QUANTITATIVE METHODS IN FINANCE
Portfolio Management

Teruo Nakatsuma

Faculty of Economics, Keio University

GENERAL EDUCATION SEMINAR (PEARL)

Portfolio Analysis

Portfolio = a collection of assets that the investor holds

Types of assets: stocks, bonds, currencies, real estate

Questions to be answered

- How much we must invest our money in each asset?
- What is the best way to form a suitable portfolio?

Return On A Portfolio Of Assets

The return on a portfolio of asset is a weighted average of the return on the individual assets.

$$R_P = \sum_{n=1}^{N} w_n R_n.$$

R_P: the return on the portfolio

R_n: the return on the n-th asset

w_n: the fraction of the funds invested in the n-th asset

Measure Of Average Outcome

Expected return

$$\mu_P = \mathrm{E}(R_P) = \mathrm{E}\left(\sum_{n=1}^{N} w_n R_n\right)$$
$$= \sum_{n=1}^{N} w_n \mathrm{E}(R_n) = \sum_{n=1}^{N} w_n \mu_n.$$

The expected return on a portfolio is the weighted average of the expected return on the individual assets.

Measure Of Dispersion

Variance

$$\sigma_P^2 = \mathrm{E}[(R_P - \mu_P)^2] = \mathrm{E}\left[\left\{\sum_{n=1}^{N} w_n(R_n - \mu_n)\right\}^2\right]$$

$$= \sum_{n=1}^{N} w_n^2 \mathrm{E}[(R_n - \mu_n)^2]$$

$$+ \sum_{n=1}^{N} \sum_{m \neq n}^{N} w_n w_m \mathrm{E}[(R_n - \mu_n)(R_m - \mu_m)]$$

$$= \sum_{n=1}^{N} w_n^2 \sigma_n^2 + \sum_{n=1}^{N} \sum_{m \neq n}^{N} w_n w_m \sigma_{nm}.$$

Example: Portfolio Of Two Assets

The expected return on a portfolio of two assets is

$$\mu_P = w_1\mu_1 + w_2\mu_2,$$

and the variance is

$$\begin{aligned}\sigma_P^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\sigma_{12} \\ &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2,\end{aligned}$$

where $\sigma_{12} = \rho_{12}\sigma_1\sigma_2$ and ρ_{12} is the correlation coefficient between R_1 and R_2.

Example: Portfolio Of Three Assets

The expected return on a portfolio of three assets is

$$\mu_P = w_1\mu_1 + w_2\mu_2 + w_3\mu_3,$$

and the variance is

$$\begin{aligned}
\sigma_P^2 &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 \\
&\quad + 2w_1w_2\sigma_{12} + 2w_1w_3\sigma_{13} + 2w_2w_3\sigma_{23} \\
&= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + w_3^2\sigma_3^2 \\
&\quad + 2w_1w_2\rho_{12}\sigma_1\sigma_2 + 2w_1w_3\rho_{13}\sigma_1\sigma_3 \\
&\quad + 2w_2w_3\rho_{23}\sigma_2\sigma_3,
\end{aligned}$$

Limit Of Diversification

If $w_n = \frac{1}{N}$ for all assets, the variance of the portfolio is

$$
\begin{aligned}
\sigma_P^2 &= \sum_{n=1}^{N} \frac{\sigma_n^2}{N^2} + \sum_{n=1}^{N} \sum_{m \neq n}^{N} \frac{\sigma_{nm}}{N^2} \\
&= \frac{1}{N} \underbrace{\sum_{n=1}^{N} \frac{\sigma_n^2}{N}}_{\bar{\sigma}_n^2} + \frac{N-1}{N} \underbrace{\sum_{n=1}^{N} \sum_{m \neq n}^{N} \frac{\sigma_{nm}}{N(N-1)}}_{\bar{\sigma}_{nm}} \\
&= \frac{1}{N}(\bar{\sigma}_n^2 - \bar{\sigma}_{nm}) + \bar{\sigma}_{nm}.
\end{aligned}
$$

Thus σ_P^2 will converge to $\bar{\sigma}_{nm} \neq 0$ as N goes to infinity.

Basic Idea Of Portfolio Selection

- Expected return as the performance measure

  The investors love a higher return from their investment.

- Variance as the risk measure

  The investors want to avoid a higher variation in the value of their funds.

But we cannot have both.

We need to find a “sweet spot” in the trade-off between risk and return.

Minimum Variance Portfolio Problem I

$$
\begin{aligned}
\min_{w_1,\dots,w_N} \quad & \sigma_P^2 = \sum_{n=1}^{N} w_n^2 \sigma_n^2 + \sum_{n=1}^{N} \sum_{m \neq n}^{N} w_n w_n \sigma_{nm}, \\
\text{s.t.} \quad & \text{(a)} \ \sum_{n=1}^{N} w_n \mu_n = \mu_P, \\
& \text{(b)} \ \sum_{n=1}^{N} w_n = 1, \\
& \text{(c)} \ w_n \geqq 0, \quad (n = 1, \dots, N).
\end{aligned} \tag{1}
$$

- The constraint (a) sets the target return of the portfolio.
- The constraint (b) is required due to the definition of $w_1, \dots, w_n$.
- The constraint (c) prohibits short selling.

Minimum Variance Portfolio Problem (Matrix Form)

$$
\begin{aligned}
\min_{\boldsymbol{w}} \quad & \sigma_P^2 = \boldsymbol{w}^\intercal \boldsymbol{\Sigma} \boldsymbol{w}, \\
\text{s.t.} \quad & \text{(a) } \boldsymbol{w}^\intercal \boldsymbol{\mu} = \mu_P, \\
& \text{(b) } \boldsymbol{w}^\intercal \boldsymbol{\iota} = 1, \\
& \text{(c) } \boldsymbol{w} \geqq \boldsymbol{0},
\end{aligned}
\tag{2}
$$

where

$$
\boldsymbol{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \ \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_N \end{bmatrix}, \ \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & \cdots & \sigma_{1N} \\ \vdots & \ddots & \vdots \\ \sigma_{N1} & \cdots & \sigma_N^2 \end{bmatrix},
$$

and $\boldsymbol{\iota}$ is the $N \times 1$ vector whose elements are equal to 1.

Optimal Portfolio Selection With The Efficient Frontier

- For each given $\boldsymbol{\mu_P}$, the minimum variance $\boldsymbol{\sigma_P^2}$ is obtained by solving the minimum variance portfolio problem (2). The graph of $(\boldsymbol{\sigma_P}, \boldsymbol{\mu_P})$ is called the minimum variance frontier.
- In particular, the upper half of the minimum variance frontier is called the efficient frontier because the investor prefers a portfolio with a higher rate of return when the risk is the same.

The efficient frontier shows us the trade-off between risk and return. The investor must choose the optimal portfolio among those on the efficient frontier.

Step 1. Construct the efficient frontier.

Step 2. Set the tolerable level of risk or the target expected return.

Step 3. Pick the corresponding portfolio on the efficient frontier.

Solution

If we drop the constraint (c) which prohibits short selling, we can derive the solution of the minimum variance portfolio problem in a closed form:

$$\boldsymbol{w} = \frac{C\mu_P - A}{D}\Sigma^{-1}\mu + \frac{B - A\mu_P}{D}\Sigma^{-1}\iota$$

$$A = \mu^\intercal \Sigma^{-1}\iota, \quad B = \mu^\intercal \Sigma^{-1}\mu,$$
$$C = \iota^\intercal \Sigma^{-1}\iota, \quad D = BC - A^2.$$

The relationship between μ_P and σ_P is given by

$$\sigma_P = \sqrt{\frac{C\mu_P^2 - 2A\mu_P + B}{D}} = \sqrt{\frac{C}{D}\left(\mu_P - \frac{A}{C}\right)^2 + \frac{1}{C}}.$$

This is the minimum variance frontier.

Remarks On Minimum Variance Portfolio Selection

1. The minimum variance portfolio problem without short selling (2) has no closed-form solution.
2. Since the minimum variance portfolio problem (2) is a type of quadratic programming problem, it can be solved by a quadratic programming solver.
3. We may incorporate additional conditions (trading fees, taxes, upper and lower bounds of weights, etc.) into the minimum variance portfolio problem (2).

When μ And Σ Are Unknown I

So far we assume we know μ and Σ in (2). In reality, however, we need to estimate them with data. Let r_{nt} denote realized return on asset n at period t ($t = 1, \dots, T$). The sample mean $\bar{r}_n$ and the sample covariance s_{nm} (or the sample variance s_n^2 when $n = m$) are defined as

$$\bar{r}_n = \frac{1}{T}\sum_{t=1}^{T} r_{it}, \; s_{nm} = \frac{1}{T}\sum_{t=1}^{T}(r_{nt} - \bar{r}_n)(r_{mt} - \bar{r}_m), \; (n, m = 1, \dots, N),$$

A straightforward way is to replace μ and Σ in (2) with

$$\bar{r} = \begin{bmatrix} \bar{r}_1 \\ \vdots \\ \bar{r}_N \end{bmatrix}, \quad S = \begin{bmatrix} s_1^2 & \cdots & s_{1N} \\ \vdots & \ddots & \vdots \\ s_{N1} & \cdots & s_N^2 \end{bmatrix},$$

respectively, but a more elegant method is known in the literature.

When μ And Σ Are Unknown II

$$
\begin{aligned}
w^{\intercal} S w &= \sum_{n=1}^{N} \sum_{m=1}^{N} w_n w_m s_{nm} \\
&= \sum_{n=1}^{N} \sum_{m=1}^{N} w_n w_m \left\{ \frac{1}{T} \sum_{t=1}^{T} (r_{nt} - \bar{r}_n)(r_{mt} - \bar{r}_m) \right\} \\
&= \frac{1}{T} \sum_{t=1}^{T} \sum_{n=1}^{N} w_n (r_{nt} - \bar{r}_n) \sum_{m=1}^{N} w_n (r_{nt} - \bar{r}_n) \\
&= \frac{1}{T} \sum_{t=1}^{T} \left\{ \sum_{n=1}^{N} w_n (r_{nt} - \bar{r}_n) \right\}^2 = \frac{1}{T} \sum_{t=1}^{T} (r_{Pt} - \bar{r}_P)^2 , \\
r_{Pt} &= \sum_{n=1}^{N} w_n r_{nt}, \quad \bar{r}_P = \sum_{n=1}^{N} w_n \bar{r}_n = \frac{1}{T} \sum_{t=1}^{T} r_{Pt},
\end{aligned}
$$

When μ And Σ Are Unknown III

Remarks

- $\boldsymbol{r_{Pt}}$ is a *realized return* of the portfolio with allocation weights $\boldsymbol{w}$ at period $\boldsymbol{t}$
- $\boldsymbol{\bar{r}_P}$ is the average of realized returns of the portfolio with allocation weights $\boldsymbol{w}$ from period 1 to period $\boldsymbol{T}$.
- $\frac{1}{T}\sum_{t=1}^{T}(r_{Pt}-\bar{r}_P)^2$ is the sample variance of realized returns of the portfolio with allocation weights $\boldsymbol{w}$ from period 1 to period $\boldsymbol{T}$.
- Using $\boldsymbol{w}^\top \boldsymbol{S}\boldsymbol{w}$ implies that we use the realized variance of a portfolio in a hypothetical situation; What if we invest our money on a portfolio with allocation weights $\boldsymbol{w}$ from period 1 to period $\boldsymbol{T}$?

When $\boldsymbol{\mu}$ And $\boldsymbol{\Sigma}$ Are Unknown IV

Define $\boldsymbol{v_t} = \boldsymbol{r_{Pt}} - \boldsymbol{\bar{r}_P}$. Then the minimum variance problem with unknown $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ is given by

Minimum variance portfolio problem

$$
\begin{aligned}
\min_{\boldsymbol{w},\boldsymbol{v}} \quad & \widehat{\mathrm{Var}}[\boldsymbol{R_P}] = \frac{1}{\boldsymbol{T}}\boldsymbol{v}^\intercal \boldsymbol{v}, \\
\text{s.t.} \quad & \boldsymbol{Dw} = \boldsymbol{v}, \quad \boldsymbol{w}^\intercal \boldsymbol{\bar{r}} = \boldsymbol{\mu_P}, \quad \boldsymbol{w}^\intercal \boldsymbol{\iota} = 1, \\
& \boldsymbol{w_1} \geqq 0, \ \cdots, \ \boldsymbol{w_N} \geqq 0,
\end{aligned} \tag{3}
$$

where

$$
\boldsymbol{v} = \begin{bmatrix} \boldsymbol{v_1} \\ \vdots \\ \boldsymbol{v_T} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} \boldsymbol{r_{11}} - \boldsymbol{\bar{r}_1} & \cdots & \boldsymbol{r_{N1}} - \boldsymbol{\bar{r}_N} \\ \vdots & \ddots & \vdots \\ \boldsymbol{r_{1T}} - \boldsymbol{\bar{r}_1} & \cdots & \boldsymbol{r_{NT}} - \boldsymbol{\bar{r}_N} \end{bmatrix}.
$$

Alternative Risk Criterion

Other than the variance $\mathbf{E}[(\boldsymbol{R_P} - \boldsymbol{\mu_P})^2]$, many alternative risk criteria have been proposed in the literature.

In this lecture, we study the following three alternatives:

- Mean absolute deviation: $\mathbf{E}[|\boldsymbol{R_P} - \boldsymbol{\mu_P}|]$
- Semivariance: $\mathbf{E}[(\boldsymbol{R_P} - \boldsymbol{\mu_P})^2 | \boldsymbol{R_{Pt}} \leqq \boldsymbol{\mu_P}]$
- Expected shortfall: $\mathbf{E}[-\boldsymbol{R_P} | \boldsymbol{R_P} \leqq \mathrm{VaR}_\alpha]$, $(\mathbf{Pr}\{\boldsymbol{R_P} \leqq \mathrm{VaR}_\alpha\} = \boldsymbol{\alpha})$

Mean Absolute Deviation Optimization

The sample mean absolute deviation is

$$\varrho^{AD}(\boldsymbol{w}) = \frac{1}{T}\sum_{t=1}^{T}|r_{Pt} - \bar{r}_P|. \tag{4}$$

Define $\boldsymbol{v_t} = r_{Pt} - \bar{r}_P$ and use the same notations as in (3). Then we have

Minimum mean absolute deviation portfolio problem

$$\begin{aligned} \min_{\boldsymbol{w},\boldsymbol{v}} \quad & \varrho^{AD}(\boldsymbol{w}) = \frac{1}{T}\sum_{t=1}^{T}|v_t|, \\ \text{s.t.} \quad & \boldsymbol{D}\boldsymbol{w} = \boldsymbol{v}, \\ & \boldsymbol{w}^\top\bar{\boldsymbol{r}} = \mu_P, \quad \boldsymbol{w}^\top\boldsymbol{\iota} = 1, \\ & w_1 \geqq 0, \cdots, w_N \geqq 0, \end{aligned} \tag{5}$$

Semivariance Optimization

The sample semivariance is

$$\varrho^{SV}(\boldsymbol{w}) = \frac{1}{T}\sum_{t=1}^{T}\{[r_{Pt} - \bar{r}_P]^{-}\}^2, \tag{6}$$

where $[r_{Pt} - \bar{r}_P]^{-} = \max\{-(r_{Pt} - \bar{r}_P), 0\}$ is call the negative part.
Define $v_t = [r_{Pt} - \bar{r}_P]^{-}$. Then we have

Minimum semivariance portfolio problem

$$\begin{aligned}
\min_{\boldsymbol{w},\boldsymbol{v}} \quad & \varrho^{SV}(\boldsymbol{w}) = \frac{1}{T}\sum_{t=1}^{T} v_t^2, \\
\text{s.t.} \quad & \boldsymbol{w}^{\intercal}\bar{\boldsymbol{r}} = \mu_P, \quad \boldsymbol{w}^{\intercal}\boldsymbol{\iota} = 1, \\
& w_1 \geqq 0, \ \cdots, \ w_N \geqq 0, \ v_1 \geqq 0, \ \cdots, \ v_T \geqq 0, \\
& r_{P1} - \bar{r}_P + v_1 \geqq 0, \ \cdots, \ r_{PT} - \bar{r}_P + v_T \geqq 0,
\end{aligned} \tag{7}$$

Value-At-Risk and Expected Shortfall

The value at risk (VaR) of a portfolio is defined as

VaR

$$\Pr\{\boldsymbol{R_P} \leqq \mathrm{VaR}_{\alpha}\} = \alpha. \tag{8}$$

The VaR is the $100\times\alpha\%$ point of the distribution of $\boldsymbol{R_P}$.

The expected shortfall (ES) is defined as

Expected shortfall

$$\mathrm{ES}_{\alpha} = \mathbf{E}[-\boldsymbol{R_P}|\boldsymbol{R_P} \leqq \mathrm{VaR}_{\alpha}] \tag{9}$$

The ES is interpreted as the conditional expected loss under a severe market condition in which our portfolio suffers from a rare but huge loss ($\boldsymbol{R_P} \leqq \mathrm{VaR}_{\alpha}$).

Coherent Risk Measure I

Suppose that $\mathcal{X}$ is a set of random variables. We regard each $\boldsymbol{X} \in \mathcal{X}$ as the return (or value) of a portfolio and let $\varrho(\boldsymbol{X})$ denote a risk measure of $\boldsymbol{X}$.

Coherent risk measure

$\varrho(\cdot)$ is said to be coherent if it satisfies the following conditions:

Monotonicity: For any $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}$,

$$\mathrm{Pr}\{\boldsymbol{X} \leqq \boldsymbol{Y}\} = 1 \text{ implies } \varrho(\boldsymbol{X}) \geqq \varrho(\boldsymbol{Y}).$$

Cash invariance: For $\boldsymbol{M} \in \mathcal{X}$ such that $\mathrm{Pr}\{\boldsymbol{M} = \boldsymbol{R}\} = 1$,

$$\varrho(\boldsymbol{X} + \boldsymbol{M}) = \varrho(\boldsymbol{X}) - \boldsymbol{R}.$$

Sub-additivity: For any $\boldsymbol{X}, \boldsymbol{Y} \in \mathcal{X}$, $\varrho(\boldsymbol{X} + \boldsymbol{Y}) \leqq \varrho(\boldsymbol{X}) + \varrho(\boldsymbol{Y})$.

Positive homogeneity: For any $\boldsymbol{\lambda} \geqq \mathbf{0}$, $\varrho(\boldsymbol{\lambda}\boldsymbol{X}) = \boldsymbol{\lambda}\varrho(\boldsymbol{X})$.

Coherent Risk Measure II

Interpretations of conditions

Monotonicity: A surely profitable portfolio should be less risky.

Cash invariance: Adding the riskless asset should reduce the risk.

Sub-additivity: Diversification should not make the portfolio risker.

Positive homogeneity: The risk should be proportional to the position.

When we replace sub-additivity and positive homogeneity with

Convexity: For any $\mathbf{0} \leqq \boldsymbol{w} \leqq \mathbf{1}$,

$$\varrho(\boldsymbol{w}\boldsymbol{X} + (1-\boldsymbol{w})\boldsymbol{Y}) \leqq \boldsymbol{w}\varrho(\boldsymbol{X}) + (1-\boldsymbol{w})\varrho(\boldsymbol{Y}).$$

$\varrho(\cdot)$ is said to be a convex risk measure.

Note: ES is coherent as well as convex but VaR is neither.

Expected Shortfall Optimization

If the probability density function of $\boldsymbol{R_P}$, say $\boldsymbol{p(R_P)}$, is known, the ES is given by

$$\mathrm{ES}_{\alpha} = \frac{1}{\alpha}\int_{-\infty}^{\infty}[\boldsymbol{R_P} - \mathrm{VaR}_{\alpha}]^{-}\boldsymbol{p}(\boldsymbol{R_P})\boldsymbol{dR_P} - \mathrm{VaR}_{\alpha}. \tag{10}$$

The portfolio selection problem with the ES is formulated as

Minimum expected shortfall portfolio problem

$$\begin{aligned} &\min_{\boldsymbol{w}} \quad \frac{1}{\alpha}\int_{-\infty}^{\infty}[\boldsymbol{R_P} - \mathrm{VaR}_{\alpha}]^{-}\boldsymbol{p}(\boldsymbol{R_P})\boldsymbol{dR_P} - \mathrm{VaR}_{\alpha}, \\ &\text{s.t.} \quad \boldsymbol{w}^{\intercal}\bar{\boldsymbol{r}} = \mu_P, \quad \boldsymbol{w}^{\intercal}\boldsymbol{\iota} = 1, \quad \boldsymbol{w}_1 \geqq 0, \dots, \boldsymbol{w}_N \geqq 0. \end{aligned} \tag{11}$$

Unfortunately, (11) is a non-linear optimization problem which is difficult to solve unless we introduce additional assumptions such as the normality of asset returns.

Approximation Method

To circumvent this obstacle, Rockafellar and Uryasev (2000) proposed an approximation method. If $\boldsymbol{T}$ is large enough, the ES (10) is approximately equivalent to

$$\varrho^{ES}(\boldsymbol{w}, \boldsymbol{c}) = \frac{1}{\alpha \boldsymbol{T}} \sum_{t=1}^{\boldsymbol{T}} [\boldsymbol{r}_{\boldsymbol{P}t} - \boldsymbol{c}]^{-} - \boldsymbol{c}. \tag{12}$$

Thus, by defining $\boldsymbol{v}_t = [\boldsymbol{r}_{\boldsymbol{P}t} - \boldsymbol{c}]^{-}$, we have

Minimum expected shortfall portfolio problem

$$\begin{aligned}
\min_{\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{c}} \quad & \varrho^{ES}(\boldsymbol{w}, \boldsymbol{c}) = \frac{1}{\alpha \boldsymbol{T}} \sum_{t=1}^{\boldsymbol{T}} \boldsymbol{v}_t - \boldsymbol{c}, \\
\text{s.t.} \quad & \boldsymbol{w}^\intercal \bar{\boldsymbol{r}} = \mu_{\boldsymbol{P}}, \quad \boldsymbol{w}^\intercal \iota = 1, \\
& \boldsymbol{w}_1 \geqq 0, \ \cdots, \ \boldsymbol{w}_{\boldsymbol{N}} \geqq 0, \ \boldsymbol{v}_1 \geqq 0, \ \cdots, \ \boldsymbol{v}_{\boldsymbol{T}} \geqq 0, \\
& \boldsymbol{r}_{\boldsymbol{P}1} - \boldsymbol{c} + \boldsymbol{v}_1 \geqq 0, \ \cdots, \ \boldsymbol{r}_{\boldsymbol{P}\boldsymbol{T}} - \boldsymbol{c} + \boldsymbol{v}_{\boldsymbol{T}} \geqq 0.
\end{aligned} \tag{13}$$

Risk Parity Approach

A risk parity approach of portfolio management focuses on balancing "risk allocation" among assets, instead of balancing the target return and the risk of the portfolio.

1. The mean-variance approach tends to produce an extremely skewed portfolio.
2. It is hard to obtain a reliable estimate of the expected return on any asset.
3. Many fund managers feel increasingly uncomfortable with the traditional asset allocation techniques since they were useless during such financial turmoil as the Global Financial Crisis.

Equal-weight (1/N) Portfolio

One trivial example of such portfolios that equalize the impact of each asset onto the total risk is

$1/N$ portfolio

$$w_n^{1/N} = \frac{1}{N}, \quad (n = 1, \ldots, N). \tag{14}$$

This is often refered to as the $1/N$ portfolio.

Global Minimum Variance Portfolio

The global minimum variance portfolio is the solution of the following optimization problem:

$$\begin{aligned} \min_{\boldsymbol{w}} \quad & \boldsymbol{w}^\intercal \boldsymbol{\Sigma} \boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{w}^\intercal \boldsymbol{\iota} = \mathbf{1}, \end{aligned} \tag{15}$$

that is,

Global minimum variance portfolio

$$\boldsymbol{w}^{MV} = \frac{\mathbf{1}}{\boldsymbol{\iota}^\intercal \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}} \boldsymbol{\Sigma}^{-1} \boldsymbol{\iota}. \tag{16}$$

Note that (16) does not depend on the expected return vector $\boldsymbol{\mu}$. Thus we do not need to estimate $\boldsymbol{\mu}$.

Properties

1. Suppose the variance of the return on the n-th asset is σ_n^2 $(n = 1, \ldots, n)$. If there is no correlation among the asset returns, the allocation weight in the global minimum variance portfolio is given by
$$w_n^{MV} = \frac{\sigma_n^{-2}}{\sum_{n=1}^{N} \sigma_n^{-2}}, \quad (n = 1, \ldots, N).$$
2. Suppose $\sigma_1^2 = \cdots = \sigma_N^2$ and the correlation coefficient between any pair of assets is constant. Then the global minimum variance portfolio is equivalent to the $1/N$ portfolio.

The first-order condition for (15) is

$$\Sigma \boldsymbol{w} = \lambda \boldsymbol{\iota}, \tag{17}$$

where λ is the Lagrangian multiplier. Because the first derivative of $\sigma(\boldsymbol{w}) = \sqrt{\boldsymbol{w}^\intercal \Sigma \boldsymbol{w}}$ is given by

$$\nabla \sigma(\boldsymbol{w}) = \begin{bmatrix} \nabla_{w_1} \sigma(\boldsymbol{w}) \\ \vdots \\ \nabla_{w_N} \sigma(\boldsymbol{w}) \end{bmatrix} = \begin{bmatrix} \nabla_1 \sigma(\boldsymbol{w}) \\ \vdots \\ \nabla_N \sigma(\boldsymbol{w}) \end{bmatrix} = \frac{1}{\sigma(\boldsymbol{w})} \Sigma \boldsymbol{w},$$

the first-order condition (17) implies

$$\nabla_1 \sigma(\boldsymbol{w}) = \cdots = \nabla_N \sigma(\boldsymbol{w}).$$

$\nabla_n \sigma(\boldsymbol{w})$ $(n = 1, \dots, N)$ is called the MRC (Marginal Risk Contribution).

Then the standard deviation $\boldsymbol{\sigma}(\boldsymbol{w})$ is decomposed as

$$\begin{aligned}\boldsymbol{\sigma}(\boldsymbol{w}) &= \frac{1}{\boldsymbol{\sigma}(\boldsymbol{w})}\boldsymbol{w}^\intercal\boldsymbol{\Sigma}\boldsymbol{w} = \boldsymbol{w}^\intercal\left(\frac{1}{\boldsymbol{\sigma}(\boldsymbol{w})}\boldsymbol{\Sigma}\boldsymbol{w}\right) \\ &= \boldsymbol{w}^\intercal\nabla\boldsymbol{\sigma}(\boldsymbol{w}) \\ &= \boldsymbol{w}_1\nabla_1\boldsymbol{\sigma}(\boldsymbol{w}) + \cdots + \boldsymbol{w}_N\nabla_N\boldsymbol{\sigma}(\boldsymbol{w}), \end{aligned} \tag{18}$$

$\boldsymbol{w}_n\nabla_n\boldsymbol{\sigma}(\boldsymbol{w})$ $(\boldsymbol{n} = 1, \dots, \boldsymbol{N})$ is interpreted as the contribution of the $\boldsymbol{n}$-th asset on the total risk of the portfolio and is called the TRC (Total Risk Contribution).

Risk Parity Portfolio I

A portfolio that satisfies

$$\boldsymbol{w}_1 \nabla_1 \sigma(\boldsymbol{w}) = \cdots = \boldsymbol{w}_N \nabla_N \sigma(\boldsymbol{w}), \tag{19}$$

is called the risk parity portfolio. Dividing both sides of (18) with $\sigma(\boldsymbol{w})$, we have

$$1 = \frac{\boldsymbol{w}_1 \nabla_1 \sigma(\boldsymbol{w})}{\sigma(\boldsymbol{w})} + \cdots + \frac{\boldsymbol{w}_N \nabla_N \sigma(\boldsymbol{w})}{\sigma(\boldsymbol{w})}.$$

Under (19),

$$\frac{\boldsymbol{w}_n \nabla_n \sigma(\boldsymbol{w})}{\sigma(\boldsymbol{w})} = \frac{1}{N}, \quad (n = 1, \ldots, N).$$

$\boldsymbol{w}^{RP}$ is the solution of the system of non-linear equations:

Risk Parity Portfolio II

Risk parity portfolio

$$\Sigma \boldsymbol{w}^{RP} = \frac{\boldsymbol{\kappa}}{\boldsymbol{w}^{RP}} = \begin{bmatrix} \frac{\kappa}{w_1^{RP}} \\ \vdots \\ \frac{\kappa}{w_N^{RP}} \end{bmatrix}, \qquad \boldsymbol{\iota}^\intercal \boldsymbol{w}^{RP} = 1. \tag{20}$$

(20) has no closed-form solution, but can be solved numerically.

Properties

1. If all correlation coefficients are equal,

$$w_n^{RP} = \frac{\sigma_n^{-1}}{\sum_{n=1}^{N} \sigma_n^{-1}}, \quad (n = 1, \dots, N),$$

that is,

$$w^{RP} = \frac{\sigma^{-1}}{\iota^\intercal \sigma^{-1}}, \quad \sigma^{-1} = \begin{bmatrix} \sigma_1^{-1} \\ \vdots \\ \sigma_N^{-1} \end{bmatrix}.$$

2. Suppose $\sigma_1^2 = \cdots = \sigma_n^2$ and the correlation coefficient between any pair of assets is constant. Then the risk parity portfolio is equivalent to the $1/N$ portfolio.

Maximum Diversification Portfolio

The maximum diversification portfolio is the solution of

$$\max_{\boldsymbol{w}} \quad \frac{\boldsymbol{\sigma}^\intercal \boldsymbol{w}}{\sqrt{\boldsymbol{w}^\intercal \Sigma \boldsymbol{w}}}. \tag{21}$$

The optimal $\boldsymbol{w}$ in (21) is given by

Maximum diversification portfolio

$$\boldsymbol{w}^{MD} = \frac{1}{\boldsymbol{\iota}^\intercal \Sigma^{-1} \boldsymbol{\sigma}} \Sigma^{-1} \boldsymbol{\sigma}. \tag{22}$$

(22) equalizes the risk allocation as

$$\frac{\nabla_1 \sigma(\boldsymbol{w}^{MD})}{\sigma_1} = \cdots = \frac{\nabla_N \sigma(\boldsymbol{w}^{MD})}{\sigma_N}. \tag{23}$$

Properties

1. If all correlation coefficients are equal,

$$w_n^{MD} = \frac{\sigma_n^{-1}}{\sum_{n=1}^{N} \sigma_n^{-1}}, \quad (n = 1, \dots, N).$$

2. Suppose $\sigma_1^2 = \cdots = \sigma_n^2$ and the correlation coefficient between any pair of assets is constant. Then the risk parity portfolio is equivalent to the $1/N$ portfolio.

Passive Management Vs. Active Management

So far we have reviewed how to manage our portfolio in terms of the balance between the expected return and the risk (the variance or the expected shortfall). This style of portfolio management is called active management. Active management also involves discretionary selection of assets.

Passive management of a portfolio, on the other hand, is a investment strategy in which an investor tries to mimic a benchmark index. Passive management funds that mimic indices are called index funds. As the benchmark portfolio, index funds use stock indices, bond indices, currencies, commodities, or even hedge funds.

The goal in passive management is to minimize a discrepancy between a portfolio and the benchmark index.

Tracking Error

Let $\boldsymbol{y_t}$ denote the return on the benchmark index at time $\boldsymbol{t}$, $\boldsymbol{r_{nt}}$ denote the return on asset $\boldsymbol{n}$ ($\boldsymbol{n} = 1, \ldots, \boldsymbol{N}$) at time $\boldsymbol{t}$ ($\boldsymbol{t} = 1, \ldots, \boldsymbol{T}$), and $\boldsymbol{w_n}$ denote the allocation weight for asset $\boldsymbol{n}$. Then a discrepancy between a portfolio and the benchmark index at time $\boldsymbol{t}$ is given by

$$
\begin{aligned}
\boldsymbol{e_t} &= \boldsymbol{y_t} - \sum_{\boldsymbol{n}=1}^{\boldsymbol{N}} \boldsymbol{w_n r_{nt}} = \boldsymbol{y_t} - \begin{bmatrix} \boldsymbol{r_{1t}} & \cdots & \boldsymbol{r_{Nt}} \end{bmatrix} \begin{bmatrix} \boldsymbol{w_1} \\ \vdots \\ \boldsymbol{w_N} \end{bmatrix} \\
&= \boldsymbol{y_t} - \boldsymbol{r_t}^{\intercal} \boldsymbol{w}
\end{aligned}
\tag{24}
$$

This discrepancy is called a tracking error.

Tracking Error Minimization

A tracking error minimization problem is formulated as

Tracking error minimization

$$
\begin{aligned}
\text{Minimize} \quad & \frac{1}{T}\sum_{t=1}^{T}(y_t - r_t^\intercal w)^2, \\
\text{subject to} \quad & \sum_{n=1}^{N} w_n = 1, \quad w_1 \geqq 0, \dots, w_N \geqq 0.
\end{aligned}
\tag{25}
$$