A Variant of Gödel's Ontological Proof in a Natural Deduction Calculus (Draft)

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Introduction

Ontological arguments for the existence of God can be traced back at least to St. Anselm (1033-1109). His argument considered a maximally conceivable being, who must exist, because if it did not have the property of existence, then we could conceive of a greater being that, apart from the other properties, also has the property of existence.

A major critique of this argument is that we do not know whether the concept maximal conceivable being in fact designates anything or if it is inconsistent, like a round square (*Fitting*, 2002)[p.134]. As Bertrand Russell pointed out, the definition of maximal allows us to define properties, like having boots, which the maximal being then also must have (**?**).

Kant argued against the ontological argument on the basis that existence is not an analytic property (*Kant*, 1781). This means that existence cannot be contained in the definition of a concept, because it is generally synthetic. All that we can say is that if God exists, then God necessarily exists (**?**).

St. Anselm's argument was elaborated further by Descartes and Leibniz. Leibniz identified that establishing the possible existence of God is a critical missing step in St. Anselm's argument. To fill this gap, he argued that the properties of God, the perfections, are compatible. This implies that it is possible to have all perfections at once and therefore the existence of a maximal being with all these properties is possible.

Gödel studied Leibniz's work (**?**) and brought his ontological argument to a modern form using a modal logic with higher-order quantification over properties. In this setting, he gave precise axioms describing the notion of *positive* property and defined God as a being having all positive properties.

Gödel's notion of positive property and Leibniz's notion of perfection are not exactly the same (**?**). On a technical level, the main distinction seems to be that Gödel's positive properties are not just atomic properties, like Leibniz's perfections, but can also consist of complex combinations of atomic properties (*Fitting*, 2002)[p.139.] (TODO: Check this claim). In particular, one of Gödel's axioms states that the conjunction of any set of positive properties is positive. And from this axiom, it follows immediately that the property of being God-like is positive. While this step is intuitively and informally clear, it is not easily formalizable in a typical logical calculus, because it requires inferring that being god-like is a (possibly infinite) conjuntion of positive properties, while it has only been defined as the property holding for individuals who have all positive properties. This interplay between universal quantification (in the definition of a God-like being) and conjunction (in Gödel's mentioned axiom) is a technical difficulty that probably explains why, starting with Scott **?**, to whom Gödel confided his manuscript, this axiom of Gödel's has been replaced by another one that simply assumes the positivity of the property of being God-like.

This paper presents new natural deduction proofs of (Scott's version of) the lemmas and theorems in Gödel's ontological argument **?**. In contrast to Scott's version **?**, also shown here in natural deduction form, the version presented here is characterized by shorter and more direct derivations.

Section 2 defines the higher-order modal natural deduction calculus used to construct the formal proofs. Section **??** shows Scott's proofs, translated to natural deduction. And section **??** presents the new proofs.

2 Natural Deduction

A *derivation* is a directed acyclich graph whose nodes are formulas and whose edges correspond to applications of the inference rules shown in Figures 1 and 3. Parts of a derivation may be surrounded by boxes. A *proof* is a derivation that additionally satisfies the following conditions:

- **eigen-variable conditions:** if ρ is a $\forall_I$ inference eliminating a variable α, then any occurrence of α in the proof should be an ancestor of the occurrence of α eliminated by ρ; if ρ is a $\exists_E$ inference introducing a variable β, then any occurrence of β in the proof should be a descendant of the occurrence of β introduced by ρ.
- **eigen-box conditions:** η must be a fresh name for a box.
- **boxed assumption condition:** any assumption should be discharged within the box where it is made.
- **unboxed root condition:** the proof's root should not be inside any box.

Double lines are used to abbreviate tedious propositional reasoning steps in the derivations. Dashed lines are used to refer to a proof shown elsewhere. Dotted lines are used to indicate folding and unfolding of definitions.

Figure 1: The intuitionistic natural deduction calculus **ND**

$$\dfrac{B}{A \to B}\ {\to_I} \qquad \dfrac{\begin{array}{c}\overline{A}^{\ n} \\ \vdots \\ B\end{array}}{A \to B}\ {\to_I^n} \qquad \dfrac{A \quad A \to B}{B}\ {\to_E}$$

$$\dfrac{A \quad B}{A \wedge B}\ {\wedge_I} \qquad \dfrac{A \wedge B}{A}\ {\wedge_{E_1}} \qquad \dfrac{A \wedge B}{B}\ {\wedge_{E_2}}$$

$$\dfrac{A \vee B \quad \begin{array}{c}\overline{A} \\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B} \\ \vdots \\ C\end{array}}{C}\ {\vee_E} \qquad \dfrac{A}{A \vee B}\ {\vee_{I_1}} \qquad \dfrac{B}{A \vee B}\ {\vee_{I_2}}$$

$$\dfrac{A[\alpha]}{\forall x.A[x]}\ {\forall_I} \qquad \dfrac{\forall x.A[x]}{A[t]}\ {\forall_E} \qquad \dfrac{A[t]}{\exists x.A[x]}\ {\exists_I} \qquad \dfrac{\exists x.A[x]}{A[\beta]}\ {\exists_E}$$

We let negation be a defined concept by $\neg A \equiv A \to \bot$. The rules for negation introduction and elimination are special cases of the implication rules. Equivalence is also a defined concept with $(A \leftrightarrow B) \equiv (A \to B) \wedge (B \to A)$.

Figure 2: Classical Rule: Double negation elimination

$$\boxed{\dfrac{\neg\neg A}{A}\ {\neg\neg_E}}$$

3 Scott's Proof

The acceptance of the correctness of the ontological argument by Gödel's work boils down to the intuitive correctness of the axioms and definitions and the belief in the soundness of the deductive system. The formal argument of Gödel is based on Leibniz proof, which in turn is based on Descartes proof. These proofs have two parts; a proof that if God's existence is possible, then it is necessary and a proof that God's existence is in fact possible.

Figure 3: Rules for Modal Operators

$$\dfrac{\eta : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \dfrac{\Box A}{n : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Box_E$$

$$\dfrac{n : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \dfrac{\Diamond A}{\eta : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Diamond_E$$

3.1 Possibly, God Exists

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Theorem 1 *Positive properties are possibly exemplified:*

$$\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom 2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}}{\forall\psi.[(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}\ \forall_E}{(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \neg\rho(x)]) \rightarrow P(\neg\rho)}\ \forall_E}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \rightarrow P(\neg\rho)} \quad \dfrac{\dfrac{\text{Axiom 1}}{\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}}{P(\neg\rho) \leftrightarrow \neg P(\rho)}\ \forall_E}{\dfrac{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \rightarrow \neg P(\rho)}{\dfrac{P(\rho) \rightarrow \Diamond\exists x.\rho(x)}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}\ \forall_I}}$$

Definition 1 *A* God-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

Axiom 3 *The property of being God-like is positive:*

$$P(G)$$

Corollary 1 *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi)\to\Diamond\exists x.\varphi(x)]}}{P(G)\to\Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\to_E$$

3.2 Being God is an essence of any God

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi)\to\Box\ P(\varphi)]$$

Definition 2 *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi\ \mathit{ess}\ x\leftrightarrow\varphi(x)\wedge\forall\psi.(\psi(x)\to\Box\forall x.(\varphi(x)\to\psi(x)))$$

Theorem 2 *Being God-like is an essence of any God-like being:*

$$\forall y.[G(y)\to G\ \mathit{ess}\ y]$$

Proof Let the following derivation with the open assumption $G(x)$ be $\Pi_1[G(x)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(\psi)^1 \qquad \dfrac{\dfrac{\text{Axiom 1}}{\forall\varphi.(\neg P(\varphi)\to P(\neg\varphi))}}{\neg P(\psi)\to P(\neg\psi)}\,\forall_E}{P(\neg\psi)}\,\to_E \qquad \dfrac{\dfrac{G(x)}{\forall\varphi.(P(\varphi)\to\varphi(x))}\,\text{D1}}{P(\neg\psi)\to\neg\psi(x)}\,\forall_E}{\neg\psi(x)}\,\to_E \qquad \psi(x)^2}{\bot}\,\to_E}{\neg\neg P(\psi)}\,\to_I^1}{P(\psi)}\,\neg\neg_E}{\psi(x)\to P(\psi)}\,\to_I^2$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_2[G(x)]$:

$$\dfrac{\dfrac{\dfrac{\psi(x)^3 \qquad \dfrac{\Pi_1[G(x)]}{\psi(x)\to P(\psi)}}{P(\psi)}\,\to_E \qquad \dfrac{\dfrac{\text{Axiom 4}}{\forall\varphi.(P(\varphi)\to\Box P(\varphi))}}{P(\psi)\to\Box P(\psi)}\,\forall_E}{\Box P(\psi)}\,\to_E}{\psi(x)\to\Box P(\psi)}\,\to_I^3$$

Let the following derivation without open assumptions be Π_3:

$$
\begin{prooftree}
\AxiomC{$P(\psi)^4$}
\AxiomC{$G(x)^5$}
\dottedLine
\RightLabel{D1}
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \varphi(x))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\psi) \to \psi(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\psi(x)$}
\RightLabel{$\to_I^5$}
\UnaryInfC{$G(x) \to \psi(x)$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^4$}
\UnaryInfC{$P(\psi) \to \forall x.(G(x) \to \psi(x))$}
\end{prooftree}
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_4[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\psi(x)^6$}
\AxiomC{Π_2}
\dashedLine
\UnaryInfC{$\psi(x) \to \Box P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box P(\psi)$}
\AxiomC{$\Box P(\psi)^7$}
\RightLabel{$\Box_E$}
\UnaryInfC{$P(\psi)$}
\AxiomC{Π_3}
\dashedLine
\UnaryInfC{$P(\psi) \to \forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_E$}
\BinaryInfC{$\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\Box_I$}
\UnaryInfC{$\Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^7$}
\UnaryInfC{$\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^6$}
\UnaryInfC{$\psi(x) \to \Box\forall x.(G(x) \to \psi(x))$}
\end{prooftree}
$$

The use of the necessitation rule above is correct, because the only open assumption $\Box P(\psi)$ is boxed. In the derivation of Theorem 2 below, the assumption $G(x)$ in the subderivation $\Pi_4[G(x)^8]$ is discharged by the rule labeled 8.

$$
\begin{prooftree}
\AxiomC{$G(x)^8$}
\AxiomC{$\Pi_4[G(x)^8]$}
\dashedLine
\UnaryInfC{$\psi(x) \to \Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))$}
\RightLabel{$\wedge_I$}
\BinaryInfC{$G(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))$}
\dottedLine
\RightLabel{D2}
\UnaryInfC{$G\ ess\ x$}
\RightLabel{$\to_I^8$}
\UnaryInfC{$G(x) \to G\ ess\ x$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall y.[G(y) \to G\ ess\ y]$}
\end{prooftree}
$$

3.3 If God's existence is possible, it is necessary

Definition 3 Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1 *If there is a God, then necessarily there exists a God:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof

$$\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists z.G(z)$}
\RightLabel{$\exists_E$}
\UnaryInfC{$G(g)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$G(g)$}
\AxiomC{Theorem 2}
\dashedLine
\UnaryInfC{$\forall y.[G(y) \to G\ ess\ y]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$G(g) \to G\ ess\ g$}
\RightLabel{$\to_E$}
\BinaryInfC{$G\ ess\ g$}
\AxiomC{Axiom 5}
\dashedLine
\UnaryInfC{$P(E)$}
\AxiomC{$G(g)$}
\RightLabel{D1}
\dottedLine
\UnaryInfC{$\forall\varphi.[P(\varphi) \to \varphi(g)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(E) \to E(g)$}
\RightLabel{$\to_E$}
\BinaryInfC{$E(g)$}
\RightLabel{D3}
\dottedLine
\UnaryInfC{$\forall\varphi.[\varphi\ ess\ g \to \Box\exists x.\varphi(x)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$G\ ess\ g \to \Box\exists x.G(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box\exists x.G(x)$}
\RightLabel{$\to_I^1$}
\UnaryInfC{$\exists z.G(z) \to \Box\exists x.G(x)$}
\end{prooftree}$$

3.4 God exists

ToDo: this is proven in a way that is slightly different from Gödel's 1970.

Theorem 3 *God exists:*

$$\exists x.G(x)$$

Formal proof:

$$\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\Box\neg\Box\exists x.G(x)$}
\RightLabel{$\Box_E$, axiom T}
\UnaryInfC{$\neg\Box\exists x.G(x)$}
\AxiomC{Lemma 1}
\dashedLine
\UnaryInfC{$\neg\Box\exists x.G(x) \to \neg\exists x.G(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\neg\exists x.G(x)$}
\RightLabel{$\Box_I$}
\UnaryInfC{$\Box\neg\exists x.G(x)$}
\dottedLine
\UnaryInfC{$\neg\Diamond\exists x.G(x)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{$\neg\Diamond\exists x.G(x)$}
\AxiomC{Corollary 1}
\dashedLine
\UnaryInfC{$\Diamond\exists x.G(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\bot$}
\RightLabel{$\to_I^1$}
\UnaryInfC{$\neg\Box\neg\Box\exists x.G(x)$}
\dottedLine
\UnaryInfC{$\neg\Box\Diamond\neg\exists x.G(x)$}
\AxiomC{Axiom B}
\dashedLine
\UnaryInfC{$\neg\exists x.G(x) \to \Box\Diamond\neg\exists x.G(x)$}
\doubleLine
\UnaryInfC{$\neg\Box\Diamond\neg\exists x.G(x) \to \neg\neg\exists x.G(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\neg\neg\exists x.G(x)$}
\RightLabel{$\neg\neg$ E}
\UnaryInfC{$\exists x.G(x)$}
\end{prooftree}$$

Note that the last step is classical and we do not prove the existential statement by providing an object for which the statement holds. This proof makes section 9 superflous and the use of axiom "M" unnecessary.

The system used contains the $\Box_E$-rule with the restriction that we have a $\Box_I$ below it. This is equivalent to modal sytem K that contains axiom K and necessitation rule N. We aslo use axiom B ($A \to \Box\Diamond A$). No other modal axioms are needed.

3.5 Necessarily, God exists

We can also prove that god exists necessarily in our system by simply introducing box on a theorem by rule N in modal sytem K. So we get:

Corollary 2 *Necessarily, God exists:*

$$\Box\exists x.G(x)$$

3.6 God exists

This section is superfluous.

Axiom 6 (M) *What is necessary is the case:*

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Corollary 3 *There exists a God:*

$$\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\text{Theorem 2}}{\Box\exists x.G(x)} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$

4 Proof system equivalent to system K

We show that the proof system with boxed parts of derivations is equivalent to the system K. The modal system K consists of the axiom K and the necessitation rule N.

Axiom 7 (The transitivity axiom K)

$$\Box(A \to B) \to (\Box A \to \Box B)$$

Axiom 8 (The necessitation rule, $\Box_I$) *If A is a theorem, then $\Box A$ is a theorem.*

Lemma 2 *The axiom K is derivable in the system.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(A \to B)^2}{A \to B}\,\Box_E \qquad \dfrac{\Box A^1}{A}\,\Box_E}{B}\,\to_E}{\Box B}\,\Box_I}{\Box A \to \Box B}\,\to_I^1}{\Box(A \to B) \to (\Box A \to \Box B)}\,\to_I^2$$

Lemma 3 *Assuming the axiom K and the necessitation rule $\Box_I$, the open formula $\Box A$ and the existence of a derivation of B from the open assumption A, then we can derive $\Box B$ without the rules for boxed parts of derivations.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}A^1\\ \vdots\\ B\end{matrix}}{A \to B}\,{\to_I^1}}{\Box(A \to B)}\,\Box_I \qquad \dfrac{\text{Axiom K}}{\Box(A \to B) \to (\Box A \to \Box B)}}{\Box A \to \Box B}\,{\to_E} \qquad \Box A^1}{\Box B}\,{\to_E}$$

5 Modal collapse

A major criticism against Gödel's proof is that its axioms lead to the so-called *modal collapse* (*Sobel*, 1987): it is possible to prove that everything that is the case is so necessarily, and hence actuality, possibility and necessity coincide (*Sobel*, 2001)[Ch. 4, section 6, theorems 9 and 10]. As this is an undesirable consequence, many solutions to the problem of the modal collapse have been proposed.

Anderson's solution (**?**) modifies the definitions of God-like being and essence, and eliminates half of an axiom. This not only avoids the modal collapse, but also makes two of Gödel's five axioms derivable from the others (**?**) under some implicit additional assumptions (**?**). Another solution involving more substantial modifications is that of Bjørdal (**??**).

On another track, Fitting has argued that greater care has to be taken with the semantics of higher-order modal logics. Quantified variables may be rigid or flexible; and properties may be treated as intensional or extensional. Making the right choices may prevent the modal collapse (*Fitting*, 2002)[Sections 11.9 and 11.10].

Anderson (**?**)[p. 292] and Sobel (*Sobel*, 2001)[p. 133] also discuss the idea that the notion of property over which quantification is allowed might be too general and restrictions might be appropriate.

It is beyond the scope of this paper to analyze these solutions in detail or propose new solutions. The purpose of this section is simply to show natural deduction derivations of the modal collapse, thus confirming that it holds for the axioms used in the previous sections.

Theorem 4 *For all constant fomulas (without free variables), A, we have:*

$$A \to \Box A$$

Note that in intuitionistic predicate logic we have $\forall y.[B \to C] \leftrightarrow [\exists y.B \to C]$ if y is not free in C.

$$
\begin{array}{cl}
\text{Theorem 2} & \\ \hdashline
\forall y.[G(y) \to G\ ess\ y] & \\ \hline
\forall y.[G(y) \to G(y) \wedge \forall \psi.(\psi(y) \to \Box \forall x.(G(x) \to \psi(x)))] & \text{D2} \\ \hline
\forall y.[G(y) \to \forall \psi.(\psi(y) \to \Box \forall x.(G(x) \to \psi(x)))] & \text{Prop. logic} \\ \hline
\forall y.[G(y) \to (A(y) \to \Box \forall x.(G(x) \to A(x)))] & \text{Second order quantifier elimination and logical steps} \\ \hline
\forall y.[G(y) \to (A \to \Box \forall x.(G(x) \to A))] & A \text{ is constant} \\ \hline
\exists y.G(y) \to (A \to \Box \forall x.(G(x) \to A)) & \text{intuitionistic predicate logic}
\end{array}
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists y.G(y) \to (A \to \Box \forall x.(G(x) \to A)) \qquad \dfrac{\text{Theorem 3}}{\exists y.G(y)}}{A \to \Box \forall x.(G(x) \to A)}\ \to_E}{A \to \Box(\exists x.G(x) \to A)}\ \text{intuitionistic predicate logic} \qquad \dfrac{}{A}\ 1}{\Box(\exists x.G(x) \to A)}\ \to_E}{\exists x.G(x) \to A}\ \Box_E \qquad \dfrac{\text{Theorem 3}}{\exists x.G(x)}\ \to_E}{\dfrac{\dfrac{A}{\Box A}\ \Box_I}{A \to \Box A}\ \to_I^1}\ \to_E
$$

6 Conclusions

The proofs of theorem T1 and corollary C1 ($\Diamond \exists x.G(x)$) do not rely on equality and use axiom A2 ($\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box \forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$) only once. Corollary C2 ($\exists x.G(x)$), which is usually regarded as a trivial corollary of main theorem T3 ($\Box \exists x.G(x)$) using the modal logic axiom T, is here derived directly from lemma L1 and corollary C1, not relying on T. The main theorem T3 becomes derivable from C2 by a single application of the necessitation rule. Furthermore, all proofs are done in the modal logic K, except for the proof of corollary C2, which requires one use of the axiom B of modal logic KB.

ToDo: Sobel Anderson on KB: Sobel page 152

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