

Kindr Cheat Sheet

Kinematics and Dynamics for Robotics

Nomenclature

(Hyper-)complex number	Q	normal capital letter
Column vector	\mathbf{a}	bold small letter
Matrix	\mathbf{M}	bold capital letter
Identity matrix	$\mathbf{1}_{n \times m}$	$n \times m$ -matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system A with basis (unit) vectors \mathbf{e}
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	Global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	Local / body-fixed coordinate system (moves with body)
Machine precision	ϵ	

Operators

Cross product	$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge \mathbf{b} = \hat{\mathbf{a}} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
Euclidean norm	$\ \mathbf{a}\ = \sqrt{\mathbf{a}^T \mathbf{a}} = \sqrt{a_1^2 + \dots + a_n^2}$
Quaternion multiplicator	$\mathbf{q} \otimes \mathbf{p} \Leftrightarrow Q \cdot P$

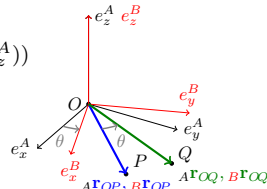
Position & Orientation

Position

Vector	\mathbf{r}_{OP}	from point O to point P
Position vector	${}^A \mathbf{r}_{OP} \in \mathbb{R}^3$	from point O to point P expr. in frame A
Homogeneous pos. vector	${}^A \bar{\mathbf{r}}_{OP} = [{}^A \mathbf{r}_{OP}^T \quad 1]^T$	from point O to point P expr. in frame A

Orientation/Rotation

- 1) Active Rotation: $\mathcal{R}^A : {}^A \mathbf{r}_{OP} \mapsto {}^A \mathbf{r}_{OQ}$ (rotates the vector \mathbf{r}_{OP})
- 2) Passive Rotation: $\mathcal{R}^P : {}^A \mathbf{r}_{OP} \mapsto {}^B \mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A)$)
- 3) Inversion: $\mathcal{R}^{A^{-1}}(\mathbf{r}) = \mathcal{R}^P(\mathbf{r})$
 $\mathcal{R}_2^A(\mathcal{R}_1^A(\mathbf{r})) = (\mathcal{R}_2^A \otimes \mathcal{R}_1^A)(\mathbf{r})$
 $= (\mathcal{R}_1^{A^{-1}} \otimes \mathcal{R}_2^{A^{-1}})(\mathbf{r})$
- 4) Concatenation: $\mathcal{R}_2^P(\mathcal{R}_1^P(\mathbf{r})) = (\mathcal{R}_2^P \otimes \mathcal{R}_1^P)(\mathbf{r})$
 $= (\mathcal{R}_1^{P^{-1}} \otimes \mathcal{R}_2^{P^{-1}})(\mathbf{r})$



Rotation Matrix	$\mathbf{R}_{BA} \in \text{SO}(3)$ ${}^A \mathbf{r}_{OQ} = \mathbf{R}_{BA} {}^A \mathbf{r}_{OP}$ ${}^B \mathbf{r}_{OP} = \mathbf{R}_{BA}^T {}^A \mathbf{r}_{OP}$	Maps the coord. of the basis vectors $({}^A \mathbf{e}_x^A, {}^A \mathbf{e}_y^A, {}^A \mathbf{e}_z^A)$ of frame A expressed in A into the coordinates of the basis vectors $({}^A \mathbf{e}_x^B, {}^A \mathbf{e}_y^B, {}^A \mathbf{e}_z^B)$ of B expressed in A . The rotation is active (alibi). ${}^A \mathbf{R}_{BA} = [{}^A \mathbf{e}_x^B \quad {}^A \mathbf{e}_y^B \quad {}^A \mathbf{e}_z^B]$
Direct Cosine Matrix	$\mathbf{C}_{BA} \in \text{SO}(3)$ ${}^B \mathbf{r}_{OP} = \mathbf{C}_{BA} {}^A \mathbf{r}_{OP}$ $\mathbf{C}_{BA} = \mathbf{R}_{BA}^T$	The coordinate transformation matrix, which transforms vectors from frame A to frame B . The rotation is passive (alias).
Rotation Quaternion	\mathbf{p}_{BA} $\mathbf{p}_{BA} \Leftrightarrow \mathbf{R}_{BA}$	The rotation is active (alibi).
Rotation Angle-axis	$(\theta, \mathbf{n})_{BA}$ $(\theta, \mathbf{n})_{BA} \Leftrightarrow \mathbf{R}_{BA}$	Rotation with unit rotation axis \mathbf{n} and angle $\theta \in [0, \pi]$. The rotation is active (alibi).
Rotation Vector	ϕ_{BA} $\phi_{BA} \Leftrightarrow \mathbf{R}_{BA}$	Rotation with rotation axis $\mathbf{n} = \frac{\phi}{\ \phi\ }$ and angle $\theta = \ \phi\ $. The rotation is active (alibi).
Euler Angles ZYX Euler Angles YPR	$(\psi, \theta, \phi)_{BA}$ $(\psi, \theta, \phi)_{BA} \Leftrightarrow \mathbf{C}_{BA}$	Tait-Bryan angles (Flight conv.): $z - y' - x''$, i.e. yaw-pitch-roll. Singularities are at $\theta = \pm \frac{\pi}{2}$. $\psi \in [-\pi, \pi], \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \phi \in [-\pi, \pi]$
Euler Angles XYZ Euler Angles RPY	$(\alpha, \beta, \gamma)_{BA}$ $(\alpha, \beta, \gamma)_{BA} \Leftrightarrow \mathbf{C}_{BA}$	Cardan angles (Glocker conv.): $x - y' - z''$, i.e. roll-pitch-yaw. Singularities are at $\beta = \pm \frac{\pi}{2}$. $\alpha \in [-\pi, \pi], \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \gamma \in [-\pi, \pi]$

Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$P = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}, \quad p_i \in \mathbb{R}$$

$$i^2 = j^2 = k^2 = ijk = -1, \quad \|P\| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2} = 1$$

Note that P_{BA} and $-P_{BA}$ represent the same rotation, but not the same unit quaternion.

Rot. quaternion as tuple: $P = (p_0, p_1, p_2, p_3) = (p_0, \vec{\mathbf{p}})$ with $\vec{\mathbf{p}} := (p_1, p_2, p_3)^T$

Rot. quaternion as vector: $\mathbf{p} = [p_0 \quad p_1 \quad p_2 \quad p_3]^T$

Conjugate: $P^* = (p_0, -\vec{\mathbf{p}})$

Inverse: $P^{-1} = P^* = (p_0, -\vec{\mathbf{p}})$

Quaternion multiplication:

$$Q \cdot P = (q_0, \vec{\mathbf{q}}) \cdot (p_0, \vec{\mathbf{p}}) = (q_0 p_0 - \vec{\mathbf{q}}^T \vec{\mathbf{p}}, q_0 \vec{\mathbf{p}} + p_0 \vec{\mathbf{q}} + \vec{\mathbf{q}} \times \vec{\mathbf{p}}) \Leftrightarrow$$

$$\mathbf{q} \otimes \mathbf{p} = \underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}} \mathbf{p} = \begin{pmatrix} q_0 & -\vec{\mathbf{q}}^T \\ \vec{\mathbf{q}} & q_0 \mathbb{1}_{3 \times 3} + \hat{\vec{\mathbf{q}}} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}} \mathbf{q} = \begin{pmatrix} p_0 & -\vec{\mathbf{p}}^T \\ \vec{\mathbf{p}} & p_0 \mathbb{1}_{3 \times 3} - \hat{\vec{\mathbf{p}}} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & p_3 & -p_2 \\ p_2 & -p_3 & p_0 & p_1 \\ p_3 & p_2 & -p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

Rotation Quaternion \Leftrightarrow Rotation Angle-Axis

$$\mathbf{p}_{BI} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{n} \sin \frac{\theta}{2} \end{bmatrix} \Leftrightarrow (\theta, \mathbf{n})_{BI} = \begin{cases} (2 \arccos(p_0), \frac{\vec{\mathbf{p}}}{\|\vec{\mathbf{p}}\|}) & \text{if } \|\vec{\mathbf{p}}\|^2 \geq \epsilon^2 \\ (0, [1 \quad 0 \quad 0]^T) & \text{otherwise} \end{cases}$$

Rotation Quaternion \Leftrightarrow Direction Cosine Matrix

$$\mathbf{C}_{AB} = \mathbf{R}_{AB}^T(\mathbf{p}_{AB}) = \mathbf{1}_{3 \times 3} + 2p_0 \hat{\vec{\mathbf{p}}} + 2\vec{\mathbf{p}}^2$$

$$= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_1 p_2 - 2p_0 p_3 & 2p_0 p_2 + 2p_1 p_3 \\ 2p_0 p_3 + 2p_1 p_2 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_2 p_3 - 2p_0 p_1 \\ 2p_1 p_3 - 2p_0 p_2 & 2p_0 p_1 + 2p_2 p_3 & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}$$

$$\mathbf{C}_{BA} = \mathbf{R}_{BA}^T = \mathbf{R}_{AB}(\mathbf{p}_{AB}) = \mathbf{1}_{3 \times 3} - 2p_0 \hat{\mathbf{p}} + 2\hat{\mathbf{p}}^2$$

$$= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_0p_3 + 2p_1p_2 & 2p_1p_3 - 2p_0p_2 \\ 2p_1p_2 - 2p_0p_3 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_0p_1 + 2p_2p_3 \\ 2p_0p_2 + 2p_1p_3 & 2p_2p_3 - 2p_0p_1 & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}$$

$$\mathbf{p}_{BA} = \mathbf{p}_{BA}(\mathbf{C}_{AB}) = \begin{bmatrix} \frac{1}{2} \sqrt{1 + \text{tr}(\mathbf{C})} \\ \frac{C_{32} - C_{23}}{4p_0} \\ \frac{C_{13} - C_{31}}{4p_0} \\ \frac{C_{21} - C_{12}}{4p_0} \end{bmatrix} \quad \text{if } \text{tr}(\mathbf{C}) > 0 \quad (\mathbf{C}_{AB} \rightarrow \mathbf{p}_{BA} \text{ is not unique})$$

Euler Angles ZYX \Leftrightarrow Direction Cosine Matrix

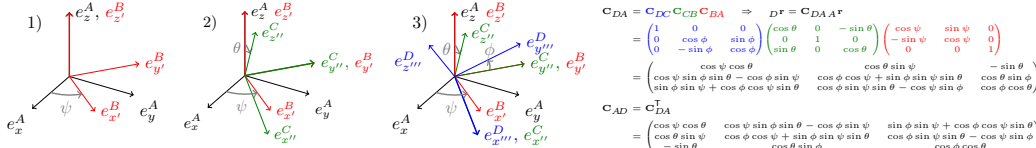


Figure 1: Rotation from I -frame to B -frame: $(z - y' - x'') - (\text{yaw-pitch-roll}) - (\psi - \theta - \phi) - (50^\circ - 25^\circ - 30^\circ)$

Euler Angles XYZ \Leftrightarrow Direction Cosine Matrix

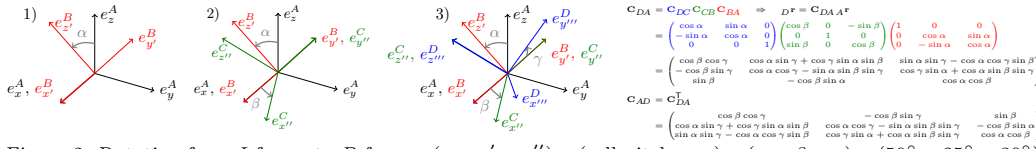


Figure 2: Rotation from I -frame to B -frame: $(x - y' - z'') - (\text{roll-pitch-yaw}) - (\alpha - \beta - \gamma) - (50^\circ - 25^\circ - 30^\circ)$

Pose

Homogeneous Transformation Matrix	\mathbf{T}_{AB}	
-----------------------------------	-------------------	--

Homogeneous Transformation Matrix

$$\mathbf{T}_{AB} = \begin{bmatrix} \mathbf{C}_{AB} & A\mathbf{r}_{AB} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

Time Derivatives of Position & Orientation

Linear Velocity

Velocity of point P expressed in frame B w.r.t. to the inertial frame I :

$${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$$

Velocity of point Q on rigid body B that rotates with ${}_B\boldsymbol{\Omega}$, where point P is on the same rigid body B :

$${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$$

Angular Velocity

${}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$	absolute angular velocity of rigid body B expressed in frame B
${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
${}_I\boldsymbol{\omega}_{IB} = {}_I\boldsymbol{\omega}_{BI}$	coord. transformation of angular velocity from frame B to frame I
${}_I\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB} {}_B\dot{\boldsymbol{\omega}}_{IB} \mathbf{C}_{IB}^T$	coord. transformation of angular velocity from frame B to frame I
${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$	composition of angular velocity

Time Derivative of Direction Cosine Matrix \Leftrightarrow Angular Velocity

$${}_I\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB} \mathbf{C}_{IB}^T = \dot{\mathbf{C}}_{BI}^T \mathbf{C}_{BI} \\ {}_B\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^T \dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI} \dot{\mathbf{C}}_{BI}^T \quad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB} {}_B\dot{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Matrix \Leftrightarrow Angular Velocity

$${}_I\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{BI} \mathbf{R}_{BI}^T = \dot{\mathbf{R}}_{IB}^T \mathbf{R}_{IB} \quad \dot{\mathbf{R}}_{IB} = \mathbf{R}_{IB} {}_I\dot{\boldsymbol{\omega}}_{IB}^T \quad \dot{\mathbf{R}}_{BI} = {}_I\dot{\boldsymbol{\omega}}_{IB} \mathbf{R}_{BI} \\ {}_B\dot{\boldsymbol{\omega}}_{IB} = \mathbf{R}_{IB} \dot{\mathbf{R}}_{IB}^T = \mathbf{R}_{BI}^T \dot{\mathbf{R}}_{BI} \quad \dot{\mathbf{R}}_{IB} = {}_B\dot{\boldsymbol{\omega}}_{IB}^T \mathbf{R}_{IB} \quad \dot{\mathbf{R}}_{BI} = \mathbf{R}_{BI} {}_B\dot{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Quaternion \Leftrightarrow Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \quad \dot{\mathbf{p}}_{BI} = \frac{1}{2}\mathbf{H}(\mathbf{p}_{BI})^T {}_I\boldsymbol{\omega}_{IB} \\ {}_B\boldsymbol{\omega}_{IB} = 2\dot{\mathbf{H}}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI} \quad \dot{\mathbf{p}}_{BI} = \frac{1}{2}\dot{\mathbf{H}}(\mathbf{p}_{BI})^T {}_B\boldsymbol{\omega}_{IB} \\ \mathbf{H}(\mathbf{p}) = \begin{bmatrix} -\hat{\mathbf{p}} & \hat{\mathbf{p}} + p_0\mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \quad \dot{\mathbf{H}}(\mathbf{p}) = \begin{bmatrix} -\hat{\mathbf{p}} & -\hat{\mathbf{p}} + p_0\mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4} \\ = \begin{bmatrix} -p_1 & p_0 & -p_3 & p_2 \\ -p_2 & p_3 & p_0 & -p_1 \\ -p_3 & -p_2 & p_1 & p_0 \end{bmatrix} = \begin{bmatrix} -p_1 & p_0 & p_3 & -p_2 \\ -p_2 & -p_3 & p_0 & p_1 \\ -p_3 & p_2 & -p_1 & p_0 \end{bmatrix}$$

Time Derivative of Angle-Axis \Leftrightarrow Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}} \sin \theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos \theta) \\ {}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}} \sin \theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos \theta) \\ \dot{\theta} = \mathbf{n}^T {}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2} \frac{\sin \theta}{1 - \cos \theta} \hat{\mathbf{n}}^2 - \frac{1}{2} \dot{\mathbf{n}} \right) {}_I\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\} \\ \dot{\theta} = \mathbf{n}^T {}_B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\frac{1}{2} \frac{\sin \theta}{1 - \cos \theta} \hat{\mathbf{n}}^2 + \frac{1}{2} \dot{\mathbf{n}} \right) {}_B\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \{0\}$$

Time Derivative of Rotation Vector \Leftrightarrow Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbf{1}_{3 \times 3} + \hat{\phi} \left(\frac{1 - \cos \|\phi\|}{\|\phi\|^2} \right) + \hat{\phi}^2 \left(\frac{\|\phi\| - \sin \|\phi\|}{\|\phi\|^3} \right) \right) \dot{\phi} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ {}_B\boldsymbol{\omega}_{IB} = \left(\mathbf{1}_{3 \times 3} - \hat{\phi} \left(\frac{1 - \cos \|\phi\|}{\|\phi\|^2} \right) + \hat{\phi}^2 \left(\frac{\|\phi\| - \sin \|\phi\|}{\|\phi\|^3} \right) \right) \dot{\phi} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ \dot{\phi} = \left(\mathbf{1}_{3 \times 3} - \frac{1}{2} \hat{\phi} + \hat{\phi}^2 \frac{1}{\|\phi\|^2} \left(1 - \frac{\|\phi\|}{2} \frac{\sin \|\phi\|}{1 - \cos \|\phi\|} \right) \right) {}_I\boldsymbol{\omega}_{IB} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\} \\ \dot{\phi} = \left(\mathbf{1}_{3 \times 3} + \frac{1}{2} \hat{\phi} + \hat{\phi}^2 \frac{1}{\|\phi\|^2} \left(1 - \frac{\|\phi\|}{2} \frac{\sin \|\phi\|}{1 - \cos \|\phi\|} \right) \right) {}_B\boldsymbol{\omega}_{IB} \quad \forall \|\phi\| \in \mathbb{R} \setminus \{0\}$$

Time Derivative of Euler Angles ZYX \Leftrightarrow Angular Velocity

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \cos \psi \sin \theta & \sin \psi \sin \theta & 1 \\ \cos \theta & \cos \theta & 0 \\ -\sin \psi & \cos \psi & 0 \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z} \\ \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin \phi}{\cos \theta} & \frac{\cos \phi}{\cos \theta} \\ 0 & \cos \theta & -\sin \phi \\ 1 & \frac{\sin \phi \sin \theta}{\cos \theta} & \frac{\cos \phi \sin \theta}{\cos \theta} \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall \theta \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z} \\ {}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & -\sin \psi & \cos \psi \cos \theta \\ 0 & \cos \psi & \cos \theta \sin \psi \\ 1 & 0 & -\sin \theta \end{bmatrix} \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix} \\ {}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} -\sin \theta & 0 & 1 \\ \cos \theta \sin \phi & \cos \phi & 0 \\ \cos \phi \cos \theta & -\sin \phi & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \theta \\ \phi \end{bmatrix}$$

Time Derivative of Euler Angles XYZ \Leftrightarrow Angular Velocity

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\sin \alpha \sin \beta}{\cos \beta} & \frac{-\cos \alpha \sin \beta}{\cos \beta} \\ 0 & \cos \alpha & \sin \alpha \\ 0 & \frac{-\sin \alpha}{\cos \beta} & \frac{\cos \alpha}{\cos \beta} \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall \beta \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z} \\ \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{\cos \gamma}{\cos \beta} & \frac{-\sin \gamma}{\cos \beta} & 0 \\ \sin \gamma & \cos \gamma & 0 \\ \frac{-\cos \gamma \sin \beta}{\cos \beta} & \frac{\sin \beta \sin \gamma}{\cos \beta} & 1 \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall \beta \in \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi \right\}, k \in \mathbb{Z} \\ {}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 1 & 0 & \sin \beta \\ 0 & \cos \alpha & -\cos \beta \sin \alpha \\ 0 & \sin \alpha & \cos \alpha \cos \beta \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \\ {}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} \cos \beta \cos \gamma & \sin \gamma & 0 \\ -\cos \beta \sin \gamma & \cos \gamma & 0 \\ \sin \beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$$

Dynamics of a Multi-Rigid-Body System

n	Number of bodies in system
n_j	Number of DoFs of the joints
n_q	Number of generalized coordinates
n_u	Number of generalized velocities
\mathbf{M}	Mass matrix
\mathbf{g}	Gyroscopic and Coriolis forces
\mathbf{f}	Generalized external forces and torques
\mathbf{h}	Combined force vector
\mathbf{J}_P	Jacobi matrix for translation of point P
\mathbf{J}_R	Jacobi matrix for rotation
\mathbf{F}_Q^A	External forces on point Q
\mathbf{M}^A	External torques
m	Mass
Θ	Intertia tensor
$(\dots)^-$	Variable before impact
$(\dots)^+$	Variable after impact
$(\dots)^\pm$	Variable before/after impact
Δt	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
\mathbf{W}	Generalized force directions for contact forces
λ	Lebesgue-measurable contact forces
Λ	Purely atomic impact impulses
\mathbf{P}	Contact percussions

Generalized Coordinates of a Floating-Base System with Rotational Joints

$$\mathbf{q} = \begin{pmatrix} I^{\mathbf{r}_{OB}} \\ \mathbf{p}_{BI} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} I^{\mathbf{v}_B} \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} I^{\mathbf{a}_B} \\ {}_B\dot{\boldsymbol{\psi}}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j \times n_j} \end{pmatrix} \Leftrightarrow \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j \times n_j} \end{pmatrix}$$

Equations of Motion with Contacts and no Impulses

$$\mathbf{M} = \sum_{i=1}^n \left[(\mathbf{J}_S^\top m \mathbf{J}_S + \mathbf{J}_R^\top \Theta_S \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\ddot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\lambda} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \quad \mathbf{g} = \sum_{i=1}^n \left[(\mathbf{J}_S^\top m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\top (\Theta_S \dot{\mathbf{J}}_R \mathbf{u} + \boldsymbol{\Omega} \times \Theta_S \boldsymbol{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^n \left[(\mathbf{J}_Q^\top \mathbf{F}_Q^A + \mathbf{J}_R^\top \mathbf{M}^A) \right]_i$$

Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta \mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{array}{l} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\Lambda \\ \mathbf{M} \underbrace{(\dot{\mathbf{u}}dt + (\mathbf{u}^+ - \mathbf{u}^-)d\eta)}_{d\mathbf{u}} - \mathbf{h}d\mathbf{t} = \mathbf{W} \underbrace{(\lambda d\mathbf{t} + \Lambda d\eta)}_{d\mathbf{P}} \end{array} \right.$$

Copyright © 2014 Autonomous Systems Lab, ETH Zurich (asl.ethz.ch)
Contact: Christian Gehring (gehrinch@ethz.ch)