

# Robotics Cheat Sheet

## Nomenclature

(Hyper-)complex number	$Q$	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbf{1}_{n \times m}$	$n \times m$ -matrix
Coordinate system (CS)	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	Cartesian right-hand system $B$ with basis (unit) vectors $\mathbf{e}$
Machine precision	$\epsilon$	

## Operators

Cross product	$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge \mathbf{b} = \hat{\mathbf{a}} \mathbf{b} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$
Euclidean norm	$\ \mathbf{a}\  = \sqrt{\mathbf{a}^T \mathbf{a}} = \sqrt{a_1^2 + \dots + a_n^2}$
Quaternion multiplicator	$\mathbf{q} \otimes \mathbf{p} \Leftrightarrow Q \cdot P$

## Pose (Position & Orientation)

### Position

Position vector	$I\mathbf{r}_{OP} \in \mathbb{R}^3$	vector from point $O$ to point $P$ expr. in CS $I$
Homogeneous pos. vector	$I\bar{\mathbf{r}}_{OP} = [I\mathbf{r}_{OP}^T \ 1]^T$	vector from point $O$ to point $P$ expr. in CS $I$

### Orientation

Rotation matrix	$\mathbf{R}_{BA} \in \text{SO}(3)$	Rotates the coord. of the basis vectors ( ${}_A\mathbf{e}_x^A, {}_A\mathbf{e}_y^A, {}_A\mathbf{e}_z^A$ ) of frame $A$ expressed in $A$ into the coordinates of the basis vectors ( ${}_A\mathbf{e}_x^B, {}_A\mathbf{e}_y^B, {}_A\mathbf{e}_z^B$ ) of $B$ expressed in $A$ . The rotation is active (alibi). ${}_A\mathbf{R}_{BA} = [{}_A\mathbf{e}_x^B \ {}_A\mathbf{e}_y^B \ {}_A\mathbf{e}_z^B]$
Direct Cosine matrix	$\mathbf{C}_{BA} \in \text{SO}(3)$ ${}_B\mathbf{r}_{OP} = \mathbf{C}_{BA} {}_A\mathbf{r}_{OP}$ $\mathbf{C}_{BA} = \mathbf{R}_{BA}^T$	The coordinate transformation matrix, which transforms vectors from frame $A$ to frame $B$ . The rotation is passive (alias).
Rotation Quaternion	$\mathbf{p}_{BA}$ $\mathbf{p}_{BA} \Leftrightarrow \mathbf{R}_{BA}$	The rotation is active (alibi).
Rotation Angle-axis	$(\theta, \mathbf{n})_{BA}$ $(\theta, \mathbf{n})_{BA} \Leftrightarrow \mathbf{R}_{BA}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta \in [-\pi, \pi)$ . The rotation is active (alibi).
Rotation vector	$\phi_{BA}$ $\phi_{BA} \Leftrightarrow \mathbf{R}_{BA}$	Rotation with rotation axis $\mathbf{n} = \frac{\phi}{\ \phi\ }$ and angle $\theta = \ \phi\ $ . The rotation is active (alibi).
Euler Angles ZYX Euler Angles YPR	$(\psi, \theta, \phi)_{BA}$ $(\psi, \theta, \phi)_{BA} \Leftrightarrow \mathbf{C}_{BA}$	Tait-Bryan angles (Flight conv.): $z - y' - x''$ , i.e. yaw-pitch-roll. Singularities are at $\theta = \pm \frac{\pi}{2}$ . $\psi \in [-\pi, \pi), \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}), \phi \in [-\pi, \pi)$
Euler Angles XYZ Euler Angles RPY	$(\alpha, \beta, \gamma)_{BA}$ $(\alpha, \beta, \gamma)_{BA} \Leftrightarrow \mathbf{C}_{BA}$	Cardan angles (Glocker conv.): $x - y' - z''$ , i.e. roll-pitch-yaw. Singularities are at $\beta = \pm \frac{\pi}{2}$ . $\alpha \in [-\pi, \pi), \beta \in [-\frac{\pi}{2}, \frac{\pi}{2}), \gamma \in [-\pi, \pi)$

### Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$$P = p_0 + p_1 i + p_2 j + p_3 k \in \mathbb{H}, \quad p_i \in \mathbb{R}$$

$$i^2 = j^2 = k^2 = ijk = -1, \quad \|P\| = \sqrt{p_0^2 + p_1^2 + p_2^2 + p_3^2} = 1$$

Note that  $P_{BA}$  and  $-P_{BA}$  represent the same rotation, but not the same unit quaternion.

Rot. quaternion as tuple:  $P = (p_0, p_1, p_2, p_3) = (p_0, \vec{\mathbf{p}})$  with  $\vec{\mathbf{p}} := (p_1, p_2, p_3)^T$

Rot. quaternion as vector:  $\mathbf{p} = [p_0 \ p_1 \ p_2 \ p_3]^T$

Conjugate:  $P^* = (p_0, -\vec{\mathbf{p}})$

Inverse:  $P^{-1} = P^* = (p_0, -\vec{\mathbf{p}})$

Quaternion multiplication:

$$Q \cdot P = (q_0, \vec{\mathbf{q}}) \cdot (p_0, \vec{\mathbf{p}}) = (q_0 p_0 - \vec{\mathbf{q}}^T \vec{\mathbf{p}}, q_0 \vec{\mathbf{p}} + p_0 \vec{\mathbf{q}} + \vec{\mathbf{q}} \times \vec{\mathbf{p}}) \Leftrightarrow$$

$$\mathbf{q} \otimes \mathbf{p} = \mathbf{Q}(\mathbf{q})\mathbf{p} = \begin{pmatrix} q_0 & -\vec{\mathbf{q}}^T \\ \vec{\mathbf{q}} & q_0 \mathbf{1}_{3 \times 3} + \hat{\vec{\mathbf{p}}} \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{pmatrix} = \mathbf{Q}(\mathbf{p})^T \mathbf{q}$$

### Rotation Quaternion $\Leftrightarrow$ Rotation Angle-Axis

$$\mathbf{p}_{BI} = \begin{bmatrix} \cos \frac{\theta}{2} \\ \mathbf{n} \sin \frac{\theta}{2} \end{bmatrix} \Leftrightarrow (\theta, \mathbf{n})_{BI} = \begin{cases} (2 \arccos(a_0), \frac{\vec{\mathbf{p}}}{\|\vec{\mathbf{p}}\|}) & \text{if } \|\vec{\mathbf{p}}\|^2 \geq \epsilon^2 \\ (0, [1 \ 0 \ 0]^T) & \text{otherwise} \end{cases}$$

### Rotation Quaternion $\Leftrightarrow$ Direction Cosine Matrix

$$\mathbf{C}_{AB} = \mathbf{R}_{AB}^T (\mathbf{p}_{AB}) = \mathbf{1}_{3 \times 3} + 2p_0 \hat{\vec{\mathbf{p}}} + 2\vec{\mathbf{p}}^2$$

$$= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_1 p_2 - 2p_0 p_3 & 2p_0 p_2 + 2p_1 p_3 \\ 2p_0 p_3 + 2p_1 p_2 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_2 p_3 - 2p_0 p_1 \\ 2p_1 p_3 - 2p_0 p_2 & 2p_0 p_1 + 2p_2 p_3 & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}$$

$$\mathbf{C}_{BA} = \mathbf{R}_{BA}^T (\mathbf{p}_{AB}) = \mathbf{1}_{3 \times 3} - 2p_0 \hat{\vec{\mathbf{p}}} + 2\vec{\mathbf{p}}^2$$

$$= \begin{bmatrix} p_0^2 + p_1^2 - p_2^2 - p_3^2 & 2p_0 p_3 + 2p_1 p_2 & 2p_1 p_3 - 2p_0 p_2 \\ 2p_1 p_2 - 2p_0 p_3 & p_0^2 - p_1^2 + p_2^2 - p_3^2 & 2p_0 p_1 + 2p_2 p_3 \\ 2p_0 p_2 + 2p_1 p_3 & 2p_2 p_3 - 2p_0 p_1 & p_0^2 - p_1^2 - p_2^2 + p_3^2 \end{bmatrix}$$

$$\mathbf{p}_{BA} = \mathbf{p}_{BA}(\mathbf{C}_{AB}) = \begin{bmatrix} \frac{1}{2} \sqrt{1 + \text{tr}(\mathbf{C})} \\ \frac{C_{32} - C_{23}}{4p_0} \\ \frac{C_{13} - C_{31}}{4p_0} \\ \frac{C_{21} - C_{12}}{4p_0} \end{bmatrix} \quad \text{if } \text{tr}(\mathbf{C}) > 0 \ (\mathbf{C}_{AB} \rightarrow \mathbf{p}_{BA} \text{ is not unique})$$

### Euler Angles ZYX $\Leftrightarrow$ Direction Cosine Matrix

$$\mathbf{C}_{DA} = \mathbf{C}_{DC} \mathbf{C}_{CB} \mathbf{C}_{BA} \Rightarrow D^r = \mathbf{C}_{DA} A^r$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{pmatrix} \begin{pmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{pmatrix} \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos \psi \cos \theta & \cos \psi \sin \theta & -\sin \psi \\ \cos \phi \cos \psi + \sin \phi \sin \psi \sin \theta & \cos \phi \sin \psi + \sin \phi \cos \psi \sin \theta & \cos \phi \cos \psi \sin \theta - \cos \phi \sin \psi \sin \theta \\ \sin \phi \cos \psi + \cos \phi \sin \psi \sin \theta & \cos \phi \sin \psi + \sin \phi \cos \psi \sin \theta & \cos \phi \cos \psi \sin \theta - \cos \phi \sin \psi \sin \theta \end{pmatrix}$$

$$\mathbf{C}_{AD} = \mathbf{C}_{DA}^T$$

$$= \begin{pmatrix} \cos \psi \cos \theta & \cos \phi \cos \psi + \sin \phi \sin \psi \sin \theta & \sin \phi \cos \psi + \cos \phi \sin \psi \sin \theta \\ \cos \theta \sin \psi & \cos \phi \sin \psi + \sin \phi \cos \psi \sin \theta & \cos \phi \sin \psi \sin \theta - \cos \psi \sin \theta \\ -\sin \theta & \cos \phi \cos \psi & \cos \phi \sin \psi \end{pmatrix}$$

Figure 1: Rotation from  $I$ -frame to  $B$ -frame:  $(z - y' - x'')$  – (yaw-pitch-roll) –  $(\psi - \theta - \phi) - (50^\circ - 25^\circ - 30^\circ)$

### Euler Angles XYZ $\Leftrightarrow$ Direction Cosine Matrix

$$\mathbf{C}_{DA} = \mathbf{C}_{DC} \mathbf{C}_{CB} \mathbf{C}_{BA} \Rightarrow D^r = \mathbf{C}_{DA} A^r$$

$$= \begin{pmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \beta & 0 & -\sin \beta \\ 0 & 1 & 0 \\ \sin \beta & 0 & \cos \beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \gamma & \sin \gamma \\ 0 & -\sin \gamma & \cos \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos \beta \cos \gamma & \cos \alpha \sin \gamma + \cos \gamma \sin \alpha \sin \beta & \sin \alpha \sin \gamma - \cos \alpha \cos \gamma \sin \beta \\ -\cos \beta \sin \gamma & \cos \alpha \cos \gamma - \sin \alpha \sin \beta \sin \gamma & \cos \gamma \sin \alpha + \cos \alpha \sin \beta \sin \gamma \\ \sin \beta & -\cos \beta \sin \alpha & \cos \alpha \cos \beta \end{pmatrix}$$

$$\mathbf{C}_{AD} = \mathbf{C}_{DA}^T$$

$$= \begin{pmatrix} \cos \beta \cos \gamma & \cos \alpha \sin \gamma + \cos \gamma \sin \alpha \sin \beta & \sin \alpha \sin \gamma - \cos \alpha \cos \gamma \sin \beta \\ \cos \alpha \sin \gamma + \cos \gamma \sin \alpha \sin \beta & \cos \alpha \cos \gamma - \sin \alpha \sin \beta \sin \gamma & \cos \gamma \sin \alpha + \cos \alpha \sin \beta \sin \gamma \\ -\cos \beta \sin \gamma & \cos \gamma \sin \alpha + \cos \alpha \sin \beta \sin \gamma & \cos \alpha \cos \beta \end{pmatrix}$$

Figure 2: Rotation from  $I$ -frame to  $B$ -frame:  $(x - y' - z'')$  – (roll-pitch-yaw) –  $(\alpha - \beta - \gamma) - (50^\circ - 25^\circ - 30^\circ)$

## Pose

Homogeneous Transformation Matrix	$\mathbf{T}_{AB}$	
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### Homogeneous Transformation Matrix

$$\mathbf{T}_{AB} = \begin{bmatrix} \mathbf{C}_{AB} & {}_A\mathbf{r}_{AB} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

## Twist (Translational & Rotational Velocity)

### Translational Velocity

${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$  velocity of point  $P$  expr. in CS  $B$  w.r.t. to the inertial system  $I$   
 ${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}$  velocity of point  $Q$  on rigid body  $B$  from point  $P$  on same body

### Angular Velocity

${}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$  ang. vel. of rigid body with body-fixed CS  $B$  w.r.t. to inertial CS  $I$   
 ${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$  inverse of angular velocity  
 ${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}{}_B\boldsymbol{\omega}_{IB}$  rotation from  $B$  to  $I$   
 ${}_I\dot{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}{}_B\dot{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^\top$  rotation from  $B$  to  $I$   
 ${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$  composition of angular velocity  
 ${}_I\dot{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{R}}_{BI}\mathbf{R}_{BI}^\top = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^\top$  angular velocity of  $B$  w.r.t.  $I$  expressed in CS  $I$   
 ${}_B\dot{\boldsymbol{\omega}}_{IB} = \mathbf{R}_{IB}\dot{\mathbf{R}}_{IB}^\top = \mathbf{C}_{IB}^\top\dot{\mathbf{C}}_{IB}$  angular velocity of  $B$  w.r.t.  $I$  expressed in CS  $B$   
 ${}_B\boldsymbol{\omega}_{IB} = 2\tilde{\mathbf{H}}(\mathbf{p}_{BI})\dot{\mathbf{p}}_{BI}$  ang. velocity from time derivative of the rotation quaternion with

$$\tilde{\mathbf{H}}(\mathbf{p}) = \begin{bmatrix} -\vec{\mathbf{p}} & -\dot{\vec{\mathbf{p}}} + p_0\mathbf{1}_{3 \times 3} \end{bmatrix} \in \mathbb{R}^{3 \times 4}$$

$$= \begin{bmatrix} -p_1 & p_0 & p_3 & -p_2 \\ -p_2 & -p_3 & p_0 & p_1 \\ -p_3 & p_2 & -p_1 & p_0 \end{bmatrix}$$

$\dot{\mathbf{p}}_{BI} = \frac{1}{2}\tilde{\mathbf{H}}(\mathbf{p}_{BI})^\top {}_B\boldsymbol{\omega}_{IB}$  time derivative of the rotation quaternion

## Dynamics of a Multi-Rigid-Body System

$n$	Number of bodies in system
$n_j$	Number of DoFs of the joints
$n_q$	Number of generalized coordinates
$n_u$	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{F}_Q^A$	External forces on point Q
$\mathbf{M}^A$	External torques
$m$	Mass
$\boldsymbol{\Theta}$	Intertia tensor
$(\dots)^-$	Variable before impact
$(\dots)^+$	Variable after impact
$(\dots)^\pm$	Variable before/after impact
$\Delta t$	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\boldsymbol{\lambda}$	Lebesgue-measurable contact forces
$\boldsymbol{\Lambda}$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions

## Generalized Coordinates of a Floating-Base System with Rotational Joints

$$\mathbf{q} = \begin{pmatrix} {}^I\mathbf{r}_{OB} \\ \mathbf{p}_{BI} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \quad \mathbf{u} = \begin{pmatrix} {}^I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \quad \dot{\mathbf{u}} = \begin{pmatrix} {}^I\mathbf{a}_B \\ {}_B\dot{\boldsymbol{\psi}}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\tilde{\mathbf{H}}^\top & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j \times n_j} \end{pmatrix} \Leftrightarrow \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbf{I}_{3 \times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\tilde{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_j \times n_j} \end{pmatrix}$$

### Equation of Motion with Contacts and no Impulses

$$\mathbf{M} = \sum_{i=1}^n \left[ (\mathbf{J}_S^\top m \mathbf{J}_S + \mathbf{J}_R^\top \boldsymbol{\Theta}_S \mathbf{J}_R) \right]_i$$

$$\boxed{\mathbf{M}\ddot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \mathbf{g} = \sum_{i=1}^n \left[ (\mathbf{J}_S^\top m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\top (\boldsymbol{\Theta}_S \dot{\mathbf{J}}_R \mathbf{u} + \boldsymbol{\Omega} \times \boldsymbol{\Theta}_S \boldsymbol{\Omega})) \right]_i$$

$$\mathbf{f} = \sum_{i=1}^n \left[ (\mathbf{J}_Q^\top \mathbf{F}_Q^A + \mathbf{J}_R^\top \mathbf{M}^A) \right]_i$$

### Equation of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta \mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \left\{ \begin{array}{l} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) = \mathbf{W}\boldsymbol{\Lambda} \\ \mathbf{M}(\underbrace{\dot{\mathbf{u}}dt + (\mathbf{u}^+ - \mathbf{u}^-)d\eta}_{d\mathbf{u}}) - \mathbf{h}d\mathbf{t} = \mathbf{W}(\underbrace{\boldsymbol{\lambda}dt + \boldsymbol{\Lambda}d\eta}_{d\mathbf{P}}) \end{array} \right.$$

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