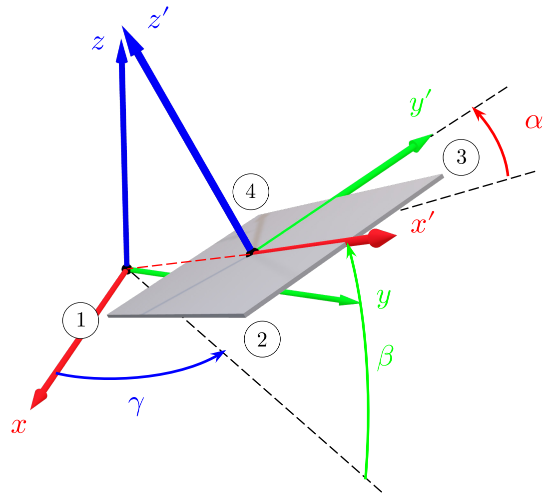
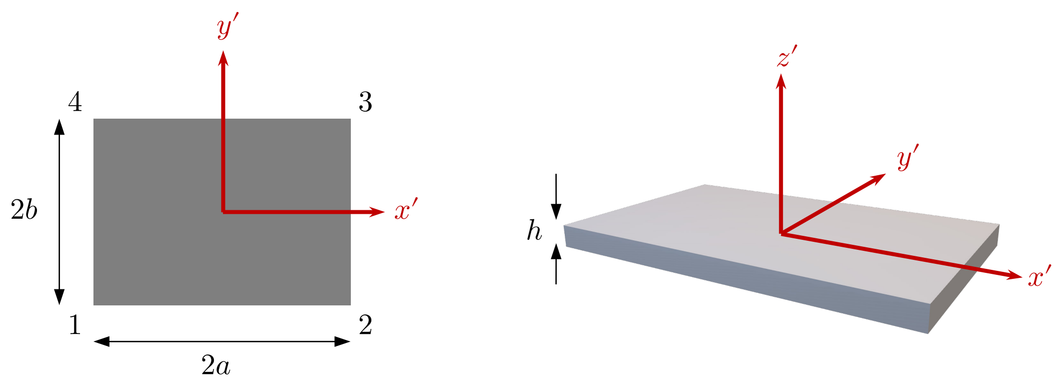
Element computations for 3D rectangular plates

(a) Local system $\{x', y', z'\}$:



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \overbrace{\begin{bmatrix} \cos\beta\cos\gamma & \cos\beta\sin\gamma & \sin\beta \\ -\sin\alpha\sin\beta\cos\gamma - \cos\alpha\sin\gamma & -\sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\cos\beta \\ -\cos\alpha\sin\beta\cos\gamma + \sin\alpha\sin\gamma & -\cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\cos\beta \end{bmatrix}}^{\mathbf{R}} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

(b) Geometric parameters:



For each element:

$$a = (x_2' - x_1')/2 = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}/2$$

$$b = (y_4' - y_1')/2 = \sqrt{(x_4 - x_1)^2 + (y_4 - y_1)^2 + (z_4 - z_1)^2}/2$$

(c) Element stiffness matrix in global coordinates:

1. Obtain the elemental stiffness matrix in local coordinates:
   - Plane stress (local degrees of freedom 1 and 2, i.e. displacement in x' and y'-directions)

$\mathbf{K}'_e([1,2,7,8,13,14,19,20],[1,2,7,8,13,14,19,20]) =$

$$
= \frac{Ebh}{12a(1-\nu^2)}\begin{bmatrix}
4 & & & & & & & \\
0 & 2(1-\nu) & & & & \text{SYM.} & & \\
-4 & 0 & 4 & & & & & \\
0 & -2(1-\nu) & 0 & 2(1-\nu) & & & & \\
-2 & 0 & 2 & 0 & 4 & & & \\
0 & -(1-\nu) & 0 & 1-\nu & 0 & 2(1-\nu) & & \\
2 & 0 & -2 & 0 & -4 & 0 & 4 & \\
0 & 1-\nu & 0 & -(1-\nu) & 0 & -2(1-\nu) & 0 & 2(1-\nu)
\end{bmatrix} +
$$

$$
+ \frac{Eah}{12b(1-\nu^2)}\begin{bmatrix}
2(1-\nu) & & & & & & & \\
0 & 4 & & & & \text{SYM.} & & \\
1-\nu & 0 & 2(1-\nu) & & & & & \\
0 & 2 & 0 & 4 & & & & \\
-(1-\nu) & 0 & -2(1-\nu) & 0 & 2(1-\nu) & & & \\
0 & -2 & 0 & -4 & 0 & 4 & & \\
-2(1-\nu) & 0 & -(1-\nu) & 0 & 1-\nu & 0 & 2(1-\nu) & \\
0 & -4 & 0 & -2 & 0 & 2 & 0 & 4
\end{bmatrix} +
$$

$$
+ \frac{Eh}{8(1-\nu^2)}\begin{bmatrix}
0 & & & & & & & \\
1+\nu & 0 & & & & & \text{SYM.} & \\
0 & 1-3\nu & 0 & & & & & \\
-(1-3\nu) & 0 & -(1+\nu) & 0 & & & & \\
0 & -(1+\nu) & 0 & 1-3\nu & 0 & & & \\
-(1+\nu) & 0 & -(1-3\nu) & 0 & 1+\nu & 0 & & \\
0 & -(1-3\nu) & 0 & 1+\nu & 0 & 1-3\nu & 0 & \\
1-3\nu & 0 & 1+\nu & 0 & -(1-3\nu) & 0 & -(1+\nu) & 0
\end{bmatrix}
$$

   - Plate bending (local degrees of freedom 3, 4 and 5, corresponding to the displacement in z'-direction and the rotation around the x' and y'-axis)

$\mathbf{K}'_e([3,4,5,9,10,11,15,16,17,21,22,23],[3,4,5,9,10,11,15,16,17,21,22,23]) =$

$$
= \frac{Eah^3}{72b^3(1-\nu^2)}\begin{bmatrix}
6 & & & & & & & & & & & \\
0 & 0 & & & & & & & & & & \\
6b & 0 & 8b^2 & & & & & \text{SYM.} & & & & \\
3 & 0 & 3b & 6 & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & & & & & & & \\
3b & 0 & 4b^2 & 6b & 0 & 8b^2 & & & & & & \\
-3 & 0 & -3b & -6 & 0 & -6b & 6 & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & \\
3b & 0 & 2b^2 & 6b & 0 & 4b^2 & -6b & 0 & 8b^2 & & & \\
-6 & 0 & -6b & -3 & 0 & -3b & 3 & 0 & -3b & 6 & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\
6b & 0 & 4b^2 & 3b & 0 & 2b^2 & -3b & 0 & 4b^2 & -6b & 0 & 8b^2
\end{bmatrix} +
$$

$$
+\frac{Ebh^3}{72a^3(1-\nu^2)}\begin{bmatrix}
6 & & & & & & & & & & & \\
6a & 8a^2 & & & & & & & & & & \\
0 & 0 & 0 & & & & & & \text{SYM.} & & & \\
-6 & -6a & 0 & 6 & & & & & & & & \\
6a & 4a^2 & 0 & -6a & 8a^2 & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & & & & & & \\
-3 & -3a & 0 & 3 & -3a & 0 & 6 & & & & & \\
3a & 2a^2 & 0 & -3a & 4a^2 & 0 & -6a & 8a^2 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & \\
3 & 3a & 0 & -3 & 3a & 0 & -6 & 6a & 0 & 6 & & \\
3a & 4a^2 & 0 & -3a & 2a^2 & 0 & -6a & 4a^2 & 0 & 6a & 8a^2 & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}+
$$

$$
+\frac{E\nu h^3}{24ab(1-\nu^2)}\begin{bmatrix}
1 & & & & & & & & & & & \\
a & 2ab & & & & & & & & & & \\
b & 0 & 0 & & & & & & \text{SYM.} & & & \\
-1 & 0 & -b & 1 & & & & & & & & \\
0 & 0 & 0 & -a & 0 & & & & & & & \\
-b & 0 & 0 & b & -2ab & 0 & & & & & & \\
1 & 0 & 0 & -1 & a & 0 & 1 & & & & & \\
0 & 0 & 0 & a & 0 & 0 & -a & 0 & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & -b & 2ab & 0 & & & \\
-1 & -a & 0 & 1 & 0 & 0 & -1 & 0 & b & 1 & & \\
-a & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a & 0 & \\
0 & 0 & 0 & 0 & 0 & 0 & b & 0 & 0 & -b & -2ab & 0
\end{bmatrix}+
$$

$$
+\frac{Eh^3}{360ab(1+\nu)}\begin{bmatrix}
21 & & & & & & & & & & & \\
3a & 8a^2 & & & & & & & & & & \\
3b & 0 & 8b^2 & & & & & & & \text{SYM.} & & \\
-21 & -3a & -3b & 21 & & & & & & & & \\
3a & -2a^2 & 0 & -3a & 8a^2 & & & & & & & \\
-3b & 0 & -8b^2 & 3b & 0 & 8b^2 & & & & & & \\
21 & 3a & 3b & -21 & 3a & -3b & 21 & & & & & \\
-3a & 2a^2 & 0 & 3a & -8a^2 & 0 & -3a & 8a^2 & & & & \\
-3b & 0 & 2b^2 & 3b & 0 & -2b^2 & -3b & 0 & 8b^2 & & & \\
-21 & -3a & -3b & 21 & -3a & 3b & -21 & 3a & 3b & 21 & & \\
-3a & -8a^2 & 0 & 3a & 2a^2 & 0 & -3a & -2a^2 & 0 & 3a & 8a^2 & \\
3b & 0 & -2b^2 & -3b & 0 & 2b^2 & 3b & 0 & -8b^2 & -3b & 0 & 8b^2
\end{bmatrix}
$$

- Torsion moment in z'-direction (local degree of freedom 6, i.e. rotation around z'-axis)

$$
\mathbf{K}'_e([6,12,18,24],[6,12,18,24]) = 4abE\begin{bmatrix}
1 & & \text{SYM.} & \\
0 & 1 & & \\
0 & 0 & 1 & \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

2. Extend the rotation matrix to 4×6 degrees of freedom:

$$\mathbf{R}_e(1:3,1:3,e) = \mathbf{R};\quad \mathbf{R}_e(4:6,4:6,e) = \mathbf{R};$$

$$\mathbf{R}_e(7:9,7:9,e) = \mathbf{R};\quad \mathbf{R}_e(10:12,10:12,e) = \mathbf{R};$$

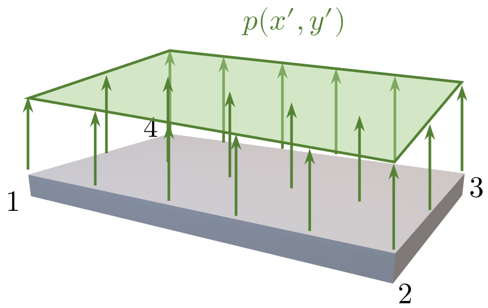
$$\mathbf{R}_e(13:15,13:15,e) = \mathbf{R};\quad \mathbf{R}_e(16:18,16:18,e) = \mathbf{R};$$

$$\mathbf{R}_e(19:21,19:21,e) = \mathbf{R};\quad \mathbf{R}_e(22:24,22:24,e) = \mathbf{R};$$

3. Extend the rotation matrix to 2×6 degrees of freedom:

$$\mathbf{K}_e(:,:,e) = \left(\mathbf{R}_e(:,:,e)\right)^{\mathrm{T}} \mathbf{K}'_e \mathbf{R}_e(:,:,e)$$

(d) Equivalent element force vector for uniform distributed loads (per unit area):



1. Assuming the value of the distributed load, $\mathbf{p}(\mathbf{x})$, is known at each node and in the global coordinates system, i.e. $\mathbf{p}_i = \mathbf{p}(x_i, y_i, z_i)$, compute the mean value for the corresponding element:

$$\bar{\mathbf{p}}_e = \begin{bmatrix} \bar{p}_{e,x} \\ \bar{p}_{e,y} \\ \bar{p}_{e,z} \end{bmatrix} = \frac{1}{4}\left(\begin{bmatrix} p_{1,x} \\ p_{1,y} \\ p_{1,z} \end{bmatrix} + \begin{bmatrix} p_{2,x} \\ p_{2,y} \\ p_{2,z} \end{bmatrix} + \begin{bmatrix} p_{3,x} \\ p_{3,y} \\ p_{3,z} \end{bmatrix} + \begin{bmatrix} p_{4,x} \\ p_{4,y} \\ p_{4,z} \end{bmatrix} \right)$$

where $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ and $\mathbf{p}_4$ are que values of $\mathbf{p}(\mathbf{x})$ evaluated at the nodes 1 and 2 of the corresponding element.

2. Obtain the components in the local coordinates system:

$$\bar{\mathbf{p}}'_e = \mathbf{R}\bar{\mathbf{p}}_e$$

3. Compute the equivalent element force vector in local coordinates:
   - In-plane distributed load in x' and y'-direction:

$$\mathbf{f}'_e([1,2,7,8,13,14,19,20]) = \bar{p}_{e,x} ab \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \bar{p}_{e,y} ab \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

- Distributed pressure in z'-direction

$$\mathbf{f}'_e([3,4,5,9,10,11,15,16,17,21,22,23]) = ab\bar{p}'_{e,z} \begin{bmatrix} 1 \\ b/3 \\ a/3 \\ 1 \\ -b/3 \\ a/3 \\ 1 \\ -b/3 \\ -a/3 \\ 1 \\ b/3 \\ -a/3 \end{bmatrix}$$

4. Compute the equivalent elemental force vector in global coordinates:

$$\mathbf{f}_e(:,e) = \left(\mathbf{R}_e(:,:,e)\right)^{\mathrm{T}} \mathbf{f}'_e$$

(e) Postprocessing parameters for each element:

1. Assuming the displacements and rotations of the structure are known (solution vector $\mathbf{u}$), get the corresponding terms for the element, $\mathbf{u}_e$.
2. Obtain the internal displacement and rotations vector in local coordinates:

$$\mathbf{u}'_{int,e}(:,e) = \mathbf{R}_e(:,:,e)\mathbf{u}_e$$

3. Obtain the plane stress tensor components:

$$\sigma'_x(:,e) = \frac{E}{1-\nu^2} \begin{bmatrix} -2b & -2a\nu & 2b & 0 & 0 & 0 & 0 & 2a\nu \\ -2b & 0 & 2b & -2a\nu & 0 & 2a\nu & 0 & 0 \\ 0 & 0 & 0 & -2a\nu & 2b & 2a\nu & -2b & 0 \\ 0 & -2a\nu & 0 & 0 & 2b & 0 & -2b & 2a\nu \end{bmatrix} \mathbf{u}'_{int,e}([1,2,7,8,13,14,19,20],e)$$

$$\sigma'_y(:,e) = \frac{E}{1-\nu^2} \begin{bmatrix} -2b\nu & -2a & 2b\nu & 0 & 0 & 0 & 0 & 2a \\ -2b\nu & 0 & 2b\nu & -2a & 0 & 2a & 0 & 0 \\ 0 & 0 & 0 & -2a & 2b\nu & 2a & -2b\nu & 0 \\ 0 & -2a & 0 & 0 & 2b\nu & 0 & -2b\nu & 2a \end{bmatrix} \mathbf{u}'_{int,e}([1,2,7,8,13,14,19,20],e)$$

$$\tau'_{xy}(:,e) = \frac{E}{2(1+\nu)} \begin{bmatrix} -2a & -2b & 0 & 2b & 0 & 0 & 2a & 0 \\ 0 & -2b & -2a & 2b & 2a & 0 & 0 & 0 \\ 0 & 0 & -2a & 0 & 2a & 2b & 0 & -2b \\ -2a & 0 & 0 & 0 & 0 & 2b & 2a & -2b \end{bmatrix} \mathbf{u}'_{int,e}([1,2,7,8,13,14,19,20],e)$$

4. Obtain the Von Mises stress tensor for each element node:

$$\sigma_{VM}(i,e) = \sqrt{\left(\sigma'_x(i,e)\right)^2 + \left(\sigma'_y(i,e)\right)^2 - \left(\sigma'_x(i,e)\right)\left(\sigma'_y(i,e)\right) + 3\left(\tau'_{xy}(i,e)\right)^2}$$

5. Obtain the hydrostatic pressure for each element node:

$$p_h(i,e) = \frac{1}{3}\left(\sigma'_x(i,e) + \sigma'_y(i,e)\right)$$