Kindr Library – Kinematics and Dynamics for Robotics

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Nomenclature

(Hyper-)complex number	Q	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbb{1}_{n\times m}$	$n \times m$-matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system A with basis (unit) vectors $\mathbf{e}$
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	local / body-fixed coordinate system (moves with body)
Rotation	$\Phi \in SO(3)$	generic rotation (for all parameterizations)
Machine precision	ϵ	

Operators

Cross product/skew/unskew	$\mathbf{a}\times\mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = (\mathbf{a})^\wedge \mathbf{b} = \hat{\mathbf{a}}\mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\mathbf{a} = \hat{\mathbf{a}}^\vee, \quad \hat{\mathbf{a}} = -\hat{\mathbf{a}}^\mathsf{T}, \quad \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$
Euclidean norm	$\|\mathbf{a}\| = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2 + \ldots + a_n^2}$
Exponential map for matrix	$\exp : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}, \mathbf{A} \mapsto e^{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{R}^{3\times 3}$
Logarithmic map for matrix	$\log : \mathbb{R}^{3\times 3} \to \mathbb{R}^{3\times 3}, \mathbf{A} \mapsto \log \mathbf{A}, \quad \mathbf{A} \in \mathbb{R}^{3\times 3}$

Position & Orientation

Position

Vector	$\mathbf{r}_{OP}$	from point O to point P
Position vector	${}_B\mathbf{r}_{OP} \in \mathbb{R}^3$	from point O to point P expr. in frame B
Homogeneous pos. vector	${}_B\bar{\mathbf{r}}_{OP} = \begin{bmatrix} {}_B\mathbf{r}_{OP}^\mathsf{T} & 1 \end{bmatrix}^\mathsf{T}$	from point O to point P expr. in frame B

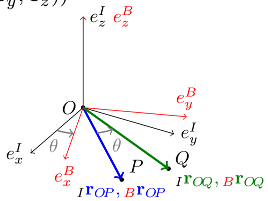
Orientation/Rotation

1) Active Rotation: $\Phi^A : {}_I\mathbf{r}_{OP} \mapsto {}_I\mathbf{r}_{OQ}$ (rotates the vector $\mathbf{r}_{OP}$)
2) Passive Rotation: $\Phi^P : {}_I\mathbf{r}_{OP} \mapsto {}_B\mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$)
3) Elementary Rotations ${}_I\mathbf{r}_{OP} = \mathbf{C}_{IB}\,{}_B\mathbf{r}_{OP}$

around z-axis: $\mathbf{C}_{IB} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

around y-axis: $\mathbf{C}_{IB} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

around x-axis: $\mathbf{C}_{IB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$



4) Inversion: $\Phi^{A^{-1}}(\mathbf{r}) = \Phi^P(\mathbf{r})$
5) Concatenation:

$$\Phi_2^A\left(\Phi_1^A(\mathbf{r})\right) = \left(\Phi_2^A \otimes \Phi_1^A\right)(\mathbf{r}) = \left(\Phi_1^{A^{-1}} \otimes \Phi_2^{A^{-1}}\right)^{-1}(\mathbf{r})$$

$$\Phi_2^P\left(\Phi_1^P(\mathbf{r})\right) = \left(\Phi_2^P \otimes \Phi_1^P\right)(\mathbf{r}) = \left(\Phi_1^{P^{-1}} \otimes \Phi_2^{P^{-1}}\right)^{-1}(\mathbf{r})$$

6) Exponential map: $\exp : \mathbb{R}^3 \to SO(3), \mathbf{v} \mapsto \exp(\hat{\mathbf{v}}), \quad \mathbf{v} \in \mathbb{R}^3$
7) Logarithmic map: $\log : SO(3) \to \mathbb{R}^3, \Phi \mapsto \log(\Phi)^\vee, \quad \Phi \in SO(3)$
8) Box plus: $\Phi_2 = \Phi_1 \boxplus \mathbf{v} = \exp(\mathbf{v}) \otimes \Phi_1, \quad \Phi_1, \Phi_2 \in SO(3), \mathbf{v} \in \mathbb{R}^3$
9) Box minus: $\mathbf{v} = \Phi_1 \boxminus \Phi_2 = \log(\Phi_1 \otimes \Phi_2^{-1}), \quad \Phi_1, \Phi_2 \in SO(3), \mathbf{v} \in \mathbb{R}^3$
10) Discrete integration: $\Phi_{IB}^{k+1} = \Phi_{IB}^k \boxplus ({}_I\boldsymbol{\omega}_{IB}^k \Delta t), \quad \Phi_{BI}^{k+1} = \Phi_{BI}^k \boxplus (-{}_B\boldsymbol{\omega}_{IB}^k \Delta t)$
11) Discrete differential: ${}_I\boldsymbol{\omega}_{IB}^k = (\Phi_{IB}^{k+1} \boxminus \Phi_{IB}^k)/\Delta t, \quad {}_B\boldsymbol{\omega}_{IB}^k = -(\Phi_{BI}^{k+1} \boxminus \Phi_{BI}^k)/\Delta t$
12) (Spherical) linear interpolation $t \in [0,1]$:

$$\Phi_t = \Phi_0 \boxplus ((\Phi_1 \boxminus \Phi_0)t), \quad \Phi_t = \Phi(t), \Phi_0 = \Phi(0), \Phi_1 = \Phi(1)$$
$$= (\Phi_1 \otimes \Phi_0^{-1})^t \otimes \Phi_0$$

Rotation Parameterizations

Rotation Matrix	$\mathbf{C}_{IB} \in SO(3)$ ${}_I\mathbf{r}_{OP} = \mathbf{C}_{IB}\,{}_B\mathbf{r}_{OP}$ $\mathbf{C}_{IB} = \mathbf{C}_{BI}^\mathsf{T}$	The rotation matrix (Direction Cosine Matrix) is a coordinate transformation matrix, which transforms vectors from frame B to frame I.
Rotation Quaternion	$\mathbf{q}_{IB} = [q_0\ q_1\ q_2\ q_3]^\mathsf{T}$	Hamiltonian unit quaternion (hypercomplex number) $Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, \quad \|Q\| = 1$
Angle-axis	$(\theta, \mathbf{n})_{IB}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta \in [0, \pi]$.
Rotation Vector	$\boldsymbol{\phi}_{IB}$	Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|}$ and angle $\theta = \|\boldsymbol{\phi}\|$.
Euler Angles ZYX Euler Angles YPR	$[z, y, x]_{IB}^\mathsf{T}$	Tait-Bryan angles (Flight conv.): $z - y' - x''$, i.e. yaw-pitch-roll. Singularities are at $y = \pm\frac{\pi}{2}$. $z \in [-\pi, \pi), y \in [-\frac{\pi}{2}, \frac{\pi}{2}), x \in [-\pi, \pi)$
Euler Angles XYZ Euler Angles RPY	$[x, y, z]_{IB}^\mathsf{T}$	Cardan angles: $x - y' - z''$, i.e. roll-pitch-yaw. Singularities are at $y = \pm\frac{\pi}{2}$. $x \in [-\pi, \pi), y \in [-\frac{\pi}{2}, \frac{\pi}{2}), z \in [-\pi, \pi)$

Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1, \quad \|Q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$$

Tuple: $Q = (q_0, q_1, q_2, q_3) = (q_0, \check{\mathbf{q}})$ with $\check{\mathbf{q}} := (q_1, q_2, q_3)^\mathsf{T}$
4×1-vector: $\mathbf{q} = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^\mathsf{T}$
Conjugate: $Q^* = (q_0, -\check{\mathbf{q}})$
Inverse: $Q^{-1} = Q^* = (q_0, -\check{\mathbf{q}})$
Quaternion multiplication:

$$Q \cdot P = (q_0, \check{\mathbf{q}}) \cdot (p_0, \check{\mathbf{p}}) = (q_0 p_0 - \check{\mathbf{q}}^\mathsf{T}\check{\mathbf{p}}, q_0\check{\mathbf{p}} + p_0\check{\mathbf{q}} + \check{\mathbf{q}} \times \check{\mathbf{p}}) \quad \Leftrightarrow$$

$$\mathbf{q} \otimes \mathbf{p} = \underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}} \mathbf{p} = \begin{pmatrix} q_0 & -\check{\mathbf{q}}^\mathsf{T} \\ \check{\mathbf{q}} & q_0 \mathbb{1}_{3\times 3} + \hat{\check{\mathbf{q}}} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}} \mathbf{q} = \begin{pmatrix} p_0 & -\check{\mathbf{p}}^\mathsf{T} \\ \check{\mathbf{p}} & p_0 \mathbb{1}_{3\times 3} - \hat{\check{\mathbf{p}}} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Note that Q_{IB} and $-Q_{IB}$ represent the same rotation, but not the same unit quaternion.

Rotation Quaternion ⇔ Rotation Vector

$$\mathbf{q}_{IB} = \begin{cases} \begin{bmatrix} \cos\left(\frac{1}{2}\|\boldsymbol{\phi}\|\right), \frac{\boldsymbol{\phi}^\mathsf{T}}{\|\boldsymbol{\phi}\|} \sin\left(\frac{1}{2}\|\boldsymbol{\phi}\|\right) \end{bmatrix}^\mathsf{T} & \text{if } \|\boldsymbol{\phi}\| \geq \epsilon \\ \begin{bmatrix} 1, \frac{1}{2}\boldsymbol{\phi}^\mathsf{T} \end{bmatrix}^\mathsf{T} & \text{otherwise} \end{cases} \quad \Leftrightarrow \quad \boldsymbol{\phi}_{IB} = \begin{cases} 2\,\mathrm{atan2}\left(\|\check{\mathbf{q}}\|, q_0\right) \frac{\check{\mathbf{q}}}{\|\check{\mathbf{q}}\|} & \text{if } \|\check{\mathbf{q}}\| \geq \epsilon \\ 2\check{\mathbf{q}} & \text{otherwise} \end{cases}$$

Rotation Quaternion ⇔ Angle-Axis

$$\mathbf{q}_{IB} = \begin{bmatrix} \cos\frac{\theta}{2} \\ \mathbf{n}\sin\frac{\theta}{2} \end{bmatrix} \quad \Leftrightarrow \quad (\theta, \mathbf{n})_{IB} = \begin{cases} \left(2\arccos(q_0), \frac{\check{\mathbf{q}}}{\|\check{\mathbf{q}}\|}\right) & \text{if } \|\check{\mathbf{q}}\| \geq \epsilon \\ \left(0, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^\mathsf{T}\right) & \text{otherwise} \end{cases}$$

Rotation Quaternion ⇔ Rotation Matrix

$$\mathbf{C}_{IB} = \mathbb{1}_{3\times 3} + 2q_0\hat{\check{\mathbf{q}}} + 2\hat{\check{\mathbf{q}}}^2 = (2q_0^2 - 1)\mathbb{1}_{3\times 3} + 2q_0\hat{\check{\mathbf{q}}} + 2\check{\mathbf{q}}\check{\mathbf{q}}^\mathsf{T}$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1q_2 - 2q_0q_3 & 2q_0q_2 + 2q_1q_3 \\ 2q_0q_3 + 2q_1q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2q_3 - 2q_0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_0q_1 + 2q_2q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$\mathbf{C}_{IB}^{-1} = \mathbf{C}_{BI} = \mathbb{1}_{3\times 3} - 2q_0\hat{\check{\mathbf{q}}} + 2\hat{\check{\mathbf{q}}}^2$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0q_3 + 2q_1q_2 & 2q_1q_3 - 2q_0q_2 \\ 2q_1q_2 - 2q_0q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_0q_1 + 2q_2q_3 \\ 2q_0q_2 + 2q_1q_3 & 2q_2q_3 - 2q_0q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

Rotation Matrix ⇔ Rotation Vector

$$\mathbf{C}_{IB} = \begin{cases} \mathbb{1} + \frac{\sin(\|\boldsymbol{\phi}\|)\hat{\boldsymbol{\phi}}}{\|\boldsymbol{\phi}\|} + \frac{(1-\cos(\|\boldsymbol{\phi}\|))\hat{\boldsymbol{\phi}}^2}{\|\boldsymbol{\phi}\|^2} & \text{if } \|\boldsymbol{\phi}\| \geq \epsilon \\ \mathbb{1} + \hat{\boldsymbol{\phi}} & \text{otherwise} \end{cases}$$

Euler Angles ZYX ⇔ Rotation Matrix

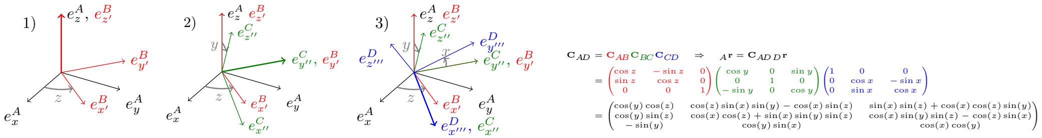


Figure 1: Rotation from A-frame to D-frame: $(z - y' - x'')$ – (yaw-pitch-roll) – $(50° - 25° - 30°)$

Euler Angles XYZ ⇔ Rotation Matrix

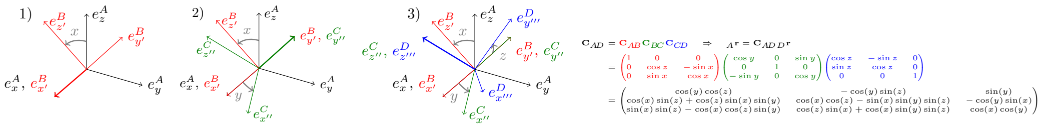


Figure 2: Rotation from A-frame to D-frame: $(x - y' - z'')$ – (roll-pitch-yaw) – $(50° - 25° - 30°)$

Pose

Homogeneous Transformation Matrix

$$\begin{bmatrix} {}_I\mathbf{r}_{IP} \\ 1 \end{bmatrix} = \mathbf{T}_{IB} \begin{bmatrix} {}_B\mathbf{r}_{BP} \\ 1 \end{bmatrix}, \quad \mathbf{T}_{IB} = \begin{bmatrix} \mathbf{C}_{IB} & {}_I\mathbf{r}_{IB} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}, \quad \mathbf{T}_{IB}^{-1} = \mathbf{T}_{BI} = \begin{bmatrix} \mathbf{C}_{IB}^\mathsf{T} & -\mathbf{C}_{IB}^\mathsf{T}\, {}_I\mathbf{r}_{IB} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

Time Derivatives of Position & Orientation

Linear Velocity

Velocity of point P expressed in a rotating frame B w.r.t. to the inertial frame I using a moving point A:
${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$
Velocity of point Q on rigid body B that rotates with ${}_B\boldsymbol{\Omega}$, where point P is on the same rigid body B:
${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$

Angular Velocity

${}_B\boldsymbol{\omega}_{IB} =: {}_B\boldsymbol{\Omega}$	(local) absolute angular velocity of rigid body B expr. in frame B
${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$	(global) angular velocity from frame B to frame I
${}_I\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^\mathsf{T}$	coord. transformation of angular velocity from frame B to frame I
${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$	composition of (relative) angular velocity

Derivatives

The derivation of the following identities can be found in [1].

$$\frac{\partial}{\partial t}\Phi_{BI}(t) = -{}_B\boldsymbol{\omega}_{IB}(t)$$
$$\frac{\partial}{\partial \mathbf{r}}(\Phi(\mathbf{r})) = \mathbf{C}(\Phi)$$
$$\frac{\partial}{\partial \Phi}(\Phi(\mathbf{r})) = -(\Phi(\mathbf{r}))^\wedge$$
$$\frac{\partial}{\partial \Phi}(\Phi^{-1}) = -\mathbf{C}(\Phi)^\mathsf{T}$$

$$\frac{\partial}{\partial \Phi_1}(\Phi_1 \otimes \Phi_2) = \mathbb{1}$$
$$\frac{\partial}{\partial \Phi_2}(\Phi_1 \otimes \Phi_2) = \mathbf{C}(\Phi_1)$$
$$\frac{\partial}{\partial \mathbf{v}}(\exp(\mathbf{v})) = \boldsymbol{\Gamma}(\mathbf{v})$$
$$\frac{\partial}{\partial \Phi}(\log(\Phi)) = \boldsymbol{\Gamma}^{-1}(\log\Phi)$$

$$\frac{\partial}{\partial \Phi_1}(\Phi_1 \boxminus \Phi_2) = \boldsymbol{\Gamma}^{-1}(\Phi_1 \boxminus \Phi_2)$$
$$\boldsymbol{\Gamma}(\mathbf{v})\mathbf{v} = \mathbf{v}$$
$$\boldsymbol{\Gamma}^{-1}(\mathbf{v})\mathbf{v} = \mathbf{v}$$
$$\boldsymbol{\Gamma}(-\mathbf{v}) = \boldsymbol{\Gamma}(\mathbf{v})^\mathsf{T}$$

Jacobian of exponential map: $\boldsymbol{\Gamma}(\mathbf{v}) = \begin{cases} \mathbb{1} + \frac{1-\cos(\|\mathbf{v}\|)\hat{\mathbf{v}}}{\|\mathbf{v}\|^2} + \frac{(\|\mathbf{v}\|-\sin(\|\mathbf{v}\|))\hat{\mathbf{v}}^2}{\|\mathbf{v}\|^3} & \text{if } \|\mathbf{v}\| \neq 0 \\ \mathbb{1} + \frac{1}{2}\hat{\mathbf{v}} & \text{otherwise} \end{cases}$

Time Derivative of Rotation Matrix ⇔ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^\mathsf{T} = \dot{\mathbf{C}}_{BI}^\mathsf{T}\mathbf{C}_{BI} \quad \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = {}_I\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}$$
$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^\mathsf{T}\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\dot{\mathbf{C}}_{BI}^\mathsf{T} \quad \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Quaternion ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} \quad \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \tfrac{1}{2}\mathbf{H}(\mathbf{q}_{IB})^\mathsf{T}\,{}_I\boldsymbol{\omega}_{IB}$$
$${}_B\boldsymbol{\omega}_{IB} = 2\bar{\mathbf{H}}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} \quad \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \tfrac{1}{2}\bar{\mathbf{H}}(\mathbf{q}_{IB})^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}$$
$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} -\check{\mathbf{q}} & \hat{\check{\mathbf{q}}} + q_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4} \qquad \bar{\mathbf{H}}(\mathbf{q}) = \begin{bmatrix} -\check{\mathbf{q}} & -\hat{\check{\mathbf{q}}} + q_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4}$$
$$= \begin{bmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \qquad = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

Time Derivative of Angle-Axis ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$
$${}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$
$$\dot{\theta} = \mathbf{n}^\mathsf{T}\,{}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 - \tfrac{1}{2}\hat{\mathbf{n}}\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$
$$\dot{\theta} = \mathbf{n}^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 + \tfrac{1}{2}\hat{\mathbf{n}}\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$

Time Derivative of Rotation Vector ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} + \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$${}_B\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} - \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} - \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} + \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

Time Derivative of Euler Angles ZYX ⇔ Angular Velocity

$$\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin(x)}{\cos(y)} & \frac{\cos(x)}{\cos(y)} \\ 0 & \cos(x) & -\sin(x) \\ 1 & \frac{\sin(x)\sin(y)}{\cos(y)} & \frac{\cos(x)\sin(y)}{\cos(y)} \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall y \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$
$$\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall y \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$
$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} -\sin(y) & 0 & 1 \\ \cos(y)\sin(x) & \cos(x) & 0 \\ \cos(x)\cos(y) & -\sin(x) & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix}$$
$${}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix}$$

Dynamics of a Multi-Rigid-Body System

$DoFs$	Degrees of Freedom
n	Number of bodies in system
n_j	Number of DoFs of the joints
n_q	Number of generalized coordinates
n_u	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{f}_Q^A$	External forces on point Q
$\mathbf{t}^A$	External torques
m	Mass
$\mathbf{\Theta}$	Inertia tensor
$(...)^-$	Variable before impact
$(...)^+$	Variable after impact
$(...)^\pm$	Variable before/after impact
Δt	Time step duration
$\Delta\mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\boldsymbol{\lambda}$	Lebesgue-measurable contact forces
$\mathbf{\Lambda}$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions
COM	Center of mass

Generalized Coordinates of a Floating-Base System with Rotational Joints

Recommended set of generalized coordinates $\mathbf{q}$ with quaternion $\mathbf{q}_{IB}$ and generalized velocities $\mathbf{u}$:

$$\mathbf{q} = \begin{pmatrix} {}_I\mathbf{r}_{IB} \\ \mathbf{q}_{IB} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot\varphi_1 \\ \vdots \\ \dot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ {}_B\boldsymbol{\psi}_{IB} \\ \ddot\varphi_1 \\ \vdots \\ \ddot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3\times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\mathsf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3x3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix}$$

$$\begin{bmatrix} {}_I\mathbf{v}_{IQ} \\ {}_I\boldsymbol{\omega}_{IQ} \end{bmatrix} = {}_I\mathbf{J}_Q(\mathbf{q})\cdot\mathbf{u}, \quad {}_I\mathbf{J}_Q(\mathbf{q}) = \begin{bmatrix} {}_I\mathbf{J}_P(\mathbf{q}) \\ {}_I\mathbf{J}_R(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{3\times 3} & -\mathbf{C}_{IB}\cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB}\cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \\ \mathbf{0}_{3\times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB}\cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix}$$

Equations of Motion with Contacts and no Impulses

Projected Newton-Euler Equations

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and}$$

$$\mathbf{M} = \sum_{i=1}^{n}\left[\left(\mathbf{J}_{\mathrm{COM}}^\mathsf{T} m \mathbf{J}_{\mathrm{COM}} + \mathbf{J}_R^\mathsf{T}\mathbf{\Theta}_{\mathrm{COM}}\mathbf{J}_R\right)\right]_i$$

$$\mathbf{g} = \sum_{i=1}^{n}\left[\left(\mathbf{J}_{\mathrm{COM}}^\mathsf{T} m \dot{\mathbf{J}}_{\mathrm{COM}}\mathbf{u} + \mathbf{J}_R^\mathsf{T}(\mathbf{\Theta}_{\mathrm{COM}}\dot{\mathbf{J}}_R\mathbf{u} + \mathbf{\Omega}\times\mathbf{\Theta}_{\mathrm{COM}}\mathbf{\Omega})\right)\right]_i$$

$$\mathbf{f} = \sum_{i=1}^{n}\left[\left(\mathbf{J}_Q^\mathsf{T}\mathbf{f}_Q^A + \mathbf{J}_R^\mathsf{T}\mathbf{t}^A\right)\right]_i$$

Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta\mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{aligned} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) \quad &= \mathbf{W}\mathbf{\Lambda} \\ \mathbf{M}\underbrace{(\dot{\mathbf{u}}\mathrm{d}t + (\mathbf{u}^+ - \mathbf{u}^-)\mathrm{d}\eta)}_{\mathrm{d}\mathbf{u}} - \mathbf{h}\mathrm{d}t \quad &= \mathbf{W}\underbrace{(\boldsymbol{\lambda}\mathrm{d}t + \mathbf{\Lambda}\mathrm{d}\eta)}_{\mathrm{d}\mathbf{P}} \end{aligned} \right.$$

Transformation of Equations of Motion

Transformation from $\bar{\mathbf{M}}(\bar{\mathbf{q}}), \bar{\mathbf{h}}(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ to $\mathbf{M}(\mathbf{q}), \mathbf{h}(\mathbf{q}, \mathbf{u})$, where $\bar{\mathbf{u}} = \mathbf{B}\mathbf{u}$:

$$\mathbf{M} = \mathbf{B}^\mathsf{T}\bar{\mathbf{M}}\mathbf{B}$$

$$\mathbf{h} = \mathbf{B}^\mathsf{T}\bar{\mathbf{h}} - \mathbf{B}^\mathsf{T}\bar{\mathbf{M}}\dot{\mathbf{B}}\mathbf{u}$$

Appendix I: Euler Angles ZYX Velocities to Angular Velocity Mapping

Given a set of Euler angles $\boldsymbol{\chi} = \begin{bmatrix} z & y & x \end{bmatrix}^T$ and velocities $\dot{\boldsymbol{\chi}} = \begin{bmatrix} \dot{z} & \dot{y} & \dot{x} \end{bmatrix}^T$, we wish to find the mapping $\mathbf{E}(\boldsymbol{\chi}) \in \mathbb{R}^{3\times 3}$ that maps $\dot{\boldsymbol{\chi}}$ to ${}_I\boldsymbol{\omega}_{IB}$:

$$ {}_I\boldsymbol{\omega}_{IB} = \mathbf{E}(\boldsymbol{\chi}) \cdot \dot{\boldsymbol{\chi}} \tag{1} $$

The columns of $\mathbf{E}(\boldsymbol{\chi})$ are the components of the unit vectors around which the rotational velocities are applied expressed in fixed frame. These are obtained by selecting the columns of a rotation matrix which is built up by successive elementary rotations specified by the Euler angle parametrization.

Starting from the reference frame I, the first rotation will be an elementary rotation around ${}_I\mathbf{e}_I^z$, which is simply given by:

$$ {}_I\mathbf{e}_I^z = \mathbb{1}_{3\times 3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{2} $$

After an elementary rotation around ${}_I\mathbf{e}_I^z$, the y axis ${}_I\mathbf{e}_{I'}^y$, will be expressed by:

$$ {}_I\mathbf{e}_{I'}^y = \mathbf{C}_{II'}(z) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(z) \\ \cos(z) \\ 0 \end{bmatrix} \tag{3} $$

After an elementary rotation around ${}_I\mathbf{e}_{I'}^y$, the x axis ${}_I\mathbf{e}_{I''}^x$ will be expressed by:

$$ {}_I\mathbf{e}_{I''}^x = \mathbf{C}_{II'}(z) \cdot \mathbf{C}_{I'I''}(y) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(y) & 0 & \sin(y) \\ 0 & 1 & 0 \\ -\sin(y) & 0 & \cos(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(y)\cos(z) \\ \cos(y)\sin(z) \\ -\sin(z) \end{bmatrix} \tag{4} $$

Finally, the mapping $\mathbf{E}(\boldsymbol{\chi})$ will be computed as:

$$ \mathbf{E}(\boldsymbol{\chi}) = \begin{bmatrix} {}_I\mathbf{e}_I^z & {}_I\mathbf{e}_{I'}^y & {}_I\mathbf{e}_{I''}^x \end{bmatrix} = \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix} \tag{5} $$

It is easy to find that $det(\mathbf{E}(\boldsymbol{\chi})) = -\cos(y)$. The mapping then becomes singular when $y = \pi/2 + k\pi, \forall k \in \mathbb{Z}$. This means that although we can always describe an angular velocity using Euler angle velocities, the inverse is not always possible. The inverse mapping is given by:

$$ \bar{\mathbf{E}}(\boldsymbol{\chi}) = \mathbf{E}^{-1}(\boldsymbol{\chi}) = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} \tag{6} $$

If we compute the rotation matrix $\mathbf{C}_{IB} = \mathbf{C}_{IB}(z, y, x) = \mathbf{C}_z(z) \cdot \mathbf{C}_y(y) \cdot \mathbf{C}_x(x)$, we can also derive the following mappings:

$$ {}_B\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}^T \cdot \mathbf{E}(\boldsymbol{\chi}) \cdot \dot{\boldsymbol{\chi}} \tag{7} $$

$$ \dot{\boldsymbol{\chi}} = \bar{\mathbf{E}}(\boldsymbol{\chi}) \cdot \mathbf{C}_{IB} \cdot {}_B\boldsymbol{\omega}_{IB} \tag{8} $$

The mapping described by (8) is valid $\forall y \neq \pi/2 + k\pi, \forall k \in \mathbb{Z}$.

Appendix II: Jacobians

We wish to derive the relationship between the generalized velocities $\mathbf{u}$ and the operational space velocities ${}_I\mathbf{v}_Q$ of a point Q, which is fixed at the end of a kinematic chain that stems from a floating base B. The position vector ${}_I\mathbf{r}_{IQ} = {}_I\mathbf{r}_{IQ}(\mathbf{q})$ of a point w.r.t. the inertial frame I is given by:

$$ {}_I\mathbf{r}_{IQ}(\mathbf{q}) = {}_I\mathbf{r}_{IB}(\mathbf{q}) + \mathbf{C}_{IB}(\mathbf{q}) \cdot {}_B\mathbf{r}_{BQ}(\mathbf{q}), \tag{9} $$

where the rotation matrix $\mathbf{C}_{IB}(\mathbf{q})$ describes the orientation of the floating base B w.r.t. the inertial frame I, ${}_I\mathbf{r}_{IB}(\mathbf{q})$ represents the position of the floating base B w.r.t. the inertial frame I expressed in the inertial frame and $\mathbf{q} = \mathbf{q}(t)$ is a function of time t.
Time differentiation yields:

$$ \begin{aligned} {}_I\mathbf{v}_Q &= {}_I\mathbf{v}_B + \dot{\mathbf{C}}_{IB} \cdot {}_B\mathbf{r}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B + \mathbf{C}_{IB} \cdot {}_B\hat{\boldsymbol{\omega}}_{IB} \cdot {}_B\mathbf{r}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B - \mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} \cdot {}_B\boldsymbol{\omega}_{IB} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B - \mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} \cdot {}_B\boldsymbol{\omega}_{IB} + \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \cdot \dot{\mathbf{q}}_j \\ &= \begin{bmatrix} \mathbb{1}_{3\times 3} & -\mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \end{bmatrix} \cdot \mathbf{u} \end{aligned} \tag{10} $$

If we attach a frame at ${}_I\mathbf{r}_{IQ}$, we can derive a similar mapping for angular velocities. The orientation of frame Q w.r.t. the inertial frame I is described by:

$$ \mathbf{C}_{IQ} = \mathbf{C}_{IB} \cdot \mathbf{C}_{BQ} \tag{11} $$

Time differentiation of both sides of (11) yields:

$$ \begin{aligned} {}_I\hat{\boldsymbol{\omega}}_{IQ} \cdot \mathbf{C}_{IQ} &= {}_I\hat{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IB} \cdot \mathbf{C}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\hat{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{BQ} \\ &= {}_I\hat{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IQ} + \mathbf{C}_{IB} \cdot \mathbf{C}_{BI} \cdot {}_I\hat{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{BI}^{\mathsf{T}} \cdot \mathbf{C}_{BQ} \\ &= {}_I\hat{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IQ} + {}_I\hat{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{IQ}, \end{aligned} \tag{12} $$

which gives finally:

$$ \begin{aligned} {}_I\boldsymbol{\omega}_{IQ} &= {}_I\boldsymbol{\omega}_{IB} + {}_I\boldsymbol{\omega}_{BQ} \\ &= \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix} \cdot \mathbf{u} \end{aligned} \tag{13} $$

Hence, the mapping from generalized velocities $\mathbf{u}$ to the operational space twist $\begin{bmatrix} {}_I\mathbf{v}_Q^T & {}_I\boldsymbol{\omega}_{IQ}^T \end{bmatrix}^T$ of frame Q is realized by the spatial Jacobian:

$$ \begin{aligned} {}_I\mathbf{J}_Q(\mathbf{q}) &= \begin{bmatrix} {}_I\mathbf{J}_P \\ {}_I\mathbf{J}_R \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{1}_{3\times 3} & -\mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \\ \mathbf{0}_{3\times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix} \end{aligned} \tag{14} $$

Appendix III: Hessians and Time Derivatives of Jacobians

Consider a kinematic chain which connects two rigid bodies. We represent the set of indexes of the rigid bodies in this chain with U_A. As shown in [2], the $i-th$ column of the spatial Hessian matrix of the spatial Jacobian J w.r.t. the j-th configuration variable q_j can be computed as:

$$ \frac{\partial \mathbf{J}_i}{\partial q_j} = \begin{bmatrix} \frac{\partial \mathbf{J}_{P_i}}{\partial q_j} \\ \frac{\partial \mathbf{J}_{R_i}}{\partial q_j} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{J}_{R_j} \times \mathbf{J}_{P_i} \\ \mathbf{J}_{R_j} \times \mathbf{J}_{R_i} \end{bmatrix} & i \geq j \\ \begin{bmatrix} \mathbf{J}_{R_i} \times \mathbf{J}_{P_j} \\ \mathbf{0} \end{bmatrix} & i < j \\ \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} & i, j \notin U_A \end{cases} \tag{15} $$

The Hessian $\mathbf{H}_k(\mathbf{q})$ is then expressed by:

$$\mathbf{H}_k(\mathbf{q}) = \begin{bmatrix} \frac{\partial \mathbf{J}_1}{\partial q_k} & \frac{\partial \mathbf{J}_2}{\partial q_k} & \dots & \frac{\partial \mathbf{J}_n}{\partial q_k}. \end{bmatrix} \tag{16}$$

Knowledge of the Hessian matrix w.r.t. each configuration variable q_k enables the computation of the time derivative of $\mathbf{J}(\mathbf{q}(t))$. Its generic element i, j can be computed as:

$$\frac{dJ_{i,j}(\mathbf{q})}{dt} = \frac{\partial J_{i,j}(\mathbf{q})}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \sum_{k=1}^{n} \frac{\partial J_{i,j}(\mathbf{q})}{\partial q_k} \cdot \frac{dq_k}{dt} = \sum_{k=1}^{n} H_{k_{i,j}}(\mathbf{q}) \cdot \frac{dq_k}{dt}, \tag{17}$$

which yields:

$$\frac{d\mathbf{J}(\mathbf{q})}{dt} = \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt}, = \sum_{k=1}^{n} \mathbf{H}_k(\mathbf{q}) \cdot \frac{dq_k}{dt}. \tag{18}$$

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[2] M. Iwamura and M. Nagao, "A method for computing the Hessian tensor of loop closing conditions in multibody systems," *Multibody System Dynamics*, vol. 30, no. 2, pp. 173–184, 2013.