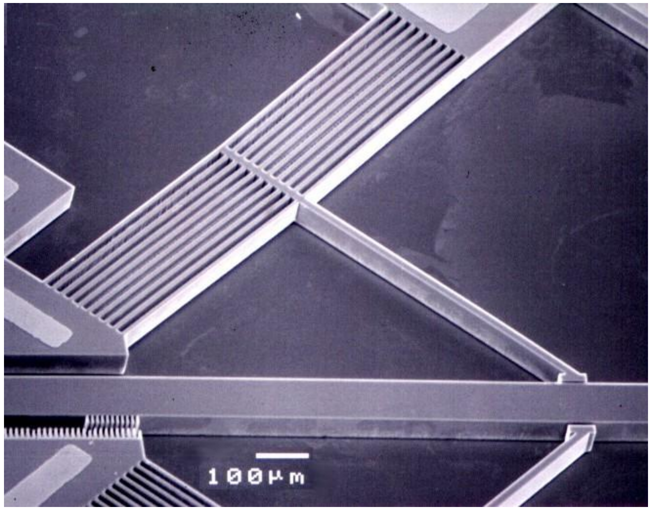
Beam Bending

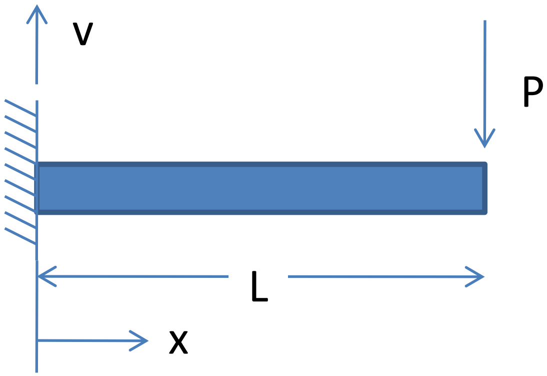
Assumptions – For simple beams

1) Cross section constant along length
2) Cross sectional dimension small compared to length
3) Stress and strain vary linearly across depth
4) Small displacements and curvature
5) Material Properties constant

Each assumption may be relaxed, but math becomes more complicated

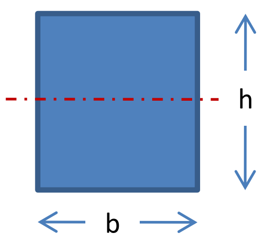


General Beam

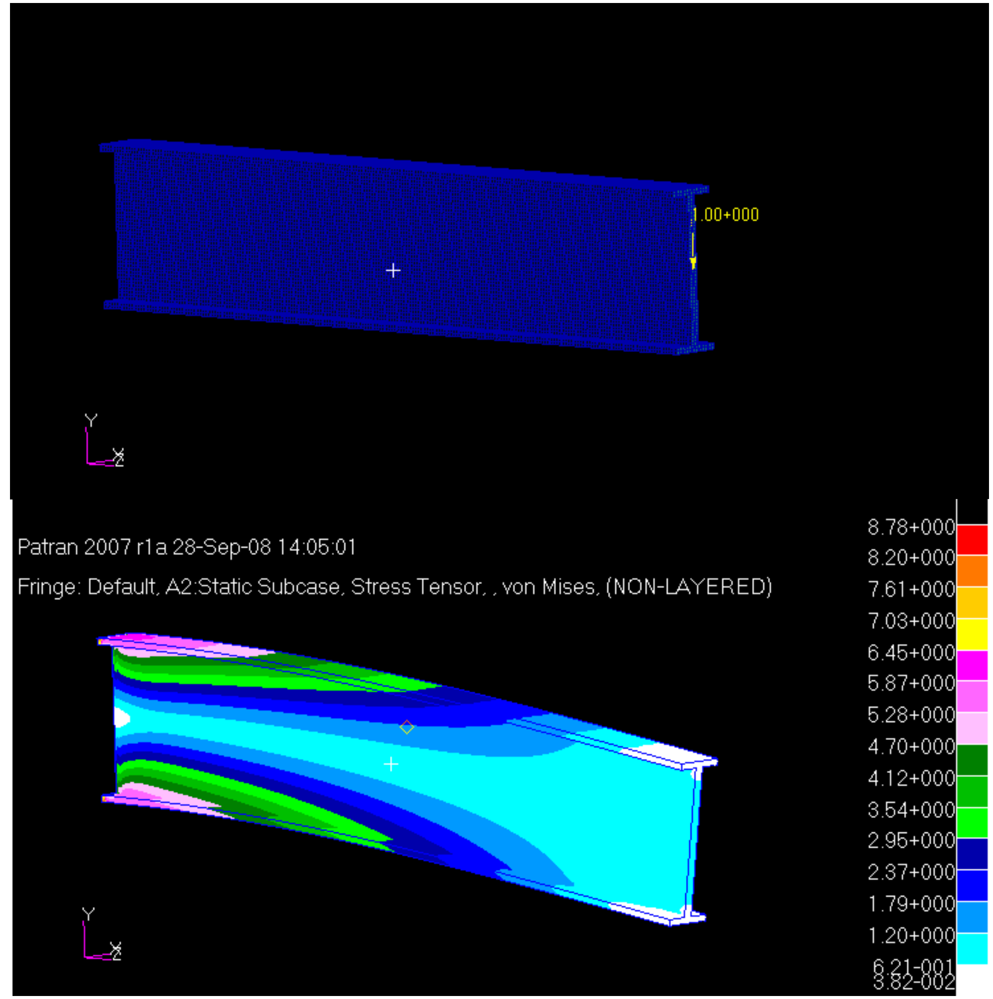


$$v = -\frac{PL^3}{3EI} \text{ at } x = L$$

$$\sigma = \frac{Mc}{I},\ M = PL \quad \text{(maximum moment)}$$



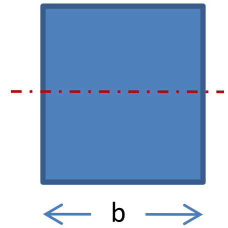
$$I = \frac{1}{12} bh^3$$



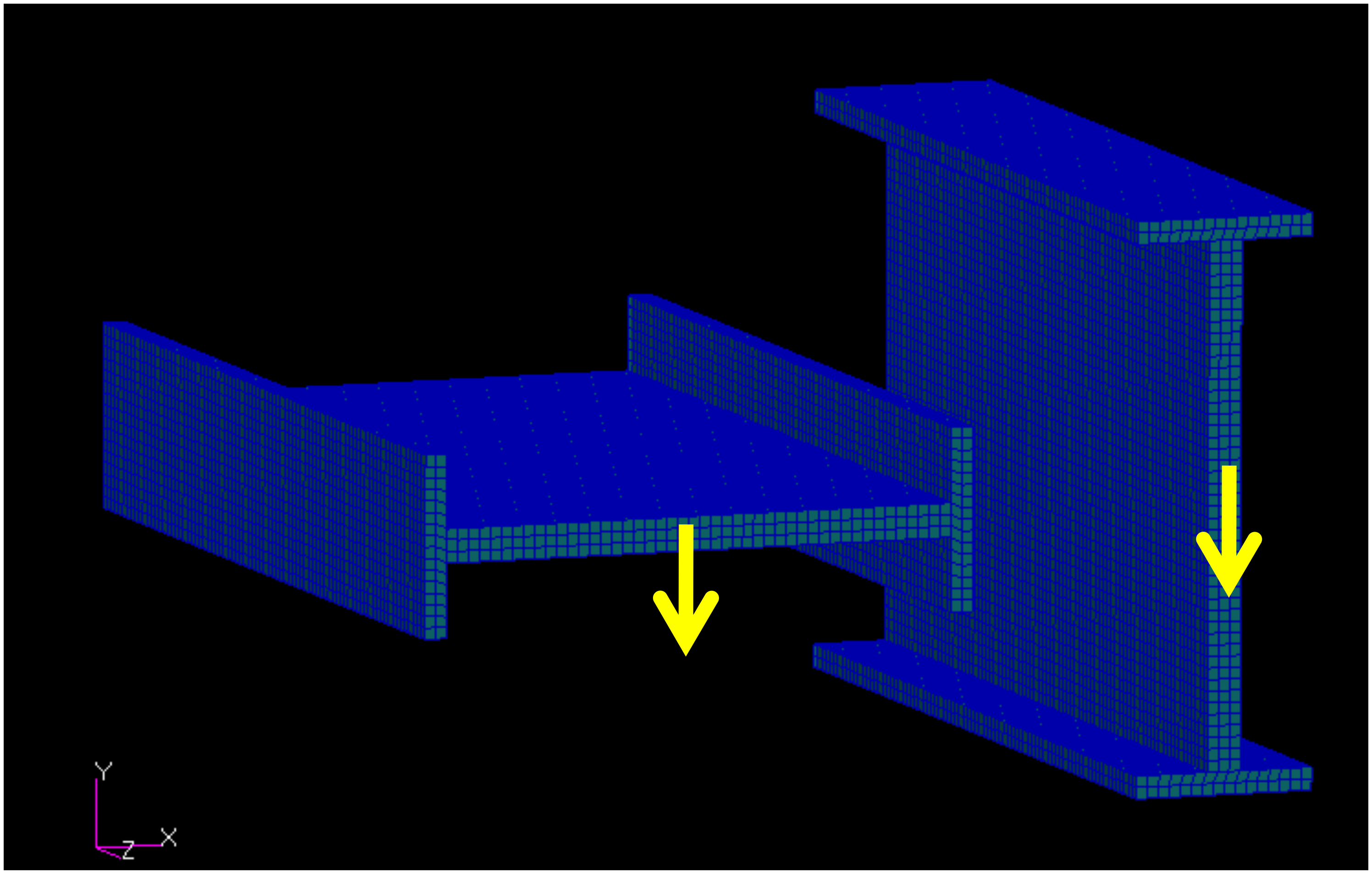
Stiffness and Stress

$$v = -\frac{PL^3}{3EI} \text{ at } x = L$$

$$\sigma = \frac{Mc}{I}, \; M = PL$$



$$I = \frac{1}{12} bh^3$$

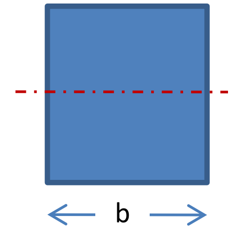


Aluminum Beam 10inches Deep, 6 inches High, 282lbf load

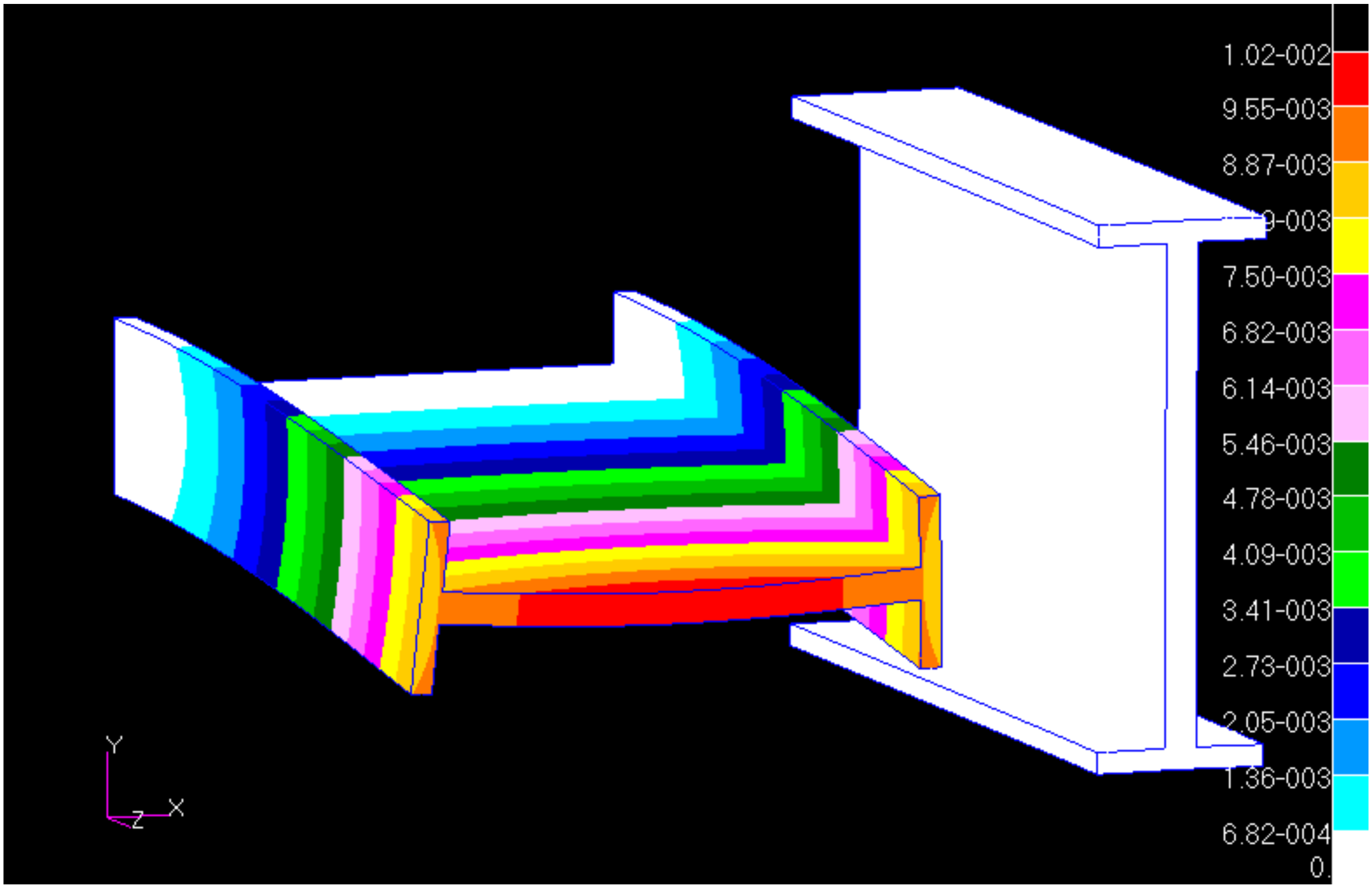
Stiffness and Stress

$$v = -\frac{PL^3}{3EI} \text{ at } x = L$$

$$\sigma = \frac{Mc}{I},\ M = PL$$

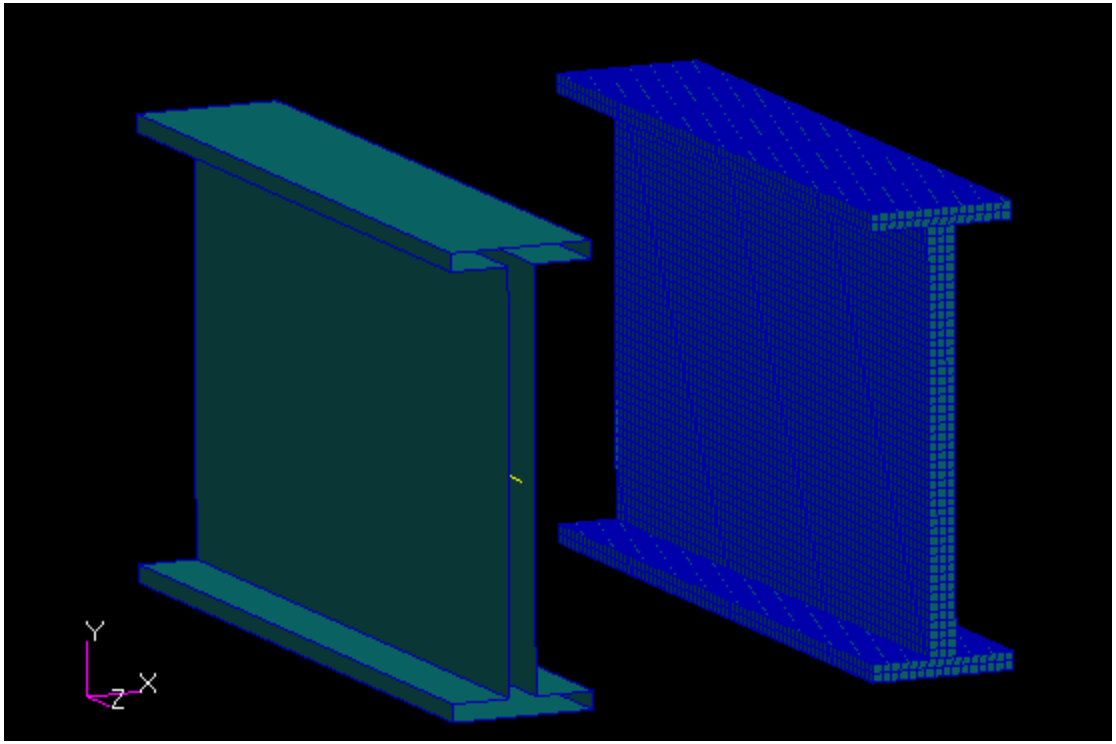


$$I = \frac{1}{12} b h^3$$

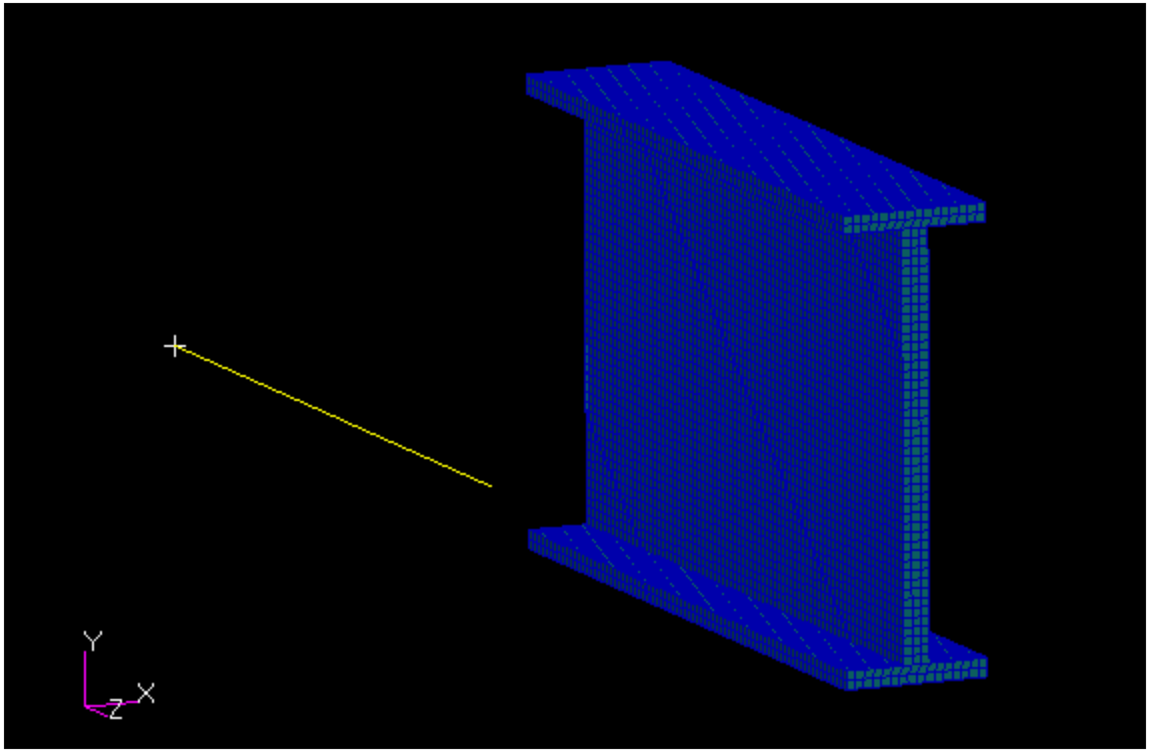


Aluminum Beam 10inches Deep, 6 inches High, 282lbf load

Why Beams?

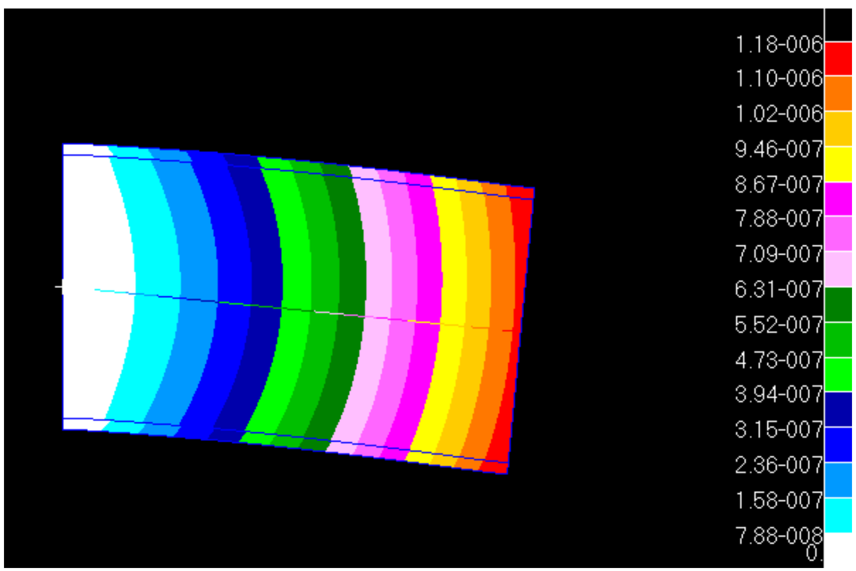


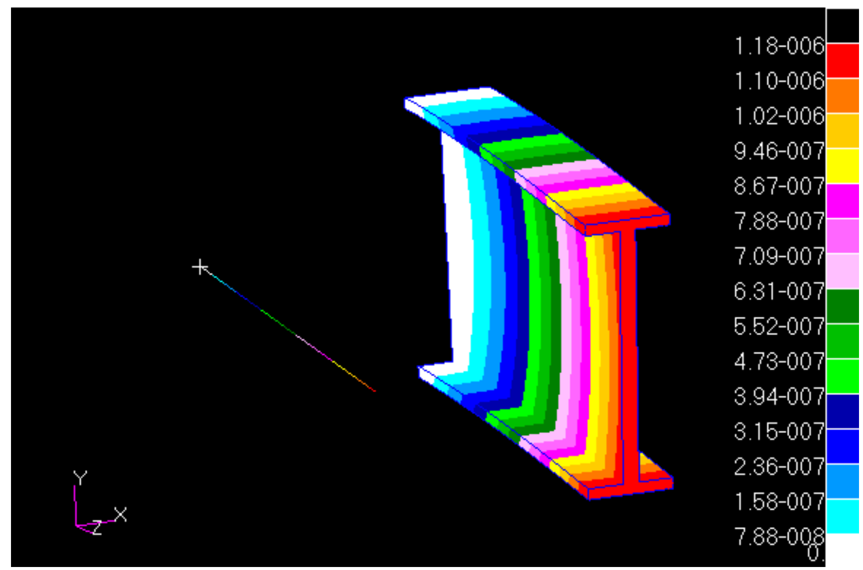
Why Beams?



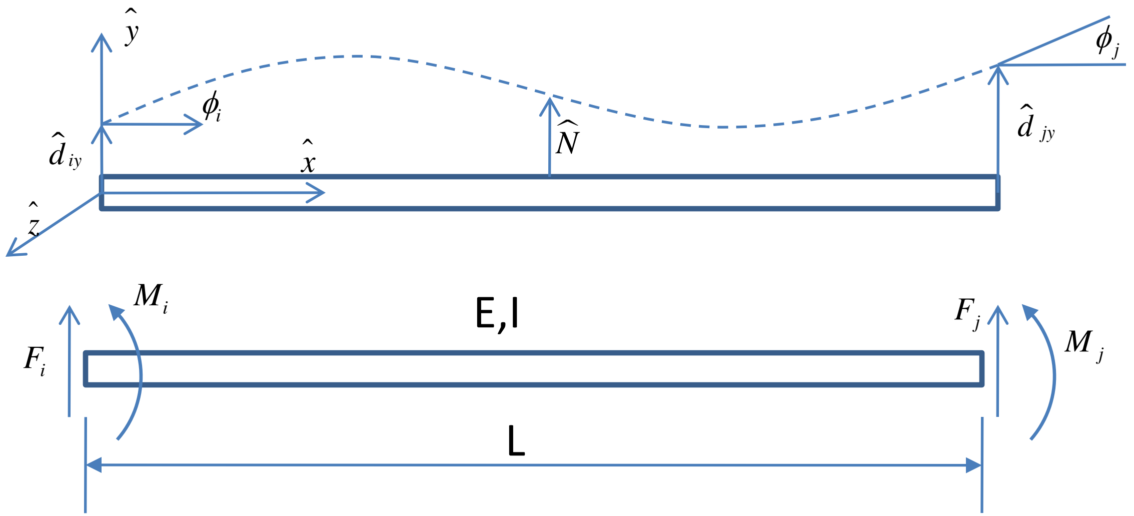
	Solid Mesh	Beam	Difference
Deflection (in)	1.15E-06	1.13E-06	-2%
Stress (psi)	2.13E+00	2.09E+00	-2%
# Elements	16159	1	16158
# Nodes	22841	2	22839
Solve time (s)	54.109	0.468	116 x
Disk space (Mb)	468.4	5.2	89 x

There ***are*** differences in boundary conditions
Stress values taken ½ inch from fixed end.





Beam: Sign Convention



Note: Different from strength of materials

- Displacement / Forces are positive along axis $\hat{y}$
- Rotations / Moments are positive about axis $\hat{z}$

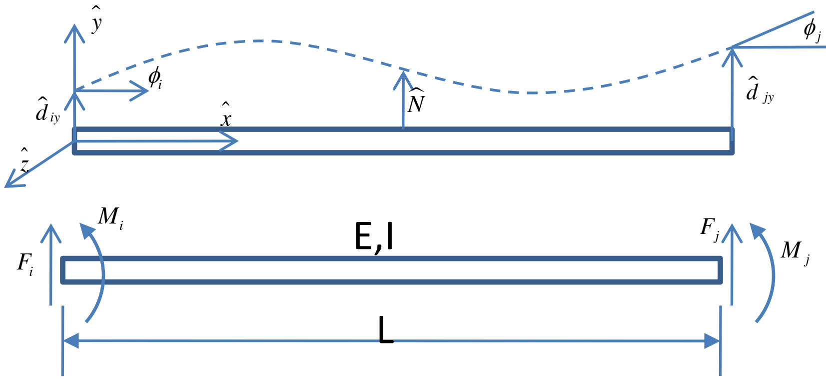
Assume Displacement Function:

$$\hat{v}\left(\hat{x}\right) = a_1 \hat{x}^3 + a_2 \hat{x}^2 + a_3 \hat{x} + a_4$$

Then slope is the rotation

$$\hat{\phi}\left(\hat{x}\right) = \frac{\partial \hat{v}}{\partial \hat{x}} = 3a_1 \hat{x}^2 + 2a_2 \hat{x} + a_3$$

These equations must be satisfied at the nodes



$$\hat{v}(\hat{x}=0)=\hat{d}_{iy}=a_4$$

$$\hat{\phi}(\hat{x}=0)=\hat{\phi}_i=a_3$$

$$\hat{v}(\hat{x}=L)=\hat{d}_{jy}=a_1L^3+a_2L^2+a_3L+a_4$$

$$\hat{\phi}(\hat{x}=L)=\hat{\phi}_j=3a_1L^2+2a_2L+a_3$$

Expressing in Matrix Form

$$\begin{Bmatrix}\hat{d}_{iy}\\ \hat{\phi}_i\\ \hat{d}_{jy}\\ \hat{\phi}_j\end{Bmatrix}=\begin{bmatrix}0&0&0&1\\0&0&1&0\\L^3&L^2&L&1\\3L^2&2L&1&0\end{bmatrix}\begin{Bmatrix}a_1\\a_2\\a_3\\a_4\end{Bmatrix}$$

$$\{d\}=[c]\{a\}$$

$$\{a\}=[c]^{-1}\{d\}$$

Substitute into the assumed form of the displacement equation

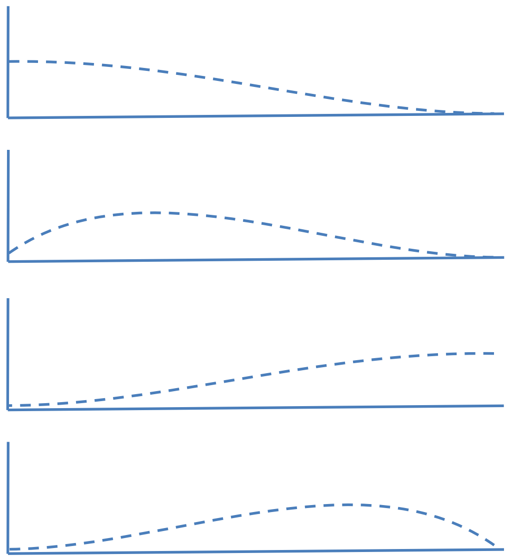
$$\hat{v}(\hat{x}) = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} \hat{d}_{iy} \\ \hat{\phi}_i \\ \hat{d}_{jy} \\ \hat{\phi}_j \end{Bmatrix}$$

$$N_1 = \frac{1}{L^3}\left(2\hat{x}^3 - 3\hat{x}^2 L + L^3\right)$$

$$N_2 = \frac{1}{L^3}\left(\hat{x}^3 L - 2\hat{x}^2 L^2 + \hat{x} L^3\right)$$

$$N_3 = \frac{1}{L^3}\left(-2\hat{x}^3 + 3\hat{x}^2 L\right)$$

$$N_4 = \frac{1}{L^3}\left(\hat{x}^3 L - \hat{x}^2 L^2\right)$$



Unit value of displacement or slope with all other values zero

From Bending Theory

$$\hat{m} = EI_1 \frac{\partial^2 v}{\partial x^2} = EI_1 \begin{bmatrix} \frac{\partial^2 N_1}{\partial x^2} & \frac{\partial^2 N_2}{\partial x^2} & \frac{\partial^2 N_3}{\partial x^2} & \frac{\partial^2 N_4}{\partial x^2} \end{bmatrix} \begin{Bmatrix} \hat{d}_{iy} \\ \hat{\phi}_i \\ \hat{d}_{jy} \\ \hat{\phi}_j \end{Bmatrix}$$

$$\hat{M}_i = -\hat{m}(x=0) = \frac{EI_1}{L^3} \begin{bmatrix} 6L & 4L^2 & -6L & 2L^2 \end{bmatrix} \begin{Bmatrix} \hat{d}_{iy} \\ \hat{\phi}_i \\ \hat{d}_{jy} \\ \hat{\phi}_j \end{Bmatrix}$$

$$\hat{F}_i = EI_1 \frac{\partial^3 v}{\partial x^3} = \text{Shear Force}$$

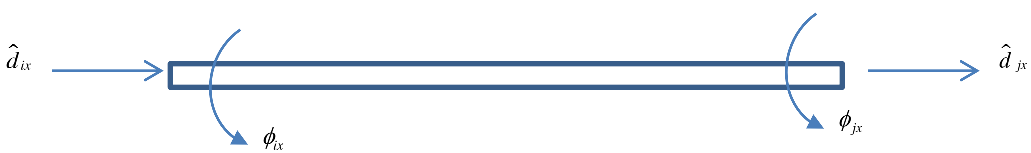
Writing in Full Matrix Form (Bending in the XY Plane)

$$\begin{Bmatrix} \hat{F}_i \\ \hat{M}_i \\ \hat{F}_j \\ \hat{M}_j \end{Bmatrix} = \underbrace{\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}}_{\left[\hat{k}\right]} \begin{Bmatrix} \hat{d}_{iy} \\ \hat{\phi}_i \\ \hat{d}_{jy} \\ \hat{\phi}_j \end{Bmatrix}$$

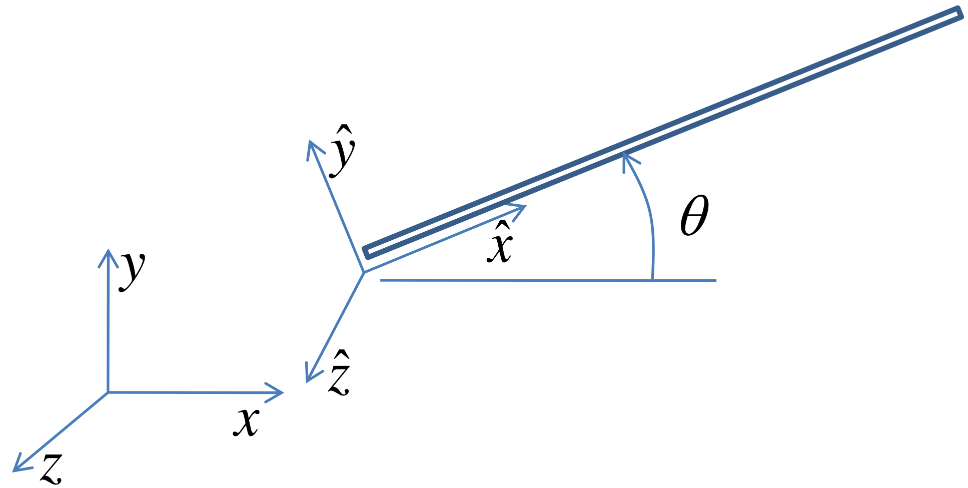
Bending in the XY Plane

Bending in the YZ Plane

Truss = Axial Behavior + Torsion



Beam Matrix given in Element Coordinate System



To convert to any other system

Use Transformation

$$[k]=[T]^T\left[\hat{k}\right][T]$$

where

$$T=\begin{bmatrix} \tilde{T} & 0 & 0 & 0 \\ 0 & \tilde{T} & 0 & 0 \\ 0 & 0 & \tilde{T} & 0 \\ 0 & 0 & & \tilde{T} \end{bmatrix}$$ 12 x 12

$$\tilde{T}=\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$$

l_1 m_1 n_1 are direction cosines of $\hat{x}$ with respect to xyz

l_2 m_2 n_2 are direction cosines of $\hat{y}$ with respect to xyz

l_3 m_3 n_3 are direction cosines of $\hat{z}$ with respect to xyz

Complete Beam has 6 stiffnesses at each node $\rightarrow$ 12 x 12 stiffness matrix

$$d_{ix} = \delta_1 = T_1 = 1$$

$$d_{iy} = \delta_2 = T_2 = 2$$

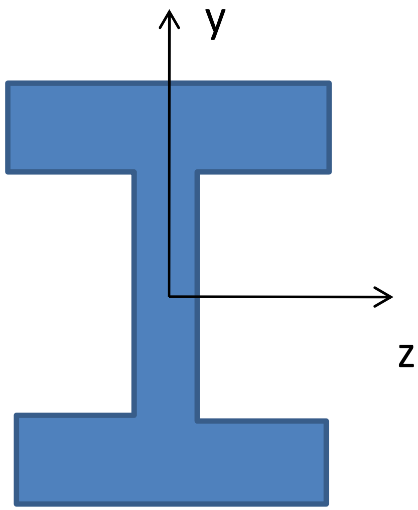
$$d_{iz} = \delta_3 = T_3 = 3$$

$$\phi_{jx} = \theta_1 = R_1 = 4$$

$$\phi_{jy} = \theta_2 = R_2 = 5$$

$$\phi_{jz} = \theta_3 = R_3 = 6$$

Rotations positive by Right-Hand-Rule about axis



I_1 = inertia for bending in the xy plane = I_Z
I_2 = inertia for bending in the xz plane = I_Y

Torsion:

$$k_\phi = \frac{JG}{L}$$

$$k = \frac{AE}{L}$$

$$G = \text{ Shear modulus } = \frac{E}{2(1+\upsilon)}$$

$$J = \text{ Torsional Factor}$$

$$J = I1 + I2 \text{ for circular cross sections only (See tables otherwise)}$$

Stiffness,
Symmetric Section
No Shear Deformation

End A						End B						
δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	
$\frac{AE}{L}$						$-\frac{AE}{L}$						δ_1
	$\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$		$-\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$	δ_2
		$\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$				$-\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$		δ_3
			$\frac{GJ}{L}$						$-\frac{GJ}{L}$			θ_1
				$\frac{4EI_2}{L}$				$\frac{6EI_2}{L^2}$		$\frac{2EI_2}{L}$		θ_2
					$\frac{4EI_1}{L}$		$-\frac{6EI_1}{L^2}$				$\frac{2EI_1}{L}$	θ_3
			Symmetry			$\frac{AE}{L}$						δ_1
							$\frac{12EI_1}{L^3}$				$-\frac{6EI_1}{L^2}$	δ_2
								$\frac{12EI_2}{L^3}$		$\frac{6EI_2}{L^2}$		δ_3
									$\frac{GJ}{L}$			θ_1
										$\frac{4EI_2}{L}$		θ_2
											$\frac{4EI_1}{L}$	θ_3

Simple Beam Element (CBAR)

CBAR Element Characteristics

The CBAR element is a general purpose beam that supports tension and compression, torsion, bending in two perpendicular planes, and shear in two perpendicular planes. The CBAR uses two grid points, and can provide stiffness to all six DOFs of each grid point. The displacement components of the grid points are three translations and three rotations.

The characteristics and limitations of the CBAR element are summarized as follows:

- Its formulation is derived from classical beam theory (plane sections remain plane).
- It must be straight and prismatic (properties cannot vary along the length).
- The shear center and neutral axis must coincide (the CBAR element cannot model warping of open sections).
- Torsional stiffening of out-of-plane cross-sectional warping is neglected.
- It includes optional transverse shear effects (important for short beams).
- The principal axis of inertia need not coincide with the element axis.
- The neutral axis may be offset from the grid points (an internal rigid link is created). This is useful for modeling stiffened plates or gridworks.
- A pin flag capability is available to provide a moment or force release at either end of the element (this permits the modeling of linkages or mechanisms).

CBAR Format

Two formats of the CBAR entry are available, as shown below:

Format:

1	2	3	4	5	6	7	8	9	10
CBAR	EID	PID	GA	GB	X1	X2	X3		
	PA	PB	W1A	W2A	W3A	W1B	W2B	W3B	

Alternate Format:

CBAR	EID	PID	GA	GB	G0				
	PA	PB	W1A	W2A	W3A	W1B	W2B	W3B	

Field	Contents
EID	Unique element identification number. (Integer > 0)
PID	Property identification number of a PBAR entry. (Integer > 0 or blank; Default is EID unless BAROR entry has nonzero entry in field 3)
GA, GB	Grid point identification numbers of connection points. (Integer > 0; GA ≠ GB).
X1, X2, X3	Components of orientation vector $\vec{v}$, from GA, in the displacement coordinate system at GA. (Real).
G0	Alternate method to supply the orientation vector $\vec{v}$ using grid point G0. Direction of $\vec{v}$ is from GA to G0. (Integer > 0; G0 ≠ GA or GB)
PA, PB	Pin flags for bar ends A and B, respectively. Used to remove connections between the grid point and selected degrees of freedom of the bar. The degrees of freedom are defined in the element's coordinate system. The bar must have stiffness associated with the PA and PB degrees of freedom to be released by the pin flags. For example, if PA = 4 is specified, the PBAR entry must have a value for J, the torsional stiffness. (Up to 5 of the unique Integers 1 through 6 anywhere in the field with no embedded blanks; Integer > 0)
W1A, W2A, W3A W1B, W2B, W3B	Components of offset vectors $\vec{w}_a$ and $\vec{w}_b$, respectively in displacement coordinate systems at points GA and GB, respectively. (Real or blank).

PID in field 3 points to a PBAR element property entry. Grid points GA and GB are connected by the element. X1, X2, and X3 are the components of orientation vector $\vec{v}$. Vector $\vec{v}$ describes how the beam's cross section is oriented with respect to the rest of the model. The continuation entry, which is optional, contains data for pin flags and offsets.

CBAR Element Coordinate System

The CBAR element coordinate system is a common source of difficulty for the new user. It is also critical to your model, since the beam's cross sectional moments of inertia are defined (on the PBAR entry) using the element coordinate system. For example, consider two possible orientations of the same rectangular beam (no offsets), as shown in Figure 6-3.

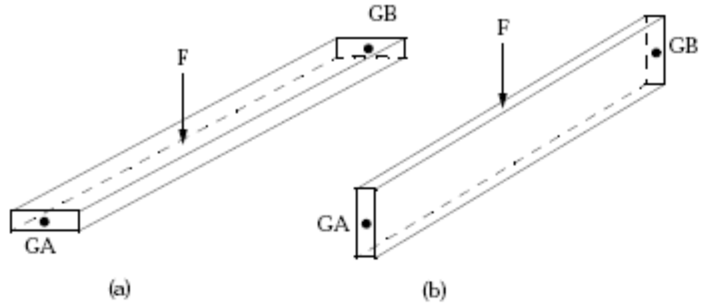
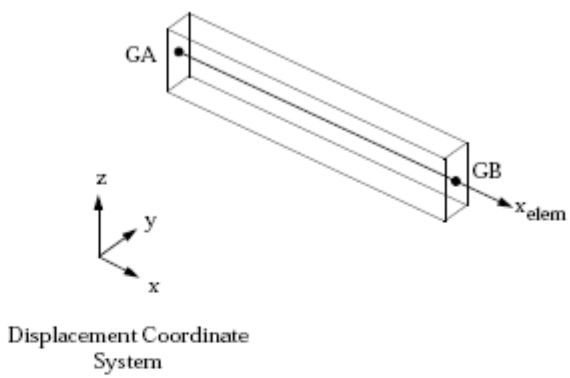


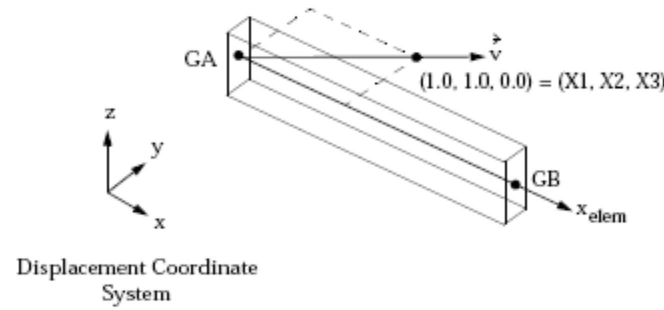
Figure 6-3 Two Orientations of the Same Beam

Although beams (a) and (b) are physically identical, their different orientations result in dramatically different load carrying abilities. In this sense, they are completely different structures. Therefore, it is critical to orient beam elements correctly.

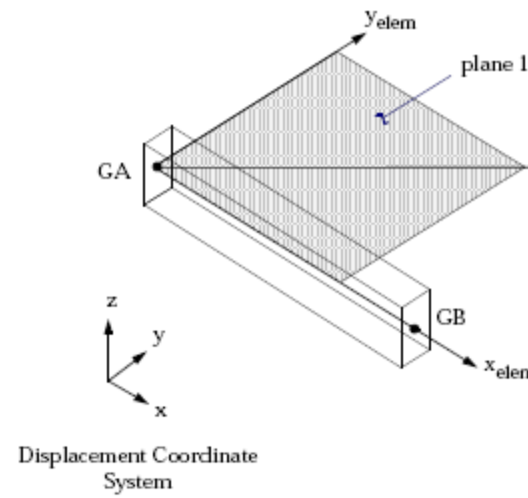
STEP 1 The element x-axis is automatically defined as the direction from GA to GB. The axis begins at GA:



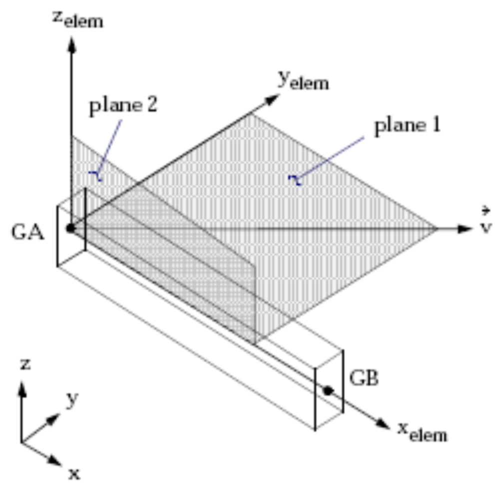
STEP 2 Next, we choose a direction for the beam's orientation vector $\vec{v}$. Vector $\vec{v}$ starts at GA and contains the point (X1, X2, X3). X1, X2, and X3-which are defined in the displacement coordinate system of GA-are entered on fields 6, 7, and 8 of the CBAR entry. The direction of $\vec{v}$ with respect to the cross section is arbitrary, but $\vec{v}$ is normally aligned with one of the beam's principal planes of inertia. A choice of (1.0, 1.0, 0.0) gives the following $\vec{v}$:



STEP3 The plane formed by the element x-axis and orientation vector $\vec{v}$ is called plane 1. The element y-axis lies in plane 1 and is perpendicular to the element x-axis, as shown below:



STEP 4 Finally, plane 2 is perpendicular to plane 1, and the element z-axis is formed by the cross product of the x and y element axes. Plane 2 contains the element x and z axes.



Displacement Coordinate System

CBAR Force and Moment Conventions

The CBAR element force and moment conventions are shown in Figure 6-5 and Figure 6-6.

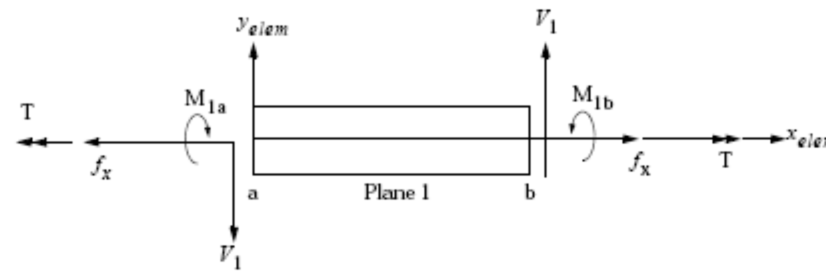


Figure 6-5 CBAR Element Internal Forces and Moments (x-y Plane)

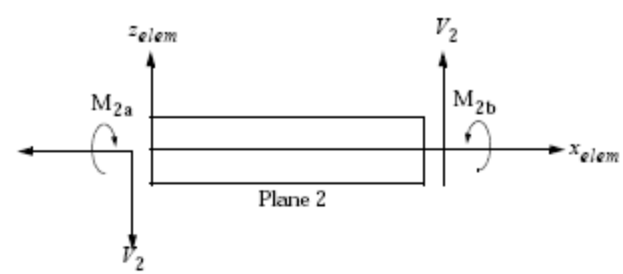


Figure 6-6 CBAR Element Internal Forces and Moments (x-z Plane)

Bar Element Property (PBAR)

The PBAR entry defines the properties of a CBAR element. The format of the PBAR entry is as follows:

1	2	3	4	5	6	7	8	9	10
PBAR	PID	MID	A	I1	I2	J	NSM		
	C1	C2	D1	D2	E1	E2	F1	F2	
	K1	K2	I12						

Field	Contents
PID	Property identification number. (Integer > 0)
MID	Material identification number. (Integer > 0)
A	Area of bar cross section. (Real)
I1, I2, I12	Area moments of inertia. (Real; $I1 \geq 0.0$, $I2 \geq 0.0$, $I1 \cdot I2 > I12^2$)
J	Torsional constant. (Real)
NSM	Nonstructural mass per unit length. (Real)
K1, K2	Area factor for shear. (Real)
Ci, Di, Ei, Fi	Stress recovery coefficients. (Real; Default = 0.0)

PID is the property's identification number from field 3 of the CBAR entry. MID references a MAT1 material property entry. I1 and I2 are area moments of inertia:

I1 = area moment of inertia for bending in plane 1 (same as Izz, bending about the z element axis)

I2 = area moment of inertia for bending in plane 2 (same as Iyy, bending about the y element axis)

J is the cross section's torsional constant (see Table 6-2). K1 and K2 depend on the shape of the cross section. K1 contributes to the shear resisting transverse force in plane 1 and K2 contributes to the shear resisting transverse force in plane 2.

Table 6-3 Area Factors for Shear

Shape of Cross Section	Value of K
Rectangular	K1 = K2 = 5/6
Solid Circular	K1 = K2 = 9/10
Thin-wall Hollow Circular	K1 = K2 = 1/2
Wide Flange Beams:	
Minor Axis	$\approx A_f / 1.2A$
Major Axis	$\approx A_w / A$

where:

A = Beam cross-sectional area

A_f = Area of flange

A_w = Area of web

The first continuation entry defines stress recovery coefficient points (Ci, Di, Ei, Fi) on the beam's cross section. These points are in the y-z plane of the element coordinate system as shown in

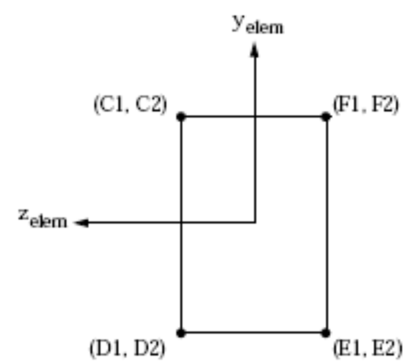
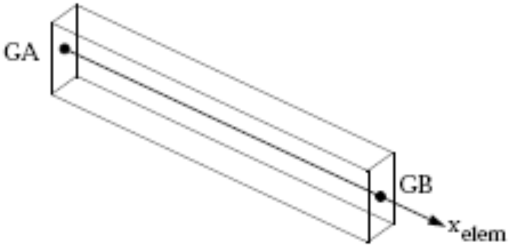
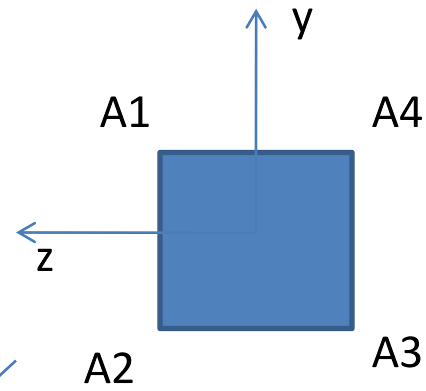


Figure 6-7 Stress Recovery Points on Beam Cross Section

By defining stress recovery points, you are providing c in the equation $\sigma = Mc/I$, thereby allowing MSC.Nastran to calculate stresses in the beam or on its surface.





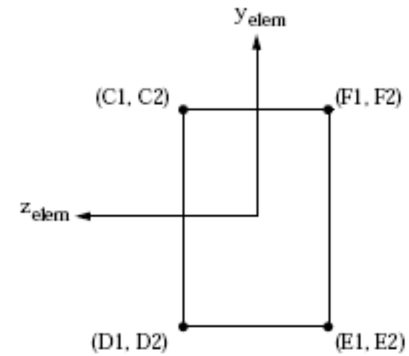


Figure 6-7 Stress Recovery Points on Beam Cross Section

1=Y
2=Z

```
...
grid,11,,0.,0.
grid,12,,16.,0.
cbar,21,20,11,12,0.,1.,0.
pbar,20,40,1.,0.083333
,0.5,0.5,-0.5,0.5,-0.5,-0.5,0.5,-0.5,
...
```

Max Stress generally at corners

$$\sigma = \frac{Mc}{I}$$

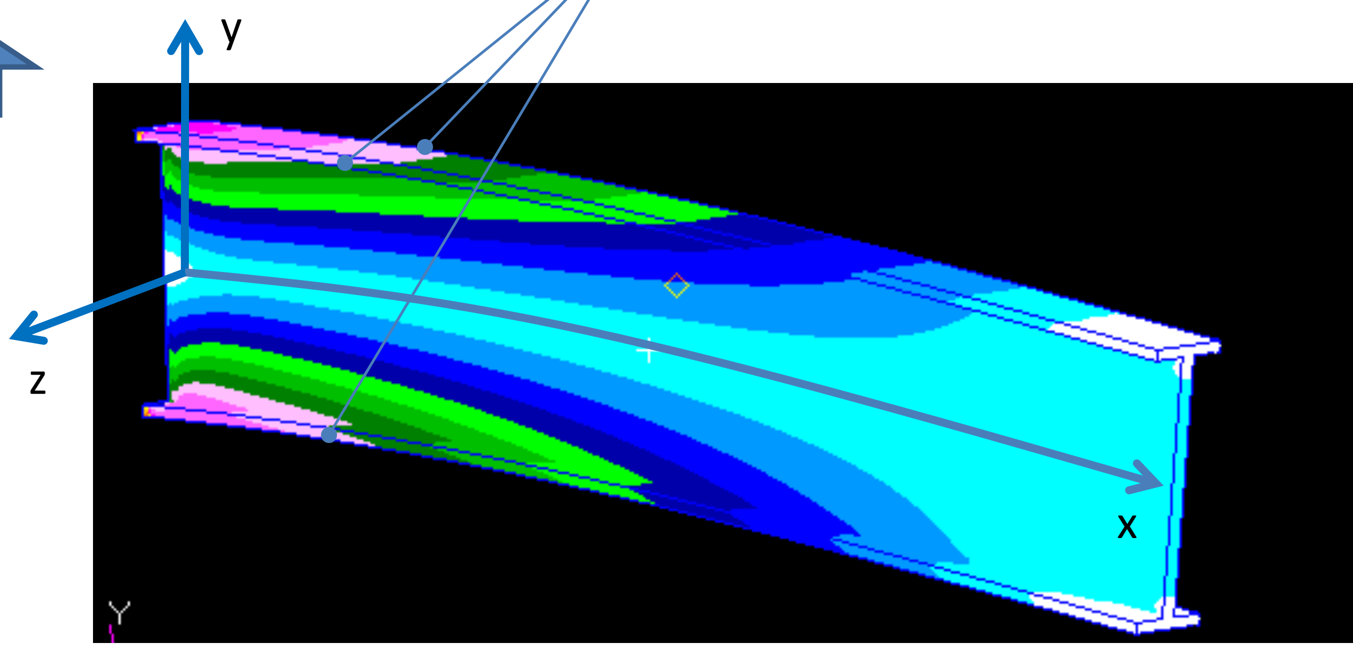
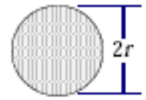
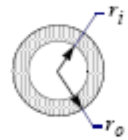
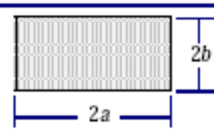
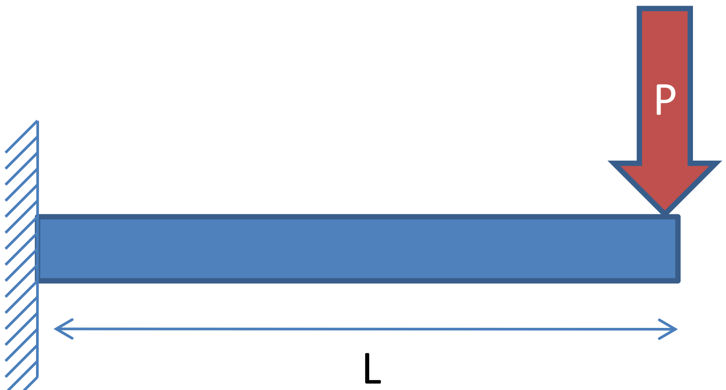
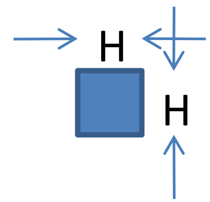


Table 6-2 Torsional Constant J for Line Elements

Type of Section	Formula for J	Cross Section
Solid Circular	$J = \frac{1}{2}\pi r^4$	
Hollow Circular	$J = \frac{1}{2}\pi(r_o^{\,4} - r_i^{\,4})$	
Solid Square	$J = 2.25\ a^4$	
Solid Rectangular	$J = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$	





$$\delta_T = \frac{PL^3}{3EI} + \frac{PL}{kAG}$$

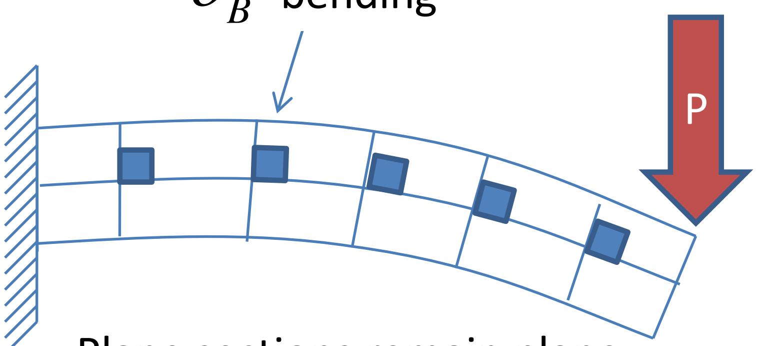
$$I = \frac{1}{12} H^3$$

$$A = H^2$$

$$K = 0.833$$

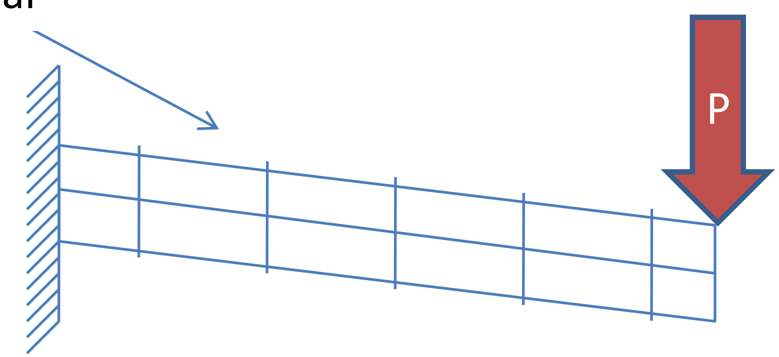
δ_B bending

δ_S shear



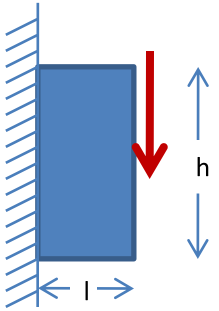
Plane sections remain plane
⊥ To mid-plane

Bar – Default → k=∞ → no shear
Beam – Default → k=1 → full shear

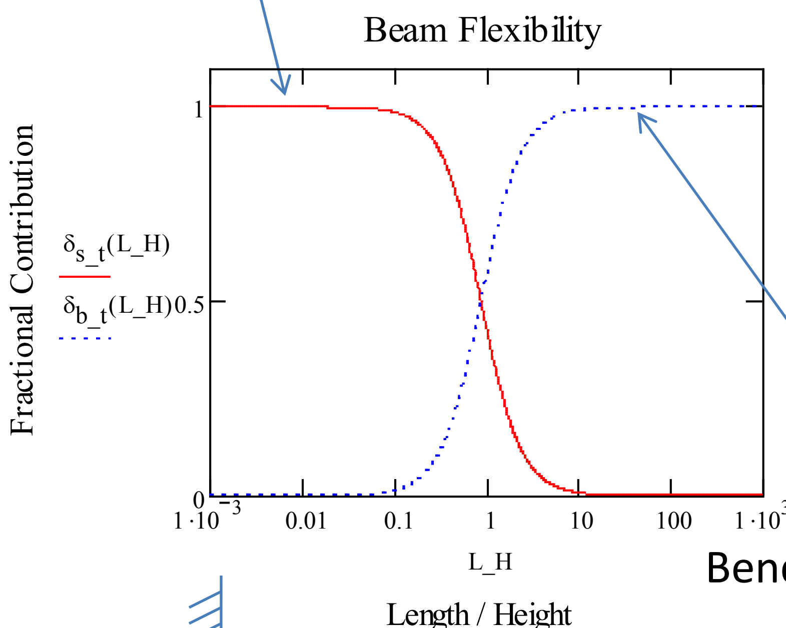


Plane sections do not remain plane
NOT ⊥ To mid-plane

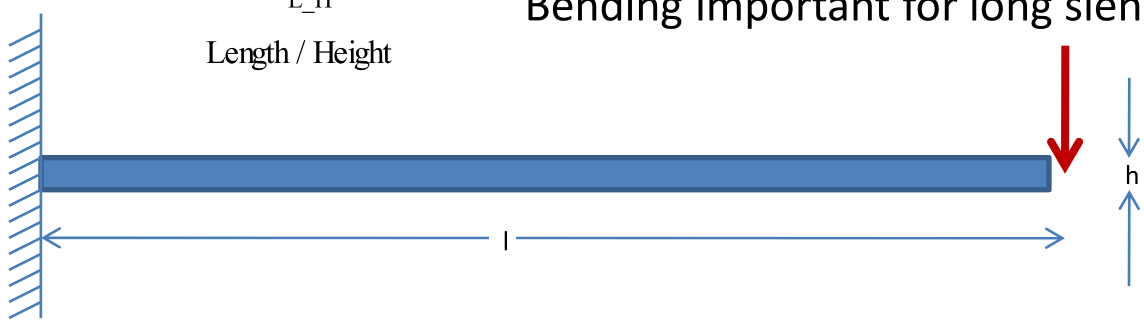
Shear important for short deep beams



Shear important for short deep beams



Bending important for long slender beams



$E := 3\,10^{6}\,\text{psi}$

$P := 1\,\text{lbf}$ $\qquad \nu := 0.3$

$K := 0.833$ $\qquad G := \dfrac{E}{2\cdot(1+\nu)}$

$B := 1\,\text{in}$

$$\delta_S(L\_H) := \frac{P\cdot L\_H}{K\cdot B\cdot G} \qquad \delta_B(L\_H) := \frac{P\cdot L\_H^3}{3\cdot E\cdot \frac{1}{12}\,B}$$

$$\delta_T(L\_H) := \delta_S(L\_H) + \delta_B(L\_H)$$

$$\delta_{s\_t}(L\_H) := \frac{\delta_S(L\_H)}{\delta_T(L\_H)} \qquad \delta_{b\_t}(L\_H) := \frac{\delta_B(L\_H)}{\delta_T(L\_H)}$$

Shear Factors: K

Bar – Default → $K_1 = K_2 = 0$ → no shear ***flexibility***

Beam – Default → $K_1 = K_2 = 1$ → full shear flexibility

$$K_i = \frac{\text{Area Active with Full Shear in } i \text{ direction}}{\text{Total Cross Sectional Area}}$$

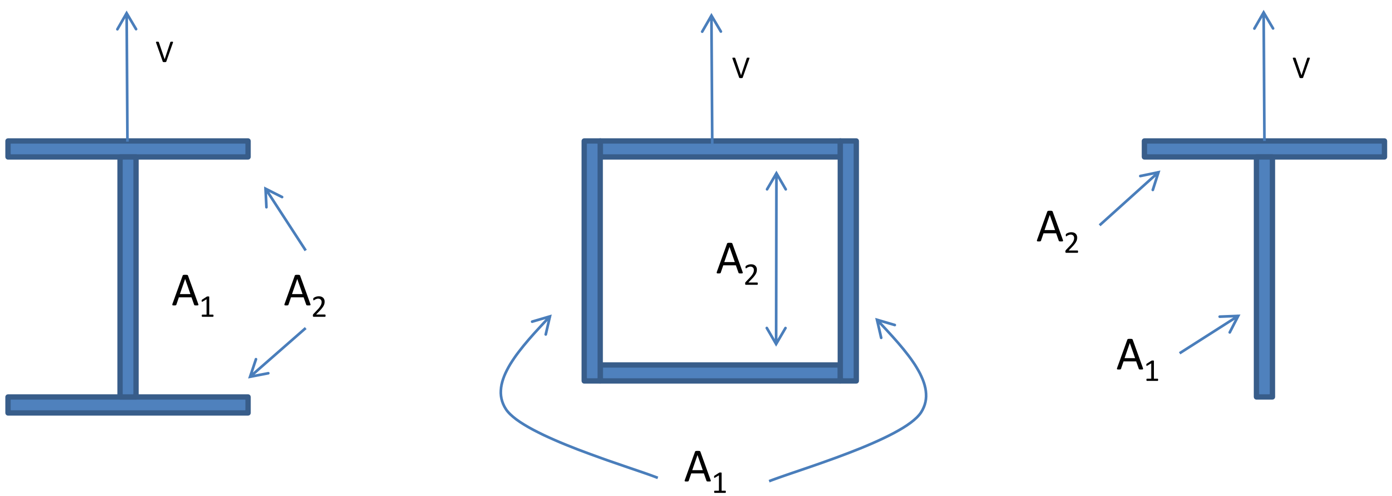
$$0 \le K_i \le 1$$

$$\text{Solid Circle: } K_1 = K_2 = 0.89$$

$$\text{Solid Rectangle: } K_1 = K_2 = 0.8333$$

$$K_1 = \frac{A_1}{A_1 + A_2}$$

$$K_2 = \frac{A_2}{A_1 + A_2}$$



BAR

5. The transverse shear stiffnesses per unit length in planes 1 and 2 are K1*A*G and K2*A*G, respectively, where G is the shear modulus. The default values for K1 and K2 are infinite; in other words, the transverse shear flexibilities are set equal to zero. K1 and K2 are ignored if I12 $\neq$ 0. K1 and K2 must be blank if A = 0.0.

BEAM

12. The shear stiffness factors K_1 and K_2 adjust the effective transverse shear cross-section area according to the Timoshenko beam theory. Their default values of 1.0 approximate the effects of shear deformation. To neglect shear deformation (i.e., to obtain the Bernoulli-Euler beam theory), the values of K_1 and K_2 should be set to 0.0.

Torsion Factor: J

For Circular Cross Section (Solid or Hollow) ***ONLY***

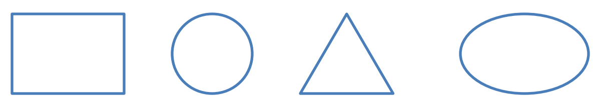
$$J = I_1 + I_2$$

All others: $J \neq I_1 + I_2$

For **Open** thin walled sections:

$$J = \frac{1}{2}\sum_i L_i t_i^3$$

For **Closed** thin walled sections:



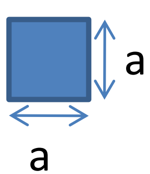
$$J = \frac{4t\tilde{A}^2}{\tilde{c}}$$

$\tilde{A} = \text{Area enclosed by midline}$

$\tilde{c} = \text{Circumference enclosed by midline}$

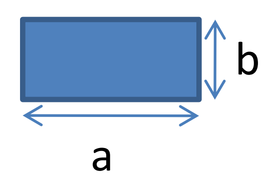
For solid Square section:

$$J = 0.1406\,a^4$$



For solid Rectangle section:

$$J = a\,b^3\left[\frac{1}{3} - 0.210\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$$



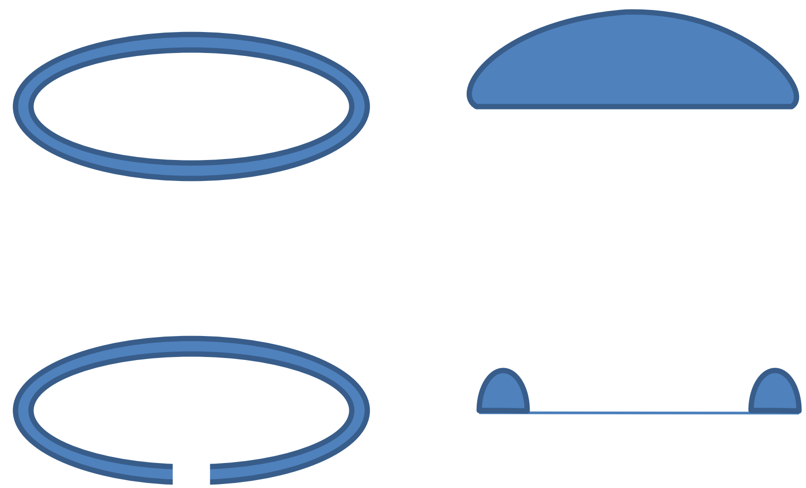
$$b < a$$

Paper Tube

Closed Section

$r_o := 0.8\text{in} \qquad r_i := 0.7\text{in}$

$$J_c := \frac{\pi}{2}\cdot\left(r_o^{\,4} - r_i^{\,4}\right) \qquad J_c = 0.266\ \text{in}^4$$



Area under membrane

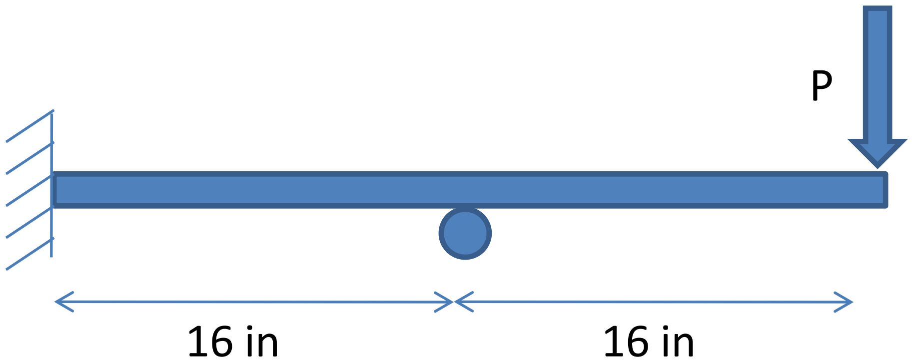
Open Section

$t := 0.1\text{in} \qquad r := 0.75\text{in} \qquad L := 2\cdot\pi\cdot r$

$$J_o := \frac{1}{3}\cdot L\cdot t^3 \qquad J_o = 1.571 \times 10^{-3}\ \text{in}^4$$

$$\frac{J_c}{J_o} = 169.5$$

Beam Problem Example



P = 10 #

E = 30E6 psi

A = 1 in^2 (square)

σ_{allow} = 2000 psi

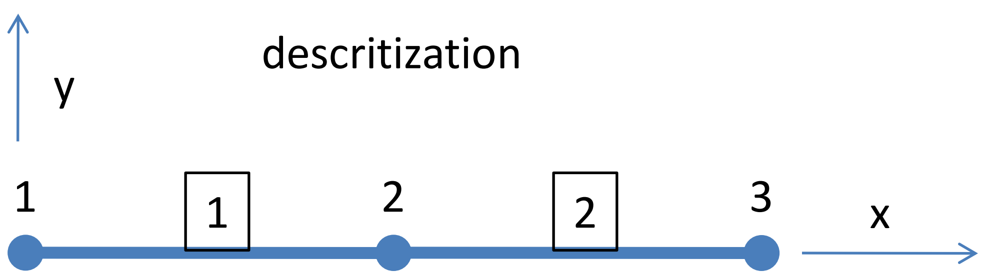
$$I = \frac{1}{12} b h^3 = \frac{1}{12} 1^4 = 0.083333$$

Consider only δ_y θ_z

Compute:

Nodal Displacements

Element Forces and Stresses



Stiffness,
Symmetric Section
No Shear Deformation

End A						End B						
δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	
$\frac{AE}{L}$						$-\frac{AE}{L}$						δ_1
	$\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$		$-\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$	δ_2
		$\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$				$-\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$		δ_3
			$\frac{GJ}{L}$						$-\frac{GJ}{L}$			θ_1
				$\frac{4EI_2}{L}$				$\frac{6EI_2}{L^2}$		$\frac{2EI_2}{L}$		θ_2
					$\frac{4EI_1}{L}$		$-\frac{6EI_1}{L^2}$				$\frac{2EI_1}{L}$	θ_3
		Symmetry				$\frac{AE}{L}$						δ_1
							$\frac{12EI_1}{L^3}$				$-\frac{6EI_1}{L^2}$	δ_2
								$\frac{12EI_2}{L^3}$		$\frac{6EI_2}{L^2}$		δ_3
									$\frac{GJ}{L}$			θ_1
										$\frac{4EI_2}{L}$		θ_2
											$\frac{4EI_1}{L}$	θ_3

Stiffness,
Symmetric Section
No Shear Deformation

End A						End B						
δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	
$\frac{AE}{L}$						$-\frac{AE}{L}$						δ_1
	$\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$		$-\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$	δ_2
		$\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$				$-\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$		δ_3
			$\frac{GJ}{L}$						$-\frac{GJ}{L}$			θ_1
				$\frac{4EI_2}{L}$				$\frac{6EI_2}{L^2}$		$\frac{2EI_2}{L}$		θ_2
					$\frac{4EI_1}{L}$		$-\frac{6EI_1}{L^2}$				$\frac{2EI_1}{L}$	θ_3
		Symmetry				$\frac{AE}{L}$						δ_1
							$\frac{12EI_1}{L^3}$				$-\frac{6EI_1}{L^2}$	δ_2
								$\frac{12EI_2}{L^3}$		$\frac{6EI_2}{L^2}$		δ_3
									$\frac{GJ}{L}$			θ_1
										$\frac{4EI_2}{L}$		θ_2
											$\frac{4EI_1}{L}$	θ_3

End A						End B						
δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	δ_1	δ_2	δ_3	θ_1	θ_2	θ_3	
$\frac{AE}{L}$						$-\frac{AE}{L}$						δ_1
	$\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$		$-\frac{12EI_1}{L^3}$				$\frac{6EI_1}{L^2}$	δ_2
		$\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$				$-\frac{12EI_2}{L^3}$		$-\frac{6EI_2}{L^2}$		δ_3
			$\frac{GJ}{L}$						$-\frac{GJ}{L}$			θ_1
				$\frac{4EI_2}{L}$				$\frac{6EI_2}{L^2}$		$\frac{2EI_2}{L}$		θ_2
					$\frac{4EI_1}{L}$		$-\frac{6EI_1}{L^2}$				$\frac{2EI_1}{L}$	θ_3
						$\frac{AE}{L}$						δ_1
							$\frac{12EI_1}{L^3}$				$-\frac{6EI_1}{L^2}$	δ_2
								$\frac{12EI_2}{L^3}$		$\frac{6EI_2}{L^2}$		δ_3
									$\frac{GJ}{L}$			θ_1
										$\frac{4EI_2}{L}$		θ_2
											$\frac{4EI_1}{L}$	θ_3

$$
[k_1]\{\delta\} = \begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \delta_{y1} \\ \theta_{z1} \\ \delta_{y2} \\ \theta_{z2} \end{Bmatrix}
$$

$$[k_1]\{\delta_1\}=\begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}\begin{Bmatrix} \delta_{y1} \\ \theta_{z1} \\ \delta_{y2} \\ \theta_{z2} \end{Bmatrix} \qquad [k_2]\{\delta_2\}=\begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}\begin{Bmatrix} \delta_{y2} \\ \theta_{z2} \\ \delta_{y3} \\ \theta_{z3} \end{Bmatrix}$$

For this problem (k_1 and k_2 are identical except for subscripts)

Assemble System Matrix

$$\begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & 0 \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 0 \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 2\times 12\frac{EI}{L^3} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 2\times 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ 0 & 0 & -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 0 & 0 & 6\frac{EI}{L^2} & 2\frac{EI}{L} & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix}\begin{Bmatrix} \delta_{y1} \\ \theta_{z1} \\ \delta_{y2} \\ \theta_{z2} \\ \delta_{y3} \\ \theta_{z3} \end{Bmatrix}=\begin{Bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \\ F_{y3} \\ M_{z3} \end{Bmatrix}$$

$$\{d_s\} = \begin{Bmatrix} \delta_{y1} \\ \theta_{z1} \\ \delta_{y2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \{d_F\} = \begin{Bmatrix} \theta_{z2} \\ \delta_{y3} \\ \theta_{z3} \end{Bmatrix} \quad \{F_F\} = \begin{Bmatrix} M_{z2} = 0 \\ F_{y3} = -10 \\ M_{z3} = 0 \end{Bmatrix}$$

$$K_{FF} d_F - K_{SS} d_S = F_F$$

$$\begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} & 0 & 0 \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & 0 \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 2\times 12\frac{EI}{L^3} & 0 & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & 0 & & & \\ 0 & 0 & -12\frac{EI}{L^3} & & \boxed{K_{FF}} & \\ 0 & 0 & 6\frac{EI}{L^2} & & & \end{bmatrix} \begin{Bmatrix} \delta_{y1} \\ \theta_{z1} \\ \delta_{y2} \\ \boxed{d_F} \end{Bmatrix} = \begin{Bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ \boxed{F_F} \end{Bmatrix}$$

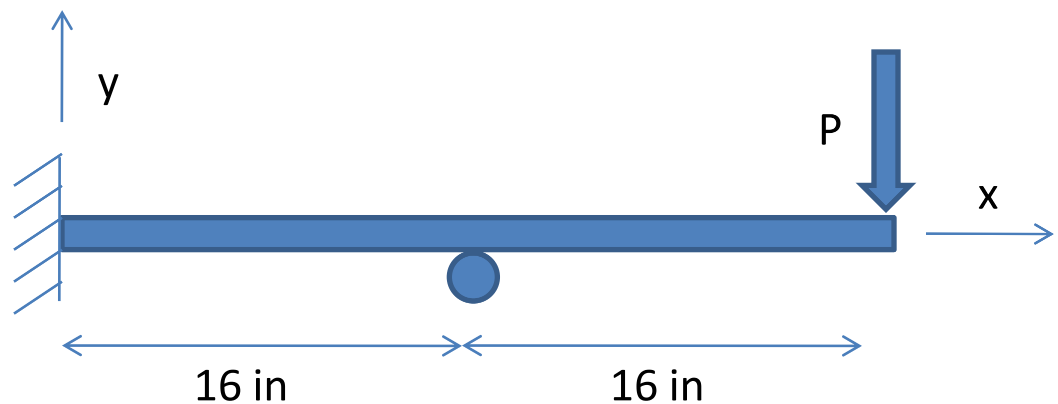
$$\begin{bmatrix} 8\dfrac{EI}{L} & -6\dfrac{EI}{L^2} & 2\dfrac{EI}{L} \\ -6\dfrac{EI}{L^2} & 12\dfrac{EI}{L^3} & -6\dfrac{EI}{L^2} \\ 2\dfrac{EI}{L} & -6\dfrac{EI}{L^2} & 4\dfrac{EI}{L} \end{bmatrix} \begin{Bmatrix} \theta_{z2} \\ \delta_{y3} \\ \theta_{z3} \end{Bmatrix} = \begin{Bmatrix} M_{z2} \\ F_{y3} \\ M_{z3} \end{Bmatrix}$$

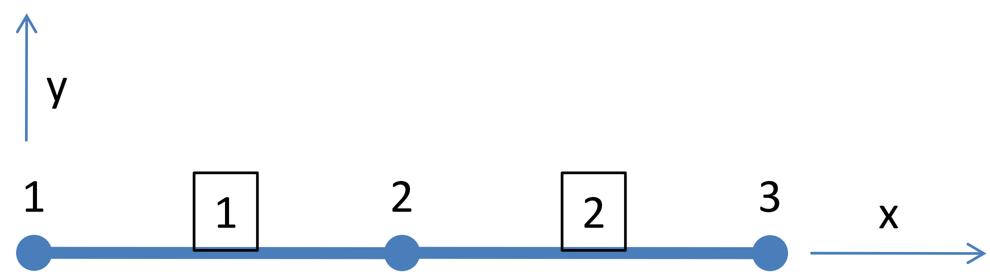
$$K_{FF} d_F - \cancel{K_{SS} d_S} = F_F$$

$$\begin{bmatrix} 1.25\mathrm{E}+06 & -5.86\mathrm{E}+04 & 3.13\mathrm{E}+05 \\ -5.86\mathrm{E}+04 & 7.32\mathrm{E}+03 & -5.86\mathrm{E}+04 \\ 3.13\mathrm{E}+05 & -5.86\mathrm{E}+04 & 6.25\mathrm{E}+05 \end{bmatrix} \begin{Bmatrix} \theta_{z2} \\ \delta_{y3} \\ \theta_{z3} \end{Bmatrix} = \begin{Bmatrix} 0 \\ -10 \\ 0 \end{Bmatrix}$$

Displacements and Rotations

$$\begin{Bmatrix} \theta_{z2} \\ \delta_{y3} \\ \theta_{z3} \end{Bmatrix} = \begin{Bmatrix} -2.56\mathrm{E}-04 \\ -9.56\mathrm{E}-03 \\ -7.68\mathrm{E}-04 \end{Bmatrix}$$





Element Equations

$$[k_1]\{\delta_1\} = \begin{bmatrix} 12\frac{EI}{L^3} & 6\frac{EI}{L^2} & -12\frac{EI}{L^3} & 6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 4\frac{EI}{L} & -6\frac{EI}{L^2} & 2\frac{EI}{L} \\ -12\frac{EI}{L^3} & -6\frac{EI}{L^2} & 12\frac{EI}{L^3} & -6\frac{EI}{L^2} \\ 6\frac{EI}{L^2} & 2\frac{EI}{L} & -6\frac{EI}{L^2} & 4\frac{EI}{L} \end{bmatrix} \begin{Bmatrix} \delta_{y1} \\ \theta_{z1} \\ \delta_{y2} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} F_{y1} \\ M_{z1} \\ F_{y2} \\ M_{z2} \end{Bmatrix}$$

Element 1 and Element 2 are identical (with different subscripts)

E1	7.32E+03	5.86E+04	-7.32E+03	5.86E+04	0.00E+00	δy1	-1.50E+01	Fy1	
	5.86E+04	6.25E+05	-5.86E+04	3.13E+05	0.00E+00	θz1	-8.00E+01	Mz1	
	-7.32E+03	-5.86E+04	7.32E+03	-5.86E+04	0.00E+00	δy2	1.50E+01	Fy2	
	5.86E+04	3.13E+05	-5.86E+04	6.25E+05	-2.56E-04	θz2	-1.60E+02	Mz2	Max Moment
E2	7.32E+03	5.86E+04	-7.32E+03	5.86E+04	0.00E+00	δy1	1.00E+01	Fy2	
	5.86E+04	6.25E+05	-5.86E+04	3.13E+05	-2.56E-04	θz1	1.60E+02	Mz2	
	-7.32E+03	-5.86E+04	7.32E+03	-5.86E+04	-9.56E-03	δy2	-1.00E+01	Fy3	
	5.86E+04	3.13E+05	-5.86E+04	6.25E+05	-7.68E-04	θz2	1.14E-13	Mz3	

$$\sigma_{peak} = \frac{Mc}{I} = \frac{160 \times 0.5}{1/12} = 960\,\text{psi}$$

$$\text{M.S.} = \frac{\sigma_{allow}}{\sigma_{peak}} - 1 = \frac{2000}{960} - 1 = 1.08$$