

Kindr Library – Kinematics and Dynamics for Robotics

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Nomenclature

(Hyper-)complex number	Q	normal capital letter
Column vector	\mathbf{a}	bold small letter
Matrix	\mathbf{M}	bold capital letter
Identity matrix	$\mathbf{1}_{n \times m}$	$n \times m$ -matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system A with basis (unit) vectors \mathbf{e}
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	local / body-fixed coordinate system (moves with body)
Rotation	$\Phi \in \text{SO}(3)$	generic rotation (for all parameterizations)
Machine precision	ϵ	

Operators

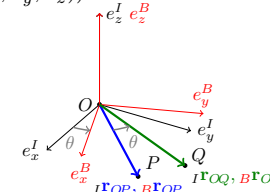
Cross product/skew/unskew	$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = (\mathbf{a})^\wedge \mathbf{b} = \hat{\mathbf{a}} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ $\mathbf{a} = \hat{\mathbf{a}}^\vee, \quad \hat{\mathbf{a}} = -\hat{\mathbf{a}}^\top, \quad \mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
Euclidean norm	$\ \mathbf{a}\ = \sqrt{\mathbf{a}^\top \mathbf{a}} = \sqrt{a_1^2 + \dots + a_n^2}$
Exponential map for matrix	$\exp: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \mathbf{A} \mapsto e^{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{R}^{3 \times 3}$
Logarithmic map for matrix	$\log: \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}, \mathbf{A} \mapsto \log \mathbf{A}, \quad \mathbf{A} \in \mathbb{R}^{3 \times 3}$

Position & Orientation

Position

Vector	\mathbf{r}_{OP}	from point O to point P
Position vector	${}_B \mathbf{r}_{OP} \in \mathbb{R}^3$	from point O to point P expr. in frame B
Homogeneous pos. vector	${}_B \bar{\mathbf{r}}_{OP} = [{}_B \mathbf{r}_{OP}^\top \quad 1]^\top$	from point O to point P expr. in frame B

Orientation/Rotation

- Active Rotation: $\Phi^A: {}_I \mathbf{r}_{OP} \mapsto {}_I \mathbf{r}_{OQ}$ (rotates the vector \mathbf{r}_{OP})
- Passive Rotation: $\Phi^P: {}_I \mathbf{r}_{OP} \mapsto {}_B \mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$)
- Elementary Rotations ${}_I \mathbf{r}_{OP} = \mathbf{C}_{IB} {}_B \mathbf{r}_{OP}$
 around z-axis: $\mathbf{C}_{IB} = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
 around y-axis: $\mathbf{C}_{IB} = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$
 around x-axis: $\mathbf{C}_{IB} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$

- Inversion: $\Phi^{A^{-1}}(\mathbf{r}) = \Phi^P(\mathbf{r})$
 $\Phi_2^A(\Phi_1^A(\mathbf{r})) = (\Phi_2^A \otimes \Phi_1^A)(\mathbf{r}) = (\Phi_1^{A^{-1}} \otimes \Phi_2^{A^{-1}})^{-1}(\mathbf{r})$
- Concatenation:
 $\Phi_2^P(\Phi_1^P(\mathbf{r})) = (\Phi_2^P \otimes \Phi_1^P)(\mathbf{r}) = (\Phi_1^{P^{-1}} \otimes \Phi_2^{P^{-1}})^{-1}(\mathbf{r})$
- Exponential map: $\exp: \mathbb{R}^3 \rightarrow \text{SO}(3), \mathbf{v} \mapsto \exp(\hat{\mathbf{v}}), \quad \mathbf{v} \in \mathbb{R}^3$
- Logarithmic map: $\log: \text{SO}(3) \rightarrow \mathbb{R}^3, \Phi \mapsto \log(\Phi)^\vee, \quad \Phi \in \text{SO}(3)$
- Box plus: $\Phi_2 = \Phi_1 \boxplus \mathbf{v} = \exp(\mathbf{v}) \otimes \Phi_1, \quad \Phi_1, \Phi_2 \in \text{SO}(3), \mathbf{v} \in \mathbb{R}^3$
- Box minus: $\mathbf{v} = \Phi_1 \boxminus \Phi_2 = \log(\Phi_1 \otimes \Phi_2^{-1}), \quad \Phi_1, \Phi_2 \in \text{SO}(3), \mathbf{v} \in \mathbb{R}^3$
- Discrete integration: $\Phi_{IB}^{k+1} = \Phi_{IB}^k \boxplus ({}_I \boldsymbol{\omega}_{IB}^k \Delta t), \quad \Phi_{BI}^{k+1} = \Phi_{BI}^k \boxminus (-{}_B \boldsymbol{\omega}_{IB}^k \Delta t)$
- Discrete differential: ${}_I \boldsymbol{\omega}_{IB}^k = (\Phi_{IB}^{k+1} \boxminus \Phi_{IB}^k) / \Delta t, \quad {}_B \boldsymbol{\omega}_{IB}^k = -(\Phi_{BI}^{k+1} \boxminus \Phi_{BI}^k) / \Delta t$
 $\Phi_t = \Phi_0 \boxplus ((\Phi_1 \boxminus \Phi_0)t), \quad \Phi_t = \Phi(t), \Phi_0 = \Phi(0), \Phi_1 = \Phi(1)$
 $= (\Phi_1 \otimes \Phi_0^{-1})^t \otimes \Phi_0$
- (Spherical) linear interpolation $t \in [0, 1]$:

Rotation Parameterizations

Rotation Matrix	$\mathbf{C}_{IB} \in \text{SO}(3)$	The rotation matrix (Direction Cosine Matrix)
	${}_I \mathbf{r}_{OP} = \mathbf{C}_{IB} {}_B \mathbf{r}_{OP}$	is a coordinate transformation matrix, which transforms vectors from frame B to frame I .
	$\mathbf{C}_{IB} = \mathbf{C}_{BI}^\top$	
Rotation Quaternion	$\mathbf{q}_{IB} = [q_0 \ q_1 \ q_2 \ q_3]^\top$	Hamiltonian unit quaternion (hypercomplex number)
		$Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, \quad \ Q\ = 1$
Angle-axis	$(\theta, \mathbf{n})_{IB}$	Rotation with unit rotation axis \mathbf{n} and angle $\theta \in [0, \pi]$.
Rotation Vector	$\boldsymbol{\phi}_{IB}$	Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\ \boldsymbol{\phi}\ }$ and angle $\theta = \ \boldsymbol{\phi}\ $.
Euler Angles ZYX	$[z, y, x]_{IB}^\top$	Tait-Bryan angles (Flight conv.): $z - y' - x''$, i.e. yaw-pitch-roll. Singularities are at $y = \pm \frac{\pi}{2}$.
Euler Angles YPR		$z \in [-\pi, \pi], y \in [-\frac{\pi}{2}, \frac{\pi}{2}], x \in [-\pi, \pi]$
Euler Angles XYZ	$[x, y, z]_{IB}^\top$	Cardan angles: $x - y' - z''$, i.e. roll-pitch-yaw.
Euler Angles RPY		Singularities are at $y = \pm \frac{\pi}{2}$. $x \in [-\pi, \pi], y \in [-\frac{\pi}{2}, \frac{\pi}{2}], z \in [-\pi, \pi]$

Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1, \quad \|Q\| = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$$

Tuple: $Q = (q_0, q_1, q_2, q_3) = (q_0, \tilde{\mathbf{q}})$ with $\tilde{\mathbf{q}} := (q_1, q_2, q_3)^\top$

4 × 1-vector: $\mathbf{q} = [q_0 \ q_1 \ q_2 \ q_3]^\top$

Conjugate: $Q^* = (q_0, -\tilde{\mathbf{q}})$

Inverse: $Q^{-1} = Q^* = (q_0, -\tilde{\mathbf{q}})$

Quaternion multiplication:

$$Q \cdot P = (q_0, \tilde{\mathbf{q}}) \cdot (p_0, \tilde{\mathbf{p}}) = (q_0 p_0 - \tilde{\mathbf{q}}^\top \tilde{\mathbf{p}}, q_0 \tilde{\mathbf{p}} + p_0 \tilde{\mathbf{q}} + \tilde{\mathbf{q}} \times \tilde{\mathbf{p}}) \Leftrightarrow$$

$$\mathbf{q} \otimes \mathbf{p} = \underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}} \mathbf{p} = \begin{pmatrix} q_0 & -\tilde{\mathbf{q}}^\top \\ \tilde{\mathbf{q}} & q_0 \mathbf{1}_{3 \times 3} + \hat{\tilde{\mathbf{q}}} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}} \mathbf{q} = \begin{pmatrix} p_0 & -\tilde{\mathbf{p}}^\top \\ \tilde{\mathbf{p}} & p_0 \mathbf{1}_{3 \times 3} - \hat{\tilde{\mathbf{p}}} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Note that Q_{IB} and $-Q_{IB}$ represent the same rotation, but not the same unit quaternion.

Rotation Quaternion \Leftrightarrow Rotation Vector

$$\mathbf{q}_{IB} = \begin{cases} \left[\cos(\frac{1}{2}\|\boldsymbol{\phi}\|), \frac{\boldsymbol{\phi}^\top}{\|\boldsymbol{\phi}\|} \sin(\frac{1}{2}\|\boldsymbol{\phi}\|) \right]^\top & \text{if } \|\boldsymbol{\phi}\| \geq \epsilon \\ [1, \frac{1}{2}\boldsymbol{\phi}^\top]^\top & \text{otherwise} \end{cases} \Leftrightarrow \boldsymbol{\phi}_{IB} = \begin{cases} 2 \operatorname{atan2}(\|\tilde{\mathbf{q}}\|, q_0) \frac{\tilde{\mathbf{q}}}{\|\tilde{\mathbf{q}}\|} & \text{if } \|\tilde{\mathbf{q}}\| \geq \epsilon \\ 2\tilde{\mathbf{q}} & \text{otherwise} \end{cases}$$

Rotation Quaternion \Leftrightarrow Angle-Axis

$$\mathbf{q}_{IB} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{n} \sin \frac{\theta}{2} \end{bmatrix} \Leftrightarrow (\theta, \mathbf{n})_{IB} = \begin{cases} (2 \arccos(q_0), \frac{\tilde{\mathbf{q}}}{\|\tilde{\mathbf{q}}\|}) & \text{if } \|\tilde{\mathbf{q}}\| \geq \epsilon \\ (0, [1 \ 0 \ 0]^\top) & \text{otherwise} \end{cases}$$

Rotation Quaternion \Leftrightarrow Rotation Matrix

$$\begin{aligned} \mathbf{C}_{IB} &= \mathbf{1}_{3 \times 3} + 2q_0 \hat{\tilde{\mathbf{q}}} + 2\tilde{\mathbf{q}}^2 = (2q_0^2 - 1)\mathbf{1}_{3 \times 3} + 2q_0 \hat{\tilde{\mathbf{q}}} + 2\tilde{\mathbf{q}}\tilde{\mathbf{q}}^\top \\ &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_1 q_2 - 2q_0 q_3 & 2q_0 q_2 + 2q_1 q_3 \\ 2q_0 q_3 + 2q_1 q_2 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_2 q_3 - 2q_0 q_1 \\ 2q_1 q_3 - 2q_0 q_2 & 2q_0 q_1 + 2q_2 q_3 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \\ \mathbf{C}_{IB}^{-1} &= \mathbf{C}_{BI} = \mathbf{1}_{3 \times 3} - 2q_0 \hat{\tilde{\mathbf{q}}} + 2\tilde{\mathbf{q}}^2 \\ &= \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2q_0 q_3 + 2q_1 q_2 & 2q_1 q_3 - 2q_0 q_2 \\ 2q_1 q_2 - 2q_0 q_3 & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2q_0 q_1 + 2q_2 q_3 \\ 2q_0 q_2 + 2q_1 q_3 & 2q_2 q_3 - 2q_0 q_1 & q_0^2 - q_1^2 - q_2^2 + q_3^2 \end{bmatrix} \end{aligned}$$

Rotation Matrix \Leftrightarrow Rotation Vector

$$\mathbf{C}_{IB} = \begin{cases} \mathbf{1} + \frac{\sin(\|\boldsymbol{\phi}\|)\hat{\boldsymbol{\phi}}}{\|\boldsymbol{\phi}\|} + \frac{(1-\cos(\|\boldsymbol{\phi}\|))\hat{\boldsymbol{\phi}}^2}{\|\boldsymbol{\phi}\|^2} & \text{if } \|\boldsymbol{\phi}\| \geq \epsilon \\ \mathbf{1} + \hat{\boldsymbol{\phi}} & \text{otherwise} \end{cases}$$

Euler Angles ZYX \Leftrightarrow Rotation Matrix

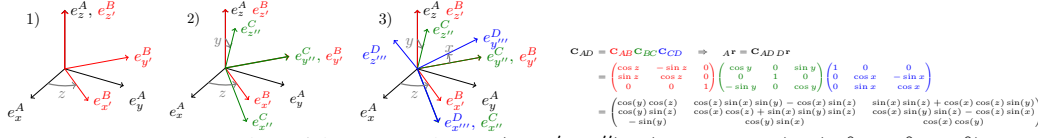


Figure 1: Rotation from A-frame to D-frame: $(z - y' - x'')$ - (yaw-pitch-roll) - $(50^\circ - 25^\circ - 30^\circ)$

Euler Angles XYZ \Leftrightarrow Rotation Matrix

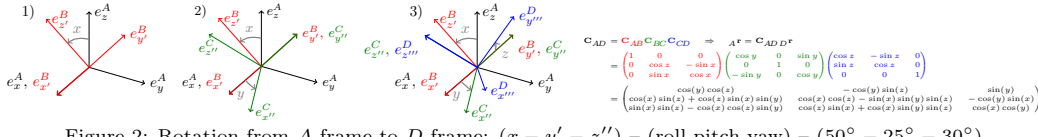


Figure 2: Rotation from A-frame to D-frame: $(x - y' - z'')$ - (roll-pitch-yaw) - $(50^\circ - 25^\circ - 30^\circ)$

Homogeneous Transformation Matrix

$$\begin{bmatrix} I\mathbf{r}_{IP} \\ 1 \end{bmatrix} = \mathbf{T}_{IB} \begin{bmatrix} B\mathbf{r}_{BP} \\ 1 \end{bmatrix}, \quad \mathbf{T}_{IB} = \begin{bmatrix} \mathbf{C}_{IB} & I\mathbf{r}_{IB} \\ \mathbf{0}^\top & 1 \end{bmatrix}, \quad \mathbf{T}_{IB}^{-1} = \mathbf{T}_{BI} = \begin{bmatrix} \mathbf{C}_{IB}^\top & -\mathbf{C}_{IB}^\top I\mathbf{r}_{IB} \\ \mathbf{0}^\top & 1 \end{bmatrix}$$

Time Derivatives of Position & Orientation

Linear Velocity

Velocity of point P expressed in a rotating frame B w.r.t. to the inertial frame I using a moving point A :

$$B\mathbf{v}_P = B\mathbf{v}_A + B\dot{\mathbf{r}}_{AP} + B\boldsymbol{\omega}_{IB} \times B\mathbf{r}_{AP}$$

Velocity of point Q on rigid body B that rotates with $B\boldsymbol{\Omega}$, where point P is on the same rigid body B :

$$B\mathbf{v}_Q = B\mathbf{v}_P + B\boldsymbol{\Omega} \times B\mathbf{r}_{PQ}, \quad B\boldsymbol{\Omega} = B\boldsymbol{\omega}_{IB}$$

Angular Velocity

$B\boldsymbol{\omega}_{IB} =: B\boldsymbol{\Omega}$	(local) absolute angular velocity of rigid body B expr. in frame B
$B\boldsymbol{\omega}_{IB} = -B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
$I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB} B\boldsymbol{\omega}_{IB}$	(global) angular velocity from frame B to frame I
$I\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB} B\dot{\boldsymbol{\omega}}_{IB} \mathbf{C}_{IB}^\top$	coord. transformation of angular velocity from frame B to frame I
$D\boldsymbol{\omega}_{AD} = D\boldsymbol{\omega}_{AB} + D\boldsymbol{\omega}_{BC} + D\boldsymbol{\omega}_{CD}$	composition of (relative) angular velocity

Derivatives

The derivation of the following identities can be found in [1].

$\frac{\partial}{\partial t} \Phi_{BI}(t) = -B\boldsymbol{\omega}_{IB}(t)$	$\frac{\partial}{\partial \mathbf{v}} (\exp(\mathbf{v})) = \boldsymbol{\Gamma}(\mathbf{v})$	$\Phi_1 \boxplus \Phi_2 = -(\Phi_2 \boxplus \Phi_1)$
$\frac{\partial}{\partial \mathbf{r}} (\Phi(\mathbf{r})) = \mathbf{C}(\Phi)$	$\frac{\partial}{\partial \Phi} (\log(\Phi)) = \boldsymbol{\Gamma}^{-1}(\log \Phi)$	$\mathbf{C}(\Phi_1 \Phi_2) = \mathbf{C}(\Phi_1) \mathbf{C}(\Phi_2)$
$\frac{\partial}{\partial \Phi} (\Phi(\mathbf{r})) = -(\Phi(\mathbf{r}))^\wedge$	$\frac{\partial}{\partial \Phi_1} (\Phi_1 \boxplus \Phi_2) = \boldsymbol{\Gamma}^{-1}(\Phi_1 \boxplus \Phi_2)$	$(\mathbf{C}(\Phi))^{-1} = \mathbf{C}(\Phi^{-1})$
$\frac{\partial}{\partial \Phi} (\Phi^{-1}) = -\mathbf{C}(\Phi)^\top$	$\frac{\partial}{\partial \Phi_2} (\Phi_1 \boxplus \Phi_2) = \boldsymbol{\Gamma}^{-1}(-(\Phi_1 \boxplus \Phi_2))$	$\boldsymbol{\Gamma}(\mathbf{v})\mathbf{v} = \mathbf{v}$
$\frac{\partial}{\partial \Phi_1} (\Phi_1 \otimes \Phi_2) = \mathbf{1}$	$\frac{d}{d\mathbf{v}_1} (\hat{\mathbf{v}}_1 \mathbf{v}_2) = -\hat{\mathbf{v}}_2$	$\boldsymbol{\Gamma}^{-1}(\mathbf{v})\mathbf{v} = \mathbf{v}$
$\frac{\partial}{\partial \Phi_2} (\Phi_1 \otimes \Phi_2) = \mathbf{C}(\Phi_1)$	$\frac{d}{d\mathbf{v}_2} (\hat{\mathbf{v}}_1 \mathbf{v}_2) = \hat{\mathbf{v}}_1$	$\boldsymbol{\Gamma}(-\mathbf{v}) = \boldsymbol{\Gamma}(\mathbf{v})^\top$
		$\boldsymbol{\Gamma}^{-1}(-\mathbf{v}) = \boldsymbol{\Gamma}(\mathbf{v}) + \hat{\mathbf{v}}$
		$\boldsymbol{\Gamma}^{-1}(\mathbf{v}) \cdot \mathbf{C}(\mathbf{v}) = \boldsymbol{\Gamma}^{-1}(\mathbf{v}) + \hat{\mathbf{v}}$
		$\boldsymbol{\Gamma}^{-1}(\mathbf{v}) \approx \mathbf{1} - \frac{1}{2} \hat{\mathbf{v}}, \quad \ \mathbf{v}\ \approx 0$

$$\text{Jacobian of exponential map: } \boldsymbol{\Gamma}(\mathbf{v}) = \begin{cases} \mathbf{1} + \frac{1-\cos(\|\mathbf{v}\|)\hat{\mathbf{v}}}{\|\mathbf{v}\|^2} + \frac{(\|\mathbf{v}\|-\sin(\|\mathbf{v}\|))\hat{\mathbf{v}}^2}{\|\mathbf{v}\|^3} & \text{if } \|\mathbf{v}\| \neq 0 \\ \mathbf{1} + \frac{1}{2} \hat{\mathbf{v}} & \text{otherwise} \end{cases}$$

Time Derivative of Rotation Matrix \Leftrightarrow Angular Velocity

$$I\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB} \mathbf{C}_{IB}^\top = \dot{\mathbf{C}}_{BI}^\top \mathbf{C}_{BI} \Leftrightarrow \dot{\mathbf{C}}_{IB} = I\dot{\boldsymbol{\omega}}_{IB} \mathbf{C}_{IB}$$

$$B\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^\top \dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI} \dot{\mathbf{C}}_{BI}^\top \Leftrightarrow \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB} B\dot{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Quaternion \Leftrightarrow Angular Velocity

$$\begin{aligned} I\dot{\boldsymbol{\omega}}_{IB} &= 2\mathbf{H}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} & \Leftrightarrow & \dot{\mathbf{q}}_{IB} = \frac{1}{2}\mathbf{H}(\mathbf{q}_{IB})^\top I\dot{\boldsymbol{\omega}}_{IB} \\ B\dot{\boldsymbol{\omega}}_{IB} &= 2\tilde{\mathbf{H}}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} & \Leftrightarrow & \dot{\mathbf{q}}_{IB} = \frac{1}{2}\tilde{\mathbf{H}}(\mathbf{q}_{IB})^\top B\dot{\boldsymbol{\omega}}_{IB} \\ \mathbf{H}(\mathbf{q}) &= \begin{bmatrix} -\hat{\mathbf{q}} & \hat{\mathbf{q}} + q_0 \mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} & \tilde{\mathbf{H}}(\mathbf{q}) &= \begin{bmatrix} -\hat{\mathbf{q}} & -\hat{\mathbf{q}} + q_0 \mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \end{aligned}$$

Time Derivative of Angle-Axis \Leftrightarrow Angular Velocity

$$I\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{n}}\dot{\theta} + \dot{\mathbf{n}} \sin \theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos \theta)$$

$$B\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{n}}\dot{\theta} + \dot{\mathbf{n}} \sin \theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos \theta)$$

$$\dot{\theta} = \mathbf{n}^\top I\dot{\boldsymbol{\omega}}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2} \frac{\sin \theta}{1 - \cos \theta} \hat{\mathbf{n}}^2 - \frac{1}{2} \hat{\mathbf{n}} \right) I\dot{\boldsymbol{\omega}}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\}$$

$$\dot{\theta} = \mathbf{n}^\top B\dot{\boldsymbol{\omega}}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2} \frac{\sin \theta}{1 - \cos \theta} \hat{\mathbf{n}}^2 + \frac{1}{2} \hat{\mathbf{n}} \right) B\dot{\boldsymbol{\omega}}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\}$$

Time Derivative of Rotation Vector \Leftrightarrow Angular Velocity

$$I\dot{\boldsymbol{\omega}}_{IB} = \left(\mathbf{1}_{3 \times 3} + \hat{\boldsymbol{\phi}} \left(\frac{1 - \cos \|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2} \right) + \hat{\boldsymbol{\phi}}^2 \left(\frac{\|\boldsymbol{\phi}\| - \sin \|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3} \right) \right) \dot{\boldsymbol{\phi}} \quad \forall \|\boldsymbol{\phi}\| \in \mathbb{R} \setminus \{0\}$$

$$B\dot{\boldsymbol{\omega}}_{IB} = \left(\mathbf{1}_{3 \times 3} - \hat{\boldsymbol{\phi}} \left(\frac{1 - \cos \|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2} \right) + \hat{\boldsymbol{\phi}}^2 \left(\frac{\|\boldsymbol{\phi}\| - \sin \|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3} \right) \right) \dot{\boldsymbol{\phi}} \quad \forall \|\boldsymbol{\phi}\| \in \mathbb{R} \setminus \{0\}$$

$$\dot{\boldsymbol{\phi}} = \left(\mathbf{1}_{3 \times 3} - \frac{1}{2} \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2 \frac{1}{\|\boldsymbol{\phi}\|^2} \left(1 - \frac{\|\boldsymbol{\phi}\|}{2} \frac{\sin \|\boldsymbol{\phi}\|}{1 - \cos \|\boldsymbol{\phi}\|} \right) \right) I\dot{\boldsymbol{\omega}}_{IB} \quad \forall \|\boldsymbol{\phi}\| \in \mathbb{R} \setminus \{0\}$$

$$\dot{\boldsymbol{\phi}} = \left(\mathbf{1}_{3 \times 3} + \frac{1}{2} \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2 \frac{1}{\|\boldsymbol{\phi}\|^2} \left(1 - \frac{\|\boldsymbol{\phi}\|}{2} \frac{\sin \|\boldsymbol{\phi}\|}{1 - \cos \|\boldsymbol{\phi}\|} \right) \right) B\dot{\boldsymbol{\omega}}_{IB} \quad \forall \|\boldsymbol{\phi}\| \in \mathbb{R} \setminus \{0\}$$

Time Derivative of Euler Angles ZYX \Leftrightarrow Angular Velocity

$$\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin(x)}{\cos(y)} & \frac{\cos(x)}{\cos(y)} \\ 0 & \cos(x) & -\sin(x) \\ 1 & \frac{\sin(x) \sin(y)}{\cos(y)} & \frac{\cos(x) \sin(y)}{\cos(y)} \end{bmatrix} B\dot{\boldsymbol{\omega}}_{IB} \quad \forall y \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$$\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \cos(z) \sin(y) & \sin(y) \sin(z) & 1 \\ \cos(y) & \cos(z) & 0 \\ -\sin(z) & \sin(z) & 0 \end{bmatrix} I\dot{\boldsymbol{\omega}}_{IB} \quad \forall y \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$$B\dot{\boldsymbol{\omega}}_{IB} = \begin{bmatrix} -\sin(y) & 0 & 1 \\ \cos(y) \sin(x) & \cos(x) & 0 \\ \cos(x) \cos(y) & -\sin(x) & 0 \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix}$$

$$I\dot{\boldsymbol{\omega}}_{IB} = \begin{bmatrix} 0 & -\sin(z) & \cos(y) \cos(z) \\ 0 & \cos(z) & \cos(y) \sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix} \begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix}$$

Dynamics of a Multi-Rigid-Body System

$DoFs$	Degrees of Freedom
n	Number of bodies in system
n_j	Number of DoFs of the joints
n_q	Number of generalized coordinates
n_u	Number of generalized velocities
\mathbf{M}	Mass matrix
\mathbf{g}	Gyroscopic and Coriolis forces
\mathbf{f}	Generalized external forces and torques
\mathbf{h}	Combined force vector
\mathbf{J}_P	Jacobi matrix for translation of point P
\mathbf{J}_R	Jacobi matrix for rotation
\mathbf{f}_Q^A	External forces on point Q
\mathbf{t}_Q^A	External torques
m	Mass
Θ	Inertia tensor
$(\dots)^-$	Variable before impact
$(\dots)^+$	Variable after impact
$(\dots)^\pm$	Variable before/after impact
Δt	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
\mathbf{W}	Generalized force directions for contact forces
λ	Lebesgue-measurable contact forces
Λ	Purely atomic impact impulses
\mathbf{P}	Contact percussions
COM	Center of mass

Generalized Coordinates of a Floating-Base System with Rotational Joints

Recommended set of generalized coordinates \mathbf{q} with quaternion \mathbf{q}_{IB} and generalized velocities \mathbf{u} :

$$\mathbf{q} = \begin{pmatrix} {}^I\mathbf{r}_{IB} \\ \mathbf{q}_{IB} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} {}^I\mathbf{v}_B \\ {}^B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} {}^I\mathbf{a}_B \\ {}^B\dot{\boldsymbol{\psi}}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j \times n_j} \end{pmatrix} \Leftrightarrow \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j \times n_j} \end{pmatrix}$$

$$\begin{bmatrix} {}^I\mathbf{v}_{IQ} \\ {}^I\boldsymbol{\omega}_{IQ} \end{bmatrix} = {}^I\mathbf{J}_Q(\mathbf{q}) \cdot \mathbf{u}, \quad {}^I\mathbf{J}_Q(\mathbf{q}) = \begin{bmatrix} {}^I\mathbf{J}_P(\mathbf{q}) \\ {}^I\mathbf{J}_R(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} \mathbb{1}_{3 \times 3} & -\mathbf{C}_{IB} \cdot {}^B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}^B\mathbf{J}_{Pq_j}(\mathbf{q}_j) \\ \mathbf{0}_{3 \times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}^B\mathbf{J}_{Rq_j}(\mathbf{q}_j) \end{bmatrix}$$

Equations of Motion with Contacts and no Impulses

Projected Newton-Euler Equations

$$\mathbf{M} = \sum_{i=1}^n \left[(\mathbf{J}_{\text{COM}}^\top m \mathbf{J}_{\text{COM}} + \mathbf{J}_R^\top \Theta_{\text{COM}} \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\lambda} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \quad \mathbf{g} = \sum_{i=1}^n \left[(\mathbf{J}_{\text{COM}}^\top m \dot{\mathbf{J}}_{\text{COM}} \mathbf{u} + \mathbf{J}_R^\top (\Theta_{\text{COM}} \dot{\mathbf{J}}_R \mathbf{u} + \boldsymbol{\Omega} \times \Theta_{\text{COM}} \boldsymbol{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^n \left[(\mathbf{J}_Q^\top \mathbf{f}_Q^A + \mathbf{J}_R^\top \mathbf{t}^A) \right]_i$$

Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta \mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{array}{l} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\Lambda \\ \mathbf{M} \underbrace{(\dot{\mathbf{u}}dt + (\mathbf{u}^+ - \mathbf{u}^-)d\eta)}_{d\mathbf{u}} - \mathbf{h}dt = \mathbf{W} \underbrace{(\lambda dt + \Lambda d\eta)}_{d\mathbf{P}} \end{array} \right.$$

Transformation of Equations of Motion

Transformation from $\bar{\mathbf{M}}(\bar{\mathbf{q}}), \bar{\mathbf{h}}(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ to $\mathbf{M}(\mathbf{q}), \mathbf{h}(\mathbf{q}, \mathbf{u})$, where $\bar{\mathbf{u}} = \mathbf{B}\mathbf{u}$:

$$\mathbf{M} = \mathbf{B}^\top \bar{\mathbf{M}} \mathbf{B}$$

$$\mathbf{h} = \mathbf{B}^\top \bar{\mathbf{h}} - \mathbf{B}^\top \bar{\mathbf{M}} \dot{\mathbf{B}} \mathbf{u}$$

Appendix I: Euler Angles ZYX Velocities to Angular Velocity Mapping

Given a set of Euler angles $\chi = [z \quad y \quad x]^T$ and velocities $\dot{\chi} = [\dot{z} \quad \dot{y} \quad \dot{x}]^T$, we wish to find the mapping $\mathbf{E}(\chi) \in \mathbb{R}^{3 \times 3}$ that maps $\dot{\chi}$ to ${}_I\boldsymbol{\omega}_{IB}$:

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{E}(\chi) \cdot \dot{\chi} \quad (1)$$

The columns of $\mathbf{E}(\chi)$ are the components of the unit vectors around which the rotational velocities are applied expressed in fixed frame. These are obtained by selecting the columns of a rotation matrix which is built up by successive elementary rotations specified by the Euler angle parametrization.

Starting from the reference frame I , the first rotation will be an elementary rotation around ${}_I\mathbf{e}_I^z$, which is simply given by:

$${}_I\mathbf{e}_I^z = \mathbb{I}_{3 \times 3} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2)$$

After an elementary rotation around ${}_I\mathbf{e}_I^z$, the y axis ${}_I\mathbf{e}_{I'}^y$, will be expressed by:

$${}_I\mathbf{e}_{I'}^y = \mathbf{C}_{II'}(z) \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sin(z) \\ \cos(z) \\ 0 \end{bmatrix} \quad (3)$$

After an elementary rotation around ${}_I\mathbf{e}_{I'}^y$, the x axis ${}_I\mathbf{e}_{I''}^x$, will be expressed by:

$${}_I\mathbf{e}_{I''}^x = \mathbf{C}_{II'}(z) \cdot \mathbf{C}_{I'I''}(y) \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(z) & -\sin(z) & 0 \\ \sin(z) & \cos(z) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(y) & 0 & \sin(y) \\ 0 & 1 & 0 \\ -\sin(y) & 0 & \cos(z) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(y)\cos(z) \\ \cos(y)\sin(z) \\ -\sin(z) \end{bmatrix} \quad (4)$$

Finally, the mapping $\mathbf{E}(\chi)$ will be computed as:

$$\mathbf{E}(\chi) = [{}_I\mathbf{e}_I^z \quad {}_I\mathbf{e}_{I'}^y \quad {}_I\mathbf{e}_{I''}^x] = \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix} \quad (5)$$

It is easy to find that $\det(\mathbf{E}(\chi)) = -\cos(y)$. The mapping then becomes singular when $y = \pi/2 + k\pi, \forall k \in \mathbb{Z}$. This means that although we can always describe an angular velocity using Euler angle velocities, the inverse is not always possible. The inverse mapping is given by:

$$\bar{\mathbf{E}}(\chi) = \mathbf{E}^{-1}(\chi) = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ \frac{\cos(y)}{-\sin(z)} & \frac{\cos(y)}{\cos(z)} & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} \quad (6)$$

If we compute the rotation matrix $\mathbf{C}_{IB} = \mathbf{C}_{IB}(z, y, x) = \mathbf{C}_z(z) \cdot \mathbf{C}_y(y) \cdot \mathbf{C}_x(x)$, we can also derive the following mappings:

$${}_B\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}^T \cdot \mathbf{E}(\chi) \cdot \dot{\chi} \quad (7)$$

$$\dot{\chi} = \bar{\mathbf{E}}(\chi) \cdot \mathbf{C}_{IB} \cdot {}_B\boldsymbol{\omega}_{IB} \quad (8)$$

The mapping described by (8) is valid $\forall y \neq \pi/2 + k\pi, \forall k \in \mathbb{Z}$.

Appendix II: Jacobians

We wish to derive the relationship between the generalized velocities \mathbf{u} and the operational space velocities ${}_I\mathbf{v}_Q$ of a point Q , which is fixed at the end of a kinematic chain that stems from a floating base B . The position vector ${}_I\mathbf{r}_{IQ} = {}_I\mathbf{r}_{IQ}(\mathbf{q})$ of a point w.r.t. the inertial frame I is given by:

$${}_I\mathbf{r}_{IQ}(\mathbf{q}) = {}_I\mathbf{r}_{IB}(\mathbf{q}) + \mathbf{C}_{IB}(\mathbf{q}) \cdot {}_B\mathbf{r}_{BQ}(\mathbf{q}), \quad (9)$$

where the rotation matrix $\mathbf{C}_{IB}(\mathbf{q})$ describes the orientation of the floating base B w.r.t. the inertial frame I , ${}_I\mathbf{r}_{IB}(\mathbf{q})$ represents the position of the floating base B w.r.t. the inertial frame I expressed in the inertial frame and $\mathbf{q} = \mathbf{q}(t)$ is a function of time t .

Time differentiation yields:

$$\begin{aligned} {}_I\mathbf{v}_Q &= {}_I\mathbf{v}_B + \dot{\mathbf{C}}_{IB} \cdot {}_B\mathbf{r}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B + \mathbf{C}_{IB} \cdot {}_B\dot{\boldsymbol{\omega}}_{IB} \cdot {}_B\mathbf{r}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B - \mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} \cdot {}_B\boldsymbol{\omega}_{IB} + \mathbf{C}_{IB} \cdot {}_B\dot{\mathbf{r}}_{BQ} \\ &= {}_I\mathbf{v}_B - \mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} \cdot {}_B\boldsymbol{\omega}_{IB} + \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \cdot \dot{\mathbf{q}}_j \\ &= \begin{bmatrix} \mathbb{I}_{3 \times 3} & -\mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \end{bmatrix} \cdot \mathbf{u} \end{aligned} \quad (10)$$

If we attach a frame at ${}_I\mathbf{r}_{IQ}$, we can derive a similar mapping for angular velocities. The orientation of frame Q w.r.t. the inertial frame I is described by:

$$\mathbf{C}_{IQ} = \mathbf{C}_{IB} \cdot \mathbf{C}_{BQ} \quad (11)$$

Time differentiation of both sides of (11) yields:

$$\begin{aligned} {}_I\dot{\boldsymbol{\omega}}_{IQ} \cdot \mathbf{C}_{IQ} &= {}_I\dot{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IB} \cdot \mathbf{C}_{BQ} + \mathbf{C}_{IB} \cdot {}_B\dot{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{BQ} \\ &= {}_I\dot{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IQ} + \mathbf{C}_{IB} \cdot \mathbf{C}_{BI} \cdot {}_I\dot{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{BI}^T \cdot \mathbf{C}_{BQ} \\ &= {}_I\dot{\boldsymbol{\omega}}_{IB} \cdot \mathbf{C}_{IQ} + {}_I\dot{\boldsymbol{\omega}}_{BQ} \cdot \mathbf{C}_{IQ}, \end{aligned} \quad (12)$$

which gives finally:

$$\begin{aligned} {}_I\boldsymbol{\omega}_{IQ} &= {}_I\boldsymbol{\omega}_{IB} + {}_I\boldsymbol{\omega}_{BQ} \\ &= \begin{bmatrix} \mathbb{I}_{3 \times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix} \cdot \mathbf{u} \end{aligned} \quad (13)$$

Hence, the mapping from generalized velocities \mathbf{u} to the operational space twist $\begin{bmatrix} {}_I\mathbf{v}_Q^T & {}_I\boldsymbol{\omega}_{IQ}^T \end{bmatrix}^T$ of frame Q is realized by the spatial Jacobian:

$$\begin{aligned} {}_I\mathbf{J}_Q(\mathbf{q}) &= \begin{bmatrix} {}_I\mathbf{J}_P \\ {}_I\mathbf{J}_R \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{I}_{3 \times 3} & -\mathbf{C}_{IB} \cdot {}_B\hat{\mathbf{r}}_{BQ} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{P_{q_j}}(\mathbf{q}_j) \\ \mathbf{0}_{3 \times 3} & \mathbf{C}_{IB} & \mathbf{C}_{IB} \cdot {}_B\mathbf{J}_{R_{q_j}}(\mathbf{q}_j) \end{bmatrix} \end{aligned} \quad (14)$$

Appendix III: Hessians and Time Derivatives of Jacobians

Consider a kinematic chain which connects two rigid bodies. We represent the set of indexes of the rigid bodies in this chain with U_A . As shown in [2], the i -th column of the spatial Hessian matrix of the spatial Jacobian \mathbf{J} w.r.t. the j -th configuration variable q_j can be computed as:

$$\frac{\partial \mathbf{J}_i}{\partial q_j} = \begin{bmatrix} \frac{\partial \mathbf{J}_{P_i}}{\partial q_j} \\ \frac{\partial \mathbf{J}_{R_i}}{\partial q_j} \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{J}_{R_j} \times \mathbf{J}_{P_i} \\ \mathbf{J}_{R_j} \times \mathbf{J}_{R_i} \\ \mathbf{0} \end{bmatrix} & i \geq j \\ \begin{bmatrix} \mathbf{J}_{R_i} \times \mathbf{J}_{P_j} \\ \mathbf{0} \end{bmatrix} & i < j \\ \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} & i, j \notin U_A \end{cases} \quad (15)$$

The Hessian $\mathbf{H}_k(\mathbf{q})$ is then expressed by:

$$\mathbf{H}_k(\mathbf{q}) = \begin{bmatrix} \frac{\partial \mathbf{J}_1}{\partial q_k} & \frac{\partial \mathbf{J}_2}{\partial q_k} & \dots & \frac{\partial \mathbf{J}_n}{\partial q_k} \end{bmatrix} \quad (16)$$

Knowledge of the Hessian matrix w.r.t. each configuration variable q_k enables the computation of the time derivative of $\mathbf{J}(\mathbf{q}(t))$. Its generic element i, j can be computed as:

$$\frac{dJ_{i,j}(\mathbf{q})}{dt} = \frac{\partial J_{i,j}(\mathbf{q})}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt} = \sum_{k=1}^n \frac{\partial J_{i,j}(\mathbf{q})}{\partial q_k} \cdot \frac{dq_k}{dt} = \sum_{k=1}^n H_{k_{i,j}}(\mathbf{q}) \cdot \frac{dq_k}{dt}, \quad (17)$$

which yields:

$$\frac{d\mathbf{J}(\mathbf{q})}{dt} = \frac{\partial \mathbf{J}(\mathbf{q})}{\partial \mathbf{q}} \cdot \frac{d\mathbf{q}}{dt}, = \sum_{k=1}^n \mathbf{H}_k(\mathbf{q}) \cdot \frac{dq_k}{dt}. \quad (18)$$

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