Kindr Cheat Sheet v1.0

Kinematics and Dynamics for Robotics

Nomenclature

(Hyper-)complex number	Q	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbb{1}_{n\times m}$	$n \times m$-matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system A with basis (unit) vectors $\mathbf{e}$
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	local / body-fixed coordinate system (moves with body)
Rotation	$\Phi \in SO(3)$	generic rotation (for all parameterizations)
Machine precision	ϵ	

Operators

Cross product	$\mathbf{a}\times\mathbf{b} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} \times \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge\mathbf{b} = \hat{\mathbf{a}}\mathbf{b} = \begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$
Skew/unskew	$\mathbf{a} = \hat{\mathbf{a}}^\vee$
Euclidean norm	$\lVert\mathbf{a}\rVert = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2 + \ldots + a_n^2}$
Exponential map for matrix	$\exp\mathrm{M} : \mathbb{R}^3 \to \mathbb{R}^3, \mathbf{A} \mapsto e^{\mathbf{A}}, \quad \mathbf{A} \in \mathbb{R}^{3\times 3}$
Logarithmic map for matrix	$\log\mathrm{M} : \mathbb{R}^3 \to \mathbb{R}^3, \mathbf{A} \mapsto \log\mathbf{A}, \quad \mathbf{A} \in \mathbb{R}^{3\times 3}$

Position & Orientation

Position

Vector	$\mathbf{r}_{OP}$	from point O to point P
Position vector	${}_B\mathbf{r}_{OP} \in \mathbb{R}^3$	from point O to point P expr. in frame B
Homogeneous pos. vector	${}_B\bar{\mathbf{r}}_{OP} = \begin{bmatrix}{}_B\mathbf{r}_{OP}^\mathsf{T} & 1\end{bmatrix}^\mathsf{T}$	from point O to point P expr. in frame B

Orientation/Rotation

1) Active Rotation: $\Phi^A : {}_I\mathbf{r}_{OP} \mapsto {}_I\mathbf{r}_{OQ}$ (rotates the vector $\mathbf{r}_{OP}$)

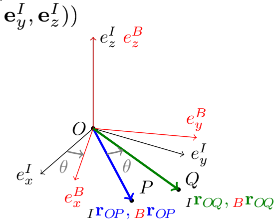
2) Passive Rotation: $\Phi^P : {}_I\mathbf{r}_{OP} \mapsto {}_B\mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$)

3) Elementary Rotations ${}_I\mathbf{r}_{OP} = \mathbf{C}_{IB}\,{}_B\mathbf{r}_{OP}$

around z-axis: $\mathbf{C}_{IB} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

around y-axis: $\mathbf{C}_{IB} = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$

around x-axis: $\mathbf{C}_{IB} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$



4) Inversion: $\Phi^{A^{-1}}(\mathbf{r}) = \Phi^P(\mathbf{r})$

5) Concatenation:

$$\Phi_2^A\left(\Phi_1^A(\mathbf{r})\right) = \left(\Phi_2^A \otimes \Phi_1^A\right)(\mathbf{r}) = \left(\Phi_1^{A^{-1}} \otimes \Phi_2^{A^{-1}}\right)^{-1}(\mathbf{r})$$

$$\Phi_2^P\left(\Phi_1^P(\mathbf{r})\right) = \left(\Phi_2^P \otimes \Phi_1^P\right)(\mathbf{r}) = \left(\Phi_1^{P^{-1}} \otimes \Phi_2^{P^{-1}}\right)^{-1}(\mathbf{r})$$

6) Exponential map: $\exp : \mathbb{R}^3 \to SO(3), \mathbf{v} \mapsto \exp\mathrm{M}(\hat{\mathbf{v}}), \quad \mathbf{v} \in \mathbb{R}^3$

7) Logarithmic map: $\log : SO(3) \to \mathbb{R}^3, \Phi \mapsto \log\mathrm{M}(\Phi)^\vee, \quad \Phi \in SO(3)$

8) Box plus: $\Phi_2 = \Phi_1 \boxplus \mathbf{v} = \exp(\mathbf{v}) \otimes \Phi_1, \quad \Phi_1, \Phi_2 \in SO(3), \mathbf{v} \in \mathbb{R}^3$

9) Box minus: $\mathbf{v} = \Phi_1 \boxminus \Phi_2 = \log\left(\Phi_1 \otimes \Phi_2^{-1}\right), \quad \Phi_1, \Phi_2 \in SO(3), \mathbf{v} \in \mathbb{R}^3$

10) Discrete integration: $\Phi_{IB}^{k+1} = \Phi_{IB}^k \boxplus ({}_I\boldsymbol{\omega}_{IB}^k \Delta t), \quad \Phi_{BI}^{k+1} = \Phi_{BI}^k \boxplus (-{}_B\boldsymbol{\omega}_{IB}^k \Delta t)$

11) Discrete differential: ${}_I\boldsymbol{\omega}_{IB}^k = (\Phi_{IB}^{k+1} \boxminus \Phi_{IB}^k)/\Delta t, \quad {}_B\boldsymbol{\omega}_{IB}^k = -(\Phi_{BI}^{k+1} \boxminus \Phi_{BI}^k/\Delta t)$

12) (Spherical) linear interpolation $t \in [0,1]$:

$$\Phi_t = \Phi_0 \boxplus ((\Phi_1 \boxminus \Phi_0)t), \quad \Phi_t = \Phi(t), \Phi_0 = \Phi(0), \Phi_1 = \Phi(1)$$

$$= (\Phi_1 \otimes \Phi_0^{-1})^t \otimes \Phi_0$$

Rotation Parameterizations

Rotation Matrix	$\mathbf{C}_{AB} \in SO(3)$ ${}_A\mathbf{r}_{OP} = \mathbf{C}_{AB}\,{}_B\mathbf{r}_{OP}$ $\mathbf{C}_{AB} = \mathbf{C}_{BA}^\mathsf{T}$	The rotation matrix (Direction Cosine Matrix) is a coordinate transformation matrix, which transforms vectors from frame B to frame A.
Rotation Quaternion	$\mathbf{q}_{AB}$ $\mathbf{q} = [q_0\; q_1\; q_2\; q_3]^\mathsf{T}$	Hamiltonian unit quaternion (hypercomplex number) $Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, \quad \lVert Q\rVert = 1$
Angle-axis	$(\theta, \mathbf{n})_{AB}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta \in [0, \pi]$.
Rotation Vector	$\boldsymbol{\phi}_{AB}$	Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\lVert\boldsymbol{\phi}\rVert}$ and angle $\theta = \lVert\boldsymbol{\phi}\rVert$.
Euler Angles ZYX Euler Angles YPR	$(\psi, \theta, \phi)_{AB}$	Tait-Bryan angles (Flight conv.): $z - y' - x''$, i.e. yaw-pitch-roll. Singularities are at $\theta = \pm\frac{\pi}{2}$. $\psi \in [-\pi, \pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}), \phi \in [-\pi, \pi)$
Euler Angles XYZ Euler Angles RPY	$(\alpha, \beta, \gamma)_{AB}$	Cardan angles (Glocker conv.): $x - y' - z''$, i.e. roll-pitch-yaw. Singularities are at $\beta = \pm\frac{\pi}{2}$. $\alpha \in [-\pi, \pi), \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}), \gamma \in [-\pi, \pi)$

Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$Q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}, \quad q_i \in \mathbb{R}, i^2 = j^2 = k^2 = ijk = -1, \quad \lVert Q\rVert = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$

Note that Q_{IB} and $-Q_{IB}$ represent the same rotation, but not the same unit quaternion.

Rot. quaternion as tuple: $Q = (q_0, q_1, q_2, q_3) = (q_0, \check{\mathbf{q}})$ with $\check{\mathbf{q}} := (q_1, q_2, q_3)^\mathsf{T}$

Rot. quaternion as vector: $\mathbf{q} = \begin{bmatrix}q_0 & q_1 & q_2 & q_3\end{bmatrix}^\mathsf{T}$

Conjugate: $Q^* = (q_0, -\check{\mathbf{q}})$

Inverse: $Q^{-1} = Q^* = (q_0, -\check{\mathbf{q}})$

Quaternion multiplication:

$Q \cdot P = (q_0, \check{\mathbf{q}}) \cdot (p_0, \check{\mathbf{p}}) = (q_0 p_0 - \check{\mathbf{q}}^\mathsf{T}\check{\mathbf{p}}, q_0\check{\mathbf{p}} + p_0\check{\mathbf{q}} + \check{\mathbf{q}} \times \check{\mathbf{p}}) \quad \Leftrightarrow$

$$\mathbf{q} \otimes \mathbf{p} = \underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}}\mathbf{p} = \begin{pmatrix}q_0 & -\check{\mathbf{q}}^\mathsf{T}\\ \check{\mathbf{q}} & q_0\mathbb{1}_{3\times3} + \hat{\check{\mathbf{q}}}\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \begin{pmatrix}q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}}\mathbf{q} = \begin{pmatrix}p_0 & -\check{\mathbf{p}}^\mathsf{T}\\ \check{\mathbf{p}} & p_0\mathbb{1}_{3\times3} - \hat{\check{\mathbf{p}}}\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}p_0 & -p_1 & -p_2 & -p_3\\ p_1 & p_0 & p_3 & -p_2\\ p_2 & -p_3 & p_0 & p_1\\ p_3 & p_2 & -p_1 & p_0\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}$$

Rotation Quaternion ⇔ Rotation Angle-Axis

$$\mathbf{q}_{IB} = \begin{bmatrix}\cos\frac{\theta}{2}\\ \mathbf{n}\sin\frac{\theta}{2}\end{bmatrix} \quad \Leftrightarrow \quad (\theta, \mathbf{n})_{IB} = \begin{cases}\left(2\arccos(p_0), \frac{\check{\mathbf{q}}}{\lVert\check{\mathbf{q}}\rVert}\right) & \text{if } \lVert\check{\mathbf{q}}\rVert^2 \geq \epsilon^2\\ \left(0, \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\mathsf{T}\right) & \text{otherwise}\end{cases}$$

Rotation Quaternion ⇔ Direction Cosine Matrix

$$\mathbf{C}_{IB} = \mathbb{1}_{3\times3} + 2q_0\hat{\check{\mathbf{q}}} + 2\hat{\check{\mathbf{q}}}^2 = (2q_0^2 - 1)\mathbb{1}_{3\times3} + 2q_0\hat{\check{\mathbf{q}}} + 2\check{\mathbf{q}}\check{\mathbf{q}}^\mathsf{T}$$

$$= \begin{bmatrix}p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_1p_2 - 2p_0p_3 & 2p_0p_2 + 2p_1p_3\\ 2p_0p_3 + 2p_1p_2 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_2p_3 - 2p_0p_1\\ 2p_1p_3 - 2p_0p_2 & 2p_0p_1 + 2p_2p_3 & p_0^2 - p_1^2 - p_2^2 + p_3^2\end{bmatrix}$$

$$\mathbf{C}_{IB}^{-1} = \mathbf{C}_{BI} = \mathbb{1}_{3\times3} - 2p_0\hat{\check{\mathbf{p}}} + 2\hat{\check{\mathbf{p}}}^2$$

$$= \begin{bmatrix}p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_0p_3 + 2p_1p_2 & 2p_1p_3 - 2p_0p_2\\ 2p_1p_2 - 2p_0p_3 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_0p_1 + 2p_2p_3\\ 2p_0p_2 + 2p_1p_3 & 2p_2p_3 - 2p_0p_1 & p_0^2 - p_1^2 - p_2^2 + p_3^2\end{bmatrix}$$

Euler Angles ZYX $\Leftrightarrow$ Direction Cosine Matrix

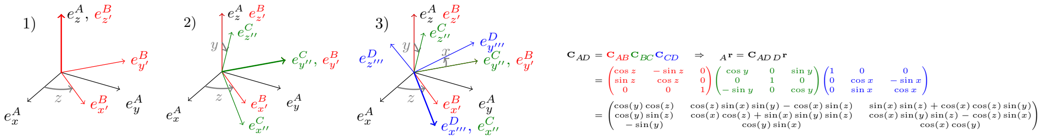


Figure 1: Rotation from A-frame to D-frame: $(z - y' - x'')$ – (yaw-pitch-roll) – $(50° - 25° - 30°)$

Euler Angles XYZ $\Leftrightarrow$ Direction Cosine Matrix

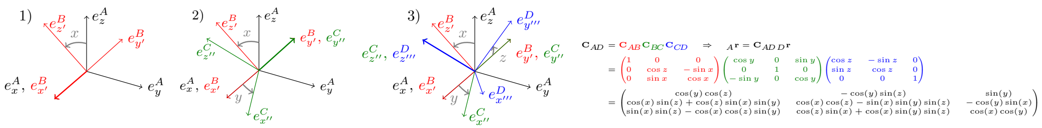


Figure 2: Rotation from A-frame to D-frame: $(x - y' - z'')$ – (roll-pitch-yaw) – $(50° - 25° - 30°)$

Pose

Homogeneous Transformation Matrix

$$\begin{bmatrix} {}_I\mathbf{r}_{IP} \\ 1 \end{bmatrix} = \mathbf{T}_{IB}\begin{bmatrix} {}_B\mathbf{r}_{BP} \\ 1 \end{bmatrix}, \quad \mathbf{T}_{IB} = \begin{bmatrix} \mathbf{C}_{IB} & {}_I\mathbf{r}_{IB} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \quad \mathbf{T}_{IB}^{-1} = \mathbf{T}_{BI} = \begin{bmatrix} \mathbf{C}_{IB}^\mathsf{T} & -\mathbf{C}_{IB}^\mathsf{T}\,{}_I\mathbf{r}_{IB} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

Time Derivatives of Position & Orientation

Linear Velocity

Velocity of point P expressed in a rotating frame B w.r.t. to the inertial frame I:

$${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$$

Velocity of point Q on rigid body B that rotates with ${}_B\boldsymbol{\Omega}$, where point P is on the same rigid body B:

$${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$$

Angular Velocity

${}_B\boldsymbol{\omega}_{IB} =: {}_B\boldsymbol{\Omega}$	(local) absolute angular velocity of rigid body B expr. in frame B
${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$	(global) angular velocity from frame B to frame I
${}_I\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^\mathsf{T}$	coord. transformation of angular velocity from frame B to frame I
${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$	composition of (relative) angular velocity

Time Derivative of Direction Cosine Matrix $\Leftrightarrow$ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^\mathsf{T} = \dot{\mathbf{C}}_{BI}^\mathsf{T}\mathbf{C}_{BI} \quad \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = {}_I\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}$$

$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^\mathsf{T}\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\dot{\mathbf{C}}_{BI}^\mathsf{T} \quad \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Quaternion $\Leftrightarrow$ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} \quad \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \tfrac{1}{2}\mathbf{H}(\mathbf{q}_{IB})^\mathsf{T}\,{}_I\boldsymbol{\omega}_{IB}$$

$${}_B\boldsymbol{\omega}_{IB} = 2\bar{\mathbf{H}}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} \quad \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \tfrac{1}{2}\bar{\mathbf{H}}(\mathbf{q}_{IB})^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}$$

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} -\check{\mathbf{q}} & \hat{\check{\mathbf{q}}} + q_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4} \qquad \bar{\mathbf{H}}(\mathbf{q}) = \begin{bmatrix} -\check{\mathbf{q}} & -\hat{\check{\mathbf{q}}} + q_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4}$$

$$= \begin{bmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \qquad = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

Time Derivative of Angle-Axis $\Leftrightarrow$ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$

$${}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1-\cos\theta)$$

$$\dot{\theta} = \mathbf{n}^\mathsf{T}\,{}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-cos\theta}\hat{\mathbf{n}}^2 - \tfrac{1}{2}\hat{\mathbf{n}}\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\theta\in\mathbb{R}\backslash\{0\}$$

$$\dot{\theta} = \mathbf{n}^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-cos\theta}\hat{\mathbf{n}}^2 + \tfrac{1}{2}\hat{\mathbf{n}}\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\theta\in\mathbb{R}\backslash\{0\}$$

Time Derivative of Rotation Vector $\Leftrightarrow$ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} + \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\|\in\mathbb{R}\backslash\{0\}$$

$${}_B\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} - \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\|\in\mathbb{R}\backslash\{0\}$$

$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} - \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\|\in\mathbb{R}\backslash\{0\}$$

$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} + \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\|\in\mathbb{R}\backslash\{0\}$$

Time Derivative of Euler Angles ZYX $\Leftrightarrow$ Angular Velocity

$$\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sin(x)}{\cos(y)} & \frac{\cos(x)}{\cos(y)} \\ 0 & \cos(x) & -\sin(x) \\ 1 & \frac{\sin(x)\sin(y)}{\cos(y)} & \frac{\cos(x)\sin(y)}{\cos(y)} \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall y\in\mathbb{R}\backslash\{\tfrac{\pi}{2}+k\pi\}, k\in\mathbb{Z}$$

$$\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} \frac{\cos(z)\sin(y)}{\cos(y)} & \frac{\sin(y)\sin(z)}{\cos(y)} & 1 \\ -\sin(z) & \cos(z) & 0 \\ \frac{\cos(z)}{\cos(y)} & \frac{\sin(z)}{\cos(y)} & 0 \end{bmatrix} {}_I\boldsymbol{\omega}_{IB} \quad \forall y\in\mathbb{R}\backslash\{\tfrac{\pi}{2}+k\pi\}, k\in\mathbb{Z}$$

$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} -\sin(y) & 0 & 1 \\ \cos(y)\sin(x) & \cos(x) & 0 \\ \cos(x)\cos(y) & -\sin(x) & 0 \end{bmatrix}\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix}$$

$${}_I\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & -\sin(z) & \cos(y)\cos(z) \\ 0 & \cos(z) & \cos(y)\sin(z) \\ 1 & 0 & -\sin(y) \end{bmatrix}\begin{bmatrix} \dot{z} \\ \dot{y} \\ \dot{x} \end{bmatrix}$$

Dynamics of a Multi-Rigid-Body System

n	Number of bodies in system
n_j	Number of DoFs of the joints
n_q	Number of generalized coordinates
n_u	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{F}_Q^A$	External forces on point Q
$\mathbf{M}^A$	External torques
m	Mass
$\boldsymbol{\Theta}$	Inertia tensor
$(...)^-$	Variable before impact
$(...)^+$	Variable after impact
$(...)^\pm$	Variable before/after impact
Δt	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\boldsymbol{\lambda}$	Lebesgue-measurable contact forces
$\boldsymbol{\Lambda}$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions

Generalized Coordinates of a Floating-Base System with Rotational Joints

Recommended set of generalized coordinates $\mathbf{q}$ with quaternion $\mathbf{q}_{IB}$ and generalized velocities $\mathbf{u}$:

$$\mathbf{q} = \begin{pmatrix} {}_I\mathbf{r}_{IB} \\ \mathbf{q}_{IB} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \qquad \mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot\varphi_1 \\ \vdots \\ \dot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \qquad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ {}_B\boldsymbol{\psi}_{IB} \\ \ddot\varphi_1 \\ \vdots \\ \ddot\varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3\times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\mathsf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3x3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j\times n_j} \end{pmatrix}$$

Equations of Motion with Contacts and no Impulses

Projected Newton-Euler Equations

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \quad \begin{aligned} \mathbf{M} &= \sum_{i=1}^{n} \left[(\mathbf{J}_S^\mathsf{T} m \mathbf{J}_S + \mathbf{J}_R^\mathsf{T} \boldsymbol{\Theta}_S \mathbf{J}_R) \right]_i \\ \mathbf{g} &= \sum_{i=1}^{n} \left[(\mathbf{J}_S^\mathsf{T} m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\mathsf{T} (\boldsymbol{\Theta}_S \dot{\mathbf{J}}_R \mathbf{u} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \boldsymbol{\Omega})) \right]_i \\ \mathbf{f} &= \sum_{i=1}^{n} \left[(\mathbf{J}_Q^\mathsf{T} \mathbf{F}_Q^A + \mathbf{J}_R^\mathsf{T} \mathbf{M}^A) \right]_i \end{aligned}$$

Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta\mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{aligned} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) \quad &= \mathbf{W}\boldsymbol{\Lambda} \\ \mathbf{M}\underbrace{(\dot{\mathbf{u}}\mathrm{d}t + (\mathbf{u}^+ - \mathbf{u}^-)\mathrm{d}\eta)}_{\mathrm{d}\mathbf{u}} - \mathbf{h}\mathrm{d}t \quad &= \mathbf{W}\underbrace{(\boldsymbol{\lambda}\mathrm{d}t + \boldsymbol{\Lambda}\mathrm{d}\eta)}_{\mathrm{d}\mathbf{P}} \end{aligned} \right.$$

Transformation of Equations of Motion

Transformation from $\bar{\mathbf{M}}(\bar{\mathbf{q}}), \bar{\mathbf{h}}(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ to $\mathbf{M}(\mathbf{q}), \mathbf{h}(\mathbf{q}, \mathbf{u})$, where $\bar{\mathbf{u}} = \mathbf{B}\mathbf{u}$:

$$\mathbf{M} = \mathbf{B}^\mathsf{T}\bar{\mathbf{M}}\mathbf{B}$$
$$\mathbf{h} = \mathbf{B}^\mathsf{T}\bar{\mathbf{h}} - \mathbf{B}^\mathsf{T}\bar{\mathbf{M}}\dot{\mathbf{B}}\mathbf{u}$$

Contact: Christian Gehring (gehrinch@ethz.ch)