

# Kindr Library – Kinematics and Dynamics for Robotics

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## Nomenclature

(Hyper-)complex number	$Q$	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbf{1}_{n \times m}$	$n \times m$ -matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system $A$ with basis (unit) vectors $\mathbf{e}$
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	local / body-fixed coordinate system (moves with body)
Rotation	$\Phi \in \text{SO}(3)$	generic rotation (for all parameterizations)
Machine precision	$\epsilon$	

## Operators

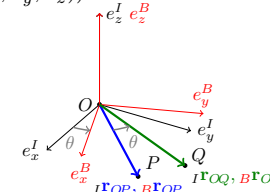
Cross product	$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge \mathbf{b} = \hat{\mathbf{a}} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
Skew/unskew	$\mathbf{a} = \hat{\mathbf{a}}^\vee$
Euclidean norm	$\ \mathbf{a}\  = \sqrt{\mathbf{a}^T \mathbf{a}} = \sqrt{a_1^2 + \dots + a_n^2}$
Exponential map for matrix	$\exp : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \mathbf{A} \mapsto e^{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{R}^{3 \times 3}$
Logarithmic map for matrix	$\log : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \mathbf{A} \mapsto \log \mathbf{A}, \quad \mathbf{A} \in \mathbb{R}^{3 \times 3}$

## Position & Orientation

### Position

Vector	$\mathbf{r}_{OP}$	from point $O$ to point $P$
Position vector	${}_B \mathbf{r}_{OP} \in \mathbb{R}^3$	from point $O$ to point $P$ expr. in frame $B$
Homogeneous pos. vector	${}_B \bar{\mathbf{r}}_{OP} = [{}_B \mathbf{r}_{OP}^\top \quad 1]^\top$	from point $O$ to point $P$ expr. in frame $B$

### Orientation/Rotation

- Active Rotation:  $\Phi^A : {}_I \mathbf{r}_{OP} \mapsto {}_I \mathbf{r}_{OQ}$  (rotates the vector  $\mathbf{r}_{OP}$ )
- Passive Rotation:  $\Phi^P : {}_I \mathbf{r}_{OP} \mapsto {}_B \mathbf{r}_{OP}$  (rotates the frame  $(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$ )
- Elementary Rotations  ${}_I \mathbf{r}_{OP} = \mathbf{C}_{IB} {}_B \mathbf{r}_{OP}$   
 around z-axis:  $\mathbf{C}_{IB} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$   
 around y-axis:  $\mathbf{C}_{IB} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$   
 around x-axis:  $\mathbf{C}_{IB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$ 

- Inversion:  $\Phi^{A^{-1}}(\mathbf{r}) = \Phi^P(\mathbf{r})$   
 $\Phi_2^A(\Phi_1^A(\mathbf{r})) = (\Phi_2^A \otimes \Phi_1^A)(\mathbf{r}) = (\Phi_1^{A^{-1}} \otimes \Phi_2^{A^{-1}})^{-1}(\mathbf{r})$
- Concatenation:  
 $\Phi_2^P(\Phi_1^P(\mathbf{r})) = (\Phi_2^P \otimes \Phi_1^P)(\mathbf{r}) = (\Phi_1^{P^{-1}} \otimes \Phi_2^{P^{-1}})^{-1}(\mathbf{r})$
- Exponential map:  $\exp : \mathbb{R}^3 \rightarrow \text{SO}(3), \mathbf{v} \mapsto \exp(\hat{\mathbf{v}}), \quad \mathbf{v} \in \mathbb{R}^3$
- Logarithmic map:  $\log : \text{SO}(3) \rightarrow \mathbb{R}^3, \Phi \mapsto \log(\Phi)^\vee, \quad \Phi \in \text{SO}(3)$
- Box plus:  $\Phi_2 = \Phi_1 \boxplus \mathbf{v} = \exp(\mathbf{v}) \otimes \Phi_1, \quad \Phi_1, \Phi_2 \in \text{SO}(3), \mathbf{v} \in \mathbb{R}^3$
- Box minus:  $\mathbf{v} = \Phi_1 \boxminus \Phi_2 = \log(\Phi_1 \otimes \Phi_2^{-1}), \quad \Phi_1, \Phi_2 \in \text{SO}(3), \mathbf{v} \in \mathbb{R}^3$
- Discrete integration:  $\Phi_{IB}^{k+1} = \Phi_{IB}^k \boxplus ({}_I \boldsymbol{\omega}_{IB}^k \Delta t), \quad \Phi_{BI}^{k+1} = \Phi_{BI}^k \boxminus (-{}_B \boldsymbol{\omega}_{IB}^k \Delta t)$
- Discrete differential:  ${}_I \boldsymbol{\omega}_{IB}^k = (\Phi_{IB}^{k+1} \boxminus \Phi_{IB}^k) / \Delta t, \quad {}_B \boldsymbol{\omega}_{IB}^k = -(\Phi_{BI}^{k+1} \boxminus \Phi_{BI}^k) / \Delta t$   
 $\Phi_t = \Phi_0 \boxplus ((\Phi_1 \boxminus \Phi_0)t), \quad \Phi_t = \Phi(t), \Phi_0 = \Phi(0), \Phi_1 = \Phi(1)$   
 $= (\Phi_1 \otimes \Phi_0^{-1})^t \otimes \Phi_0$
- (Spherical) linear interpolation  $t \in [0, 1]$ :

## Rotation Parameterizations

Rotation Matrix	$\mathbf{C}_{IB} \in \text{SO}(3)$ ${}_I \mathbf{r}_{OP} = \mathbf{C}_{IB} {}_B \mathbf{r}_{OP}$ is a coordinate transformation matrix, $\mathbf{C}_{IB} = \mathbf{C}_{BI}^\top$	The rotation matrix (Direction Cosine Matrix) which transforms vectors from frame $B$ to frame $I$ .
Rotation Quaternion	$\mathbf{q}_{IB}$ $\mathbf{q} = [q_0 \ q_1 \ q_2 \ q_3]^\top$	Hamiltonian unit quaternion (hypercomplex number) $Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, \quad \ \mathbf{Q}\  = 1$
Angle-axis	$(\theta, \mathbf{n})_{IB}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta \in [0, \pi]$ .
Rotation Vector	$\boldsymbol{\phi}_{IB}$	Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\ \boldsymbol{\phi}\ }$ and angle $\theta = \ \boldsymbol{\phi}\ $ .
Euler Angles ZYX Euler Angles YPR	$[z, y, x]_{IB}^\top$	Tait-Bryan angles (Flight conv.): $z - y' - x''$ , i.e. yaw-pitch-roll. Singularities are at $y = \pm \frac{\pi}{2}$ . $z \in [-\pi, \pi], y \in [-\frac{\pi}{2}, \frac{\pi}{2}], x \in [-\pi, \pi]$
Euler Angles XYZ Euler Angles RPY	$[x, y, z]_{IB}^\top$	Cardan angles: $x - y' - z''$ , i.e. roll-pitch-yaw. Singularities are at $y = \pm \frac{\pi}{2}$ . $x \in [-\pi, \pi], y \in [-\frac{\pi}{2}, \frac{\pi}{2}], z \in [-\pi, \pi]$

## Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1, \quad \|\mathbf{Q}\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$$

Note that  $Q_{IB}$  and  $-Q_{IB}$  represent the same rotation, but not the same unit quaternion.

Rot. quaternion as tuple:  $Q = (q_0, q_1, q_2, q_3) = (q_0, \hat{\mathbf{q}})$  with  $\hat{\mathbf{q}} := (q_1, q_2, q_3)^\top$

Rot. quaternion as vector:  $\mathbf{q} = [q_0 \quad q_1 \quad q_2 \quad q_3]^\top$

Conjugate:  $Q^* = (q_0, -\hat{\mathbf{q}})$

Inverse:  $Q^{-1} = Q^* = (q_0, -\hat{\mathbf{q}})$

Quaternion multiplication:

$$Q \cdot P = (q_0, \hat{\mathbf{q}}) \cdot (p_0, \hat{\mathbf{p}}) = (q_0 p_0 - \hat{\mathbf{q}}^\top \hat{\mathbf{p}}, q_0 \hat{\mathbf{p}} + p_0 \hat{\mathbf{q}} + \hat{\mathbf{q}} \times \hat{\mathbf{p}}) \Leftrightarrow$$

$$\underbrace{\mathbf{q} \otimes \mathbf{p}}_{\text{quaternion matrix}} = \underbrace{\begin{pmatrix} q_0 & -\hat{\mathbf{q}}^\top \\ \hat{\mathbf{q}} & q_0 \mathbf{1}_{3 \times 3} + \hat{\mathbf{q}} \end{pmatrix}}_{\text{quaternion matrix}} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}} \mathbf{q} = \begin{pmatrix} p_0 & -\hat{\mathbf{p}}^\top \\ \hat{\mathbf{p}} & p_0 \mathbf{1}_{3 \times 3} - \hat{\mathbf{p}} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

## Rotation Quaternion $\Leftrightarrow$ Rotation Angle-Axis

$$\mathbf{q}_{IB} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{n} \sin \frac{\theta}{2} \end{bmatrix} \Leftrightarrow (\theta, \mathbf{n})_{IB} = \begin{cases} (2 \arccos(p_0), \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}) & \text{if } \|\hat{\mathbf{q}}\|^2 \geq \epsilon^2 \\ (0, [1 \quad 0 \quad 0]^\top) & \text{otherwise} \end{cases}$$

## Rotation Quaternion $\Leftrightarrow$ Rotation Matrix

$$\mathbf{C}_{IB} = \mathbf{1}_{3 \times 3} + 2q_0 \hat{\mathbf{q}} + 2\hat{\mathbf{q}}^2 = (2q_0^2 - 1)\mathbf{1}_{3 \times 3} + 2q_0 \hat{\mathbf{q}} + 2\hat{\mathbf{q}} \hat{\mathbf{q}}^\top$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1 q_2 - 2q_0 q_3 & 2q_0 q_2 + 2q_1 q_3 \\ 2q_0 q_3 + 2q_1 q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_0 q_1 + 2q_2 q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

$$\mathbf{C}_{IB}^{-1} = \mathbf{C}_{BI} = \mathbf{1}_{3 \times 3} - 2q_0 \hat{\mathbf{q}} + 2\hat{\mathbf{q}}^2$$

$$= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0 q_3 + 2q_1 q_2 & 2q_1 q_3 - 2q_0 q_2 \\ 2q_1 q_2 - 2q_0 q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_0 q_1 + 2q_2 q_3 \\ 2q_0 q_2 + 2q_1 q_3 & 2q_2 q_3 - 2q_0 q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix}$$

## Euler Angles ZYX $\Leftrightarrow$ Direction Cosine Matrix

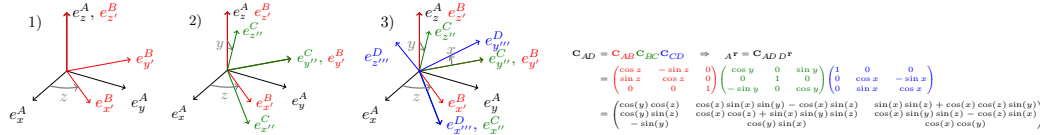


Figure 1: Rotation from A-frame to D-frame:  $(z - y' - x'') - (\text{yaw-pitch-roll}) - (50^\circ - 25^\circ - 30^\circ)$

## Euler Angles XYZ $\Leftrightarrow$ Direction Cosine Matrix

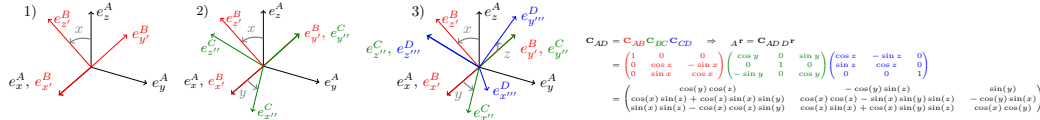


Figure 2: Rotation from A-frame to D-frame:  $(x - y' - z'') - (\text{roll-pitch-yaw}) - (50^\circ - 25^\circ - 30^\circ)$

## Pose

### Homogeneous Transformation Matrix

$$\begin{bmatrix} I\mathbf{r}_{IP} \\ 1 \end{bmatrix} = \mathbf{T}_{IB} \begin{bmatrix} B\mathbf{r}_{BP} \\ 1 \end{bmatrix}, \quad \mathbf{T}_{IB} = \begin{bmatrix} \mathbf{C}_{IB} & I\mathbf{r}_{IB} \\ \mathbf{0}^T & 1 \end{bmatrix}, \quad \mathbf{T}_{IB}^{-1} = \mathbf{T}_{BI} = \begin{bmatrix} \mathbf{C}_{IB}^T & -\mathbf{C}_{IB}^T I\mathbf{r}_{IB} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

## Time Derivatives of Position & Orientation

### Linear Velocity

Velocity of point  $P$  expressed in a rotating frame  $B$  w.r.t. to the inertial frame  $I$  using a moving point  $A$ :

$$B\mathbf{v}_P = B\mathbf{v}_A + B\dot{\mathbf{r}}_{AP} + B\boldsymbol{\omega}_{IB} \times B\mathbf{r}_{AP}$$

Velocity of point  $Q$  on rigid body  $B$  that rotates with  $B\boldsymbol{\Omega}$ , where point  $P$  is on the same rigid body  $B$ :

$$B\mathbf{v}_Q = B\mathbf{v}_P + B\boldsymbol{\Omega} \times B\mathbf{r}_{PQ}, \quad B\boldsymbol{\Omega} = B\boldsymbol{\omega}_{IB}$$

### Angular Velocity

$$\begin{aligned} B\boldsymbol{\omega}_{IB} &=: B\boldsymbol{\Omega} && \text{(local) absolute angular velocity of rigid body } B \text{ expr. in frame } B \\ B\boldsymbol{\omega}_{BI} &= -B\boldsymbol{\omega}_{IB} && \text{inverse of angular velocity} \\ I\boldsymbol{\omega}_{IB} &= \mathbf{C}_{IB} B\boldsymbol{\omega}_{IB} && \text{(global) angular velocity from frame } B \text{ to frame } I \\ I\dot{\boldsymbol{\omega}}_{IB} &= \mathbf{C}_{IB} B\dot{\boldsymbol{\omega}}_{IB} \mathbf{C}_{IB}^T && \text{coord. transformation of angular velocity from frame } B \text{ to frame } I \\ D\boldsymbol{\omega}_{AD} &= D\boldsymbol{\omega}_{AB} + D\boldsymbol{\omega}_{BC} + D\boldsymbol{\omega}_{CD} && \text{composition of (relative) angular velocity} \end{aligned}$$

## Derivatives

The derivation of the following identities can be found in [1].

$$\left. \begin{aligned} \frac{\partial}{\partial t} \Phi_{BI}(t) &= -B\boldsymbol{\omega}_{IB}(t) \\ \frac{\partial}{\partial \mathbf{r}} (\Phi(\mathbf{r})) &= \mathbf{C}(\Phi) \\ \frac{\partial}{\partial \Phi} (\Phi(\mathbf{r})) &= -(\Phi(\mathbf{r}))^\wedge \\ \frac{\partial}{\partial \Phi} (\Phi^{-1}) &= -\mathbf{C}(\Phi)^T \end{aligned} \right| \begin{aligned} \frac{\partial}{\partial \Phi_1} (\Phi_1 \otimes \Phi_2) &= \mathbf{1} \\ \frac{\partial}{\partial \Phi_2} (\Phi_1 \otimes \Phi_2) &= \mathbf{C}(\Phi_1) \\ \frac{\partial}{\partial \mathbf{v}} (\exp(\mathbf{v})) &= \boldsymbol{\Gamma}(\mathbf{v}) \\ \frac{\partial}{\partial \mathbf{v}} (\log(\Phi)) &= \boldsymbol{\Gamma}^{-1}(\log(\Phi)) \end{aligned} \quad \frac{\partial}{\partial \Phi_1} (\Phi_1 \boxplus \Phi_2) = \boldsymbol{\Gamma}^{-1}(\Phi_1 \boxplus \Phi_2)$$

$$\text{Jacobian of exponential map: } \boldsymbol{\Gamma}(\mathbf{v}) = \begin{cases} \mathbf{1} + \frac{1 - \cos(\|\mathbf{v}\|)\hat{\mathbf{v}}}{\|\mathbf{v}\|^2} + \frac{(\|\mathbf{v}\| - \sin(\|\mathbf{v}\|))\hat{\mathbf{v}}^2}{\|\mathbf{v}\|^3} & \text{if } \|\mathbf{v}\| \neq 0 \\ \mathbf{1} + \frac{1}{2}\hat{\mathbf{v}} & \text{otherwise} \end{cases}$$

## Time Derivative of Rotation Matrix $\Leftrightarrow$ Angular Velocity

$$\begin{aligned} I\dot{\boldsymbol{\omega}}_{IB} &= \dot{\mathbf{C}}_{IB} \mathbf{C}_{IB}^T = \dot{\mathbf{C}}_{BI}^T \mathbf{C}_{BI} && \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = I\dot{\boldsymbol{\omega}}_{IB} \mathbf{C}_{IB} \\ B\dot{\boldsymbol{\omega}}_{IB} &= \mathbf{C}_{IB}^T \dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI} \dot{\mathbf{C}}_{BI}^T && \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB} B\dot{\boldsymbol{\omega}}_{IB} \end{aligned}$$

## Time Derivative of Rotation Quaternion $\Leftrightarrow$ Angular Velocity

$$\begin{aligned} I\boldsymbol{\omega}_{IB} &= 2\mathbf{H}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} && \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \frac{1}{2}\mathbf{H}(\mathbf{q}_{IB})^T I\boldsymbol{\omega}_{IB} \\ B\boldsymbol{\omega}_{IB} &= 2\tilde{\mathbf{H}}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} && \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \frac{1}{2}\tilde{\mathbf{H}}(\mathbf{q}_{IB})^T B\boldsymbol{\omega}_{IB} \\ \mathbf{H}(\mathbf{q}) &= \begin{bmatrix} -\hat{\mathbf{q}} & \dot{\hat{\mathbf{q}}} + q_0 \mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} && \mathbf{H}(\mathbf{q}) = \begin{bmatrix} -\hat{\mathbf{q}} & -\dot{\hat{\mathbf{q}}} + q_0 \mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \\ &= \begin{bmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} && = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix} \end{aligned}$$

## Time Derivative of Angle-Axis $\Leftrightarrow$ Angular Velocity

$$\begin{aligned} I\boldsymbol{\omega}_{IB} &= \dot{\mathbf{n}}\theta + \dot{\mathbf{n}}\sin\theta + \mathbf{n}\dot{\mathbf{n}}(1 - \cos\theta) \\ B\boldsymbol{\omega}_{IB} &= \dot{\mathbf{n}}\theta + \dot{\mathbf{n}}\sin\theta - \mathbf{n}\dot{\mathbf{n}}(1 - \cos\theta) \\ \dot{\theta} &= \mathbf{n}^T I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left( -\frac{1}{2} \frac{\sin\theta}{1 - \cos\theta} \hat{\mathbf{n}}^2 - \frac{1}{2} \hat{\mathbf{n}} \right) I\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\} \\ \dot{\theta} &= \mathbf{n}^T B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left( -\frac{1}{2} \frac{\sin\theta}{1 - \cos\theta} \hat{\mathbf{n}}^2 + \frac{1}{2} \hat{\mathbf{n}} \right) B\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\} \end{aligned}$$

## Time Derivative of Rotation Vector $\Leftrightarrow$ Angular Velocity

$$\begin{aligned} I\boldsymbol{\omega}_{IB} &= \left( \mathbf{1}_{3 \times 3} + \hat{\phi} \left( \frac{1 - \cos\|\phi\|}{\|\phi\|^2} \right) + \hat{\phi}^2 \left( \frac{\|\phi\| - \sin\|\phi\|}{\|\phi\|^3} \right) \right) \dot{\phi} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ B\boldsymbol{\omega}_{IB} &= \left( \mathbf{1}_{3 \times 3} - \hat{\phi} \left( \frac{1 - \cos\|\phi\|}{\|\phi\|^2} \right) + \hat{\phi}^2 \left( \frac{\|\phi\| - \sin\|\phi\|}{\|\phi\|^3} \right) \right) \dot{\phi} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ \dot{\phi} &= \left( \mathbf{1}_{3 \times 3} - \frac{1}{2} \hat{\phi} + \hat{\phi}^2 \frac{1}{\|\phi\|^2} \left( 1 - \frac{\|\phi\|}{2} \frac{\sin\|\phi\|}{1 - \cos\|\phi\|} \right) \right) I\boldsymbol{\omega}_{IB} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ \dot{\phi} &= \left( \mathbf{1}_{3 \times 3} + \frac{1}{2} \hat{\phi} + \hat{\phi}^2 \frac{1}{\|\phi\|^2} \left( 1 - \frac{\|\phi\|}{2} \frac{\sin\|\phi\|}{1 - \cos\|\phi\|} \right) \right) B\boldsymbol{\omega}_{IB} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \end{aligned}$$

## Time Derivative of Euler Angles ZYX $\Leftrightarrow$ Angular Velocity

$$\begin{aligned} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} 0 & \frac{\sin(x)}{\cos(y)} & \frac{\cos(x)}{\cos(y)} \\ 0 & \cos(x) & -\sin(x) \\ 1 & \frac{\sin(x)\sin(y)}{\cos(y)} & \frac{\cos(x)\sin(y)}{\cos(y)} \end{bmatrix} B\boldsymbol{\omega}_{IB} \quad \forall y \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z} \\ \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} &= \begin{bmatrix} \frac{\cos(y)}{\cos(z)\sin(y)} & \frac{\cos(y)}{\sin(y)\sin(z)} & 1 \\ \frac{\cos(y)}{\cos(z)} & \frac{\cos(y)}{\cos(z)} & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} I\boldsymbol{\omega}_{IB} \quad \forall y \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z} \\ B\boldsymbol{\omega}_{IB} &= \begin{bmatrix} -\sin(y) & 0 & 1 \\ \cos(y)\sin(x) & \cos(x) & 0 \\ \cos(x)\cos(y) & -\sin(x) & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} \\ I\boldsymbol{\omega}_{IB} &= \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} \end{aligned}$$

## Dynamics of a Multi-Rigid-Body System

$DoFs$	Degrees of Freedom
$n$	Number of bodies in system
$n_j$	Number of DoFs of the joints
$n_q$	Number of generalized coordinates
$n_u$	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{f}_Q^A$	External forces on point Q
$\mathbf{t}_Q^A$	External torques
$m$	Mass
$\Theta$	Inertia tensor
$(\dots)^-$	Variable before impact
$(\dots)^+$	Variable after impact
$(\dots)^\pm$	Variable before/after impact
$\Delta t$	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\lambda$	Lebesgue-measurable contact forces
$\Lambda$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions
COM	Center of mass

### Generalized Coordinates of a Floating-Base System with Rotational Joints

Recommended set of generalized coordinates  $\mathbf{q}$  with quaternion  $\mathbf{q}_{IB}$  and generalized velocities  $\mathbf{u}$ :

$$\mathbf{q} = \begin{pmatrix} {}^I\mathbf{r}_{IB} \\ \mathbf{q}_{IB} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} {}^I\mathbf{v}_B \\ {}^B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} {}^I\mathbf{a}_B \\ {}^B\dot{\boldsymbol{\psi}}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j \times n_j} \end{pmatrix} \Leftrightarrow \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j \times n_j} \end{pmatrix}$$

$$\begin{bmatrix} {}^I\mathbf{v}_{IQ} \\ {}^I\boldsymbol{\omega}_{IQ} \end{bmatrix} = {}^I\mathbf{J}_Q(\mathbf{q}) \cdot \mathbf{u}, \quad {}^I\mathbf{J}_Q(\mathbf{q}) = \begin{bmatrix} {}^I\mathbf{J}_P(\mathbf{q}) \\ {}^I\mathbf{J}_R(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{3 \times 3} & -\mathbf{C}_{IB} \cdot {}^B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}^B\mathbf{J}_{Pq_j}(\mathbf{q}_j) \\ \mathbf{0}_{3 \times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}^B\mathbf{J}_{Rq_j}(\mathbf{q}_j) \end{bmatrix}$$

### Equations of Motion with Contacts and no Impulses

Projected Newton-Euler Equations

$$\mathbf{M} = \sum_{i=1}^n \left[ (\mathbf{J}_{\text{COM}}^\top m \mathbf{J}_{\text{COM}} + \mathbf{J}_R^\top \Theta_{\text{COM}} \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\lambda} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \quad \mathbf{g} = \sum_{i=1}^n \left[ (\mathbf{J}_{\text{COM}}^\top m \dot{\mathbf{J}}_{\text{COM}} \mathbf{u} + \mathbf{J}_R^\top (\Theta_{\text{COM}} \dot{\mathbf{J}}_R \mathbf{u} + \boldsymbol{\Omega} \times \Theta_{\text{COM}} \boldsymbol{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^n \left[ (\mathbf{J}_Q^\top \mathbf{f}_Q^A + \mathbf{J}_R^\top \mathbf{t}^A) \right]_i$$

### Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta \mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{array}{l} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\Lambda \\ \mathbf{M} \underbrace{(\dot{\mathbf{u}}dt + (\mathbf{u}^+ - \mathbf{u}^-)d\eta)}_{d\mathbf{u}} - \mathbf{h}dt = \mathbf{W} \underbrace{(\lambda dt + \Lambda d\eta)}_{d\mathbf{P}} \end{array} \right.$$

## Transformation of Equations of Motion

Transformation from  $\bar{\mathbf{M}}(\bar{\mathbf{q}}), \bar{\mathbf{h}}(\bar{\mathbf{q}}, \bar{\mathbf{u}})$  to  $\mathbf{M}(\mathbf{q}), \mathbf{h}(\mathbf{q}, \mathbf{u})$ , where  $\bar{\mathbf{u}} = \mathbf{B}\mathbf{u}$ :

$$\mathbf{M} = \mathbf{B}^\top \bar{\mathbf{M}} \mathbf{B}$$

$$\mathbf{h} = \mathbf{B}^\top \bar{\mathbf{h}} - \mathbf{B}^\top \bar{\mathbf{M}} \dot{\mathbf{B}} \mathbf{u}$$

## Appendix I: Euler Angles ZYX Velocities to Angular Velocity Mapping

Given a set of Euler angles  $\chi = [z \quad y \quad x]^T$  and velocities  $\dot{\chi} = [\dot{z} \quad \dot{y} \quad \dot{x}]^T$ , we wish to find the mapping  $\mathbf{E}(\chi) \in \mathbb{R}^{3 \times 3}$  that maps  $\dot{\chi}$  to  ${}_I\boldsymbol{\omega}_{IB}$ :

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{E}(\chi) \cdot \dot{\chi} \quad (1)$$

The columns of  $\mathbf{E}(\chi)$  are the components of the unit vectors around which the rotational velocities are applied expressed in fixed frame. These are obtained by selecting the columns of a rotation matrix which is built up by successive elementary rotations specified by the Euler angle parametrization.

Starting from the reference frame  $I$ , the first rotation will be an elementary rotation around  ${}_I\mathbf{e}_I^z$ , which is simply given by:

$${}_I\mathbf{e}_I^z = \mathbb{I}_{3 \times 3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2)$$

After an elementary rotation around  ${}_I\mathbf{e}_I^z$ , the y axis  ${}_I\mathbf{e}_{I'}^y$ , will be expressed by:

$${}_I\mathbf{e}_{I'}^y = \mathbf{C}_{II'}(z) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(z) \\ \cos(z) \\ 0 \end{bmatrix} \quad (3)$$

After an elementary rotation around  ${}_I\mathbf{e}_{I'}^y$ , the x axis  ${}_I\mathbf{e}_{I''}^x$ , will be expressed by:

$${}_I\mathbf{e}_{I''}^x = \mathbf{C}_{II'}(z) \cdot \mathbf{C}_{I'I''}(y) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(y) & 0 & \sin(y) \\ 0 & 1 & 0 \\ -\sin(y) & 0 & \cos(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(y)\cos(z) \\ \cos(y)\sin(z) \\ -\sin(z) \end{bmatrix} \quad (4)$$

Finally, the mapping  $\mathbf{E}(\chi)$  will be computed as:

$$\mathbf{E}(\chi) = [{}_I\mathbf{e}_I^z \quad {}_I\mathbf{e}_{I'}^y \quad {}_I\mathbf{e}_{I''}^x] = \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix} \quad (5)$$

It is easy to find that  $\det(\mathbf{E}(\chi)) = -\cos(y)$ . The mapping then becomes singular when  $y = \pi/2 + k\pi, \forall k \in \mathbb{Z}$ . This means that although we can always describe an angular velocity using Euler angle velocities, the inverse is not always possible. The inverse mapping is given by:

$$\bar{\mathbf{E}}(\chi) = \mathbf{E}^{-1}(\chi) = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} \quad (6)$$

If we compute the rotation matrix  $\mathbf{C}_{IB} = \mathbf{C}_{IB}(z, y, x) = \mathbf{C}_z(z) \cdot \mathbf{C}_y(y) \cdot \mathbf{C}_x(x)$ , we can also derive the following mappings:

$${}_B\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}^T \cdot \mathbf{E}(\chi) \cdot \dot{\chi} \quad (7)$$

$$\dot{\chi} = \bar{\mathbf{E}}(\chi) \cdot \mathbf{C}_{IB} \cdot {}_B\boldsymbol{\omega}_{IB} \quad (8)$$

The mapping described by (8) is valid  $\forall y \neq \pi/2 + k\pi, \forall k \in \mathbb{Z}$ .

## Appendix II: Jacobians

We wish to derive the relationship between the generalized velocities  $\mathbf{u}$  and the operational space velocities  ${}_I\mathbf{v}_Q$  of a point  $Q$ , which is fixed at the end of a kinematic chain that stems from a floating base  $B$ . The position vector  ${}_I\mathbf{r}_{IQ} = {}_I\mathbf{r}_{IQ}(\mathbf{q})$  of a point w.r.t. the inertial frame  $I$  is given by:

$${}_I\mathbf{r}_{IQ}(\mathbf{q}) = {}_I\mathbf{r}_{IB}(\mathbf{q}) + \mathbf{C}_{IB}(\mathbf{q}) \cdot {}_B\mathbf{r}_{BQ}(\mathbf{q}), \quad (9)$$

where the rotation matrix  $\mathbf{C}_{IB}(\mathbf{q})$  describes the orientation of the floating base  $B$  w.r.t. the inertial frame  $I$ ,  ${}_I\mathbf{r}_{IB}(\mathbf{q})$  represents the position of the floating base  $B$  w.r.t. the inertial frame  $I$  expressed in the inertial frame and  $\mathbf{q} = \mathbf{q}(t)$  is a function of time  $t$ .

Time differentiation yields:

$$\begin{aligned} {}_I\mathbf{v}_Q &= {}_I\mathbf{v}_B + \dot{\mathbf{C}}_{IB} \cdot {}_B\mathbf{r}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B + \mathbf{C}_{IB} \cdot {}_B\dot{\boldsymbol{\omega}}_{IB} \cdot {}_B\mathbf{r}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B - \mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} \cdot {}_B\boldsymbol{\omega}_{IB} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B - \mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} \cdot {}_B\boldsymbol{\omega}_{IB} + \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \cdot \dot{\mathbf{q}}_j \\ &= \begin{bmatrix} \mathbb{I}_{3 \times 3} & -\mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \end{bmatrix} \cdot \mathbf{u} \end{aligned} \quad (10)$$

If we attach a frame at  ${}_I\mathbf{r}_{IQ}$ , we can derive a similar mapping for angular velocities. The orientation of frame  $Q$  w.r.t. the inertial frame  $I$  is described by:

$$\mathbf{C}_{IQ} = \mathbf{C}_{IB} \cdot \mathbf{C}_{BQ} \quad (11)$$

Time differentiation of both sides of (11) yields:

$$\begin{aligned} {}_I\dot{\boldsymbol{\omega}}_{IQ} \cdot \mathbf{C}_{IQ} &= {}_I\dot{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IB} \cdot \mathbf{C}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{BQ} \\ &= {}_I\dot{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IQ} + \mathbf{C}_{IB} \cdot \mathbf{C}_{BI} \cdot {}_I\dot{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{BI}^T \cdot \mathbf{C}_{BQ} \\ &= {}_I\dot{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IQ} + {}_I\dot{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{IQ}, \end{aligned} \quad (12)$$

which gives finally:

$$\begin{aligned} {}_I\boldsymbol{\omega}_{IQ} &= {}_I\boldsymbol{\omega}_{IB} + {}_I\boldsymbol{\omega}_{BQ} \\ &= \begin{bmatrix} \mathbb{I}_{3 \times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix} \cdot \mathbf{u} \end{aligned} \quad (13)$$

Hence, the mapping from generalized velocities  $\mathbf{u}$  to the operational space twist  $\begin{bmatrix} {}_I\mathbf{v}_Q^T & {}_I\boldsymbol{\omega}_{IQ}^T \end{bmatrix}^T$  of frame  $Q$  is realized by the spatial Jacobian:

$$\begin{aligned} {}_I\mathbf{J}_Q(\mathbf{q}) &= \begin{bmatrix} {}_I\mathbf{J}_P \\ {}_I\mathbf{J}_R \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{I}_{3 \times 3} & -\mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \\ \mathbf{0}_{3 \times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix} \end{aligned} \quad (14)$$

## Appendix III: Hessians and Time Derivatives of Jacobians

Consider a kinematic chain which connects two rigid bodies. We represent the set of indexes of the rigid bodies in this chain with  $U_A$ . As shown in [2], the  $i$ -th column of the spatial Hessian matrix of the spatial Jacobian  $\mathbf{J}$  w.r.t. the  $j$ -th configuration variable  $q_j$  can be computed as:

$$\frac{\partial \mathbf{J}_i}{\partial q_j} = \begin{bmatrix} \frac{\partial \mathbf{J}_{P_i}}{\partial q_j} \\ \frac{\partial \mathbf{J}_{R_i}}{\partial q_j} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{J}_{R_j} \times \mathbf{J}_{P_i} \\ \mathbf{J}_{R_j} \times \mathbf{J}_{R_i} \\ \mathbf{J}_{R_i} \times \mathbf{J}_{P_j} \\ \mathbf{0} \end{bmatrix} & i \geq j \\ \begin{bmatrix} \mathbf{J}_{R_i} \times \mathbf{J}_{P_j} \\ \mathbf{0} \end{bmatrix} & i < j \\ \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} & i, j \notin U_A \end{cases} \quad (15)$$

The Hessian  $\mathbf{H}_k(\mathbf{q})$  is then expressed by:

$$\mathbf{H}_k(\mathbf{q}) = \begin{bmatrix} \frac{\partial \mathbf{J}_1}{\partial q_k} & \frac{\partial \mathbf{J}_2}{\partial q_k} & \dots & \frac{\partial \mathbf{J}_n}{\partial q_k} \end{bmatrix} \quad (16)$$

Knowledge of the Hessian matrix w.r.t. each configuration variable  $q_k$  enables the computation of the time derivative of  $\mathbf{J}(\mathbf{q}(t))$ . Its generic element  $i, j$  can be computed as:

$$\frac{dJ_{i,j}(\mathbf{q})}{dt} = \frac{\partial J_{i,j}(\mathbf{q})}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \sum_{k=1}^n \frac{\partial J_{i,j}(\mathbf{q})}{\partial q_k} \cdot \frac{dq_k}{dt} = \sum_{k=1}^n H_{k_{i,j}}(\mathbf{q}) \cdot \frac{dq_k}{dt}, \quad (17)$$

which yields:

$$\frac{d\mathbf{J}(\mathbf{q})}{dt} = \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt}, = \sum_{k=1}^n \mathbf{H}_k(\mathbf{q}) \cdot \frac{dq_k}{dt}. \quad (18)$$

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