QUANTITATIVE METHODS IN FINANCE
Fixed Income Instruments

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Interest Rate I

Suppose $W(t)$ is the wealth at time t ($t \geqq 0$) and r is an interest rate.

- Simple interest rate

$$W(t) = (1 + rt)W(0).$$

- One-year compound interest rate

$$W(t) = (1 + r)^t W(0).$$

- $\frac{1}{M}$-year compound interest rate

$$W(t) = \left(1 + \frac{r}{M}\right)^{Mt} W(0).$$

Interest Rate II

Napier's constant is defined as

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x .$$

Then

$$\begin{aligned}
&\lim_{M \to \infty} \left(1 + \frac{r}{M}\right)^M W(0) \\
&= \left\{ \lim_{M \to \infty} \left(1 + \frac{r}{M}\right)^{\frac{M}{r}} \right\}^r W(0) \\
&= \left\{ \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x \right\}^r W(0) \\
&= e^r W(0), \quad x = \frac{M}{r}.
\end{aligned}$$

Interest Rate III

Therefore

$$
\begin{aligned}
W(t) &= \left\{ \lim_{M\to\infty} \left(1 + \frac{r}{M}\right)^{M} \right\}^{t} W(0) \\
&= \left\{ \lim_{M\to\infty} \left(1 + \frac{r}{M}\right)^{\frac{M}{r}} \right\}^{rt} W(0) \\
&= e^{rt} W(0),
\end{aligned}
$$

which is called the continuous compound interest rate.

Present Value I

The present value of the wealth at time t is defined as

Present value

$$PV(0) = B(t)W(t),$$

where

$$B(t) = \begin{cases} \dfrac{1}{(1+r)^t}, & \text{(1-year compound)}, \\ \dfrac{1}{\left(1+\frac{r}{M}\right)^{Mt}}, & \left(\frac{1}{M}\text{-year compound}\right), \\ e^{-rt}, & \text{(continuous compound)}. \end{cases}$$

The present value $PV(0)$ is interpreted as the amount of money you need to put in your bank account now ($t = 0$) to receive $W(t)$ at time t. In

Present Value II

this context, the interest rate $\boldsymbol{r}$ is called the discount rate and $\boldsymbol{B}(\boldsymbol{t})$ is called the discount factor.

Conversely,

$$\boldsymbol{W}(\boldsymbol{t}) = \frac{\boldsymbol{PV}(0)}{\boldsymbol{B}(\boldsymbol{t})},$$

is called the future value. The future value is simply regarded as the amount of money you expect to receive at time $\boldsymbol{t}$.

Cash Flow

A cash flow is, as its name suggests, a flow of cash payments. Suppose a project or enterprise (e.g., gold mine, oil well, power plant, factory, shopping mall) will produce cash payments for T years.

Let $C(t)$ denote the amount of payment at time t $(0 \leqq t \leqq T)$. If $C(t) > 0$, it is treated as gain or profit (inflow); otherwise, it is loss (outflow). Suppose the number of payments is N but intervals between them is not necessarily irregular. Let t_n $(t = 1, \dots, N)$ denote the time of the n-th payment. As a convention, we suppose $t_0 = 0$ and $t_N = T$.

Then the cash flow is represented as

$$\{C(t_1), \dots, C(t_n), \dots, C(t_N)\}.$$

Furthermore, let $B(t_n)$ denote the discount factor for $C(t_n)$.

Table: Cash flow, discount factor and present value

Time	t_1	$\cdots$	t_n	$\cdots$	t_N
Cash Flow	$C(t_1)$	$\cdots$	$C(t_n)$	$\cdots$	$C(t_N)$
Discount Factor	$B(t_1)$	$\cdots$	$B(t_n)$	$\cdots$	$B(t_N)$
Present Value	$B(t_1)C(t_1)$	$\cdots$	$B(t_n)C(t_n)$	$\cdots$	$B(t_N)C(t_N)$

The present value of the cash flow is defined as the sum of all present values, i.e.,

$$\begin{aligned} PV(0) &= B(t_1)C(t_1) + \cdots + B(t_n)C(t_n) + \cdots + B(t_N)C(t_N) \\ &= \sum_{n=1}^{N} B(t_n)C(t_n). \end{aligned}$$

Net Present Value I

In practice, we need an initial investment to start the new project that will produce the cash flow $\{C(t_1), \ldots, C(t_N)\}$. Let $C(0)$ denote the amount of the initial investment. In case of investment in a new factory, for example, $C(0)$ represents the amount of fund necessary for the purchase equipments and machineries. Thus $C(0)$ must be negative. The net present value (NPV) of a cash flow is defined as

$$\begin{aligned} NPV(0) &= C(0) + PV(0) \\ &= C(0) + \sum_{n=1}^{N} B(t_n)C(t_n) = \sum_{n=0}^{N} B(t_n)C(t_n), \end{aligned}$$

since $t_0 = 0$ and $B(t_0) = B(0) = 1$.

Net Present Value II

Suppose intervals between payments are equal, that is,

$$t_n = \frac{n}{N} T = n\Delta, \quad \Delta = \frac{T}{N}, \quad (n = 0, 1, \dots, N).$$

Then $B(t_n) = \beta(r)^n$ where

$$\beta(r) = \begin{cases} \dfrac{1}{(1+r)^{\Delta}}, & \text{(one -year compound)} \\ \dfrac{1}{\left(1+\frac{r}{M}\right)^{M\Delta}}, & \left(\frac{1}{M}\text{-year compound}\right) \\ e^{-r\Delta}, & \text{(continuous compound).} \end{cases}$$

Therefore the NPV is expressed as a polynomial function:

$$NPV(0) = c_0 + c_1 x + \cdots + c_n x^n + \cdots + c_N x^N,$$

where $x = \beta(r)$ and $c_n = C(t_n)$ $(n = 0, 1, \dots, N)$.

Internal Rate Of Return I

The internal rate of return (IRR) is the discount rate r^* that makes the NPV of the cash flow equal to zero:

$$0 = c_0 + c_1\beta(r^*) + \cdots + c_n\beta(r^*)^n + \cdots + c_N\beta(r^*)^N.$$

The solution r^* is obtained by solving the polynomial equation:

$$0 = c_0 + c_1 x + \cdots + c_n x^n + \cdots + c_N x^N,$$

with respect to x, and transform the solution x^* with

$$r^* = \begin{cases} \left(\dfrac{1}{x^*}\right)^{\frac{1}{\Delta}} - 1, & \text{(1-year compound)} \\ M\left\{\left(\dfrac{1}{x^*}\right)^{\frac{1}{M\Delta}} - 1\right\}, & \left(\frac{1}{M}\text{-year compound}\right) \\ -\dfrac{\log x^*}{\Delta}, & \text{(continuous compound)} \end{cases}$$

Internal Rate Of Return II

To simplify mathematical expressions, we suppose $\Delta = 1$ and $M = 1$, i.e., each payment occurs at the end of each year.

Special cases

Case 1: When $N = 1$,

$$0 = c_0 + c_1 x^* \quad \Rightarrow \quad r^* = \frac{c_1 - (-c_0)}{-c_0}.$$

Case 2: When $N \to \infty$ and $c_n = c$ for all $n = 1, 2, 3, 4$,

$$0 = c_0 + \sum_{n=1}^{\infty} c(x^*)^n = c_0 + c\frac{x^*}{1 - x^*} \quad \Rightarrow \quad r^* = \frac{c}{-c_0}.$$

Internal Rate Of Return III

In general a higher-order polynomial equation could have multiple solutions and many of them are complex numbers. To ensure that the polynomial equation has the unique real-valued solution, the following two conditions must be satisfied:

(A1) $c_0 < 0$.

(A2) $c_n \geqq 0$ $(n = 1, \dots, N)$ and $c_n > 0$ for some n.

To ensure that the IRR is positive, we need the extra condition:

(A3) $\sum_{n=0}^{N} c_n > 0$.

Bond

A bond is an instrument of borrowing. A typical bond promises to pay the predetermined amount of cash, the principal or the face value, to the bond holder on the predetermined future date, the maturity date. The length of time until the maturity date is called the time to maturity. A coupon-bearing bond promises to pay the predetermined amount of cash periodically to the bond holder. This payment is called the coupon. A bond without coupon payment is called a zero-coupon bond.

Issuers of bonds

- Sovereign bond: a bond issued by a national government
- Municipal bond: a bond issued by a local government
- Corporate bond: a bond issued by a corporation

Bond Vs. Bank Loan

As instruments to raise capital, bonds seem similar to bank loans (both are categorized as debts in the balance sheet), though there are notable differences between them:

1. Bonds are often issued at auctions (primary market).
2. Bonds can be transferred from one holder to others.
3. Bonds are tradable in the financial market (secondary market) until the maturity date.
4. When the bond issuer fails to pay either coupon or principal, it is deemed a default. Credit ratings are indicators for the default risk of bond issuers, which are assigned by credit rating agencies (e.g., S&P, Moody's, Fitch). So-called junk bonds are issued by less credit-worthy corporations. Since investors view junk bonds as risky assets, they are traded at a large discount in the secondary market.

Cash Flows Of Bonds

Zero-coupon bond

t	0	1	2	$\cdots$	$T-1$	T
$C(t)$	$-V$	0	0	$\cdots$	0	F

Coupon-bearing bond

t	0	1	2	$\cdots$	$T-1$	T
$C(t)$	$-P$	C	C	$\cdots$	C	$C+F$

- C — coupon
- F — face value
- T — time to maturity
- V — price of the zero-coupon bond
- P — price of the coupon-bearing bond

Yield To Maturity Of A Zero-Coupon Bond I

Let $V(t)$ denote the price of a zero-coupon bond that will mature at time t $(0 < t \leqq T)$. The yield to maturity (YTM) or simply yield, denoted by $y(t)$, is defined as

Zero-coupon bond yield

$$y(t) = \begin{cases} \left(\dfrac{F}{V(t)}\right)^{\frac{1}{t}} - 1, & \text{(1-year compound)} \\ \dfrac{1}{t} \log \dfrac{F}{V(t)}, & \text{(continuous compound)} \end{cases}$$

Basically, the yield $y(t)$ is equivalent to the internal rate of return on the cash flow of the zero-coupon bond. $y(t)$ as a function of time to maturity is called the zero yield curve or simply yield curve.

Yield To Maturity Of A Zero-Coupon Bond II

Conversely, once the yield curve $\boldsymbol{y(t)}$ is known, the price of a zero-coupon bond is given by

Zero-coupon bond price

$$\boldsymbol{V}(\boldsymbol{t}) = \begin{cases} \dfrac{\boldsymbol{F}}{\{1+\boldsymbol{y}(\boldsymbol{t})\}^{\boldsymbol{t}}}, & \text{(1-year compound)} \\ \\ \boldsymbol{e}^{-\boldsymbol{y}(\boldsymbol{t})\boldsymbol{t}}\boldsymbol{F}, & \text{(continuous compound)} \end{cases}$$

The price of a zero-coupon bond is equal to the present value of the face value $\boldsymbol{F}$ discounted by the yield $\boldsymbol{y}(\boldsymbol{t})$. Thus a zero-coupon bond is also called a discount bond.

Yield To Maturity Of A Coupon-Bearing Bond

The yield to maturity of a coupon-bearing bond is the internal rate of return on its cash flow. In this context, $y(t)$ must be constant for any t, though y is time-varying in reality. Therefore the yield $y(t)$ is the real-valued solution of the following polynomial equation.

Coupon-bearing bond yield

$$P(0) = C \sum_{t=1}^{T-1} B(t, y) + (C + F) B(T, y),$$

$$B(t, y) = \begin{cases} \dfrac{1}{(1+y)^t}, & \text{(1-year compound)} \\ e^{-yt}, & \text{(continuous compound)} \end{cases}$$

In general, unlike a zero-coupon bond, no closed-form solution is available in case of a coupon-bearing bond. Moreover, y depends on coupon C as well as face value F, price $P(0)$ and time to maturity T.

Relationship Between Bond Price And Bond Yield

Suppose coupons and the principal are paid at the end of each year.

Bond price and bond yield

$$V(y) = B(t, y)F, \quad (t = 1, \dots, T)$$

$$P(y) = C\sum_{t=1}^{T-1} B(t, y) + (C + F)B(T, y).$$

1. Both $V(t)$ and $P(y)$ are decreasing functions of the yield y.
2. The yield will go up when the price goes down.
3. The yield is uniquely determined since (A1) and (A2) are always satisfied.
4. The yield is negative if

$$\sum_{t=0}^{T} C(t) = TC + F - P < 0.$$

Price Sensitivity To A Yield Curve Shift I

The yield curve frequently shifts due to business cycles, market sentiments, interventions by central banks and other numerous factors. In this lecture, we concentrate on a parallel shift in the yield curve: $y(t) + \lambda$ for all t. We consider the following measurement of sensitivity to a yield curve shift:

$$\text{sensitivity to yield curve shift} = \frac{\lim_{\lambda \to 0} \frac{P(y+\lambda) - P(y)}{\lambda}}{P(y)},$$

where y may be time-varying and "$\lim_{\lambda \to 0}$" means that the shift λ is infinitesimally small. To obtain the exact formula, we need the differential of the bond price with respect to the yield. The differential of $f(x)$ is

$$\nabla_x f(x) \triangleq \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}.$$

Price Sensitivity To A Yield Curve Shift II

Using the above notation, we have

$$\lim_{\lambda\to 0}\frac{P(y+\lambda)-P(y)}{\lambda}=\nabla_y P(y)=C\sum_{t=1}^{T}\nabla_y B(t,y)+F\nabla_y B(T,y)$$

$$=\begin{cases}-C\sum_{t=1}^{T}\dfrac{tB(t,y)}{1+y}-F\dfrac{TB(T,y)}{1+y}, & \text{(1-year compound)}\\ -C\sum_{t=1}^{T}tB(t,y)-FTB(T,y), & \text{(continuous compound)}\end{cases}$$

since

$$\nabla_y B(t,y)=\begin{cases}-\dfrac{t}{(1+y)^{t+1}}, & \text{(1-year compound)}\\ -te^{-yt}, & \text{(continuous compound)}\end{cases}$$

Price Sensitivity To A Yield Curve Shift III

Hence the price sensitivity to a infinitesimally small shift in the yield curve is given by

$$\frac{\nabla_{\boldsymbol{y}} P(\boldsymbol{y})}{P(\boldsymbol{y})} = \begin{cases} -\frac{D(\boldsymbol{y})}{1+\boldsymbol{y}}, & \text{(1-year compound)} \\ -D(\boldsymbol{y}), & \text{(continuous compound)} \end{cases}$$

$$D(\boldsymbol{y}) = \frac{C \sum_{t=1}^{T} tB(t, \boldsymbol{y}) + FTB(T, \boldsymbol{y})}{P(\boldsymbol{y})}.$$

$D(\boldsymbol{y})$ is called the duration.

Remarks

1. $D(\boldsymbol{y}) > 0$. Thus the price will increase if $\boldsymbol{\lambda} < \mathbf{0}$, and vice versa.
2. $D(\boldsymbol{y}) = t$ for a zero-coupon bond with time to maturity t.

Convexity

The convexity is defined as

$$C(y) = \frac{\nabla_y^2 P(y)}{P(y)} = \frac{S(y) + (1 + D(y))D(y)}{(1+y)^2}, \text{ (1-year compound)}$$

$$S(y) = \frac{C\sum_{t=1}^{T}(t - D(y))^2 B(t,y) + F(T - D(y))^2 B(T,y)}{P(y)}.$$

1. In case of continuous compound, omit $(1+y)^2$ in the first equation.
2. $S(y)$ is called the dispersion.
3. $C(y) > 0$. Thus the rate of change in the bond price is a convex function of the shift λ.
4. Suppose there are two bonds with the same duration. Then the bond with large convexity will suffer less from the upward shift $(\lambda > 0)$ while it will gain more from the downward shift $(\lambda < 0)$.

Yield Curve Estimation I

In practice, long-term bonds are exclusively coupon-bearing ones. Thus we need to estimate the yield curve with price data of coupon-bearing bonds. Suppose N bonds with various maturities $T_1, \ldots, T_N$ are traded in the market. They must be issued by the same agent. The price of bond n is P_n and its cash flow is $\{C_n(1), \ldots, C_n(\bar{T})\}$ where $\bar{T}$ is the longest maturity in the market. Since bond n will mature at T_n, $C_n(t) = 0$ for $t > T_n$.

Then the bond price is given by

$$P_n = \sum_{t=1}^{T_n} B(t)C_n(t) = \sum_{t=1}^{\bar{T}} B(t)C_n(t), \quad (n = 1, \ldots, N)$$

where $B(t)$ is the discount factor.

Yield Curve Estimation II

The discount factors $B(1), \ldots, B(\bar{T})$ are obtained as the solution of the following system of equations:

$$\begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_N \end{bmatrix} = \begin{bmatrix} C_1(1) & C_1(2) & \cdots & C_1(\bar{T}) \\ C_2(1) & C_2(2) & \cdots & C_2(\bar{T}) \\ \vdots & & \ddots & \vdots \\ C_N(1) & C_N(2) & \cdots & C_N(\bar{T}) \end{bmatrix} \begin{bmatrix} B(1) \\ B(2) \\ \vdots \\ B(\bar{T})) \end{bmatrix}.$$

Finally, the yield curve $y(t)$ is given by

$$y(t) = \begin{cases} \left(\dfrac{1}{B(t)}\right)^{\frac{1}{t}} - 1, & \text{(1-year compound)} \\[2ex] \dfrac{1}{t}\log\dfrac{1}{B(t)}, & \text{(continuous compound)} \end{cases}$$

Defaultable Bond And Credit Spread I

Suppose the probability of default is π and the recovery rate is ρ ($0 \leqq \pi, \rho < 1$). Then the payment of a defaultable zero-coupon bond is

$$\begin{cases} \rho F, & \text{(the issuer defaults with } \pi) \\ F, & \text{(the issuer does not default with } 1-\pi) \end{cases}$$

The expected payment is $(1-\pi)F + \pi\rho F = (1-(1-\rho)\pi)F$.

The "fair price" of the zero-coupon bond is defined as

$$\tilde{V} = \begin{cases} \dfrac{(1-(1-\rho)\pi)F}{(1+\tilde{y})^T}, & \text{(1-year compound)} \\ e^{-\tilde{y}T}(1-(1-\rho)\pi)F, & \text{(continuous compound)} \end{cases}$$

Defaultable Bond And Credit Spread II

The yield on the defaultable bond is given by

$$\tilde{y} = \begin{cases} \dfrac{1+y}{(1-(1-\rho)\pi)^{\frac{1}{T}}} - 1, & \text{(1-year compound)} \\ y - \dfrac{1}{T}\log(1-(1-\rho)\pi), & \text{(continous compound)} \end{cases}$$

where y is the yield on riskless bond with the same maturity.

Since $1-(1-\rho)\pi < 1$, $\tilde{y} - y$ is positive. The difference between $\tilde{y}$ and y is called the credit spread.

1. The credit spread is increased when the probability of default is increased.
2. The credit spread is increased when the recovery rate is decreased.