Kindr Cheat Sheet

Kinematics and Dynamics for Robotics

Nomenclature

(Hyper-)complex number	Q	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbb{1}_{n\times m}$	$n\times m$-matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system A with basis (unit) vectors $\mathbf{e}$
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	Global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	Local / body-fixed coordinate system (moves with body)
Rotation	$\mathcal{R}\in SO(3)$	generic rotation (for all parameterizations)
Machine precision	ϵ	

Operators

Cross product	$\mathbf{a}\times\mathbf{b} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge\mathbf{b} = \hat{\mathbf{a}}\mathbf{b} = \begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$
Skew/unskew	$\mathbf{a} = \hat{\mathbf{a}}^\vee$
Euclidean norm	$\lVert\mathbf{a}\rVert = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2+\ldots+a_n^2}$
Exponential map for matrix	$\text{expM}: \mathbb{R}^3\to\mathbb{R}^3, \mathbf{A}\mapsto e^{\mathbf{A}}, \quad \mathbf{A}\in\mathbb{R}^{3\times3}$
Logarithmic map for matrix	$\text{logM}: \mathbb{R}^3\to\mathbb{R}^3, \mathbf{A}\mapsto \log\mathbf{A}, \quad \mathbf{A}\in\mathbb{R}^{3\times3}$

Position & Orientation

Position

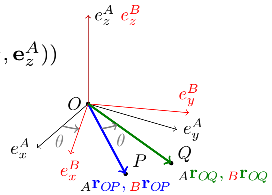
Vector	$\mathbf{r}_{OP}$	from point O to point P
Position vector	${}_A\mathbf{r}_{OP}\in\mathbb{R}^3$	from point O to point P expr. in frame A
Homogeneous pos. vector	${}_A\bar{\mathbf{r}}_{OP} = \begin{bmatrix}{}_A\mathbf{r}_{OP}^\intercal & 1\end{bmatrix}^\intercal$	from point O to point P expr. in frame A

Orientation/Rotation

1) Active Rotation: $\mathcal{R}^A: {}_A\mathbf{r}_{OP}\mapsto {}_A\mathbf{r}_{OQ}$ (rotates the vector $\mathbf{r}_{OP}$)
2) Passive Rotation: $\mathcal{R}^P: {}_A\mathbf{r}_{OP}\mapsto {}_B\mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A)$)
3) Inversion: ${\mathcal{R}^A}^{-1}(\mathbf{r}) = \mathcal{R}^P(\mathbf{r})$
4) Concatenation:

$$\mathcal{R}_2^A\left(\mathcal{R}_1^A(\mathbf{r})\right) = \left(\mathcal{R}_2^A\otimes\mathcal{R}_1^A\right)(\mathbf{r}) = \left({\mathcal{R}_1^A}^{-1}\otimes{\mathcal{R}_2^A}^{-1}\right)(\mathbf{r})$$

$$\mathcal{R}_2^P\left(\mathcal{R}_1^P(\mathbf{r})\right) = \left(\mathcal{R}_2^P\otimes\mathcal{R}_1^P\right)(\mathbf{r}) = \left({\mathcal{R}_1^P}^{-1}\otimes{\mathcal{R}_2^P}^{-1}\right)(\mathbf{r})$$



5) Exponential map: $\exp: \mathbb{R}^3\to SO(3), \mathbf{v}\mapsto \text{expM}(\hat{\mathbf{v}}), \quad \mathbf{v}\in\mathbb{R}^3$
6) Logarithmic map: $\log: SO(3)\to\mathbb{R}^3, \mathcal{R}\mapsto \text{logM}(\mathcal{R})^\vee, \quad \mathcal{R}\in SO(3)$
7) Box plus: $\mathcal{R}_2 = \mathcal{R}_1 \boxplus \mathbf{v} = \exp(\mathbf{v})\otimes\mathcal{R}_1, \quad \mathcal{R}_1,\mathcal{R}_2\in SO(3), \mathbf{v}\in\mathbb{R}^3$
8) Box minus: $\mathbf{v} = \mathcal{R}_1\boxminus\mathcal{R}_2 = \log\left(\mathcal{R}_1\otimes\mathcal{R}_2^{-1}\right), \quad \mathcal{R}_1,\mathcal{R}_2\in SO(3), \mathbf{v}\in\mathbb{R}^3$
9) Discrete integration: $\mathcal{R}^{k+1} = \mathcal{R}^k\boxplus\left({}_B\boldsymbol{\omega}_{IB}\Delta t\right)$

Rotation Matrix	$\mathbf{R}_{BA}\in SO(3)$ ${}_A\mathbf{r}_{OQ} = \mathbf{R}_{BA}\,{}_A\mathbf{r}_{OP}$ ${}_B\mathbf{r}_{OP} = \mathbf{R}_{BA}^\intercal\,{}_A\mathbf{r}_{OP}$	Maps the coord. of the basis vectors $({}_A\mathbf{e}_x^A, {}_A\mathbf{e}_y^A, {}_A\mathbf{e}_z^A)$ of frame A expressed in A into the coordinates of the basis vectors $({}_A\mathbf{e}_x^B, {}_A\mathbf{e}_y^B, {}_A\mathbf{e}_z^B)$ of B expressed in A. The rotation is active (alibi). ${}_A\mathbf{R}_{BA} = \begin{bmatrix}{}_A\mathbf{e}_x^B & {}_A\mathbf{e}_y^B & {}_A\mathbf{e}_z^B\end{bmatrix}$
Direct Cosine Matrix	$\mathbf{C}_{BA}\in SO(3)$ ${}_B\mathbf{r}_{OP} = \mathbf{C}_{BA}\,{}_A\mathbf{r}_{OP}$ $\mathbf{C}_{BA} = \mathbf{R}_{BA}^\intercal$	The coordinate tranformation matrix, which transforms vectors from frame A to frame B. The rotation is passive (alias).
Rotation Quaternion	$\mathbf{p}_{BA}$ $\mathbf{p}_{BA}\Leftrightarrow\mathbf{R}_{BA}$	The rotation is active (alibi).
Rotation Angle-axis	$(\theta,\mathbf{n})_{BA}$ $(\theta,\mathbf{n})_{BA}\Leftrightarrow\mathbf{R}_{BA}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta\in[0,\pi]$. The rotation is active (alibi).
Rotation Vector	$\boldsymbol{\phi}_{BA}$ $\boldsymbol{\phi}_{BA}\Leftrightarrow\mathbf{R}_{BA}$	Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\lVert\boldsymbol{\phi}\rVert}$ and angle $\theta = \lVert\boldsymbol{\phi}\rVert$. The rotation is active (alibi).
Euler Angles ZYX Euler Angles YPR	$(\psi,\theta,\phi)_{BA}$ $(\psi,\theta,\phi)_{BA}\Leftrightarrow\mathbf{C}_{BA}$	Tait-Bryan angles (Flight conv.): $z-y'-x''$, i.e. yaw-pitch-roll. Singularities are at $\theta=\pm\frac{\pi}{2}$. $\psi\in[-\pi,\pi), \theta\in[-\frac{\pi}{2},\frac{\pi}{2}), \phi\in[-\pi,\pi)$
Euler Angles XYZ Euler Angles RPY	$(\alpha,\beta,\gamma)_{BA}$ $(\alpha,\beta,\gamma)_{BA}\Leftrightarrow\mathbf{C}_{BA}$	Cardan angles (Glocker conv.): $x-y'-z''$, i.e. roll-pitch-yaw. Singularities are at $\beta=\pm\frac{\pi}{2}$. $\alpha\in[-\pi,\pi), \beta\in[-\frac{\pi}{2},\frac{\pi}{2}), \gamma\in[-\pi,\pi)$

Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$\boxed{\begin{aligned}&P = p_0 + p_1 i + p_2 j + p_3 k\in\mathbb{H}, \quad p_i\in\mathbb{R}\\ &i^2 = j^2 = k^2 = ijk = -1, \quad \lVert P\rVert = \sqrt{p_0^2+p_1^2+p_2^2+p_3^2} = 1\end{aligned}}$$

Note that P_{BA} and $-P_{BA}$ represent the same rotation, but not the same unit quaternion.
Rot. quaternion as tuple: $P = (p_0,p_1,p_2,p_3) = (p_0,\vec{\mathbf{p}})$ with $\vec{\mathbf{p}} := (p_1,p_2,p_3)^\intercal$
Rot. quaternion as vector: $\mathbf{p} = \begin{bmatrix}p_0 & p_1 & p_2 & p_3\end{bmatrix}^\intercal$
Conjugate: $P^* = (p_0,-\vec{\mathbf{p}})$
Inverse: $P^{-1} = P^* = (p_0,-\vec{\mathbf{p}})$
Quaternion multiplication:

$$Q\cdot P = (q_0,\vec{\mathbf{q}})\cdot(p_0,\vec{\mathbf{p}}) = (q_0p_0 - \vec{\mathbf{q}}^\intercal\vec{\mathbf{p}}, q_0\vec{\mathbf{p}} + p_0\vec{\mathbf{q}} + \vec{\mathbf{q}}\times\vec{\mathbf{p}}) \quad\Leftrightarrow$$

$$\mathbf{q}\otimes\mathbf{p} = \underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}}\mathbf{p} = \begin{pmatrix}q_0 & -\vec{\mathbf{q}}^\intercal\\ \vec{\mathbf{q}} & q_0\mathbb{1}_{3\times3}+\hat{\vec{\mathbf{q}}}\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \begin{pmatrix}q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}}\mathbf{q} = \begin{pmatrix}p_0 & -\vec{\mathbf{p}}^\intercal\\ \vec{\mathbf{p}} & p_0\mathbb{1}_{3\times3}-\hat{\vec{\mathbf{p}}}\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}p_0 & -p_1 & -p_2 & -p_3\\ p_1 & p_0 & p_3 & -p_2\\ p_2 & -p_3 & p_0 & p_1\\ p_3 & p_2 & -p_1 & p_0\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}$$

Rotation Quaternion ⇔ Rotation Angle-Axis

$$\mathbf{p}_{BI} = \begin{bmatrix}\cos\frac{\theta}{2}\\ \mathbf{n}\sin\frac{\theta}{2}\end{bmatrix} \quad\Leftrightarrow\quad (\theta,\mathbf{n})_{BI} = \begin{cases}\left(2\arccos(p_0), \frac{\vec{\mathbf{p}}}{\lVert\vec{\mathbf{p}}\rVert}\right) & \text{if } \lVert\vec{\mathbf{p}}\rVert^2\geq\epsilon^2\\ \left(0, \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\intercal\right) & \text{otherwise}\end{cases}$$

Rotation Quaternion ⇔ Direction Cosine Matrix

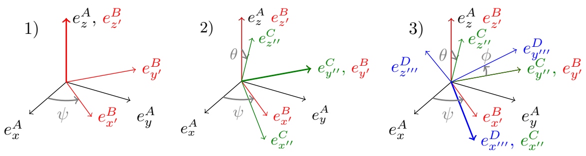
$$\mathbf{C}_{AB} = \mathbf{R}_{AB}^\intercal(\mathbf{p}_{AB}) = \mathbb{1}_{3\times3} + 2p_0\hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2$$

$$= \begin{bmatrix}p_0^2+p_1^2-p_2^2-p_3^2 & 2p_1p_2-2p_0p_3 & 2p_0p_2+2p_1p_3\\ 2p_0p_3+2p_1p_2 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_2p_3-2p_0p_1\\ 2p_1p_3-2p_0p_2 & 2p_0p_1+2p_2p_3 & p_0^2-p_1^2-p_2^2+p_3^2\end{bmatrix}$$

$$\mathbf{C}_{BA} = \mathbf{R}_{BA}^{\mathsf{T}} = \mathbf{R}_{AB}(\mathbf{p}_{AB}) = \mathbb{1}_{3\times3} - 2p_0\hat{\vec{\mathbf{p}}} + 2\hat{\vec{\mathbf{p}}}^2 = \begin{bmatrix} p_0^2+p_1^2-p_2^2-p_3^2 & 2p_0p_3+2p_1p_2 & 2p_1p_3-2p_0p_2 \\ 2p_1p_2-2p_0p_3 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_0p_1+2p_2p_3 \\ 2p_0p_2+2p_1p_3 & 2p_2p_3-2p_0p_1 & p_0^2-p_1^2-p_2^2+p_3^2 \end{bmatrix}$$

$$\mathbf{p}_{BA} = \mathbf{p}_{BA}(\mathbf{C}_{AB}) = \begin{bmatrix} \frac{1}{2}\sqrt{1+\mathrm{tr}(\mathbf{C})} \\ \frac{C_{32}-C_{23}}{4p_0} \\ \frac{C_{13}-C_{31}}{4p_0} \\ \frac{C_{21}-C_{12}}{4p_0} \end{bmatrix} \quad \text{if } \mathrm{tr}(\mathbf{C}) > 0 \ (\mathbf{C}_{AB} \to \mathbf{p}_{BA} \text{ is not unique})$$

Euler Angles ZYX ⇔ Direction Cosine Matrix



$$\mathbf{C}_{DA} = \mathbf{C}_{DC}\mathbf{C}_{CB}\mathbf{C}_{BA} \quad \Rightarrow \quad {}_D\mathbf{r} = \mathbf{C}_{DA}\,{}_A\mathbf{r}$$

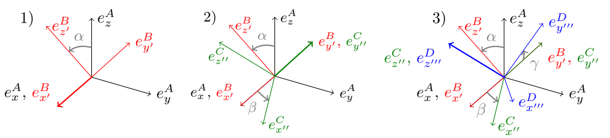
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\psi\cos\theta & \cos\theta\sin\psi & -\sin\theta \\ \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\theta\sin\phi \\ \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi & \cos\phi\cos\theta \end{pmatrix}$$

$$\mathbf{C}_{AD} = \mathbf{C}_{DA}^{\mathsf{T}} = \begin{pmatrix} \cos\psi\cos\theta & \cos\psi\sin\phi\sin\theta - \cos\phi\sin\psi & \sin\phi\sin\psi + \cos\phi\cos\psi\sin\theta \\ \cos\theta\sin\psi & \cos\phi\cos\psi + \sin\phi\sin\psi\sin\theta & \cos\phi\sin\psi\sin\theta - \cos\psi\sin\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\phi\cos\theta \end{pmatrix}$$

Figure 1: Rotation from I-frame to B-frame: $(z - y' - x'')$ – (yaw-pitch-roll) – $(\psi - \theta - \phi)$ – $(50° - 25° - 30°)$

Euler Angles XYZ ⇔ Direction Cosine Matrix



$$\mathbf{C}_{DA} = \mathbf{C}_{DC}\mathbf{C}_{CB}\mathbf{C}_{BA} \quad \Rightarrow \quad {}_D\mathbf{r} = \mathbf{C}_{DA}\,{}_A\mathbf{r}$$

$$= \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{pmatrix}$$

$$= \begin{pmatrix} \cos\beta\cos\gamma & \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta \\ -\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma \\ \sin\beta & -\cos\beta\sin\alpha & \cos\alpha\cos\beta \end{pmatrix}$$

$$\mathbf{C}_{AD} = \mathbf{C}_{DA}^{\mathsf{T}} = \begin{pmatrix} \cos\beta\cos\gamma & -\cos\beta\sin\gamma & \sin\beta \\ \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta & \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha \\ \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta & \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta \end{pmatrix}$$

Figure 2: Rotation from I-frame to B-frame: $(x - y' - z'')$ – (roll-pitch-yaw) – $(\alpha - \beta - \gamma)$ – $(50° - 25° - 30°)$

Pose

Homogeneous Transformation Matrix	$\mathbf{T}_{AB}$	

Homogeneous Transformation Matrix

$$\mathbf{T}_{AB} = \begin{bmatrix} \mathbf{C}_{AB} & {}_A\mathbf{r}_{AB} \\ \mathbf{0}^{\mathsf{T}} & 1 \end{bmatrix}$$

Time Derivatives of Position & Orientation

Linear Velocity

Velocity of point P expressed in frame B w.r.t. to the inertial frame I:

$${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$$

Velocity of point Q on rigid body B that rotates with ${}_B\boldsymbol{\Omega}$, where point P is on the same rigid body B:

$${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$$

Angular Velocity

${}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$	absolute angular velocity of rigid body B expressed in frame B
${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$	coord. transformation of angular velocity from frame B to frame I
${}_I\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^{\mathsf{T}}$	coord. transformation of angular velocity from frame B to frame I
${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$	composition of angular velocity

Time Derivative of Direction Cosine Matrix ⇔ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^{\mathsf{T}} = \dot{\mathbf{C}}_{BI}^{\mathsf{T}}\mathbf{C}_{BI}$$

$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^{\mathsf{T}}\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\dot{\mathbf{C}}_{BI}^{\mathsf{T}} \qquad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Matrix ⇔ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{BI}\mathbf{R}_{BI}^{\mathsf{T}} = \dot{\mathbf{R}}_{IB}^{\mathsf{T}}\mathbf{R}_{IB} \qquad \dot{\mathbf{R}}_{IB} = \mathbf{R}_{IB}\,{}_I\hat{\boldsymbol{\omega}}_{IB}^{\mathsf{T}} \qquad \dot{\mathbf{R}}_{BI} = {}_I\hat{\boldsymbol{\omega}}_{IB}\mathbf{R}_{BI}$$

$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{R}_{IB}\dot{\mathbf{R}}_{IB}^{\mathsf{T}} = \mathbf{R}_{BI}^{\mathsf{T}}\dot{\mathbf{R}}_{BI} \qquad \dot{\mathbf{R}}_{IB} = {}_B\hat{\boldsymbol{\omega}}_{IB}^{\mathsf{T}}\mathbf{R}_{IB} \qquad \dot{\mathbf{R}}_{BI} = \mathbf{R}_{BI}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Quaternion ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \qquad \dot{\mathbf{p}}_{BI} = \tfrac{1}{2}\mathbf{H}(\mathbf{p}_{BI})^{\mathsf{T}}\,{}_I\boldsymbol{\omega}_{IB}$$

$${}_B\boldsymbol{\omega}_{IB} = 2\bar{\mathbf{H}}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \qquad \dot{\mathbf{p}}_{BI} = \tfrac{1}{2}\bar{\mathbf{H}}(\mathbf{p}_{BI})^{\mathsf{T}}\,{}_B\boldsymbol{\omega}_{IB}$$

$$\mathbf{H}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & \hat{\vec{\mathbf{p}}} + p_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4} = \begin{bmatrix} -p_1 & p_0 & -p_3 & p_2 \\ -p_2 & p_3 & p_0 & -p_1 \\ -p_3 & -p_2 & p_1 & p_0 \end{bmatrix}$$

$$\bar{\mathbf{H}}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & -\hat{\vec{\mathbf{p}}} + p_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4} = \begin{bmatrix} -p_1 & p_0 & p_3 & -p_2 \\ -p_2 & -p_3 & p_0 & p_1 \\ -p_3 & p_2 & -p_1 & p_0 \end{bmatrix}$$

Time Derivative of Angle-Axis ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$

$${}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$

$$\dot{\theta} = \mathbf{n}^{\mathsf{T}}\,{}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 - \tfrac{1}{2}\hat{\mathbf{n}}\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$

$$\dot{\theta} = \mathbf{n}^{\mathsf{T}}\,{}_B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-\cos\theta}\hat{\mathbf{n}}^2 + \tfrac{1}{2}\hat{\mathbf{n}}\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$

Time Derivative of Rotation Vector ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} + \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

$${}_B\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} - \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} - \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} + \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

Time Derivative of Euler Angles ZYX ⇔ Angular Velocity

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{\cos\psi\sin\theta}{\cos\theta} & \frac{\sin\psi\sin\theta}{\cos\theta} & 1 \\ -\sin\psi & \cos\psi & 0 \\ \frac{\cos\psi}{\cos\theta} & \frac{\sin\psi}{\cos\theta} & 0 \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin\phi}{\cos\theta} & \frac{\cos\phi}{\cos\theta} \\ 0 & \cos\phi & -\sin\phi \\ 1 & \frac{\sin\phi\sin\theta}{\cos\theta} & \frac{\cos\phi\sin\theta}{\cos\theta} \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$${}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & -\sin\psi & \cos\psi\cos\theta \\ 0 & \cos\psi & \cos\theta\sin\psi \\ 1 & 0 & -\sin\theta \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\phi & \cos\phi & 0 \\ \cos\phi\cos\theta & -\sin\phi & 0 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Time Derivative of Euler Angles XYZ ⇔ Angular Velocity

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin\alpha\sin\beta}{\cos\beta} & -\frac{\cos\alpha\sin\beta}{\cos\beta} \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \frac{-\sin\alpha}{\cos\beta} & \frac{\cos\alpha}{\cos\beta} \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall\beta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\cos\gamma}{\cos\beta} & \frac{-\sin\gamma}{\cos\beta} & 0 \\ \sin\gamma & \cos\gamma & 0 \\ \frac{-\cos\gamma\sin\beta}{\cos\beta} & \frac{\sin\beta\sin\gamma}{\cos\beta} & 1 \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall\beta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$${}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 1 & 0 & \sin\beta \\ 0 & \cos\alpha & -\cos\beta\sin\alpha \\ 0 & \sin\alpha & \cos\alpha\cos\beta \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

Dynamics of a Multi-Rigid-Body System

n	Number of bodies in system
n_j	Number of DoFs of the joints
n_q	Number of generalized coordinates
n_u	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{F}_Q^A$	External forces on point Q
$\mathbf{M}^A$	External torques
m	Mass
$\mathbf{\Theta}$	Intertia tensor
$(...)^-$	Variable before impact
$(...)^+$	Variable after impact
$(...)^\pm$	Variable before/after impact
Δt	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\boldsymbol{\lambda}$	Lebesgue-measurable contact forces
$\mathbf{\Lambda}$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions

Generalized Coordinates of a Floating-Base System with Rotational Joints

$$\mathbf{q} = \begin{pmatrix} {}_I\mathbf{r}_{OB} \\ \mathbf{p}_{BI} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot\varphi_1 \\ \vdots \\ \dot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ {}_B\boldsymbol{\psi}_{IB} \\ \ddot\varphi_1 \\ \vdots \\ \ddot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3\times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\mathsf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3x3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix}$$

Equations of Motion with Contacts and no Impulses

$$\mathbf{M} = \sum_{i=1}^{n} \left[(\mathbf{J}_S^\mathsf{T} m \mathbf{J}_S + \mathbf{J}_R^\mathsf{T} \mathbf{\Theta}_S \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and} \quad \mathbf{g} = \sum_{i=1}^{n} \left[(\mathbf{J}_S^\mathsf{T} m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\mathsf{T} (\mathbf{\Theta}_S \dot{\mathbf{J}}_R \mathbf{u} + \mathbf{\Omega} \times \mathbf{\Theta}_S \mathbf{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^{n} \left[(\mathbf{J}_Q^\mathsf{T} \mathbf{F}_Q^A + \mathbf{J}_R^\mathsf{T} \mathbf{M}^A) \right]_i$$

Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta\mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{aligned} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) \quad &= \mathbf{W}\mathbf{\Lambda} \\ \mathbf{M}\underbrace{(\dot{\mathbf{u}}\mathrm{d}t + (\mathbf{u}^+ - \mathbf{u}^-)\mathrm{d}\eta)}_{\mathrm{d}\mathbf{u}} - \mathbf{h}\mathrm{d}t \quad &= \mathbf{W}\underbrace{(\boldsymbol{\lambda}\mathrm{d}t + \mathbf{\Lambda}\mathrm{d}\eta)}_{\mathrm{d}\mathbf{P}} \end{aligned} \right.$$

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