Graphical Analysis

Direction field

Ride a time machine and go back to the 1940s. You have a nonlinear ODE for a complicated biological process. Unfortunately, you can not solve the equation analytically, and desktop computers have not been developed yet. How will you investigate the model?

The best option for you is to use the skill that even our cave ancestors had. Yes, we will draw figures to understand the dynamics of the model. Though it may sound silly at the age of supercomputers, graphical analysis of ODEs is a powerful tool.

Take a simple nonlinear system,

$$\frac{dx}{dt} = x(1-x) \tag{1}$$

We want to draw a graph showing the time evolution of x; even a rough sketch would also work.

Take t in the horizontal axes and x in the vertical axes. Take a point (t_i, x_i) and draw a tiny bar at this point such that the slope of the bar is equal to dx/dt at that point. Add an arrowhead to the bar indicating the sign of dx/dt at this point.

For example, take the point (0, 0.2). At this point is dx/dt = 0.2×(1-0.2) = 0.16 Therefore, draw a small bar having slope 0.16 at this point. As the sign of dx/dt is positive, at this position, x will increase with time. So, put an arrowhead, to the bar, pointing up.

Cover the whole t-x space with points arranged at regular intervals forming an array and draw small arrows at each of those points. The resulting figure will look like Figure 1. This figure is called the Direction Field of the ODE.

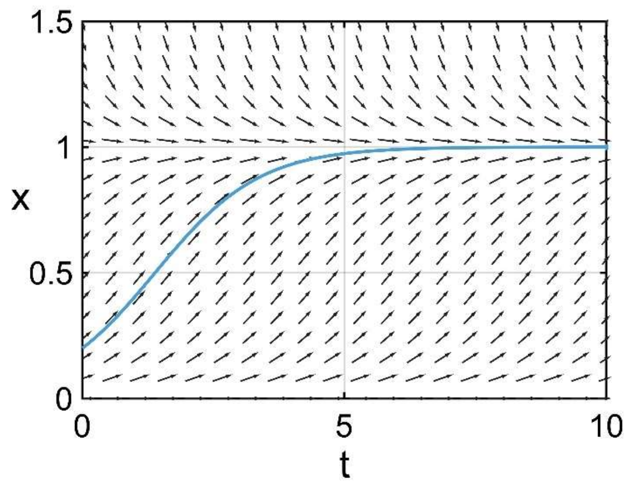


Figure 1: The direction field of $dx/dt = x(1\text{-}x)$. The Integral curve for the initial condition $t = 0$, $x = 0.2$ is shown in blue.

We can use the direction field to sketch the time evolution of the dependent variable of an ODE. For example, start at (0, 0.2) in the direction field in Figure 1 and follow the arrow at this point to reach the next closest point in the direction field. This way, we can hop from one point to another, following the arrows and connect them by a smooth line (the blue line in Figure 1). This line is the Integral Curve for the ODE.

In principle, this integral curve is the one we would get by integrating the equation analytically or by a numerical method. In reality, the smoothness of the integral curve will depend upon the density of arrows and our drawing skills. Even if the curve is not smooth, we can get a gross idea of the behavior of x. That is the strength of the graphical analysis.

Starting with different initial values, we can draw different integral curves. When two integral curves are parallel to each other, they are called Isoclines. When the slope of an integral curve is zero at any point in the direction field, it is called a Nullcline.

Equation 1 is an autonomous equation. Therefore, the slope of an arrow in the direction field is independent of t. That is evident in Figure 1, as all the arrows for a particular value of x are parallel to each other. A consequence of such parallel arrows is that x can not have oscillation. To have an oscillatory behavior, for some values of x, the arrows need to change direction with time. In general, a first-order autonomous ODE cannot have oscillation.

Steady states

You must have noticed an interesting feature in Figure 1. For $x = 0$ and $x = 1$, all the arrows are horizontal, with slope zero. Therefore, at these two values, x does not change with time. These two are called the Steady States or Fixed Points of the ODE. In mathematical literature, a fixed point is often called an Equilibrium. In Biology, most of the systems are away from the thermodynamic equilibrium. To avoid any confusion with the thermodynamic aspects, we will stick to "steady state" or "fixed point" and would not use 'equilibrium.'

Steady states of an ODE are calculated by setting the derivative equal to zero and then solving the equation. For Equation 1, if $\frac{dx}{dt} = 0$, then

$$x(1-x) = 0$$

$$\therefore \quad x = 0 \ \text{ or } \ 1$$

Therefore, $x = 0$ and $x = 1$ are the two steady states for Equation 1.

Steady states have a unique dynamical property. If not disturbed, once in a steady state, a system will stay there forever. Check the integral curve in Figure 1. It starts at (0, 0.2), and with the increase in t, x increases and then reaches the steady state $x = 1$ at around $t = 5$. Subsequently, with the progress of time, it remains at that steady state.

However, what will happen if we slightly disturb or perturb a system at a steady state? Suppose following Equation 1, x has reached the steady $x = 1$. If we forcefully change the value of x to 0.9, what will happen to the system? Will x return, after some time, to the same steady state or it will diverge away from it? This question can be answered by investigating the stability of the steady state.

Equation 1 has two steady states, $x = 1$ and 0. We will now check the stability of both the steady states. Start with $x = 1$. Take a value for x, slightly smaller than 1. Say, $x = 0.9$. For $x = 0.9$, $dx/dt = 0.09$. As this value is positive, the arrow in the direction field at x = 0.9 would point upwards. That means if we change x from 1 to 0.9, with time, x will evolve towards $x = 1$, the steady state.

Take a value of x slightly higher than 1. Say, x = 1.1. For x = 1.1, dx/dt = −0.11. So, the arrow in the direction field at x = 1.1 would point downwards. If we change x from 1 to 1.1, with time, x will eventually go back to x = 1.

Therefore, any perturbation of the system near the steady state x = 1 will eventually die out, and x will return to the same steady state. That is why this steady state is called a Stable Steady State (Figure 2). To be precise, this is an Asymptotically Stable Steady State. We will explain it later.

Now, check the behavior of x near the other steady state x = 0. Take a value for x, slightly smaller than 0. Say x = −0.1. For x = −0.1, dx/dt = −0.11. So, the arrow in the direction field at x = −0.1 would point downwards. That means if we change x from 0 to −0.1, with time x would move away from this steady state.

Take a value of x slightly higher than 0. Say x = 0.1. For x = 0.1, dx/dt = 0.09. As this value is positive, the arrow in the direction field at x = 0.1 would point upward. Here also, with time, x would move away from the steady state. Therefore, if we perturb the system from the steady state x = 0, the system would diverge away from this steady state. That is why this steady state is called an Unstable Steady State (Figure 2).

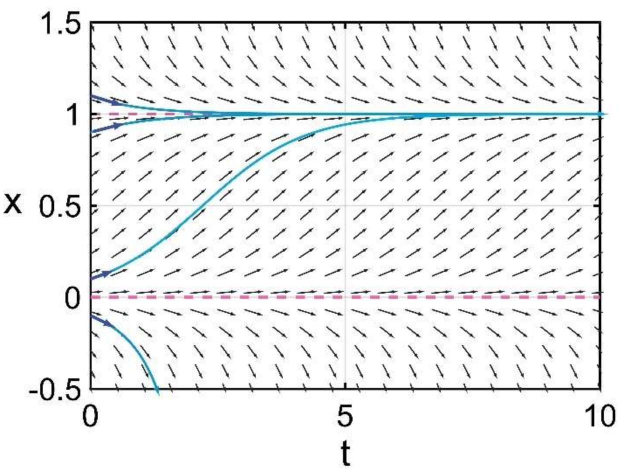


Figure 2: Different steady states of dx/dt = x(1-x). Steady states at x = 0 and 1 are shown in pink dotted lines. Trajectories starting from different initial conditions are shown in blue.

There can be another type of steady state called a semi-stable or half-stable steady state. From one side of a semi-stable steady state, the arrows in the direction fields point to the steady state, and from the other side, they move away from it.

As shown in Figure 3, the steady state of $dx/dt = (x+1)^2$ is −1. For $x < -1$, dx/dt is positive. Therefore, for $x < -1$, the arrows in the direction field point towards the steady state. dx/dt is positive also for $x > -1$, and the arrows in this region point up and away from the steady state. So, if we perturb the system from the steady state, from one side, the system will move towards the steady state, and from another side, it will diverge away from the steady state. Therefore this steady state is semi-stable.

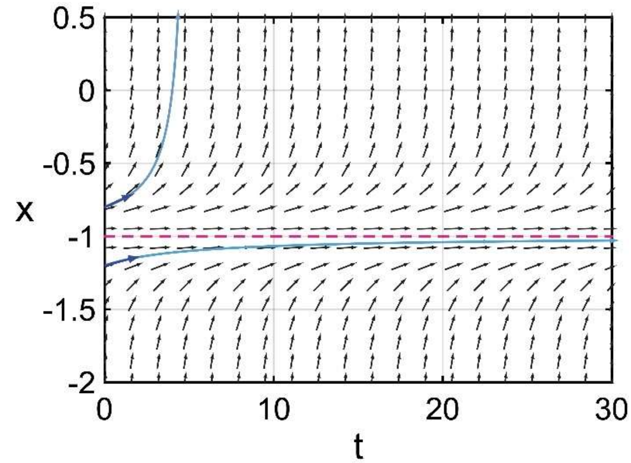


Figure 3: Semi-stable steady state. The direction field of $dx/dt = (x+1)^2$ is shown here. The steady state, $x = -1$, is shown by the pink dotted line. Trajectories starting from two initial conditions are shown in blue.

Stability analysis of steady states

Though the direction field is useful, it is tedious to draw it by hand. If you want to identify steady states and stability of steady states, there is an easier method. Identify the steady states algebraically by setting the derivative equal to zero. Then use the tabular method shown below to calculate the stability of each steady state.

In this method, we take values slightly bigger and smaller than the steady state value of the dependent variable. Then we calculate the derivatives at these two points. If the sign of the derivative is positive, place an arrow pointing up. Otherwise, use an arrow pointing down. For the steady state, as the derivative is zero, draw a horizontal arrow. As shown in the table, if both arrows, from higher and lower values, point to each other, the steady state is stable. When they point opposite to each other, the steady state is unstable. Otherwise, when they are in the same direction, the steady state is semi-stable. Repeat this calculation for all the steady states of the system.

Stable Steady State

Value of x	dx/dt at x	Arrow
$x_{ss} + \delta$	dx/dt < 0	↓
x_{ss}	dx/dt = 0	→
$x_{ss} - \delta$	dx/dt > 0	↑

Unstable Steady State

Value of x	dx/dt at x	Arrow
$x_{ss} + \delta$	dx/dt > 0	↑
x_{ss}	dx/dt = 0	→
$x_{ss} - \delta$	dx/dt < 0	↓

Semi-stable Steady State

Value of x	dx/dt at x	Arrow
$x_{ss} + \delta$	dx/dt < 0	↓
x_{ss}	dx/dt = 0	→
$x_{ss} - \delta$	dx/dt < 0	↓

Value of x	dx/dt at x	Arrow
$x_{ss} + \delta$	dx/dt > 0	↑
x_{ss}	dx/dt = 0	→
$x_{ss} - \delta$	dx/dt > 0	↑

There is another way to identify the stability of a steady state. Let's take the following ODE,

$$\frac{dx}{dt} = f(x) \tag{2}$$

Let x_{ss} is a steady state for this system. Suppose the system is at this steady state, and now we perturb it slightly to $x = x_{ss} + \eta$, such that η is very small. We can check whether this perturbation will grow or shrink with time by taking the time derivative of η. From Equation 2, at x = $x_{ss} + \eta$,

$$\frac{d\eta}{dt} = \frac{d}{dt}(x - x_{ss})$$

Since x_{ss} is a constant,

$$\frac{d\eta}{dt} = \frac{dx}{dt} = f(x)$$

$$\frac{d\eta}{dt} = f(x+\eta) \tag{3}$$

We can expand $f(x + \eta)$ by Taylor series at x_{ss} to calculate dη/dt,

$$f(x_{ss} + \eta) = f(x_{ss}) + \eta f'(x_{ss}) + \frac{\eta^2}{2} f''(x_{ss}) + \frac{\eta^3}{3} f'''(x_{ss}) + \ldots. \tag{4}$$

As η is very small, we can neglect the higher-order terms in Equation 4. Further, at the steady state $f(x_{ss}) = 0$. Therefore, we can write,

$$f(x_{ss} + \eta) = \eta f'(x_{ss})$$
$$\frac{d\eta}{dt} = \eta f'(x_{ss}) \tag{5}$$

$f'(x_{ss})$ is the derivative of $f(x)$ at x_{ss} and is constant. So, Equation 5 is a linear ODE, and its solution is,

$$\eta(t) = \eta e^{f'(x_{ss})t} \tag{6}$$

If $f'(x_{ss}) < 0$, the perturbation η will decrease exponentially with time. Therefore, the steady state x_{ss} would be stable. On the other hand, $f'(x_{ss}) > 0$, η will increase exponentially with time, and the system would diverge away from the steady state. So, for $f'(x_{ss}) > 0$, the steady state is unstable.

However, we can not decide the stability using this method, when $f'(x_{ss}) = 0$.

The same conclusions would be drawn when $x = x_{ss} - \eta$.

Pay attention to Equation 6. For a stable steady state, the perturbation will die exponentially, and the system would move towards the steady state exponentially. The exponential function is asymptotic. So the system perturbed from the steady state would reach the steady state asymptotically as $t \to \infty$. That is why this type of steady state is called an asymptotically stable steady state.

Analysis of a system of ODEs

The direction field is suitable for the analysis of a one-dimensional system. However, we can not use the same technique for a system of ODEs. Consider the following simplified model for predator-prey interaction,

$$\frac{dx}{dt} = x - xy \tag{7}$$

$$\frac{dy}{dt} = xy - y \tag{8}$$

Here, x and y are the number of prey and predator, respectively. The death rate of prey depends upon the number of predators, and the birth rate of predators depends upon the availability of prey.

Equation 7 and 8 form a two-dimensional system of ODEs. We can draw separate direction fields for x and y. However, this is a coupled system. The arrows in the direction field of x or y will depend upon the value of both x and y. These are a bit difficult to draw.

We have a better alternative, to see the time evolution of both x and y in one plot. Imagine that we have a three-dimensional plot with x, y, and t axes, and we are looking at the x-y plane only. With time both x and y will change, and we will see a trajectory on the x-y plane. In practice, we will drop the time axes and draw only the x-y plane. We will call it a phase plane plot.

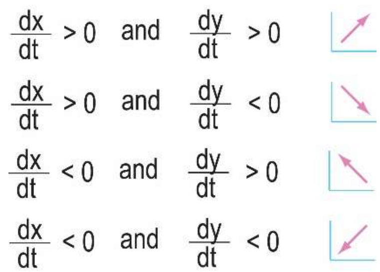
In a direction field, we place arrows equivalent to the derivative of the dependent variable. These arrows give us the trajectory of the system, starting from an initial position. We will use a similar approach to draw the phase plane plot.

Going back to our predator-prey model, let us first draw x and y axes and then place a large number of equally spaced grid points to cover the x-y plane. At each of these points, we will draw a small arrow having a slope equal to dy/dx. This arrow will give us the direction of change in x and y simultaneously.

For the given system of ODEs (Equation 7-8), the slope of an arrow at a point (x, y) would be,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{xy - y}{x - xy} \tag{9}$$

We will use the following rule to decide the direction of the arrowhead,



In this way, we can fill the *x*-*y* plane with arrows to draw the phase plane plot. For a particular set of initial values of *x* and *y*, we can draw the trajectory by joining successive arrows on phase plane, starting from the initial values.

The phase plane plot for our predator-prey model is shown in Figure 4. The arrows represent dy/dx at different positions. The blue line shows the trajectory of the system for the initial value of (x = 2, y = 1). A phase plane plot showing the steady states, arrows at regular intervals, and often a few trajectories is called a phase portrait.

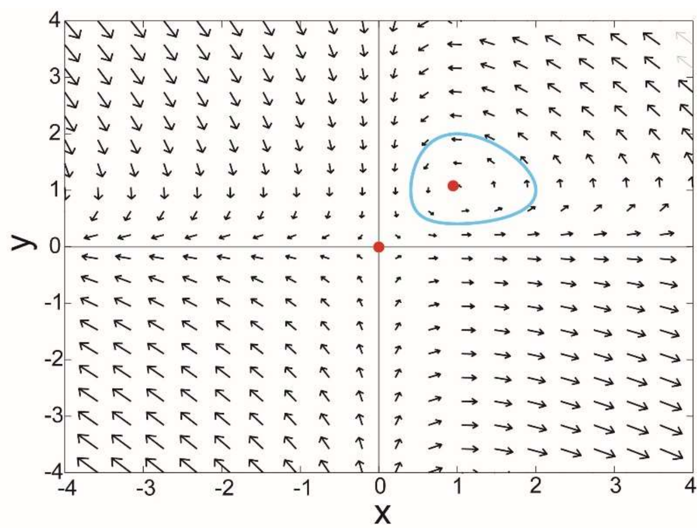


Figure 4: Phase portrait for the predator-prey model. The blue line is the trajectory for the initial condition, t = 0, x = 2, y = 1. The red dots are the steady states (x = 0, y = 0) and (x = 1, y = 1).

A phase plane plot is a vector field. A two-dimensional vector field for *x* and *y* is visualized by drawing scaled arrows at the equally spaced points on the *x*-*y* plane. Each of these arrows represents a vector,

$$\mathbf{v} = \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$$

where, $f(x,y)$ and $g(x,y)$ are functions of x and y.

Equivalently, for the predator-prey system,

$$\mathbf{v} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} x - xy \\ xy - y \end{bmatrix}$$

As a phase plane plot is equivalent to a vector field, we can use mathematical tools and concepts of a vector field for phase plane analysis. You can also imagine as if a phase portrait shows the movement of a liquid in the phase space. The arrows or vectors represent the velocity vectors of the liquid particles. This line of thought is useful in understanding different types of phase portraits.

Steady states for a system of ODEs

For a system of ODEs, a steady state is a point on the phase plane, where time- derivatives of all the dependent variables are zero. For our predator-prey model (Equations 7 and 8),

$$\begin{aligned} &\frac{dx}{dt} = 0 \\ &\frac{dy}{dt} = 0 \\ \therefore \quad & x - xy = 0 \\ & xy - y = 0 \end{aligned}$$

Solving these two equations simultaneously we get the following steady states for the system: (x = 0, y = 0) and (x = 1, y = 1). Red dots show these two positions in Figure 4.

If a system is at a steady state, it will stay there forever, unless disturbed. One of the steady states of the predator-prey model is (0, 0). If the number of predator and prey are both zero, nothing happens, and the total count of both the animals would remain 0. The other steady state is (1, 1). If we start with one prey and one predator, as per our model, their number will remain unchanged (though in reality the predator would eat the prey and after some time it would die of hunger).

The dynamics of prey and predator numbers gets interesting only when we are away from these two steady states. Assume, initially x = 2 and y = 1. Following the arrows on the phase plane, we get the trajectory of the system starting from this position (blue line in Figure 4). Note that this trajectory is a closed orbit.

A trajectory on the phase plane gives us the time evolution of dependent variables. Follow the blue line in Figure 4. Starting at (x =2, y = 1), with time, the number of prey will decrease, along with an increase in the number of the predator. That is reasonable. With an abundance of food, predators will reproduce more. However, eventually, the number of prey will drop below the requirement to feed all the predators. So after some time, the number of predators will decrease, and the number of prey will start increasing. Eventually, the trajectory will take us back to the initial position (x = 2, y = 1). Therefore, there will be a cycle of increase and decrease in numbers of prey and predator.

For any positive value of x and y, if $x, y \neq 1$, there will be a closed orbit in the phase plane around the steady state (x =1, y =1), and x and y will have oscillation. So, x = 1 and y =1 is a new type of steady state. It is called a center type steady state. We will discuss this and many other types of steady states in the next chapter.

You can explore the oscillatory behavior of the predator-prey model by numerical simulation. The JSim code for the model is given below. Do simulations with the following initial conditions: at t = 0, (x = 1, y = 1) or (x = 2, y =1) or (x = 4, y = 1). The results of these simulations are shown in Figure 5.

```
math predator_prey_model{
      realDomain t;
      t.min=0;t.delta=0.01;t.max=10;
      real x(t), y(t);
      when (t=t.min){x=2; y=1;}
      x:t = x-x*y;
      y:t = x*y-y;
}
```

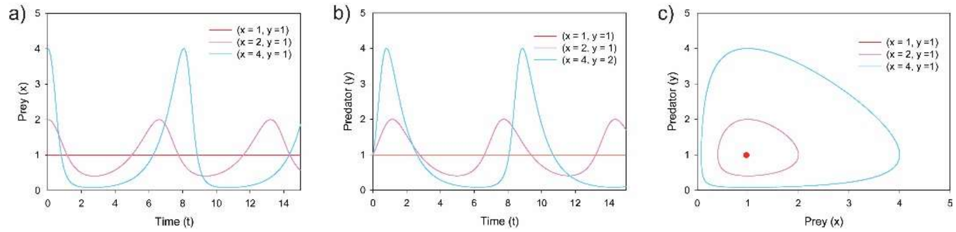


Figure 5: Simulation of the predator-prey model with three initial conditions. Time evolution of x and y are shown in a) and b). The trajectories on the phase plane are shown in c).

Nullclines and phase plane plot

Drawing a complete phase plane plot with a large number of arrows is not an easy task. While drawing the phase plane manually, one needs to be judicious and draw a sketch that broadly captures the dynamics of the system. Figure 6 shows how one can systematically perform the phase plane analysis for the predator-prey model to identify the steady states and get an idea of the possible trajectories.

The first step is to draw the nullclines. On a nullcline of a dependent variable, its derivative is equal to zero. Therefore, on x-nullcline $dx/dt = 0$. Depending upon the differential equation, a variable can have one or more nullclines.

We can identify the nullclines algebraically. For the x-nullcline of the predator-prey model:

$$\frac{dx}{dt} = x - xy = 0$$

$$x(1-y) = 0$$

$\therefore x = 0$ or $y = 1$.

Therefore, in this system, x has two nullclines, $x = 0$ and $y = 1$. We get the y-nullclines by setting $dy/dt = 0$ and those are $x = 1$ and $y = 0$. All the nullclines are also shown in Figure 6a.

Notice, that at the intersection of an x-nullcline with a y-nullcline, both dx/dt and dy/dt are equal to zero. Therefore, any such point of intersection is a steady state. In Figure 6a, there are two intersections between x- and y-nullclines. Those two are the steady states for the model, (x = 0, y =0) and (x =1, y =1).

The next step would be to draw arrows at the right places. We will place the arrows near the steady states. To begin with, draw small bars on nullclines, near steady states, without arrowheads (Figure 6b). On x-nullclines dx/dt = 0. Therefore, arrows on x-nullclines will be vertical. On the other hand, arrows on y-nullclines will be horizontal.

Now place the arrowheads on these bars. Let us take the bar placed on x-nullcline on the right-side of the steady state (1, 1). Say its position is (1.5, 1). At this point, dx/dt = 0 as it is on x-nullcline. But $dy/dt = xy - y$ = 0.5. As the sign of dy/dt is positive, the arrow at this position will point up.

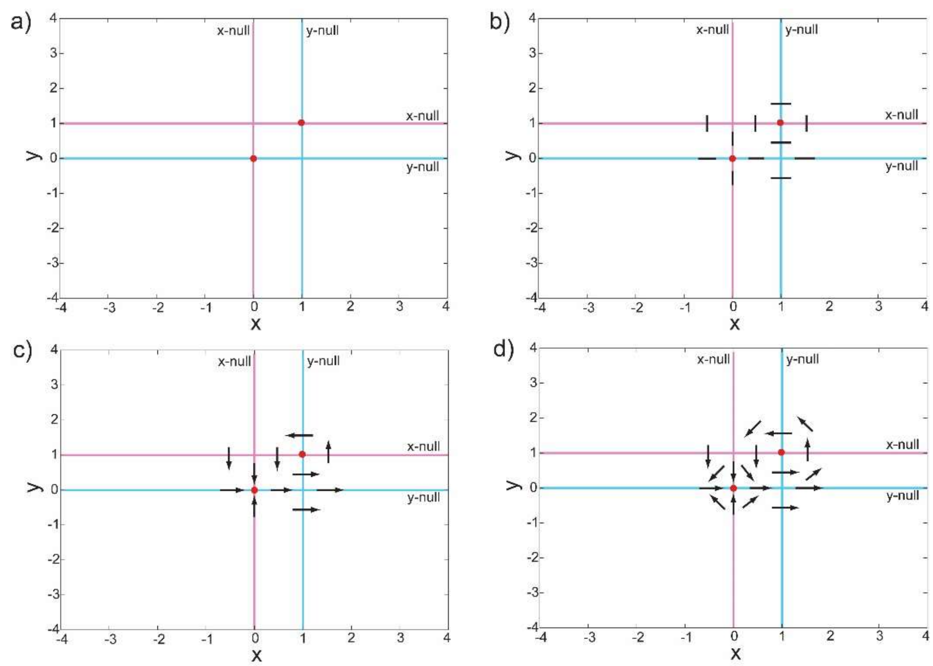


Figure 6: Steps in drawing the phase plane plot for the predator-prey model. x- and y-nullclines are shown in pink and blue, respectively. Steady states are marked in red.

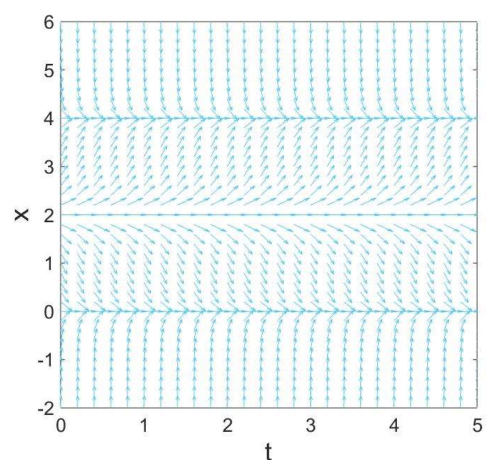
Directions for other arrows are decided similarly and marked in the plot (Figure 6c). Subsequently, few extra arrows are placed around the steady states to make the figure more comprehensive (Figure 6d). The directions of those arrows are decided by the sign of dx/dt and dy/dt at those positions. For example, one arrow is placed at (0.5, 0.5). Here, dx/dt = 0.25 and dy/dt = −0.25. Therefore, the arrow is pointing to the lower-right quadrant.

Figure 6d shows a simplified phase plane plot for the predator-prey model. One can draw this manually. Though we have very few arrows, we can identify the steady states and also get a feel of the behavior of the system near the steady states. For example, from Figure 6d, it is evident that the system possibly has a closed orbit around the steady state (1, 1). Though we may not confirm the stability of a steady state by such a rough sketch, the nullclines are very useful to identify all the possible steady states of the system. Often, you will find that drawing the nullclines to get the steady states is much easier than calculating the steady states algebraically.

Exercises

1. We are drawing a direction field for the ODE $\frac{dx}{dt} = xt$. What is the direction of the arrow or vector at $x = 1$ and $t = 1$?

2. Identify the steady states in the following direction field and comment on the stability of those steady states.



3. For $\frac{dx}{dt} = x\left(\frac{k}{1+x}\right)$, prove that the dynamics of x is not oscillatory.

4. For the following ODE, identify the steady states and comment on their stability.

$$\frac{dx}{dt} = \frac{4x}{1+x/3} - 2x$$

5. For the following system of ODEs, draw x and y nullclines on the phase plane. Identify and mark the steady states in this plot. Also mention values of x and y at the steady states.

$$\frac{dx}{dt} = (x-2)^2 - y$$
$$\frac{dy}{dt} = (y-4)$$

6. For the following system of ODEs, draw the phase plane plot with nullclines and few vectors (small arrows). Mark the steady states in the plot. What type of trajectories do you expect for this system?

$$\frac{dx}{dt} = 1 - y$$
$$\frac{dy}{dt} = x + 1$$