Kindr Cheat Sheet v1.0

Kinematics and Dynamics for Robotics

Nomenclature

(Hyper-)complex number	Q	normal capital letter
Column vector	$\mathbf{a}$	bold small letter
Matrix	$\mathbf{M}$	bold capital letter
Identity matrix	$\mathbb{1}_{n\times m}$	$n\times m$-matrix
Coordinate system (CS)	$\mathbf{e}_x^A, \mathbf{e}_y^A, \mathbf{e}_z^A$	Cartesian right-hand system A with basis (unit) vectors $\mathbf{e}$
Inertial frame	$\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I$	global / inertial / world coordinate system (never moves)
Body-fixed frame	$\mathbf{e}_x^B, \mathbf{e}_y^B, \mathbf{e}_z^B$	local / body-fixed coordinate system (moves with body)
Rotation	$\mathcal{R}\in SO(3)$	generic rotation (for all parameterizations)
Machine precision	ϵ	

Operators

Cross product	$\mathbf{a}\times\mathbf{b} = \begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}\times\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \Leftrightarrow (\mathbf{a})^\wedge\mathbf{b} = \hat{\mathbf{a}}\mathbf{b} = \begin{bmatrix}0 & -a_3 & a_2\\ a_3 & 0 & -a_1\\ -a_2 & a_1 & 0\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$
Skew/unskew	$\mathbf{a} = \hat{\mathbf{a}}^\vee$
Euclidean norm	$\lVert\mathbf{a}\rVert = \sqrt{\mathbf{a}^T\mathbf{a}} = \sqrt{a_1^2+\ldots+a_n^2}$
Exponential map for matrix	$\text{expM}: \mathbb{R}^3\to\mathbb{R}^3, \mathbf{A}\mapsto e^{\mathbf{A}}, \quad \mathbf{A}\in\mathbb{R}^{3\times3}$
Logarithmic map for matrix	$\text{logM}: \mathbb{R}^3\to\mathbb{R}^3, \mathbf{A}\mapsto \log\mathbf{A}, \quad \mathbf{A}\in\mathbb{R}^{3\times3}$

Position & Orientation

Position

Vector	$\mathbf{r}_{OP}$	from point O to point P
Position vector	${}_B\mathbf{r}_{OP}\in\mathbb{R}^3$	from point O to point P expr. in frame B
Homogeneous pos. vector	${}_B\bar{\mathbf{r}}_{OP} = \begin{bmatrix}{}_B\mathbf{r}_{OP}^\mathsf{T} & 1\end{bmatrix}^\mathsf{T}$	from point O to point P expr. in frame B

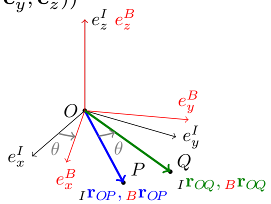
Orientation/Rotation

1) Active Rotation: $\mathcal{R}^A : {}_A\mathbf{r}_{OP} \mapsto {}_A\mathbf{r}_{OQ}$ (rotates the vector $\mathbf{r}_{OP}$)
2) Passive Rotation: $\mathcal{R}^P : {}_A\mathbf{r}_{OP} \mapsto {}_B\mathbf{r}_{OP}$ (rotates the frame $(\mathbf{e}_x^I, \mathbf{e}_y^I, \mathbf{e}_z^I)$)
3) Elementary Rotations ${}_I\mathbf{r}_{OP} = \mathbf{C}_{IB}\,{}_B\mathbf{r}_{OP}$

around z-axis: $\mathbf{C}_{IB} = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$

around y-axis: $\mathbf{C}_{IB} = \begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$

around x-axis: $\mathbf{C}_{IB} = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}$



4) Inversion: $\mathcal{R}^{A^{-1}}(\mathbf{r}) = \mathcal{R}^P(\mathbf{r})$
5) Concatenation:

$$\mathcal{R}_2^A\left(\mathcal{R}_1^A(\mathbf{r})\right) = \left(\mathcal{R}_2^A\otimes\mathcal{R}_1^A\right)(\mathbf{r}) = \left(\mathcal{R}_1^{A^{-1}}\otimes\mathcal{R}_2^{A^{-1}}\right)^{-1}(\mathbf{r})$$

$$\mathcal{R}_2^P\left(\mathcal{R}_1^P(\mathbf{r})\right) = \left(\mathcal{R}_2^P\otimes\mathcal{R}_1^P\right)(\mathbf{r}) = \left(\mathcal{R}_1^{P^{-1}}\otimes\mathcal{R}_2^{P^{-1}}\right)^{-1}(\mathbf{r})$$

6) Exponential map: $\exp: \mathbb{R}^3\to SO(3), \mathbf{v}\mapsto \text{expM}(\hat{\mathbf{v}}), \quad \mathbf{v}\in\mathbb{R}^3$
7) Logarithmic map: $\log: SO(3)\to\mathbb{R}^3, \mathcal{R}\mapsto \text{logM}(\mathcal{R})^\vee, \quad \mathcal{R}\in SO(3)$
8) Box plus: $\mathcal{R}_2 = \mathcal{R}_1 \boxplus \mathbf{v} = \exp(\mathbf{v})\otimes\mathcal{R}_1, \quad \mathcal{R}_1,\mathcal{R}_2\in SO(3), \mathbf{v}\in\mathbb{R}^3$
9) Box minus: $\mathbf{v} = \mathcal{R}_1 \boxminus \mathcal{R}_2 = \log(\mathcal{R}_1\otimes\mathcal{R}_2^{-1}), \quad \mathcal{R}_1,\mathcal{R}_2\in SO(3), \mathbf{v}\in\mathbb{R}^3$
10) Discrete integration: $\mathcal{R}_{IB}^{k+1} = \mathcal{R}_{IB}^k \boxplus ({}_I\boldsymbol{\omega}_{IB}^k\Delta t), \quad \mathcal{R}_{BI}^{k+1} = \mathcal{R}_{BI}^k \boxplus (-{}_B\boldsymbol{\omega}_{IB}^k\Delta t)$
11) Discrete differential: ${}_I\boldsymbol{\omega}_{IB}^k = (\mathcal{R}_{IB}^{k+1}\boxminus\mathcal{R}_{IB}^k)/\Delta t, \quad {}_B\boldsymbol{\omega}_{IB}^k = -(\mathcal{R}_{BI}^{k+1}\boxminus\mathcal{R}_{BI}^k/\Delta t)$
12) (Spherical) linear interpolation $t\in[0,1]$: $\mathcal{R}_t = \mathcal{R}_0 \boxplus ((\mathcal{R}_1\boxminus\mathcal{R}_0)t), \quad \mathcal{R}_t = \mathcal{R}(t), \mathcal{R}_0 = \mathcal{R}(0), \mathcal{R}_1 = \mathcal{R}(1)$
$= (\mathcal{R}_1\otimes\mathcal{R}_0^{-1})^t\otimes\mathcal{R}_0$

Rotation Parameterizations

Rotation Matrix	$\mathbf{C}_{BA}\in SO(3)$ ${}_B\mathbf{r}_{OP} = \mathbf{C}_{BA}\,{}_A\mathbf{r}_{OP}$ $\mathbf{C}_{BA} = \mathbf{C}_{AB}^\mathsf{T}$	The rotation matrix (Direction Cosine Matrix) is a coordinate transformation matrix, which transforms vectors from frame A to frame B.
Rotation Quaternion	$\mathbf{q}_{BA}$ $\mathbf{q} = [q_0\ q_1\ q_2\ q_3]^\mathsf{T}$	Hamiltonian unit quaternion (hypercomplex number) $Q = q_0 + q_1 i + q_2 j + q_3 k \in\mathbb{H}, \quad q_i\in\mathbb{R}, \quad \lVert Q\rVert = 1$
Angle-axis	$(\theta,\mathbf{n})_{BA}$	Rotation with unit rotation axis $\mathbf{n}$ and angle $\theta\in[0,\pi]$.
Rotation Vector	$\boldsymbol{\phi}_{BA}$	Rotation with rotation axis $\mathbf{n} = \frac{\boldsymbol{\phi}}{\lVert\boldsymbol{\phi}\rVert}$ and angle $\theta = \lVert\boldsymbol{\phi}\rVert$.
Euler Angles ZYX Euler Angles YPR	$(\psi,\theta,\phi)_{BA}$	Tait-Bryan angles (Flight conv.): $z-y'-x''$, i.e. yaw-pitch-roll. Singularities are at $\theta = \pm\frac{\pi}{2}$. $\psi\in[-\pi,\pi), \theta\in[-\frac{\pi}{2},\frac{\pi}{2}), \phi\in[-\pi,\pi)$
Euler Angles XYZ Euler Angles RPY	$(\alpha,\beta,\gamma)_{BA}$	Cardan angles (Glocker conv.): $x-y'-z''$, i.e. roll-pitch-yaw. Singularities are at $\beta = \pm\frac{\pi}{2}$. $\alpha\in[-\pi,\pi), \beta\in[-\frac{\pi}{2},\frac{\pi}{2}), \gamma\in[-\pi,\pi)$

Rotation Quaternion

A rotation quaternion is a Hamiltonian unit quaternion:

$Q = q_0 + q_1 i + q_2 j + q_3 k \in\mathbb{H}, \quad q_i\in\mathbb{R}, i^2 = j^2 = k^2 = ijk = -1, \quad \lVert Q\rVert = \sqrt{q_0^2+q_1^2+q_2^2+q_3^2} = 1$

Note that Q_{AB} and $-Q_{AB}$ represent the same rotation, but not the same unit quaternion.

Rot. quaternion as tuple: $Q = (q_0, q_1, q_2, q_3) = (q_0, \check{\mathbf{q}})$ with $\check{\mathbf{q}} := (q_1, q_2, q_3)^\mathsf{T}$

Rot. quaternion as vector: $\mathbf{q} = \begin{bmatrix}q_0 & q_1 & q_2 & q_3\end{bmatrix}^\mathsf{T}$

Conjugate: $Q^* = (q_0, -\check{\mathbf{q}})$

Inverse: $Q^{-1} = Q^* = (q_0, -\check{\mathbf{q}})$

Quaternion multiplication:

$Q\cdot P = (q_0,\check{\mathbf{q}})\cdot(p_0,\check{\mathbf{p}}) = (q_0 p_0 - \check{\mathbf{q}}^\mathsf{T}\check{\mathbf{p}}, q_0\check{\mathbf{p}} + p_0\check{\mathbf{q}} + \check{\mathbf{q}}\times\check{\mathbf{p}}) \quad \Leftrightarrow$

$$\mathbf{q}\otimes\mathbf{p} = \underbrace{\mathbf{Q}(\mathbf{q})}_{\text{quaternion matrix}}\mathbf{p} = \begin{pmatrix}q_0 & -\check{\mathbf{q}}^\mathsf{T}\\ \check{\mathbf{q}} & q_0\mathbb{1}_{3\times3} + \hat{\check{\mathbf{q}}}\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix} = \begin{pmatrix}q_0 & -q_1 & -q_2 & -q_3\\ q_1 & q_0 & -q_3 & q_2\\ q_2 & q_3 & q_0 & -q_1\\ q_3 & -q_2 & q_1 & q_0\end{pmatrix}\begin{pmatrix}p_0\\p_1\\p_2\\p_3\end{pmatrix}$$

$$= \underbrace{\bar{\mathbf{Q}}(\mathbf{p})}_{\text{conjugate quat. matrix}}\mathbf{q} = \begin{pmatrix}p_0 & -\check{\mathbf{p}}^\mathsf{T}\\ \check{\mathbf{p}} & p_0\mathbb{1}_{3\times3} - \hat{\check{\mathbf{p}}}\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix} = \begin{pmatrix}p_0 & -p_1 & -p_2 & -p_3\\ p_1 & p_0 & p_3 & -p_2\\ p_2 & -p_3 & p_0 & p_1\\ p_3 & p_2 & -p_1 & p_0\end{pmatrix}\begin{pmatrix}q_0\\q_1\\q_2\\q_3\end{pmatrix}$$

Rotation Quaternion ⇔ Rotation Angle-Axis

$$\mathbf{q}_{IB} = \begin{bmatrix}\cos\frac{\theta}{2}\\ \mathbf{n}\sin\frac{\theta}{2}\end{bmatrix} \quad\Leftrightarrow\quad (\theta,\mathbf{n})_{IB} = \begin{cases}\left(2\arccos(p_0), \frac{\check{\mathbf{q}}}{\lVert\check{\mathbf{q}}\rVert}\right) & \text{if } \lVert\check{\mathbf{q}}\rVert^2\geq\epsilon^2\\ \left(0, \begin{bmatrix}1 & 0 & 0\end{bmatrix}^\mathsf{T}\right) & \text{otherwise}\end{cases}$$

Rotation Quaternion ⇔ Direction Cosine Matrix

$$\mathbf{C}_{IB} = \mathbb{1}_{3\times3} + 2q_0\hat{\check{\mathbf{q}}} + 2\hat{\check{\mathbf{q}}}^2 = (2q_0^2-1)\mathbb{1}_{3\times3} + 2q_0\hat{\check{\mathbf{q}}} + 2\check{\mathbf{q}}\check{\mathbf{q}}^\mathsf{T}$$

$$= \begin{bmatrix}p_0^2+p_1^2-p_2^2-p_3^2 & 2p_1p_2-2p_0p_3 & 2p_0p_2+2p_1p_3\\ 2p_0p_3+2p_1p_2 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_2p_3-2p_0p_1\\ 2p_1p_3-2p_0p_2 & 2p_0p_1+2p_2p_3 & p_0^2-p_1^2-p_2^2+p_3^2\end{bmatrix}$$

$$\mathbf{C}_{IB}^{-1} = \mathbf{C}_{BI} = \mathbb{1}_{3\times3} - 2p_0\hat{\check{\mathbf{p}}} + 2\hat{\check{\mathbf{p}}}^2$$

$$= \begin{bmatrix}p_0^2+p_1^2-p_2^2-p_3^2 & 2p_0p_3+2p_1p_2 & 2p_1p_3-2p_0p_2\\ 2p_1p_2-2p_0p_3 & p_0^2-p_1^2+p_2^2-p_3^2 & 2p_0p_1+2p_2p_3\\ 2p_0p_2+2p_1p_3 & 2p_2p_3-2p_0p_1 & p_0^2-p_1^2-p_2^2+p_3^2\end{bmatrix}$$

Euler Angles ZYX ⇔ Direction Cosine Matrix

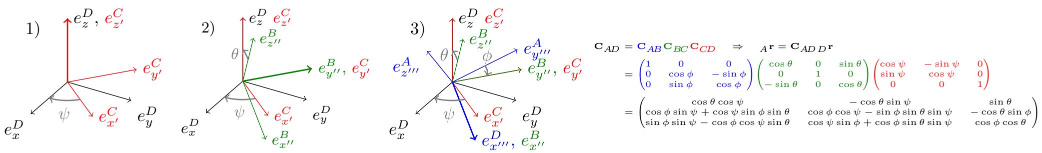


Figure 1: Rotation from A-frame to D-frame: $(z - y' - x'')$ – (yaw-pitch-roll) – $(\psi - \theta - \phi)$ – $(50° - 25° - 30°)$

Euler Angles XYZ ⇔ Direction Cosine Matrix

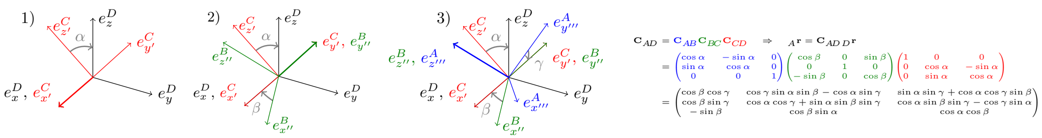


Figure 2: Rotation from A-frame to D-frame: $(x - y' - z'')$ – (roll-pitch-yaw) – $(\alpha - \beta - \gamma)$ – $(50° - 25° - 30°)$

Pose

Homogeneous Transformation Matrix

$$\begin{bmatrix} {}_I\mathbf{r}_{IP} \\ 1 \end{bmatrix} = \mathbf{T}_{IB}\begin{bmatrix} {}_B\mathbf{r}_{BP} \\ 1 \end{bmatrix}, \quad \mathbf{T}_{IB} = \begin{bmatrix} \mathbf{C}_{IB} & {}_I\mathbf{r}_{IB} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix} \quad \mathbf{T}_{IB}^{-1} = \mathbf{T}_{BI} = \begin{bmatrix} \mathbf{C}_{IB}^\mathsf{T} & -\mathbf{C}_{IB}^\mathsf{T}\,{}_I\mathbf{r}_{IB} \\ \mathbf{0}^\mathsf{T} & 1 \end{bmatrix}$$

Time Derivatives of Position & Orientation

Linear Velocity

Velocity of point P expressed in frame B w.r.t. to the inertial frame I:

$${}_B\mathbf{v}_P = {}_B\mathbf{v}_A + {}_B\dot{\mathbf{r}}_{AP} + {}_B\boldsymbol{\omega}_{IB} \times {}_B\mathbf{r}_{AP}$$

Velocity of point Q on rigid body B that rotates with ${}_B\boldsymbol{\Omega}$, where point P is on the same rigid body B:

$${}_B\mathbf{v}_Q = {}_B\mathbf{v}_P + {}_B\boldsymbol{\Omega} \times {}_B\mathbf{r}_{PQ}, \quad {}_B\boldsymbol{\Omega} = {}_B\boldsymbol{\omega}_{IB}$$

Angular Velocity

${}_B\boldsymbol{\omega}_{IB} =: {}_B\boldsymbol{\Omega}$	(local) absolute angular velocity of rigid body B expr. in frame B
${}_B\boldsymbol{\omega}_{IB} = -{}_B\boldsymbol{\omega}_{BI}$	inverse of angular velocity
${}_I\boldsymbol{\omega}_{IB} = \mathbf{C}_{IB}\,{}_B\boldsymbol{\omega}_{IB}$	(global) angular velocity from frame B to frame I
${}_I\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}^\mathsf{T}$	coord. transformation of angular velocity from frame B to frame I
${}_D\boldsymbol{\omega}_{AD} = {}_D\boldsymbol{\omega}_{AB} + {}_D\boldsymbol{\omega}_{BC} + {}_D\boldsymbol{\omega}_{CD}$	composition of (relative) angular velocity

Time Derivative of Direction Cosine Matrix ⇔ Angular Velocity

$${}_I\hat{\boldsymbol{\omega}}_{IB} = \dot{\mathbf{C}}_{IB}\mathbf{C}_{IB}^\mathsf{T} = \dot{\mathbf{C}}_{BI}^\mathsf{T}\mathbf{C}_{BI} \quad \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = {}_I\hat{\boldsymbol{\omega}}_{IB}\mathbf{C}_{IB}$$
$${}_B\hat{\boldsymbol{\omega}}_{IB} = \mathbf{C}_{IB}^\mathsf{T}\dot{\mathbf{C}}_{IB} = \mathbf{C}_{BI}\dot{\mathbf{C}}_{BI}^\mathsf{T} \quad \Leftrightarrow \quad \dot{\mathbf{C}}_{IB} = \mathbf{C}_{IB}\,{}_B\hat{\boldsymbol{\omega}}_{IB}$$

Time Derivative of Rotation Quaternion ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = 2\mathbf{H}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} \quad \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \tfrac{1}{2}\mathbf{H}(\mathbf{q}_{IB})^\mathsf{T}\,{}_I\boldsymbol{\omega}_{IB}$$
$${}_B\boldsymbol{\omega}_{IB} = 2\bar{\mathbf{H}}(\mathbf{q}_{IB})\dot{\mathbf{q}}_{IB} \quad \Leftrightarrow \quad \dot{\mathbf{q}}_{IB} = \tfrac{1}{2}\bar{\mathbf{H}}(\mathbf{q}_{IB})^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}$$

$$\mathbf{H}(\mathbf{q}) = \begin{bmatrix} -\check{\mathbf{q}} & \hat{\check{\mathbf{q}}} + q_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4} \qquad \bar{\mathbf{H}}(\mathbf{q}) = \begin{bmatrix} -\check{\mathbf{q}} & -\hat{\check{\mathbf{q}}} + q_0\mathbb{1}_{3\times3} \end{bmatrix} \in \mathbb{R}^{3\times4}$$
$$= \begin{bmatrix} -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{bmatrix} \qquad = \begin{bmatrix} -q_1 & q_0 & q_3 & -q_2 \\ -q_2 & -q_3 & q_0 & q_1 \\ -q_3 & q_2 & -q_1 & q_0 \end{bmatrix}$$

Time Derivative of Angle-Axis ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta + \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos\theta)$$
$${}_B\boldsymbol{\omega}_{IB} = \mathbf{n}\dot{\theta} + \dot{\mathbf{n}}\sin\theta - \hat{\mathbf{n}}\dot{\mathbf{n}}(1 - \cos\theta)$$
$$\dot{\theta} = \mathbf{n}^\mathsf{T}\,{}_I\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-cos\theta}\hat{\mathbf{n}}^2 - \tfrac{1}{2}\hat{\mathbf{n}}\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$
$$\dot{\theta} = \mathbf{n}^\mathsf{T}\,{}_B\boldsymbol{\omega}_{IB}, \quad \dot{\mathbf{n}} = \left(-\tfrac{1}{2}\tfrac{\sin\theta}{1-cos\theta}\hat{\mathbf{n}}^2 + \tfrac{1}{2}\hat{\mathbf{n}}\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{0\}$$

Time Derivative of Rotation Vector ⇔ Angular Velocity

$${}_I\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} + \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$${}_B\boldsymbol{\omega}_{IB} = \left(\mathbb{1}_{3\times3} - \hat{\boldsymbol{\phi}}\left(\frac{1-\cos\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^2}\right) + \hat{\boldsymbol{\phi}}^2\left(\frac{\|\boldsymbol{\phi}\|-\sin\|\boldsymbol{\phi}\|}{\|\boldsymbol{\phi}\|^3}\right)\right)\dot{\boldsymbol{\phi}} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} - \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_I\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$
$$\dot{\boldsymbol{\phi}} = \left(\mathbb{1}_{3\times3} + \tfrac{1}{2}\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\phi}}^2\frac{1}{\|\boldsymbol{\phi}\|^2}\left(1 - \frac{\|\boldsymbol{\phi}\|}{2}\frac{\sin\|\boldsymbol{\phi}\|}{1-\cos\|\boldsymbol{\phi}\|}\right)\right){}_B\boldsymbol{\omega}_{IB} \quad \forall\|\boldsymbol{\phi}\| \in \mathbb{R}\backslash\{0\}$$

Time Derivative of Euler Angles ZYX ⇔ Angular Velocity

$$\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \frac{-\cos\psi\sin\theta}{\cos\theta} & \frac{(\sin\theta\sin\psi)}{\cos\theta} & 1 \\ \sin\psi & \cos\psi & 0 \\ \frac{\cos\psi}{\cos\theta} & \frac{-\sin\psi}{\cos\theta} & 0 \end{bmatrix} {}_B\boldsymbol{\omega}_{IB} \quad \forall\theta \in \mathbb{R}\backslash\{\tfrac{\pi}{2} + k\pi\}, k \in \mathbb{Z}$$

$${}_B\boldsymbol{\omega}_{IB} = \begin{bmatrix} 0 & \sin\psi & \cos\theta\cos\psi \\ 0 & \cos\psi & -\cos\theta\sin\psi \\ 1 & 0 & \sin\theta \end{bmatrix}\begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix}$$

Dynamics of a Multi-Rigid-Body System

n	Number of bodies in system
n_j	Number of DoFs of the joints
n_q	Number of generalized coordinates
n_u	Number of generalized velocities
$\mathbf{M}$	Mass matrix
$\mathbf{g}$	Gyroscopic and Coriolis forces
$\mathbf{f}$	Generalized external forces and torques
$\mathbf{h}$	Combined force vector
$\mathbf{J}_P$	Jacobi matrix for translation of point P
$\mathbf{J}_R$	Jacobi matrix for rotation
$\mathbf{F}_Q^A$	External forces on point Q
$\mathbf{M}^A$	External torques
m	Mass
$\mathbf{\Theta}$	Intertia tensor
$(...)^-$	Variable before impact
$(...)^+$	Variable after impact
$(...)^\pm$	Variable before/after impact
Δt	Time step duration
$\Delta \mathbf{u}$	Velocity change over one time step
$\mathbf{W}$	Generalized force directions for contact forces
$\boldsymbol{\lambda}$	Lebesgue-measurable contact forces
$\mathbf{\Lambda}$	Purely atomic impact impulses
$\mathbf{P}$	Contact percussions

Generalized Coordinates of a Floating-Base System with Rotational Joints

Recommended set of generalized coordinates $\mathbf{q}$ with quaternion $\mathbf{q}_{IB}$ and generalized velocities $\mathbf{u}$:

$$\mathbf{q} = \begin{pmatrix} {}_I\mathbf{r}_{IB} \\ \mathbf{q}_{IB} \\ \varphi_1 \\ \vdots \\ \varphi_{n_j} \end{pmatrix} \in \mathbb{R}^{7+n_j} = \mathbb{R}^{n_q} \qquad \mathbf{u} = \begin{pmatrix} {}_I\mathbf{v}_B \\ {}_B\boldsymbol{\omega}_{IB} \\ \dot{\varphi}_1 \\ \vdots \\ \dot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j} = \mathbb{R}^{n_u} \qquad \dot{\mathbf{u}} = \begin{pmatrix} {}_I\mathbf{a}_B \\ {}_B\boldsymbol{\psi}_{IB} \\ \ddot{\varphi}_1 \\ \vdots \\ \ddot{\varphi}_{n_j} \end{pmatrix} \in \mathbb{R}^{6+n_j}$$

$$\dot{\mathbf{q}} = \mathbf{F}\mathbf{u}, \quad \mathbf{F} = \begin{pmatrix} \mathbb{1}_{3\times 3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{2}\bar{\mathbf{H}}^\mathsf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j \times n_j} \end{pmatrix} \quad \Leftrightarrow \quad \mathbf{u} = \bar{\mathbf{F}}\dot{\mathbf{q}}, \quad \bar{\mathbf{F}} = \begin{pmatrix} \mathbb{1}_{3x3} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\bar{\mathbf{H}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbb{1}_{n_j \times n_j} \end{pmatrix}$$

Equations of Motion with Contacts and no Impulses

Projected Newton-Euler Equations

$$\boxed{\mathbf{M}\dot{\mathbf{u}} - \mathbf{h} = \mathbf{W}\boldsymbol{\lambda}} \text{ with } \mathbf{h} := \mathbf{f} - \mathbf{g}, \text{ and } \quad \begin{aligned} \mathbf{M} &= \sum_{i=1}^{n} \left[(\mathbf{J}_S^\mathsf{T} m \mathbf{J}_S + \mathbf{J}_R^\mathsf{T} \mathbf{\Theta}_S \mathbf{J}_R) \right]_i \\ \mathbf{g} &= \sum_{i=1}^{n} \left[(\mathbf{J}_S^\mathsf{T} m \dot{\mathbf{J}}_S \mathbf{u} + \mathbf{J}_R^\mathsf{T} (\mathbf{\Theta}_S \dot{\mathbf{J}}_R \mathbf{u} + \mathbf{\Omega} \times \mathbf{\Theta}_S \mathbf{\Omega})) \right]_i \\ \mathbf{f} &= \sum_{i=1}^{n} \left[(\mathbf{J}_Q^\mathsf{T} \mathbf{F}_Q^A + \mathbf{J}_R^\mathsf{T} \mathbf{M}^A) \right]_i \end{aligned}$$

Equations of Motion with Contacts and Impulses

$$\boxed{\mathbf{M}\Delta\mathbf{u} - \mathbf{h}\Delta t = \mathbf{W}\mathbf{P}} \quad \begin{cases} \mathbf{M}(\mathbf{u}^+ - \mathbf{u}^-) &= \mathbf{W}\mathbf{\Lambda} \\ \mathbf{M}\underbrace{(\dot{\mathbf{u}}\mathrm{d}t + (\mathbf{u}^+ - \mathbf{u}^-)\mathrm{d}\eta)}_{\mathrm{d}\mathbf{u}} - \mathbf{h}\mathrm{d}t &= \mathbf{W}\underbrace{(\boldsymbol{\lambda}\mathrm{d}t + \mathbf{\Lambda}\mathrm{d}\eta)}_{\mathrm{d}\mathbf{P}} \end{cases}$$

Transformation of Equations of Motion

Transformation from $\bar{\mathbf{M}}(\bar{\mathbf{q}}), \bar{\mathbf{h}}(\bar{\mathbf{q}}, \bar{\mathbf{u}})$ to $\mathbf{M}(\mathbf{q}), \mathbf{h}(\mathbf{q}, \mathbf{u})$, where $\bar{\mathbf{u}} = \mathbf{B}\mathbf{u}$:

$$\begin{aligned} \mathbf{M} &= \mathbf{B}^\mathsf{T}\bar{\mathbf{M}}\mathbf{B} \\ \mathbf{h} &= \mathbf{B}^\mathsf{T}\bar{\mathbf{h}} - \mathbf{B}^\mathsf{T}\bar{\mathbf{M}}\dot{\mathbf{B}}\mathbf{u} \end{aligned}$$

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