Deformation modes of four-nodes finite elements

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1 Plane stress

For plane-stress conditions the stiffness matrix of a four-nodes element reads

$$K = \frac{E}{24(1-\nu^2)}\hat{K} \tag{1}$$

with

$$\hat{K} = \begin{bmatrix}
12-4\nu & 3\nu+3 & -2\nu-6 & 9\nu-3 & 2\nu-6 & -3\nu-3 & 4\nu & 3-9\nu\\
3\nu+3 & 12-4\nu & 3-9\nu & 4\nu & -3\nu-3 & 2\nu-6 & 9\nu-3 & -2\nu-6\\
-2\nu-6 & 3-9\nu & 12-4\nu & -3\nu-3 & 4\nu & 9\nu-3 & 2\nu-6 & 3\nu+3\\
9\nu-3 & 4\nu & -3\nu-3 & 12-4\nu & 3-9\nu & -2\nu-6 & 3\nu+3 & 2\nu-6\\
2\nu-6 & -3\nu-3 & 4\nu & 3-9\nu & 12-4\nu & 3\nu+3 & -2\nu-6 & 9\nu-3\\
-3\nu-3 & 2\nu-6 & 9\nu-3 & -2\nu-6 & 3\nu+3 & 12-4\nu & 3-9\nu & 4\nu\\
4\nu & 9\nu-3 & 2\nu-6 & 3\nu+3 & -2\nu-6 & 3-9\nu & 12-4\nu & -3\nu-3\\
3-9\nu & -2\nu-6 & 3\nu+3 & 2\nu-6 & 9\nu-3 & 4\nu & -3\nu-3 & 12-4\nu
\end{bmatrix} . \tag{2}$$

The characteristic polynomial is given by

$$\det(K-\lambda I) = 0 \ ,$$

or

$$\frac{\lambda^3(E-\lambda\nu-\lambda)^2(E+\lambda\nu-\lambda)(\nu E-3E-6\nu^2\lambda+6\lambda)^2}{36(\nu-1)^3(1+\nu)^4} = 0 \ . \tag{3}$$

What gives as eigenvalues

$$\lambda \in \left\{0, \frac{E(3-\nu)}{6(1-\nu^2)}, \frac{E}{1+\nu}, \frac{E}{1-\nu}\right\} \ .$$

And the corresponding eigenvectors are

$$\begin{aligned}
\lambda_1 &= 0, & u^{(1)} &= (1,0,1,0,1,0,1,0)\\
& & u^{(2)} &= (0,1,0,1,0,1,0,1)\\
& & u^{(3)} &= (0,1,0,-1,2,-1,2,1)\\
\lambda_2 &= \tfrac{E(3-\nu)}{6(1-\nu^2)}, & u^{(4)} &= (1,0,-1,0,1,0,-1,0)\\
& & u^{(5)} &= (0,1,0,-1,0,1,0,-1)\\
\lambda_3 &= \tfrac{E}{1+\nu}, & u^{(6)} &= (1,0,0,-1,-1,0,0,1)\\
& & u^{(7)} &= (0,1,1,0,0,-1,-1,0)\\
\lambda_4 &= \tfrac{E}{1-\nu}, & u^{(8)} &= (1,1,-1,1,-1,-1,1,-1)
\end{aligned} \tag{4}$$

2 Plane strain

In the case of plane strain the stiffness matrix of a four-nodes element reads

$$K = \frac{E}{24(1-2\nu)(1+\nu)}\hat{K} \tag{5}$$

with

$$\hat{K} = \begin{bmatrix} 12-16\nu & 3 & 4\nu-6 & 12\nu-3 & 8\nu-6 & -3 & 4\nu & 3-12\nu \\ 3 & 12-16\nu & 3-12\nu & 4\nu & -3 & 8\nu-6 & 12\nu-3 & 4\nu-6 \\ 4\nu-6 & 3-12\nu & 12-16\nu & -3 & 4\nu & 12\nu-3 & 8\nu-6 & 3 \\ 12\nu-3 & 4\nu & -3 & 12-16\nu & 3-12\nu & 4\nu-6 & 3 & 8\nu-6 \\ 8\nu-6 & -3 & 4\nu & 3-12\nu & 12-16\nu & 3 & 4\nu-6 & 12\nu-3 \\ -3 & 8\nu-6 & 12\nu-3 & 4\nu-6 & 3 & 12-16\nu & 3-12\nu & 4\nu \\ 4\nu & 12\nu-3 & 8\nu-6 & 3 & 4\nu-6 & 3-12\nu & 12-16\nu & -3 \\ 3-12\nu & 4\nu-6 & 3 & 8\nu-6 & 12\nu-3 & 4\nu & -3 & 12-16\nu \end{bmatrix} \tag{6}$$

The characteristic polynomial is given by

$$\frac{\lambda^3(E-\nu\lambda-\lambda)^2(E+2\nu^2\lambda+\nu\lambda-\lambda)(4\nu E-3E-12\nu^2\lambda-6\nu\lambda+6\lambda)^2}{36(\nu+1)^5\ (2\nu-1)^3} = 0\ . \tag{7}$$

What gives as eigenvalues

$$\lambda \in \left\{0, \frac{E(3-4\nu)}{6(1-2\nu)(1+\nu)}, \frac{E}{1+\nu}, \frac{E}{(1+\nu)(1-2\nu)}\right\}\ .$$

And the corresponding eigenvectors are

$$\begin{aligned}
\lambda_1 &= 0, & u^{(1)} &= (1,0,1,0,1,0,1,0) \\
& & u^{(2)} &= (0,1,0,1,0,1,0,1) \\
& & u^{(3)} &= (0,1,0,-1,2,-1,2,1) \\
\lambda_2 &= \tfrac{E(3-4\nu)}{6(1-2\nu)(1+\nu)}, & u^{(4)} &= (1,0,-1,0,1,0,-1,0) \\
& & u^{(5)} &= (0,1,0,-1,0,1,0,-1) \\
\lambda_3 &= \tfrac{E}{1+\nu}, & u^{(6)} &= (1,0,0,-1,-1,0,0,1) \\
& & u^{(7)} &= (0,1,1,0,0,-1,-1,0) \\
\lambda_4 &= \tfrac{E}{(1+\nu)(1-2\nu)}, & u^{(8)} &= (1,1,-1,1,-1,-1,1,-1)
\end{aligned} \tag{8}$$

As expected, the eigenvectors are the same while the eigenvalues differ.