Deep Bayesian Generative Models for Knowledge Transfer and MRI Processing

MLSS'19, Moscow

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Outline

Overview

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$p(\theta)$ is the prior distribution,
$p(x|\theta)$ is the assumed model

Probabilistic Machine Learning: Challenge

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- Prior Knowledge/Structure Incorporation

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Solution:

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Approximate Inference: Approaches

MCMC

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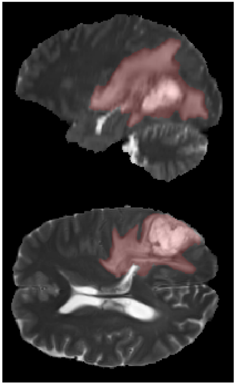
Bayesian generative models for knowledge transfer in DNN on 3D MRI data

A. Kuzina, E. Egorov, E. Burnaev. Bayesian generative models for knowledge transfer in MRI semantic segmentation problems. Journal: Frontiers in Neuroscience, section Brain Imaging Methods, 2019

https://arxiv.org/abs/1908.05480

Problem statement

Magnetic resonance imaging (MRI) — medical imaging technique used in radiology to form pictures of the body anatomy



MRI with labelled brain tumor

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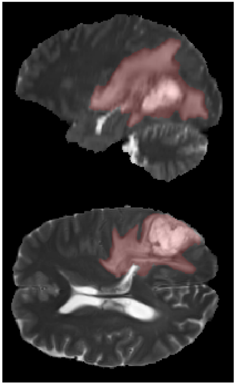
MRI semantic segmentation applications in medicine:

- Tumors (e.g. brain, liver) analysis and monitoring



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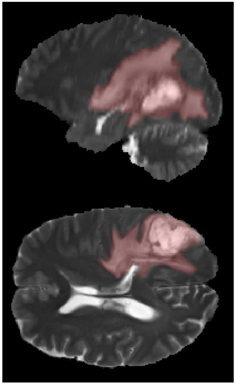

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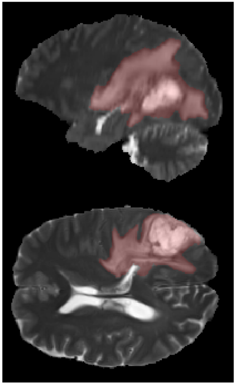

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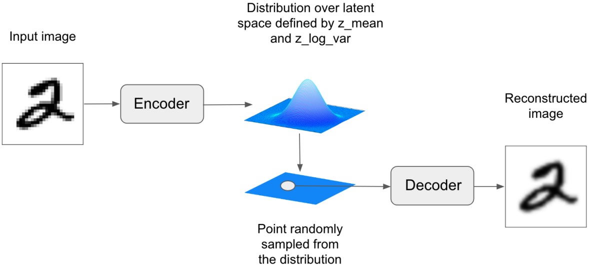
Solution for 1:
Transfer Learning under Bayesian Approach

Variational Autoencoder



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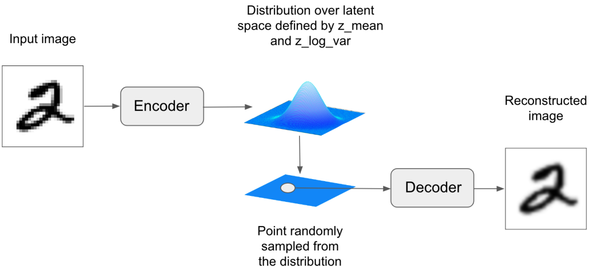
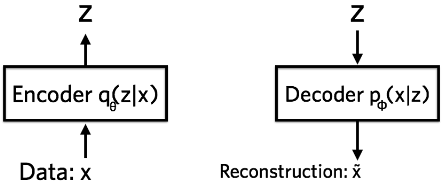
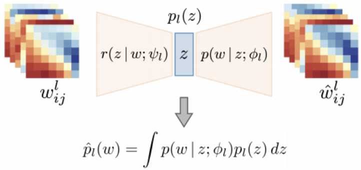




Deep Weight Prior

Main Idea

Use VAE model to learn implicit prior distribution over convolutional filters of each layer



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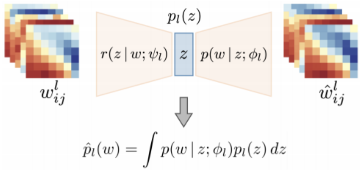

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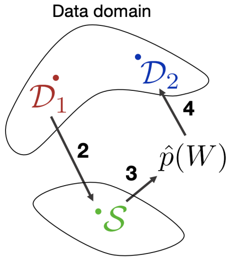

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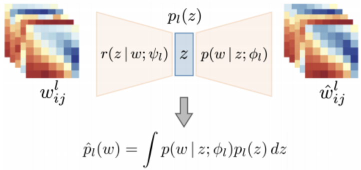

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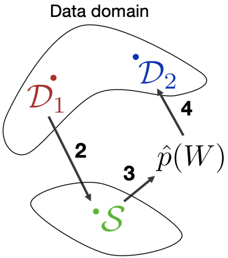

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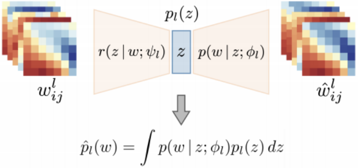

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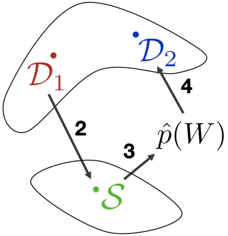

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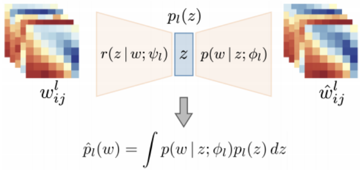

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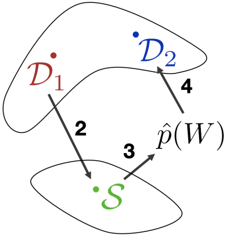
4. Use trained prior for variational inference on the target dataset ($\mathcal{D}_2$)



Experimental Set-Up

Datasets

- 285 MRI of patients with brain tumor (BRATS18)
- 170 MRI of patients with multiple sclerosis (MS)

Prepocessing:

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Binary semantic segmentation

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Metrics:
Dice Similarity Coefficient:

$$\mathrm{DSC} = \frac{2TP}{2TP + FP + FN}$$

Intersection over Union:

$$\mathrm{IOU} = \frac{TP}{TP + FP + FN}$$

Example of MRI slices and ground truth segmentation

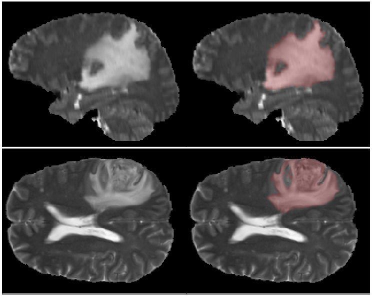

Figure: BRATS18 dataset

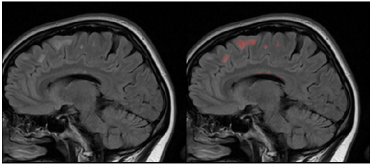

Figure: MS dataset

Experimental Set-Up [2]

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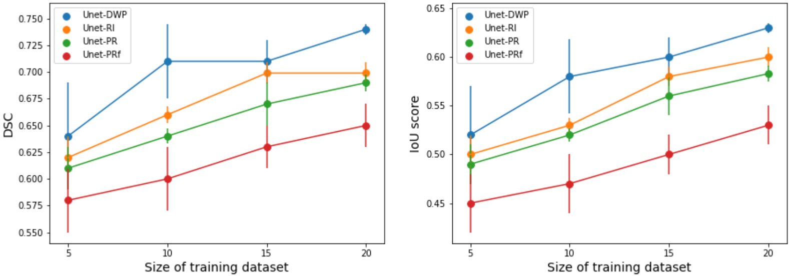
- Do variational inference with implicit prior (VAE) on subset of BRATS18 dataset (5-20 images)

Results



- **Unet-RI** (orange): without transfer learning
- **Unet-PR** (green): fine-tuning of the whole network
- **Unet-PRf** (red): fine-tuning of the input and output block

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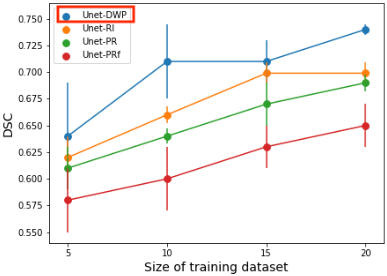

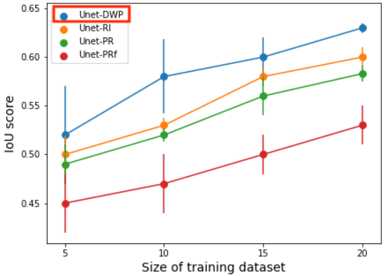

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	Task09_Spleen		
Train size	UNet-DWP (ours)	UNet-RI	UNet-PR
5	0.275	**0.284**	0.209
10	**0.328**	0.293	0.052
15	**0.389**	0.306	0.243
20	**0.353**	0.336	0.156

Table: Mean Dice Similarity Score for the subsets of `Task03_Liver` and `Task09_Spleen` datasets.

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Next Step

How to update our prior with a new dataset, i.e. perform incremental learning for VAE?

Appendix: lower bound on KL-divergence for DWP

$$\mathrm{KL}(q_\theta(W)||p(W)) = \sum_{l,i,j} \mathrm{KL}(q_{\theta^l_{ij}}(w^l_{ij})||p(w^l_{ij})) \le$$

$$\le \sum_{l,i,j} \left(-\mathrm{H}(q_{\theta^l_{ij}}(w^l_{ij})) + \mathbb{E}_{q_{\theta^l_{ij}}(w^l_{ij})} \mathrm{KL}(r(z|w^l_{i,j})||p^l(z)) + \mathbb{E}_{r(z|w^l_{i,j})} \log p(w^l_{i,j}|z) \right) = \mathrm{KL}_{bound}$$

$$\mathcal{L}(\theta) = L_D - \mathrm{KL}(q_\theta(W)||p(W)) \ge L_D - \mathrm{KL}_{bound} \to \max$$

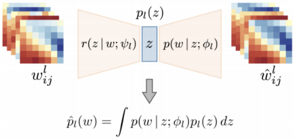
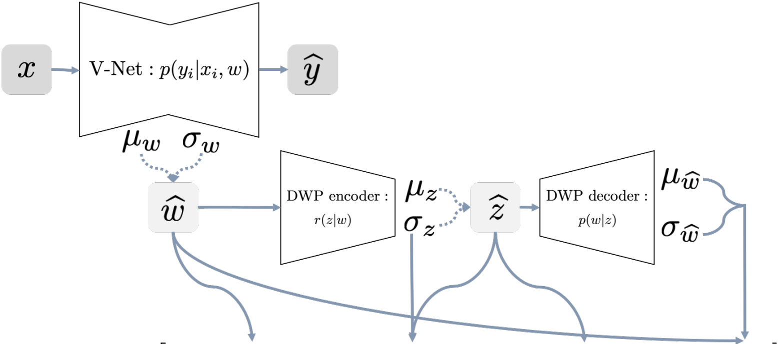


Figure: VAE for learning DWP

Appendix: DWP loss illustartion



$$\mathcal{L}^{\text{approx}} = \mathcal{L}_{\mathcal{D}^*} + \sum\nolimits_{p,k,i} \left[-\log q_{\theta^{(i)}_{pk}}(\widehat{w}^{(i)}_{p,k})) - \log r_{\psi^{(i)}}(\widehat{z}|\widehat{w}^{(i)}_{p,k}) + \log p^{(i)}(\widehat{z}) + \log p_{\phi^{(i)}}(\widehat{w}^{(i)}_{p,k})|\widehat{z}) \right]$$

Variational Inference via MaxEnt Pursuit

Evgenii Egorov, Kirill Neklyudov, Ruslan Kostoev, Evgeny Burnaev. MaxEntropy Pursuit Variational Inference. ISNN 2019: Advances in Neural Networks – ISNN, Springer, 2019 pp. 409-417

https://arxiv.org/abs/1905.07855

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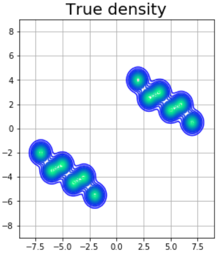
- Avoid model specific work

MPVI: Toy example



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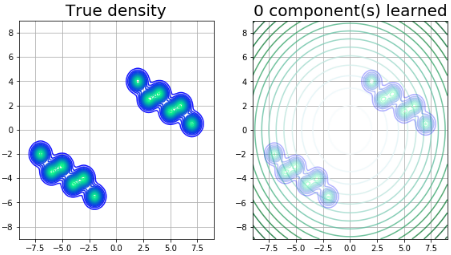

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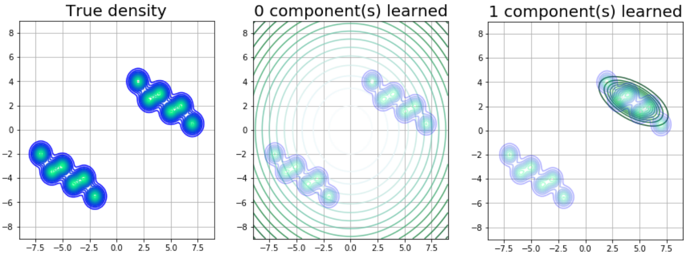

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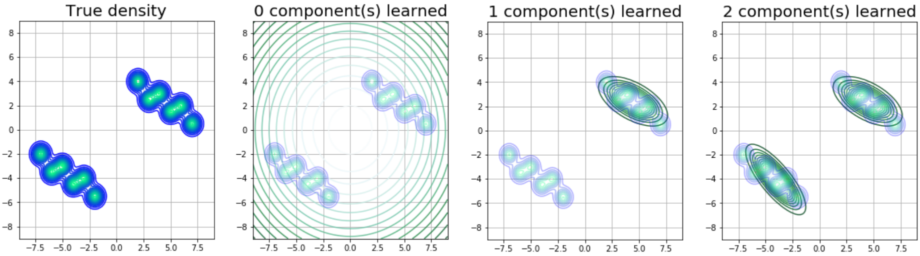

MPVI | Component Optimization: Problem

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- Here $\mathcal{F}[q]$ is ELBO (accuracy of approximation)

$$\mathcal{F}[q] = \mathbb{E}_q \left[\frac{p(x,\theta)}{q(\theta)} \right]$$

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- Considering the first order terms, we get the following optimization problem:

$$\max_{h \in Q} \mathcal{H}[h] + \lambda \left\langle h, \log \frac{p(x,\theta)}{q_t} \right\rangle .$$

MPVI | Component Optimization: Solution

$$\max_{h \in Q} \mathcal{H}[h] + \lambda \left\langle h, \log \frac{p(x,\theta)}{q_t} \right\rangle$$

Problem Proprieties

- Strictly concave over h

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Since h^* is intractable, we can find it by optimizing

$$h^* = \arg\min_{h \in Q} KL \left(h \middle\| \frac{1}{Z(\lambda)} \left[\frac{p(x,\theta)}{q_t}\right]^\lambda \right).$$

MPVI | Connection with Variational Inference

For $\lambda = 1$ **MPVI** optimization problem:

$$\arg\max_{h \in Q} \mathcal{H}[h] + \left\langle h, \log \frac{p(x, \theta)}{q_t} \right\rangle =$$

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- Term (1) corresponds to the standard **Variational Inference** objective (ELBO)

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We can note than:

- Term (1) corresponds to the standard **Variational Inference** objective (ELBO)
- Term (2) plays the role of **similarity penalty** with the current solution q_t

MPVI | Weight Optimization

After getting h for given q_t, we should select α in a

$$q_{t+1}(\theta) = (1-\alpha)q_t(\theta) + \alpha h(\theta)$$

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Theoretical solution

- Convex problem
- $\alpha^* = -\frac{\int \frac{1}{p(\theta|x)} q_t(h - q_t) d\theta}{\int \frac{1}{p(\theta|x)} (h-q_t)^2 d\theta} = -\frac{\int \frac{1}{p(x,\theta)} q_t(h - q_t) d\theta}{\int \frac{1}{p(x,\theta)} (h-q_t)^2 d\theta}$

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Implementation

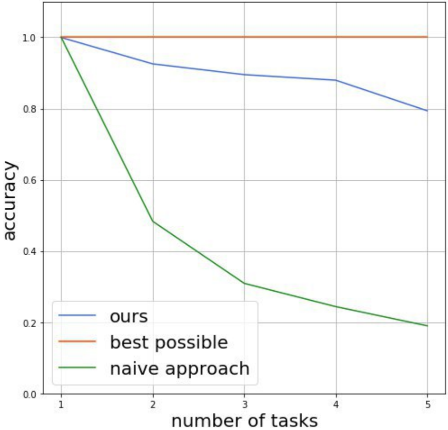
In practice we use stochastic gradient descent over α

MPVI Incremental Learning

Problem: Neural Networks suffer from Catastrophic Forgetting
Solution: $p(\theta|x, x^{\text{new}}) \approx (1-\alpha)q(\theta|x) + \alpha q(\theta|x^{\text{new}})$

Experiment

- Dataset: MNIST, 10 classes classification
- Incremental setting: pair classes arrive: 0 vs 1, 2 vs 3, etc.
- Neural Network: LeNet-5
- Prior: Factorized Normal
- Metric: Accuracy



BooVAE: incremental learning for VAE

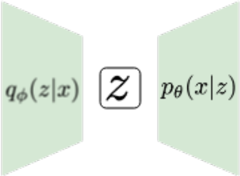
Anna Kuzina, Evgenii Egorov, Evgeny Burnaev. BooVAE: A scalable framework for continual VAE learning under boosting approach

https://arxiv.org/abs/1908.11853

Boosting for Incremental Learning: VAE

Optimal Prior for VAE[1]

$$p^*(z) = \arg\max_{p(\cdot)} \mathcal{L}(Data, p) = \frac{1}{N}\sum_{n=1}^{N} q_\phi(z|x_n)$$



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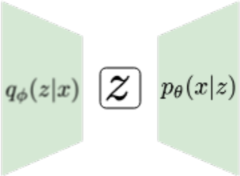

- We approximate $p^*(z)$ via boosting
- Given p_T from the previous task, learn a new component h to update the learned prior for the new task T+1:

$$KL\left(\alpha h + (1-\alpha)p_T || p^*\right) \to \min_{h,\alpha}$$

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BooVAE algorithm

Require: Dataset $\{(x_i)\}_{i=1}^N$
Require: λ, Maximal number of components K

Choose random subset $\mathcal{M} \subset \mathcal{D}$
Initialize prior $p_0 = q_{\phi^*}(z|u_0)$
$\{\theta^*, \phi^*, u_0\} = \arg\max \mathcal{L}(p_0, \theta, \phi)$
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$k = k + 1$
end if
end while
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while not converged **do**
  Update network parameters $\theta^*, \phi^* = \arg\max \mathcal{L}(p_{k-1}, \theta, \phi)$
  if $k < K$ **then**
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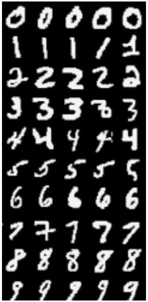
Boosting: experimental set-up [1]

- Tasks are arriving sequentially, one at a time
- **Datsets**: MNIST, fashion MNIST (10 classes each)



MNIST

Fashion MNIST

Boosting: experimental set-up [2]

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- Diversity of generated images

$$\sum_{k=1}^{K} \mathrm{KL}\left(u || \widehat{x}\right), \; u \sim \mathrm{Be}\left(\frac{1}{K}\right), \; \widehat{x} \sim \mathrm{Be}\left(\frac{N_k}{N}\right)$$

N_k — number of images from class k among N generated.

MNIST results

Below: the smaller the better

height# tasks	Standard	Standard + EWC	MoG	Boo (ours)	Boo (ours) + EWC
2	343.54 (26.38)	256.55 (8.38)	**96.50 (1.95)**	100.11 (1.39)	97.49 (0.40)
3	122.05 (2.31)	121.91 (1.31)	107.78 (3.59)	104.33 (1.34)	**102.90 (0.95)**
4	146.06 (0.32)	142.00 (2.28)	123.95 (5.05)	118.78 (1.45)	**117.07 (0.46)**
5	197.02 (5.68)	192.84 (0.12)	143.44 (8.05)	132.08 (0.64)	**130.80 (0.87)**
6	164.29 (3.78)	159.80 (3.14)	143.33 (2.49)	135.42 (1.64)	**131.83 (1.24)**
7	205.21 (5.58)	187.43 (5.20)	163.14 (9.02)	142.21 (1.85)	**137.38 (1.57)**
8	213.25 (9.22)	189.06 (4.72)	172.00 (12.93)	140.80 (2.42)	**138.47 (2.50)**
9	171.04 (3.64)	160.47 (2.53)	164.18 (9.49)	141.70 (0.97)	**140.13 (2.67)**
10	186.79 (2.32)	170.26 (2.20)	181.53 (29.02)	142.92 (1.99)	**140.68 (1.86)**

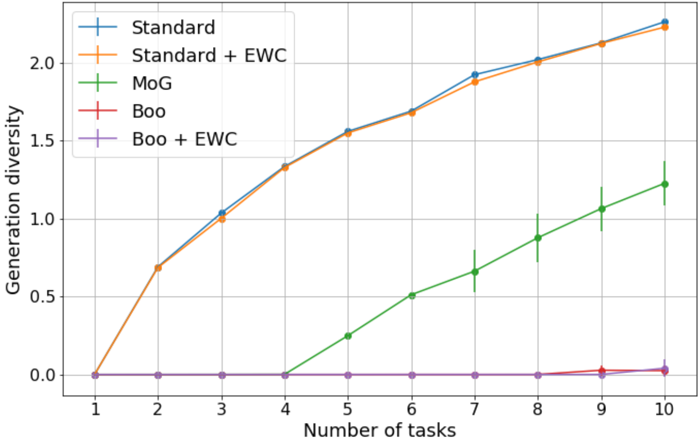
Table: NLL Results on MNIST.

Fashion MNIST results

height# tasks	Standard	Standard + EWC	MoG	Boo (ours)	Boo (ours) + EWC
2	262.22 (2.92)	271.14 (6.05)	239.43 (2.76)	**227.83 (3.34)**	229.81 (2.31)
3	289.45 (2.72)	287.45 (3.87)	266.18 (1.87)	**255.85 (1.61)**	256.47 (2.16)
4	274.08 (2.42)	272.82 (1.02)	264.35 (3.16)	**248.96 (0.85)**	249.08 (1.40)
5	272.87 (1.80)	270.44 (0.98)	264.51 (1.93)	**253.12 (1.43)**	**253.26 (1.20)**
6	487.05 (43.78)	417.81 (7.44)	282.00 (4.06)	**250.87 (2.12)**	**250.64 (1.18)**
7	274.72 (2.93)	272.09 (6.25)	292.93 (9.16)	**250.87 (0.69)**	253.50 (2.59)
8	1827.62 (489.47)	565.81 (22.94)	448.55 (103.92)	260.05 (5.25)	**250.30 (0.48)**
9	321.49 (17.36)	289.17 (2.43)	321.72 (14.11)	**256.42 (1.00)**	**256.33 (0.78)**
10	964.90 (237.27)	427.83 (21.19)	440.96 (49.75)	284.86 (21.21)	**256.58 (1.27)**

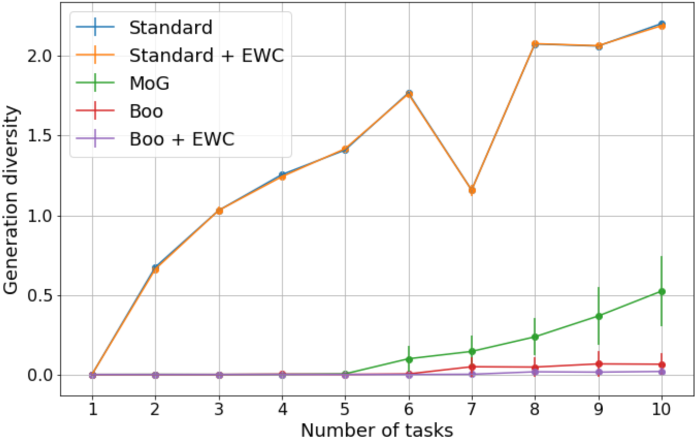
Table: NLL Results on fashion MNIST.

Diversity of generated images



MNIST: KL divergence between uniform and generated distribution

Diversity of generated images



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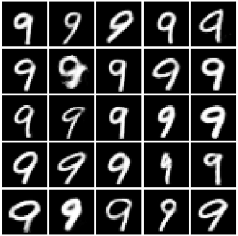
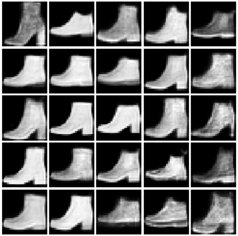
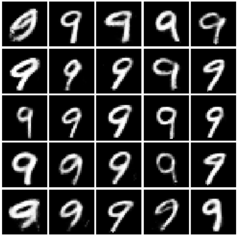
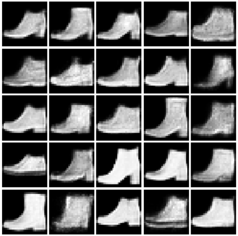




Standard prior: Samples after training on 10 tasks incrementally

Diversity of generated images





Standard + EWC: Samples after training on 10 tasks incrementally

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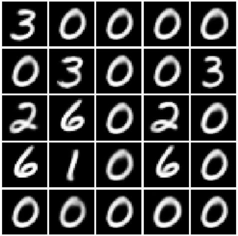
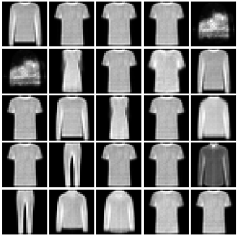
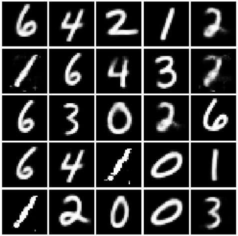
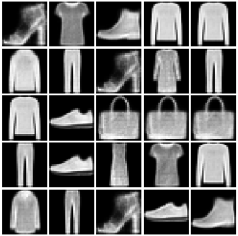




MoG: Samples after training on 10 tasks incrementally

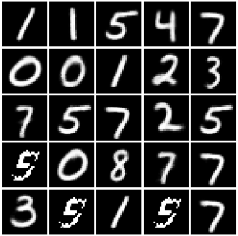
Diversity of generated images

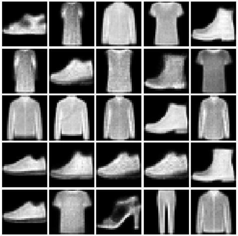




Boo: Samples after training on 10 tasks incrementally

Diversity of generated images





Boo + EWC: Samples after training on 10 tasks incrementally