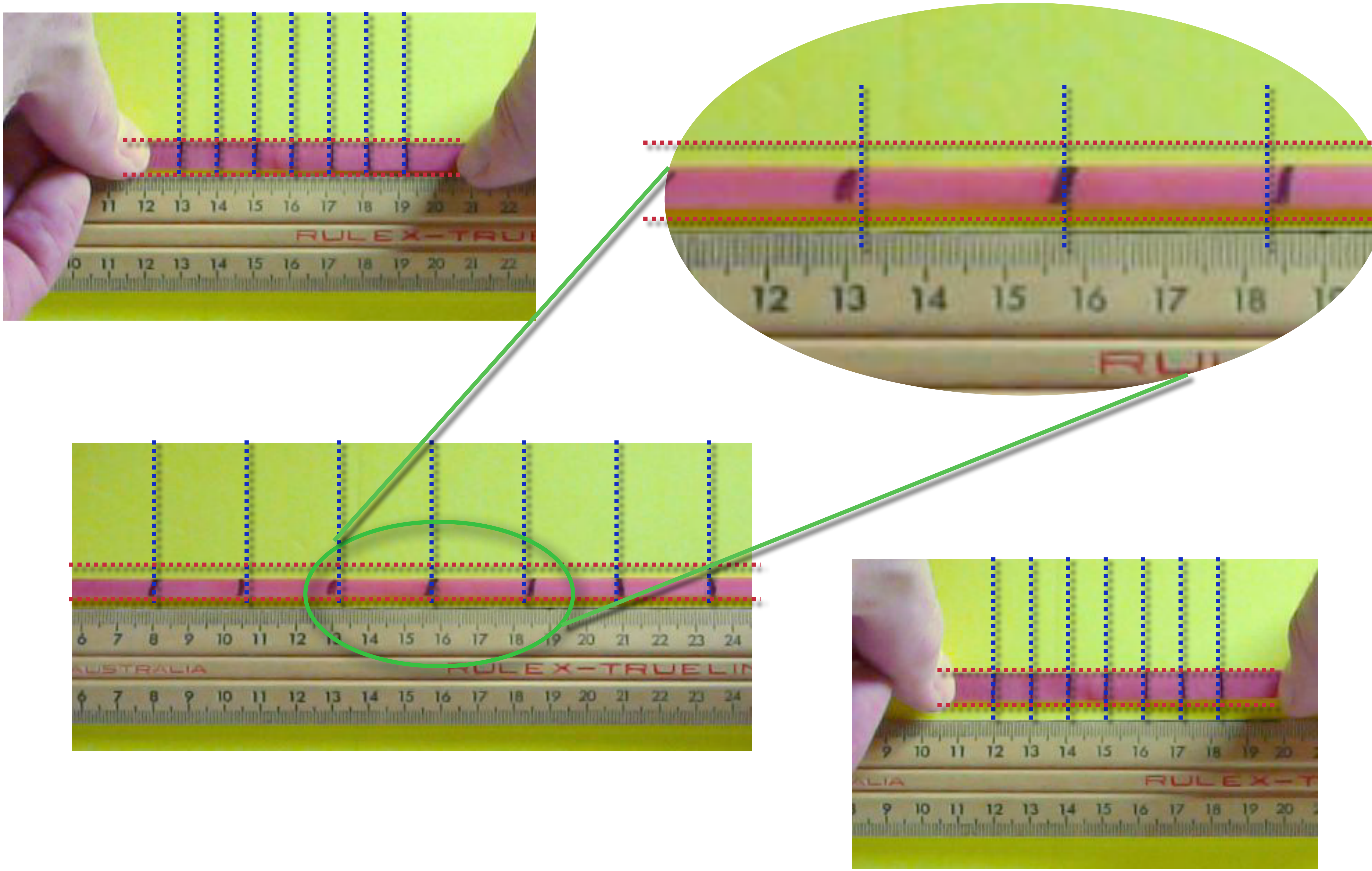
$$\sigma = E\epsilon$$



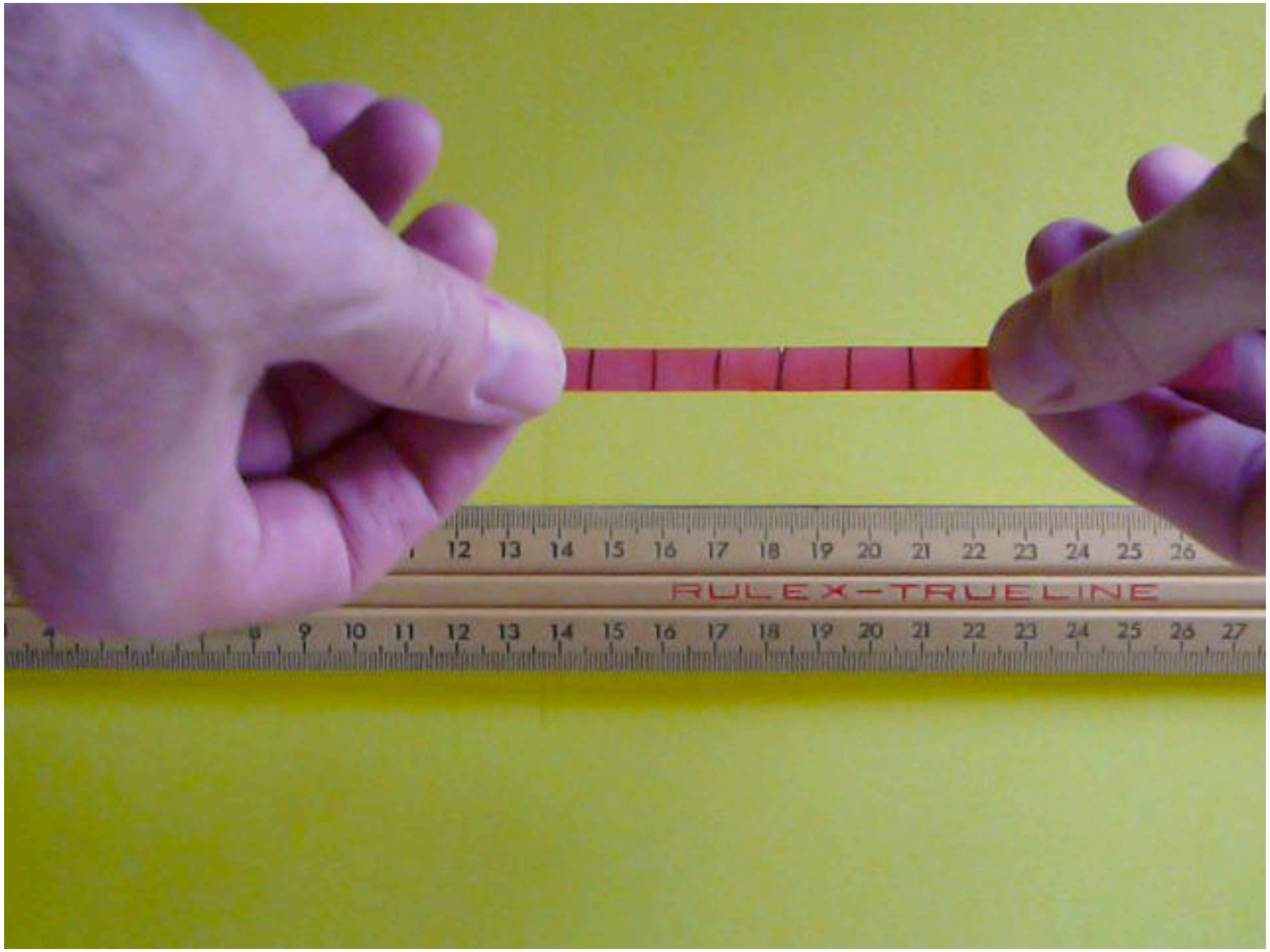
What is elasticity ?



Elastic deformation in real life



Elastic limit / failure



Stored energy release is very rapid.
Deformation occurs in less than 1/30 second

Elasticity

Lamé parameters describe elastic response.

In 1D, strain (length change) is proportional to the applied force. In general, there may also be a change in shape (other directions change length as well as the one being pulled / pushed)

$$\sigma_{xx} = 2\mu\epsilon_{xx} + \lambda(\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz})$$

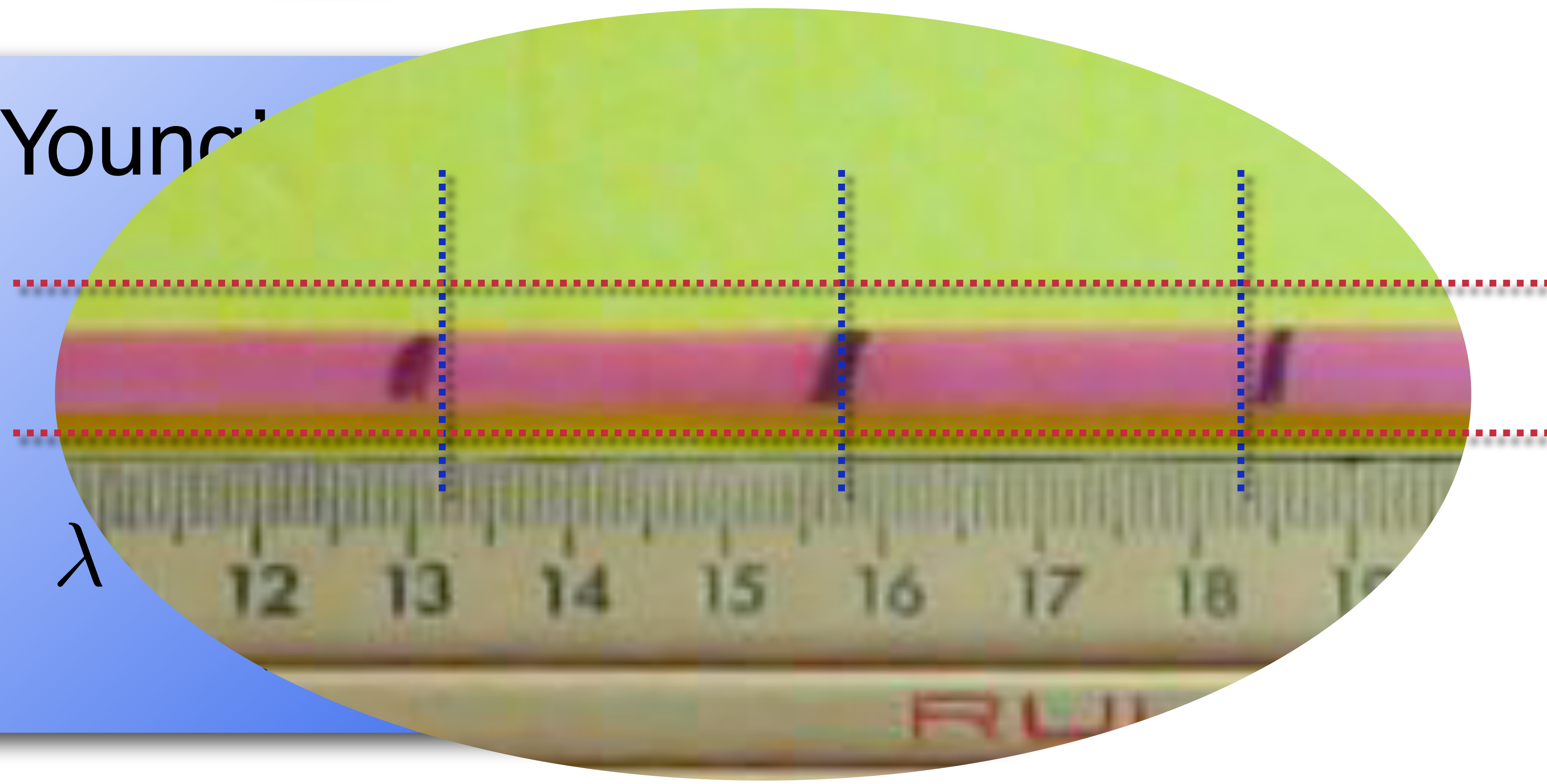
Stress — σ

Strain — ϵ

$$\mu = \frac{E}{2(1+\nu)}$$

Young

λ



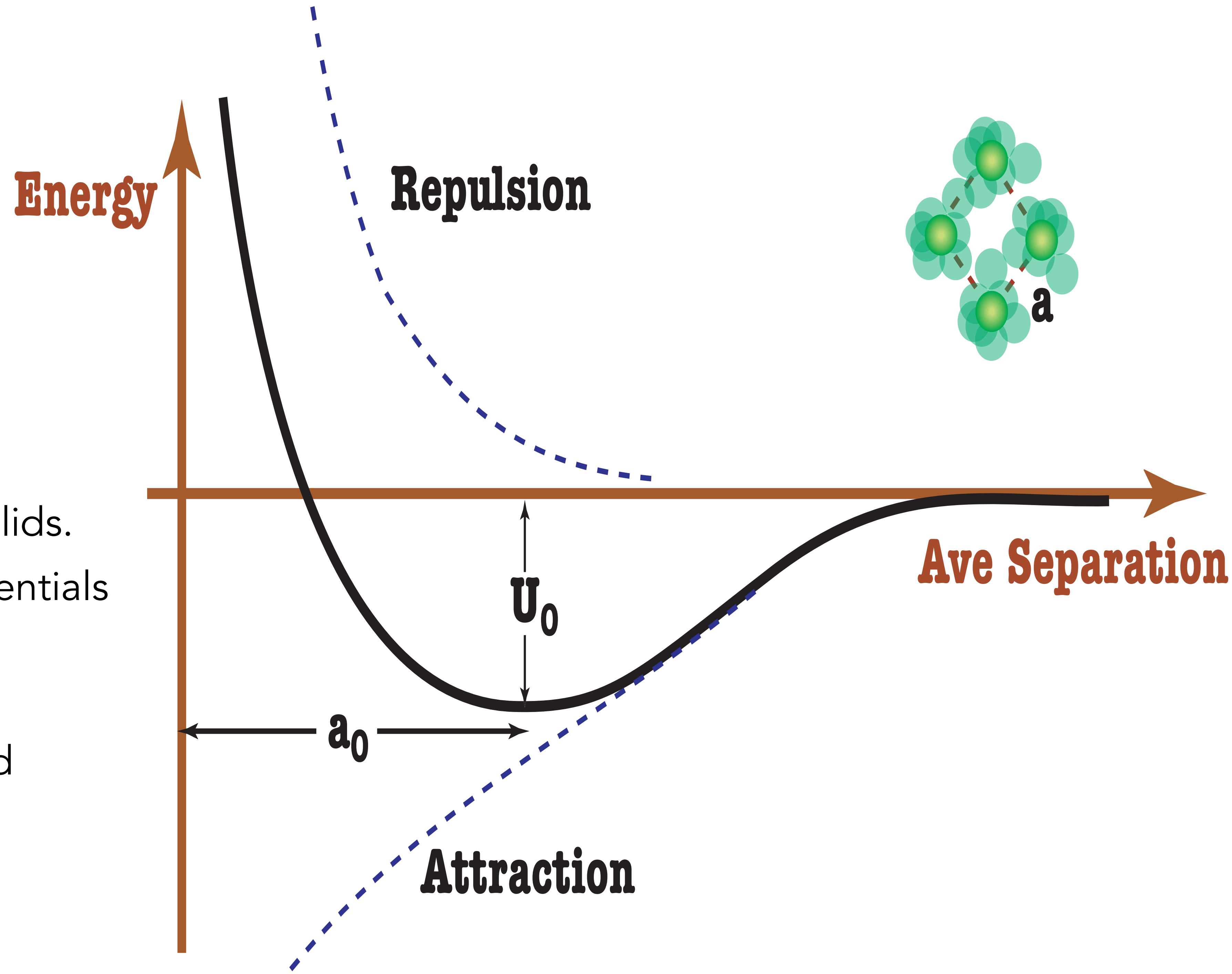
Origin of elasticity

Elastic effects are most important at low strains, modest stresses and short timescales.

The mechanism for elastic deformation comes directly from the interatomic forces in solids. In crystals the lattice energy potentials have a minimum.

Equilibrium spacing is associated with this minimum.

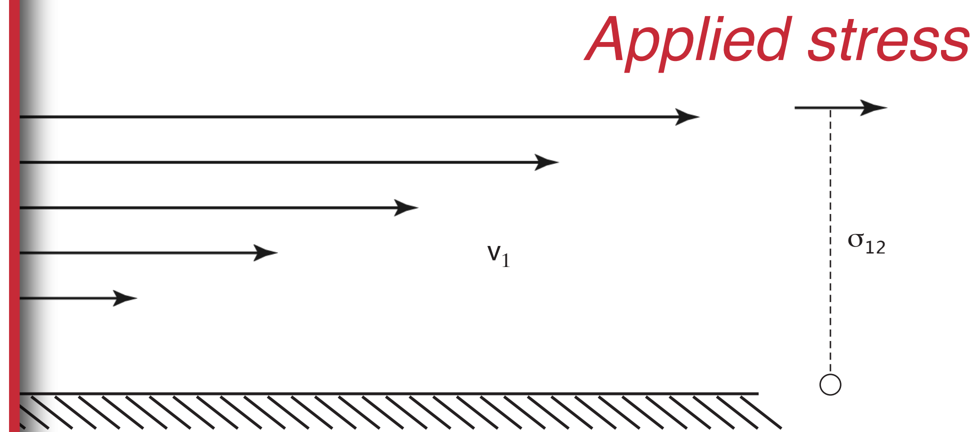
Asymmetry of the potential well is usually cited as the basis for thermal expansivity.

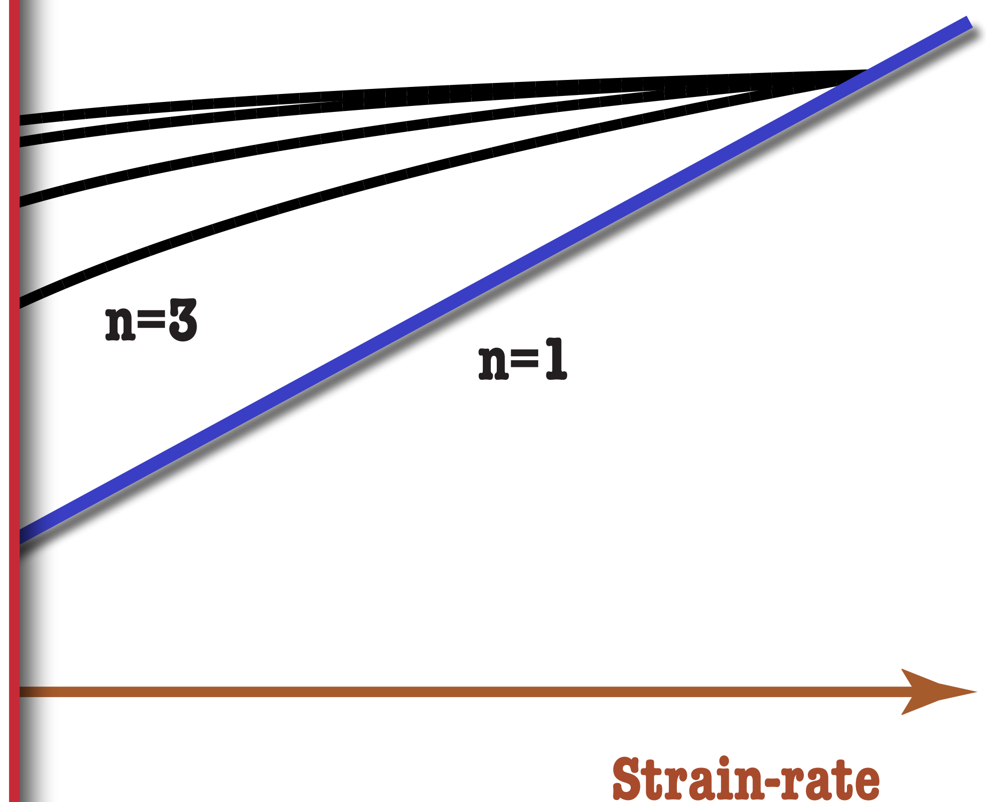


What is viscosity ?

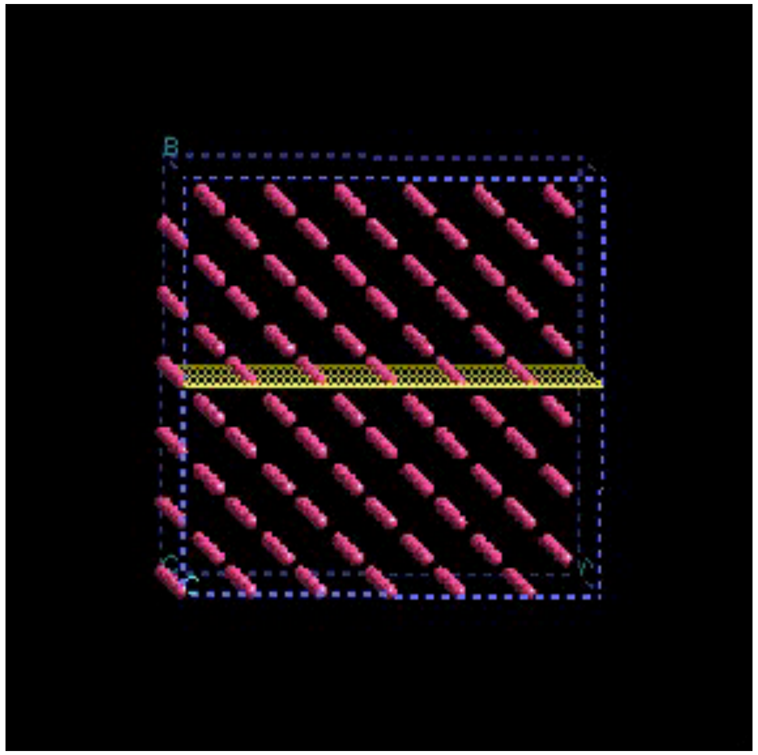
Viscous deformation is irreversible **flow**

Viscosity is a measure of the resistance of a fluid to deform under shear stress. It is commonly perceived as "thickness", or resistance to flow. Viscosity describes a fluid's internal resistance to flow and may be thought of as a measure of fluid friction. Water is runny, having a lower viscosity, while honey is "thick" having a higher viscosity

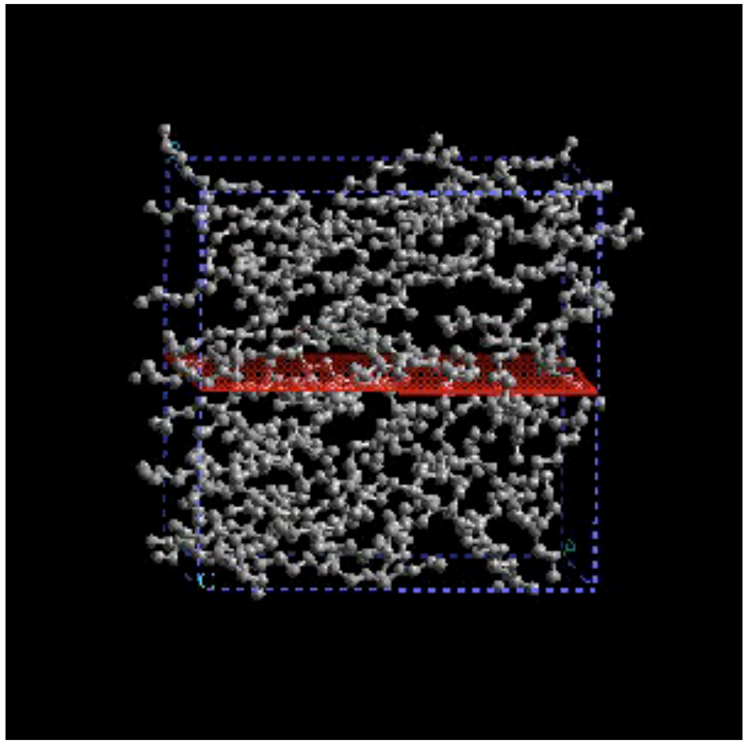




Viscous flow mechanism



Liquid sodium - simulation



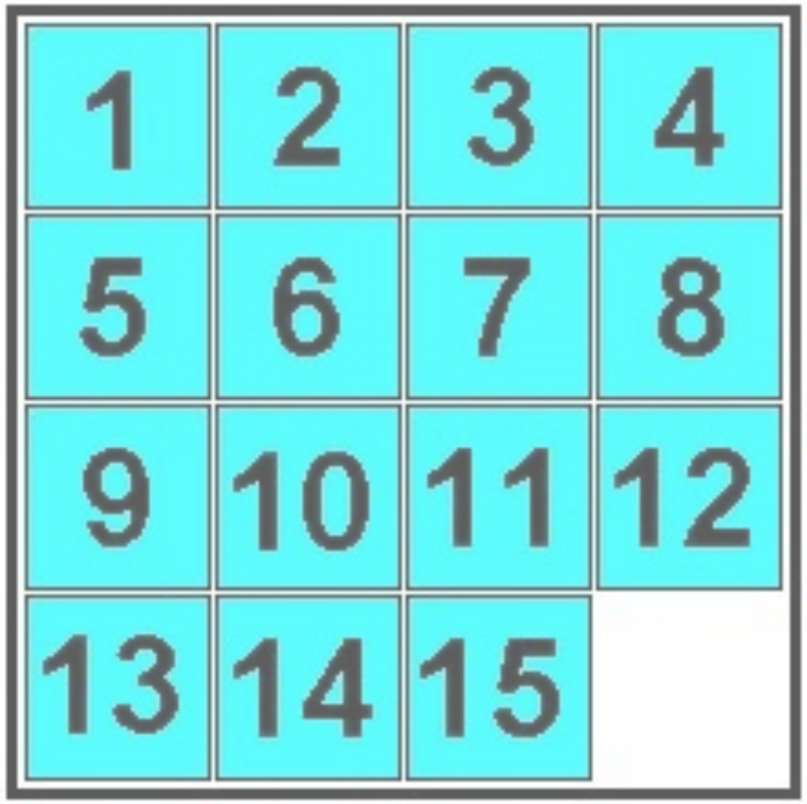
Liquid octane - simulation

In solids the process is **solid state creep** which occurs through movement of crystal defects:

- Point defects diffusing through the crystal
- Point defects diffusing through grain boundaries
- Dislocation movement

In order to achieve viscous deformation there must be some mechanism for permanently re-organizing atoms or molecules (unlike elastic deformation).

Diffusion Creep

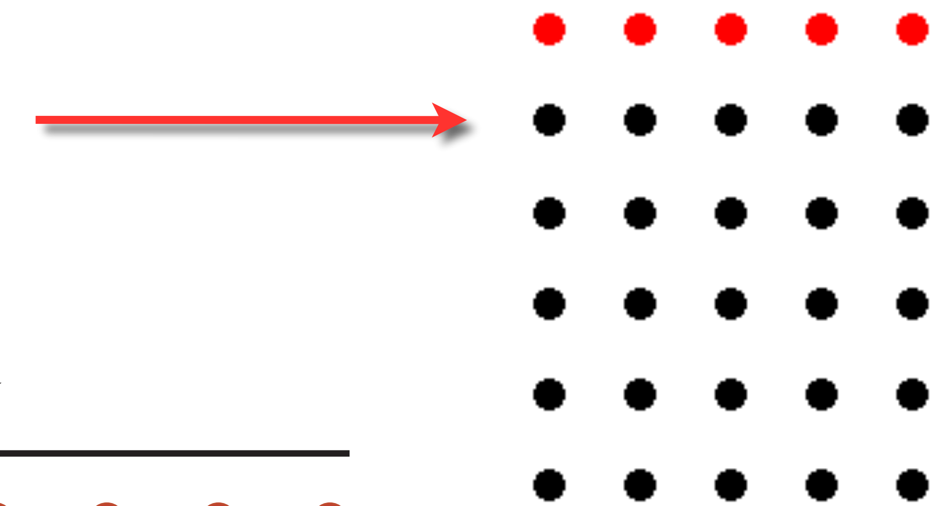


One way is a movement of point defects in the crystal lattice.

If a crystal contains lattice vacancies then diffusion of those vacancies can result in a macroscopic strain.

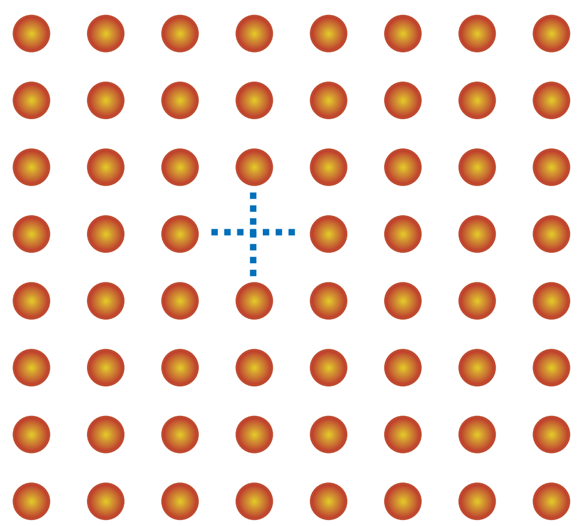
The vacancies can be considered as real "objects"

Diffusion Creep

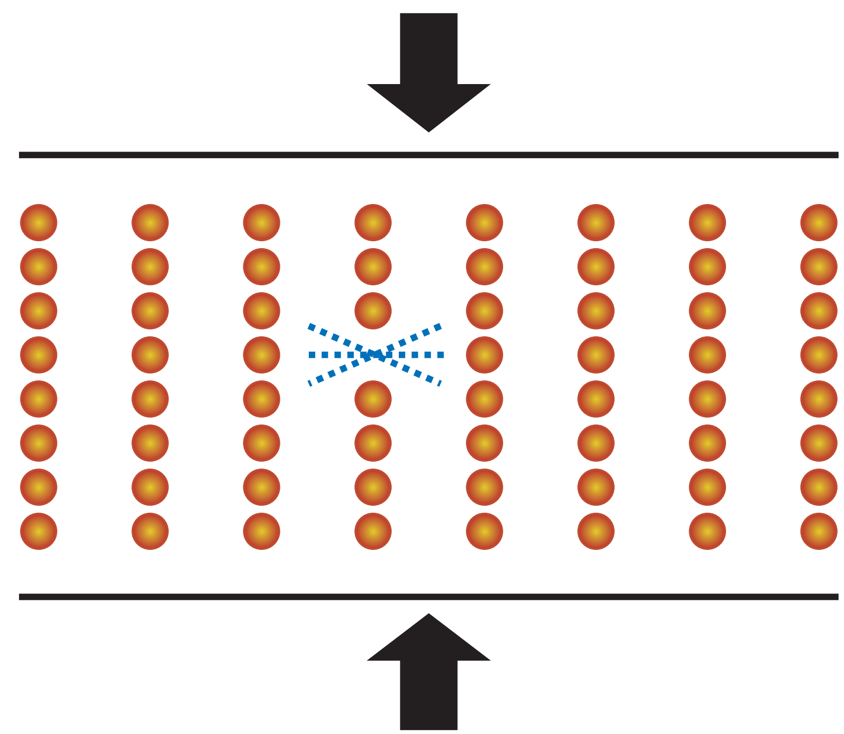


Vacancy Diffusion

Unstressed



Equal barrier in
x,y directions



Barrier higher in y direction,
preferred direction exists

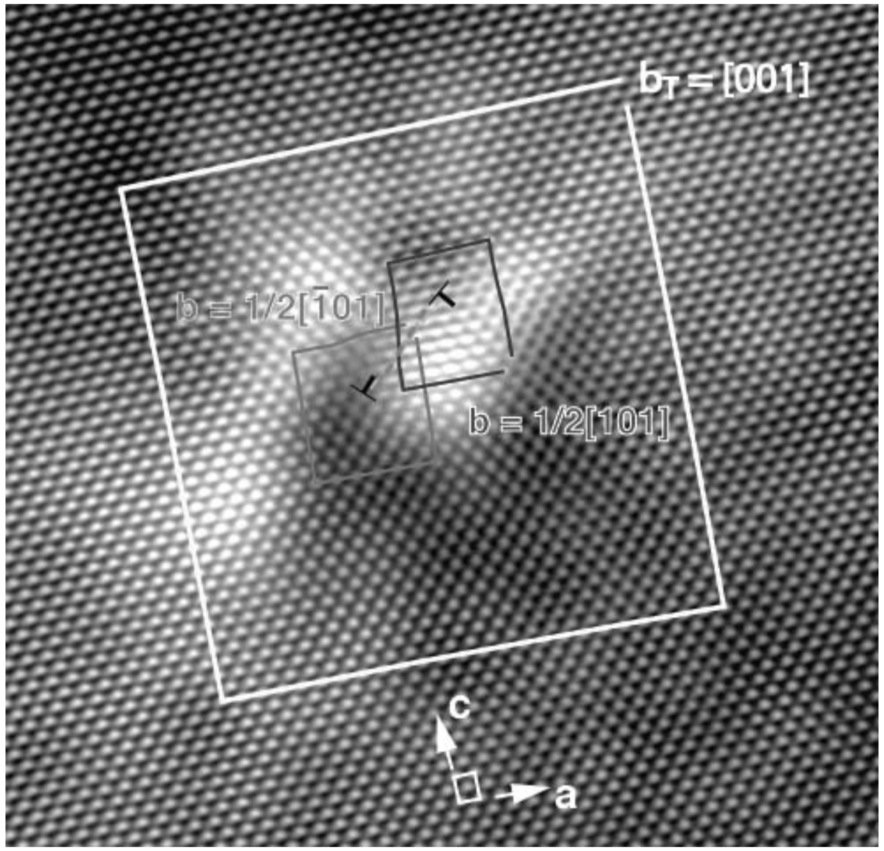
There needs to be a mechanism which "directs" diffusion of the defects to produce the observed macroscopic deformation seen in the movie. This occurs because movement of atoms is slightly easier in the directions with longer bonds

Dislocation Creep

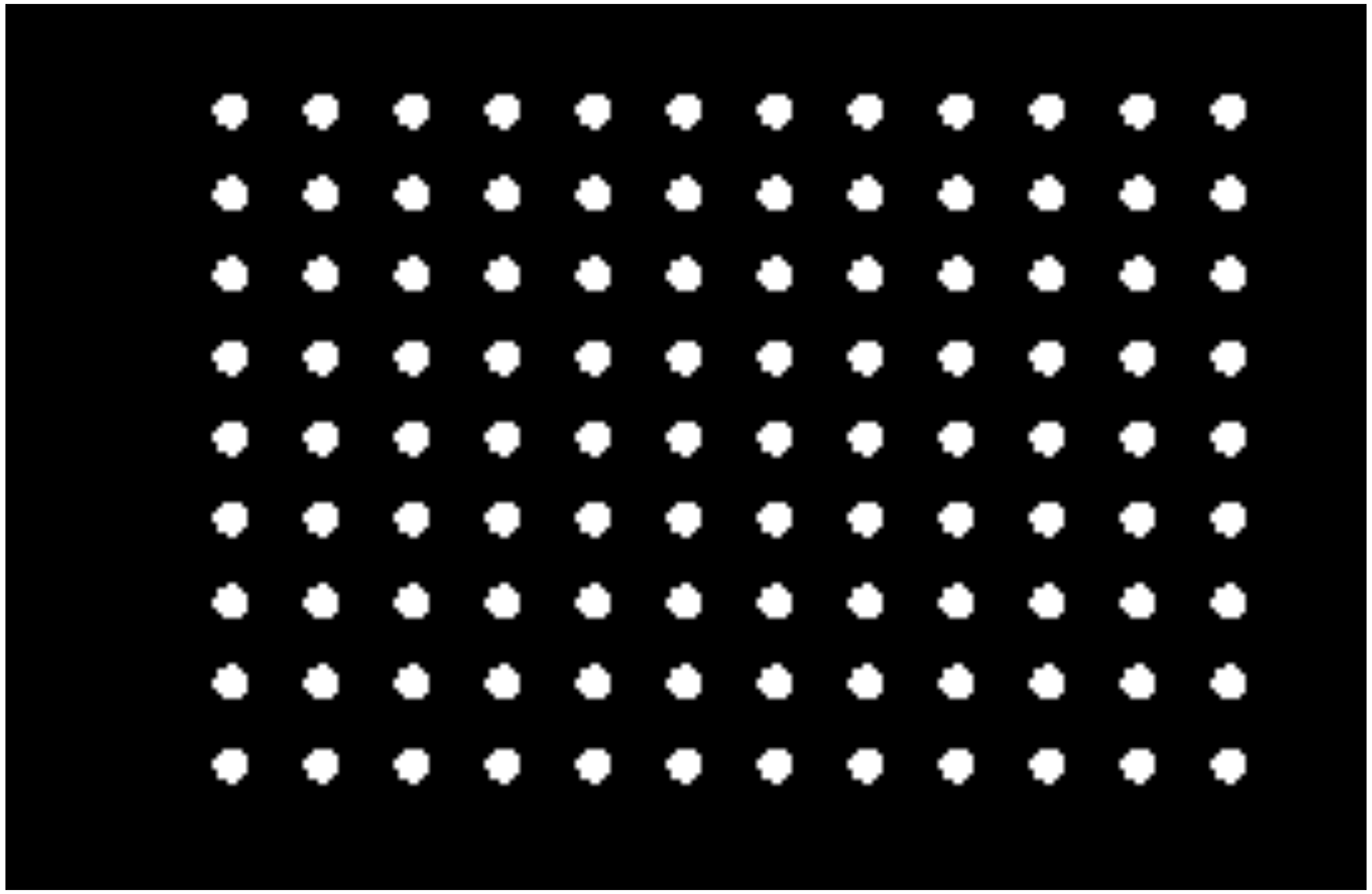
The movement of line defects (**dislocations**) also produces macroscopic deformation.

In each case we can calculate the rate of movement of defects as a function of stress (based on knowing the activation energy required to move atoms).

This movement is a strain rate — if we know strain rate as a function of stress, we know the apparent **viscosity**

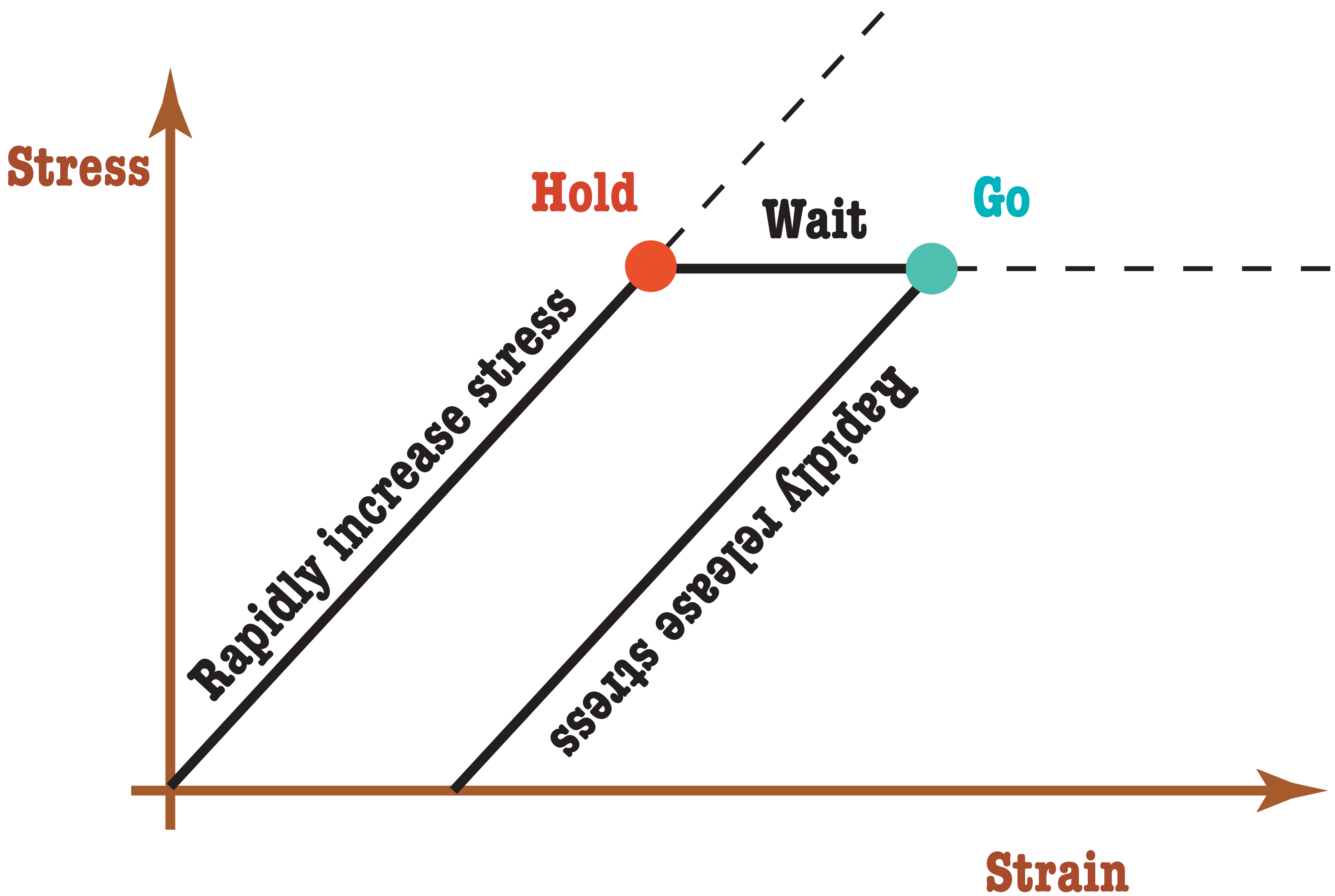


http://rses.anu.edu.au/research/annrep/ar2003/exp/index.php?p=sharp



Viscoelasticity

The reality, which is not surprising given the physical mechanisms involved, is that elastic deformation and viscous deformation occur at the same time.



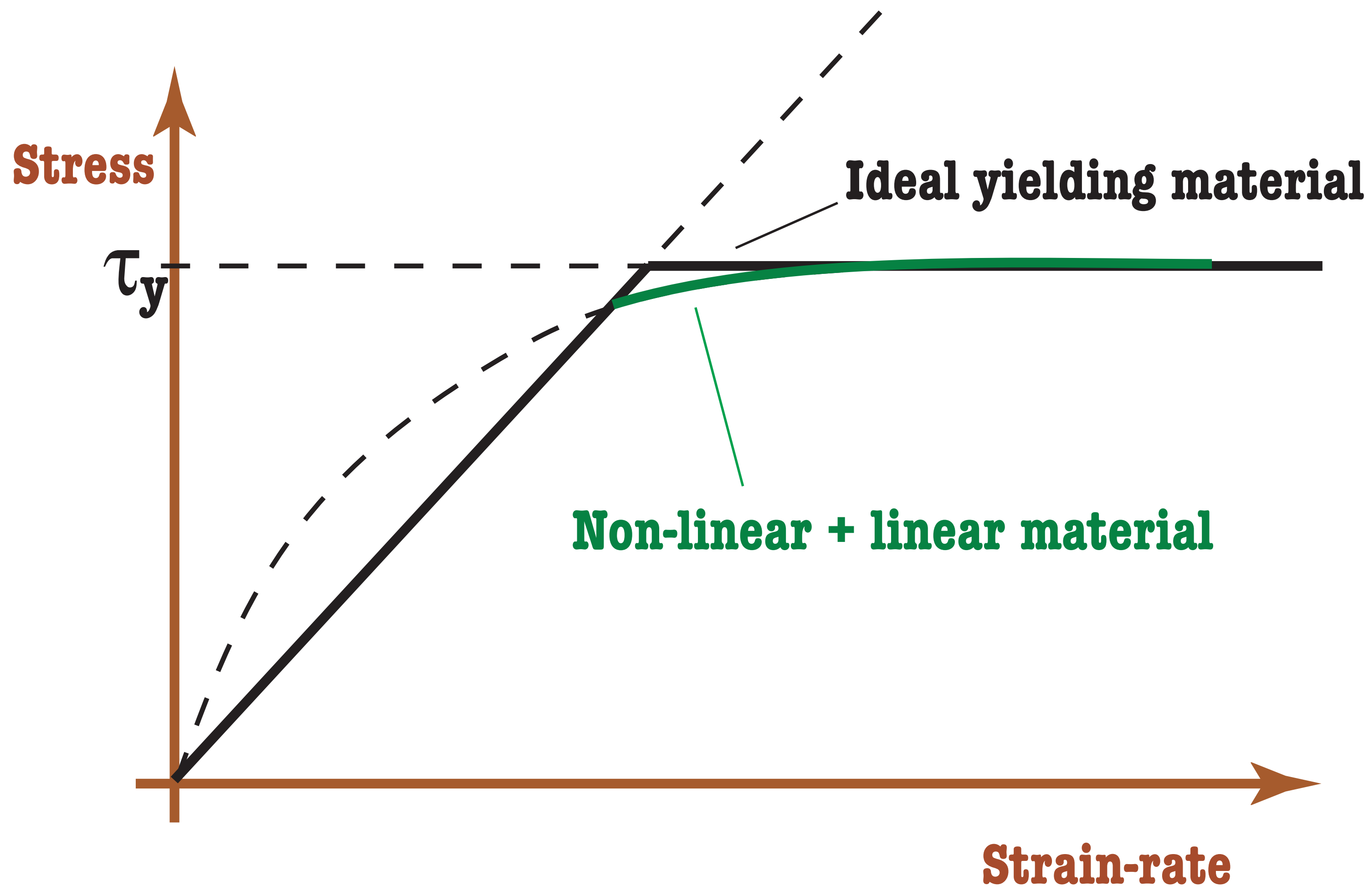
Familiar examples — food and other organic materials which contain long protein molecules in solution

Viscoelasticity

LEFT: Weissenberg effect: A viscoelastic fluid will climb a spinning rod — a viscous fluid is spun away into a hyperbolic shape

RIGHT: Tubeless siphon: a viscoelastic fluid can be "siphoned" without the use of a hose.

Brittle deformation



Our everyday experience is with materials which have a finite strength. Stretched too much or too hard, they will break.

This happens for fluids as well as solids: fluids of this type include toothpaste, gels, sewage sludge (slurries in general) and granular materials.

We can interpret yielding in terms of elastic bond breakage, crack development, frictional bonding in granular materials.

Application

We now have an idea why / how solid materials can behave as fluids —

- Movement of crystal defects
- Slip on fine-grain faults / cracks

Now we can move on to how the science of fluid dynamics applies to the Earth as a planet

- Convection equations
- How they scale to rocks instead of water or syrup
- Dimensionless numbers
- Flow patterns
- Observables

Navier Stokes equation

$$\rho_0 \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \eta \nabla^2 \mathbf{v} - \nabla p - g \rho_0 \alpha (T - T_0) \hat{\mathbf{z}}$$

$$\left(\frac{\partial T}{\partial t} + (\mathbf{v} \cdot \nabla) T \right) = \kappa \nabla^2 T$$

Depending upon the following physical parameters which vary from experiment to experiment & planet to planet

viscosity	η	$Pa.s$
temperature scale	$\Delta T = T - T_0$	K
length scale	x	m
density	ρ_0	$kg.m^{-3}$
thermal expansivity	α	K^{-1}
gravity	g	$m.s^{-2}$
thermal diffusivity	κ	$m^2 s^{-1}$

Substituting for all the existing terms in the Navier-Stokes equation

$$\frac{\rho_0 \kappa}{d^2}\frac{D}{Dt'}\left(\frac{\kappa}{d}\mathbf{v}'\right) = \frac{\eta}{d^2}\acute{\nabla}^2\left(\frac{\kappa}{d}\mathbf{v}'\right) - \frac{\eta\kappa}{d^3}\acute{\nabla} p' + g\rho_0\alpha\Delta T T'\hat{\mathbf{z}}$$

Collect up terms:

$$\frac{\rho\kappa}{\eta}\frac{D\mathbf{v}'}{Dt'} = \acute{\nabla}^2\mathbf{v}' - \acute{\nabla} p' + \frac{g\rho_0\alpha\Delta T d^3}{\kappa\eta}T'\hat{\mathbf{z}}$$

or

$$\boxed{\frac{1}{\mathrm{Pr}}\frac{D\mathbf{v}'}{Dt'} = \acute{\nabla}^2\mathbf{v}' - \acute{\nabla} p' + \mathrm{Ra}\,T'\hat{\mathbf{z}}}$$

Pr is known as the Prandtl number, and **Ra** is known as the Rayleigh number; both are non-dimensional numbers.

Rayleigh number — measures the intensity of the buoyancy forces

$$\mathrm{Ra} = \frac{g\rho\alpha\Delta T d^3}{\eta\kappa}$$

Reynolds number — measures the relative importance of intertial to viscous forces

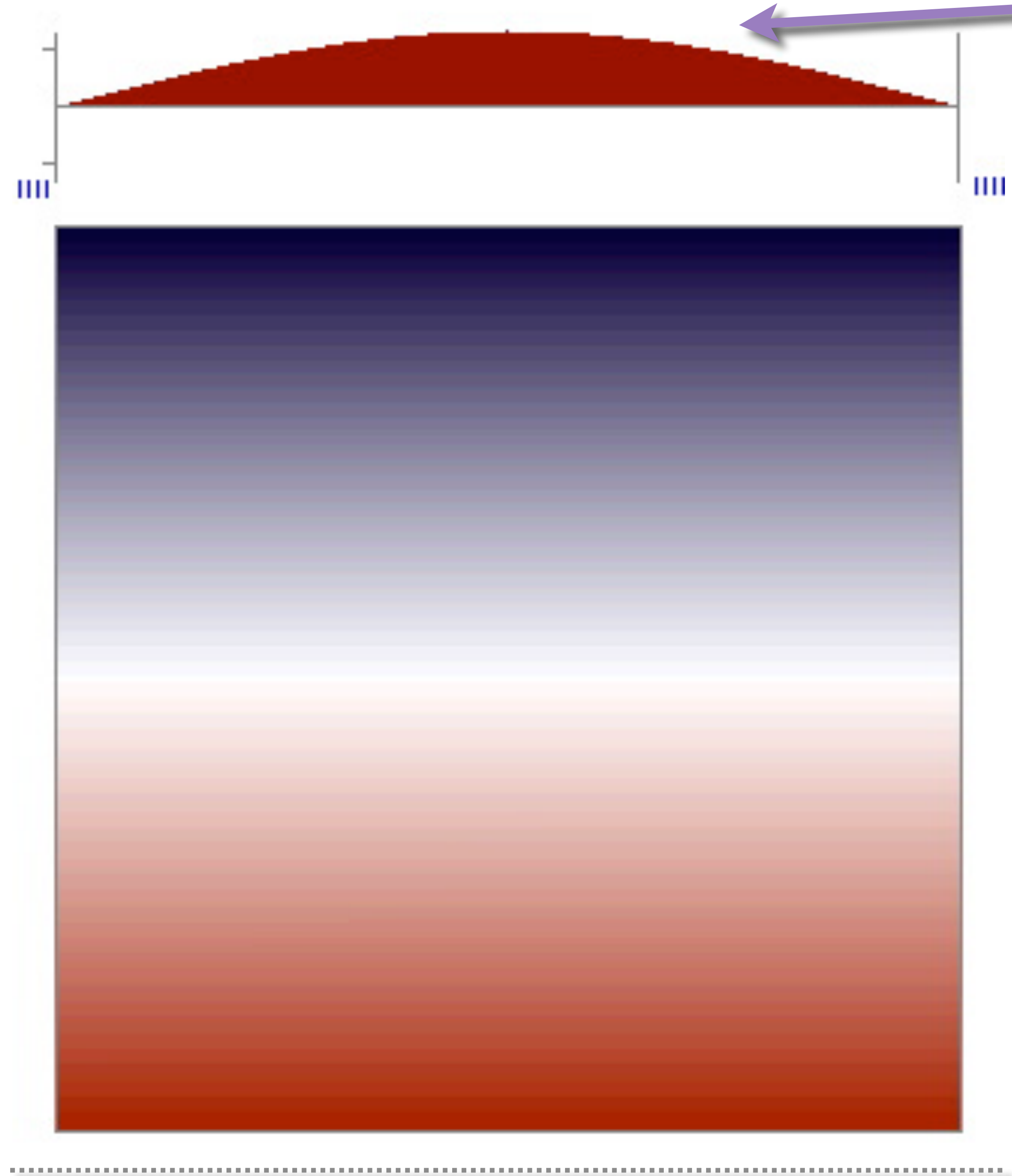
$$\mathrm{Re} = \frac{\rho V_0 d}{\eta} = \frac{V_0 d}{\nu}$$

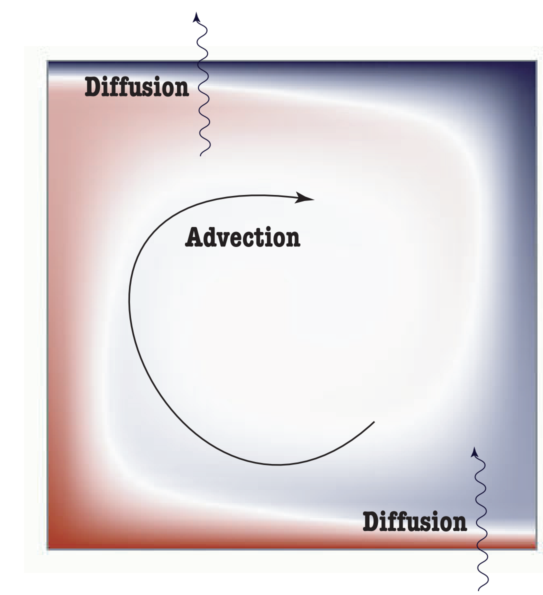
Prandtl number — measures the relative importance of stress diffusion and thermal diffusion

$$\mathrm{Pr} = \frac{\eta}{\rho\kappa} = \frac{\nu}{\kappa}$$

Thermal convection

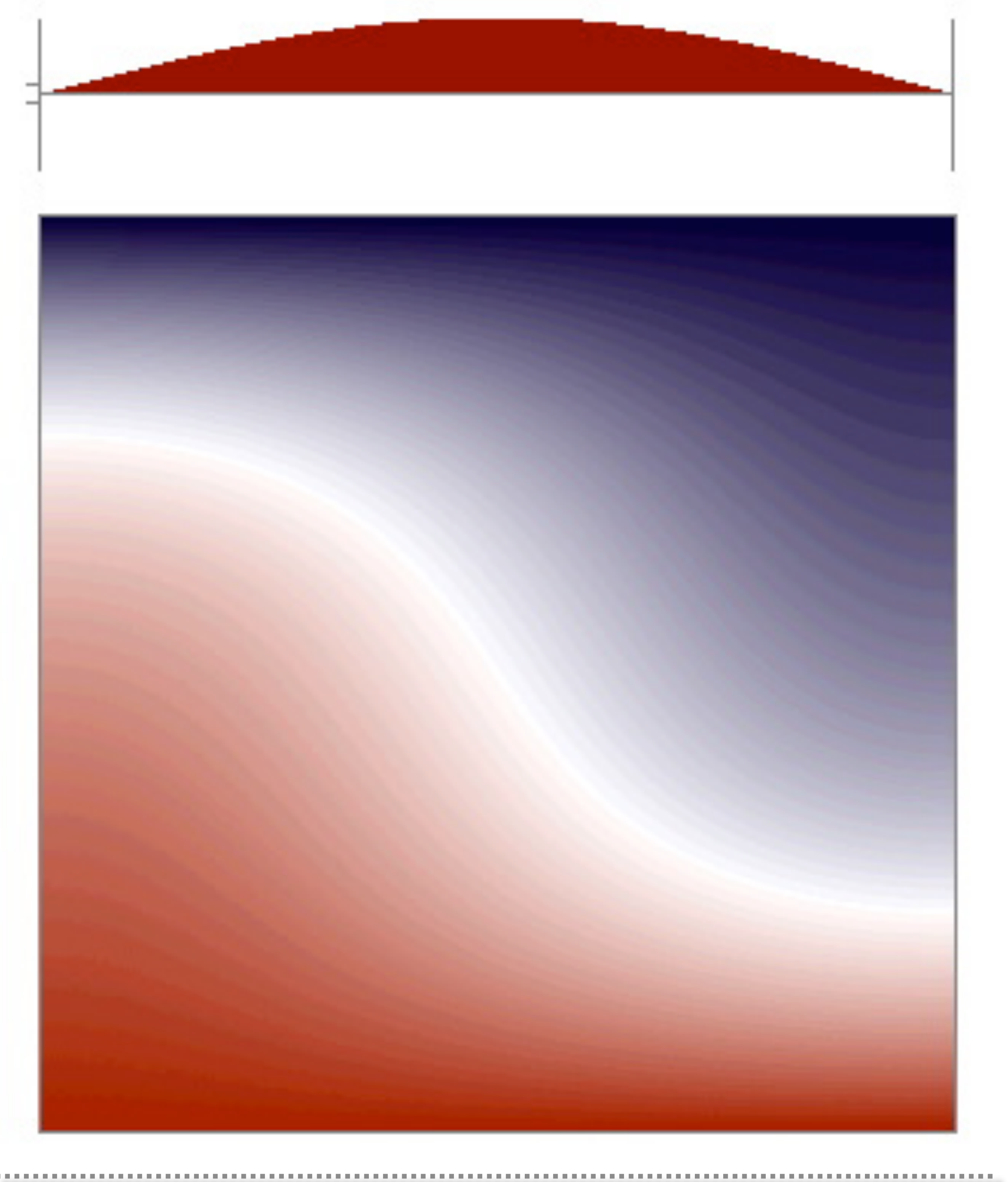
Surface velocity

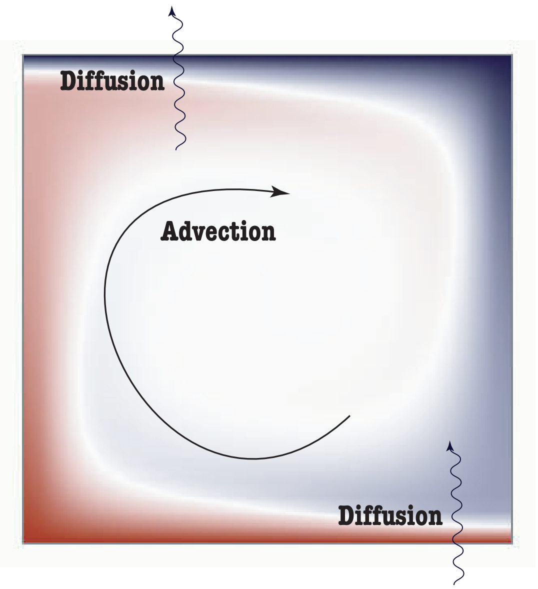




Rayleigh number = 100

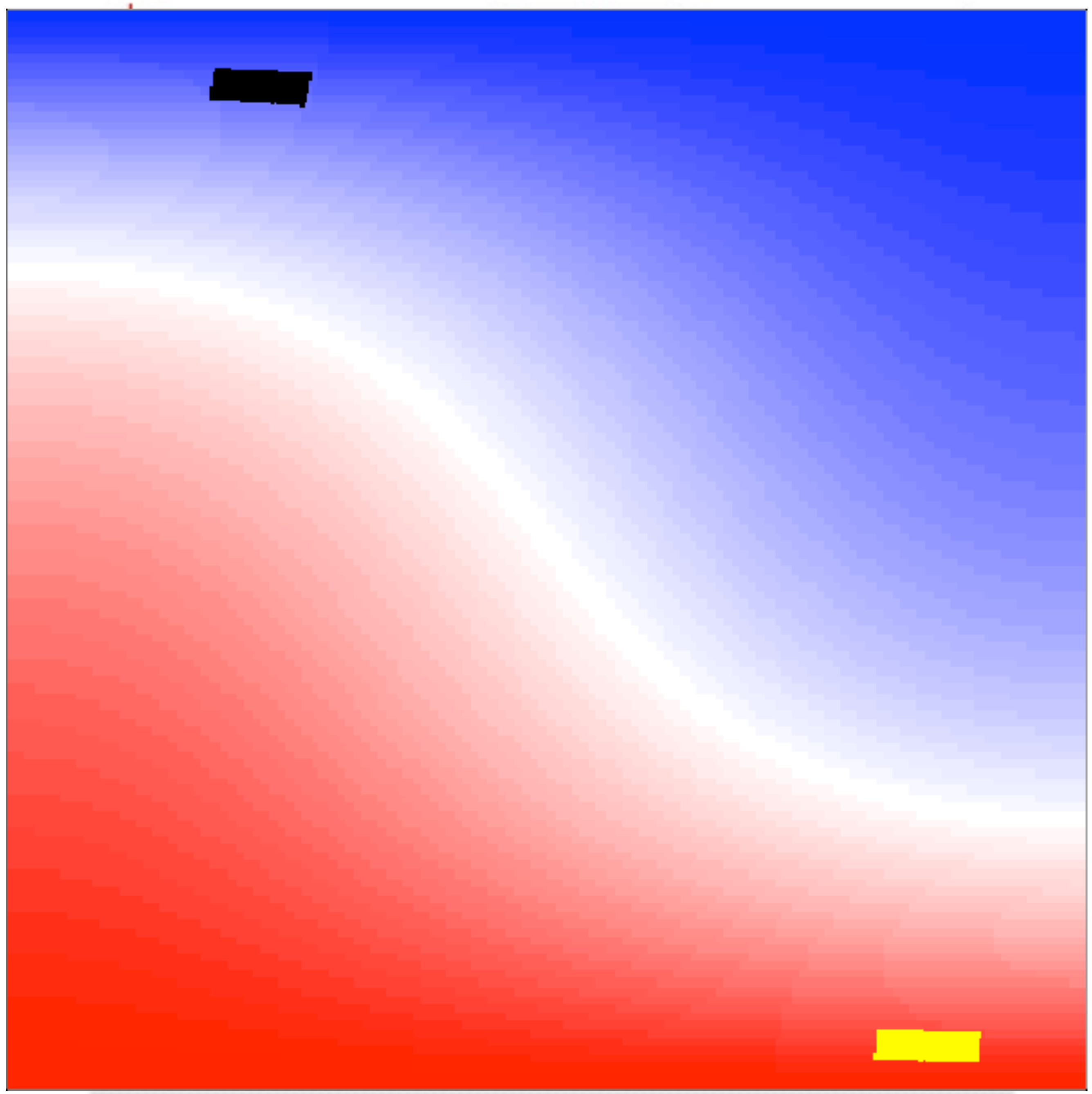
Thermal convection

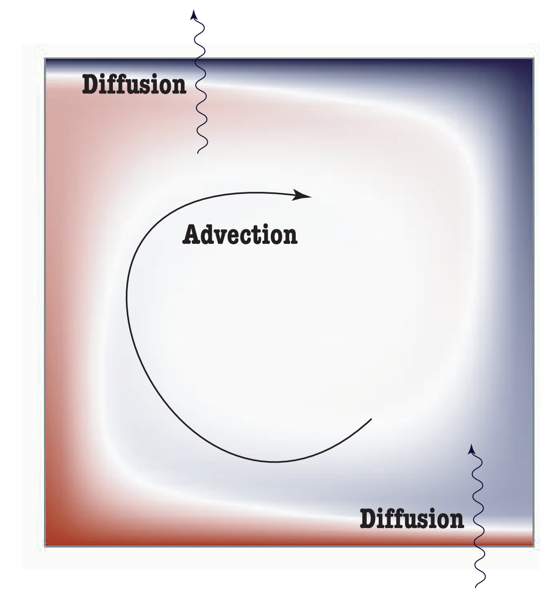




Rayleigh number = 1000

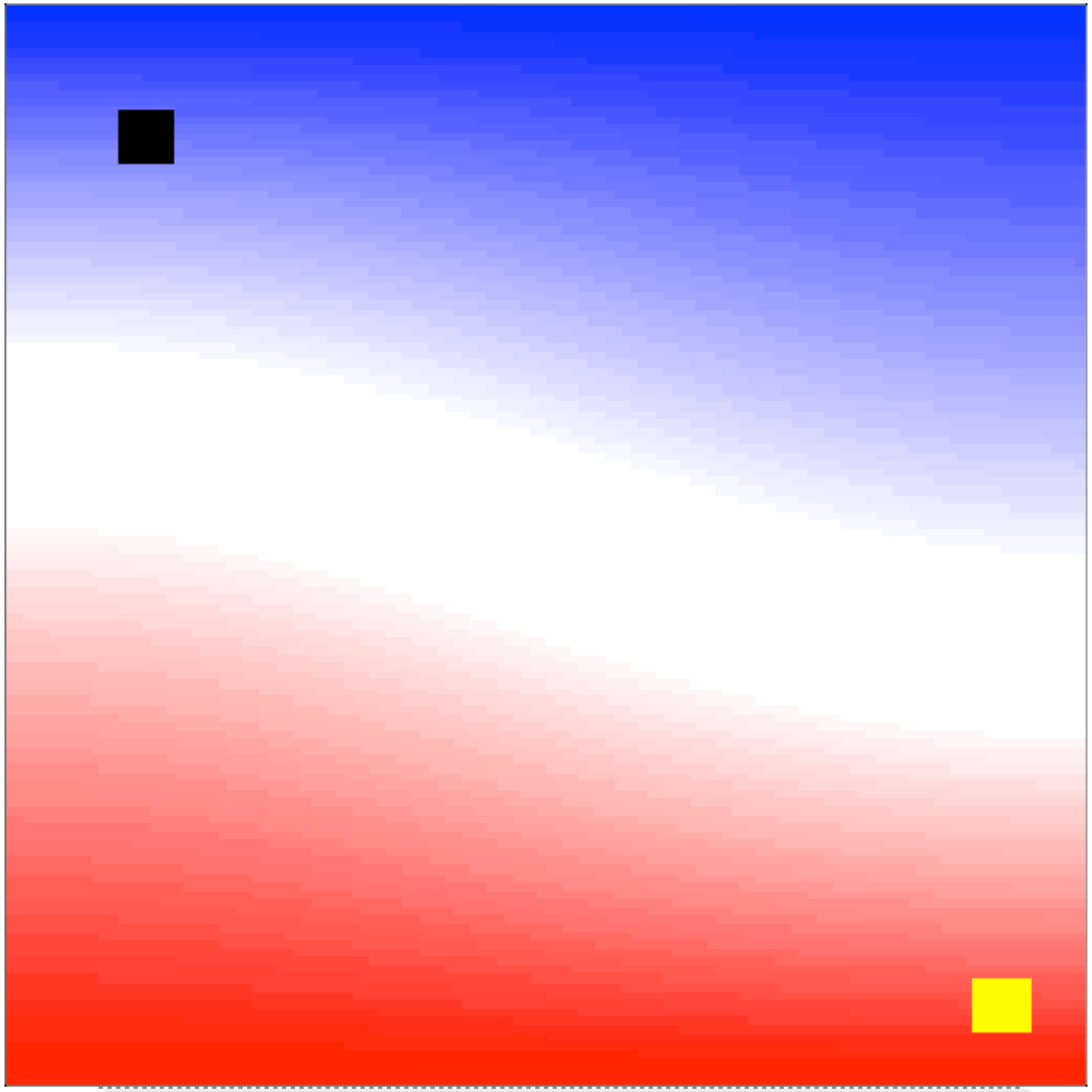
Thermal convection

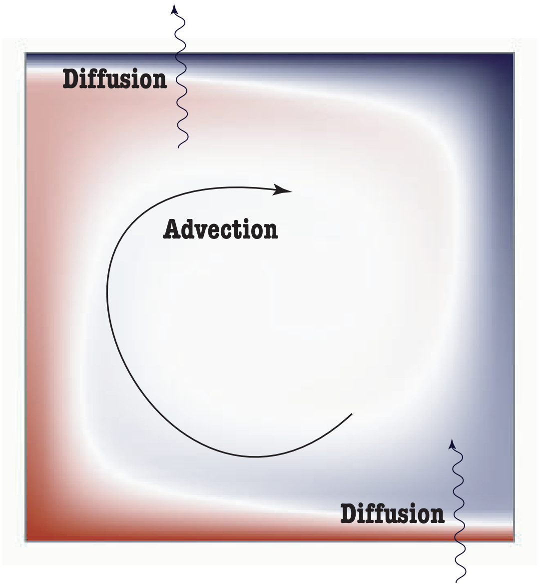




Rayleigh number = 10000

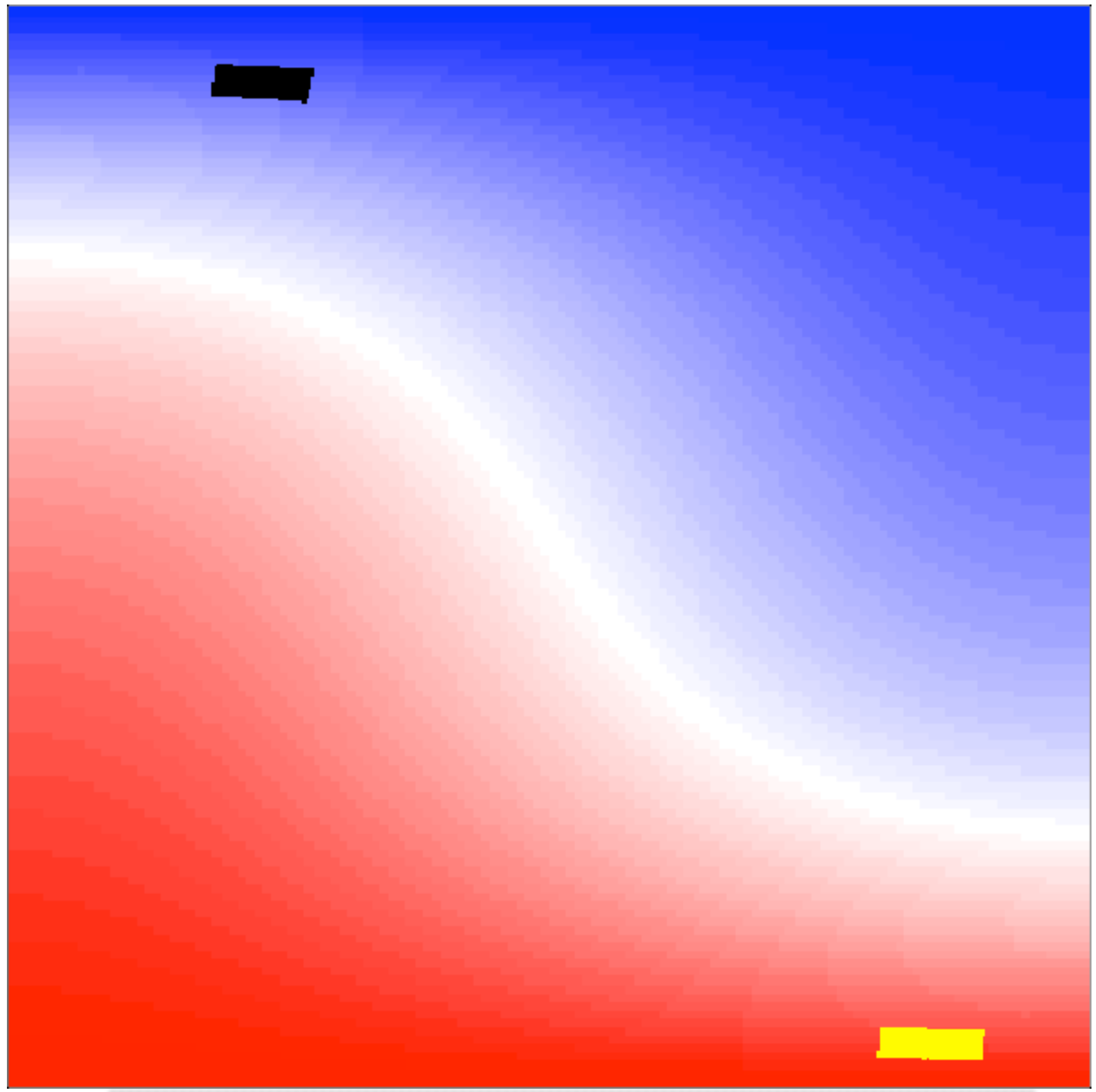
Thermal convection

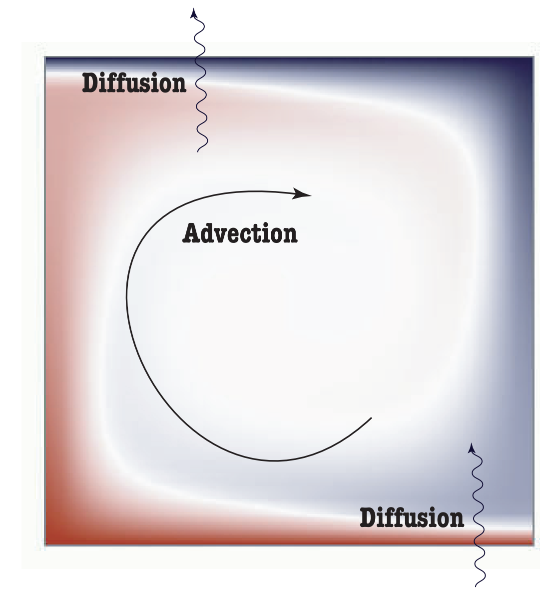




Rayleigh number = 100000

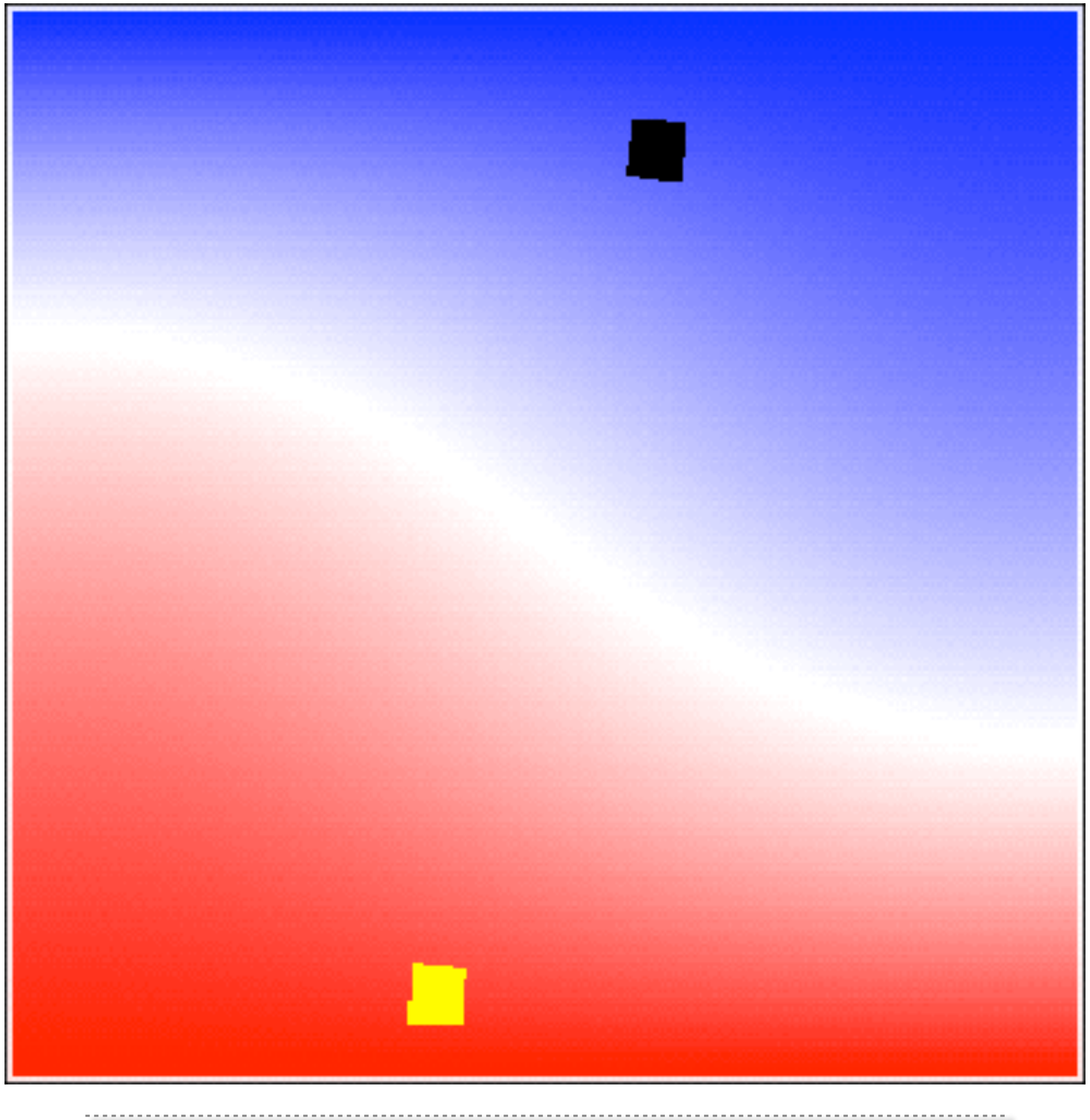
Thermal convection

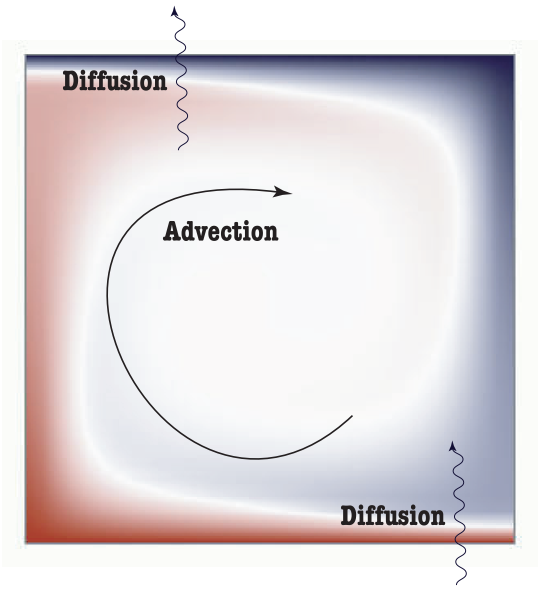




Rayleigh number = 1000000

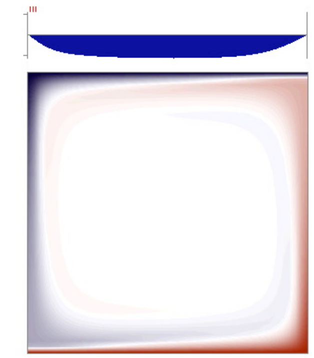
Thermal convection

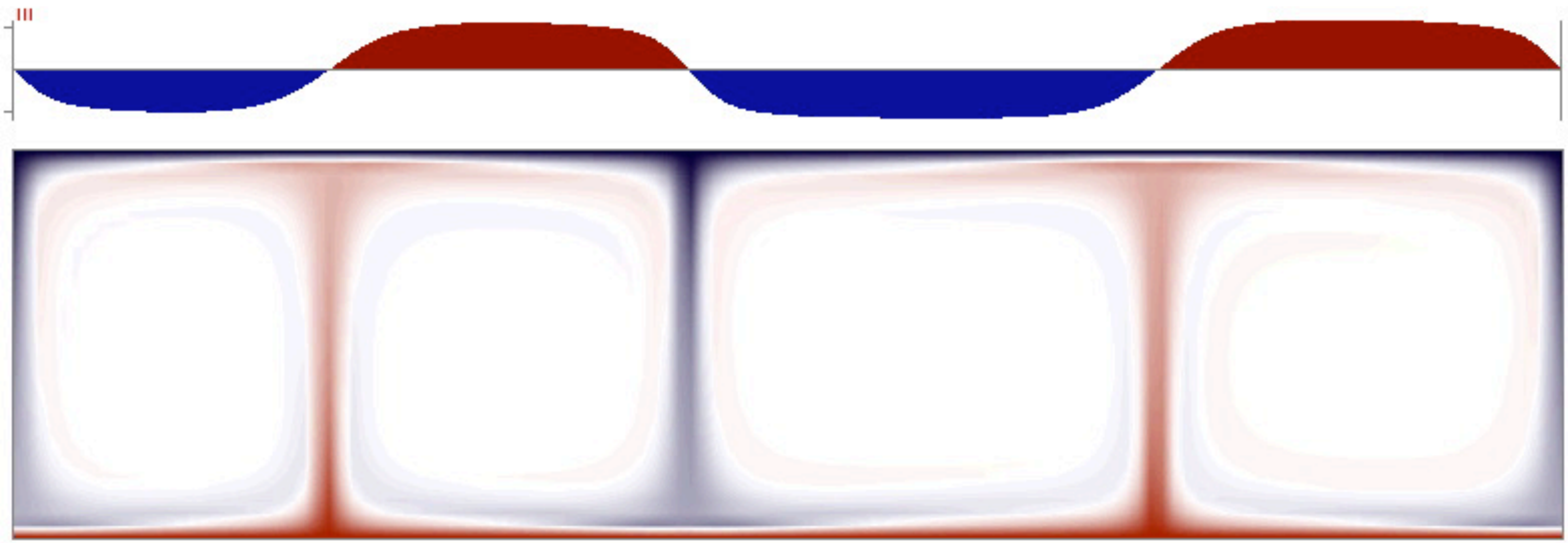




Rayleigh number = 10000000

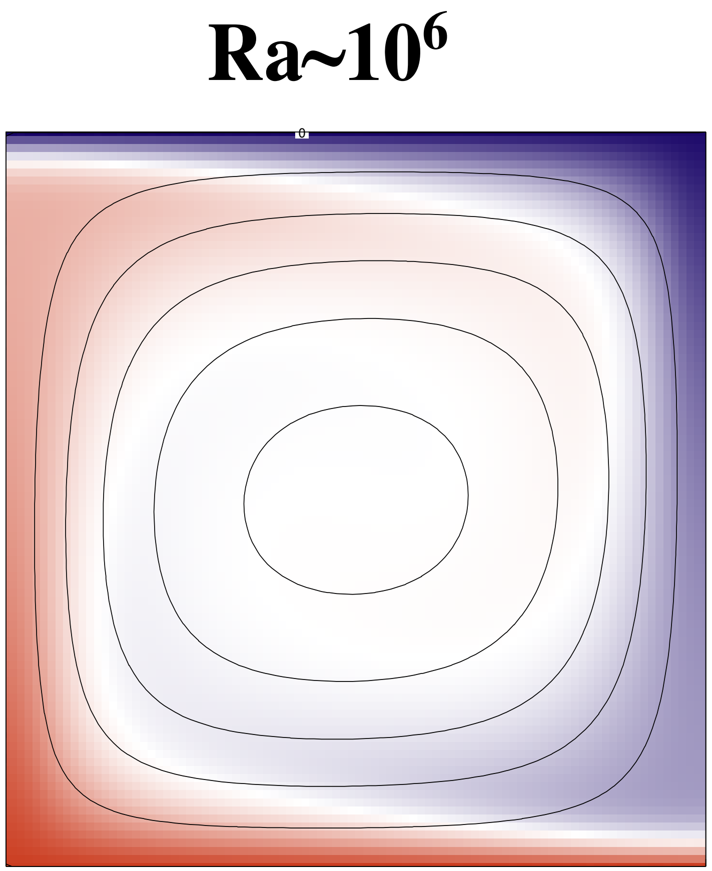
Thermal convection

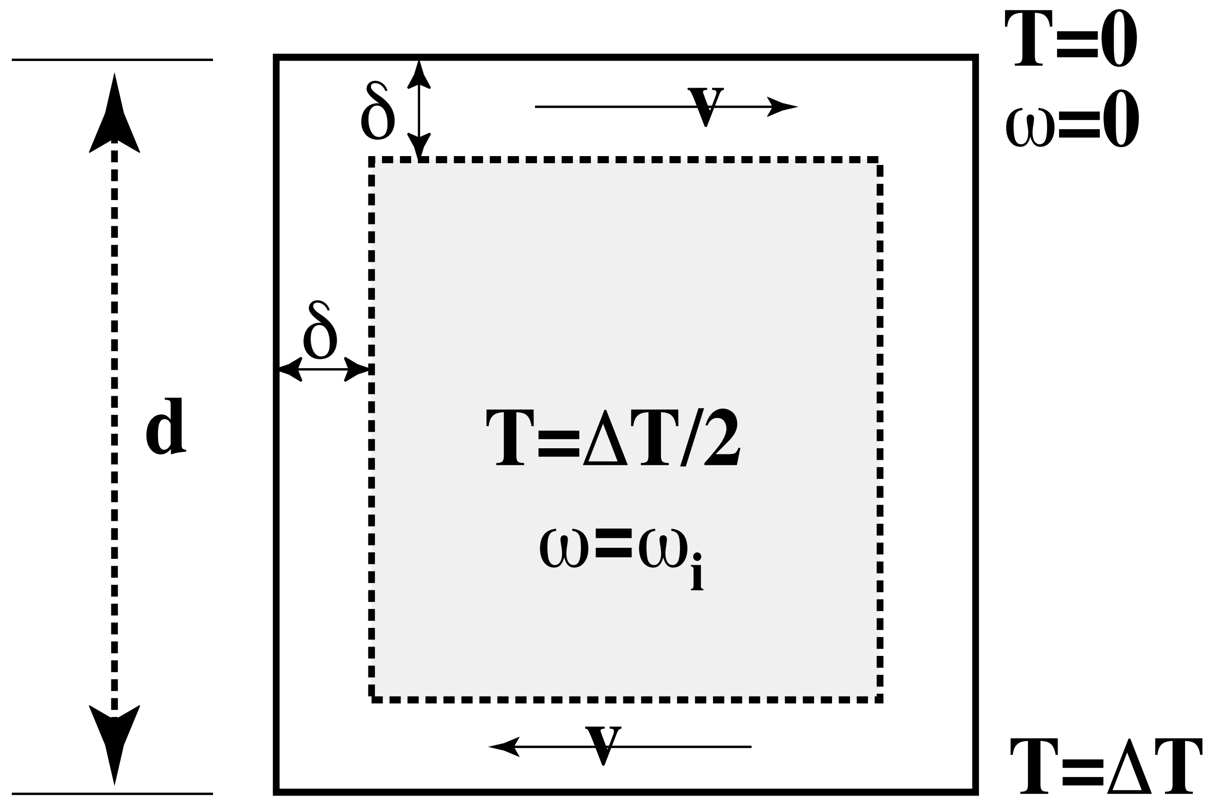




Rayleigh number = 1000000, wide aspect ratio

Thermal convection



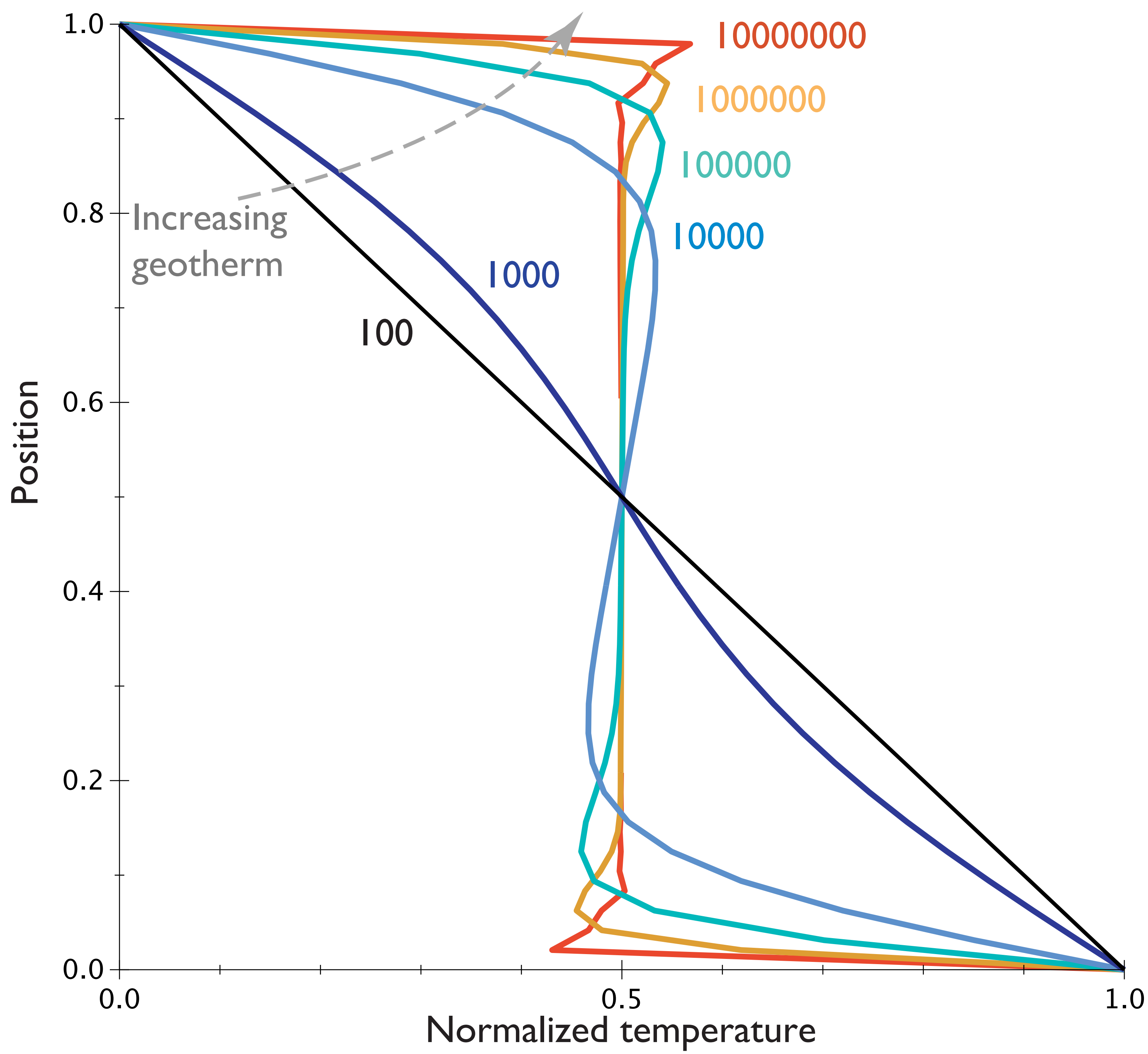


A relatively simple pattern emerges from the competition between advection of heat by fluid flow and diffusion

We will see this picture again when we consider the patterns of heat flow in the Earth's ocean floor.

Convection length and timescales are determined by the Rayleigh number. What is the Ra for the Earth ?

Thermal convection



$$Ra = \frac{g \rho \alpha \Delta T d^3}{\kappa \eta}$$

The temperature (velocity, stress etc) profiles vary systematically with Rayleigh number.

Nusselt number

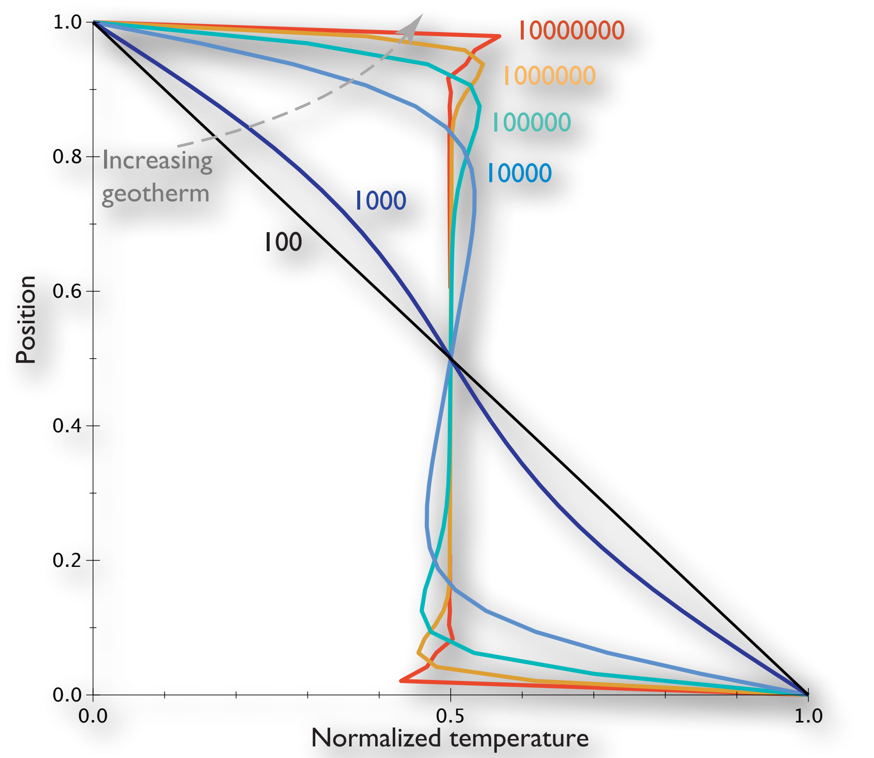
A third, independent number can be derived for the thermally driven flow equations. This is known as the Nusselt number and it relates to the heat transported by the layer:

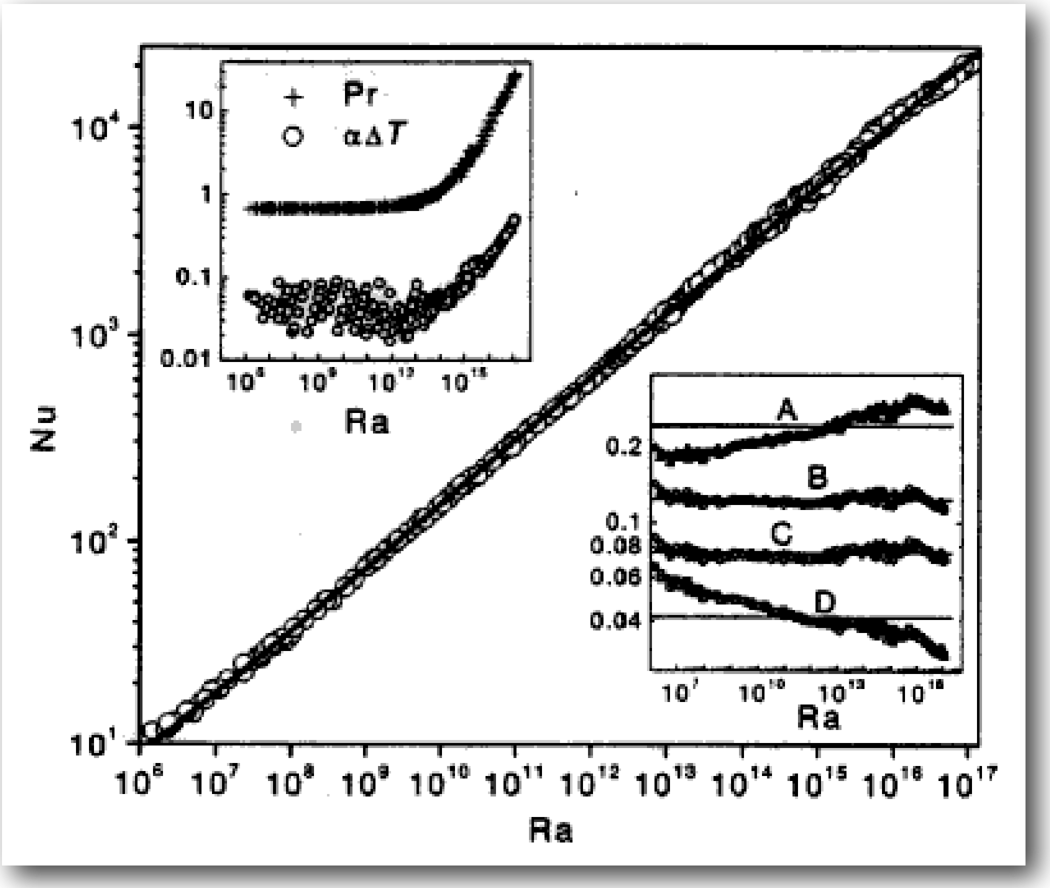
$$\mathrm{Nu} = \frac{qd}{k\Delta T}$$

The Nusselt number is the ratio of the heat transported by the convecting fluid (q) to the heat transported by an equivalent solid material having the same thermal conductivity. The convective heat transport includes a conductive contribution so the Nusselt number is always greater than one.

It is observed that

$$\mathrm{Nu} \propto \mathrm{Ra}^{1/3}$$



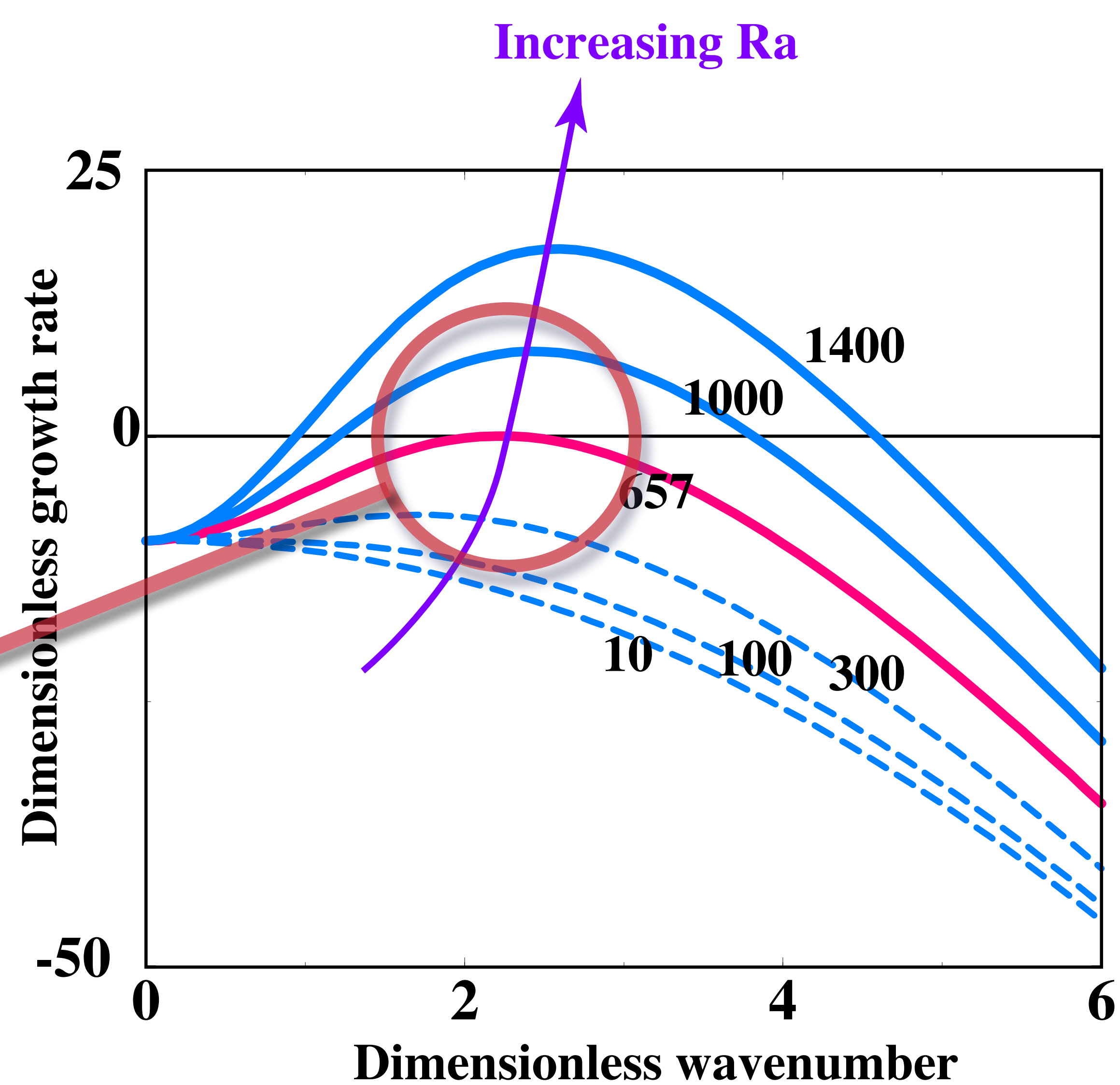


Critical Rayleigh number

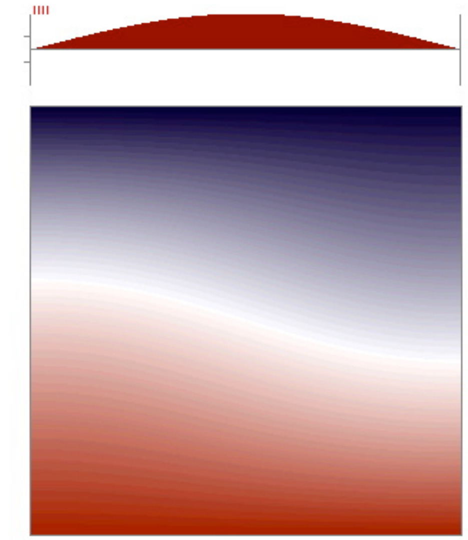
At low Rayleigh number diffusion dominates advection: fluid motion and any lateral temperature variations die away. At high **Ra**, the reverse is true: lateral temperature variations are not damped out, but grow rapidly

$$Ra_c = \frac{27}{4}\pi^4 = 657.51$$

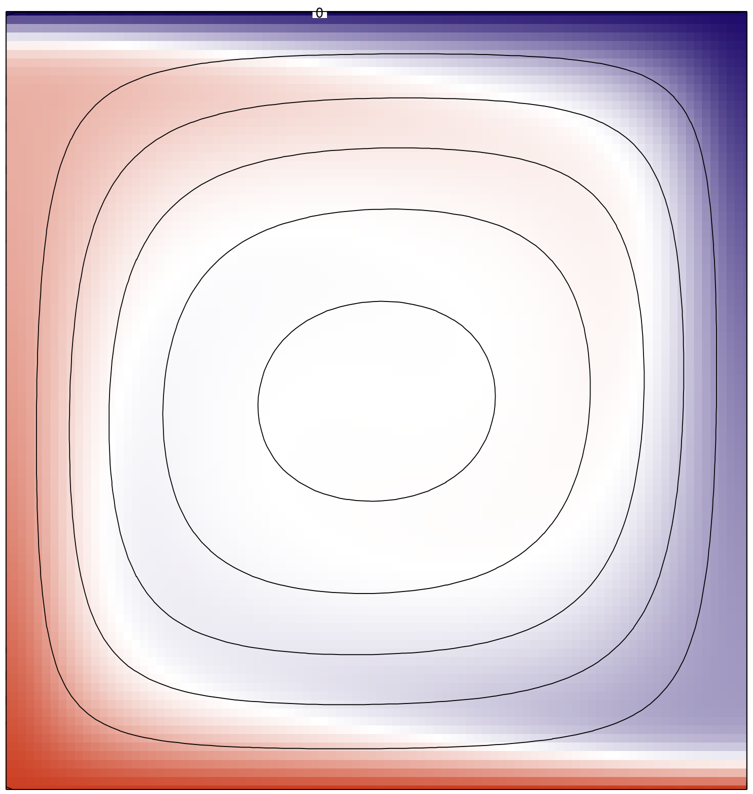
$$k = \frac{\pi}{2^{1/2}} = 2.22$$

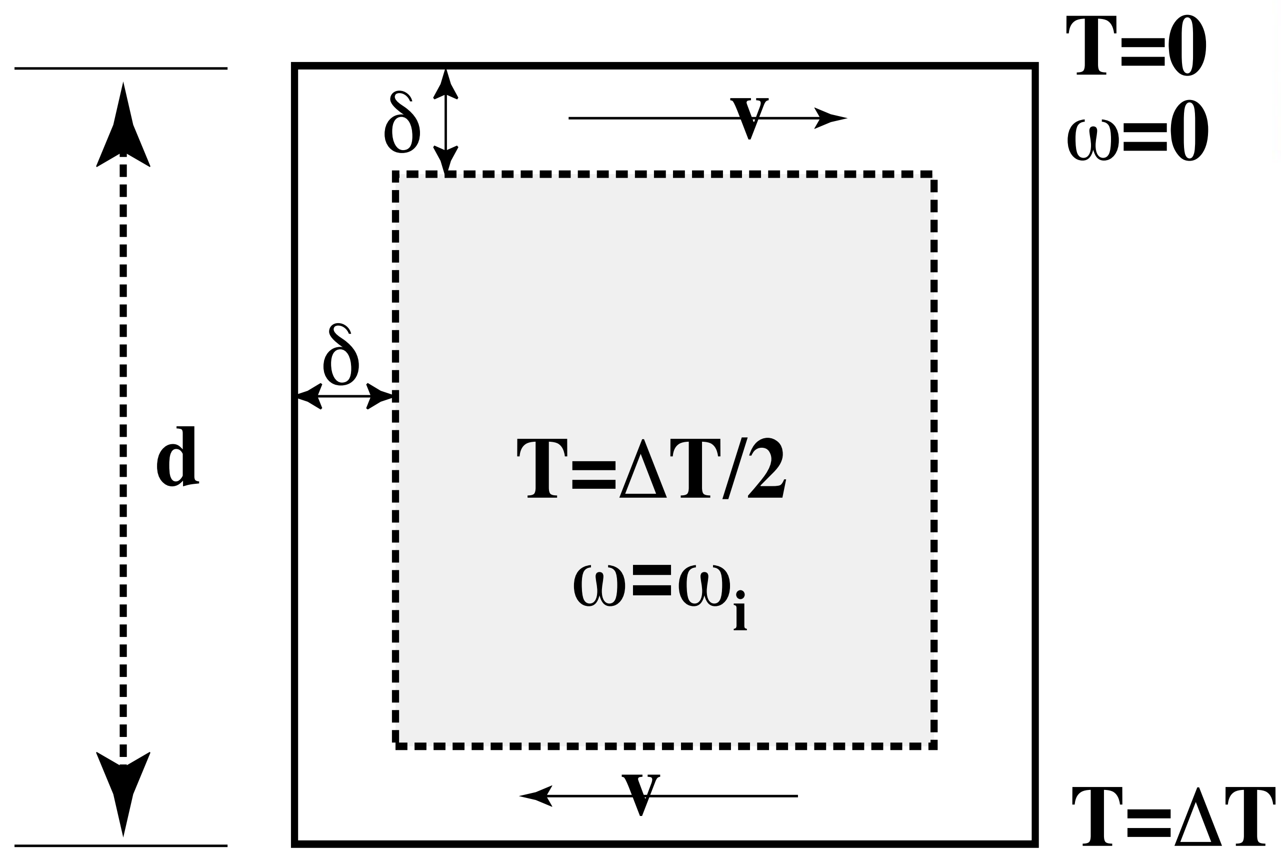


Boundary layer theory



Ra~10^6



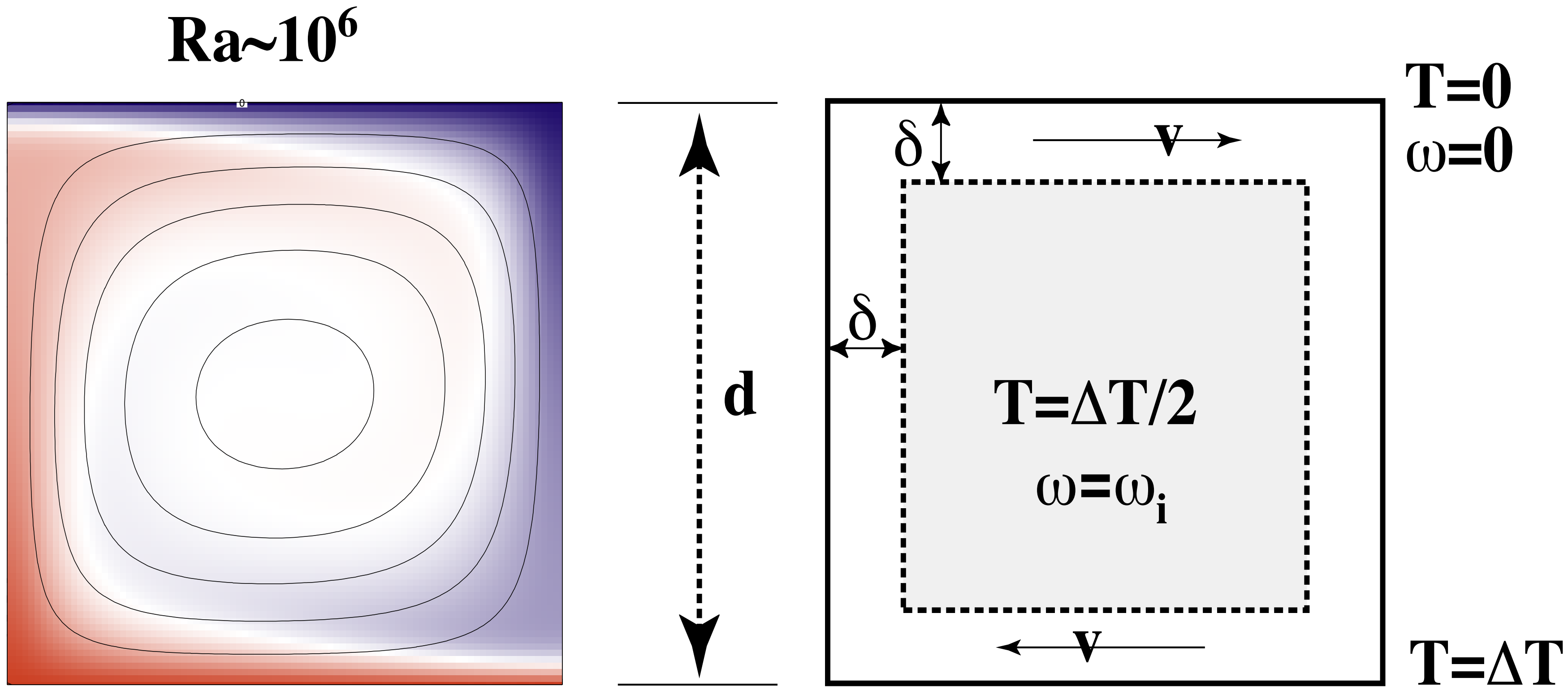


We can analyze simple, constant viscosity convection using boundary layer theory — provided we already have some idea of the geometry of fully developed convection.

Boundary layer theory

We assume the geometry of fully developed convection is as follows

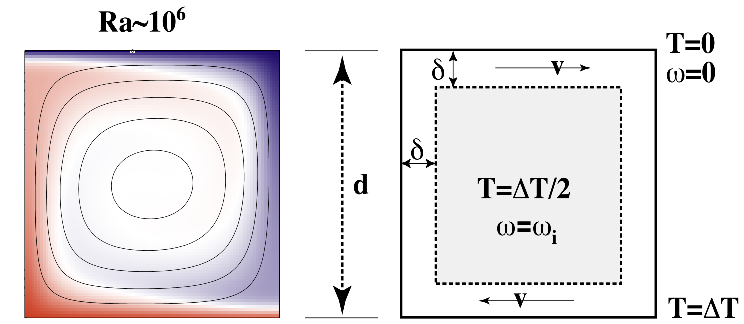
- Narrow thermal boundary layers
- Approximately isothermal core



Thin horizontal boundary layers where heat diffuses into the layer and is carried away by horizontal advection. These are connected by vertical boundary layers of similar dimension.

Gradients along boundary layers are typically on a scale of the layer depth (large so leads to small gradients). Gradients across the boundary layers are typically large because the boundary layers are thin

Boundary layer theory



For the vertical thermal boundary layer, assume that:

$$\frac{\partial v_2}{\partial x_2} \approx \frac{V}{\delta}; \qquad \frac{\partial v_2}{\partial x_1} \approx \frac{V}{d}; \qquad v_1 = 0; \qquad \frac{\partial T}{\partial x_1} \approx \frac{\Delta T}{\delta}$$

So that

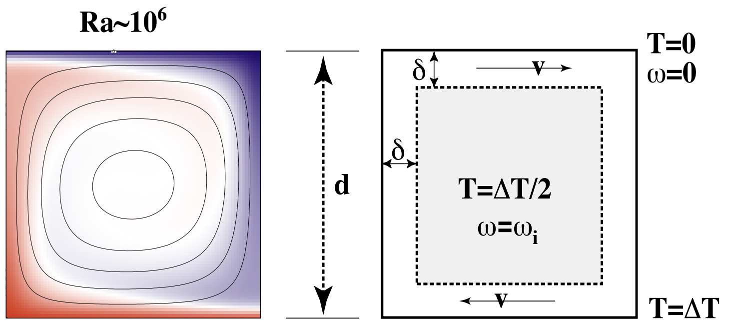
Stokes equation

$$\boxed{\frac{\partial^3 v_2}{\partial x_1^3} - \frac{\partial^3 v_1}{\partial x_2^3} + \frac{\partial^3 v_2}{\partial x_1 \partial x_2^2} - \frac{\partial^3 v_1}{\partial x_2 \partial x_1^2} = \frac{g\rho\alpha}{\eta}\frac{\partial T}{\partial x_1}}$$

becomes

$$\cancel{\frac{V}{d^3}} + \frac{V}{d\delta^2} \approx \frac{g\rho\alpha}{\eta}\frac{\Delta T}{\delta}$$

$$\boxed{\rightarrow \frac{Vd^3}{d\delta^2} \approx \frac{g\rho\alpha d^3}{\eta}\frac{\Delta T}{\delta}}$$



Now assume that the system is approximately steady so

$$\kappa \nabla^2 T = (\mathbf{v} \cdot \nabla) T$$

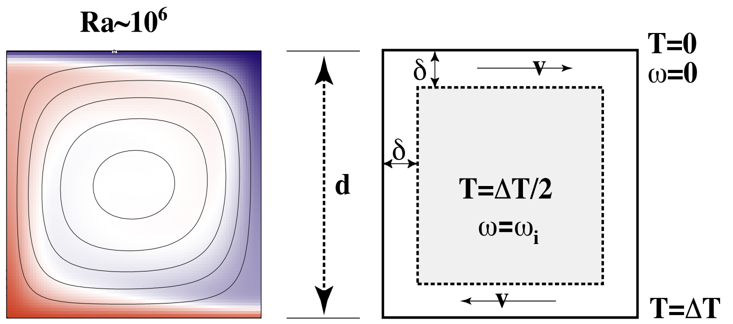
and in the horizontal thermal boundary layer, assume that:

$$v_1 \approx V; \qquad v_2 \approx 0; \qquad \frac{\partial T}{\partial x_1} \approx \frac{\Delta T}{d}; \qquad \frac{\partial T}{\partial x_2} \approx \frac{\Delta T}{\delta};$$

Which produces

$$\kappa \nabla^2 T = (\mathbf{v} \cdot \nabla) T \quad \longrightarrow \quad \boxed{\frac{v \Delta T}{d} \sim \frac{\Delta T \kappa}{\delta^2}}$$

Boundary layer theory



Eliminating $v/d \sim \kappa/\delta^2$ gives

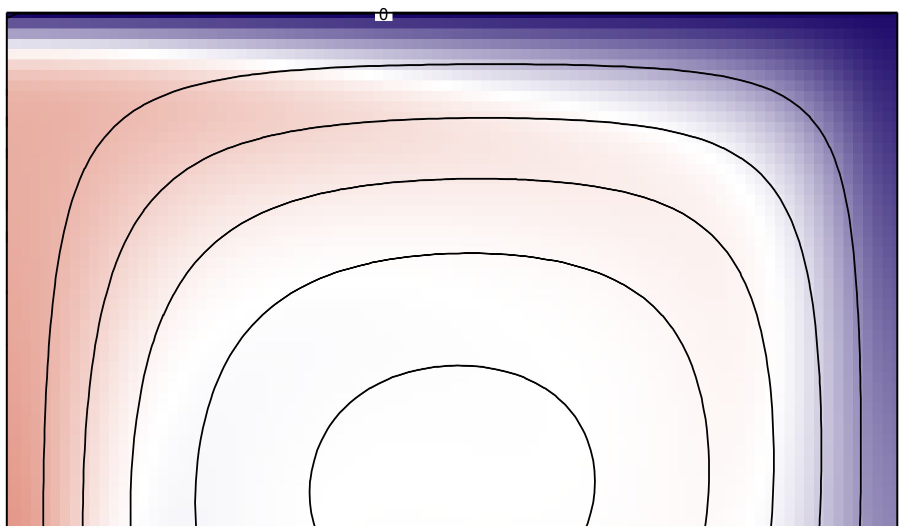
$$\boxed{\frac{\delta}{d} \sim Ra^{-1/3}}$$

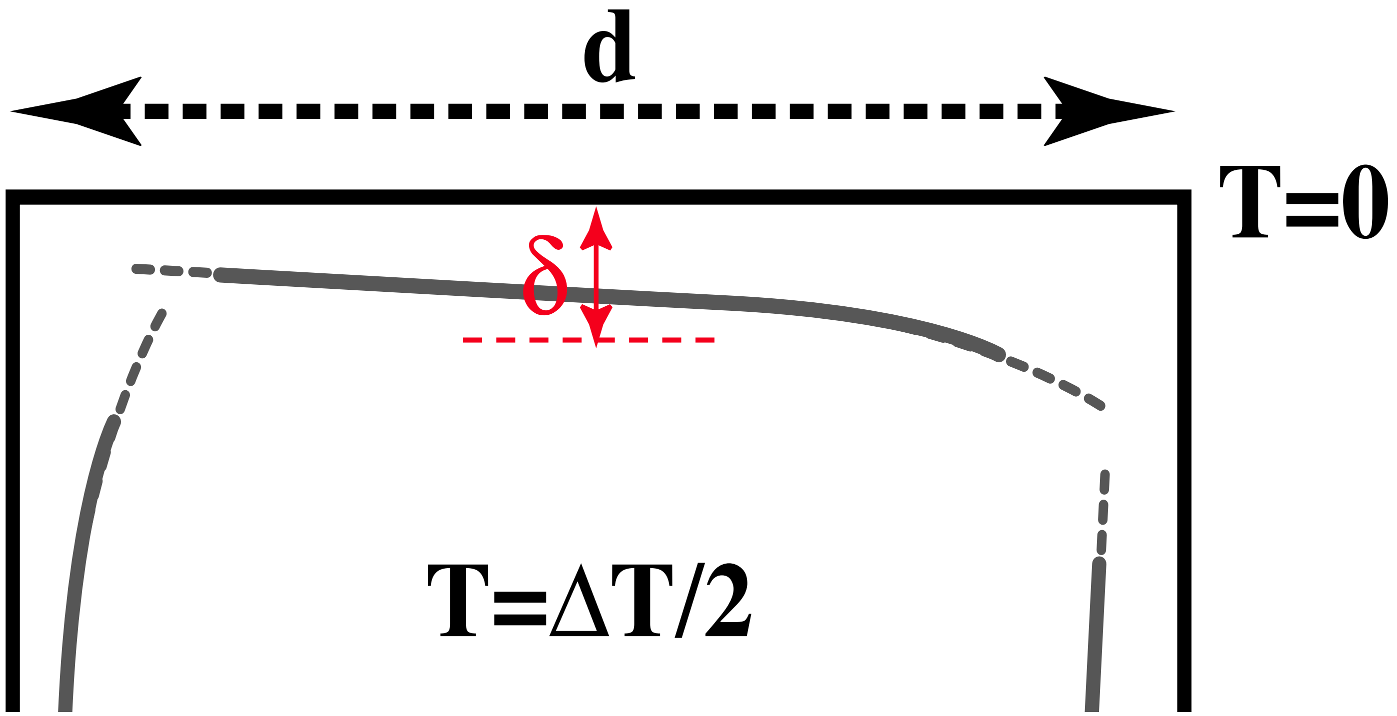
and

$$\boxed{v \sim \frac{\kappa}{d} Ra^{2/3}}$$

This theory balances transfer of momentum and temperature across and out of the boundary layer with advection of each quantity along the boundary layer to maintain a steady state.

$$\boxed{\boldsymbol{Nu \sim \frac{\rho C_p v \Delta T \delta}{(k\Delta T/d)d} \sim Ra^{1/3}}}$$



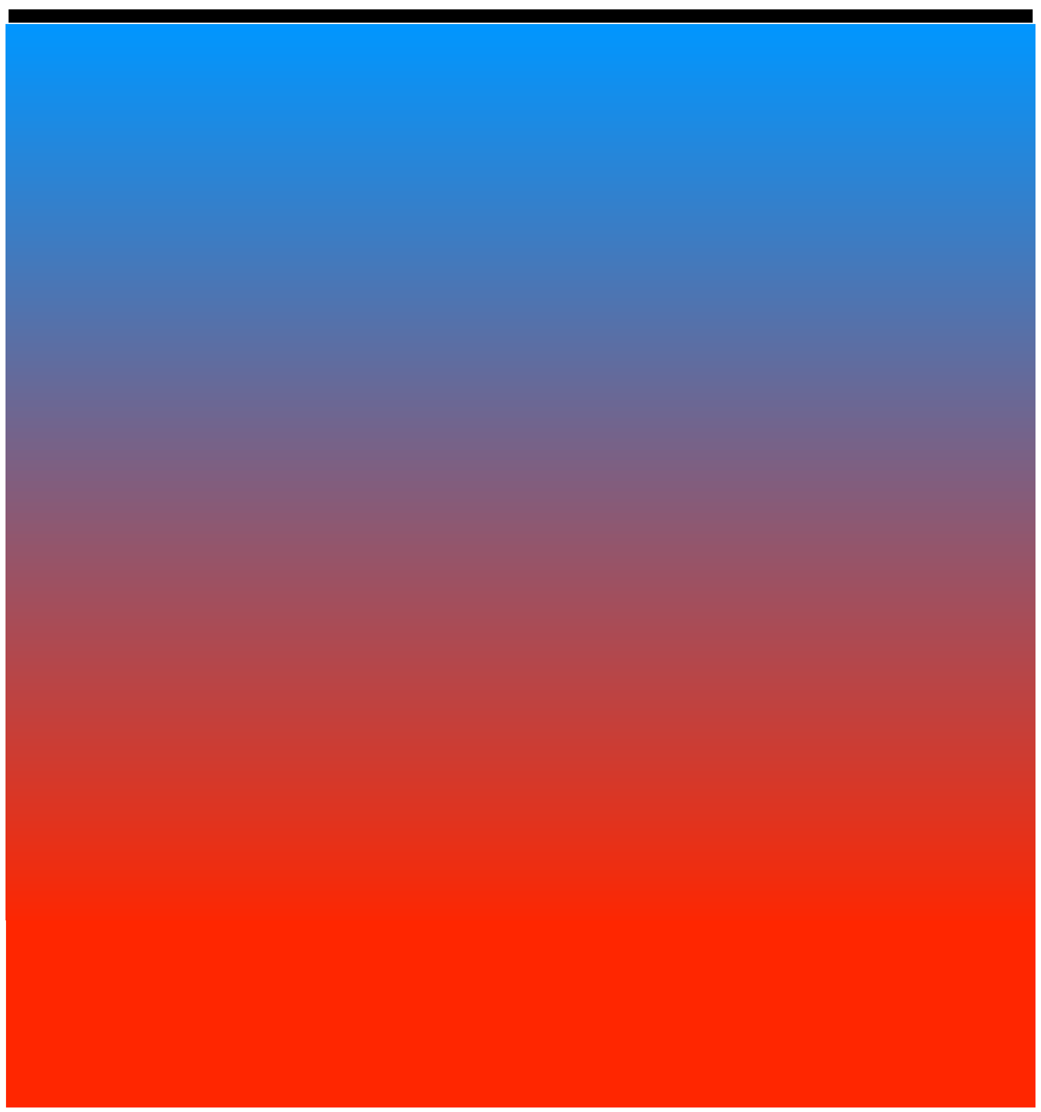


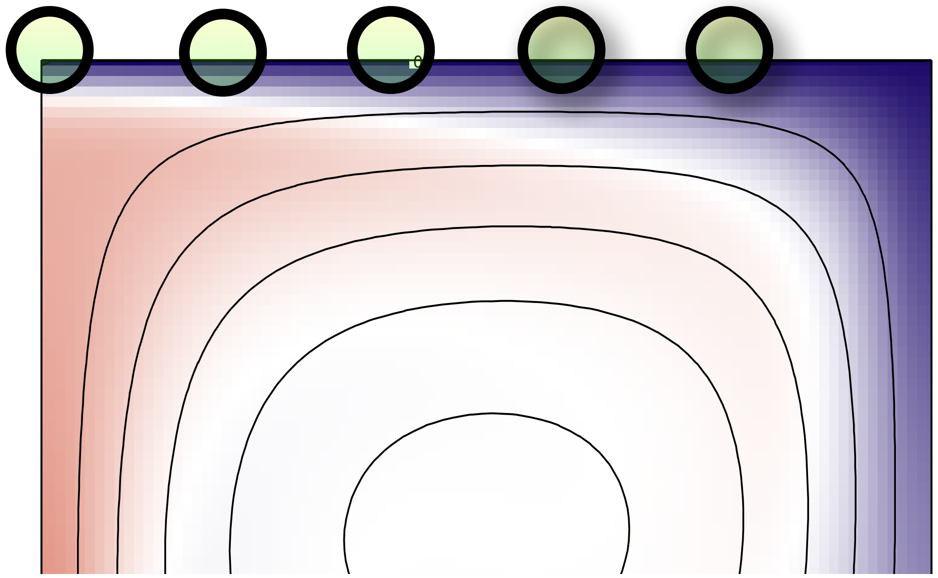
We can also try to account for the variation in the boundary layer thickness as it moves along the horizontal boundary.

Consider the boundary layer to be very thin above the upwelling plume (left side), thickening as it cools to the right. In a Lagrangian frame this is a 1D diffusion equation.

Boundary layer model & Cooling 1/2 space

Sudden change of surface temperature

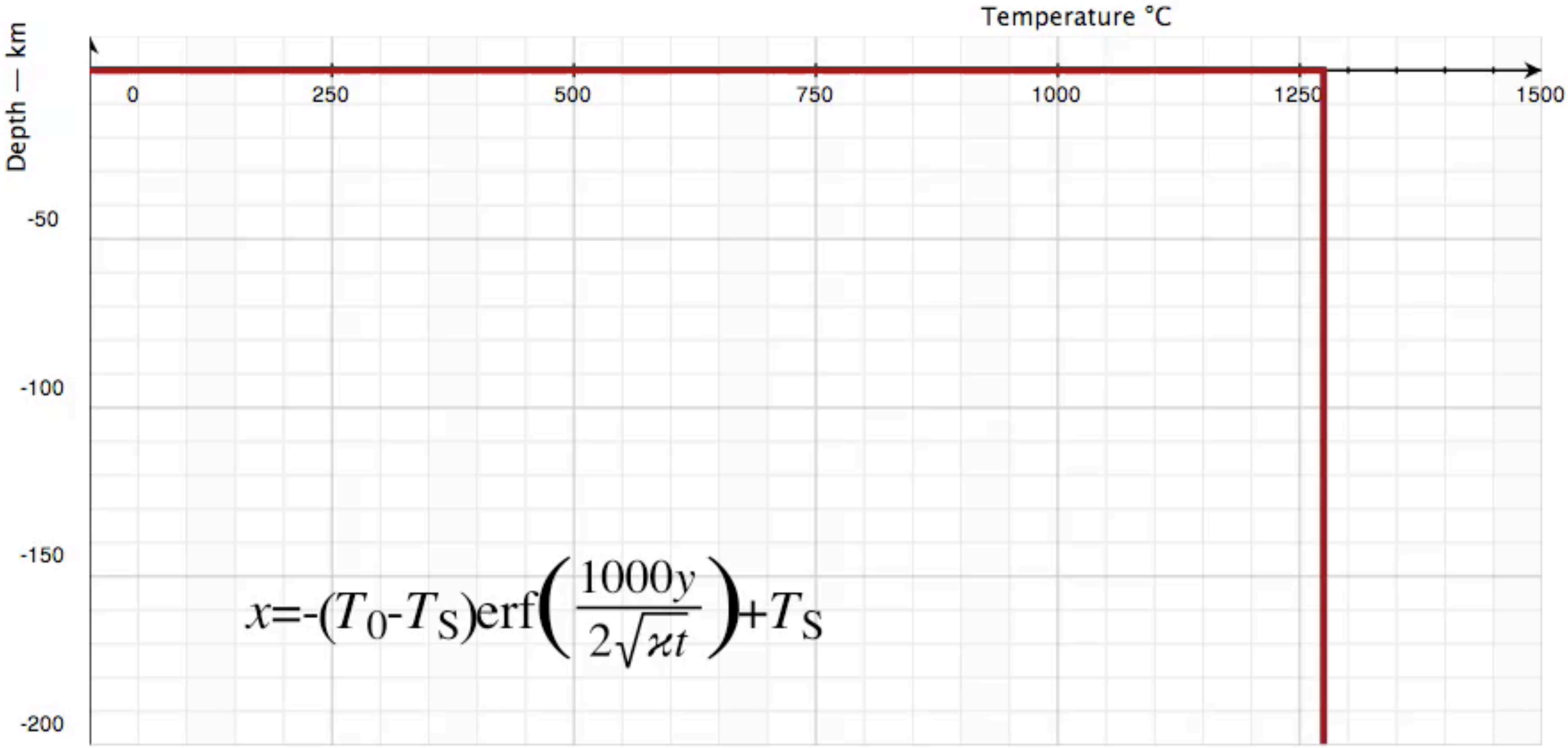




Uniformly warm/cool

Cooling lithosphere

At any given point riding along on the plate



Heating/cooling 1/2 space

The solution for the temperature in an infinite half space with a suddenly-imposed change in surface temperature at time $t = 0$:

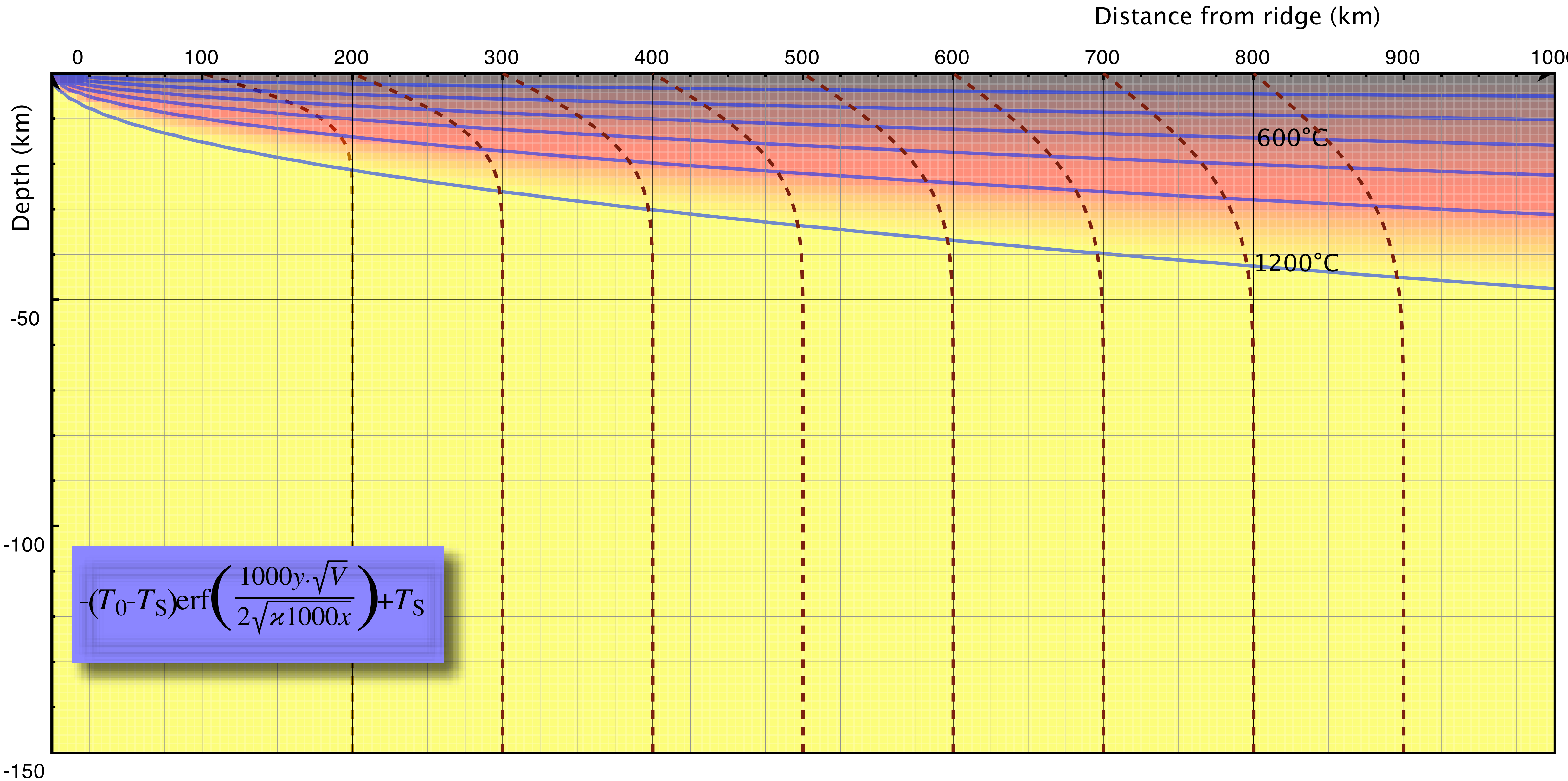
$$\boxed{\frac{T - T_0}{T_s - T_0} = \mathrm{erfc}\left(\frac{x_2}{2\sqrt{\kappa t}}\right)}$$

Heating

$$\boxed{\frac{T - T_s}{T_0 - T_s} = \mathrm{erf}\left(\frac{x_2}{2\sqrt{\kappa t}}\right)}$$

Cooling

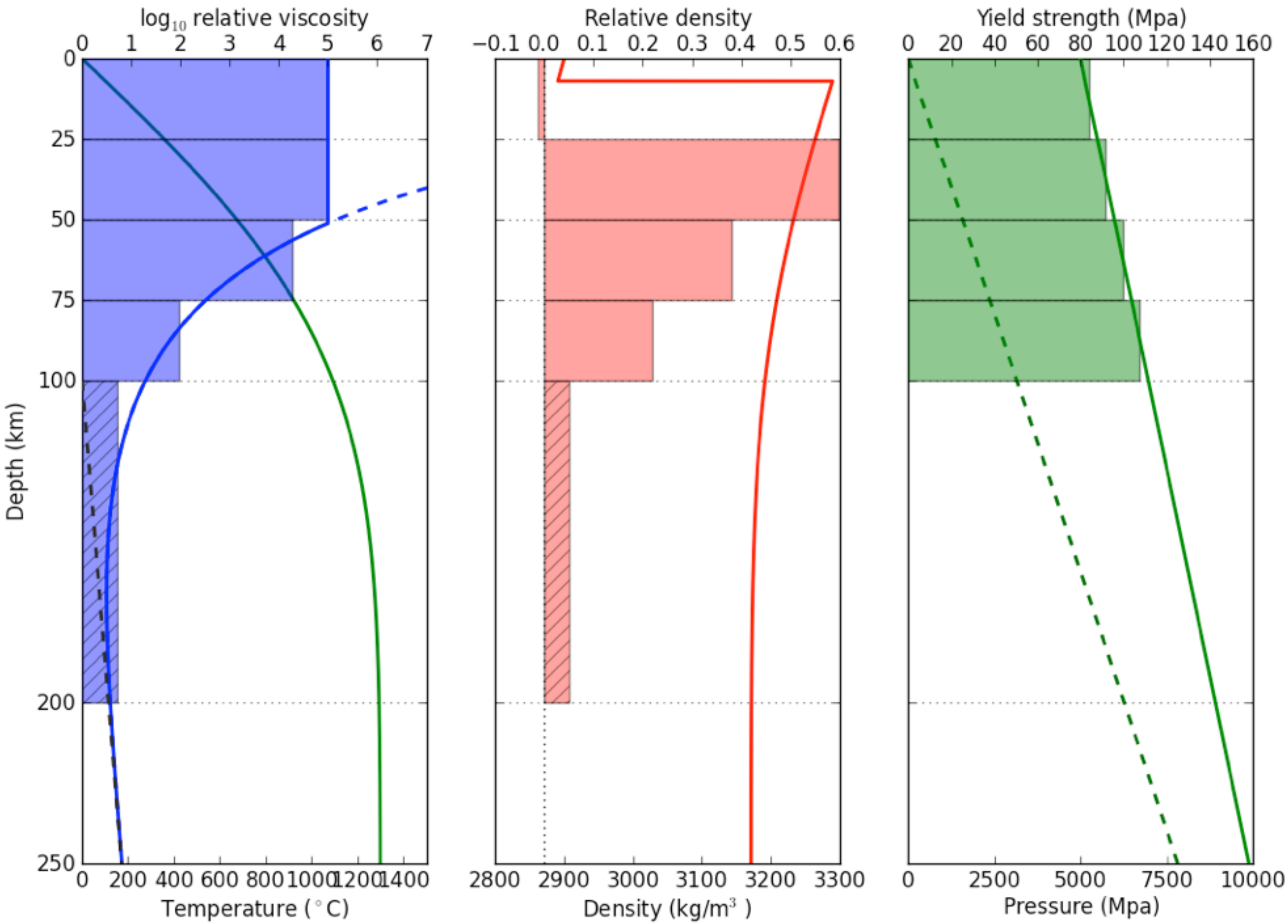
Heating/cooling 1/2 space



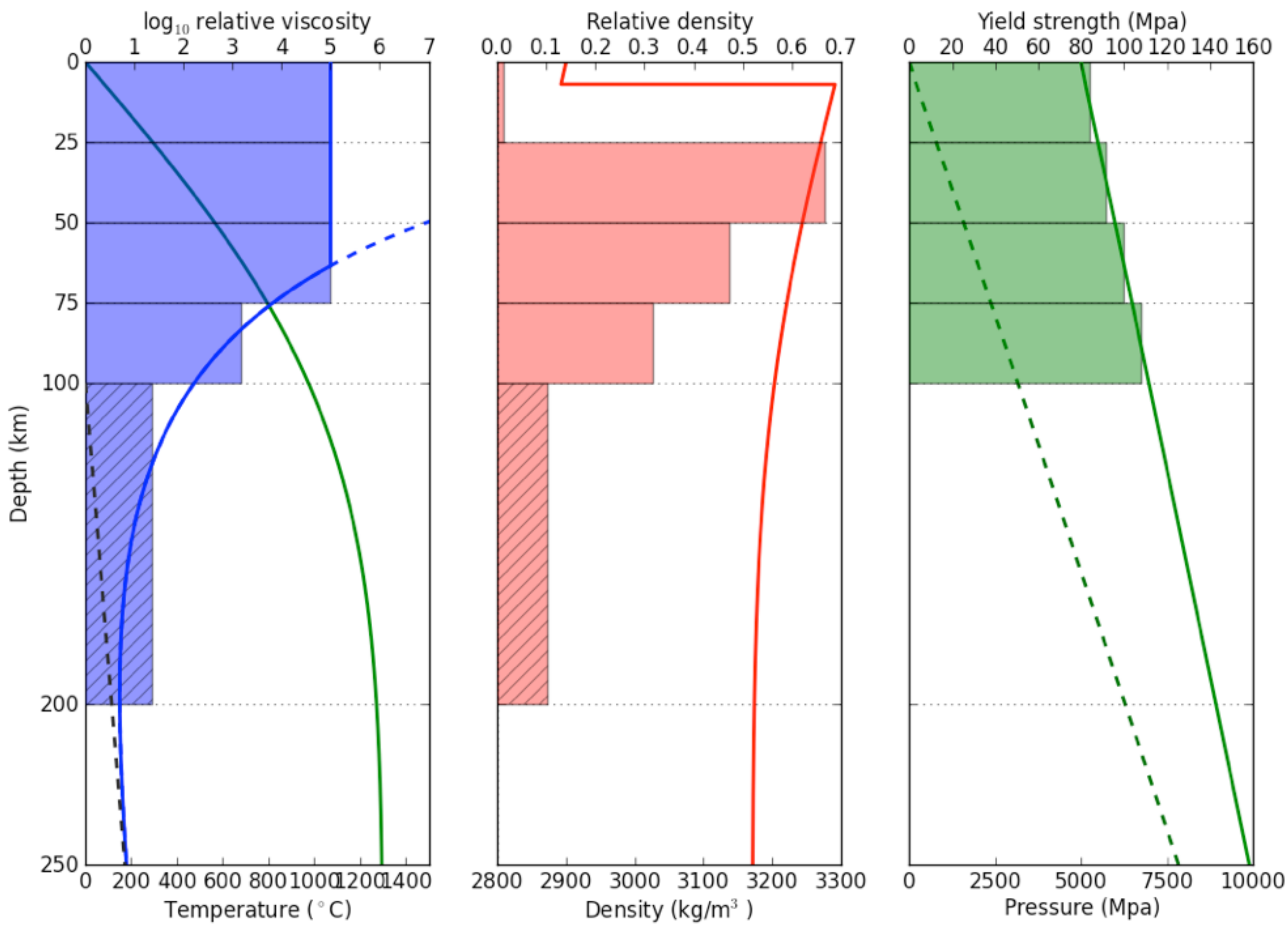
Lithosphere / boundary layer thickness — choose an isotherm

$$\delta \propto \sqrt{\kappa t} \qquad \delta \propto \sqrt{\kappa x / v}$$

80 Myr old lithosphere

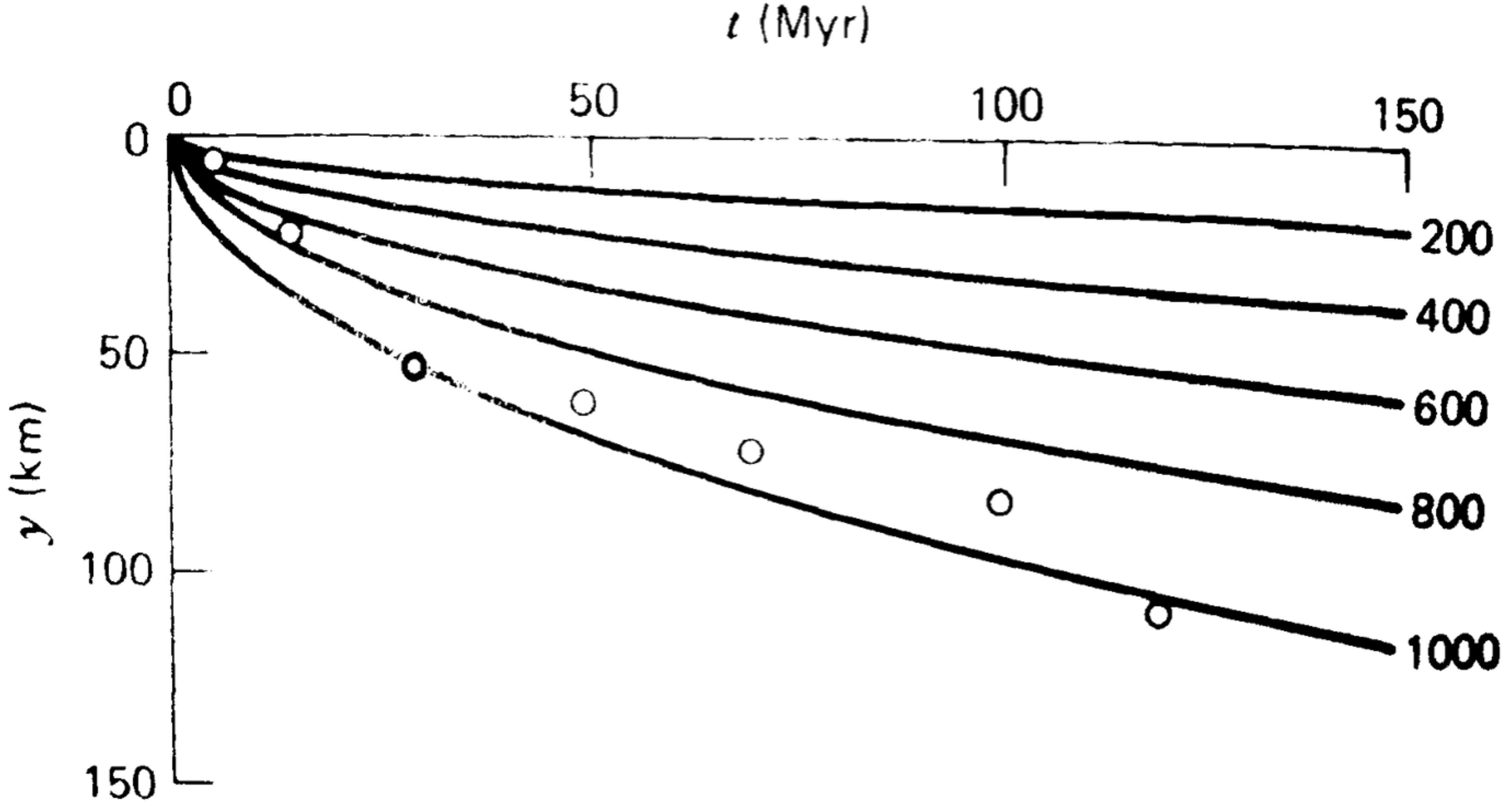


120 Myr old lithosphere



Agreement with observation

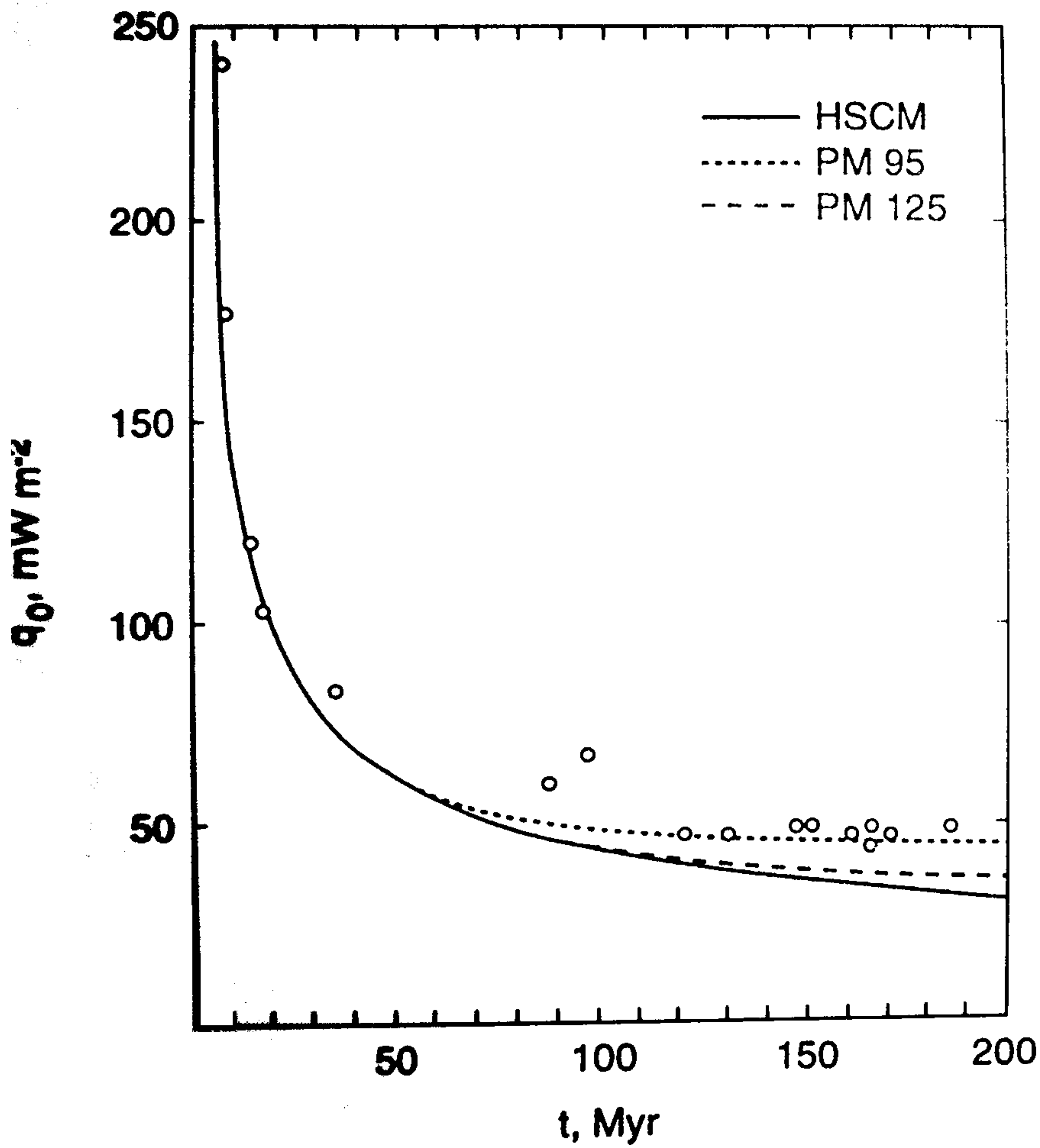
This predicts the thickness of the thermal boundary layer which can be measured by seismology.



Agreement with observation

We can differentiate the cooling half-space model to give the temperature gradient at the surface, and hence the heat flux.

The oceanic heat flux is hard to measure at very young ages, but heat flux agrees quite well (at least until about 80-100 Myr)



Sea floor depth calculation

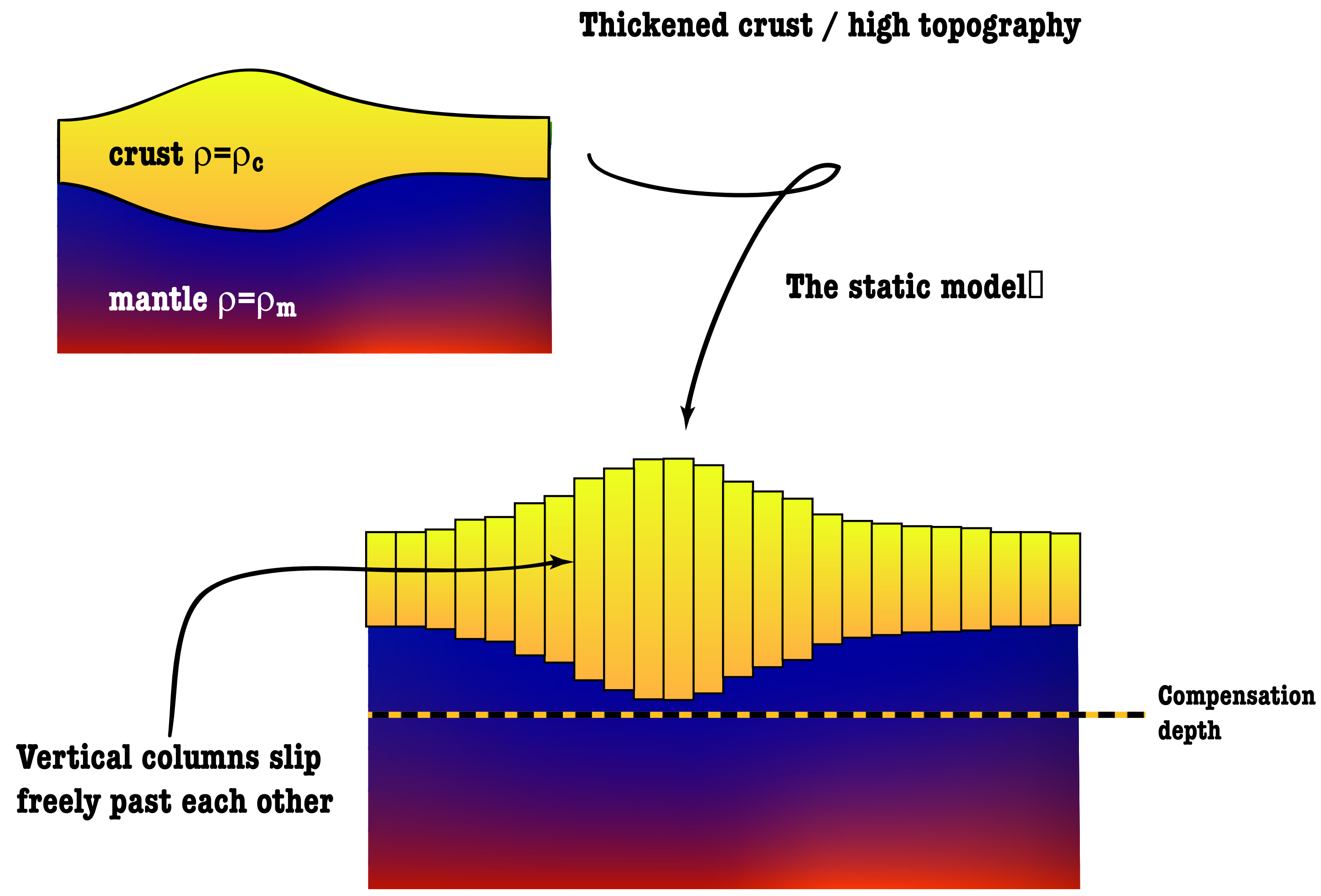
With a few more assumptions, it is possible to turn the thermal cooling model into a prediction for the depth of the sea-floor, a quantity that is much easier to measure.

As the thermal boundary layer cools, it contracts and this produces a small density change. The concept of isostasy applies here (assuming that the changes are over a sufficiently broad length-scale that elastic effects can be ignored).

Isostasy allows us to predict the change in surface height:

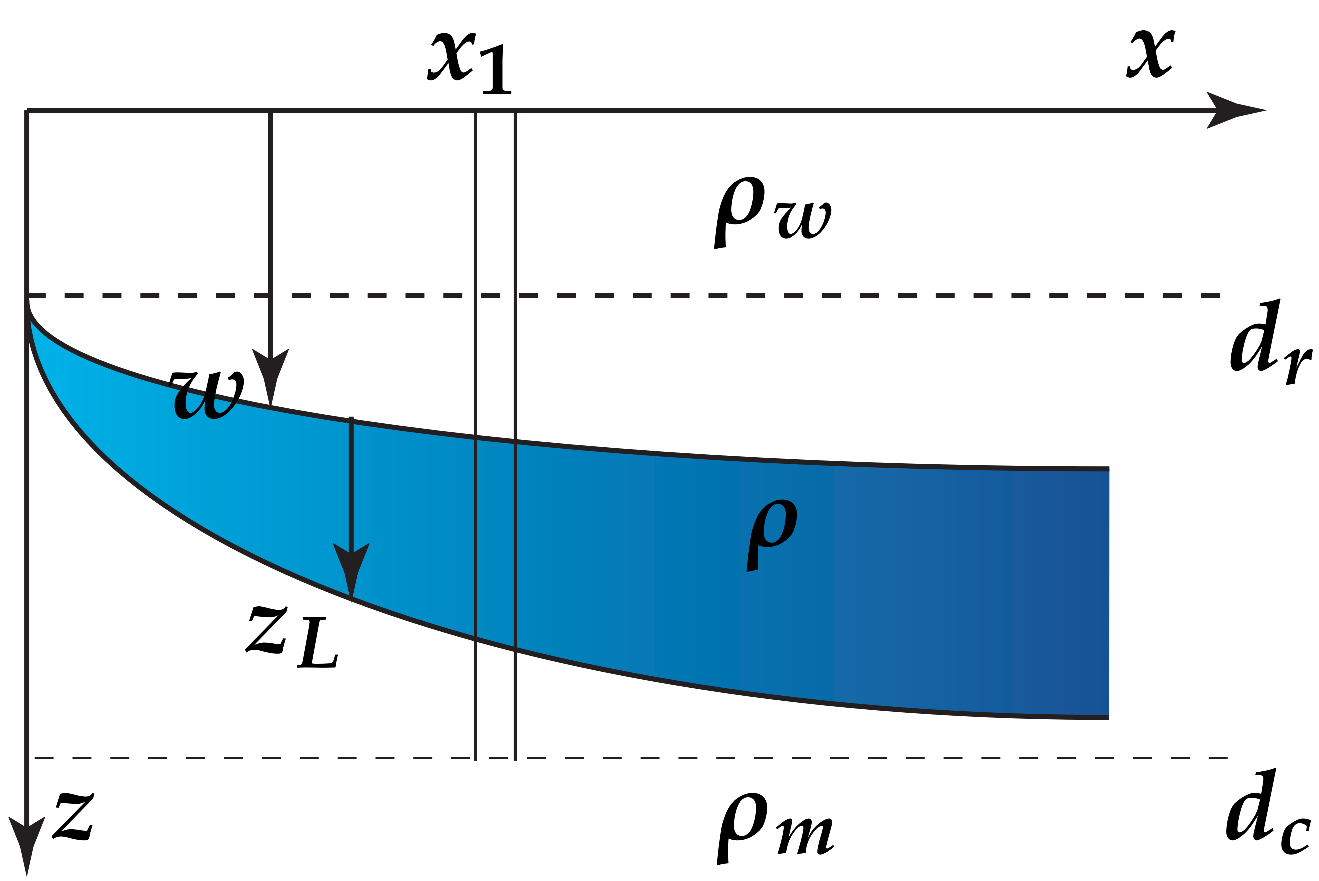
$$\int_{d_c}^{h} \rho(z)dz = C$$

(solve for h)



In the case of the Earth's ocean floor, we integrate downwards from sea level:

$$\int_{d_r}^{d_c} \rho(z)dz = \rho_m(d_c - d_r)$$



Assume no density variations beneath the boundary layer and compare the density column at any x with the column at $x = 0$.

$$w\rho_w + \int_{w}^{z_L} \rho(z)dz = \rho_m z_L$$

Rearranging, and changing the limits of the integral

$$w(\rho_w - rho_m) + \int_{d_r}^{z_L} (\rho(z) - \rho_m) dz = 0$$

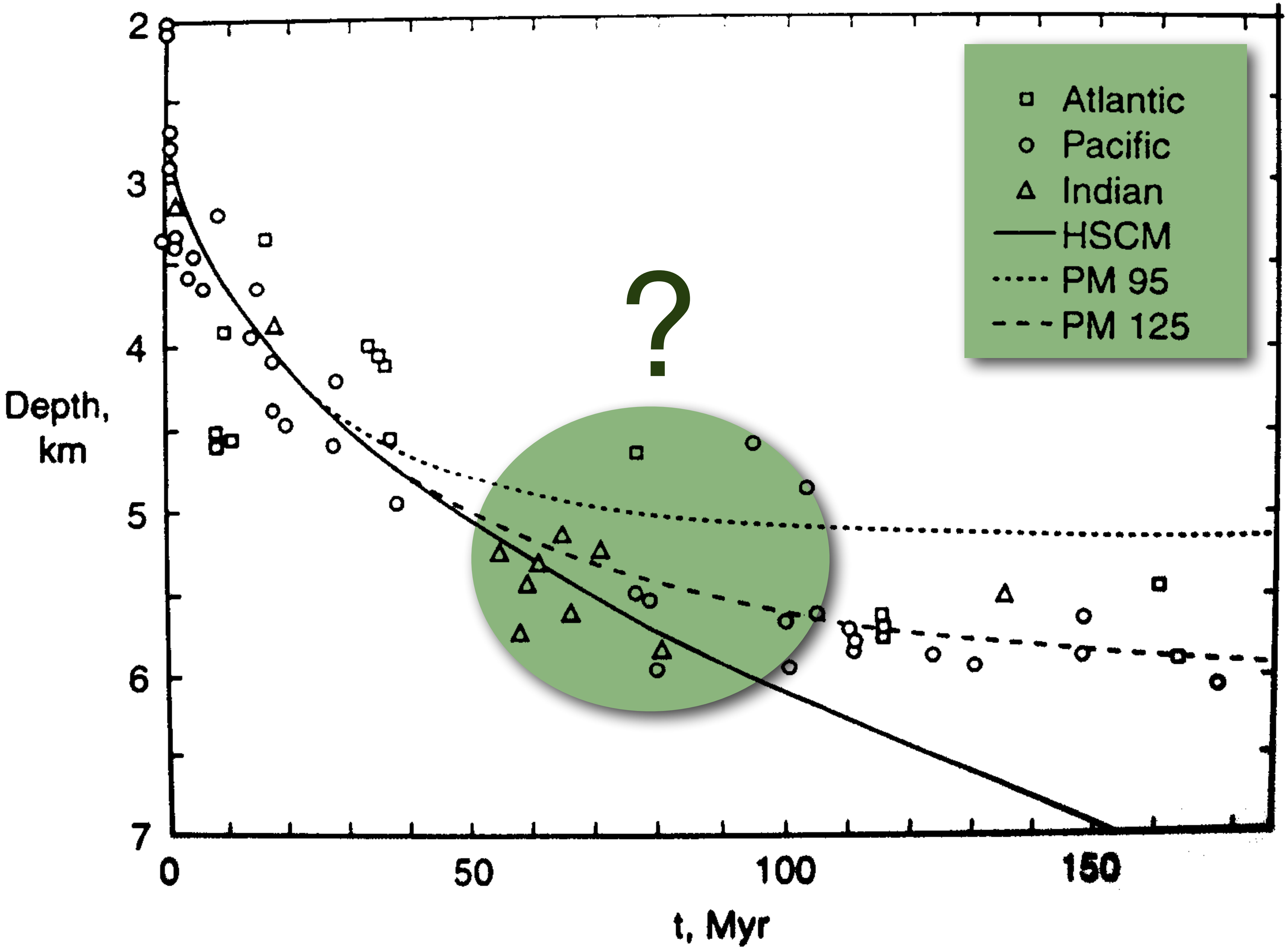
But $(\rho - \rho_m) = \rho_m \alpha (T_1 - T)$ and $T(z)$ was calculated earlier:

$$w(\rho_w - rho_m) = \rho_m \alpha (T_1 - T_0) \int_0^{\infty} \text{erfc} \left[\frac{z}{2} \left(\frac{v}{\kappa x} \right)^{1/2} \right]$$

The ocean depth as a function of seafloor age is

$$\boxed{w = \frac{2 \rho_m \alpha (T_1 - T_0)}{\rho_m - \rho_w} \left[\frac{\kappa x}{\pi v} \right]^{1/2}}$$

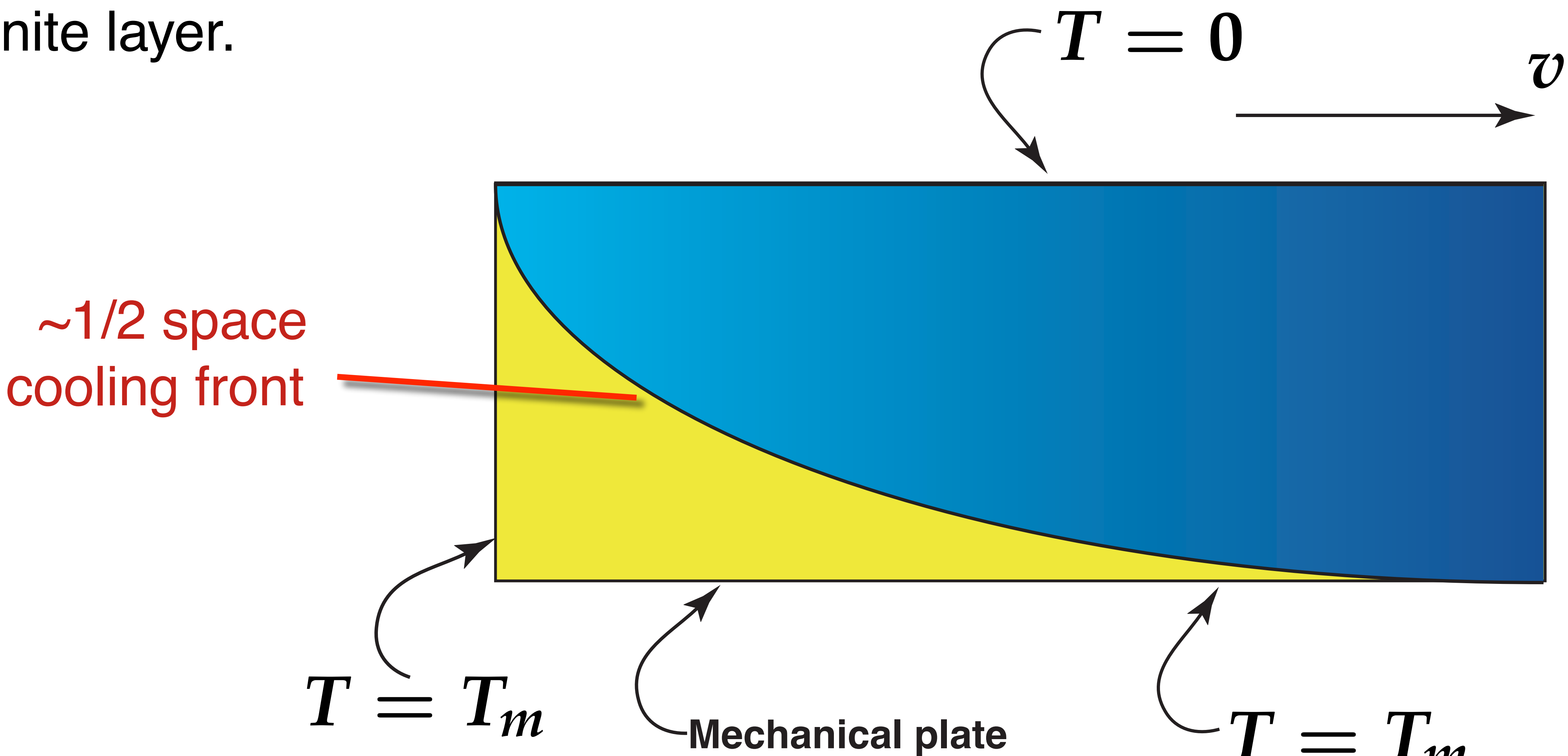
Agreement with observations



The plate model

The half-space cooling model holds until about 80Myr. After that the lithosphere thickness appears to plateau. This requires additional heat into the lithosphere at depth.

One suggestion was that the plates are the **fundamental** objects in the system which only cool to a certain thickness. They are modeled as mechanically distinct — this is not a half space but a finite layer.



Solve for a static plate initially:

$$
\begin{array}{llll}
T = T_m & \text{at} & t = 0 & 0 \leq z \leq z_{\text{plate}} \\
T = T_0 & \text{at} & z = 0 & t > 0 \\
T = T_m & \text{at} & z = z_{\text{plate}} & t > 0
\end{array}
$$

Solution has the form

$$
T = T_0 + (T_m - T_0)\left[\frac{z}{z_{\text{plate}}} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\exp\left(-\frac{\kappa n^2 \pi^2 t}{z_{\text{plate}}^2}\right)\sin\left(\frac{n\pi z}{z_{\text{plate}}}\right)\right]
$$

At small times this reverts to the cooling half space solution, and at large times to an equilibrium profile:

$$\boxed{T = T_0 + (T_m - T_0)\frac{z}{z_{\text{plate}}}}$$

The surface heat flow at large times is

$$\boxed{q_{0_\infty} = \frac{k(T_m - T_0)}{z_{\text{plate}}}}$$

i.e. the heat flux tends to a non-zero limit.

Boundary layer theory: summary

We make dramatic simplifications of the way the system behaves and those simplifications are designed to make the differential equations disappear into simple ratios (e.g. the way we approximate the boundary layer makes the temperature gradient constant in the BL).

- Relates quantities such as Nusselt number, typical velocity to Rayleigh number — scaling laws.
- Gives a 1D picture of the convection cell — ability to transfer heat or momentum as a "black-box"
- Leads to the possibility of parameterized convection — thermal history models for planets which do not require full solution of convection equations.
- Extension to compute surface topography and heat flow for plates.

Revisiting Heat Flow — Estimated Temperature at 5km depth

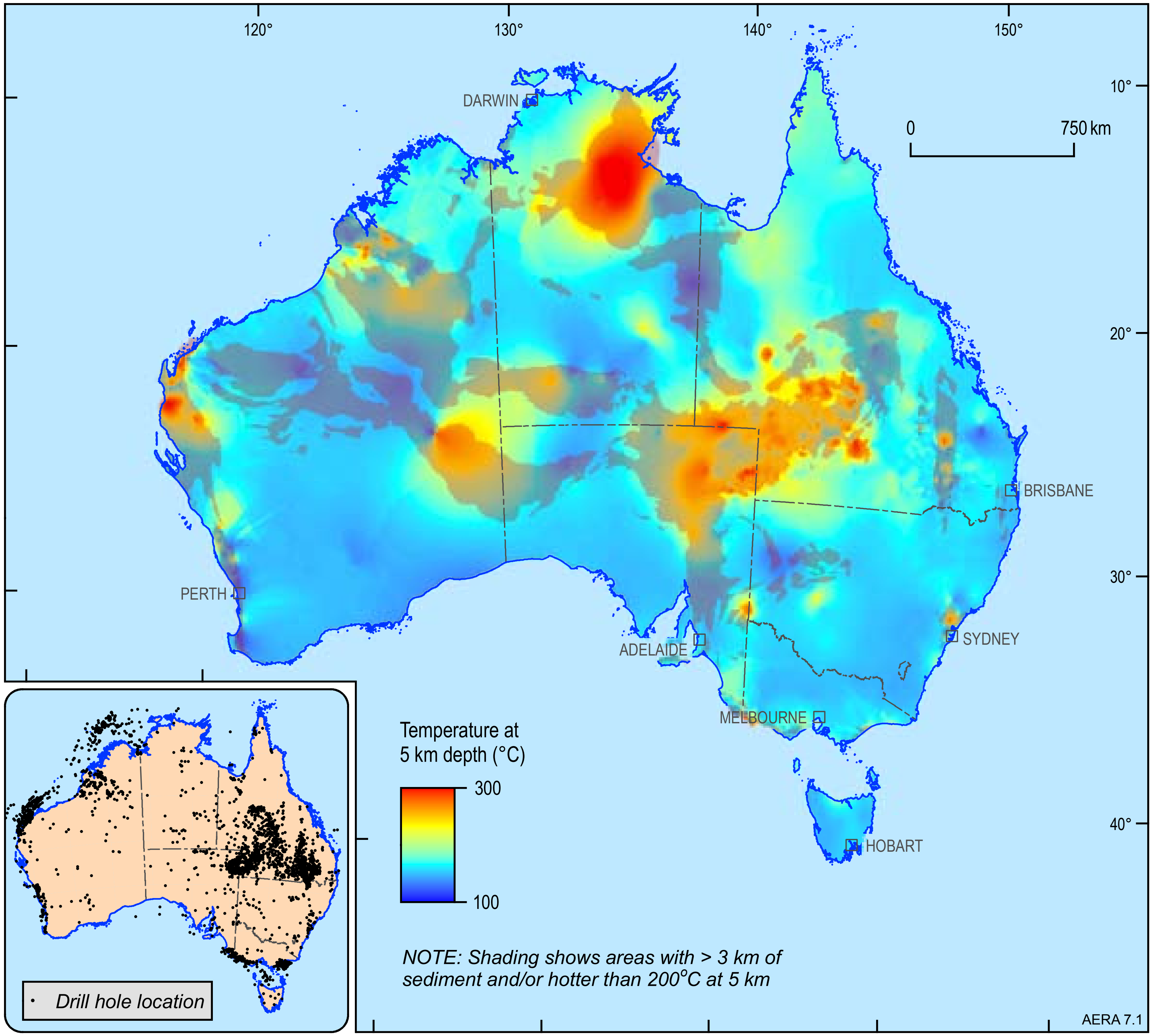


Figure 7.1 Predicted temperature at 5 km depth based mostly on bottom-hole temperature measurements in more than 5000 petroleum and water boreholes

Source: Data from Earth Energy Pty Ltd; AUSTHERM database; Geoscience Australia

Radiative cooling of a hot sphere (in space)

The Stefan-Boltzmann law tells us the rate at which a hot object loses heat to its surroundings.

$$P = \varepsilon\sigma A\left(T_{\mathrm{hot}}^4 - T_{\mathrm{ambient}}^4\right)$$

The average surface temperature of the Earth is about 15°C or 288K.

This implies a loss of heat at a rate of **2 x 10^{17} W** globally (to the chill of space)

We can make some assumptions about the energy content of the Earth to see how long this can be sustained

$$\frac{dE}{dt} = \frac{dE}{dT}\frac{dT}{dt} = \varepsilon\sigma A\left(T_{\mathrm{hot}}^4 - \cancel{T_{\mathrm{ambient}}^4}\right)$$

$$t_{\mathrm{cooling}} = \frac{\rho C_p V_\oplus}{\varepsilon\sigma A_\oplus}\int_{T_{\mathrm{early}}}^{T_{\mathrm{final}}}\frac{1}{T^4}dT = \frac{R_\oplus \rho C_p}{3\epsilon\sigma}\left(\frac{1}{T_{\mathrm{final}}^3} - \cancel{\frac{1}{T_{\mathrm{early}}^3}}\right)$$

Radiative cooling of a hot sphere (in space)

Assuming an initial temperature of 1500K (molten rock observed at the Earth's surface) gives a cooling time of ***250,000 years*** !

That's quite young !

What is missing in this analysis ?

- The incoming radiation from the sun (estimated **1.75 x 10^{17} W**) implies the Earth's surface is near equilibrium with the solar radiation and therefore we should not assume cooling to the cold depths of outer space (how long has this near-equilibrium been established ?)

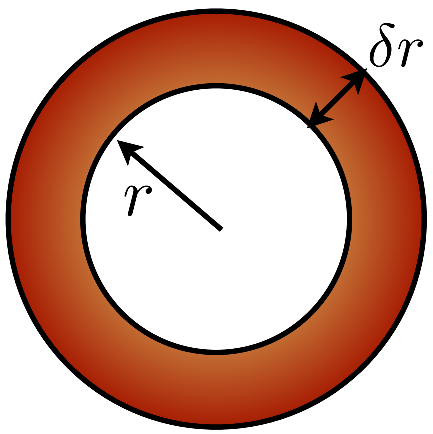
- The assumption that the Earth's interior is isothermal

- The assumption that heat loss from the interior to the surface is efficient

This analysis is useful for a molten planet (magma ocean) where the heat transport in a vigorously convecting interior is very efficient. Once the planet begins to crystallise, conductive cooling through the chilled surface layers becomes the rate-controlling process.

Solution for the uniform conduction of heat out of a sphere

What is the equilibrium heat flow across a thin spherical shell ?



- Heat flux in balances heat flux out
- Account for heat generation within the shell

$$Q_r = q_r(r).4\pi r^2$$
$$Q_{r+\delta r} = q_r(r+\delta r).4\pi(r+\delta r)^2$$

The steady solution has this form

$$T = -\frac{\rho H}{6k}r^2 + \frac{C_1}{r} + C_2$$

Which is only interesting if there is internal heating and otherwise implies constant temperature / no additional heat at depth. Steady-state, of course, tells us nothing about the age of the Earth.

Transient solution for a sphere cooling from a uniform T

The transient (i.e. interesting) equivalent to this problem is qualitatively different from the steady one. There are now variations with time and these trade-off with variations in the radial direction.

Fourier's approach was to find solutions in the form of a series of terms, each of which individually satisfies the equations, but none of which, in isolation, looks much like the actual solution.

The equation we want to solve looks like this:

$$\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial T}{\partial r}\right)$$

It says that the temperature changes due to radial temperature gradients and, equally, that radial temperature gradients are established by the changes in temperature.

NOTE*: I have left out the material "constants" like thermal conductivity, radius of the sphere, density which means I am assuming these are all 1 (and constant)*

Transient solution for a sphere cooling from a uniform T

$$\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial T}{\partial r}\right)$$

Look for solutions that have this form …

Initial temperature distribution

$$T(r,t) = \frac{2}{r}\sum_{n=1}^{\infty} \exp(n^2\pi^2 t)\sin(n\pi r)\int_0^1 f_T(r^\star) r^\star \sin(n\pi r^\star) dr^\star$$

Which is a lot simpler if we assume the initial temperature is constant everywhere

$$T(r,t) = \frac{2}{r}\sum_{n=1}^{\infty} \exp(n^2\pi^2 t)\sin(n\pi r)\underbrace{\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2\pi^2}\right)}_{\text{constant}}$$

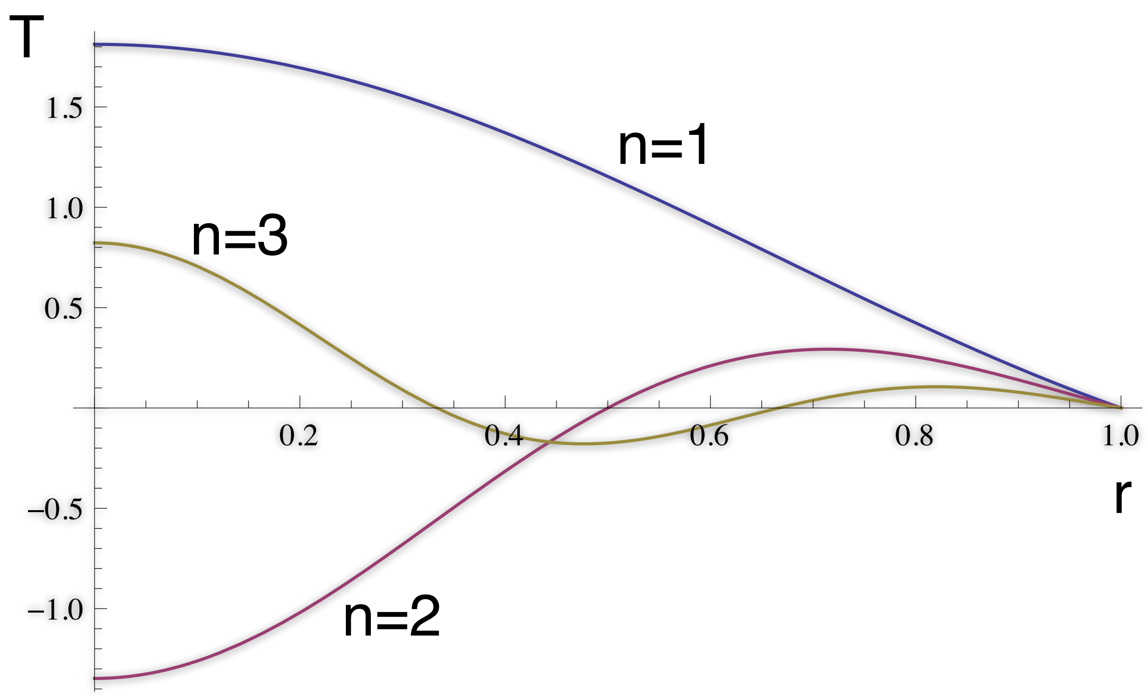
Transient solution for a sphere cooling from a uniform T

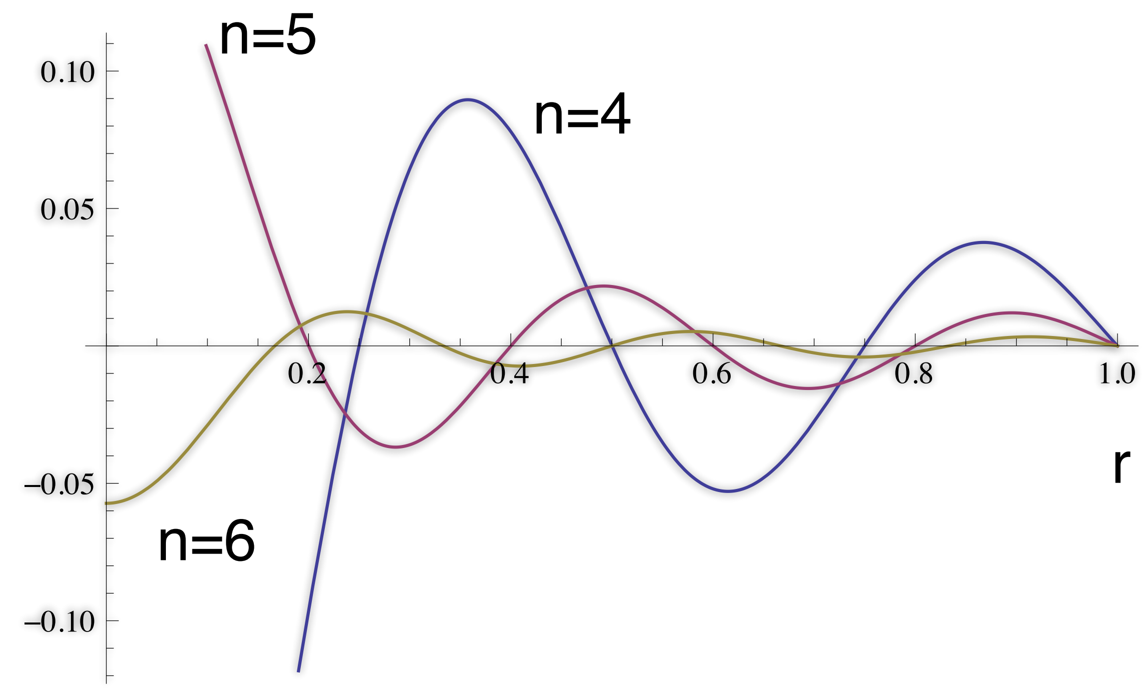
$$T(r,t) = \frac{2}{r}\sum_{n=1}^{\infty} \exp(n^2\pi^2 t)\sin(n\pi r)\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2\pi^2}\right)$$

Ignoring how we come by this (complicated-looking) solution ...

What do these terms look like ?

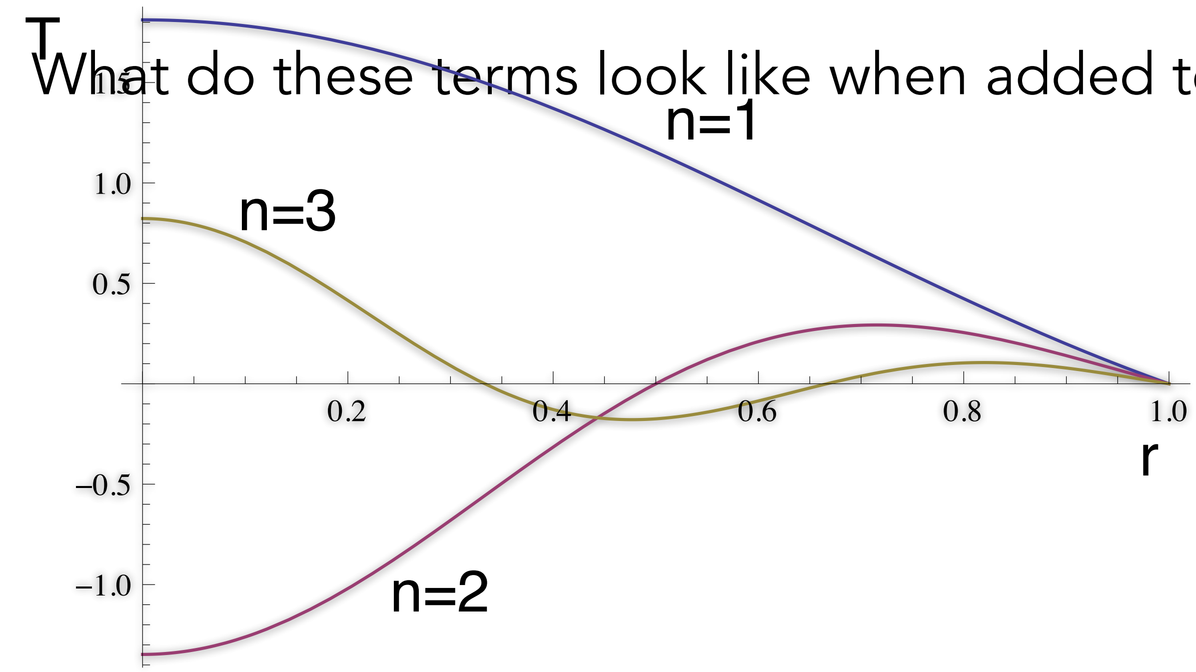
t=0.01

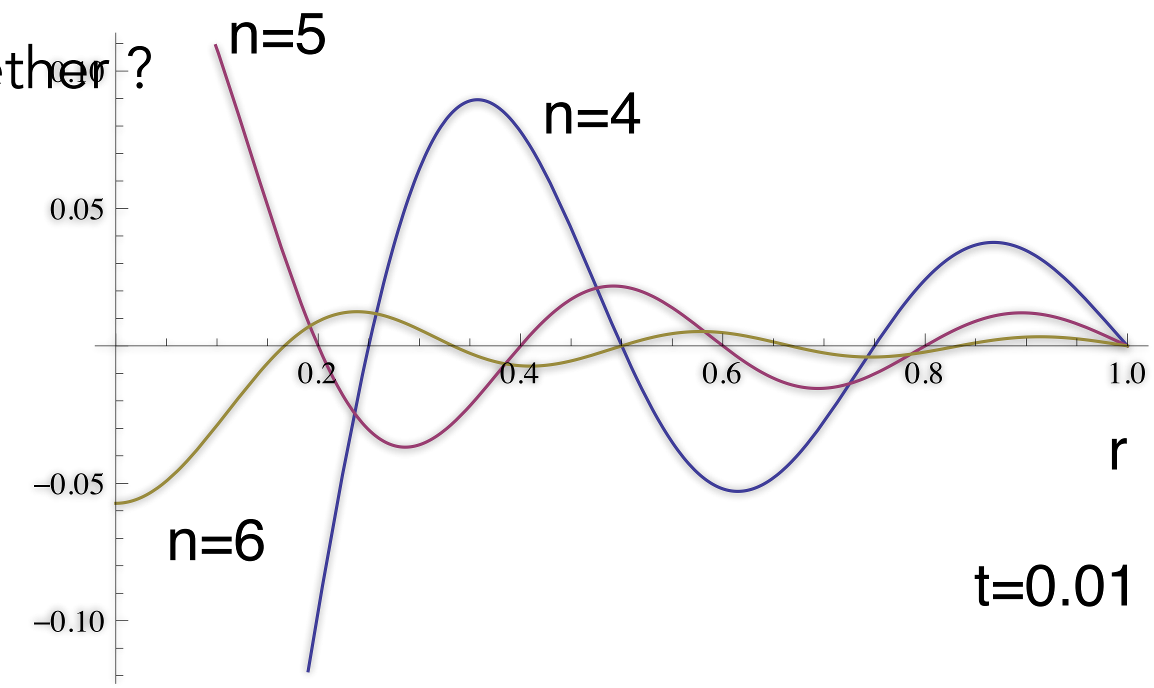


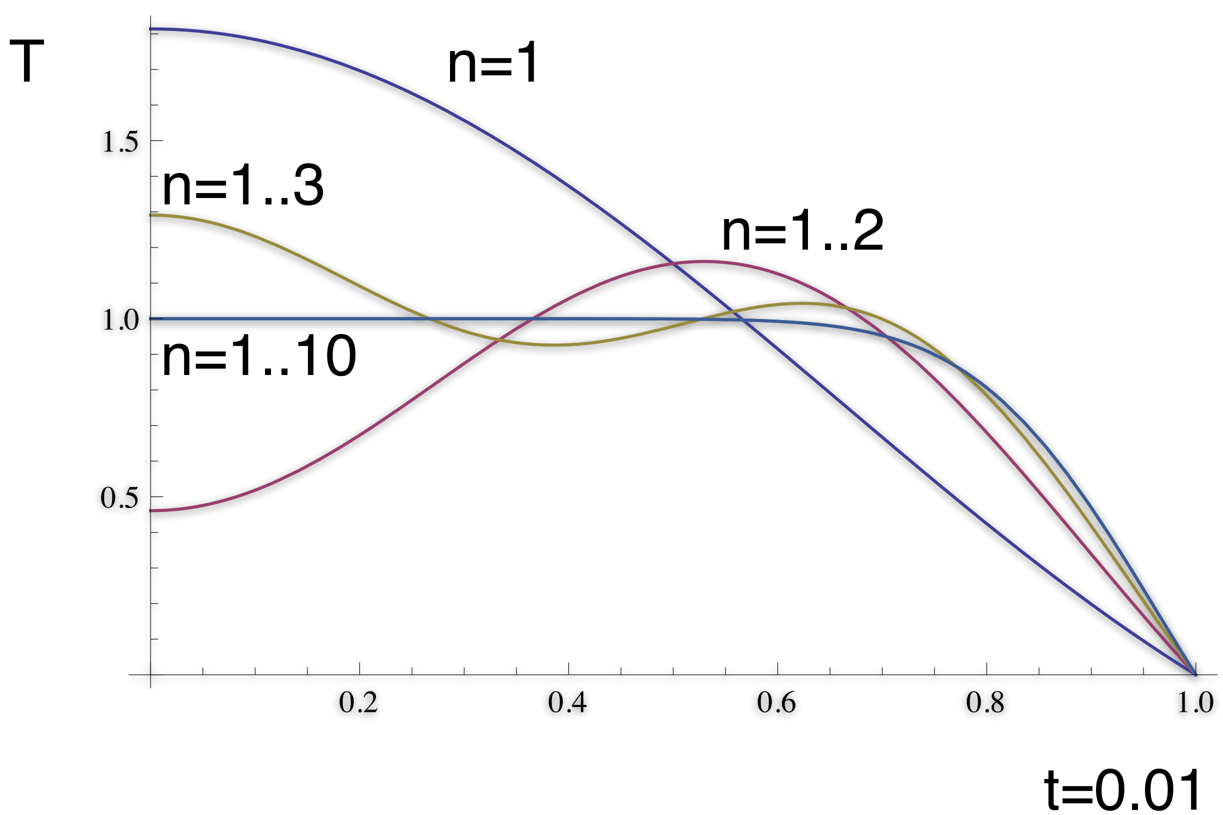


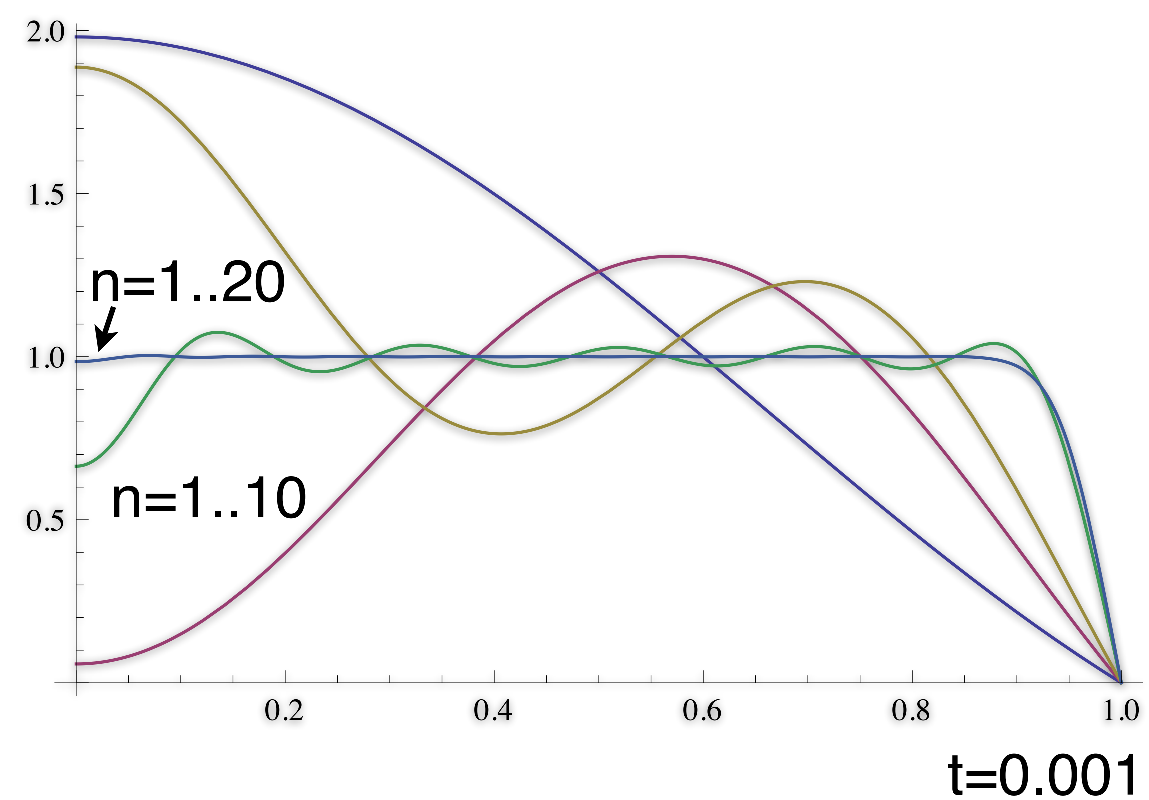
Transient solution for a sphere cooling from a uniform T

What do these terms look like when added together ?



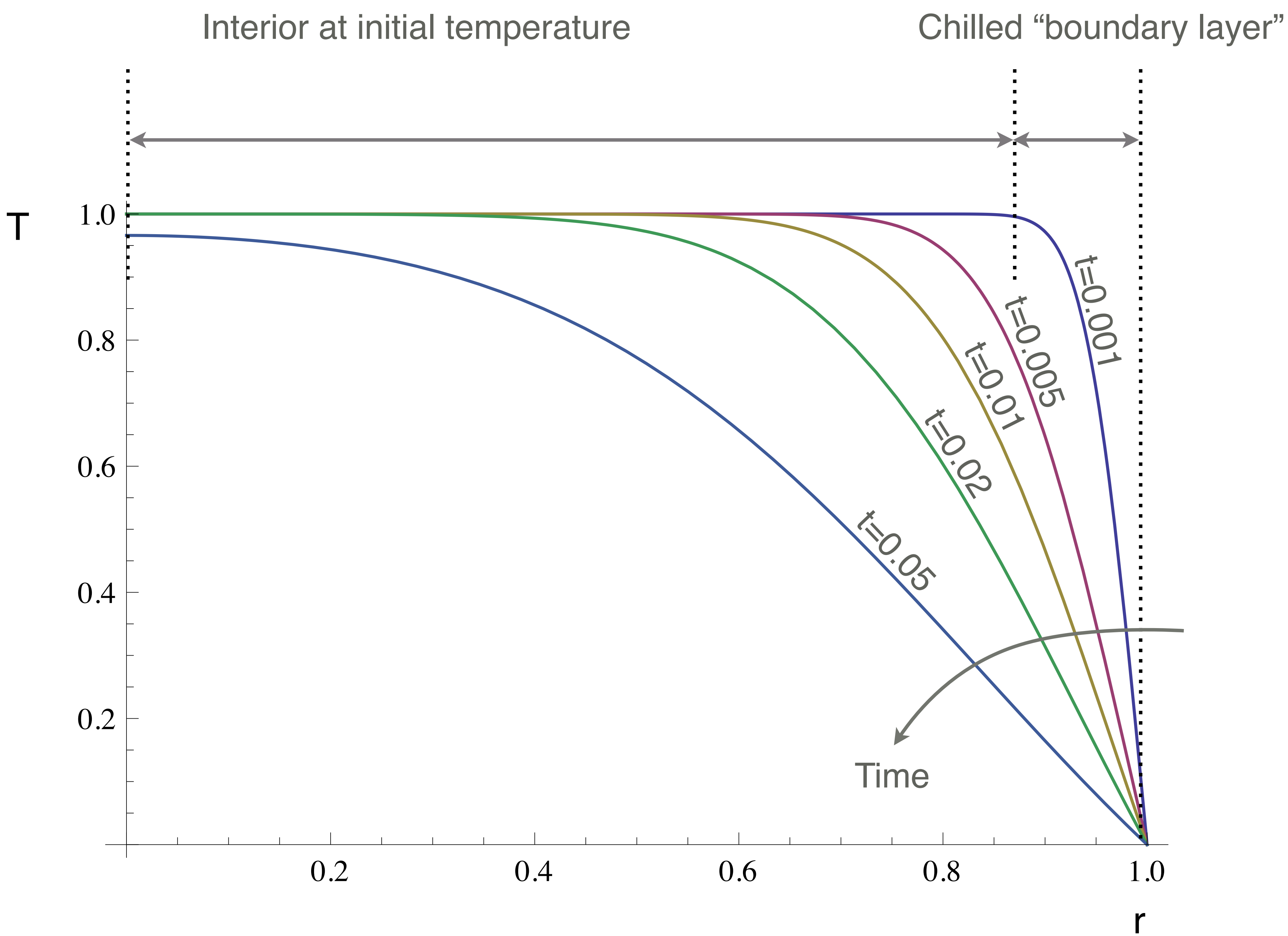






Transient solution for a sphere cooling from a uniform T

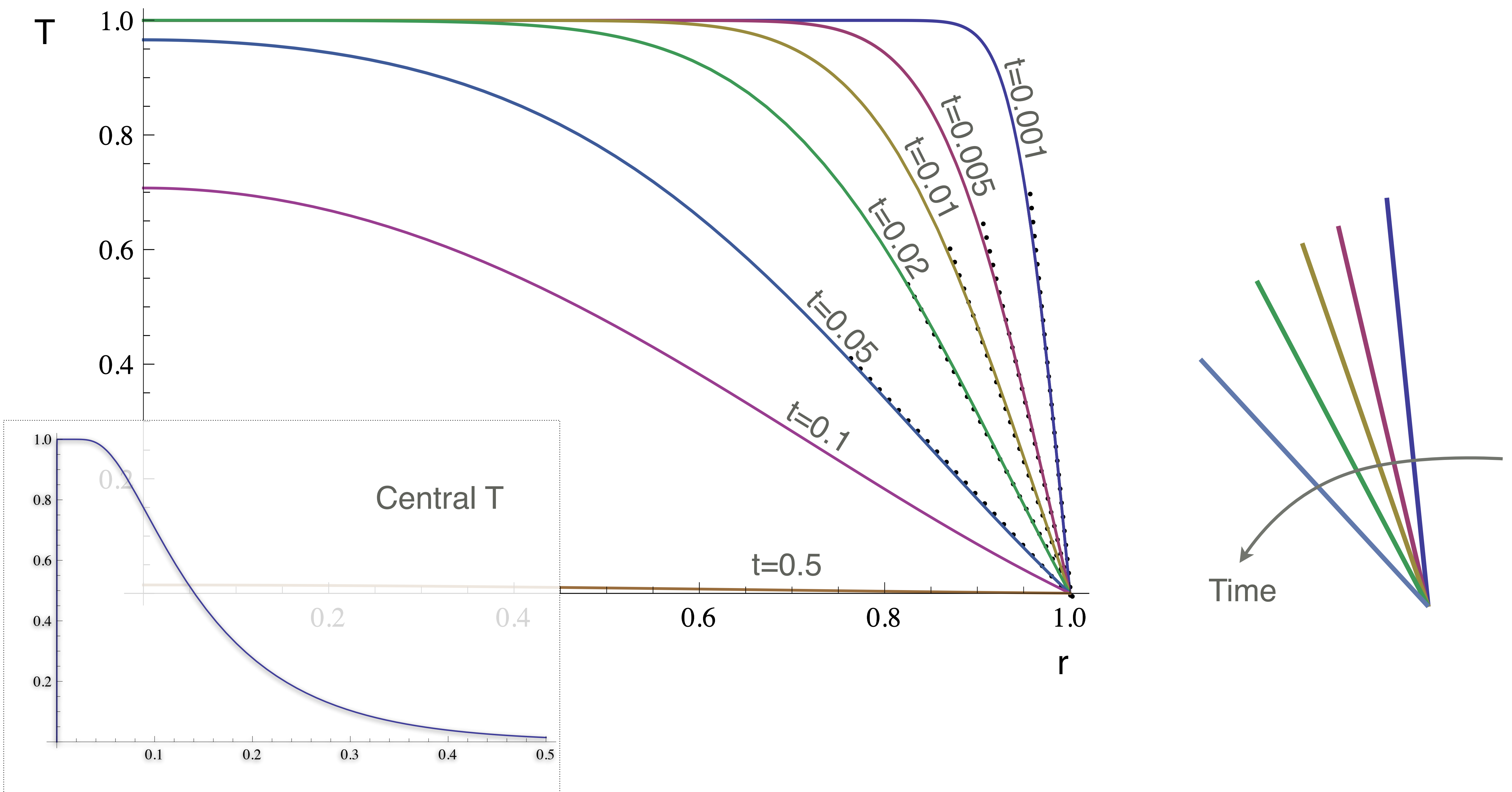
Physical interpretation



Transient solution for a sphere cooling from a uniform T

Look at the temperature gradient near the surface.

Rapidly decreasing with time — surface heat flux is a measure of cooling time !



Transient solution for a sphere cooling from a uniform T

Look at the temperature gradient near the surface.

$$T(r,t) = \frac{2}{r}\sum_{n=1}^{\infty}\exp(n^2\pi^2 t)\sin(n\pi r)\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{n^2\pi^2}\right)$$

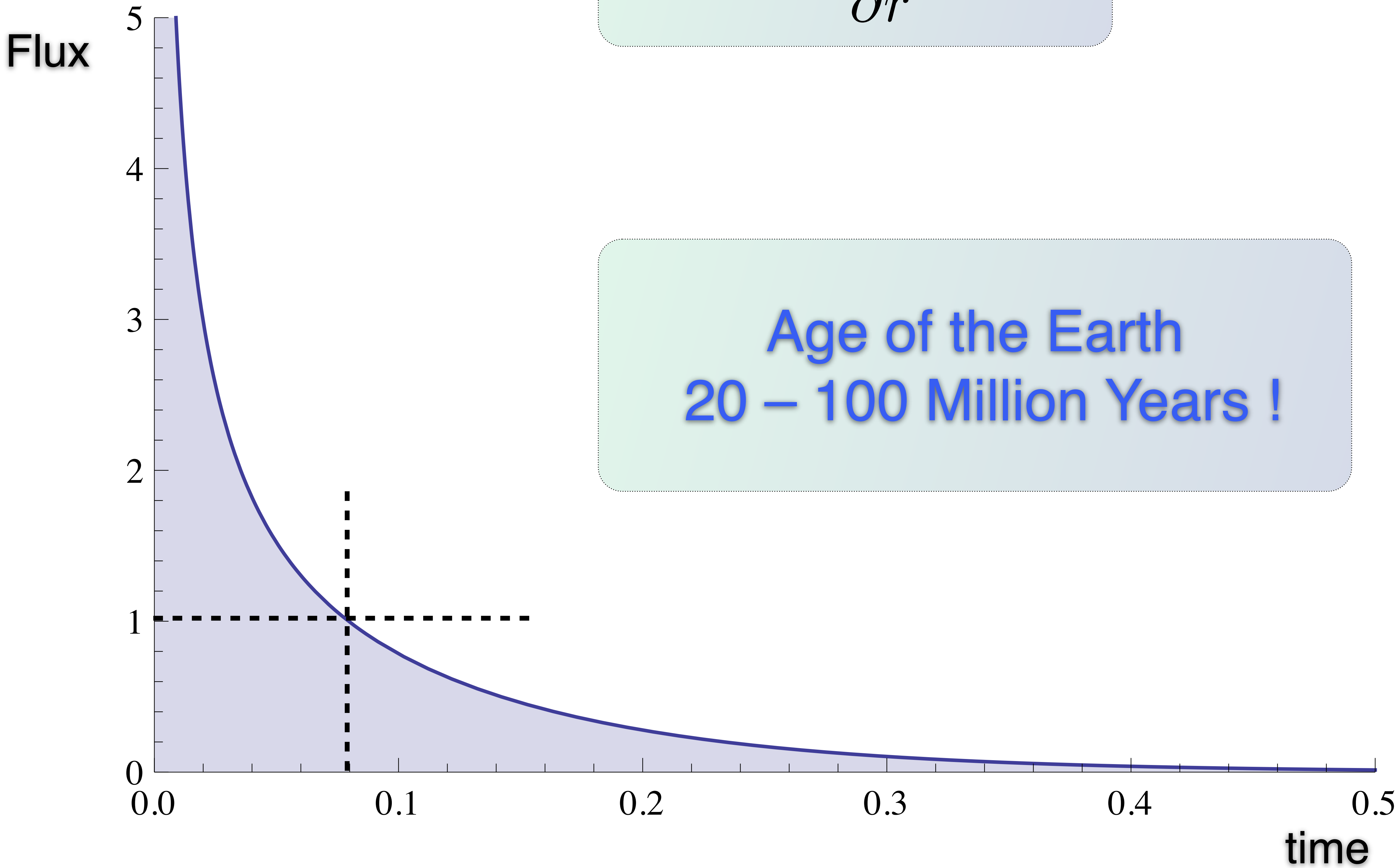
We can differentiate this w.r.t the radius and find the value at the surface (r=1)
- see next slide for a plot

$$\frac{\partial T}{\partial r}(1,t) = -\sum_{n=1}^{\infty}\frac{2e^{-n^2\pi^2 t}(n\pi\cos(n\pi) - \sin(n\pi))^2}{n^2\pi^2}$$

Transient solution for a sphere cooling from a uniform T

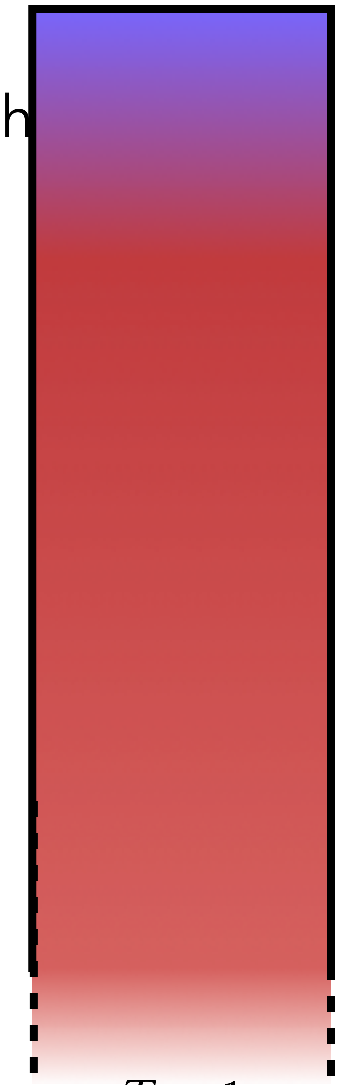
The surface heat flux is given by

$$q_0 = -\frac{\partial T}{\partial r}(1, t)$$



Age of the Earth
20 – 100 Million Years !

Cooling a semi-infinite domain from the boundary



This is a problem not unlike the cooling sphere, but with no inherent length scale in th
The equations are similar but we are working in a Cartesian geometry.

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial z^2}$$

This equation simplifies to an ODE if we make this substitution,

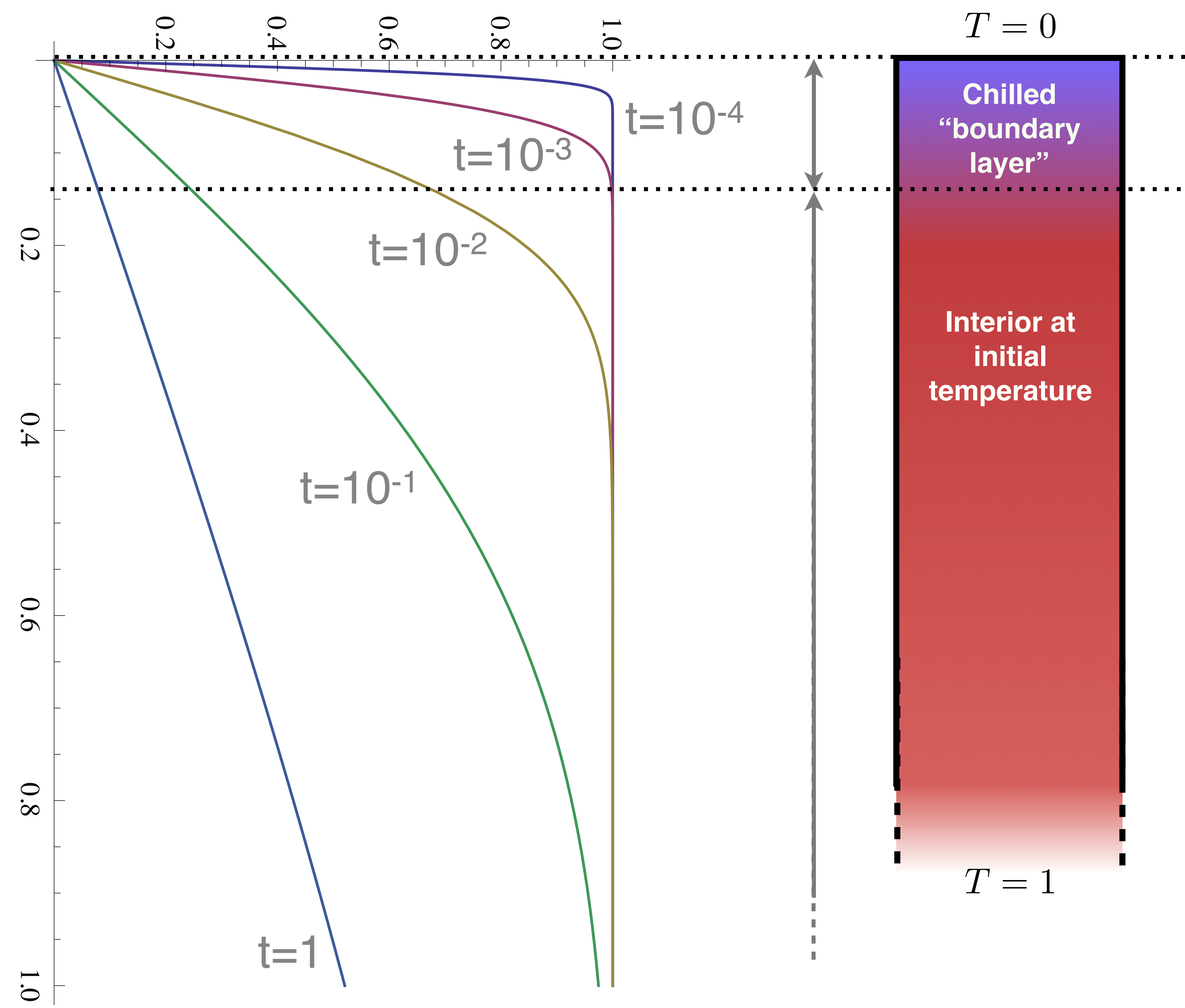
$$T^\star = 1 - T; \;\; \eta = \frac{z}{2\sqrt{t}} \qquad\qquad -\eta \frac{dT^\star}{d\eta} = \frac{1}{2}\frac{d^2 T^\star}{d\eta^2}$$

Which has the following solution

$$T^\star = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\zeta^2} d\zeta = 1 - \mathrm{erf}(\eta) = \mathrm{erfc}\left(\frac{z}{2\sqrt{t}}\right)$$

Cooling a semi-infinite domain from the boundary

$$T = \mathrm{erf}\left(\frac{z}{2\sqrt{t}}\right)$$



This is a very different solution strategy to the cooling sphere (and yet they should, in a mathematical sense, approach each other as the radius becomes large)

The solution looks very similar in character: a cooling front which propagates in from the surface.

All the curves are the same, they just differ by a scaling factor.

Observables from Convection

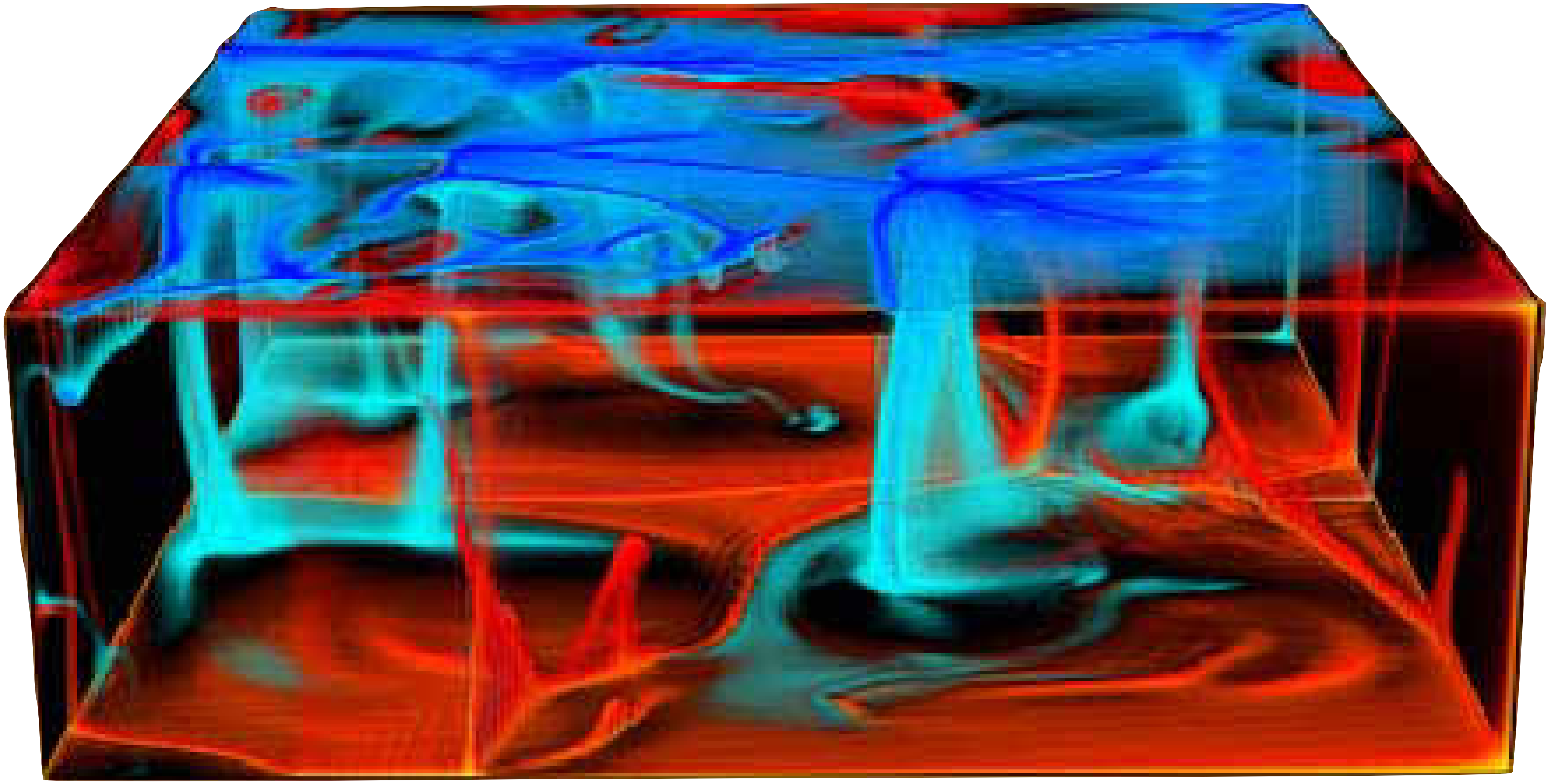
What quantitative information can we obtain about the dynamics of planetary interiors from remote sensing — and from really remote sensing (in orbit) ?

Which remote observations tell us what information ?

- Gravity measurements: density distribution
- Surface topography: surface stress field
- Surface heat flow: near surface temperature gradients
- Seismology: elastic wave speed distribution
- Morphology: deformation styles / relative ages
- E/M: conductivity / magnetic susceptibility

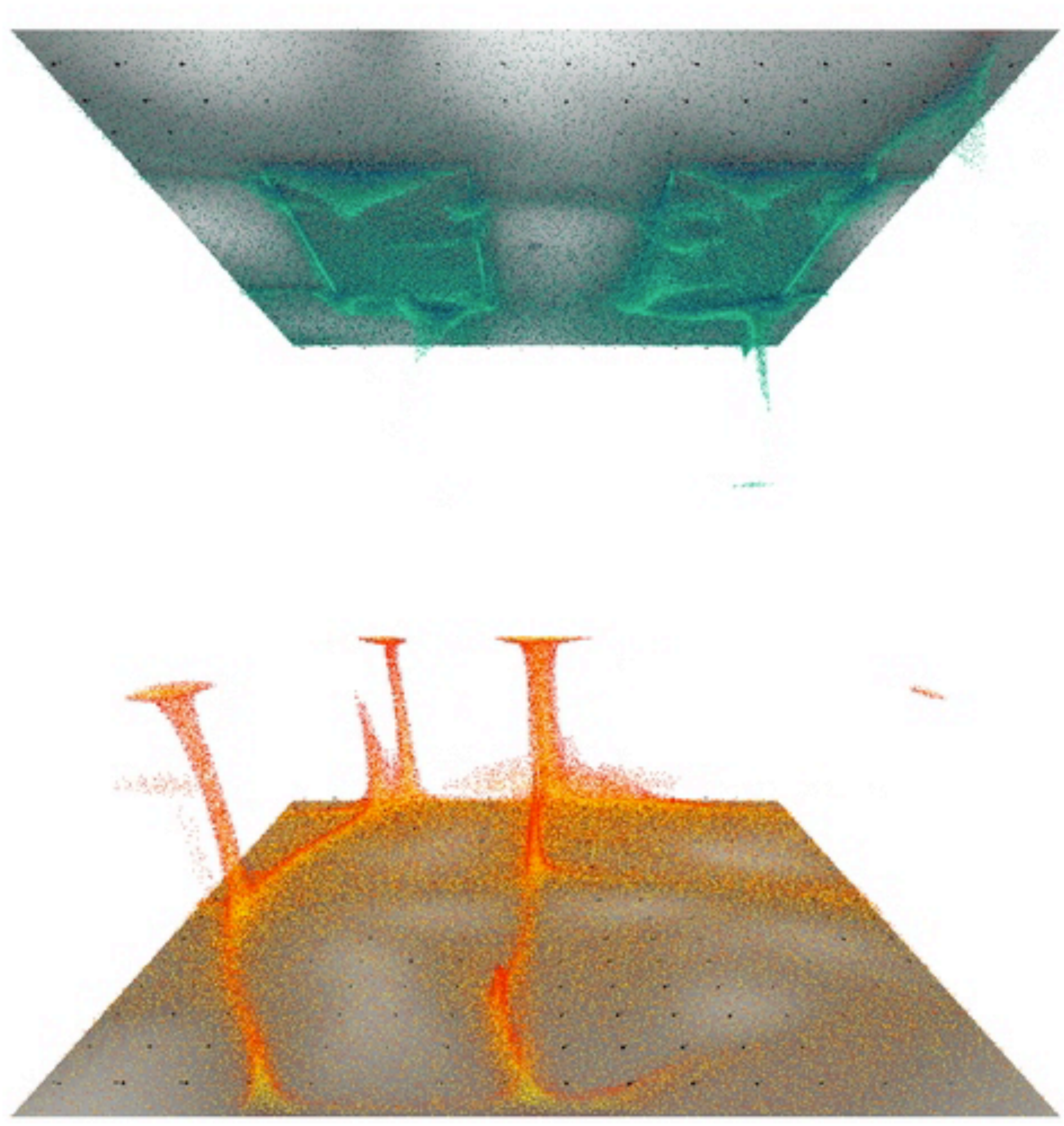
None of these tells us directly what the flow is doing at depth —how to solve this **and** the uncertainty of rheological parameters

3d convection model



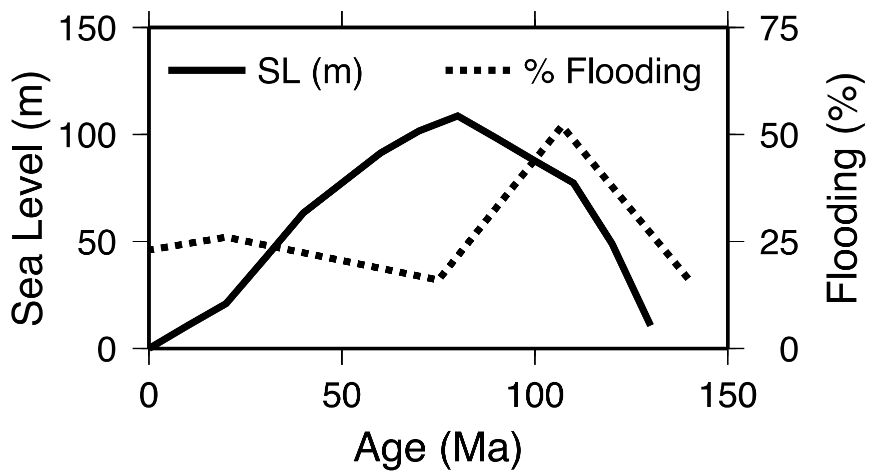
How is the complexity of 3d thermal convection manifest at the surface ? Upwellings, downwellings etc ? How do we see through the lithosphere ?

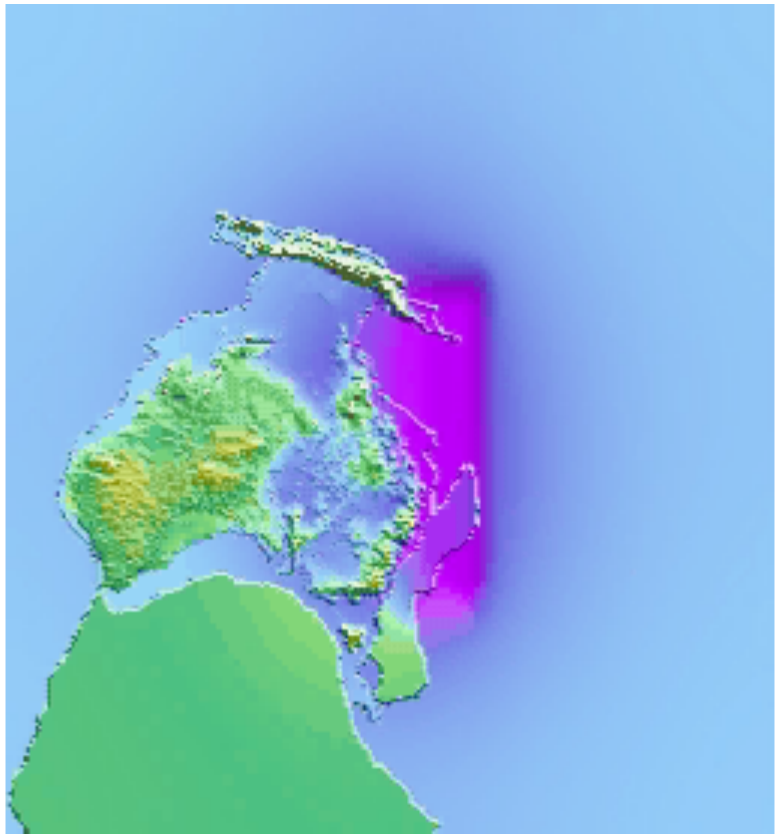
Small-scale instabilities of the (continental) lithosphere

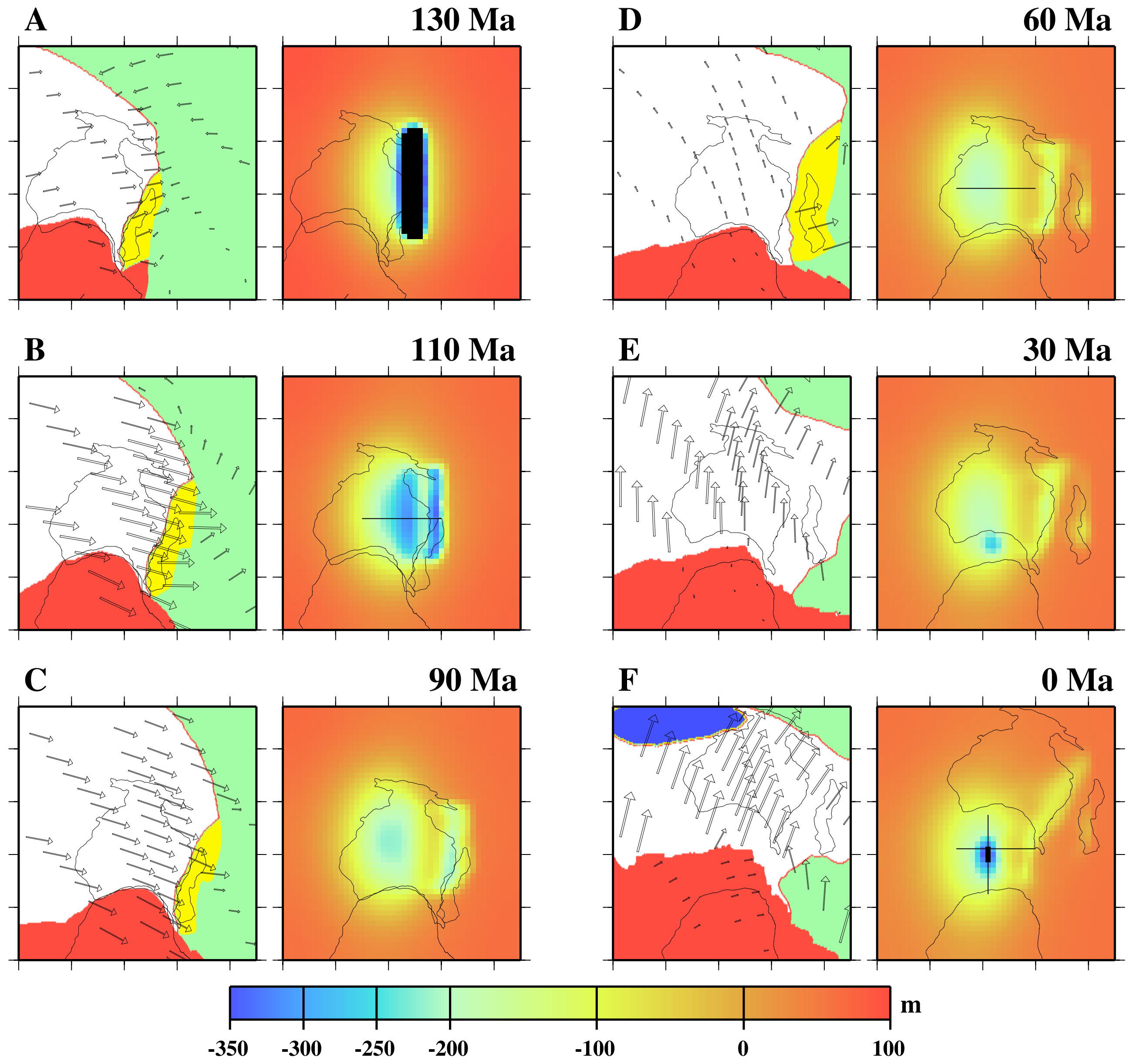


Note how the pattern of small-scale downwellings is highly influenced by steps in the lithosphere

Messages from the mantle

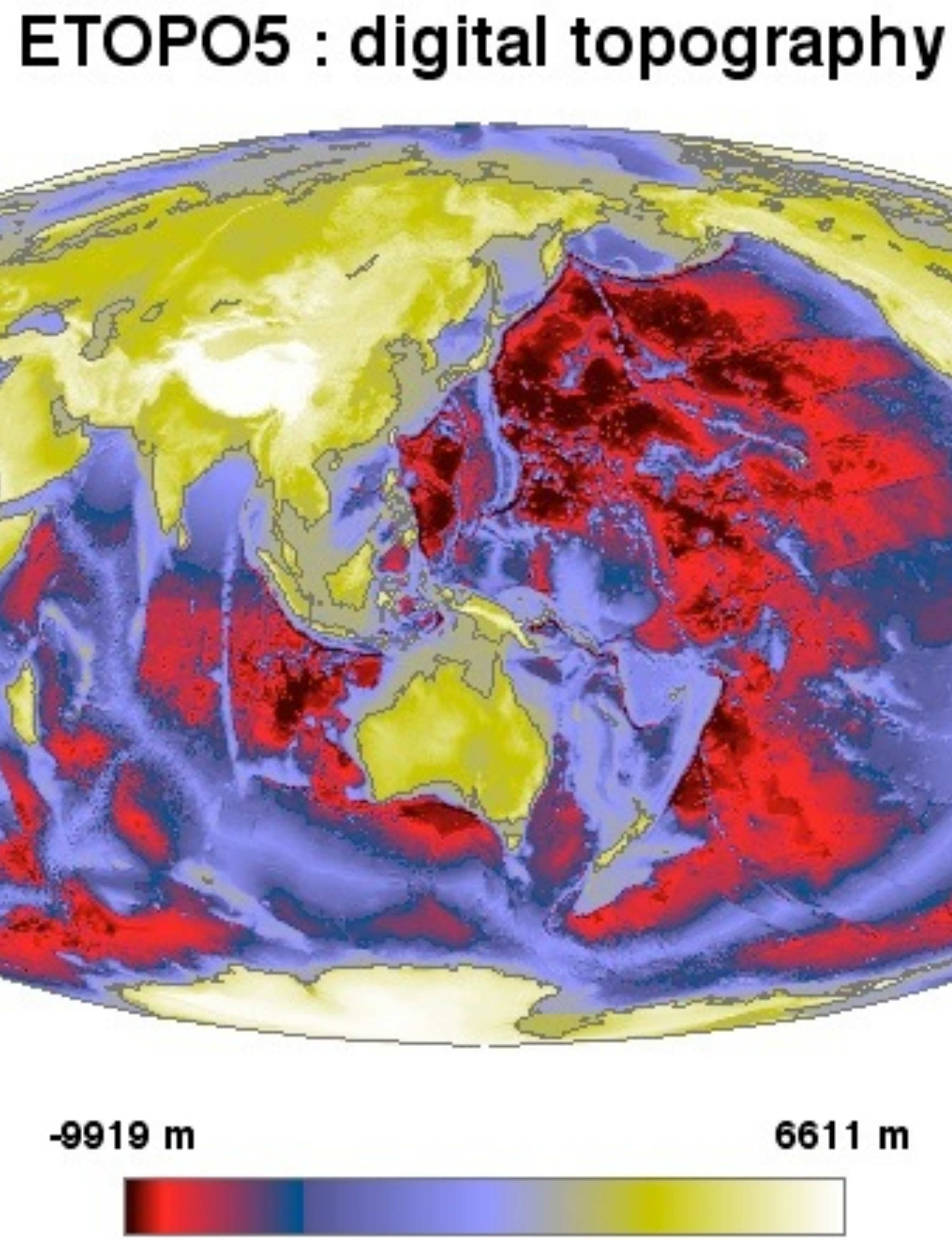






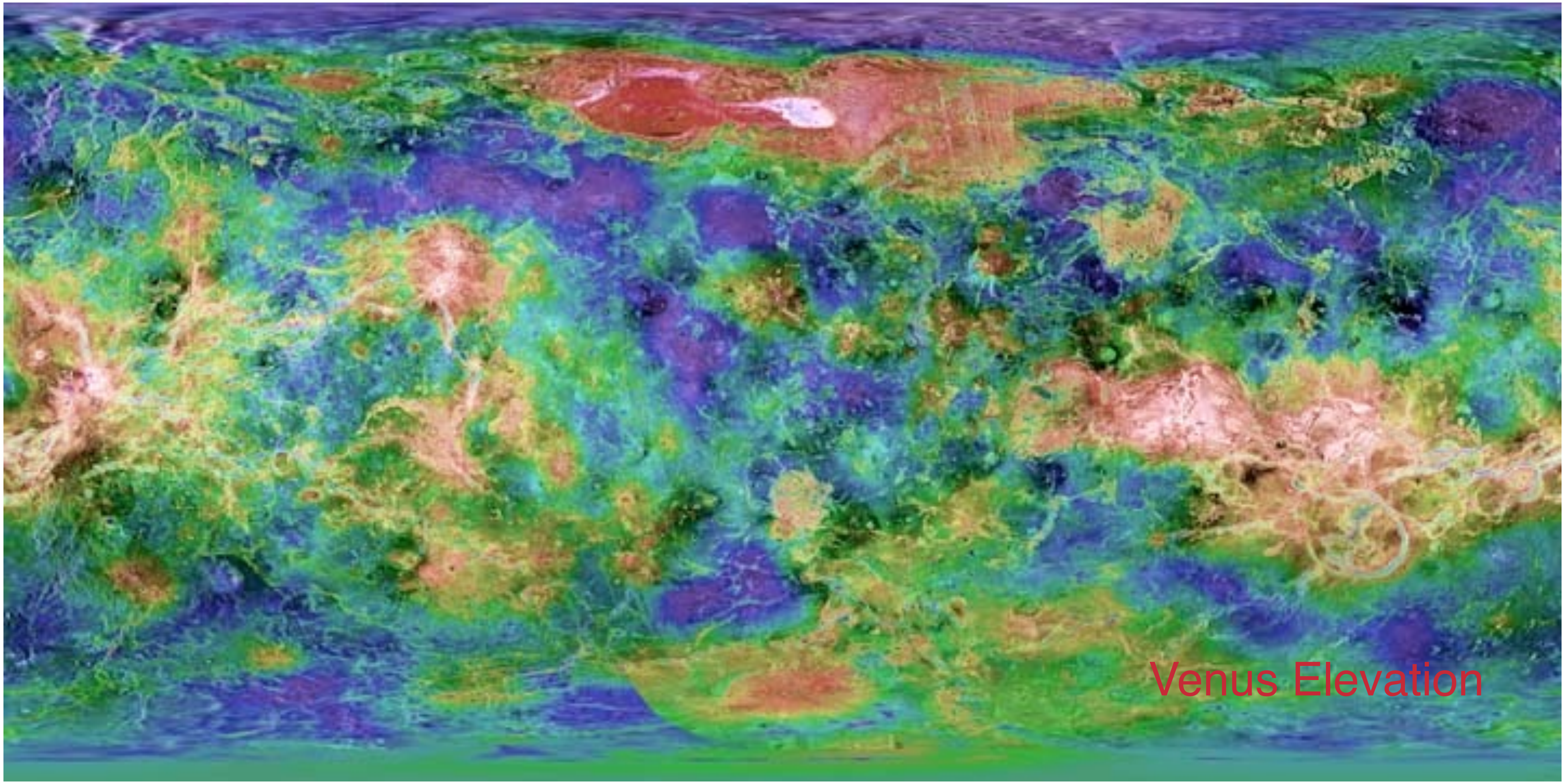
M. Gurnis, L. Moresi, and R. D. Muller. Models of mantle convection incorporating plate tectonics: the Australian region since the cretaceous. In M. A. Richards, R. Gordon, and R. van der Hilst, editors, *AGU Geophysical Monograph 21*, pages 211–238. American Geophysical Union, 2000.

Surface topography



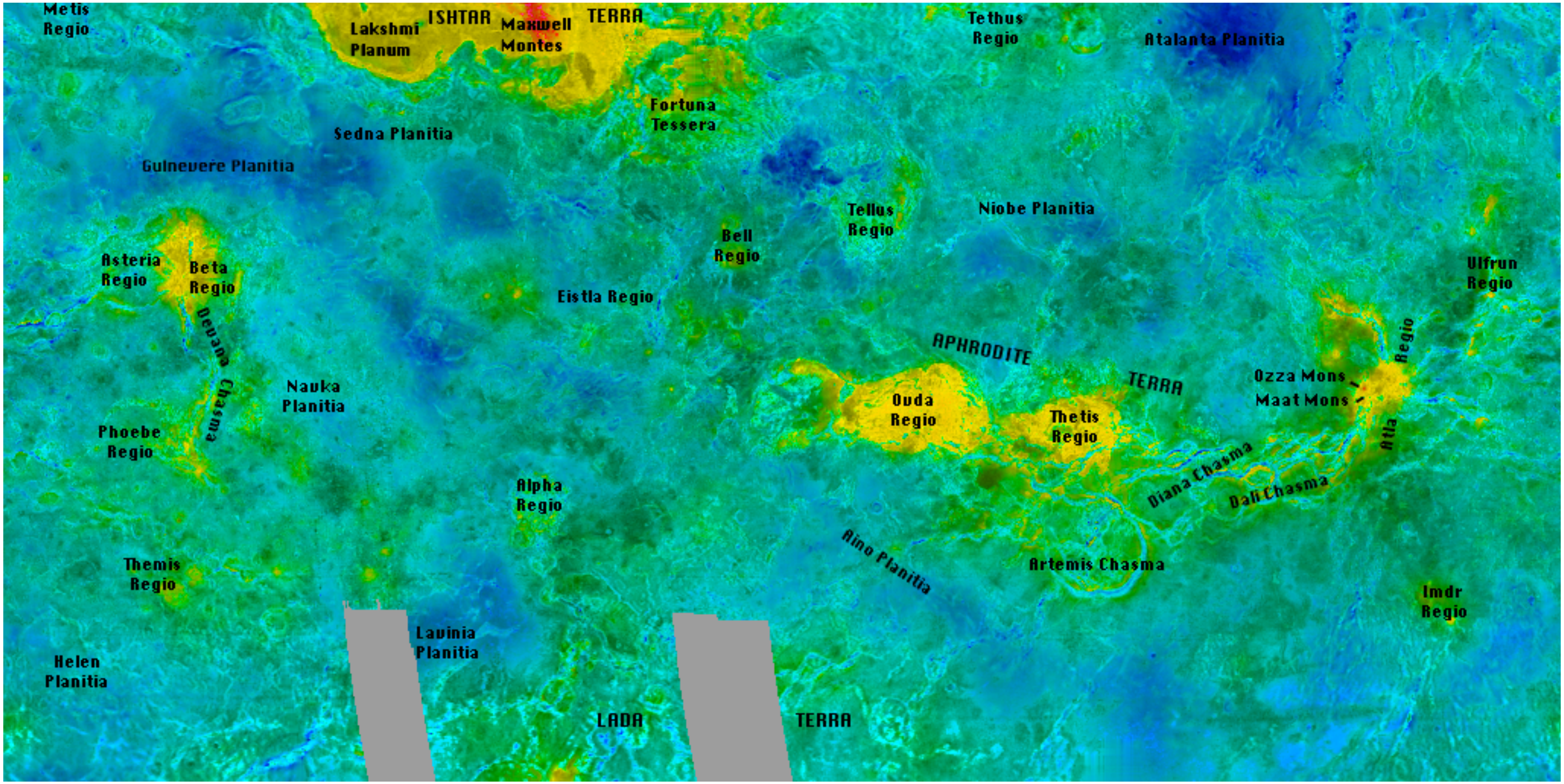
A complex blend of isostasy, strength and mantle dynamics

Surface topography — Venus

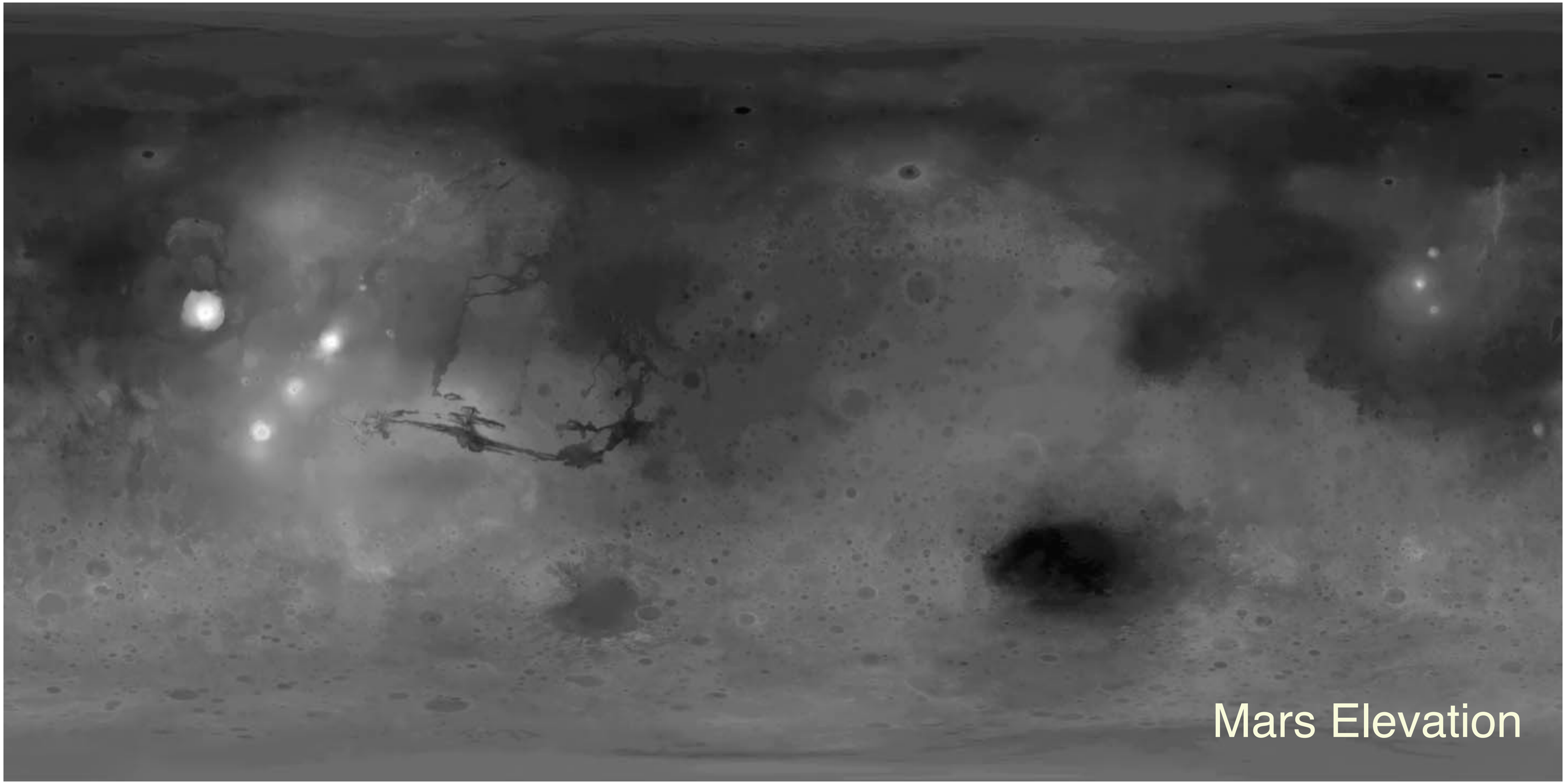


A complex blend of isostasy, strength and mantle dynamics

Surface topography — Venus

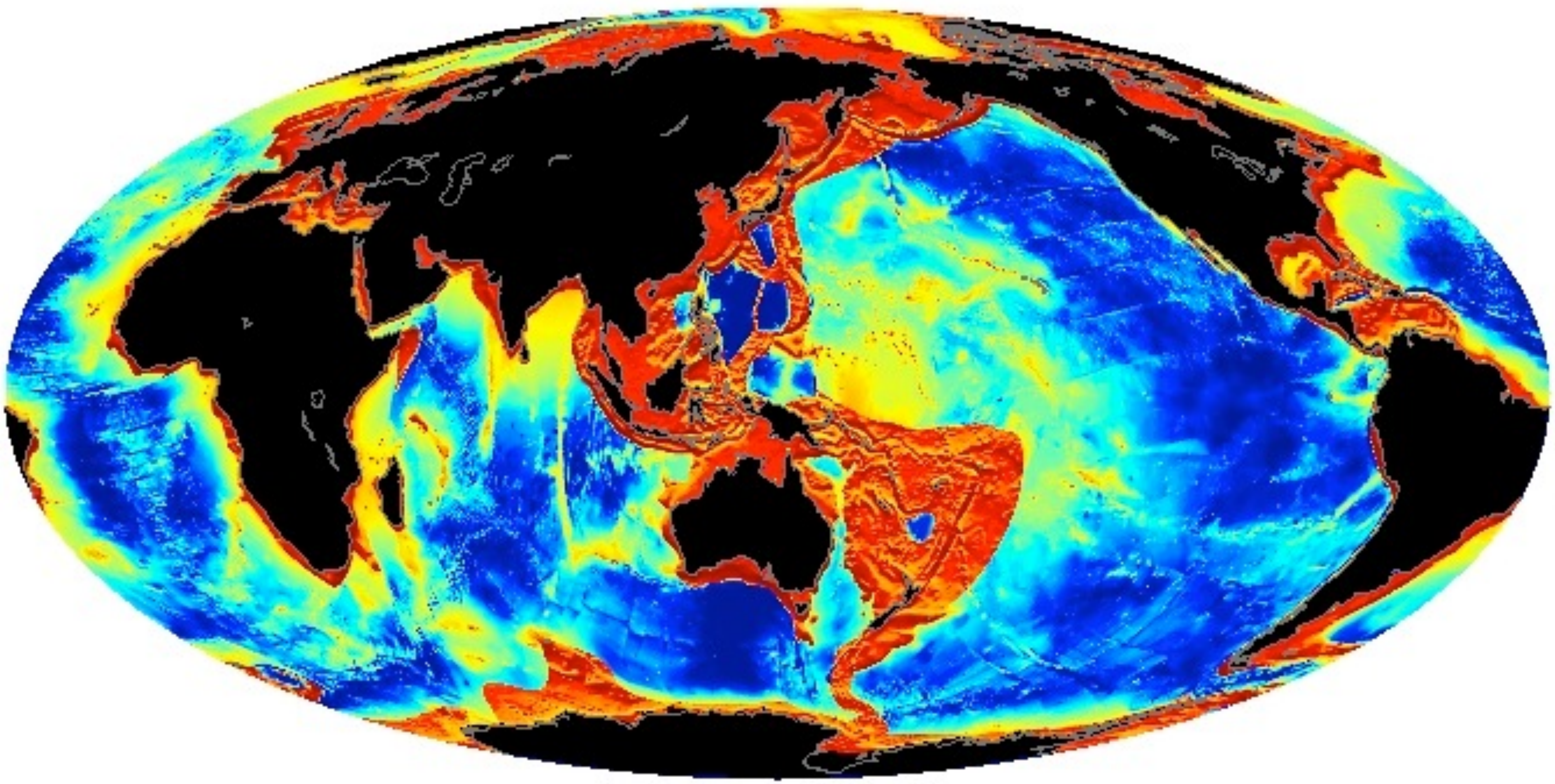


Surface topography — Mars

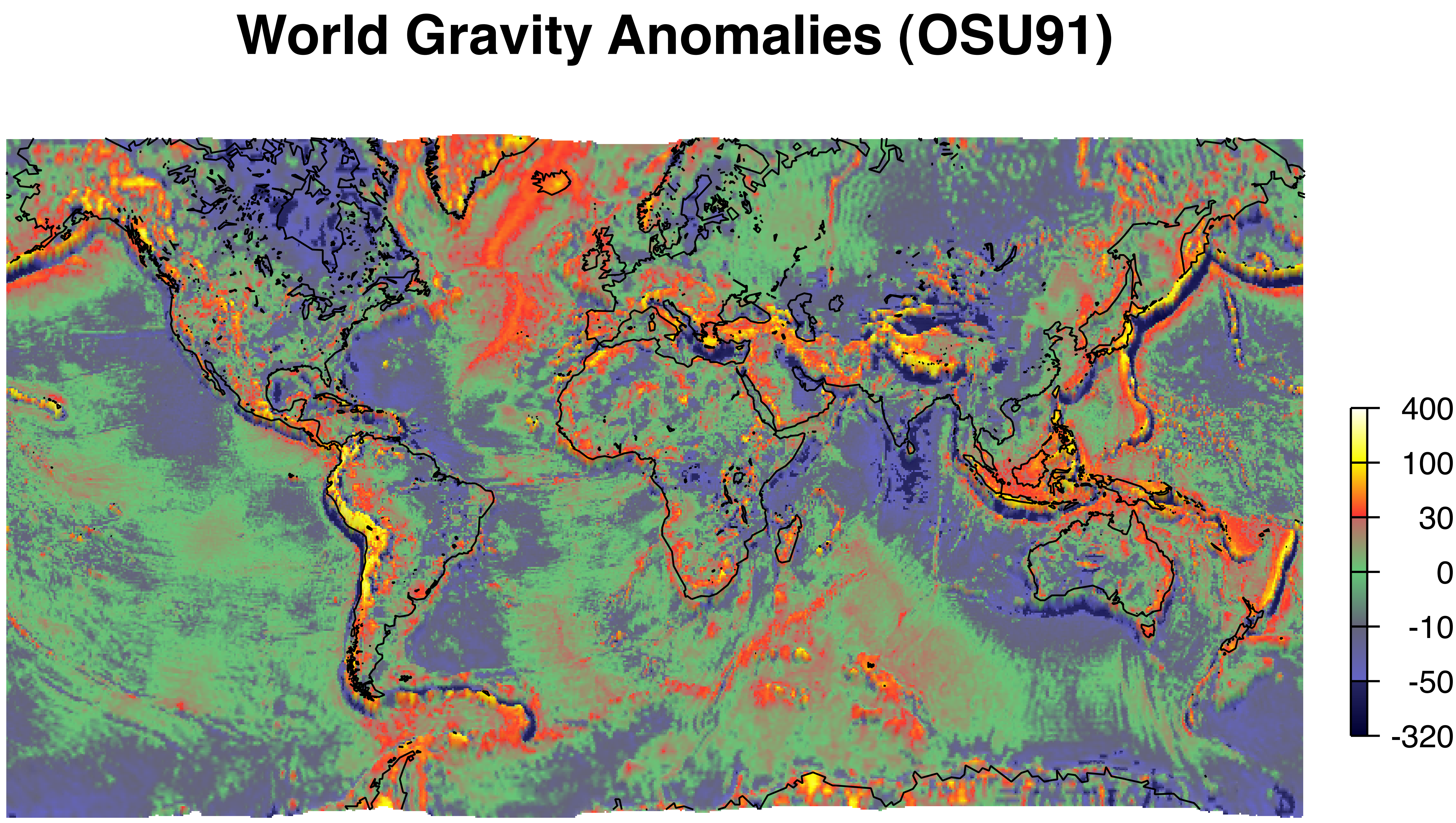


A complex blend of isostasy, strength and mantle dynamics

Surface topography

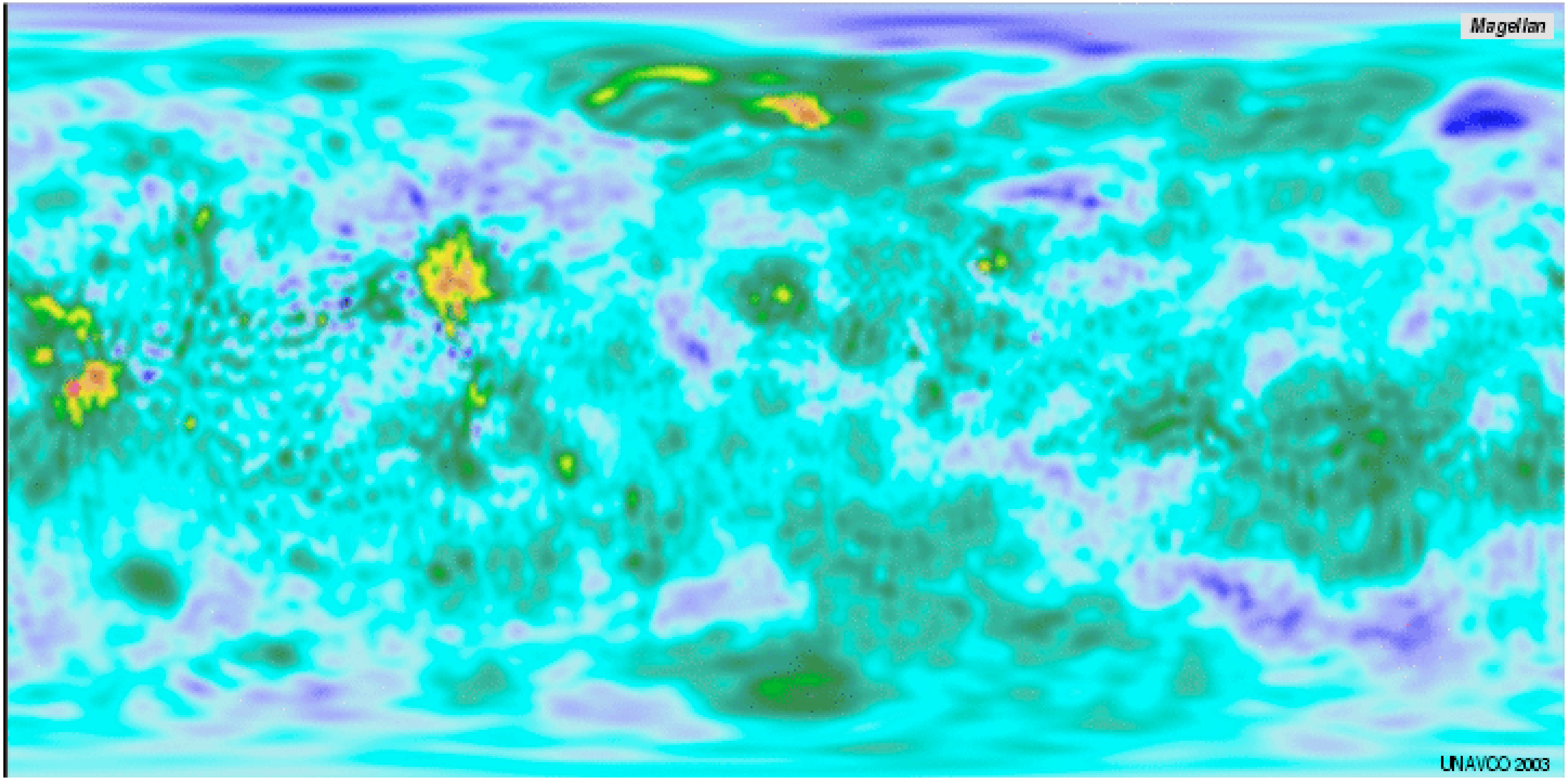


With sediments and depth-age relationships removed



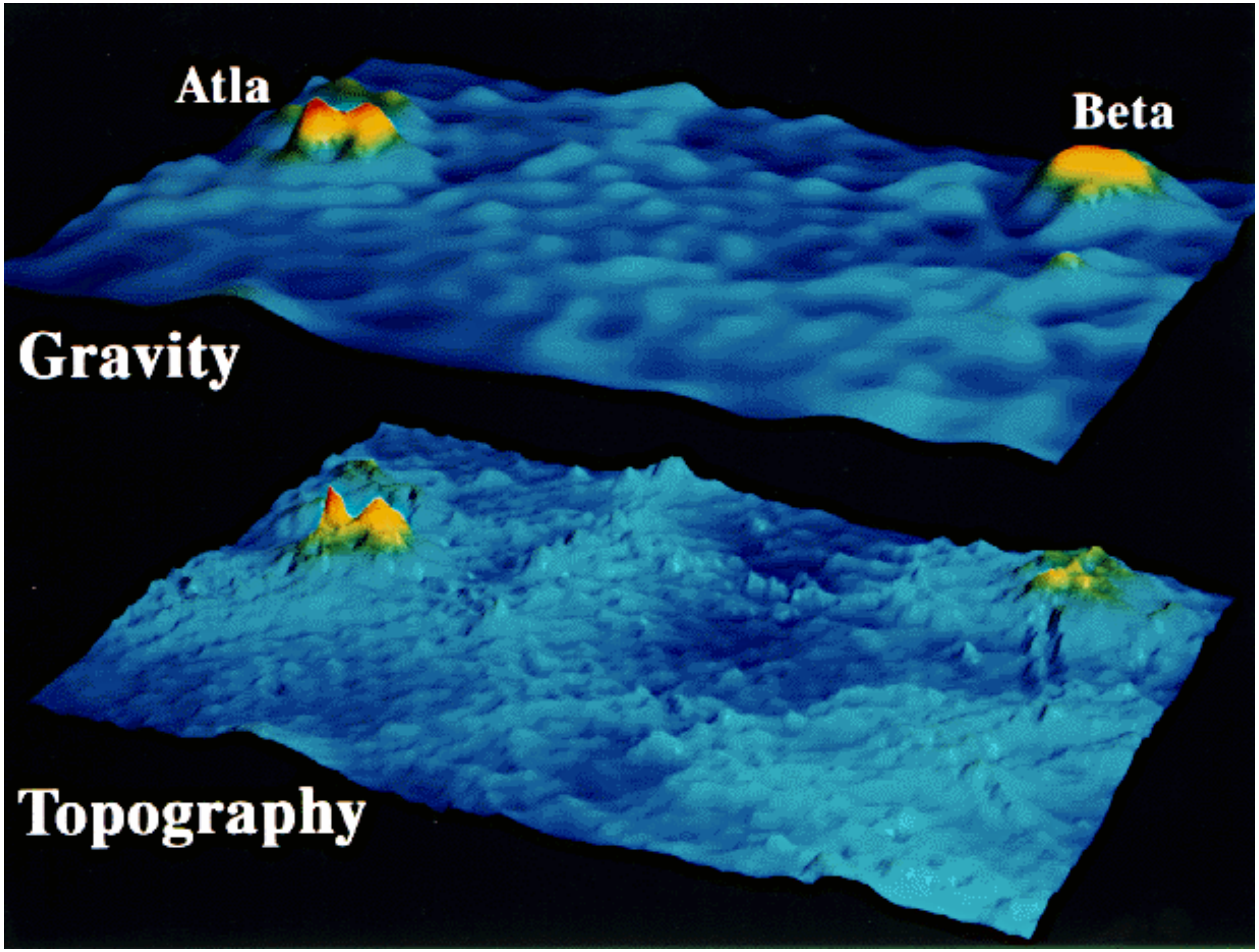
What we see (or don't see) is mainly isostasy

Gravity field



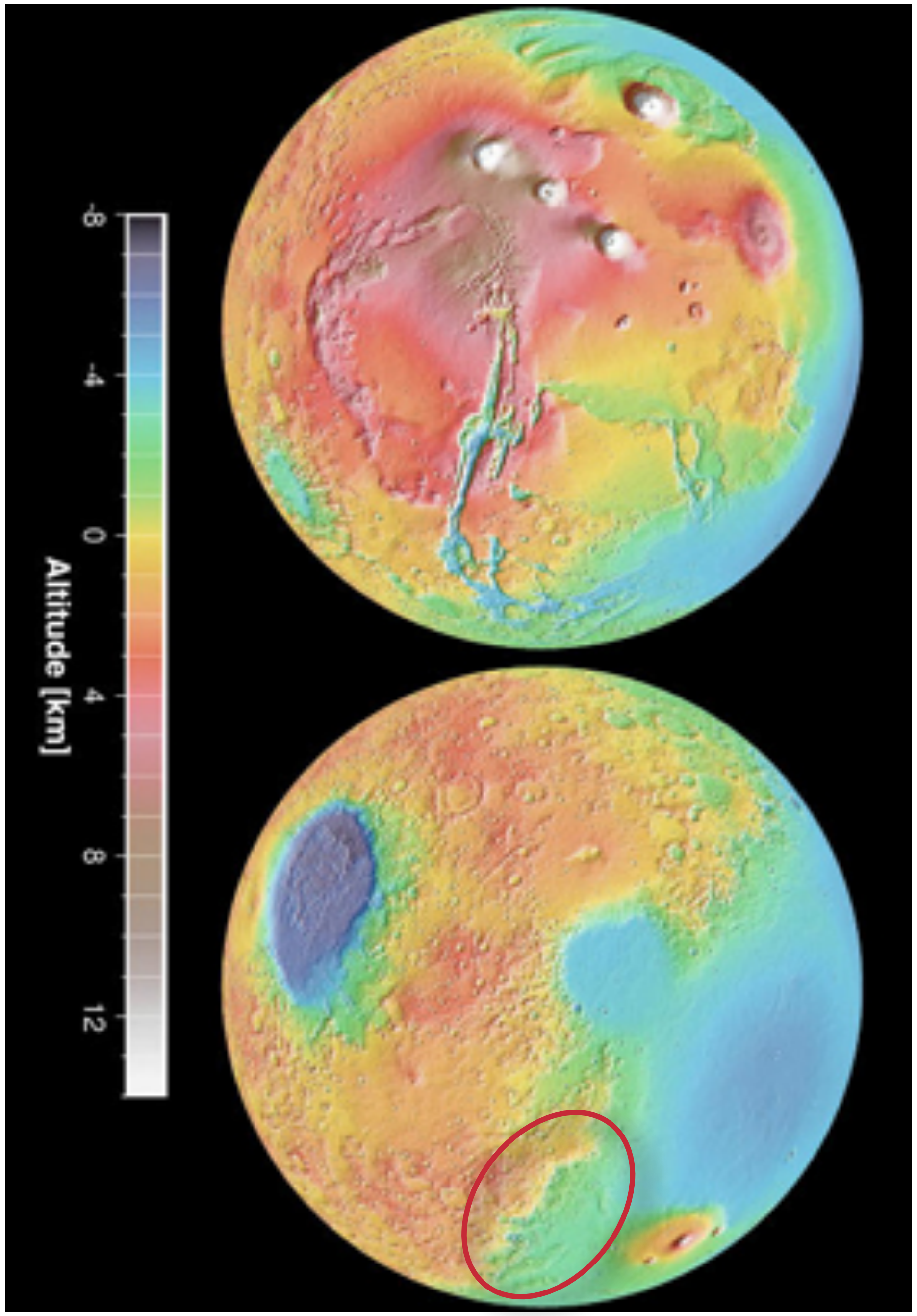
Venus is similar in some ways — whole regions of topography are invisible. But some still stand out strongly.

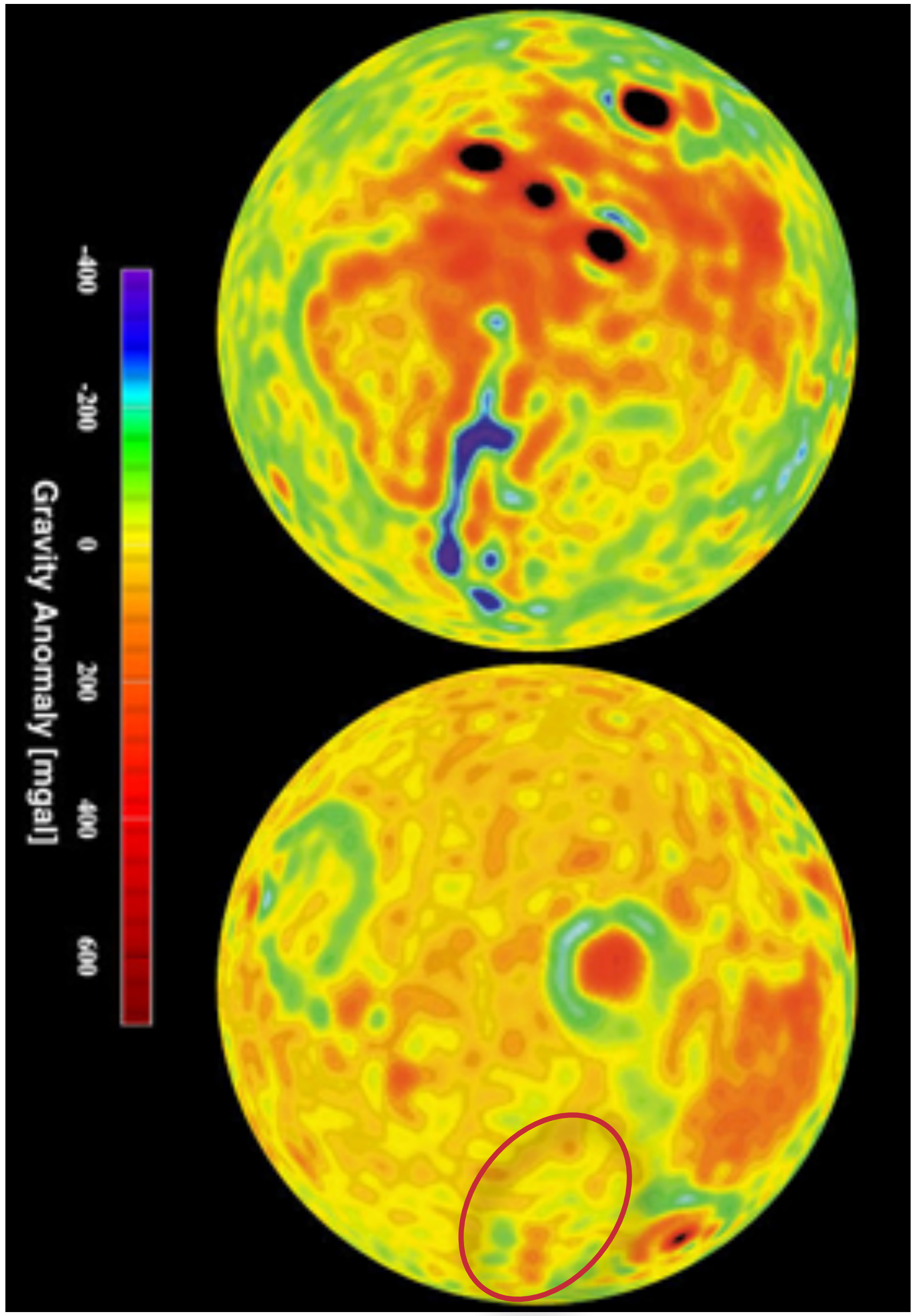
Gravity & topography



Relative magnitude of gravity and topography is generally more useful as each has different sensitivity to deep mass, crustal strength, isostatic compensation etc.

Gravity & topography

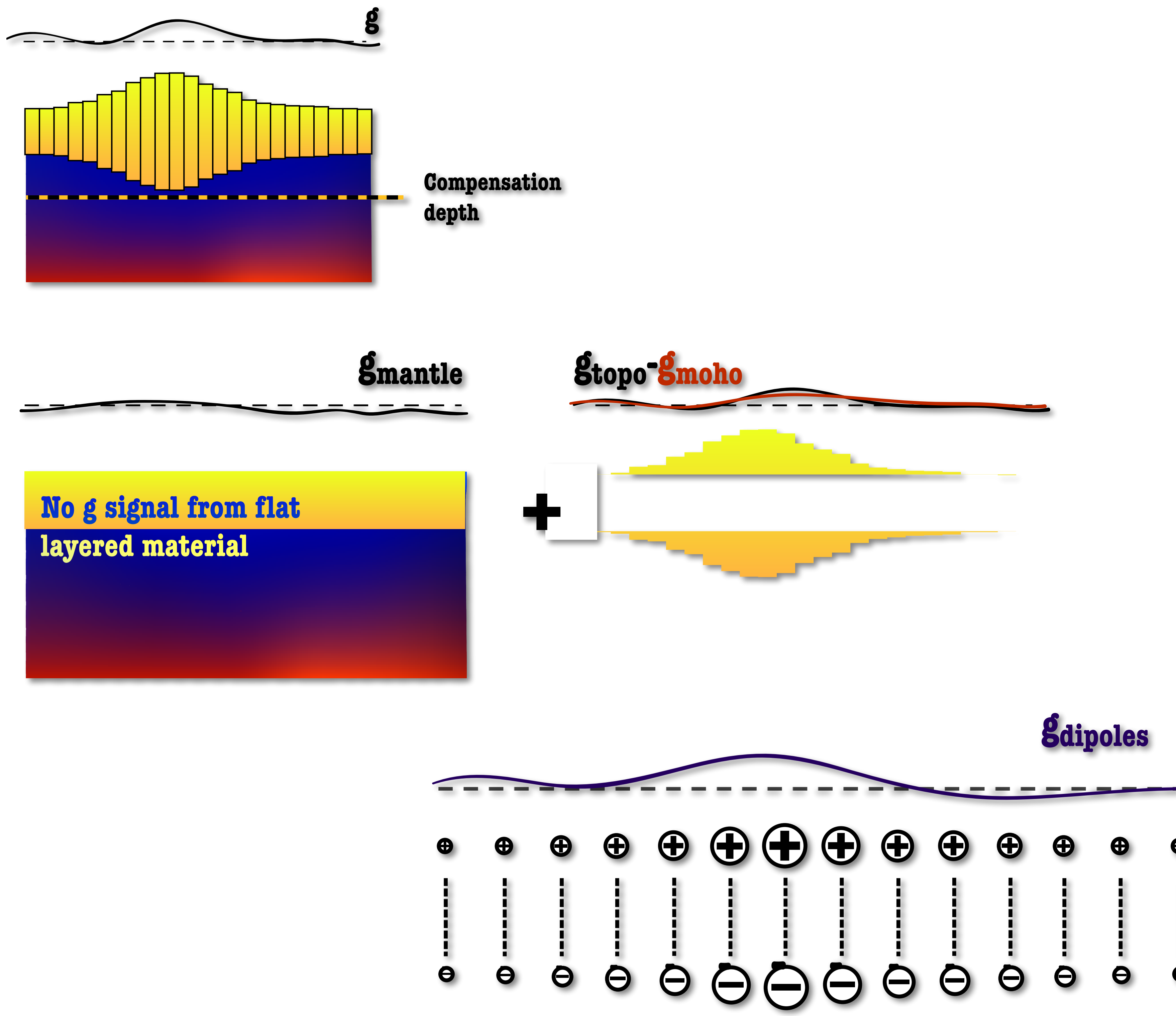




- crustal thickening,
- impact craters
- tharsis rise
- volcanic edifices.

Gravity anomaly and Isostasy

The lack of gravity (anomaly) signature associated with continental topography on Earth is explained simply by ISOSTASY — wherever there is a topographic high, there is a corresponding crustal root which produces a gravitational dipole



Geoid

Gravity and topography both reveal the goings on in the interior directly and indirectly. They have very different sensitivities to deep mass.

It is common to use **geoid in place of gravity** since this is relatively easy to obtain over the oceans. It is basically equivalent information to knowledge of the gravity field.

Geoid: *The gravitational equipotential which best coincides (in a least squares sense) with mean sea level.*

For other planets some other reference level is required.

Also, consider what happens if mean sea level changes over time.

Equipotentials

gravity — the force of nature which manifests itself as an attraction between objects with mass.

The gravitational force between two objects is

$$\boxed{F = \frac{Gm_1m_2}{r^2}}$$

If one of the bodies is planet sized and spherical then the acceleration experienced by a nearby object is

$$\boxed{g = \frac{GM}{r^2}}$$

It takes energy to lift an object against the force of gravity This produces a gravitational potential energy:

$$\Delta E = GMm\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

Energy required to take a test mass from some point above a spherical planet out to an infinite distance:

$$\boxed{U = \frac{GM}{r}}$$

Surfaces where U is constant are spherical. These surfaces are perpendicular to the direction of the gravitational force.

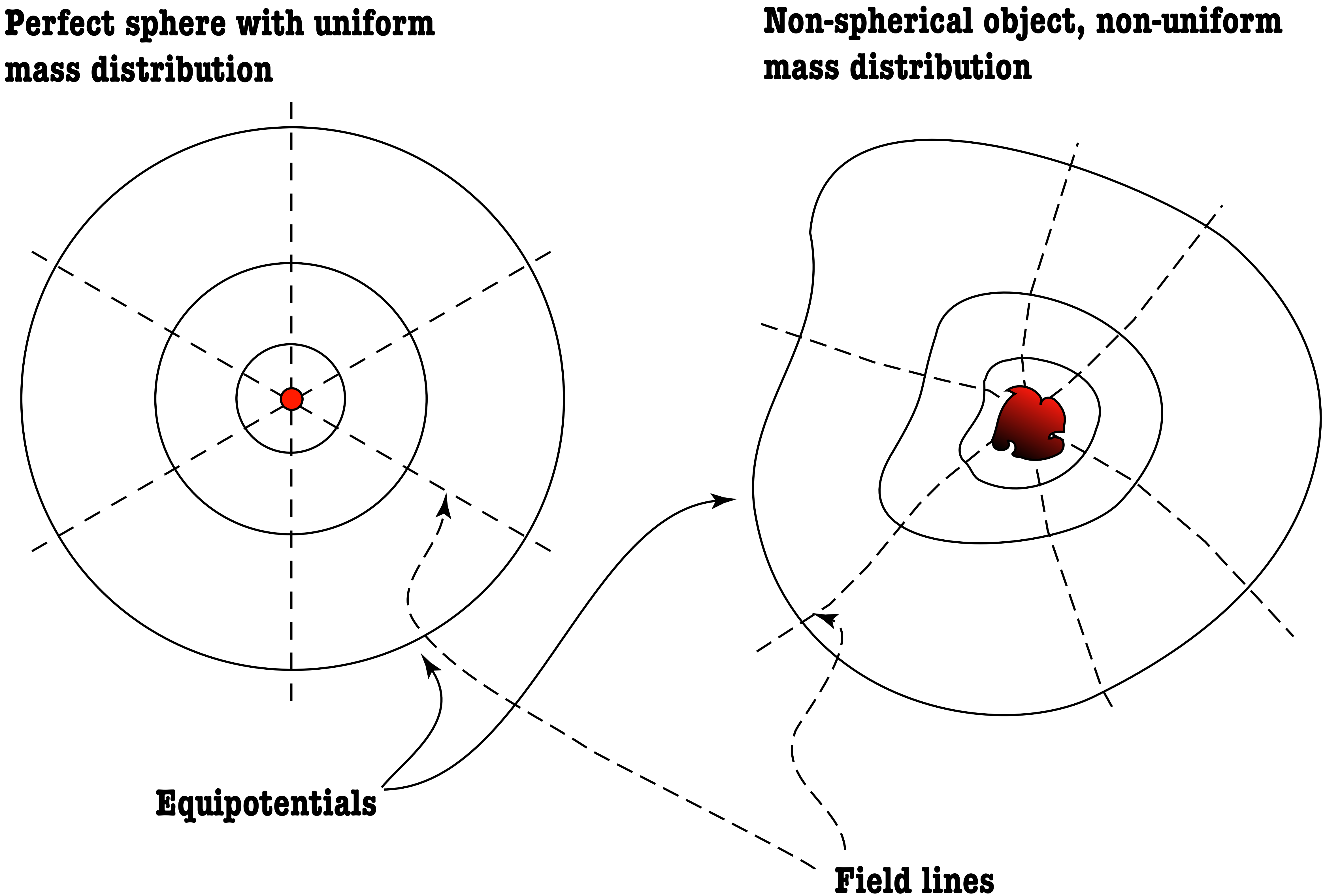
In the realistic, non-spherical planet, the **equipotentials** are not spherical and the field lines are not radial.

gradient vector

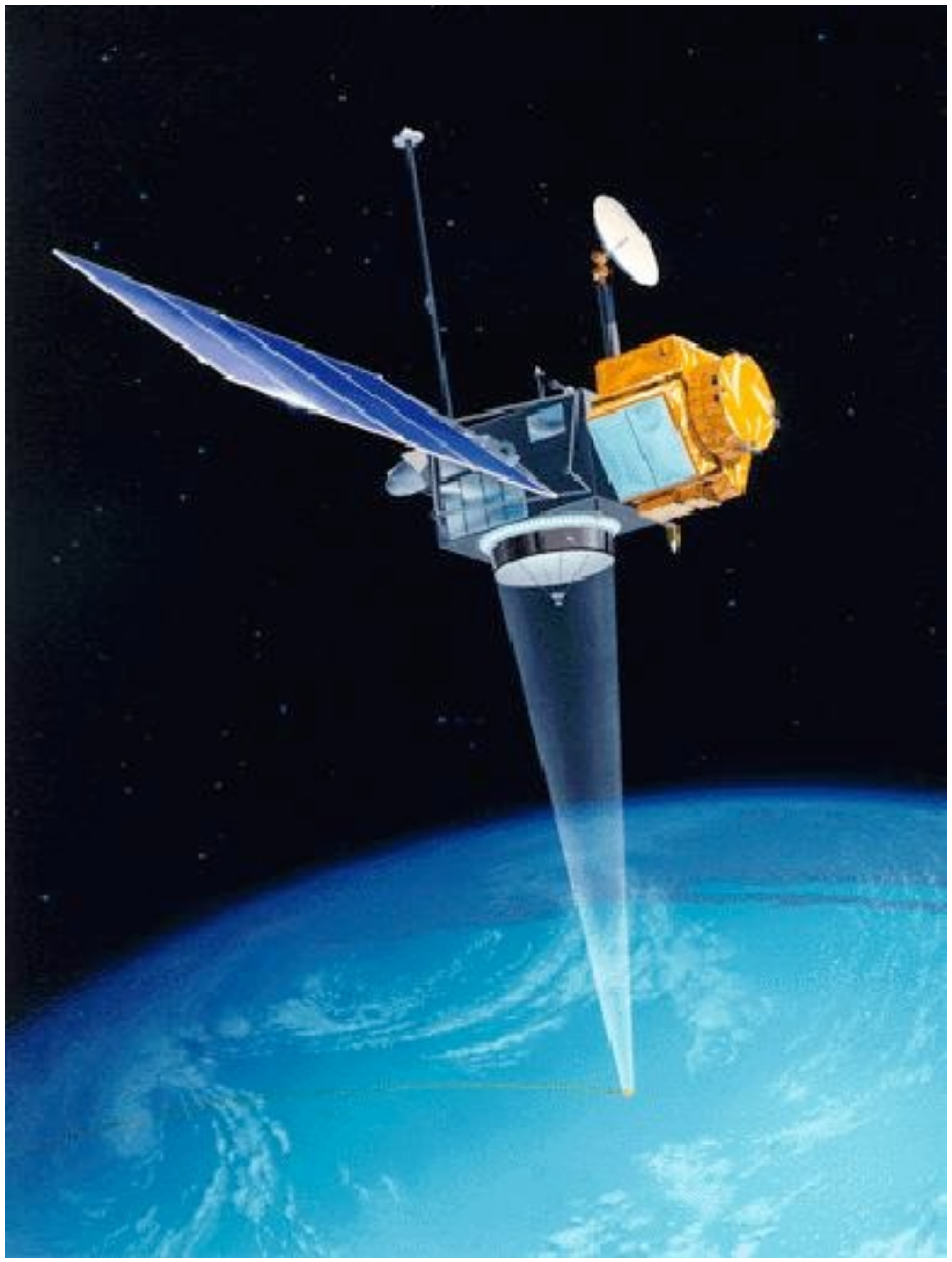
$$g = -\nabla U$$

U contains all the information about the gravitational field g plus a constant of integration.

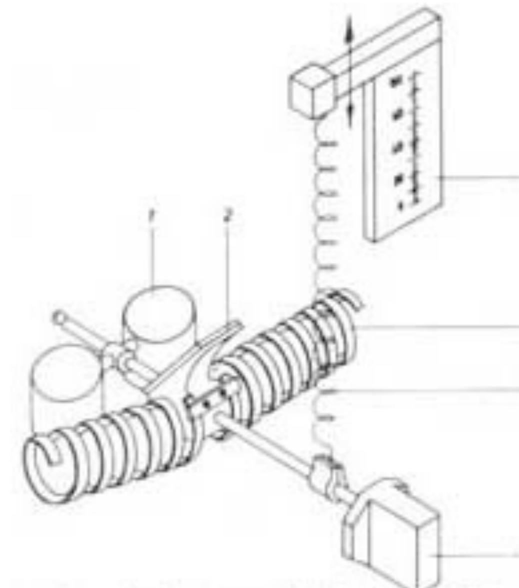
Equipotentials



Geoid/gravity measurement



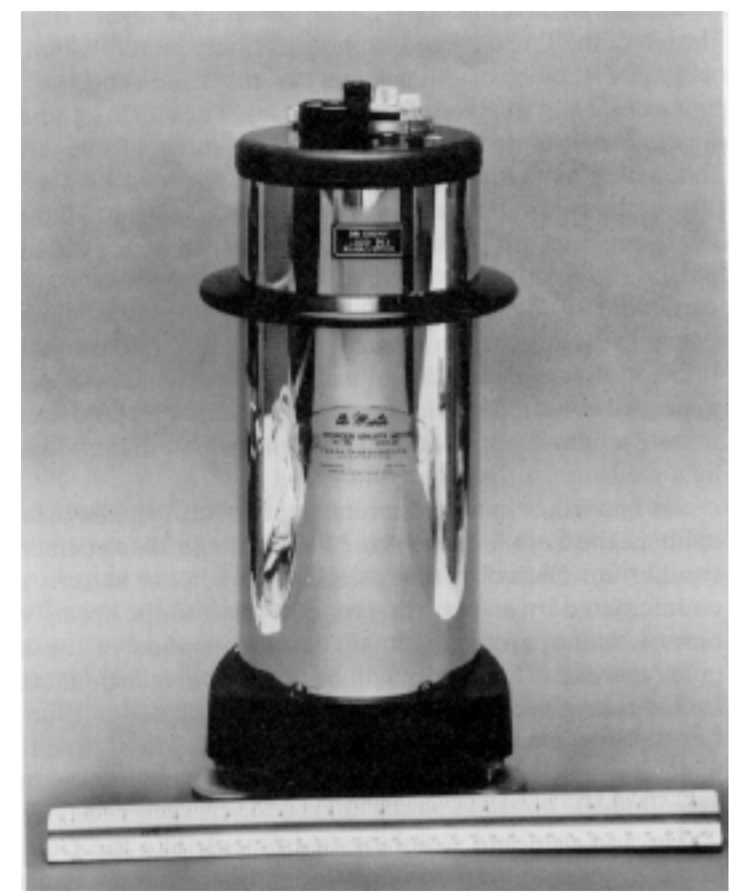
Ocean — sea surface height



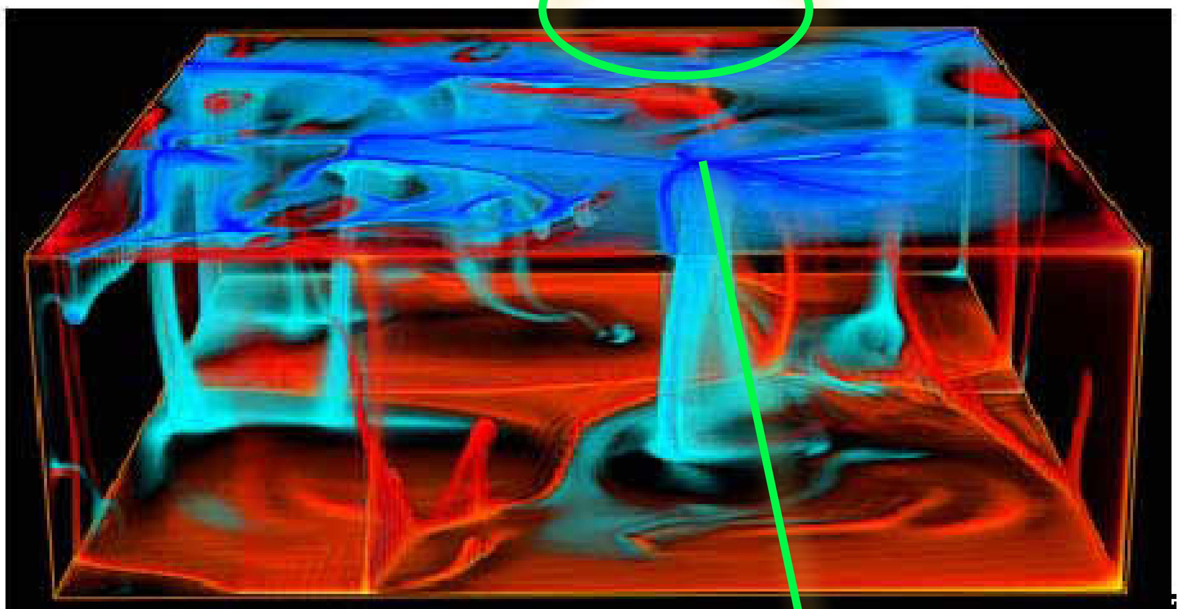
1. Capsul for barometric compensation
2. Mirror
3. Micrometric scale
4. Measuring spring
5. Main springs
6. Mass

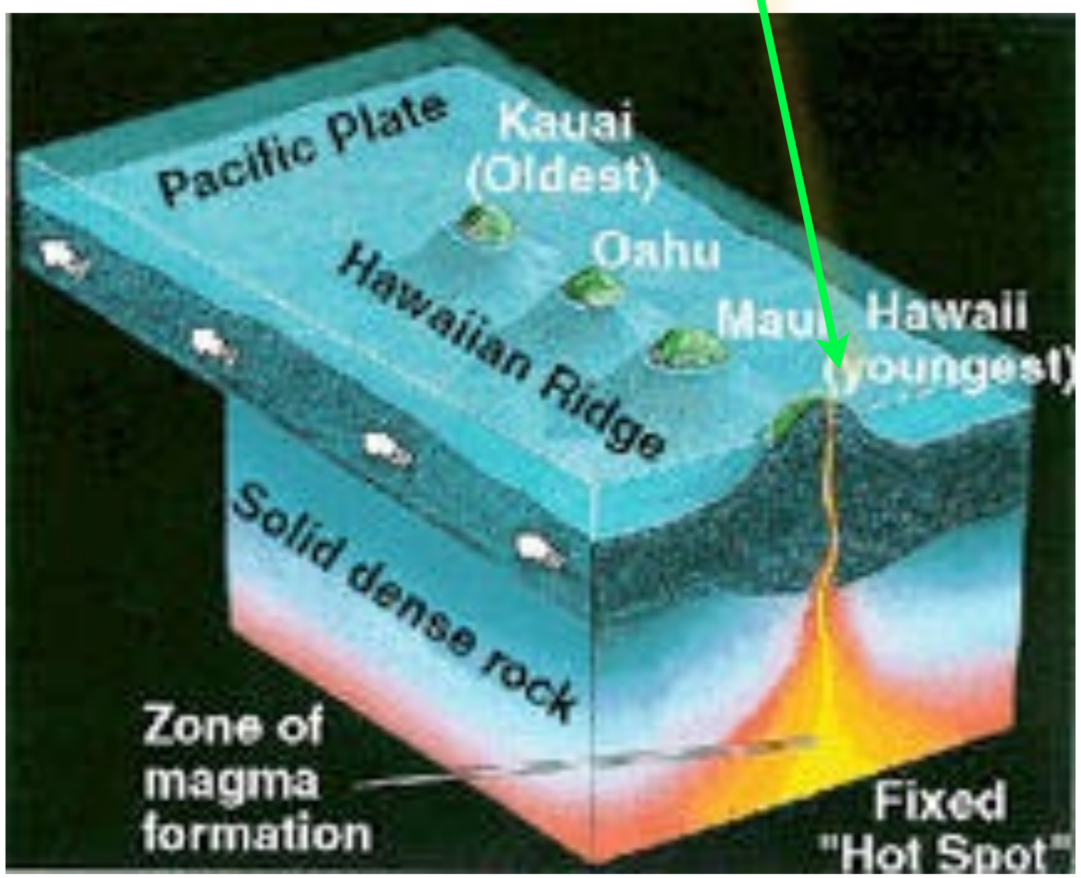
p.267 "Tides of the Planet Earth" - Paul Melchior

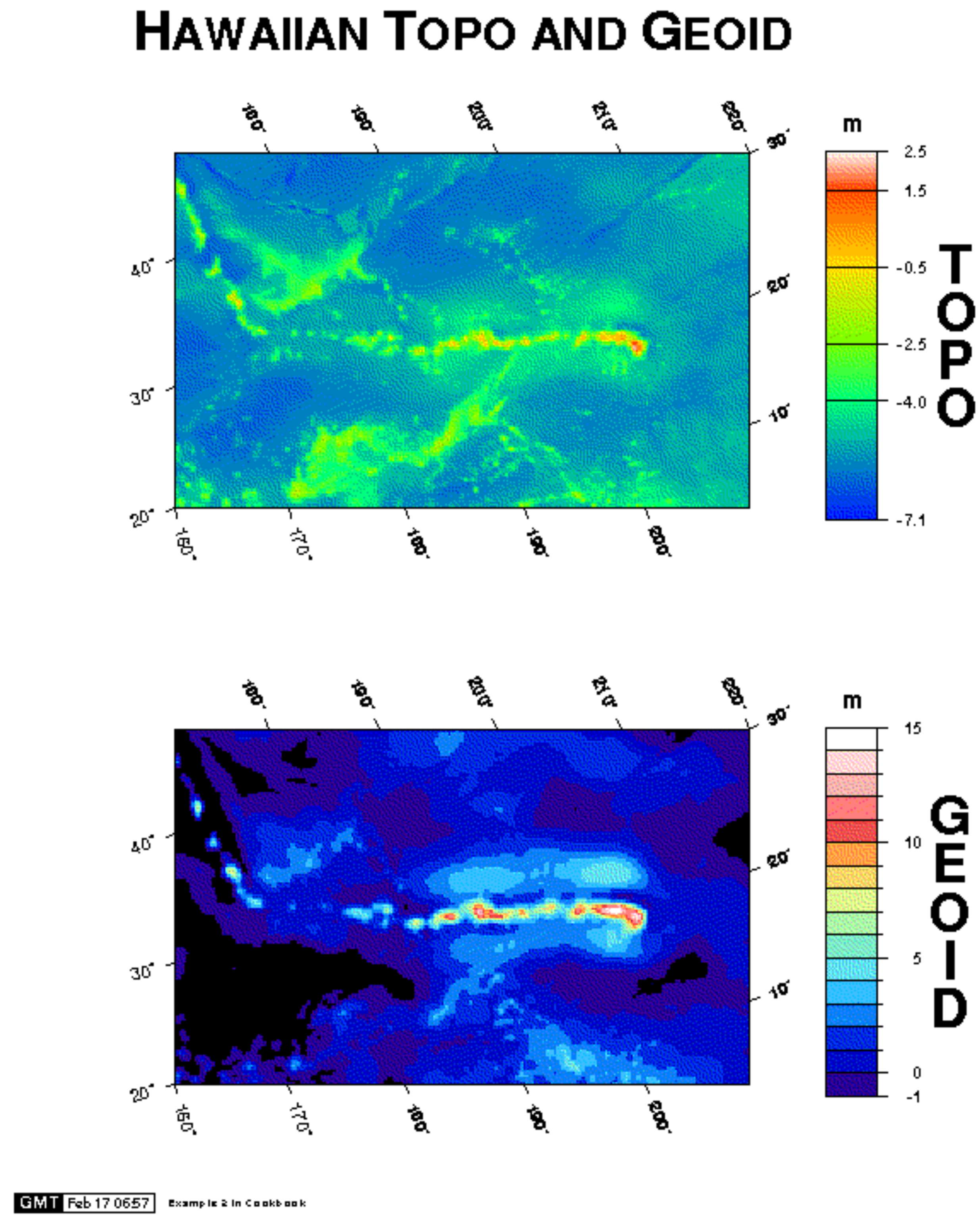
Land — strength of gravitational acceleration



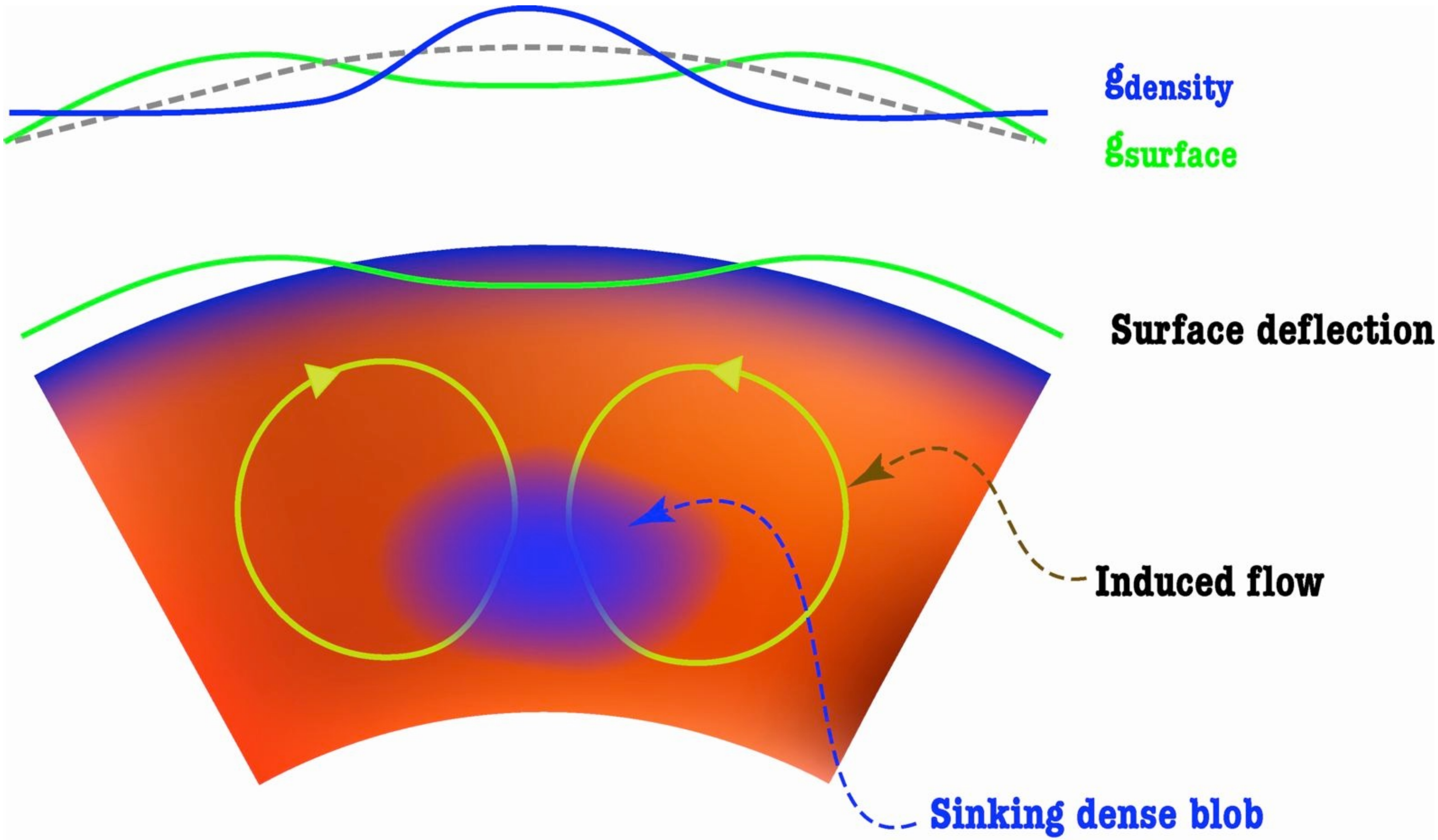
Example: mantle plumes





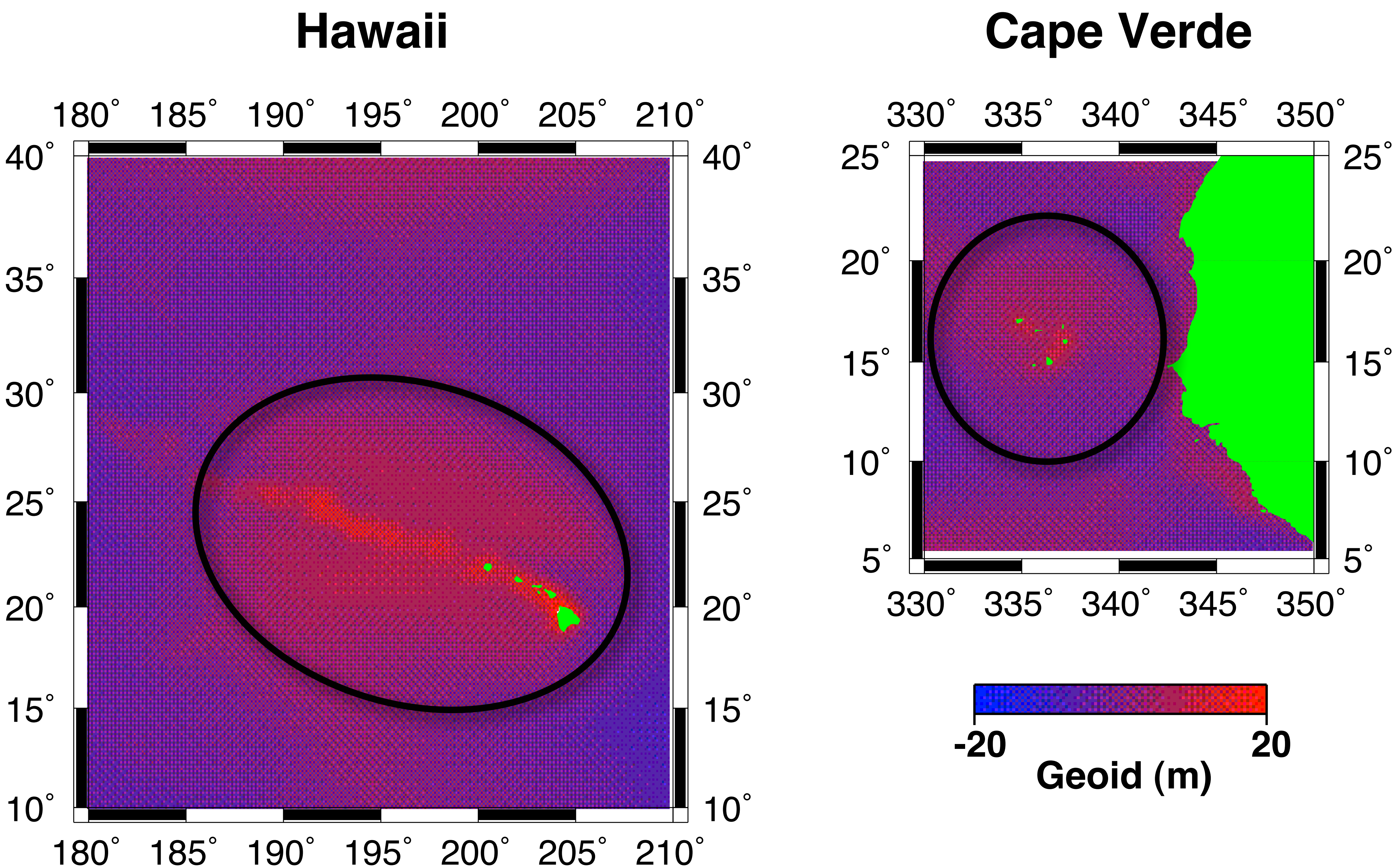


Dynamic topography / geoid



Dynamic topography (in the context of geodynamics) specifically means that topography which results from viscous flow in the interior of the Earth.

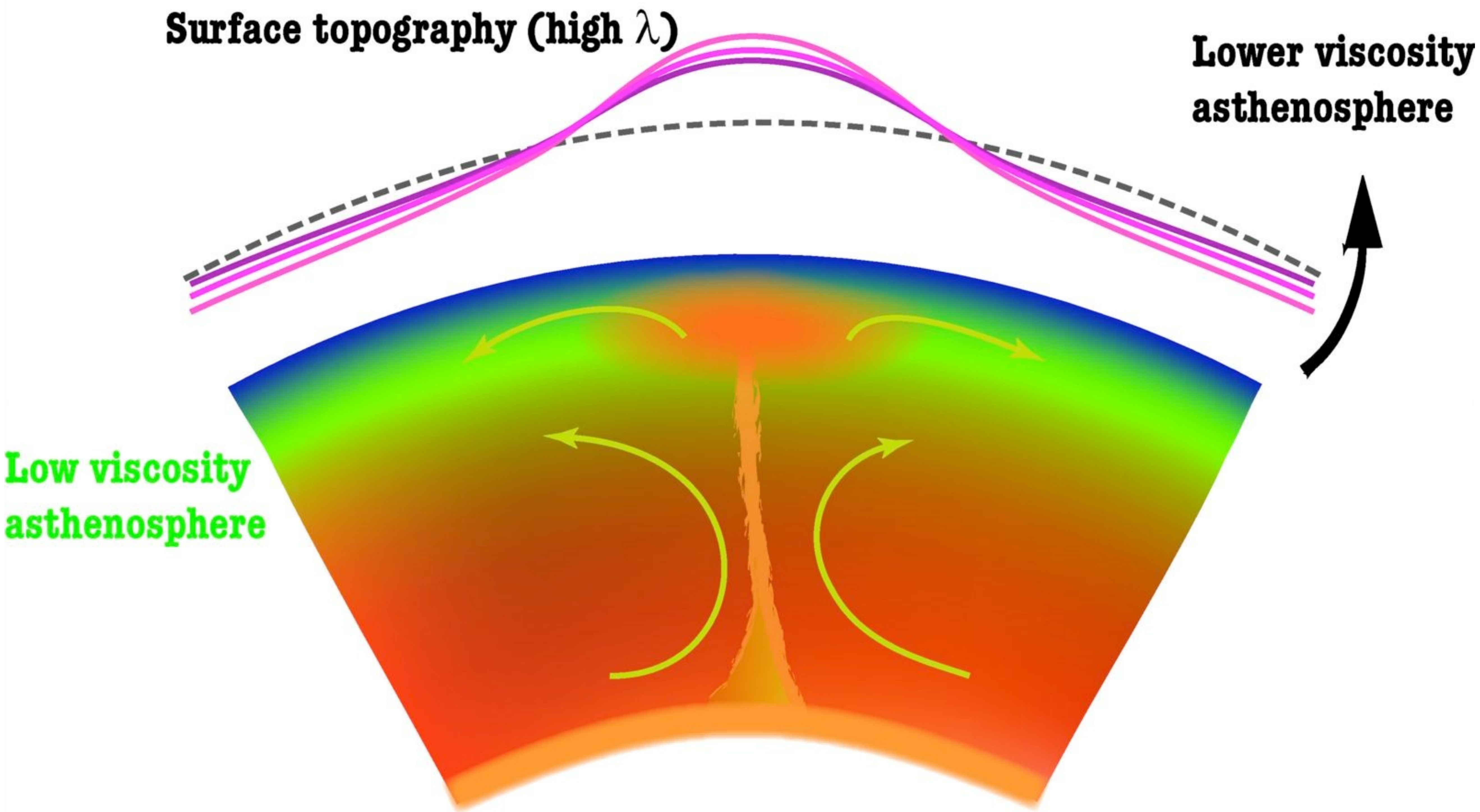
Geoid over a plume



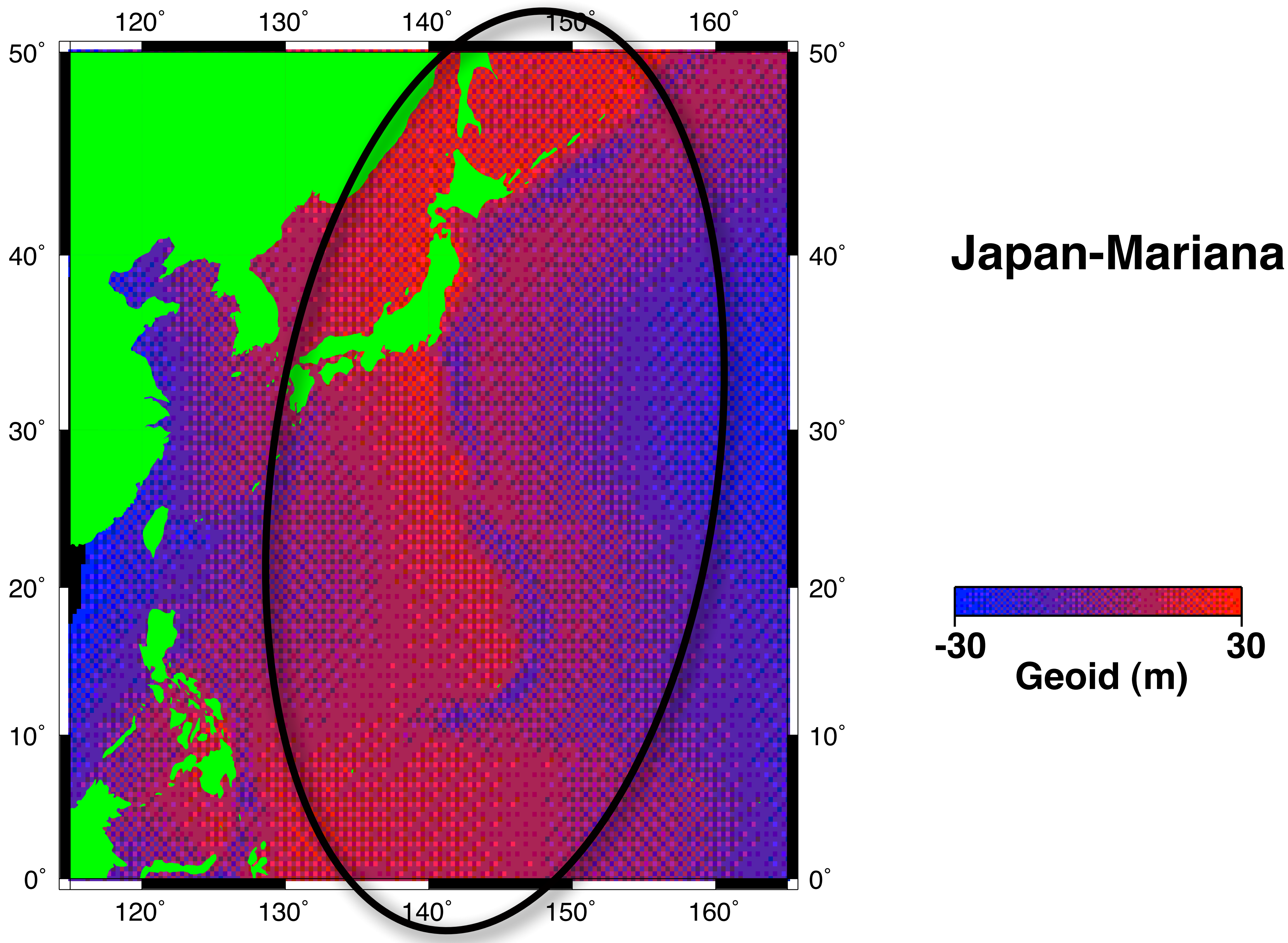
Broad scale geoid high over upwelling material

Dynamic topography / geoid of plume

Light material directly beneath the lithosphere causes significant uplift of surface

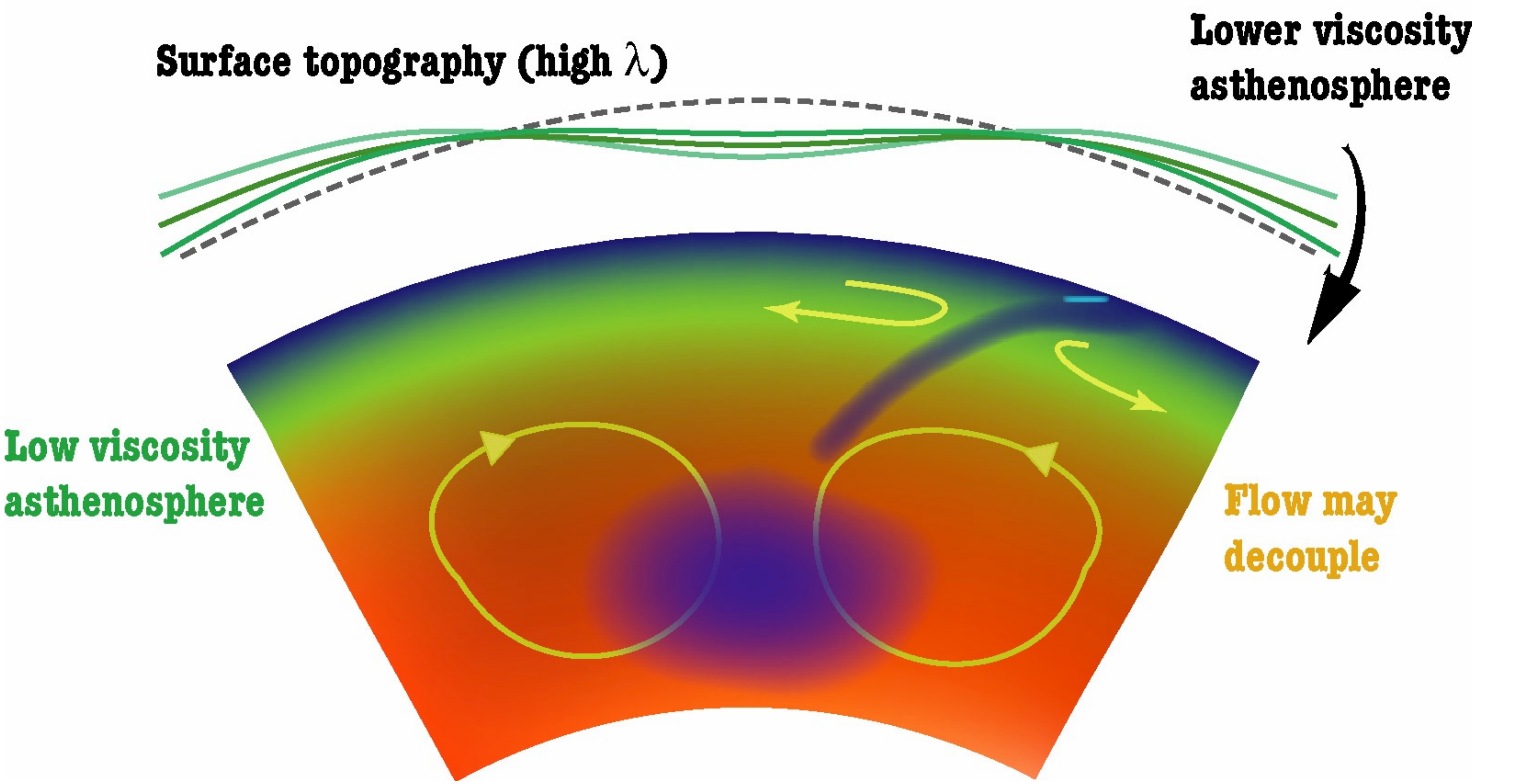


Geoid over subduction zone



Broadscale geoid high over subducted material

Geoid over subduction zone



Dynamic topography driven weakly over a large area by deep mass, strongly local to the trench by the attached slab
Geoid contribution from the deep mass doesn't care about the details of the trench

Take away points

Convection occurs in any (viscous) fluid layer when the critical Rayleigh number is exceeded. That critical value is around 650 for a flat layer and changes a little in the spherical shell or when there is internal heating.

The solid mantle has a finite (though very large) viscosity due to imperfections in the crystal lattice of the mineral grains that allow slow deformation to accumulate and relax elastic stresses.

That finite viscosity is somewhere on the order of 10^{19} Pa.s and this leads to a Rayleigh number that is well above critical.

At high Rayleigh number, the convecting fluid develops thermal boundary layers whose structure is quite easy to predict.

If the oceanic plates are thermal boundary layers, then we can predict their thickness and subsidence as they move from the ridge to the trench. This works quite well.

It is much harder to see the effects of deep convective structure except at very long wavelengths.