hw6

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1 Problem 1

Solve the following problem using KKT conditions

$$Min \;\; f = 4x_1 - 3x_2 + 2x_1^2 - 3x_1x_2 + 4x_2^2$$

$$g_1(x): \;\; 2x_1 - 1.5x_2 = 5$$

The KKT conditions can be written as:

$$\frac{\partial f}{\partial x_1} - \lambda \frac{\partial g_1}{\partial x_1} = 0 \tag{1}$$
$$\frac{\partial f}{\partial x_2} - \lambda \frac{\partial g_1}{\partial x_2} = 0 \tag{2}$$
$$g_1(x) - b_1 = 0 \tag{3}$$

which evaluates to:

$$4x_1 - 3x_2 - 2\lambda = -4 \tag{4}$$
$$-3x_1 + 8x_2 + 1.5\lambda = 3 \tag{5}$$
$$2x_1 - 1.5x_2 - 5 = 0 \tag{6}$$

This can be solved using a system of linear equations

```
In [115]: A = [4 -3 -2; -3 8 1.5; 2 -1.5 0];
          b = [-4; 3; 5];
          c = A\b
          println(c)

          fun(x1, x2) = 4x1 -3x2 + 2*x1^2 - 3x1*x2 + 4 * x2^2
          opt = fun(c[1], c[2])
          println("Optimum: ", opt)
```

```
[2.5, 0.0, 7.0]
Optimum: 22.5
```

The Values then are:

$$x_1 = 2.5 \tag{7}$$
$$x_2 = 0 \tag{8}$$
$$\lambda = 7 \tag{9}$$

which gives the objective

$$f = 22.5$$

1.0.1 (b) Change the constraint to be:

$$g_1(x) = 2x_1 - 1.5x_2 = 5.1$$

because the change in the constraint is 0.1 we expect that the change in the objective will be $0.1 * \lambda = 0.7$

```
In [116]: A = [4 -3 -2; -3 8 1.5; 2 -1.5 0];
          b2 = [-4; 3; 5.1];
          c2 = A\b2
          print(c2)

          fun(x1, x2) = 4x1 -3x2 + 2*x1^2 - 3x1*x2 + 4 * x2^2
          opt2 = fun(c2[1], c2[2])
          println("\nNew optimum: ", opt)
          delta_opt = opt2 - opt
          println("Change in optimum: ", delta_opt)
```

```
[2.55, 0.0, 7.1]
New optimum: 22.5
Change in optimum: 0.7049999999999983
```

As we can see the change in the optimum value was 0.705 which is very close to the predicted value of 0.7. This shows that the λ value accurately predicts the change in the optimum.

1.0.2 (c) Are the KKT equations for a problem with quadratic objective and a linear equality constraint always linear? Is this true for a problem with a quadratic objective and a linear inequality constraint?

If the problem has a quadratic objective and a linear equality constrant then the KKT equations will be linear, if a linear inequality constraint is present than the problem will also be linear.

1.1 Problem 3: Solve the following problem using KKT conditions

$$Min\ f(x) = x_1^2 + 2x_2^2 + 3x_3^2$$

$$g_1(x) : x_1 + 5x_2 = 12$$

$$g_2(x) : -2x_1 - x_2 - 4x_3 \leq -18$$

First we must formulate the problem to match the required format. We change g_2 to be:

$$g_2(x) : 2x_1 - x_2 + 4x_3 - 18 \leq 0$$

which in turn formulates the system of linear equations representing the KKT Conditions:

$$\begin{bmatrix} 2 & 0 & 0 & -1 & 2 \\ 0 & 4 & 0 & -5 & -1 \\ 0 & 0 & 6 & 0 & 4 \\ 2 & -1 & 4 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \lambda_2 \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 18 \\ 12 \end{bmatrix} \tag{10}$$

```
In [124]: A = [2 0 0 -1 2
               0 4 0 -5 -1
               0 0 6 0 4
               2 -1 4 0 0
               1 5 0 0 0]
          b = [0
               0
               0
               18
               12]
          x = A\b
          println("The values of the x vector are: $x")
          g2(x1, x2, x3) = 2*x1 - x2 + 4x3 - 18
          g1(x1, x2) = x1 + 5x2 - 12
          g1_x = g1(x[1], x[2])
          g2_x = g2(x[1], x[2], x[3])
          println("g1 evaluates to: $g1_x")
          println("g2 evaluates to: $g2_x")
```

```
The values of the x vector are: [4.71698, 1.4566, 2.50566, 1.91698, -3.75849]
g1 evaluates to: 3.907985046680551e-14
g2 evaluates to: 3.552713678800501e-15
```

Here we can see that:

$$x_1 = 4.717 x_2 = 1.457 x_3 = 2.506 \lambda_1 = -3.7585 \lambda_2 = 1.9170$$

We also note from the code that both of the constraints are binding. This satisfies the neccessary conditions for an optimum

To check the sufficient conditions we must check that $\nabla_x^2 L(\mathbf{x}^*, \lambda^*)$ is positive definite

$$\nabla_x^2 L(\mathbf{x}^*, \lambda^*) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix} - \lambda_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \lambda_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

```
In [126]: L = [1 0 0
               0 4 0
               0 0 6]
          isOpt = isposdef(L)
          if isOpt
              println("The values correspond to a constrained optimum")
          end
```

```
The values correspond to a constrained optimum
```

because the hessian of the lagrangian function is postive definite then we can conclude that the point is a constrained minimum.

1.2 Problem 4

$Min\ f(x) = x_1^2 + x_2^2 g_1(x) = x_1^2 + x_2^2 - 9 = 0 g_2(x) = x_1 + x_2^2 - 1 \leq 0 g_3(x) = x_1 + x_2 - 1 \leq 0$

Once again the inequality constraints must be reformatted

$$g_2(x) = -x_1 - x_2^2 + 1 \geq 0 g_3(x) = -x_1 - x_2 + 1 \geq 0$$

1.2.1 (a) Verify that [-2.3723, -1.8364] is a local optimum

Check which constraints are binding

```
In [119]: x = [-2.3723 -1.8364]
          g1(x) = x[1]^2 + x[2]^2 - 9
          g2(x) = x[1] + x[2]^2 - 1
          g3(x) = x[1] + x[2] - 1
          g1x = g1(x)
          g2x = g2(x)
          g3x = g3(x)
          println("g1 = $g1x")
          println("g2 = $g2x")
          println("g3 = $g3x")
```

```
g1 = 0.00017225000000031798
g2 = 6.49600000000028e-5
g3 = -5.2087
```

within acceptable roundoff g1 and g2 are binding constraints. We must now check if the KKT conditions are satisfied.

```
In [128]: using NLsolve
          function f!(F, x)
              F[1] = 2*x[1] - x[3] * 2.0 * x[1] + x[4]
              F[2] = 1.0 - x[3] * 2.0 * x[2] - x[4] * -2 * x[2]
              F[3] = x[1]^2.0 + x[2]^2.0 - 9.0
```

```
            F[4] = -x[1] - x[2]^2.0 + 1.0
        end

        x0 = [-2.37; -1.836; -1.0; -1.0]
        sol = nlsolve(f!, x0)
        λ1 = sol.zero[3]
        λ2 = sol.zero[4]
        println("λ1 = $λ1")
        println("λ2 = $λ2")
```

```
λ1 = 0.7785253137160885
λ2 = 1.0508005262435787
```

Because both λ values are positive the KKT conditions are satisfied. We now check the sufficient conditions.

```
In [121]: hessf = [2 0; 0 0]
          hessg1 = [2 0; 0 2]
          hessg2 = [0 0; 0 -2]
          lagrangian = hessf - λ1 * hessg1 - λ2 * hessg2
          println(lagrangian)

          isposdef(lagrangian)
```

```
[0.442949 0.0; 0.0 0.54455]
```

```
Out[121]: true
```

Because the lagrangain function is positive definite the point is a constrained optimum

1.2.2 Verify that [-2.5, -1.6583] is not a local optimum

```
In [122]: x = [-2.5, -1.6583]
          g1x = g1(x)
          g2x = g2(x)
          g3x = g3(x)
          println("g1 = $g1x")
          println("g2 = $g2x")
          println("g3 = $g3x")
```

```
g1 = -4.1109999999733304e-5
g2 = -0.7500411099999997
g3 = -5.1583000000000006
```

The only binding constraint is g1, but all of the constraints are feasible. we now solve for λ_1

$$2x_1 - \lambda_1(2x_1) = 0\lambda_1 = -2and1 - \lambda_1(2x_2) = 0\lambda_1 = 3.3$$

because we cannot solve for a value of λ_1 the point is not a local optimum

1.2.3 Drop the equality constraint from the problem. Using the countour plot above to see where the optimum lies, solve for the optimum using the KKT conditions.

We see that only g_2 is binding the system of equations that we need to solve is then

$$2x_1 + \lambda_2(1) = 01 + \lambda_2(2x_2) = 0 - x_1 - x_2^2 + 1 = 0$$

In [123]:
```
function f!(F, x)
    F[1] = 2*x[1] + x[3]
    F[2] = 1.0 - x[3] * -2 * x[2]
    F[3] = -x[1] - x[2]^2.0 + 1.0
end

x0 = [-2.37; -1.836; -1.0]
sol = nlsolve(f!, x0)
x1 = sol.zero[1]
x2 = sol.zero[2]
λ1 = sol.zero[3]
println("x1 = $x1")
println("x2 = $x2")
println("λ1 = $λ1")
println("The potential optimum point is [$x1, $x2]")

lagr = hessf - λ1 * hessg2
isOpt = isposdef(lagr)
if isOpt
    println("The lagrangian is positive definite therefore the potential optimum is a
end
```

```
x1 = -0.2258029814778883
x2 = -1.1071598716887676
λ1 = 0.4516059629557766
The potential optimum point is [-0.2258029814778883, -1.1071598716887676]
The lagrangian is positive definite therefore the potential optimum is a constrained optimum
```

The point [-0.2258, -1.1072 is the constrained optimum for the problem without the equality constraint.