Practice of Finite Element Method, SUPAERO SM3A

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Exercices extracted from J-F. Imbert, Analyse des structures par éléments finis, Cepadues Supplementary Materials (Matlab code) can be found on:

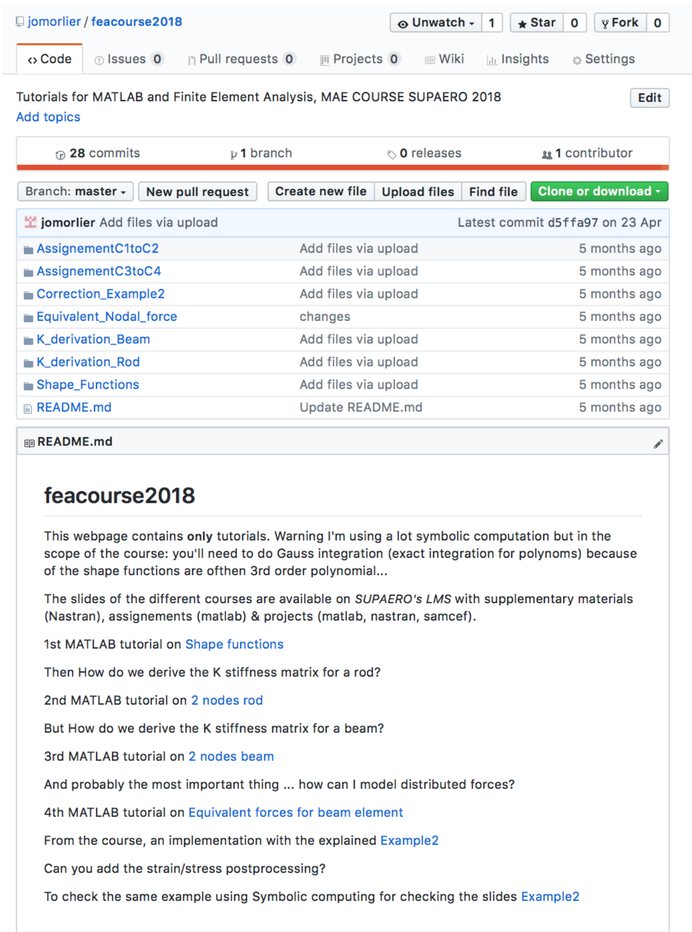

Figure 1: *A bar submitted to an affine load.*

https://github.com/jomorlier/feacourse2018

PC1. Finite elements of single beam models

Exercice 1 (20'): Equivalent loading forces

Compute the equivalent (generalized) nodal loads related to an affine load density $p(s)$ of Fig. 2.

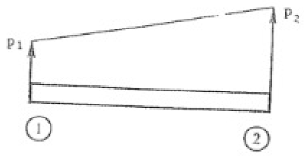

Figure 2: *A bar submitted to an affine load.*

Exercice 2 (20'): Stiffness matrix $\boldsymbol{K}$ of an in-plan bending of a straight beam

Compute the stiffness matrix $\boldsymbol{K}$ of the bending beam model in Fig. 3

1. by writing the strain complementary energy,
2. or from the compliance matrix,

when the shear strain related to the shear load:

1. is neglected (thin beam case);
2. is not neglected (thick beam case).

The beam in the (x, y)-plane is clamped at the node $|\overline{\underline{1}}|$ and is submitted to both a shear force V_y and a torque $\mathcal{M}_z$ at the node $|\overline{\underline{2}}|$.

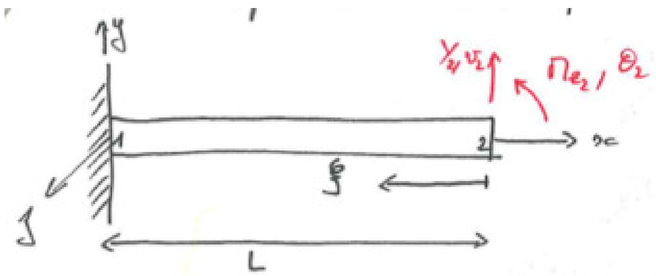

Figure 3: *In-plane bending of a straight beam.*

PC 2: Static analysis of a beam assembly structures

Exercice 1 (15' in class + homework). 1D generalized beam

Compute the stiffness matrix $\boldsymbol{K}$ of the 3D bending and strained beam model in Fig. 4.

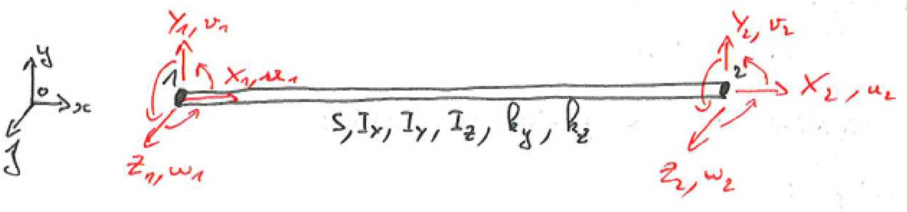

Figure 4: *General case of a straight beam FE model.*

Exercice 2: Beam-Spring structures

Consider a thin cantilever beam that is colinear to the x-axis as depicted on Fig. 5. Its Young's modulus is E, cross-sectional area A, length $(1+\alpha)L/\alpha$ (where $\alpha > 0$). While the beam is also clamped at its two extremities, an elastic support is attached to the beam at the beam material point abscissa L/α. That elastic support is modeled as a spring with negligible mass and stiffness k, and that is colinear to the y-axis. The beam is moreover submitted to a transverse linear distributed force density $-p_y\vec{y}_{\text{glo}}$ with a constant magnitude p_y.

Determine the deflection of the nodes, the forces and stresses in each member, and the reaction forces at the supports.

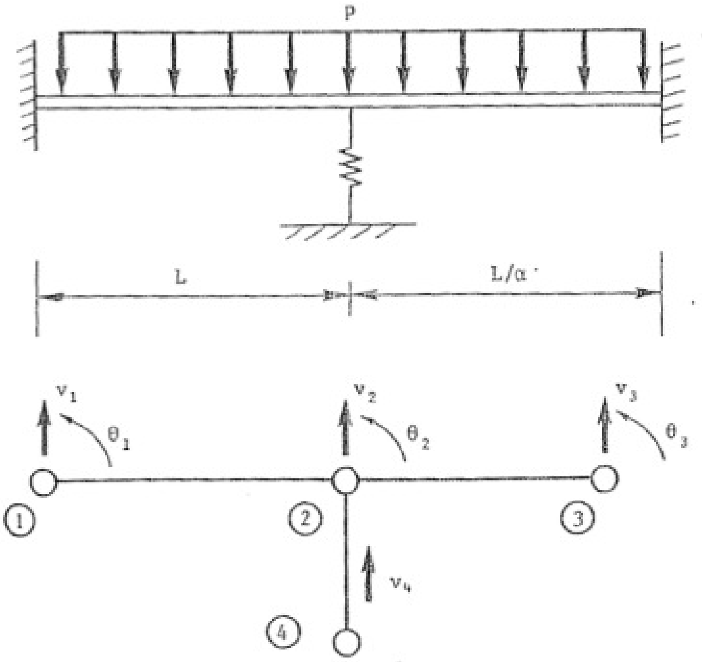

Figure 5: *The clamped beam model on an elastic support.*

Reminder & hints:

- compute the stiffness matrix of the free structure for the suggested FE system;
- write down the vectorial balance equation in (virtual) work sense:

$$\underbrace{\widehat{\boldsymbol{q}}^T \boldsymbol{F}_{eq}^{int}}_{\text{cohesive load work}} = \underbrace{\widehat{\boldsymbol{q}}^T \boldsymbol{F}_{eq}^{ext}}_{\text{external load work}} \quad ;$$

- use the boundary conditions to decompose the algebraic system of equations in two algebraic subsystems, one for the unknown displacements (degrees of freedom), the other one for the unknown reaction loads.

PC 3 (45'). 2D static FEA of a frame of beams: bad conditioning number.

Consider a frame composed of a floor and two poles (cf. Fig. 6 and table below), which is submitted to a lateral load. The poles are modeled as pure bending beams and the floor as a complete, stretched and bending beam. In reality, we can admit that the floor is infinitely rigid in bending so that the rotations at nodes $|\overline{2}|$ and $|\overline{3}|$ are zero and the floor merely turns like a single bar.

1. Express the stiffness matrix of the free-body structure.
2. Decompose the stiffness matrix of the algebraic system of equations to solve, while taking into account the boundary conditions of the clamped poles.
3. Solve the algebraic system of equations for a unit-norm lateral load while keeping for the stiffness matrix:
   (a) either four significant digits,
   (b) or six significant digits.

   Conclude.

Data:

Floor	Poles
$L = 3.45m$ $S = 3.45 \times 10^{-2} m^2$ $E = 20 \times 10^{10} N/m^2$	$L = 3.45m$ $t = 2 \times 10^{-3} m$ $R = 10^{-1} m$ $I = \pi R^3 t$ $E = 20 \times 10^{10} N/m^2$

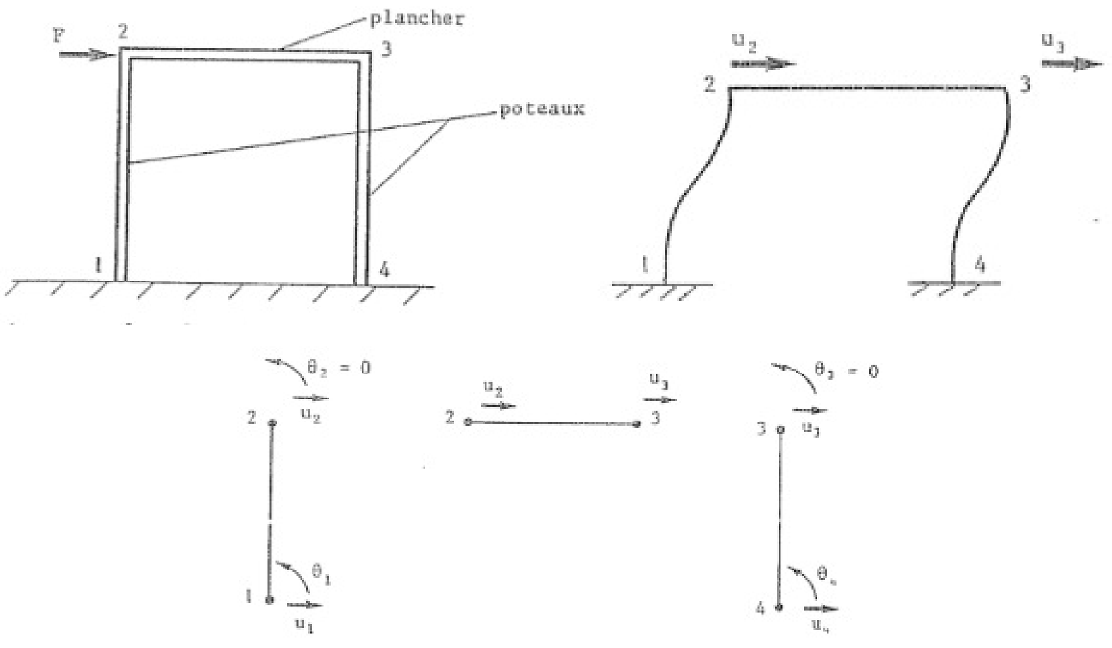

Figure 6: *The spring-beam frame and its FE modeling.*

PC4 (45'). Quadratic finite element of bars.

1. Determine the quadratic interpolating function of the displacement field (using the natural coordinate $\xi \in [-1, 1]$) in a linearly elastic bar (sketched on Fig. 7)

$$u(\xi) = \boldsymbol{N}^T \boldsymbol{q} \text{ where } \boldsymbol{N} \overset{déf}{=} \begin{bmatrix} N_1(\xi) \\ N_2(\xi) \\ N_3(\xi) \end{bmatrix} \text{ and } \boldsymbol{q} \overset{déf}{=} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}.$$

2. By using the appropriate Gauss-Legendre' quadrature formulas, give the minimal and maximal numbers of integration points that are required to accurately compute the stiffness matrix and the coherent mass matrix, while considering the following bar section cases:

   (a) linearly varying section area S;

   (b) constant section area S; also compute explicitly the stiffness matrix for this case.

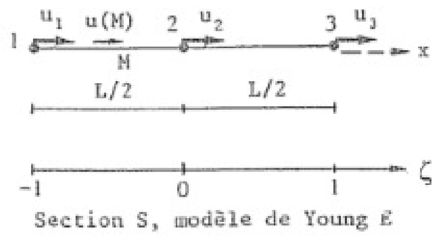

Figure 7: *Quadratic isoparametric FE of bars.*

Reminder: for computational efficiency sake, use the appropriate Gauss-Legendre's quadrature formulas (1D). The GAUSS-LEGENDRE's $2n^{\text{th}}$-order rule, *i.e.* that is based on n points of integration,

$$\int_{X_i}^{X_f} f(x)\, dx \equiv \frac{X_f - X_i}{2} \int_{-1}^{1} f\Big(\frac{X_f - X_i}{2}\xi + \frac{X_f + X_i}{2}\Big)\, d\xi \approx \frac{X_f - X_i}{2} \sum_{k=1}^{n} \varpi_k f\Big(\frac{X_f - X_i}{2}\xi_k + \frac{X_f + X_i}{2}\Big)$$

where for instance

number of points n	weights $(\varpi_k)_{k\in 1,\ldots,n}$	point locations $(\xi_k)_{k\in 1,\ldots,n}$
1 (rectangle rule)	(2)	(0)
2	$(1, 1)$	$(-\sqrt{1/3}, \sqrt{1/3})$
3	$(5/9, 8/9, 5/9)$	$(-\sqrt{3/5}, 0, \sqrt{3/5})$
4	$\Big(\frac{3\sqrt{1.2}-1}{6\sqrt{1.2}}, \frac{3\sqrt{1.2}+1}{6\sqrt{1.2}}, \frac{3\sqrt{1.2}+1}{6\sqrt{1.2}}, \frac{3\sqrt{1.2}-1}{6\sqrt{1.2}}\Big)$	$\Big(-\frac{\sqrt{3+2\sqrt{1.2}}}{\sqrt{7}}, -\frac{\sqrt{3-2\sqrt{1.2}}}{\sqrt{7}}, \frac{\sqrt{3-2\sqrt{1.2}}}{\sqrt{7}}, \frac{\sqrt{3+2\sqrt{1.2}}}{\sqrt{7}}\Big)$

is exact for any polynomial function $f(x)$ of degree $2n - 1$ in $x \in [X_i\,, X_f]$.