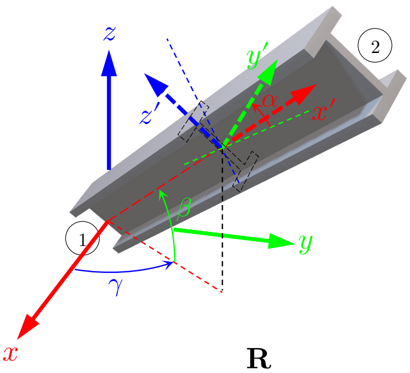
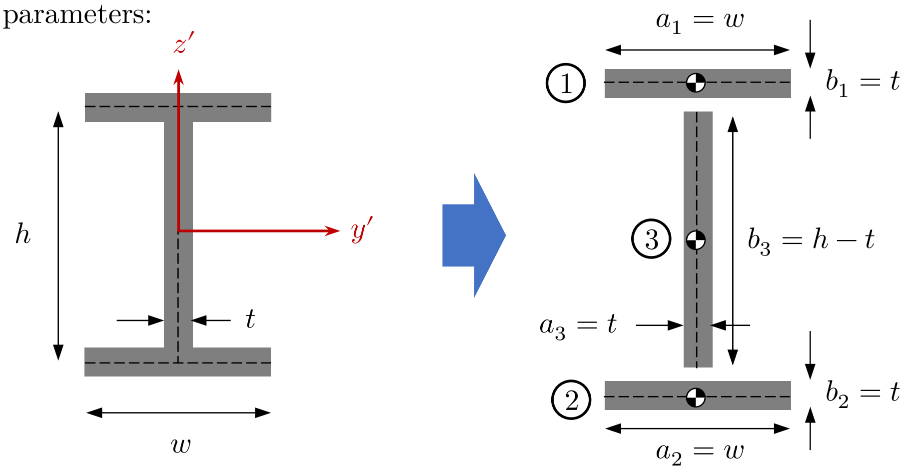
Element computations for 3D beams

(a) Local system $\{x', y', z'\}$:



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & \sin\beta \\ -\sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & -\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{bmatrix}}^{\mathbf{R}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) Geometric parameters:



1. Split the section into rectangular segments, and for each segment i obtain the coordinates of its centroid, (y'_i, z'_i) and its side lengths, $(a_i,\ b_i)$.
2. Compute the area and second moments of area for each segment:
   $A_i = a_i b_i;\quad I_{y,i} = \frac{a_i b_i^3}{12};\quad I_{z,i} = \frac{b_i a_i^3}{12}$
3. Compute the total area of the section:
   $A = \sum_i A_i$
4. Compute the centroid of the section:
   $y'_G = \frac{\sum_i y'_i A_i}{A};\quad z'_G = \frac{\sum_i z'_i A_i}{A}$
5. Apply the parallel axis theorem to compute the inertia of the section:
   $I_y = \sum_i [I_{y,i} + A_i(z'_i - z'_G)^2]\,;\quad I_z = \sum_i [I_{z,i} + A_i(y'_i - y'_G)^2]$
6. Compute the torsional constant of the section:
   $J = \sum_i 4\min(I_{y,i}, I_{z,i})$ (approximation valid for open thin-walled sections)
7. Compute the length of the beam:
   $L = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

(c) Element stiffness matrix in global coordinates:

1. Obtain the elemental stiffness matrix in local coordinates:
   - Axial load (local degree of freedom 1, i.e. displacement in x'-direction)

$$\mathbf{K}'_e([1,7],[1,7]) = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

   - Shear load in y'-direction and bending moment in z'-direction (local degrees of freedom 2 and 6, respectively, corresponding to the displacement in y'-direction and the rotation around the z'-axis)

$$\mathbf{K}'_e([2,6,8,12],[2,6,8,12]) = 2\frac{EI_z}{L^3}\begin{bmatrix} 6 & 3L & -6 & 3L \\ 3L & 2L^2 & -3L & L^2 \\ -6 & -3L & 6 & -3L \\ 3L & L^2 & -3L & 2L^2 \end{bmatrix}$$

   - Shear load in z'-direction and bending moment in y'-direction (local degrees of freedom 3 and 5, respectively, corresponding to the displacement in z'-direction and the rotation around the y'-axis)

$$\mathbf{K}'_e([3,5,9,11],[3,5,9,11]) = 2\frac{EI_y}{L^3}\begin{bmatrix} 6 & -3L & -6 & -3L \\ -3L & 2L^2 & 3L & L^2 \\ -6 & 3L & 6 & 3L \\ -3L & L^2 & 3L & 2L^2 \end{bmatrix}$$

   - Torsion moment in x'-direction (local degree of freedom 4, i.e. rotation around x'-axis)

$$\mathbf{K}'_e([4,10],[4,10]) = \frac{EJ}{2(1+\nu)L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

2. Extend the rotation matrix to 2×6 degrees of freedom:

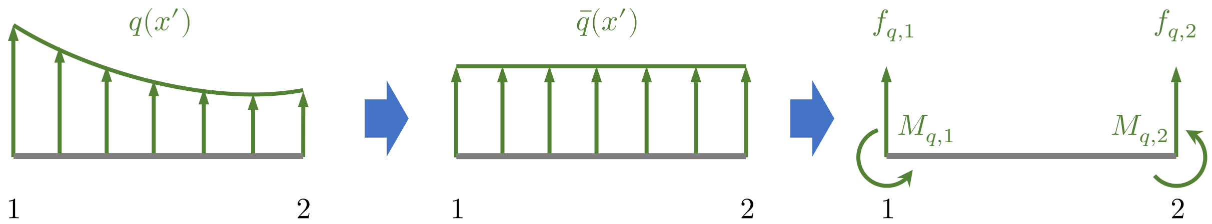
$$\mathbf{R}_e(1:3,1:3,e) = \mathbf{R}; \quad \mathbf{R}_e(4:6,4:6,e) = \mathbf{R};$$

$$\mathbf{R}_e(7:9,7:9,e) = \mathbf{R}; \quad \mathbf{R}_e(10:12,10:12,e) = \mathbf{R};$$

3. Extend the rotation matrix to 2×6 degrees of freedom:

$$\mathbf{K}_e(:,:,e) = \left(\mathbf{R}_e(:,:,e)\right)^{\mathrm{T}}\mathbf{K}'_e\mathbf{R}_e(:,:,e)$$

(d) Equivalent element force vector for distributed loads (per unit length):



1. Assuming the value of the distributed load, $\mathbf{q}(\mathbf{x})$, is known at each node and in the global coordinates system, i.e. $\mathbf{q}_i = \mathbf{q}(x_i, y_i, z_i)$, compute the mean value for the corresponding element:

$$\bar{\mathbf{q}}_e = \begin{bmatrix} \bar{q}_{e,x} \\ \bar{q}_{e,y} \\ \bar{q}_{e,z} \end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix} q_{1,x} \\ q_{1,y} \\ q_{1,z} \end{bmatrix} + \begin{bmatrix} q_{2,x} \\ q_{2,y} \\ q_{2,z} \end{bmatrix} \right)$$

   where $\mathbf{q}_1$ and $\mathbf{q}_2$ are que values of $\mathbf{q}(\mathbf{x})$ evaluated at the nodes 1 and 2 of the corresponding element.

2. Obtain the components in the local coordinates system:

$$\bar{\mathbf{q}}'_e = \mathbf{R}\bar{\mathbf{q}}_e$$

3. Compute the equivalent element force vector in local coordinates:
   - Axial distributed load in x'-direction:

$$\mathbf{f}'_e([1,7]) = \frac{\bar{q}'_{e,x} L}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

   - Shear distributed load in y'-direction

$$\mathbf{f}'_e([2,6,8,12]) = \frac{\bar{q}'_{e,y} L}{2} \begin{bmatrix} 1 \\ L/6 \\ 1 \\ -L/6 \end{bmatrix}$$

   - Shear distributed load in z'-direction

$$\mathbf{f}'_e([3,5,9,11]) = \frac{\bar{q}'_{e,z} L}{2} \begin{bmatrix} 1 \\ -L/6 \\ 1 \\ L/6 \end{bmatrix}$$

4. Compute the equivalent elemental force vector in global coordinates:

$$\mathbf{f}_e(:,e) = \left(\mathbf{R}_e(:,:,e)\right)^{\mathrm{T}} \mathbf{f}'_e$$

(e) Postprocessing parameters for each element:

1. Assuming the displacements and rotations of the structure are known (solution vector $\mathbf{u}$), get the corresponding terms for the element, $\mathbf{u}_e$.
2. Compute the internal forces vector in local coordinates:

$$\mathbf{f}'_{int,e} = \mathbf{R}_e(:,:,e)\mathbf{K}_e(:,:,e)\mathbf{u}_e$$

3. Get the internal forces and moments:
   - Axial force in x'-direction (constant):

$$N(e) = -\mathbf{f}'_{int,e}(1) = \mathbf{f}'_{int,e}(7)$$

- Shear force in y'-direction (constant):

$$Q_y(e) = -\mathbf{f}'_{int,e}(2) = \mathbf{f}'_{int,e}(8)$$

- Shear force in z'-direction (constant):

$$Q_z(e) = -\mathbf{f}'_{int,e}(3) = \mathbf{f}'_{int,e}(9)$$

- Torsion moment around x'-axis (constant):

$$T(e) = -\mathbf{f}'_{int,e}(4) = \mathbf{f}'_{int,e}(10)$$

- Bending moment around y'-axis (linear):

$$M_y(:,e) = [-\mathbf{f}'_{int,e}(5), \mathbf{f}'_{int,e}(11)]^T$$

- Bending moment around z'-axis (linear):

$$M_z(:,e) = [-\mathbf{f}'_{int,e}(6), \mathbf{f}'_{int,e}(12)]^T$$

4. Obtain the internal displacement and rotations vector in local coordinates:

$$\mathbf{u}'_{int,e}(:,e) = \mathbf{R}_e(:,:,e)\mathbf{u}_e$$

This vector contains information to compute:

- Axial deformation as:

$$\varepsilon(e) = \frac{1}{L}\left(\mathbf{u}'_{int,e}(7,e) - \mathbf{u}'_{int,e}(1,e)\right)$$

- Cubic polynomial deflection curves:

$$u'_{y,e}(x',e) = A_y x'^3 + B_y x'^2 + C_y x' + D_y$$

$$\theta'_{z,e}(x',e) = 3A_y x'^2 + 2B_y x' + C_y$$

$$u'_{z,e}(x',e) = A_z x'^3 + B_z x'^2 + C_z x' + D_z$$

$$\theta'_{y,e}(x',e) = 3A_z x'^2 + 2B_z x' + C_z$$

where:

$$\begin{bmatrix} A_y \\ B_y \\ C_y \\ D_y \end{bmatrix} = \frac{1}{L}\begin{bmatrix} 2 & L & -2 & L \\ -3L & -2L^2 & 3L & -L^2 \\ 0 & L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}\mathbf{u}'_{int,e}([2,6,8,12],e)$$

$$\begin{bmatrix} A_z \\ B_z \\ C_z \\ D_z \end{bmatrix} = \frac{1}{L}\begin{bmatrix} 2 & -L & -2 & -L \\ -3L & 2L^2 & 3L & L^2 \\ 0 & -L^3 & 0 & 0 \\ L^3 & 0 & 0 & 0 \end{bmatrix}\mathbf{u}'_{int,e}([3,5,9,11],e)$$