# Robotics Cheat Sheet

## Nomenclature

| | | |
|---|---|---|
| (Hyper-)complex number | $Q$ | normal capital letter |
| Column vector | $\mathbf{a}$ | bold small letter |
| Matrix | $\mathbf{M}$ | bold capital letter |
| Identity matrix | $\mathbb{1}_{n\times m}$ | $n\times m$-matrix |
| Coordinate system (CS) | $\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$ | Cartesian right-hand system $B$ with basis (unit) vectors $\mathbf{e}$ |
| Machine precision | $\epsilon$ | |

## Operators

| | |
|---|---|
| Cross product | $\mathbf{a}\times\mathbf{b} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge\mathbf{b} = \hat{\mathbf{a}}\mathbf{b} = \begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ |
| Euclidean norm | $\lVert\mathbf{a}\rVert = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2+\ldots+a_n^2}$ |
| Quaternion multiplicator | $\mathbf{q}\otimes\mathbf{p} \Leftrightarrow Q\cdot P$ |

## Pose (Position & Orientation)

### Position

| | | |
|---|---|---|
| Position vector | ${}_I\mathbf{r}_{OP}\in\mathbb{R}^3$ | vector from point $O$ to point $P$ expr. in CS $I$ |
| Homogeneous pos. vector | ${}_I\bar{\mathbf{r}}_{OP} = \begin{bmatrix}{}_I\mathbf{r}_{OP}^\top & 1\end{bmatrix}^\top$ | vector from point $O$ to point $P$ expr. in CS $I$ |

### Orientation

| | | |
|---|---|---|
| Rotation matrix | $\mathbf{R}_{BA}\in SO(3)$ | Rotates the coord. of the basis vectors $({}_A\mathbf{e}_x^A, {}_A\mathbf{e}_y^A, {}_A\mathbf{e}_z^A)$ of frame $A$ expressed in $A$ into the coordinates of the basis vectors $({}_A\mathbf{e}_x^B, {}_A\mathbf{e}_y^B, {}_A\mathbf{e}_z^B)$ of $B$ expressed in $A$. The rotation is active (alibi). ${}_A\mathbf{R}_{BA} = \begin{bmatrix}{}_A\mathbf{e}_x^B & {}_A\mathbf{e}_y^B & {}_A\mathbf{e}_z^B\end{bmatrix}$ |
| Direct Cosine matrix | $\mathbf{C}_{BA}\in SO(3)$ <br> ${}_B\mathbf{r}_{OP} = \mathbf{C}_{BA}\,{}_A\mathbf{r}_{OP}$ <br> $\mathbf{C}_{BA} = \mathbf{R}_{BA}^\top$ | The coordinate tranformation matrix, which transforms vectors from frame $A$ to frame $B$. The rotation is passive (alias). |
| Rotation Quaternion | $\mathbf{p}_{BA}$ <br> $\mathbf{p}_{BA}\Leftrightarrow\mathbf{R}_{BA}$ | The rotation is active (alibi). |
| Rotation Angle-axis | $(\theta,\mathbf{n})_{BA}$ <br> $(\theta,\mathbf{n})_{BA}\Leftrightarrow\mathbf{R}_{BA}$ | Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta\in[-\frac{\pi}{2},\frac{\pi}{2})$. The rotation is active (alibi). |
| Rotation vector | $\boldsymbol{\phi}_{BA}$ <br> $\boldsymbol{\phi}_{BA}\Leftrightarrow\mathbf{R}_{BA}$ | Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\lVert\boldsymbol{\phi}\rVert}$ and angle $\theta = \lVert\boldsymbol{\phi}\rVert$. The rotation is active (alibi). |
| Euler Angles ZYX <br> Euler Angles YPR | $(\psi,\theta,\phi)_{BA}$ | Tait-Bryan angles (Flight conv.): $z-y'-x''$, i.e. yaw-pitch-roll. Singularities are at $\theta=\pm\frac{\pi}{2}$. $\psi\in(-\pi,\pi), \theta\in(-\frac{\pi}{2},\frac{\pi}{2}), \phi\in(-\pi,\pi)$ |
| Euler Angles XYZ <br> Euler Angles RPY | $(\alpha,\beta,\gamma)_{BA}$ | Cardan angles (Glocker conv.): $x-y'-z''$, i.e. roll-pitch-yaw. Singularities are at $\beta=\pm\frac{\pi}{2}$. $\alpha\in(-\pi,\pi), \beta\in(-\frac{\pi}{2},\frac{\pi}{2}), \gamma\in(-\pi,\pi)$ |

### Rotation Quaternion

A rotation quaternion is defined as a Hamiltonian unit quaternion:

$$\boxed{\begin{aligned}&P = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}, \quad p_i\in\mathbb{R}\\ &i^2 = j^2 = k^2 = ijk = -1, \quad \lVert P\rVert = \sqrt{p_0^2+p_1^2+p_2^2+p_3^2} = 1\end{aligned}}$$

Note that $P_{BA}$ and $-P_{BA}$ represent the same rotation, but not the same unit quaternion.
Rot. quaternion as tuple: $P = (p_0,p_1,p_2,p_3) = (p_0,\vec{\mathbf{p}})$ with $\vec{\mathbf{p}} := (p_1,p_2,p_3)^\top$
Rot. quaternion as vector: $\mathbf{p} = \begin{bmatrix}p_0 & p_1 & p_2 & p_3\end{bmatrix}^\top$
Conjugate: $P^* = (p_0, -\vec{\mathbf{p}})$
Inverse: $P^{-1} = P^* = (p_0, -\vec{\mathbf{p}})$
Quaternion multiplication:

$$Q\cdot P = (q_0,\vec{\mathbf{q}})\cdot(p_0,\vec{\mathbf{p}}) = (q_0p_0 - \vec{\mathbf{q}}^\top\vec{\mathbf{p}},\, q_0\vec{\mathbf{p}} + p_0\vec{\mathbf{q}} + \vec{\mathbf{q}}\times\vec{\mathbf{p}}) \quad\Leftrightarrow$$

$$\mathbf{q}^{\otimes}\mathbf{p} = \mathbf{Q}(\mathbf{q})\mathbf{p} = \begin{pmatrix}q_0 & -\vec{\mathbf{q}}^\top\\ \vec{\mathbf{q}} & q_0\mathbb{1}_{3\times3} + \hat{\vec{\mathbf{p}}}\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \begin{pmatrix}q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \mathbf{Q}(\mathbf{p})^\top\mathbf{q}$$

### Rotation Quaternion ⇔ Rotation Angle-Axis

$$\mathbf{p}_{BI} = \begin{bmatrix}\cos\frac{\theta}{2}\\ \mathbf{n}\sin\frac{\theta}{2}\end{bmatrix} \quad\Leftrightarrow\quad (\theta,\mathbf{n})_{BI} = \begin{cases}(2\arccos(a_0), \frac{\vec{\mathbf{p}}}{\lVert\vec{\mathbf{p}}\rVert}) & \text{if } \lVert\vec{\mathbf{p}}\rVert^2\geq\epsilon^2\\ (0, \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\top) & \text{otherwise}\end{cases}$$

### Rotation Quaternion ⇔ Direction Cosine Matrix

$$\mathbf{C}_{AB} = \mathbf{R}_{AB}^\top(\mathbf{p}_{AB}) = \mathbb{1}_{3\times3} + 2p_0\hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2$$

$$= \begin{bmatrix}p_0^2+p_1^2-p_2^2-p_3^2 & 2p_1p_2-2p_0p_3 & 2p_0p_2+2p_1p_3\\ 2p_0p_3+2p_1p_2 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_2p_3-2p_0p_1\\ 2p_1p_3-2p_0p_2 & 2p_0p_1+2p_2p_3 & p_0^2-p_1^2-p_2^2+p_3^2\end{bmatrix}$$

$$\mathbf{C}_{BA} = \mathbf{R}_{BA}^\top = \mathbf{R}_{AB}(\mathbf{p}_{AB}) = \mathbb{1}_{3\times3} - 2p_0\hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2$$

$$= \begin{bmatrix}p_0^2+p_1^2-p_2^2-p_3^2 & 2p_0p_3+2p_1p_2 & 2p_1p_3-2p_0p_2\\ 2p_1p_2-2p_0p_3 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_0p_1+2p_2p_3\\ 2p_0p_2+2p_1p_3 & 2p_2p_3-2p_0p_1 & p_0^2-p_1^2-p_2^2+p_3^2\end{bmatrix}$$

$$\mathbf{p}_{BA} = \mathbf{p}_{BA}(\mathbf{C}_{AB}) = \begin{bmatrix}\frac{1}{2}\sqrt{1+\mathrm{tr}(\mathbf{C})}\\ \frac{C_{32}-C_{23}}{4p_0}\\ \frac{C_{13}-C_{31}}{4p_0}\\ \frac{C_{21}-C_{12}}{4p_0}\end{bmatrix} \quad \text{if } \mathrm{tr}(\mathbf{C})>0\ (\mathbf{C}_{AB}\to\mathbf{p}_{BA} \text{ is not unique})$$

### Euler Angles ZYX ⇔ Direction Cosine Matrix

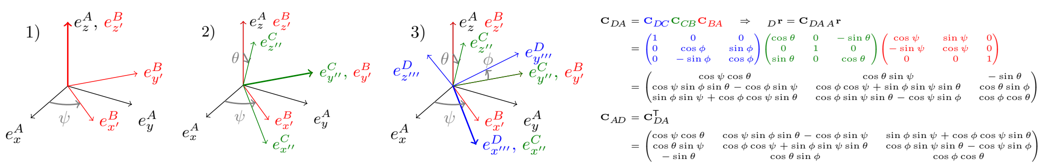

Figure 1: Rotation from $I$-frame to $B$-frame: $(z-y'-x'')$ – (yaw-pitch-roll) – $(\psi-\theta-\phi)$ – $(50°-25°-30°)$

### Euler Angles XYZ ⇔ Direction Cosine Matrix

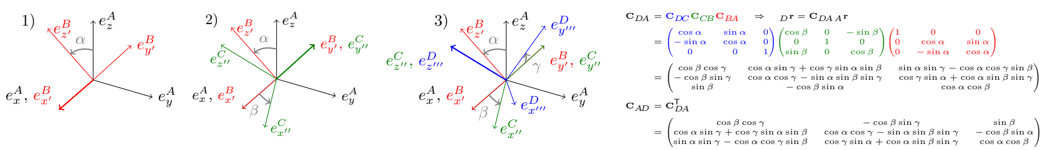

Figure 2: Rotation from $I$-frame to $B$-frame: $(x-y'-z'')$ – (roll-pitch-yaw) – $(\alpha-\beta-\gamma)$ – $(50°-25°-30°)$

## Pose

| | | |
|---|---|---|
| Homogeneous Transformation Matrix | $\mathbf{T}_{AB}$ | |

### Homogeneous Transformation Matrix

$$\mathbf{T}_{AB} = \begin{bmatrix}\mathbf{C}_{AB} & {}_A\mathbf{r}_{AB}\\ \mathbf{0}^\top & 1\end{bmatrix}$$

# Twist (Translational & Rotational Velocity)

## Translational Velocity

| | |
|---|---|
| ${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$ | velocity of point $P$ expr. in CS $B$ w.r.t. to the inertial system $I$ |
| ${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}$ | velocity of point $Q$ on rigid body $B$ from point $P$ on same body |

## Angular Velocity

| | |
|---|---|
| ${}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$ | ang. vel. of rigid body with body-fixed CS $B$ w.r.t. to inertial CS $I$ |
| ${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$ | inverse of angular velocity |
| ${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$ | rotation from $B$ to $I$ |
| ${}_I\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^\mathsf{T}$ | rotation from $B$ to $I$ |
| ${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$ | composition of angular velocity |
| ${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{BI}\mathbf{R}_{BI}^\mathsf{T} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^\mathsf{T}$ | angular velocity of $B$ w.r.t. $I$ expressed in CS $I$ |
| ${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{R}_{IB}\dot{\mathbf{R}}_{IB}^\mathsf{T} = \mathbf{C}_{IB}^\mathsf{T}\dot{\mathbf{C}}_{IB}$ | angular velocity of $B$ w.r.t. $I$ expressed in CS $B$ |
| ${}_B\boldsymbol{\omega}_{IB} = 2\bar{\mathbf{H}}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI}$ | ang. velocity from time derivative of the rotation quaternion with $\bar{\mathbf{H}}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & -\hat{\vec{\mathbf{p}}} + p_0\mathbb{1}_{3\times 3}\end{bmatrix} \in \mathbb{R}^{3\times 4}$ |
| $\dot{\mathbf{p}}_{BI} = \frac{1}{2}\bar{\mathbf{H}}(\mathbf{p}_{BI})^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}$ | time derivative of the rotation quaternion |

# Dynamics of a Multi-Rigid-Body System

| | |
|---|---|
| $n$ | Number of bodies in system |
| $n_j$ | Number of DoFs of the joints |
| $n_q$ | Number of generalized coordinates |
| $n_u$ | Number of generalized velocities |
| $\mathbf{M}$ | Mass matrix |
| $\mathbf{g}$ | Gyroscopic and Coriolis forces |
| $\mathbf{f}$ | Generalized external forces and torques |
| $\mathbf{h}$ | Combined force vector |
| $\mathbf{J}_P$ | Jacobi matrix for translation of point P |
| $\mathbf{J}_R$ | Jacobi matrix for rotation |
| $\mathbf{F}_Q^A$ | External forces on point Q |
| $\mathbf{M}^A$ | External torques |
| $m$ | Mass |
| $\boldsymbol{\Theta}$ | Intertia tensor |
| $(...)^-$ | Variable before impact |
| $(...)^+$ | Variable after impact |
| $(...)^\pm$ | Variable before/after impact |
| $\Delta t$ | Time step duration |
| $\Delta\mathbf{u}$ | Velocity change over one time step |
| $\mathbf{W}$ | Generalized force directions for contact forces |
| $\boldsymbol{\lambda}$ | Lebesgue-measurable contact forces |
| $\boldsymbol{\Lambda}$ | Purely atomic impact impulses |
| $\mathbf{P}$ | Contact percussions |

# Generalized Coordinates of a Floating-Base System with Rotational Joints

$$\mathbf{q} = \begin{pmatrix} {}_I\mathbf{r}_{OB} \\ \mathbf{p}_{BI} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \qquad \mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \qquad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ {}_B\boldsymbol{\psi}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{I}_{3\times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\mathsf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j\times n_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbf{I}_{3x3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j\times n_j} \end{pmatrix}$$

# Equation of Motion with Contacts and no Impulses

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and}$$

$$\mathbf{M} = \sum_{i=1}^{n}\Big[(\mathbf{J}_S^\mathsf{T} m\mathbf{J}_S + \mathbf{J}_R^\mathsf{T}\boldsymbol{\Theta}_S\mathbf{J}_R)\Big]_i$$

$$\mathbf{g} = \sum_{i=1}^{n}\Big[(\mathbf{J}_S^\mathsf{T} m\dot{\mathbf{J}}_S\mathbf{u} + \mathbf{J}_R^\mathsf{T}(\boldsymbol{\Theta}_S\dot{\mathbf{J}}_R\mathbf{u} + \boldsymbol{\Omega}\times\boldsymbol{\Theta}_S\boldsymbol{\Omega}))\Big]_i$$

$$\mathbf{f} = \sum_{i=1}^{n}\Big[(\mathbf{J}_Q^\mathsf{T}\mathbf{F}_Q^A + \mathbf{J}_R^\mathsf{T}\mathbf{M}^A)\Big]_i$$

# Equation of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta\mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{aligned} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) &= \mathbf{W}\boldsymbol{\Lambda} \\ \mathbf{M}\underbrace{(\dot{\mathbf{u}}\mathrm{d}t + (\mathbf{u}^+ - \mathbf{u}^-)\mathrm{d}\eta)}_{\mathrm{d}\mathbf{u}} - \mathbf{h}\mathrm{d}t &= \mathbf{W}\underbrace{(\boldsymbol{\lambda}\mathrm{d}t + \boldsymbol{\Lambda}\mathrm{d}\eta)}_{\mathrm{d}\mathbf{P}} \end{aligned} \right.$$