

# Kindr Cheat Sheet

## Kinematics and Dynamics for Robotics

### Nomenclature

(Hyper-)complex number	$Q$	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbf{1}_{n \times m}$	$n \times m$ -matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system $A$ with basis (unit) vectors $\mathbf{e}$
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	Global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	Local / body-fixed coordinate system (moves with body)
Machine precision	$\epsilon$	

### Operators

Cross product	$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge \mathbf{b} = \hat{\mathbf{a}} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
Euclidean norm	$\ \mathbf{a}\  = \sqrt{\mathbf{a}^T \mathbf{a}} = \sqrt{a_1^2 + \dots + a_n^2}$
Quaternion multiplicator	$\mathbf{q} \otimes \mathbf{p} \Leftrightarrow Q \cdot P$

### Position & Orientation

#### Position

Vector	$\mathbf{r}_{OP}$	from point $O$ to point $P$
Position vector	${}^A \mathbf{r}_{OP} \in \mathbb{R}^3$	from point $O$ to point $P$ expr. in frame $A$
Homogeneous pos. vector	${}^A \bar{\mathbf{r}}_{OP} = [{}^A \mathbf{r}_{OP}^T \quad 1]^T$	from point $O$ to point $P$ expr. in frame $A$

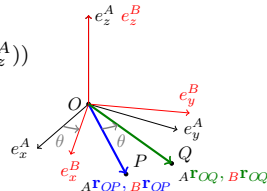
#### Orientation/Rotation

- 1) Active Rotation:  $\mathcal{R}^A : {}^A \mathbf{r}_{OP} \mapsto {}^A \mathbf{r}_{OQ}$  (rotates the vector  $\mathbf{r}_{OP}$ )
- 2) Passive Rotation:  $\mathcal{R}^P : {}^A \mathbf{r}_{OP} \mapsto {}^B \mathbf{r}_{OP}$  (rotates the frame  $(\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A)$ )
- 3) Inversion:  $\mathcal{R}^{A^{-1}}(\mathbf{r}) = \mathcal{R}^P(\mathbf{r})$   

$$\mathcal{R}_2^A(\mathcal{R}_1^A(\mathbf{r})) = (\mathcal{R}_2^A \otimes \mathcal{R}_1^A)(\mathbf{r})$$

$$= (\mathcal{R}_1^{A^{-1}} \otimes \mathcal{R}_2^{A^{-1}})(\mathbf{r})$$
- 4) Concatenation:  $\mathcal{R}_2^P(\mathcal{R}_1^P(\mathbf{r})) = (\mathcal{R}_2^P \otimes \mathcal{R}_1^P)(\mathbf{r})$   

$$= (\mathcal{R}_1^{P^{-1}} \otimes \mathcal{R}_2^{P^{-1}})(\mathbf{r})$$



Rotation Matrix	$\mathbf{R}_{BA} \in \text{SO}(3)$ ${}^A \mathbf{r}_{OQ} = \mathbf{R}_{BA} {}^A \mathbf{r}_{OP}$ ${}^B \mathbf{r}_{OP} = \mathbf{R}_{BA}^T {}^A \mathbf{r}_{OP}$	Maps the coord. of the basis vectors $({}^A \mathbf{e}_x^A, {}^A \mathbf{e}_y^A, {}^A \mathbf{e}_z^A)$ of frame $A$ expressed in $A$ into the coordinates of the basis vectors $({}^A \mathbf{e}_x^B, {}^A \mathbf{e}_y^B, {}^A \mathbf{e}_z^B)$ of $B$ expressed in $A$ . The rotation is active (alibi). ${}^A \mathbf{R}_{BA} = [{}^A \mathbf{e}_x^B \quad {}^A \mathbf{e}_y^B \quad {}^A \mathbf{e}_z^B]$
Direct Cosine Matrix	$\mathbf{C}_{BA} \in \text{SO}(3)$ ${}^B \mathbf{r}_{OP} = \mathbf{C}_{BA} {}^A \mathbf{r}_{OP}$ $\mathbf{C}_{BA} = \mathbf{R}_{BA}^T$	The coordinate transformation matrix, which transforms vectors from frame $A$ to frame $B$ . The rotation is passive (alias).
Rotation Quaternion	$\mathbf{p}_{BA}$ $\mathbf{p}_{BA} \Leftrightarrow \mathbf{R}_{BA}$	The rotation is active (alibi).
Rotation Angle-axis	$(\theta, \mathbf{n})_{BA}$ $(\theta, \mathbf{n})_{BA} \Leftrightarrow \mathbf{R}_{BA}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta \in [0, \pi]$ . The rotation is active (alibi).
Rotation Vector	$\phi_{BA}$ $\phi_{BA} \Leftrightarrow \mathbf{R}_{BA}$	Rotation with rotation axis $\mathbf{n} = \frac{\phi}{\ \phi\ }$ and angle $\theta = \ \phi\ $ . The rotation is active (alibi).
Euler Angles ZYX Euler Angles YPR	$(\psi, \phi, \theta)_{BA}$ $(\psi, \theta, \phi)_{BA} \Leftrightarrow \mathbf{C}_{BA}$	Tait-Bryan angles (Flight conv.): $z - y' - x''$ , i.e. yaw-pitch-roll. Singularities are at $\theta = \pm \frac{\pi}{2}$ . $\psi \in [-\pi, \pi], \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \phi \in [-\pi, \pi]$
Euler Angles XYZ Euler Angles RPY	$(\alpha, \beta, \gamma)_{BA}$ $(\alpha, \beta, \gamma)_{BA} \Leftrightarrow \mathbf{C}_{BA}$	Cardan angles (Glocker conv.): $x - y' - z''$ , i.e. roll-pitch-yaw. Singularities are at $\beta = \pm \frac{\pi}{2}$ . $\alpha \in [-\pi, \pi], \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \gamma \in [-\pi, \pi]$

#### Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$P = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}, \quad p_i \in \mathbb{R}$$

$$i^2 = j^2 = k^2 = ijk = -1, \quad \|P\| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2} = 1$$

Note that  $P_{BA}$  and  $-P_{BA}$  represent the same rotation, but not the same unit quaternion.

Rot. quaternion as tuple:  $P = (p_0, p_1, p_2, p_3) = (p_0, \vec{\mathbf{p}})$  with  $\vec{\mathbf{p}} := (p_1, p_2, p_3)^T$

Rot. quaternion as vector:  $\mathbf{p} = [p_0 \quad p_1 \quad p_2 \quad p_3]^T$

Conjugate:  $P^* = (p_0, -\vec{\mathbf{p}})$

Inverse:  $P^{-1} = P^* = (p_0, -\vec{\mathbf{p}})$

Quaternion multiplication:

$$Q \cdot P = (q_0, \vec{\mathbf{q}}) \cdot (p_0, \vec{\mathbf{p}}) = (q_0 p_0 - \vec{\mathbf{q}}^T \vec{\mathbf{p}}, q_0 \vec{\mathbf{p}} + p_0 \vec{\mathbf{q}} + \vec{\mathbf{q}} \times \vec{\mathbf{p}}) \Leftrightarrow$$

$$\mathbf{q} \otimes \mathbf{p} = \mathbf{Q}(\mathbf{q})\mathbf{p} = \begin{pmatrix} q_0 & -\vec{\mathbf{q}}^T \\ \vec{\mathbf{q}} & q_0 \mathbf{1}_{3 \times 3} + \hat{\vec{\mathbf{q}}} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathbf{Q}(\mathbf{p})^T \mathbf{q}$$

#### Rotation Quaternion $\Leftrightarrow$ Rotation Angle-Axis

$$\mathbf{p}_{BI} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{n} \sin \frac{\theta}{2} \end{bmatrix} \Leftrightarrow (\theta, \mathbf{n})_{BI} = \begin{cases} (2 \arccos(p_0), \frac{\vec{\mathbf{p}}}{\|\vec{\mathbf{p}}\|}) & \text{if } \|\vec{\mathbf{p}}\|^2 \geq \epsilon^2 \\ (0, [1 \quad 0 \quad 0]^T) & \text{otherwise} \end{cases}$$

#### Rotation Quaternion $\Leftrightarrow$ Direction Cosine Matrix

$$\mathbf{C}_{AB} = \mathbf{R}_{AB}^T(\mathbf{p}_{AB}) = \mathbf{1}_{3 \times 3} + 2p_0 \hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2$$

$$= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_1 p_2 - 2p_0 p_3 & 2p_0 p_2 + 2p_1 p_3 \\ 2p_0 p_3 + 2p_1 p_2 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_2 p_3 - 2p_0 p_1 \\ 2p_1 p_3 - 2p_0 p_2 & 2p_0 p_1 + 2p_2 p_3 & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}$$

$$\mathbf{C}_{BA} = \mathbf{R}_{BA}^T = \mathbf{R}_{AB}(\mathbf{p}_{AB}) = \mathbf{1}_{3 \times 3} - 2p_0 \hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2$$

$$= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_0 p_3 + 2p_1 p_2 & 2p_1 p_3 - 2p_0 p_2 \\ 2p_1 p_2 - 2p_0 p_3 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_0 p_1 + 2p_2 p_3 \\ 2p_0 p_2 + 2p_1 p_3 & 2p_2 p_3 - 2p_0 p_1 & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}$$

$$\mathbf{p}_{BA} = \mathbf{p}_{BA}(\mathbf{C}_{AB}) = \begin{bmatrix} \frac{1}{2}\sqrt{1+\text{tr}(\mathbf{C})} \\ \frac{C_{32}-C_{23}}{4p_0} \\ \frac{C_{13}-C_{31}}{4p_0} \\ \frac{C_{21}-C_{12}}{4p_0} \end{bmatrix} \quad \text{if } \text{tr}(\mathbf{C}) > 0 \quad (\mathbf{C}_{AB} \rightarrow \mathbf{p}_{BA} \text{ is not unique})$$

### Euler Angles ZYX $\Leftrightarrow$ Direction Cosine Matrix

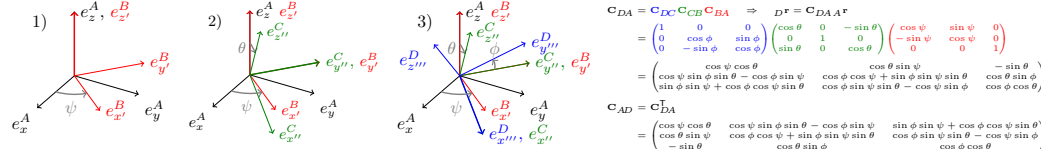


Figure 1: Rotation from  $I$ -frame to  $B$ -frame:  $(z - y' - x'') - (\text{yaw-pitch-roll}) - (\psi - \theta - \phi) - (50^\circ - 25^\circ - 30^\circ)$

### Euler Angles XYZ $\Leftrightarrow$ Direction Cosine Matrix

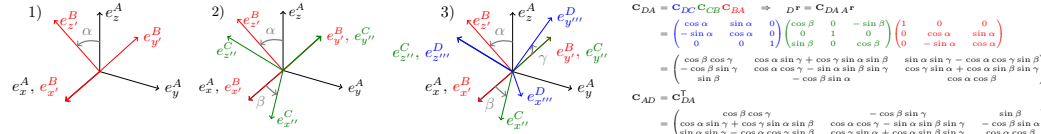


Figure 2: Rotation from  $I$ -frame to  $B$ -frame:  $(x - y' - z'') - (\text{roll-pitch-yaw}) - (\alpha - \beta - \gamma) - (50^\circ - 25^\circ - 30^\circ)$

### Pose

Homogeneous Transformation Matrix	$\mathbf{T}_{AB}$	
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### Homogeneous Transformation Matrix

$$\mathbf{T}_{AB} = \begin{bmatrix} \mathbf{C}_{AB} & A^{\mathbf{r}}_{AB} \\ \mathbf{0}^{\mathbf{T}} & 1 \end{bmatrix}$$

## Time Derivatives of Position & Orientation

### Linear Velocity

Velocity of point  $P$  expressed in frame  $B$  w.r.t. to the inertial frame  $I$ :

$$\mathbf{B}\mathbf{v}_P = \mathbf{B}\mathbf{v}_A + \mathbf{B}\dot{\mathbf{r}}_{AP} + \mathbf{B}\boldsymbol{\omega}_{IB} \times \mathbf{B}\mathbf{r}_{AP}$$

Velocity of point  $Q$  on rigid body  $B$  that rotates with  ${}_B\boldsymbol{\Omega}$ , where point  $P$  is on the same rigid body  $B$ :

$$\mathbf{B}\mathbf{v}_Q = \mathbf{B}\mathbf{v}_P + \mathbf{B}\boldsymbol{\Omega} \times \mathbf{B}\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = \mathbf{B}\boldsymbol{\omega}_{IB}$$

### Angular Velocity

${}_B\boldsymbol{\Omega} = \mathbf{B}\boldsymbol{\omega}_{IB}$	absolute angular velocity of rigid body $B$ expressed in frame $B$
$\mathbf{B}\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
$\mathbf{I}\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\mathbf{B}\boldsymbol{\omega}_{IB}$	coord. transformation of angular velocity from frame $B$ to frame $I$
$\mathbf{I}\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\mathbf{B}\dot{\boldsymbol{\omega}}_{IB} + \mathbf{C}_{IB}^{\mathbf{T}}{}_B\boldsymbol{\omega}_{IB}$	coord. transformation of angular velocity from frame $B$ to frame $I$
$\mathbf{D}\boldsymbol{\omega}_{AD} = \mathbf{D}\boldsymbol{\omega}_{AB} + \mathbf{D}\boldsymbol{\omega}_{BC} + \mathbf{D}\boldsymbol{\omega}_{CD}$	composition of angular velocity

### Time Derivative of Direction Cosine Matrix $\Leftrightarrow$ Angular Velocity

$$\mathbf{I}\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^{\mathbf{T}} = \dot{\mathbf{C}}_{BI}^{\mathbf{T}}\mathbf{C}_{BI} \\ \mathbf{B}\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^{\mathbf{T}}\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\dot{\mathbf{C}}_{BI}^{\mathbf{T}} \quad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\mathbf{B}\dot{\boldsymbol{\omega}}_{IB}$$

### Time Derivative of Rotation Matrix $\Leftrightarrow$ Angular Velocity

$$\mathbf{I}\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{BI}\mathbf{R}_{BI}^{\mathbf{T}} = \dot{\mathbf{R}}_{BI}^{\mathbf{T}}\mathbf{R}_{BI} \quad \dot{\mathbf{R}}_{IB} = \mathbf{R}_{IB}\mathbf{I}\dot{\boldsymbol{\omega}}_{IB}^{\mathbf{T}} \quad \dot{\mathbf{R}}_{BI} = \mathbf{I}\dot{\boldsymbol{\omega}}_{IB}\mathbf{R}_{BI} \\ \mathbf{B}\dot{\boldsymbol{\omega}}_{IB} = \mathbf{R}_{IB}\dot{\mathbf{R}}_{IB}^{\mathbf{T}} = \mathbf{R}_{BI}^{\mathbf{T}}\dot{\mathbf{R}}_{BI} \quad \dot{\mathbf{R}}_{IB} = \mathbf{B}\dot{\boldsymbol{\omega}}_{IB}^{\mathbf{T}}\mathbf{R}_{IB} \quad \dot{\mathbf{R}}_{BI} = \mathbf{R}_{BI}\mathbf{B}\dot{\boldsymbol{\omega}}_{IB}$$

### Time Derivative of Rotation Quaternion $\Leftrightarrow$ Angular Velocity

$$\mathbf{I}\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \quad \dot{\mathbf{p}}_{BI} = \frac{1}{2}\mathbf{H}(\mathbf{p}_{BI})^{\mathbf{T}}\mathbf{I}\boldsymbol{\omega}_{IB} \\ \mathbf{B}\boldsymbol{\omega}_{IB} = 2\dot{\mathbf{H}}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \quad \dot{\mathbf{p}}_{BI} = \frac{1}{2}\dot{\mathbf{H}}(\mathbf{p}_{BI})^{\mathbf{T}}\mathbf{B}\boldsymbol{\omega}_{IB} \\ \mathbf{H}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & \vec{\mathbf{p}} + p_0\mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \quad \dot{\mathbf{H}}(\mathbf{p}) = \begin{bmatrix} -\vec{\dot{\mathbf{p}}} & -\vec{\dot{\mathbf{p}}} + p_0\mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \\ = \begin{bmatrix} -p_1 & p_0 & -p_3 & p_2 \\ -p_2 & p_3 & p_0 & -p_1 \\ -p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \quad = \begin{bmatrix} -p_1 & p_0 & p_3 & -p_2 \\ -p_2 & -p_3 & p_0 & p_1 \\ -p_3 & p_2 & -p_1 & p_0 \end{bmatrix}$$

### Time Derivative of Angle-Axis $\Leftrightarrow$ Angular Velocity

$$\mathbf{I}\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos\theta) \\ \mathbf{B}\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos\theta) \\ \dot{\theta} = \mathbf{n}^{\mathbf{T}}\mathbf{I}\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2}\frac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 - \frac{1}{2}\dot{\mathbf{n}}\right)\mathbf{I}\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\} \\ \dot{\theta} = \mathbf{n}^{\mathbf{T}}\mathbf{B}\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2}\frac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 + \frac{1}{2}\dot{\mathbf{n}}\right)\mathbf{B}\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\}$$

### Time Derivative of Rotation Vector $\Leftrightarrow$ Angular Velocity

$$\mathbf{I}\boldsymbol{\omega}_{IB} = \left(\mathbf{1}_{3 \times 3} + \hat{\phi} \left(\frac{1 - \cos\|\phi\|}{\|\phi\|^2}\right) + \hat{\phi}^2 \left(\frac{\|\phi\| - \sin\|\phi\|}{\|\phi\|^3}\right)\right)\dot{\phi} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ \mathbf{B}\boldsymbol{\omega}_{IB} = \left(\mathbf{1}_{3 \times 3} - \hat{\phi} \left(\frac{1 - \cos\|\phi\|}{\|\phi\|^2}\right) + \hat{\phi}^2 \left(\frac{\|\phi\| - \sin\|\phi\|}{\|\phi\|^3}\right)\right)\dot{\phi} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\}$$

### Time Derivative of Euler Angles ZYX $\Leftrightarrow$ Angular Velocity

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos\psi\sin\theta & \sin\psi\sin\theta & 1 \\ \cos\psi & \cos\theta & 0 \\ -\sin\psi & \sin\psi & 0 \end{bmatrix} \mathbf{I}\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z} \\ \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & \sin\phi & \cos\phi \\ 0 & \cos\theta & -\sin\phi \\ 1 & \sin\phi\sin\theta & \cos\phi\sin\theta \end{bmatrix} \mathbf{B}\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z} \\ \mathbf{I}\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & -\sin\psi & \cos\psi\cos\theta \\ 0 & \cos\psi & \cos\theta\sin\psi \\ 1 & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \\ \mathbf{B}\boldsymbol{\omega}_{IB} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

### Time Derivative of Euler Angles XYZ $\Leftrightarrow$ Angular Velocity

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin\alpha\sin\beta}{\cos\beta} & -\frac{\cos\alpha\sin\beta}{\cos\beta} \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\frac{\sin\alpha}{\cos\beta} & \frac{\cos\alpha}{\cos\beta} \end{bmatrix} \mathbf{I}\boldsymbol{\omega}_{IB} \quad \forall \beta \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z} \\ \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\cos\gamma}{\cos\beta} & -\frac{\sin\gamma}{\cos\beta} & 0 \\ \sin\gamma & \cos\gamma & 0 \\ -\frac{\cos\gamma\sin\beta}{\cos\beta} & \frac{\sin\beta\sin\gamma}{\cos\beta} & 1 \end{bmatrix} \mathbf{B}\boldsymbol{\omega}_{IB} \quad \forall \beta \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z} \\ \mathbf{I}\boldsymbol{\omega}_{IB} = \begin{bmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\cos\beta\sin\alpha \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} \\ \mathbf{B}\boldsymbol{\omega}_{IB} = \begin{bmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

# Dynamics of a Multi-Rigid-Body System

$n$	Number of bodies in system
$n_j$	Number of DoFs of the joints
$n_q$	Number of generalized coordinates
$n_u$	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{F}_Q^A$	External forces on point Q
$\mathbf{M}^A$	External torques
$m$	Mass
$\Theta$	Intertia tensor
$(\dots)^-$	Variable before impact
$(\dots)^+$	Variable after impact
$(\dots)^\pm$	Variable before/after impact
$\Delta t$	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\lambda$	Lebesgue-measurable contact forces
$\Lambda$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions

## Generalized Coordinates of a Floating-Base System with Rotational Joints

$$\mathbf{q} = \begin{pmatrix} I^{\mathbf{r}OB} \\ \mathbf{p}_{BI} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} I^{\mathbf{v}B} \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} I^{\mathbf{a}B} \\ {}_B\dot{\boldsymbol{\psi}}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j \times n_j} \end{pmatrix} \Leftrightarrow \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j \times n_j} \end{pmatrix}$$

## Equations of Motion with Contacts and no Impulses

$$\mathbf{M} = \sum_{i=1}^n \left[ (\mathbf{J}_S^\top m \mathbf{J}_S + \mathbf{J}_R^\top \Theta_S \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\lambda} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \quad \mathbf{g} = \sum_{i=1}^n \left[ (\mathbf{J}_S^\top m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\top (\Theta_S \dot{\mathbf{J}}_R \mathbf{u} + \boldsymbol{\Omega} \times \Theta_S \boldsymbol{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^n \left[ (\mathbf{J}_Q^\top \mathbf{F}_Q^A + \mathbf{J}_R^\top \mathbf{M}^A) \right]_i$$

## Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta \mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{array}{l} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\Lambda \\ \mathbf{M}(\underbrace{\dot{\mathbf{u}}dt + (\mathbf{u}^+ - \mathbf{u}^-)d\eta}_{d\mathbf{u}}) - \mathbf{h}d\mathbf{t} = \mathbf{W}(\underbrace{\lambda d\mathbf{t} + \Lambda d\eta}_{d\mathbf{P}}) \end{array} \right.$$

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