# Kindr Cheat Sheet

Kinematics and Dynamics for Robotics

## Nomenclature

| | | |
|---|---|---|
| (Hyper-)complex number | $Q$ | normal capital letter |
| Column vector | $\mathbf{a}$ | bold small letter |
| Matrix | $\mathbf{M}$ | bold capital letter |
| Identity matrix | $\mathbb{1}_{n\times m}$ | $n\times m$-matrix |
| Coordinate system (CS) | $\mathbf{e}_x^A,\mathbf{e}_y^A,\mathbf{e}_z^A$ | Cartesian right-hand system $A$ with basis (unit) vectors $\mathbf{e}$ |
| Inertial frame | $\mathbf{e}_x^I,\mathbf{e}_y^I,\mathbf{e}_z^I$ | Global / inertial / world coordinate system (never moves) |
| Body-fixed frame | $\mathbf{e}_x^B,\mathbf{e}_y^B,\mathbf{e}_z^B$ | Local / body-fixed coordinate system (moves with body) |
| Rotation | $\mathcal{R}\in SO(3)$ | generic rotation (for all parameterizations) |
| Machine precision | $\epsilon$ | |

## Operators

| | |
|---|---|
| Cross product | $\mathbf{a}\times\mathbf{b}=\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}\Leftrightarrow(\mathbf{a})^\wedge\mathbf{b}=\hat{\mathbf{a}}\mathbf{b}=\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ |
| Skew/unskew | $\mathbf{a}=\hat{\mathbf{a}}^\vee$ |
| Euclidean norm | $\|\mathbf{a}\|=\sqrt{\mathbf{a}^T\mathbf{a}}=\sqrt{a_1^2+\ldots+a_n^2}$ |
| Exponential map for matrix | $\text{expM}:\mathbb{R}^3\to\mathbb{R}^3,\mathbf{A}\mapsto e^{\mathbf{A}},\quad \mathbf{A}\in\mathbb{R}^{3\times3}$ |
| Logarithmic map for matrix | $\text{logM}:\mathbb{R}^3\to\mathbb{R}^3,\mathbf{A}\mapsto\log\mathbf{A},\quad \mathbf{A}\in\mathbb{R}^{3\times3}$ |

## Position & Orientation

### Position

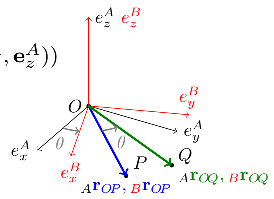
| | | |
|---|---|---|
| Vector | $\mathbf{r}_{OP}$ | from point $O$ to point $P$ |
| Position vector | ${}_A\mathbf{r}_{OP}\in\mathbb{R}^3$ | from point $O$ to point $P$ expr. in frame $A$ |
| Homogeneous pos. vector | ${}_A\bar{\mathbf{r}}_{OP}=\begin{bmatrix}{}_A\mathbf{r}_{OP}^\mathsf{T} & 1\end{bmatrix}^\mathsf{T}$ | from point $O$ to point $P$ expr. in frame $A$ |

### Orientation/Rotation

1) Active Rotation: $\mathcal{R}^A:{}_A\mathbf{r}_{OP}\mapsto{}_A\mathbf{r}_{OQ}$ (rotates the vector $\mathbf{r}_{OP}$)
2) Passive Rotation: $\mathcal{R}^P:{}_A\mathbf{r}_{OP}\mapsto{}_B\mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^A,\mathbf{e}_y^A,\mathbf{e}_z^A)$)
3) Inversion: $\mathcal{R}^{A^{-1}}(\mathbf{r})=\mathcal{R}^P(\mathbf{r})$
4) Concatenation:

$$\mathcal{R}_2^A\left(\mathcal{R}_1^A(\mathbf{r})\right)=\left(\mathcal{R}_2^A\otimes\mathcal{R}_1^A\right)(\mathbf{r})=\left(\mathcal{R}_1^{A^{-1}}\otimes\mathcal{R}_2^{A^{-1}}\right)^{-1}(\mathbf{r})$$

$$\mathcal{R}_2^P\left(\mathcal{R}_1^P(\mathbf{r})\right)=\left(\mathcal{R}_2^P\otimes\mathcal{R}_1^P\right)(\mathbf{r})=\left(\mathcal{R}_1^{P^{-1}}\otimes\mathcal{R}_2^{P^{-1}}\right)^{-1}(\mathbf{r})$$




5) Exponential map: $\exp:\mathbb{R}^3\to SO(3),\mathbf{v}\mapsto\text{expM}(\hat{\mathbf{v}}),\quad \mathbf{v}\in\mathbb{R}^3$
6) Logarithmic map: $\log:SO(3)\to\mathbb{R}^3,\mathcal{R}\mapsto\text{logM}(\mathcal{R})^\vee,\quad \mathcal{R}\in SO(3)$
7) Box plus: $\mathcal{R}_2=\mathcal{R}_1\boxplus\mathbf{v}=\exp(\mathbf{v})\otimes\mathcal{R}_1,\quad \mathcal{R}_1,\mathcal{R}_2\in SO(3),\mathbf{v}\in\mathbb{R}^3$
8) Box minus: $\mathbf{v}=\mathcal{R}_1\boxminus\mathcal{R}_2=\log\left(\mathcal{R}_1\otimes\mathcal{R}_2^{-1}\right),\quad \mathcal{R}_1,\mathcal{R}_2\in SO(3),\mathbf{v}\in\mathbb{R}^3$
9) Discrete integration: $\mathcal{R}^{k+1}=\mathcal{R}^k\boxplus({}_B\boldsymbol{\omega}_{IB}\Delta t)$

| | | |
|---|---|---|
| Rotation Matrix | $\mathbf{R}_{BA}\in SO(3)$<br>${}_A\mathbf{r}_{OQ}=\mathbf{R}_{BA}\,{}_A\mathbf{r}_{OP}$<br>${}_B\mathbf{r}_{OP}=\mathbf{R}_{BA}^\mathsf{T}\,{}_A\mathbf{r}_{OP}$ | Maps the coord. of the basis vectors $({}_A\mathbf{e}_x^A,{}_A\mathbf{e}_y^A,{}_A\mathbf{e}_z^A)$ of frame $A$ expressed in $A$ into the coordinates of the basis vectors $({}_A\mathbf{e}_x^B,{}_A\mathbf{e}_y^B,{}_A\mathbf{e}_z^B)$ of $B$ expressed in $A$.<br>The rotation is active (alibi).<br>${}_A\mathbf{R}_{BA}=\begin{bmatrix}{}_A\mathbf{e}_x^B & {}_A\mathbf{e}_y^B & {}_A\mathbf{e}_z^B\end{bmatrix}$ |
| Direct Cosine Matrix | $\mathbf{C}_{BA}\in SO(3)$<br>${}_B\mathbf{r}_{OP}=\mathbf{C}_{BA}\,{}_A\mathbf{r}_{OP}$<br>$\mathbf{C}_{BA}=\mathbf{R}_{BA}^\mathsf{T}$ | The coordinate tranformation matrix, which transforms vectors from frame $A$ to frame $B$.<br>The rotation is passive (alias). |
| Rotation Quaternion | $\mathbf{p}_{BA}$<br>$\mathbf{p}_{BA}\Leftrightarrow\mathbf{R}_{BA}$ | The rotation is active (alibi). |
| Rotation Angle-axis | $(\theta,\mathbf{n})_{BA}$<br>$(\theta,\mathbf{n})_{BA}\Leftrightarrow\mathbf{R}_{BA}$ | Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta\in[0,\pi]$.<br>The rotation is active (alibi). |
| Rotation Vector | $\boldsymbol{\phi}_{BA}$<br>$\boldsymbol{\phi}_{BA}\Leftrightarrow\mathbf{R}_{BA}$ | Rotation with rotation axis $\mathbf{n}=\frac{\boldsymbol{\phi}}{\|\boldsymbol{\phi}\|}$ and angle $\theta=\|\boldsymbol{\phi}\|$.<br>The rotation is active (alibi). |
| Euler Angles ZYX<br>Euler Angles YPR | $(\psi,\theta,\phi)_{BA}$<br>$(\psi,\theta,\phi)_{BA}\Leftrightarrow\mathbf{C}_{BA}$ | Tait-Bryan angles (Flight conv.): $z-y'-x''$, i.e. yaw-pitch-roll. Singularities are at $\theta=\pm\frac{\pi}{2}$.<br>$\psi\in[-\pi,\pi),\theta\in[-\frac{\pi}{2},\frac{\pi}{2}),\phi\in[-\pi,\pi)$ |
| Euler Angles XYZ<br>Euler Angles RPY | $(\alpha,\beta,\gamma)_{BA}$<br>$(\alpha,\beta,\gamma)_{BA}\Leftrightarrow\mathbf{C}_{BA}$ | Cardan angles (Glocker conv.): $x-y'-z''$, i.e. roll-pitch-yaw. Singularities are at $\beta=\pm\frac{\pi}{2}$.<br>$\alpha\in[-\pi,\pi),\beta\in[-\frac{\pi}{2},\frac{\pi}{2}),\gamma\in[-\pi,\pi)$ |

### Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$\boxed{\begin{array}{l}P=p_0+p_1i+p_2j+p_3k\in\mathbb{H},\quad p_i\in\mathbb{R}\\ i^2=j^2=k^2=ijk=-1,\quad \|P\|=\sqrt{p_0^2+p_1^2+p_2^2+p_3^2}=1\end{array}}$$

Note that $P_{BA}$ and $-P_{BA}$ represent the same rotation, but not the same unit quaternion.
Rot. quaternion as tuple: $P=(p_0,p_1,p_2,p_3)=(p_0,\vec{\mathbf{p}})$ with $\vec{\mathbf{p}}:=(p_1,p_2,p_3)^\mathsf{T}$
Rot. quaternion as vector: $\mathbf{p}=\begin{bmatrix}p_0 & p_1 & p_2 & p_3\end{bmatrix}^\mathsf{T}$
Conjugate: $P^*=(p_0,-\vec{\mathbf{p}})$
Inverse: $P^{-1}=P^*=(p_0,-\vec{\mathbf{p}})$
Quaternion multiplication:

$$Q\cdot P=(q_0,\vec{\mathbf{q}})\cdot(p_0,\vec{\mathbf{p}})=(q_0p_0-\vec{\mathbf{q}}^\mathsf{T}\vec{\mathbf{p}},q_0\vec{\mathbf{p}}+p_0\vec{\mathbf{q}}+\vec{\mathbf{q}}\times\vec{\mathbf{p}})\quad\Leftrightarrow$$

$$\mathbf{q}\otimes\mathbf{p}=\underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}}\mathbf{p}=\begin{pmatrix}q_0 & -\vec{\mathbf{q}}^\mathsf{T}\\ \vec{\mathbf{q}} & q_0\mathbb{1}_{3\times3}+\hat{\vec{\mathbf{q}}}\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}=\begin{pmatrix}q_0&-q_1&-q_2&-q_3\\q_1&q_0&-q_3&q_2\\q_2&q_3&q_0&-q_1\\q_3&-q_2&q_1&q_0\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}$$

$$=\underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}}\mathbf{q}=\begin{pmatrix}p_0 & -\vec{\mathbf{p}}^\mathsf{T}\\ \vec{\mathbf{p}} & p_0\mathbb{1}_{3\times3}-\hat{\vec{\mathbf{p}}}\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}=\begin{pmatrix}p_0&-p_1&-p_2&-p_3\\p_1&p_0&p_3&-p_2\\p_2&-p_3&p_0&p_1\\p_3&p_2&-p_1&p_0\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}$$

### Rotation Quaternion ⇔ Rotation Angle-Axis

$$\mathbf{p}_{BI}=\begin{bmatrix}\cos\frac{\theta}{2}\\ \mathbf{n}\sin\frac{\theta}{2}\end{bmatrix}\quad\Leftrightarrow\quad(\theta,\mathbf{n})_{BI}=\begin{cases}\left(2\arccos(p_0),\frac{\vec{\mathbf{p}}}{\|\vec{\mathbf{p}}\|}\right) & \text{if } \|\vec{\mathbf{p}}\|^2\geq\epsilon^2\\ \left(0,\begin{bmatrix}1 & 0 & 0\end{bmatrix}^\mathsf{T}\right) & \text{otherwise}\end{cases}$$

### Rotation Quaternion ⇔ Direction Cosine Matrix

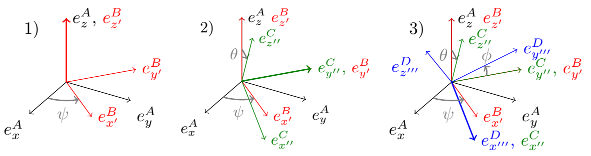
$$\mathbf{C}_{AB}=\mathbf{R}_{AB}^\mathsf{T}(\mathbf{p}_{AB})=\mathbb{1}_{3\times3}+2p_0\hat{\vec{\mathbf{p}}}+2\hat{\vec{\mathbf{p}}}^2$$

$$=\begin{bmatrix}p_0^2+p_1^2-p_2^2-p_3^2 & 2p_1p_2-2p_0p_3 & 2p_0p_2+2p_1p_3\\ 2p_0p_3+2p_1p_2 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_2p_3-2p_0p_1\\ 2p_1p_3-2p_0p_2 & 2p_0p_1+2p_2p_3 & p_0^2-p_1^2-p_2^2+p_3^2\end{bmatrix}$$

$$\mathbf{C}_{BA} = \mathbf{R}_{BA}^{\mathsf{T}} = \mathbf{R}_{AB}(\mathbf{p}_{AB}) = \mathbb{1}_{3\times 3} - 2p_0\hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2$$
$$= \begin{bmatrix} p_0^2+p_1^2-p_2^2-p_3^2 & 2p_0p_3+2p_1p_2 & 2p_1p_3-2p_0p_2 \\ 2p_1p_2-2p_0p_3 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_0p_1+2p_2p_3 \\ 2p_0p_2+2p_1p_3 & 2p_2p_3-2p_0p_1 & p_0^2-p_1^2-p_2^2+p_3^2 \end{bmatrix}$$

$$\mathbf{p}_{BA} = \mathbf{p}_{BA}(\mathbf{C}_{AB}) = \begin{bmatrix} \frac{1}{2}\sqrt{1+\mathrm{tr}(\mathbf{C})} \\ \frac{C_{32}-C_{23}}{4p_0} \\ \frac{C_{13}-C_{31}}{4p_0} \\ \frac{C_{21}-C_{12}}{4p_0} \end{bmatrix} \quad \text{if } \mathrm{tr}(\mathbf{C}) > 0 \ (\mathbf{C}_{AB} \to \mathbf{p}_{BA} \text{ is not unique})$$

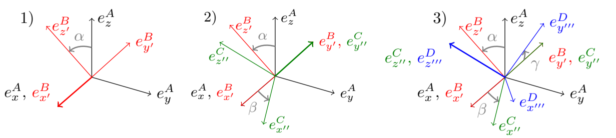
### Euler Angles ZYX ⇔ Direction Cosine Matrix



$$\mathbf{C}_{DA} = \mathbf{C}_{DC}\mathbf{C}_{CB}\mathbf{C}_{BA} \quad \Rightarrow \quad {}_D\mathbf{r} = \mathbf{C}_{DA}\,{}_A\mathbf{r}$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos\psi\cos\theta & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi & \cos\phi\cos\theta \end{pmatrix}$$
$$\mathbf{C}_{AD} = \mathbf{C}_{DA}^{\mathsf{T}}$$
$$= \begin{pmatrix} \cos\psi\cos\theta & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{pmatrix}$$

Figure 1: Rotation from $I$-frame to $B$-frame: $(z - y' - x'')$ – (yaw-pitch-roll) – $(\psi - \theta - \phi)$ – $(50° - 25° - 30°)$

### Euler Angles XYZ ⇔ Direction Cosine Matrix



$$\mathbf{C}_{DA} = \mathbf{C}_{DC}\mathbf{C}_{CB}\mathbf{C}_{BA} \quad \Rightarrow \quad {}_D\mathbf{r} = \mathbf{C}_{DA}\,{}_A\mathbf{r}$$
$$= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$
$$= \begin{pmatrix} \cos\beta\cos\gamma & \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta \\ -\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma \\ \sin\beta & -\cos\beta\sin\alpha & \cos\alpha\cos\beta \end{pmatrix}$$
$$\mathbf{C}_{AD} = \mathbf{C}_{DA}^{\mathsf{T}}$$
$$= \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha \\ \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta & \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix}$$

Figure 2: Rotation from $I$-frame to $B$-frame: $(x - y' - z'')$ – (roll-pitch-yaw) – $(\alpha - \beta - \gamma)$ – $(50° - 25° - 30°)$

## Pose

| | | |
|---|---|---|
| Homogeneous Transformation Matrix | $\mathbf{T}_{AB}$ | |

### Homogeneous Transformation Matrix

$$\mathbf{T}_{AB} = \begin{bmatrix} \mathbf{C}_{AB} & {}_A\mathbf{r}_{AB} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

# Time Derivatives of Position & Orientation

## Linear Velocity

Velocity of point $P$ expressed in frame $B$ w.r.t. to the inertial frame $I$:

${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$

Velocity of point $Q$ on rigid body $B$ that rotates with ${}_B\boldsymbol{\Omega}$, where point $P$ is on the same rigid body $B$:

${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$

## Angular Velocity

| | |
|---|---|
| ${}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$ | absolute angular velocity of rigid body $B$ expressed in frame $B$ |
| ${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$ | inverse of angular velocity |
| ${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$ | coord. transformation of angular velocity from frame $B$ to frame $I$ |
| ${}_I\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^{\mathsf{T}}$ | coord. transformation of angular velocity from frame $B$ to frame $I$ |
| ${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$ | composition of angular velocity |

### Time Derivative of Direction Cosine Matrix ⇔ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^{\mathsf{T}} = \dot{\mathbf{C}}_{BI}^{\mathsf{T}}\mathbf{C}_{BI}$$
$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^{\mathsf{T}}\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\dot{\mathbf{C}}_{BI}^{\mathsf{T}} \qquad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

### Time Derivative of Rotation Matrix ⇔ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{BI}\mathbf{R}_{BI}^{\mathsf{T}} = \dot{\mathbf{R}}_{IB}^{\mathsf{T}}\mathbf{R}_{IB} \qquad \dot{\mathbf{R}}_{IB} = \mathbf{R}_{IB}\,{}_I\hat{\boldsymbol{\omega}}_{IB}^{\mathsf{T}} \qquad \dot{\mathbf{R}}_{BI} = {}_I\hat{\boldsymbol{\omega}}_{IB}\mathbf{R}_{BI}$$
$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{R}_{IB}\dot{\mathbf{R}}_{IB}^{\mathsf{T}} = \mathbf{R}_{BI}^{\mathsf{T}}\dot{\mathbf{R}}_{BI} \qquad \dot{\mathbf{R}}_{IB} = {}_B\hat{\boldsymbol{\omega}}_{IB}^{\mathsf{T}}\mathbf{R}_{IB} \qquad \dot{\mathbf{R}}_{BI} = \mathbf{R}_{BI}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

### Time Derivative of Rotation Quaternion ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \qquad \dot{\mathbf{p}}_{BI} = \tfrac{1}{2}\mathbf{H}(\mathbf{p}_{BI})^{\mathsf{T}}\,{}_I\boldsymbol{\omega}_{IB}$$
$${}_B\boldsymbol{\omega}_{IB} = 2\bar{\mathbf{H}}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \qquad \dot{\mathbf{p}}_{BI} = \tfrac{1}{2}\bar{\mathbf{H}}(\mathbf{p}_{BI})^{\mathsf{T}}\,{}_B\boldsymbol{\omega}_{IB}$$
$$\mathbf{H}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & \hat{\vec{\mathbf{p}}} + p_0\mathbb{1}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{3\times 4} \qquad \bar{\mathbf{H}}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & -\hat{\vec{\mathbf{p}}} + p_0\mathbb{1}_{3\times 3} \end{bmatrix} \in \mathbb{R}^{3\times 4}$$
$$= \begin{bmatrix} -p_1 & p_0 & -p_3 & p_2 \\ -p_2 & p_3 & p_0 & -p_1 \\ -p_3 & -p_2 & p_1 & p_0 \end{bmatrix} \qquad = \begin{bmatrix} -p_1 & p_0 & p_3 & -p_2 \\ -p_2 & -p_3 & p_0 & p_1 \\ -p_3 & p_2 & -p_1 & p_0 \end{bmatrix}$$

### Time Derivative of Angle-Axis ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$
$${}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$
$$\dot{\theta} = \mathbf{n}^{\mathsf{T}}\,{}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 - \tfrac{1}{2}\hat{\mathbf{n}}\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$
$$\dot{\theta} = \mathbf{n}^{\mathsf{T}}\,{}_B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 + \tfrac{1}{2}\hat{\mathbf{n}}\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$

### Time Derivative of Rotation Vector ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times 3} + \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$${}_B\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times 3} - \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times 3} - \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times 3} + \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

### Time Derivative of Euler Angles ZYX ⇔ Angular Velocity

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{\cos\psi\sin\theta}{\cos\theta} & \frac{\sin\psi\sin\theta}{\cos\theta} & 1 \\ -\sin\psi & \cos\psi & 0 \\ \frac{\cos\psi}{\cos\theta} & \frac{\sin\psi}{\cos\theta} & 0 \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$
$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \\ 0 & \cos\phi & -\sin\phi \\ 1 & \frac{\sin\phi\sin\theta}{\cos\theta} & \frac{\cos\phi\sin\theta}{\cos\theta} \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$
$${}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & -\sin\psi & \cos\psi\cos\theta \\ 0 & \cos\psi & \cos\theta\sin\psi \\ 1 & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$
$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

### Time Derivative of Euler Angles XYZ ⇔ Angular Velocity

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin\alpha\sin\beta}{\cos\beta} & -\frac{\cos\alpha\sin\beta}{\cos\beta} \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \frac{-\sin\alpha}{\cos\beta} & \frac{\cos\alpha}{\cos\beta} \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall\beta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$
$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\cos\gamma}{\cos\beta} & \frac{-\sin\gamma}{\cos\beta} & 0 \\ \sin\gamma & \cos\gamma & 0 \\ \frac{-\cos\gamma\sin\beta}{\cos\beta} & \frac{\sin\beta\sin\gamma}{\cos\beta} & 1 \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall\beta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$
$${}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\cos\beta\sin\alpha \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$
$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

# Dynamics of a Multi-Rigid-Body System

| | |
|---|---|
| $n$ | Number of bodies in system |
| $n_j$ | Number of DoFs of the joints |
| $n_q$ | Number of generalized coordinates |
| $n_u$ | Number of generalized velocities |
| $\mathbf{M}$ | Mass matrix |
| $\mathbf{g}$ | Gyroscopic and Coriolis forces |
| $\mathbf{f}$ | Generalized external forces and torques |
| $\mathbf{h}$ | Combined force vector |
| $\mathbf{J}_P$ | Jacobi matrix for translation of point P |
| $\mathbf{J}_R$ | Jacobi matrix for rotation |
| $\mathbf{F}_Q^A$ | External forces on point Q |
| $\mathbf{M}^A$ | External torques |
| $m$ | Mass |
| $\mathbf{\Theta}$ | Intertia tensor |
| $(...)^-$ | Variable before impact |
| $(...)^+$ | Variable after impact |
| $(...)^\pm$ | Variable before/after impact |
| $\Delta t$ | Time step duration |
| $\Delta\mathbf{u}$ | Velocity change over one time step |
| $\mathbf{W}$ | Generalized force directions for contact forces |
| $\boldsymbol{\lambda}$ | Lebesgue-measurable contact forces |
| $\mathbf{\Lambda}$ | Purely atomic impact impulses |
| $\mathbf{P}$ | Contact percussions |

## Generalized Coordinates of a Floating-Base System with Rotational Joints

$$\mathbf{q} = \begin{pmatrix} {}_I\mathbf{r}_{OB} \\ \mathbf{p}_{BI} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \qquad \mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot\varphi_1 \\ \vdots \\ \dot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \qquad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ {}_B\boldsymbol{\psi}_{IB} \\ \ddot\varphi_1 \\ \vdots \\ \ddot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3\times3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\intercal & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3x3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix}$$

## Equations of Motion with Contacts and no Impulses

$$\mathbf{M} = \sum_{i=1}^{n} \left[ (\mathbf{J}_S^\intercal m \mathbf{J}_S + \mathbf{J}_R^\intercal \mathbf{\Theta}_S \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and} \qquad \mathbf{g} = \sum_{i=1}^{n} \left[ (\mathbf{J}_S^\intercal m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\intercal (\mathbf{\Theta}_S \dot{\mathbf{J}}_R \mathbf{u} + \mathbf{\Omega} \times \mathbf{\Theta}_S \mathbf{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^{n} \left[ (\mathbf{J}_Q^\intercal \mathbf{F}_Q^A + \mathbf{J}_R^\intercal \mathbf{M}^A) \right]_i$$

## Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta\mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \begin{cases} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) & = \mathbf{W}\mathbf{\Lambda} \\ \mathbf{M}\underbrace{(\dot{\mathbf{u}}\mathrm{d}t + (\mathbf{u}^+ - \mathbf{u}^-)\mathrm{d}\eta)}_{\mathrm{d}\mathbf{u}} - \mathbf{h}\mathrm{d}t & = \mathbf{W}\underbrace{(\boldsymbol{\lambda}\mathrm{d}t + \mathbf{\Lambda}\mathrm{d}\eta)}_{\mathrm{d}\mathbf{P}} \end{cases}$$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