Chapter 4

Stability Analysis of Linear and Nonlinear systems

Analysis of a linear system

The graphical method of analysis is useful to visualize the dynamical behavior of a system. However, we have a rigorous mathematical technique to test the stability of a steady state and behavior of the system near a steady state. We will learn to analyze a linear system of ODEs first and use a similar approach for nonlinear systems.

We will analyze a linear system of ODEs using linear algebra. Consider a system of linear ODEs, for two dependent variables:

$$\frac{dx}{dt} = a_1 x + b_1 y \tag{1}$$

$$\frac{dy}{dt} = a_2 x + b_2 y \tag{2}$$

Note that this is an autonomous, homogeneous, linear system of ODEs. In matrix notation, we can rewrite this system of ODEs as,

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{3}$$

We can further represent Equation 3 in vector notation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{4}$$

$\dot{\mathbf{x}}$ and $\mathbf{x}$ are column vectors for the derivatives and the dependent variables, respectively. $\mathbf{A}$ is called the coefficient matrix, and it is a square matrix,

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

Before we proceed with the system of ODEs, it will be better to recapitulate some essential aspects of linear algebra. Suppose we have a system of linear equations,

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{5}$$

This system of equations can be solved algebraically by arranging terms,

$$x = -\frac{b_1}{a_1} y$$

$$(a_1 b_2 - a_2 b_1) y = 0 \tag{6}$$

Therefore, this system will have a unique solution, $x = 0$ and $y = 0$, iff $(a_1b_2 - a_2b_1) \neq 0$. Note that, $(a_1b_2 - a_2b_1)$ is the determinant of the coefficient matrix of Equation 5.

In general, any linear homogeneous system of equations, of the form $\mathbf{Ax} = 0$, will have a unique solution $\mathbf{x} = 0$, iff det$\mathbf{A} \neq 0$. That is called the trivial solution for the system of equations.

Equation 4 is an autonomous, homogenous, linear system of ODEs of two dependent variables. We can find its steady state by setting $\dot{\mathbf{x}} = 0$,

$$\therefore \quad \mathbf{Ax} = 0 \tag{7}$$

Equation 7 will have a unique solution $\mathbf{x} = 0$, iff det$\mathbf{A} \neq 0$. Therefore, $x = 0$, $y = 0$ is the only steady state for the system of ODEs in Equation 1-2, when the determinant of the coefficient matrix is not equal to zero. In general, when the determinant of the coefficient matrix is not equal to zero, an autonomous, homogeneous, linear system of ODE will have only one steady state at $\mathbf{x} = 0$.

We have identified the steady state of the system. Now we will analyze the stability of the steady state. In other words, we want to understand the trajectories around this steady state on the phase plane. The concepts of eigenvalue and eigenvector will be valuable in this analysis.

Eigenvalue and eigenvector

Suppose, for a square matrix **A**, there is a relation **Av** = λ**v**, where **v** is a non-zero vector and λ is a scaler. This scaler quantity λ is called the eigenvalue of the matrix **A**. The vector **v** is the corresponding eigenvector. The number of eigenvalues depends upon the dimension of the matrix, and for every eigenvalue, we will have a corresponding eigenvector. A 2-by-2 matrix will have two eigenvalues and two eigenvectors.

It is easy to calculate the eigenvalues and eigenvectors of a two-dimensional system, like the following,

$$\mathbf{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

By definition,

$$\mathbf{Av} = \lambda \mathbf{v}$$

By rearranging terms, we can write,

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0 \tag{8}$$

Here, **I** is the identity matrix.

Equation 8 is a linear homogeneous equation for **v**. Therefore, it will have the trivial solution, **v** = 0, when det(**A**-λ**I**) ≠ 0. However, the eigenvector **v** must be a non-zero vector. So, we will consider the case where det(**A**-λ**I**) = 0.

For **A**,

$$\mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{bmatrix}$$

Therefore,

$$\det(\mathbf{A}-\lambda\mathbf{I}) = (a_1-\lambda)(\mathrm{b}_2-\lambda)-a_2b_1$$

For det(**A**-λ**I**) = 0,

$$\begin{aligned}(a_1-\lambda)(\mathrm{b}_2-\lambda)-a_2b_1 &= 0\\ \lambda^2-(a_1+b_2)\lambda+(a_1b_2-a_2b_1) &= 0\\ \lambda^2-\lambda.tr\mathbf{A}+\det\mathbf{A} &= 0\end{aligned}$$

Solving this quadratic equation of λ we get the eigenvalues,

$$\lambda = \frac{tr\mathbf{A}\pm\sqrt{tr\mathbf{A}^2-4\det\mathbf{A}}}{2}$$

We will take each of the values of λ and plugin Equation 8 to calculate the corresponding eigenvector.

Let us take a numerical example and calculate the eigenvalues and eigenvector of a square matrix.

$$\mathbf{A} = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$$

For this matrix, $tr\mathbf{A} = 1 + 1 = 2$ and $\det\mathbf{A} = 1\times1-2\times2 = -3$.

Therefore,

$$\lambda = \frac{tr\mathbf{A}\pm\sqrt{tr\mathbf{A}^2-4\det\mathbf{A}}}{2} = 1\pm2$$

As this matrix has two eigenvalues, 3 and −1, there will be two corresponding eigenvectors. For λ = 3, using Equation 8,

$$
\begin{gathered}
(\mathbf{A} - \lambda \mathbf{I})\mathbf{v} = 0 \\
\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \mathbf{v} = 0 \\
\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \mathbf{v} = 0
\end{gathered} \tag{9}
$$

Let,

$$
\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
$$

From Equation 9, we get

$$
\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0
$$

Solving this linear system of equations, we get, $x_1 = x_2$. Therefore, both components of the eigenvector **v** are the same. We can consider any real number of x_1. But, for the ease of calculations we will consider, $x_1 = x_2 = 1$.

Therefore, the eigenvector for the eigenvalue $\lambda = 3$ is,

$$
\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}
$$

Similarly, the eigenvector for the other eigenvalue $\lambda = -1$ is,

$$
\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
$$

A geometric representation of eigenvectors and eigenvalue is shown in Figure 1. The eigenvector **v** is shown as the blue arrow. The dotted line shows the span of this vector. The eigenvalue λ is a scaler quantity. So, the product of λ and **v** is a vector on the span of **v**. Remember that, **Av** = λ**v**. That means, when **v** is multiplied by **A**, **v** gets stretched or squeezed or reversed along its span.

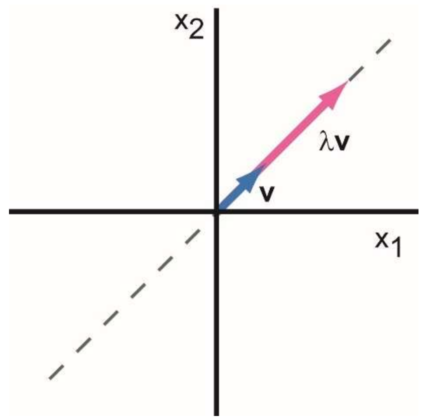

Figure 1: Graphical representation of an eigenvector. The blue arrow represents the eigenvector **v**. The span of the vector is shown by the dotted line. The red arrow is for λ**v** obtained by multiplying **v** by the eigenvalue λ.

The solution of a linear system of ODEs

Consider a linear homogenous system of ODEs, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. Let λ, and **v** are eigenvalue and eigenvector of **A**, respectively.

Assume,

$$\mathbf{x} = e^{\lambda t}\mathbf{v}$$

Differentiating both sides with respect to *t*,

$$\begin{aligned}\dot{\mathbf{x}} &= \lambda e^{\lambda t}\mathbf{v} \\ &= e^{\lambda t}\lambda\mathbf{v} = e^{\lambda t}\mathbf{A}\mathbf{v} = \mathbf{A}e^{\lambda t}\mathbf{v} = \mathbf{A}\mathbf{x}\end{aligned}$$

This shows that if λ and **v** are the eigenvalue and eigenvector of the coefficient matrix **A**, then $\mathbf{x} = e^{\lambda t}\mathbf{v}$ is a solution of the linear homogenous system of ODEs $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. This is a useful theorem.

Let us check this theorem for the following system of ODEs,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} \tag{10}$$

here, $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\dot{\mathbf{x}} = \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix}$

A has two eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = -1$. The corresponding eigenvectors are,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Therefore, this system of ODEs has two solutions,

$$\mathbf{x}_1 = e^{\lambda_1 t}\mathbf{v}_1 = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = e^{\lambda_2 t}\mathbf{v}_2 = e^{-1t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

It can be shown that if a system of linear homogeneous ODEs has n linearly independent solutions $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, then the general solution of the system is,

$$\mathbf{x} = C_1\mathbf{x}_1 + C_2\mathbf{x}_2 + \ldots + C_n\mathbf{x}_n \tag{11}$$

where, $C_1, C_2, \ldots, C_n$ are scalar constants that can be estimated from the initial condition.

For a two-dimensional system, Equation 11 can be written as,

$$\mathbf{x} = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \tag{12}$$

Therefore, the general solution for the system of ODEs in Equation 10 is,

$$\mathbf{x} = C_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{13}$$

To calculate the constants in Equation 13, we have to specify an initial condition. Consider at $t = 0$,

$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

With this initial condition, we can write Equation 13 as,

$$C_1 e^{3\times 0} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-0} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} C_1 & -C_2 \\ C_1 & C_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{14}$$

Solving Equation 14, we get $C_1 = 1.5$ and $C_2 = -0.5$.

Therefore, for this particular initial condition, the solution of the system of ODEs in Equation 10 is,

$$\mathbf{x} = 1.5 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0.5 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \tag{15}$$

Position vector and trajectory

An autonomous linear homogeneous system of ODEs, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, with $\det\mathbf{A} \neq 0$, has only one steady state at $\mathbf{x} = 0$. Equation 11 is the general solution of the system. For a particular initial condition, we can calculate the values of the scalar constants in Equation 11 and get a specific solution. That particular solution gives the trajectory of the system, from this initial position on the phase plane.

To understand how Equation 11 helps to understand the temporal dynamics of a system, let us study the linear system of Equation 10. This system has two eigenvectors,

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

These two vectors are shown in blue in Figure 2a. As these vectors are linearly independent, their linear combination can generate any vector in the x-y phase plane. Equation 12 represents a linear combination of these two vectors. For this linear combination, each eigenvector is multiplied by a scalar constant and a time-dependent scalar term. We can rewrite Equation 12 as,

$$\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2 \tag{16}$$

where, $\mathbf{u}_1 = C_1 e^{3t}\mathbf{v}_1, \quad \mathbf{u}_2 = C_2 e^{-t}\mathbf{v}_2$

For the initial condition, $x = 2$ and $y = 1$, at $t = 0$, we have shown that $C_1 = 1.5$ and $C_2 = -0.5$. Therefore,

$$\mathbf{u}_1 = C_1 e^{3\times 0}\mathbf{v}_1 = \begin{bmatrix} 1.5 \\ 1.5 \end{bmatrix}, \quad \mathbf{u}_2 = C_2 e^{-1\times 0}\mathbf{v}_2 = \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}$$

These two vectors are shown in blue in Figure 2b. Linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ generates the vector,

$$\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This vector is shown in pink in Figure 2b. Note that the tip of this vector is at ($x = 2$, $y = 1$). Therefore, vector $\mathbf{x}$, is the position vector of the system at $t = 0$.

We will find the trajectory of the system from this initial position. Consider $t = 0.1$. At this time point, the position vector will be,

$$\begin{aligned} \mathbf{x} &= \mathbf{u}_1 + \mathbf{u}_2 \\ &= C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2 \\ &= 1.5\times e^{3\times 0.1}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 0.5\times e^{-0.1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2.48 \\ 1.58 \end{bmatrix} \end{aligned}$$

$\mathbf{x}$ is shown by the pink arrow in Figure 3a. The tip of this vector is at ($x = 2.48$, $y = 1.58$), and that is the position of the system on the phase plane at $t = 0.1$. So with time, the position vector will change, and the tip of the position vector will give us the trajectory of the system starting from the initial position.

The trajectory of the system, for the given initial condition, is shown by the blue line in Figure *3*a. With time, e^{3t} increases, thereby stretching the eigenvector $\mathbf{v}_1$. On the other hand, e^{-t} decreases with time, shortening the eigenvector $\mathbf{v}_2$. With time, e^{3t} changes faster than e^{-t}, as 3t > t. That is why the trajectory is curved.

The position vector depends upon the initial condition as we estimate the values of C_1 and C_2 from the initial condition. Therefore, the trajectory of the system on the phase plane also depends upon the initial condition.

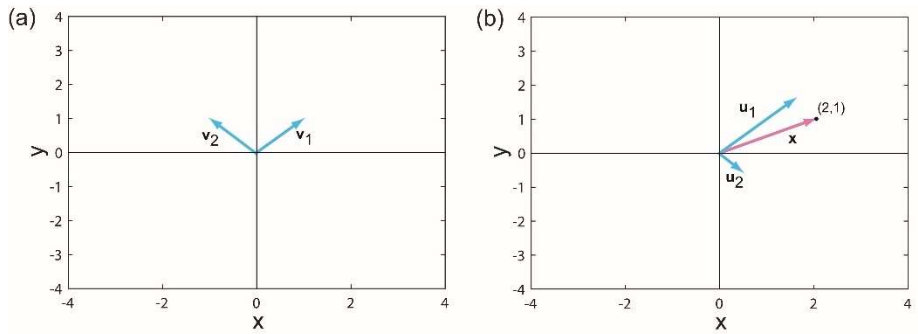

Figure 2: Visualization of the position vector. a) Two eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ of the linear system in Equation 10 are shown in blue. b) The position vector $\mathbf{x}$ at $t = 0$ is shown in pink. $\mathbf{x}$ is equal to $\mathbf{u}_1 + \mathbf{u}_2$. $\mathbf{u}_1 = 1.5\mathbf{v}_1$ and $\mathbf{u}_2 = -0.5\mathbf{v}_2$.

Let check the trajectory of the system with a few other initial conditions. Consider, at $t = 0$, $x = 1$ and $y = 1$. This initial position is on the span of the eigenvector $\mathbf{v}_1$. For this initial condition, $C_1 = 1$ and $C_2 = 0$. Using Equation 13, the position vector is,

$$\mathbf{x} = 1 \times e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \times e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore, with time the position vector will lengthen along the span of $\mathbf{v}_1$. So with time, the system will move away from the steady state (0, 0) along a straight line, with an exponentially increasing speed. That is true for any initial position on the span of $\mathbf{v}_1$. Note the pink arrows in Figure *3*b.

For the initial position ($x = -1$, $y = 1$) on the span of $\mathbf{v}_2$, C_1 and C_2 are 0 and 1, respectively. For this initial condition, the position vector is,

$$\mathbf{x} = 0 \times e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 1 \times e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

In this case, with time, the position vector will shorten along the span of $\mathbf{v}_2$. Therefore, with time the system will move towards the steady state (0, 0). Check the blue arrows in Figure *3*b.

The complete phase portrait of the system is shown in Figure 3b. In this phase portrait, there is only one straight trajectory that takes the system towards the steady state. There is another straight path that takes the system away from the steady state. All other trajectories are curved – they move close to the steady and but eventually take the system away from the steady state. A steady state with this type of phase portrait is called a Saddle.

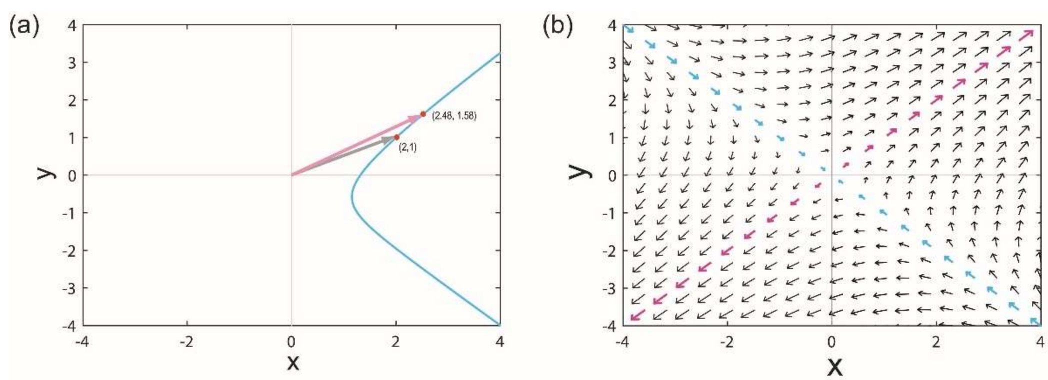

Figure 3: Position vector and trajectory. a) Position vector $\mathbf{x}$ at $t = 0$ and $t = 0.1$ are shown by grey and pink colored arrows, respectively. The blue line is the trajectory for initial condition $x = 2$ and $y = 1$ at $t = 0$. b) Complete phase portrait of the system. The steady state is at (0, 0). The pink arrows are on the span of eigenvector $\mathbf{v}_1$ and pointing away from the steady state. The blue arrows are on the span of eigenvector $\mathbf{v}_2$ and are pointing towards the steady state.

The trace-determinant plot

In general, the phase portrait of a linear homogenous system of ODEs depends upon the eigenvalues of its coefficient matrix. For a two-dimensional system, phase portraits can be categorized in specific groups based on the eigenvalues.

For a two-dimensional system, the eigenvalue of the coefficient matrix $\mathbf{A}$ is,

$$\lambda = \frac{tr\mathbf{A} \pm \sqrt{tr\mathbf{A}^2 - 4\det\mathbf{A}}}{2}$$

When, det**A** < tr**A**²/4, λ has two solutions,

$$\lambda_1 = \frac{tr\mathbf{A} + \sqrt{tr\mathbf{A}^2 - 4\det\mathbf{A}}}{2}, \quad \lambda_2 = \frac{tr\mathbf{A} - \sqrt{tr\mathbf{A}^2 - 4\det\mathbf{A}}}{2}$$

Both the eigenvalues are real numbers. When both of the eigenvalues are positives, with time, the system moves away from the steady state at (0, 0). This type of steady state is called an asymptotically unstable node or source node. A source node is possible only when, tr**A** > 0, det**A** > 0 and det**A** < tr$\mathbf{A}^2$/4 (Figure 4).

When both the eigenvalues are real negative numbers, with time, the system will move towards the steady state at (0,0). This type of steady state is called an asymptotically stable node or a sink node. A sink node is possible only when, tr**A** < 0, det**A** > 0 and det**A** < tr$\mathbf{A}^2$/4 (Figure 4).

Remember that the general solution (Equation 12) decides the trajectories of the system on the phase plane. When the eigenvalues are different, with time, $C_1e^{\lambda_1 t}$ and $C_2e^{\lambda_2 t}$ change with different rates. That gives rise to curved trajectories, as shown for the sink node and source node in Figure 4.

However, when both the eigenvalues are equal, trajectories will be straight lines, going through (0, 0). When both the eigenvalues are equal and negative, the direction of these trajectories will be towards the steady state at (0,0). This type of steady state is called a stable proper node. This will happen only when, tr**A** < 0 and det**A** = tr$\mathbf{A}^2$/4.

When eigenvalues are equal and positive, the trajectories will be straight lines coming from the steady state at (0, 0). That is called an unstable proper node. This will happen only when tr**A** > 0 and det**A** = tr$\mathbf{A}^2$/4.

When one eigenvalue is positive, and the other is negative, we get a saddle point. A saddle point is possible only when det**A** < 0 (Figure 4).

Completely different dynamics are observed when det**A** > tr$\mathbf{A}^2$/4. Under this condition, the eigenvalues are complex conjugate numbers ($\lambda = a \pm ib$). Correspondingly, the eigenvectors are also complex conjugates, $\mathbf{v} = \mathbf{p} \pm i\mathbf{q}$.

Therefore, the general solution takes the form,

$$\mathbf{x} = C_1 e^{(a+ib)t}(\mathbf{p} + i\mathbf{q}) + C_2 e^{(a-ib)t}(\mathbf{p} - i\mathbf{q}) \tag{17}$$

Using Euler's formula, Equation 17 can be written as,

$$\mathbf{x} = C_1 e^{at}[\cos(bt) + i\sin(bt)](\mathbf{p} + i\mathbf{q}) + C_2 e^{at}[\cos(bt) - i\sin(bt)](\mathbf{p} - i\mathbf{q}) \tag{18}$$

We are interested only in real solutions for our system. So we have to rearrange Equation 18 to get the real solutions.

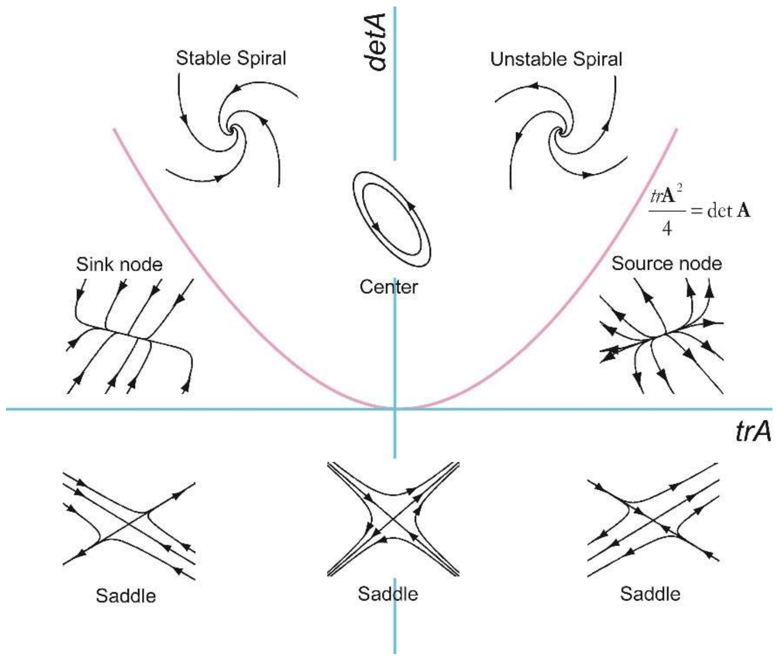

Figure 4: The trace-determinant plot. The trace and determinant of the coefficient matrix **A** of a two-dimensional linear homogeneous system of ODEs can be used to identify the stability of the steady state and the phase portrait. Different phase portraits are shown here in their respective tr**A**-det**A** positions. At the intersection of two axes, tr**A** = 0 and det**A** = 0. On the pink line tr$\mathbf{A}^2$/4 = det**A**. Above this line det**A** > tr$\mathbf{A}^2$/4.

Let,

$$\mathbf{g}(t) = e^{at}[\mathbf{p}\cos(bt) - \mathbf{q}\sin(bt)]$$
$$\mathbf{h}(t) = e^{at}[\mathbf{p}\sin(bt) + \mathbf{q}\cos(bt)]$$

Equation 18 can be written in terms of **g**(t) and **h**(t) as,

$$\mathbf{x} = C_1[\mathbf{g}(t) + i\mathbf{h}(t)] + C_2[\mathbf{g}(t) - i\mathbf{h}(t)] \tag{19}$$

To get rid of the complex number in Equation 19, assume,

$$C_1 = \frac{k_1 - ik_2}{2}, \quad C_2 = \frac{k_1 + ik_2}{2}$$

Replacing C_1 and C_2, Equation 19 can be written as,

$$\mathbf{x} = k_1\mathbf{g}(t) + k_2\mathbf{h}(t) \tag{20}$$

Like, C_1 and C_2, we can calculate the values of k_1 and k_2 from the initial condition.

When tr**A** = 0, the real part of the eigenvalues $a = 0$. That reduces Equation 20 to,

$$\mathbf{x} = [k_1\cos(bt) + k_2\sin(bt)]\mathbf{p} + [k_2\cos(bt) - k_1\sin(bt)]\mathbf{q} \tag{21}$$

Equation 21, gives rise to a closed trajectory for the system with the steady state (0, 0) at the center (Figure 4). When |**p**| = |**q**|, the trajectory would be a circle. Otherwise, it would be elliptical.

This type of phase portrait is called center-type. Here, the steady state is unique. If the system is at the steady state, it will stay there forever, if not perturbed. Once perturbed, it will move to a closed orbit around the steady state. Due to this closed orbit, we will observe periodic oscillation of the dependent variables.

When the real part of the eigenvalues, a ≠ 0, then the size of a trajectory will change with time. We can imagine as if the system is on a closed orbit, and with time it jumps to the next closest orbit. This type of movement will create a spiral. When a < 0, the trajectory will spiral into the steady state. This is called a stable spiral (Figure 4). For this type of phase portrait, the steady state is stable, and the dependent variables have damped oscillation. Initially, there will be oscillation with a higher amplitude. Eventually, with time, the oscillation will be damped, and both the variables will reach the steady state. Stable spiral can happen only when tr**A** < 0 and det**A** > tr**A**^{2}/4.

When a > 0, the trajectories will spiral out, away from the steady state. This is called an unstable spiral (Figure 4). In this case, the steady state is unstable, and the dependent variables have damped oscillation with increasing amplitude. Unstable spiral can happen only when tr**A** > 0 and det**A** > tr**A**^{2}/4.

Analysis of nonlinear systems

A linear homogeneous system of ODE with a non-zero determinant of the coefficient matrix has only one trivial steady state solution at (0, 0). However, a nonlinear system can have more than one steady states. Therefore, unlike a linear system, we cannot make a generalized statement on the stability of a nonlinear system. We have to check the stability of each of the steady states individually.

To analyze a nonlinear system, we have to identify all possible steady states. That can be done using the algebraic method of solving simultaneous equations or by drawing the nullclines on the phase plane. Subsequently, we will linearize the nonlinear system near those steady states. This linearization is based on the basic definition of the derivative and partial derivative of a function.

We will take the help of multivariate calculus for the linearization of a nonlinear system of ODEs. Consider that $f(x, y)$ is a continuous function of two variables x and y. Geometrically, this is a three-dimensional system. In a three-dimensional plot, the function $f(x, y)$ will give us a smooth surface S (the blue surface in Figure 5).

Let $z_0 = f(x_0, y_0)$. Then (x_0, y_0, z_0) is a point on S. Keeping y fixed at y_0, we can calculate the derivative of $f(x, y)$ with respect to x at x_0. That will be the partial derivative of $f(x, y)$ at that point with respect to x,

$$\frac{\partial f(x,y)}{\partial x}\Big|_{x_0} = \frac{\partial f}{\partial x} = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{(x - x_0)} \tag{22}$$

With the limit $x \to x_0$, Equation 22 can be approximated to,

$$f(x, y_0) \approx f(x_0, y_0) + (x - x_0)\frac{\partial f}{\partial x} \tag{23}$$

Let us see the geometric meaning of Equation 23. Take a curve C_x on the surface S, such that the curve goes through (x_0, y_0, z_0) and y remains constant along the curve (Figure 5). Draw a tangent T_x to this curve at (x_0, y_0, z_0). The partial derivative $\frac{\partial f}{\partial x}$ at (x_0, y_0, z_0) gives the slope of this tangent, and Equation 23 is the equation for the tangent.

Similarly, we can have a curve C_y on the surface S, such that the curve goes through (x_0, y_0, z_0) and x remains constant along the curve (Figure 5). The equation for the tangent to this curve at (x_0, y_0, z_0) will be,

$$f(x_0, y) \approx f(x_0, y_0) + (y - y_0)\frac{\partial f}{\partial y} \tag{24}$$

where, $\frac{\partial f}{\partial y}$ is the partial derivative of $f(x, y)$ with respect to y at (x_0, y_0).

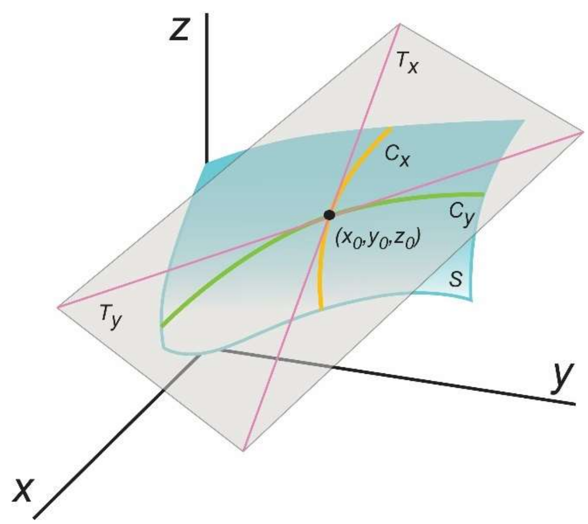

Figure 5: Geometric representation of partial derivatives of a function. The blue surface S represents a function $f(x, y)$. The grey plane goes through (x_0, y_0, z_0) and has the tangents T_x and T_y on it.

The tangents T_x and T_y are on a plane that passes through (x_0, y_0, z_0). Equation of that plane is,

$$f(x,y) = f(x_0,y_0) + (x-x_0)\frac{\partial f}{\partial x} + (y-y_0)\frac{\partial f}{\partial y} \tag{25}$$

Equation 25 is the backbone of the stability analysis of nonlinear systems.

Consider a two-dimensional system of nonlinear autonomous ODEs:

$$\frac{dx}{dt} = g(x,y) \tag{26}$$

$$\frac{dy}{dt} = h(x,y) \tag{27}$$

Let, one of the steady states for this system be (x_0, y_0). Let, $\partial g/\partial x$ and $\partial g/\partial y$ are the partial derivatives of $g(x, y)$ at (x_0, y_0). Similarly, $\partial h/\partial x$ and $\partial h/\partial y$ are the partial derivatives of $h(x, y)$ at (x_0, y_0).

Following, Equation 25, we can write,

$$g(x,y) = g(x_0,y_0) + (x-x_0)\frac{\partial g}{\partial x} + (y-y_0)\frac{\partial g}{\partial y}$$

$$h(x,y) = h(x_0,y_0) + (x-x_0)\frac{\partial h}{\partial x} + (y-y_0)\frac{\partial h}{\partial y}$$

As (x_0, y_0) is a steady state, $g(x_0, y_0) = 0$ and $h(x_0, y_0) = 0$. Therefore, we can write these two equations as,

$$g(x,y) = (x-x_0)\frac{\partial g}{\partial x} + (y-y_0)\frac{\partial g}{\partial y} \tag{28}$$

$$h(x,y) = (x-x_0)\frac{\partial h}{\partial x} + (y-y_0)\frac{\partial h}{\partial y} \tag{29}$$

in the matrix format, this system of equations is,

$$\begin{bmatrix} g(x,y) \\ h(x,y) \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \begin{bmatrix} (x-x_0) \\ (y-y_0) \end{bmatrix} \tag{30}$$

Following, Equations 26-27,

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \begin{bmatrix} (x - x_0) \\ (y - y_0) \end{bmatrix} \tag{31}$$

Let, $p = (x - x_0)$ and $q = (y - y_0)$. Therefore, $dx/dt = dp/dt$ and $dy/dt = dq/dt$. Substituting appropriate terms in Equation 31, we get

$$\begin{bmatrix} \frac{dp}{dt} \\ \frac{dq}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} \tag{32}$$

Equation 32 is a linear homogeneous system of the form $\dot{\mathbf{p}} = \mathbf{Jp}$, where $\mathbf{J}$ is the Jacobian matrix having the partial derivatives of $g(x, y)$ and $h(x, y)$ at the steady state (x_0, y_0),

$$\mathbf{J} = \begin{bmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y} \\ \frac{\partial h(x,y)}{\partial x} & \frac{\partial h(x,y)}{\partial y} \end{bmatrix}_{(x_0, y_0)}$$

As it is a linear homogenous system, the steady state solution of Equation 32 is (0, 0). Further, we can use the eigenvalue-based algebraic method to analyze the stability of the steady state and predict the phase portrait around the steady state.

Note that for the derivation of Equation 32, we have considered that (x, y) is very close to the steady state of the original nonlinear system (x_0, y_0). Therefore, the phase portrait of the original nonlinear system of Equations 26-27, near the steady state (x_0, y_0), will be given by the phase portrait of the linear system of Equation 32.

Phase portrait analysis of the predator-prey model

The predator-prey model that we introduced earlier is a nonlinear system of ODEs. We will use the Jacobian matrix-based stability analysis for this system.

$$\frac{dx}{dt} = x - xy \tag{33}$$

$$\frac{dy}{dt} = xy - y \tag{34}$$

First, identify all possible steady states. One can quickly check that this system of ODEs has two steady states – (0, 0) and (1, 1).

Now we have to construct the Jacobian matrix at each of these two steady states. In general, the Jacobian matrix for this system of ODEs is,

$$\mathbf{J} = \begin{bmatrix} \frac{\partial}{\partial x}(x - xy) & \frac{\partial}{\partial y}(x - xy) \\ \frac{\partial}{\partial x}(xy - y) & \frac{\partial}{\partial y}(xy - y) \end{bmatrix} = \begin{bmatrix} 1 - y & -x \\ y & x - 1 \end{bmatrix}$$

So, **J** at (0, 0) will be,

$$\mathbf{J} = \begin{bmatrix} 1 - y & -x \\ y & x - 1 \end{bmatrix} = \begin{bmatrix} 1 - 0 & -0 \\ 0 & 0 - 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For this matrix, tr**J** = 0 and det**J** = -1. As det**J** < 0 this steady state is a saddle (consult Figure 4).

The Jacobian matrix for the other steady (1, 1) is,

$$\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

For this matrix tr**J** = 0 and det**J** = 1. As tr**J** = 0, det**J** > tr**J**^{2}/4, this steady state would be a center type (consult Figure 4).

The complete phase portrait of this system is shown in Figure 6a. This system has two steady states with different stability and different types of trajectories around them. The steady state (1, 1) is of center type. Therefore, there are closed orbits around it. Two such closed orbits are shown in Figure 6a for initial condition (x = 2, y =1) and (x = 4 and y = 1). The oscillatory dynamics of x and y for these two initial conditions are shown in Figure 6b.

The other steady state (0, 0) is a saddle. If the system starts near to it, it may move closer for some time, but would eventually move away from this steady state.

As the system has two steady states, the overall phase portrait of the system is a combination of two phase portraits. This is quite evident in the difference in the shapes of two closed orbits shown in Figure 6a. Though the pink orbit is almost elliptical, the blue one is not. That is because the blue orbit is closer to the saddle point and affected more by the dynamics around it.

Also, you can notice the difference in the pattern of the vector field (arrows) in the positive quadrants and the rest of the quadrants. The vector field in the positive quadrant is a combination of the saddle and center type steady states. Whereas, the vector fields in the three other quadrants are typical of a saddle.

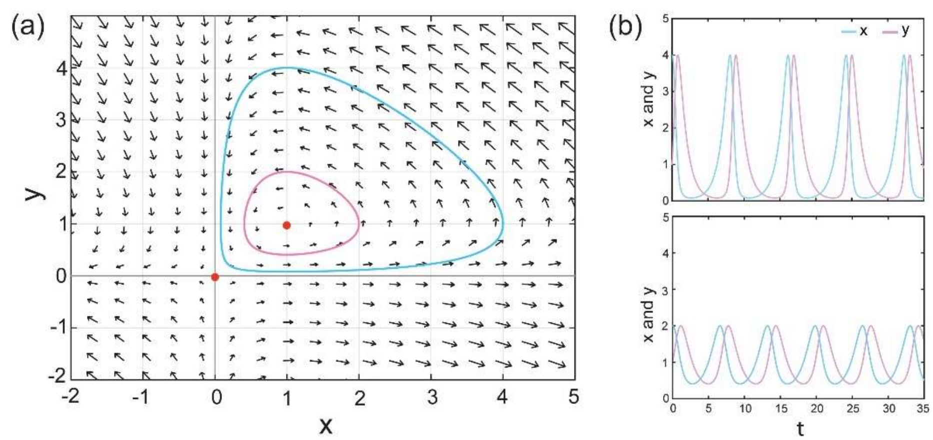

Figure 6: Phase portrait and trajectories for the Predator-prey model. a) shows the phase portrait. Two red dots represent steady states. Trajectories starting from two locations are shown by pink and blue lines. The Temporal behaviours of the dependent variables for these two trajectories are shown in (b). The upper plot is for the initial condition $x_0 = 4$, $y_0 = 1$ and the lower panel is for $x_0 = 2$, $y_0 = 1$.

Exercises

1. For the following system of ODEs, calculate the eigenvalues and corresponding eigenvectors of the coefficient matrix.

$$\frac{dx}{dt} = 2x + y \quad \frac{dy}{dt} = x + 2y$$

2. Calculate the generalized solution for the following system of ODEs.

$$\frac{dx}{dt} = -\frac{1}{2}x \quad \frac{dy}{dt} = \frac{1}{2}x - \frac{1}{4}y \quad \frac{dz}{dt} = \frac{1}{4}y - \frac{1}{6}z$$

3. We have a system made up of two types of cells X and Y. Both of the cell types are dividing and dying. X types cells also differentiate into Y type cells. The dynamics of this system in terms of size of X and Y populations is given below. Calculate the steady state of this system and comment on its stability.

$$\frac{dx}{dt} = -3x \quad \frac{dy}{dt} = 2x - 2y$$

4. Two proteins (x and y) control each other through mutual repression. The dynamical model for the system consists of the following system of ODEs. Calculate the steady state values of these two proteins. Comment on the stability of the steady state and the type of phase portrait expected for this system. Consider $x, y \geq 0$.

$$\frac{dx}{dt} = \frac{y}{1+y} - 2x \qquad \frac{dy}{dt} = \frac{x}{1+x} - y$$

5. Expression of a protein is controlled by an external signal S. The protein also controls its own expression by a negative feedback. The following system of ODEs represents the dynamics of the system, with m and P representing mRNA and protein respectively. Prove that for any value of S ≥ 0, the steady state of the system is a spiral sink.

$$\frac{dm}{dt} = \frac{S}{1+S} + \frac{1}{1+P} - m \qquad \frac{dP}{dt} = m - p$$