Study Notes: Matrices and Systems of Equations

1 Systems of Linear Equations

A **system of linear equations** is a set of equations involving the same set of variables. For example:

$$\begin{cases} 2x + 3y = 5 \\ 4x - y = 11 \end{cases}$$

This can be written in matrix form as:

$$A\mathbf{x} = \mathbf{b} \quad \text{where} \quad A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

2 Row Echelon Form (REF)

A matrix is in **row echelon form** if:

- All nonzero rows are above any all-zero rows.
- The leading entry of each nonzero row is to the right of the one in the row above it.
- Entries below a leading entry are all zero.

The **reduced row echelon form (RREF)** also requires:

- Leading entries are 1.
- Each leading 1 is the only nonzero entry in its column.

3 Matrix Arithmetic

Matrix operations include:

- **Addition/Subtraction:** Only if dimensions match.
- **Scalar multiplication:** Multiply each element by a constant.
- **Matrix multiplication:** If A is $m \times n$, B must be $n \times p$. The result is $m \times p$.

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad AB = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

4 Matrix Algebra

Important properties:

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- Identity matrix I: $AI = IA = A$
- Inverse matrix A^{-1}: $AA^{-1} = A^{-1}A = I$

5 Elementary Matrices

An **elementary matrix** results from applying one row operation to the identity matrix.

Types of elementary row operations:

- Swap two rows.
- Multiply a row by a nonzero scalar.

- Add a multiple of one row to another.

Multiplying a matrix A on the left by an elementary matrix E applies the corresponding row operation to A.

6 Partitioned Matrices

A **partitioned matrix** is divided into smaller submatrices (blocks).

Example:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Block matrices are useful in simplifying large matrix operations and in applications like parallel computing and system design.