RICHARD TIESZEN

GÖDEL AND THE INTUITION OF CONCEPTS

ABSTRACT. Gödel has argued that we can cultivate the intuition or 'perception' of abstract concepts in mathematics and logic. Gödel's ideas about the intuition of concepts are not incidental to his later philosophical thinking but are related to many other themes in his work, and especially to his reflections on the incompleteness theorems. I describe how some of Gödel's claims about the intuition of abstract concepts are related to other themes in his philosophy of mathematics. In most of this paper, however, I focus on a central question that has been raised in the literature on Gödel: what kind of account could be given of the intuition of abstract concepts? I sketch an answer to this question that uses some ideas of a philosopher to whom Gödel also turned in this connection: Edmund Husserl. The answer depends on how we understand the conscious directedness toward 'objects' and the meaning of the term 'abstract' in the context of a theory of the intentionality of cognition.

Gödel has argued that we can cultivate the intuition or 'perception' of abstract concepts in mathematics and logic (see, e.g., Gödel 1944, 1947/64, *1953/59, *1961/?, 1972).[1] Gödel's ideas about the intuition of concepts are not incidental to his later philosophical thinking but are related to many other themes in his work, and especially to his reflections on the incompleteness theorems. I will describe below how some of Gödel's claims about the intuition of abstract concepts are related to other themes in his philosophy of mathematics. In most of this paper, however, I will focus on a central question that has been raised in the literature on Gödel: what kind of account could be given of the intuition of abstract concepts? I sketch an answer to this question that uses some ideas of a philosopher to whom Gödel also turned in this connection: Edmund Husserl. It is not my goal in this paper to give a complete account of Husserl's own views on the intuition of abstract or ideal objects and, in any case, it would not be feasible. For more details about Husserl's specific views on categorical intuition and related topics see Tieszen (2003).

Gödel's comments on abstract concepts are a product of his philosophical interpretation of his incompleteness theorems. In Section 1 of this paper I present an argument, based on the incompleteness theorems, that is supposed to take us from recognition of the existence of concrete

Synthese **133:** 363–391, 2002.

sign configurations to recognition of the existence of abstract concepts. I then describe the basic features of concepts that Gödel mentions in various papers. We will see why Gödel thinks that clarification of the meaning of basic mathematical concepts is required for deciding undecided mathematical propositions, finding consistency proofs, augmenting mathematics with new axioms, and developing mathematics in general. By 1961 Gödel is arguing that it is Husserl's phenomenology that offers useful ideas about clarification of the meaning of mathematical concepts. In Section 2 I locate the place of concepts in a Husserlian view of mathematical cognition. Section 3 presents some examples that show how abstract concepts are involved in everyday perception as well as in mathematical experience. On the view I present, we ascend to mathematical concepts in the more active and reflective parts of our experience. Central to this view of concepts is the notion of intentionality. I argue that an account of the intuition of abstract concepts depends crucially on the fact that human cognition exhibits intentionality. In Section 6 it is argued that the phenomenological ontology of concepts must be understood on this basis. The alleged problem of 'epistemic access' to abstract concepts, discussed in Section 8, must also be understood on this basis. In Section 9 of the paper I discuss several views that, according to Gödel, either ignore or attempt to eliminate the intuition of concepts.

Why would anyone think that there is or could be an intuition of abstract concepts? We now proceed to the elements of an answer to this question.

1. FROM CONCRETE SIGNS TO ABSTRACT CONCEPTS

The philosophical ideas that lie behind the shift from concrete signs to abstract concepts are part of Hilbert's program, although they are certainly not all Hilbert's inventions. They have a rather deep background in the history of philosophy. Hilbert proposed to put mathematics on secure foundations after the discovery of the set-theoretic paradoxes. The idea would be to show that formalized parts of mathematics were consistent using only a special theory, let us call it C (for 'concrete mathematics'). C could not be just any theory. What would make C special is the fact that it would possess the kinds of properties needed to insure reliability or security. It should be finitary and not infinitary, for we do not understand the infinite very well and in our thinking about the infinite we may be led into inconsistencies and paradoxes. It should be possible to understand C as a theory involving only concrete and not abstract entities, for abstract entities are mysterious and we should avoid postulating them whenever possible. The concrete entities in this case are finite sign configurations.

C should be concerned with what is real and not with what is ideal. Its sentences and proofs should be surveyable in immediate intuition. Immediate intuition is a non-mysterious form of sensory perception. C should not be a creature of pure thought or pure reason, for we may be led by pure reason into antinomies and hopeless confusion. C represents the part of our mathematical thinking that is contentual and meaningful. The contrast is with parts of mathematical thinking that we may regard as purely formal and 'meaningless' in the sense that we need not consider their purported references to abstract or infinitary objects or concepts. On this kind of view, the only way to understand mathematical rigor is as *formal* rigor.

For formal theories T that contain enough mathematics to make Gödel numbering possible, Gödel's first incompleteness theorem says that if T is consistent then there is a sentence G, the Gödel sentence for T, such that $\nvdash_{\mathrm{T}}$ G and $\nvdash_{\mathrm{T}} \neg$G. The second theorem says that if T is consistent then $\nvdash_{\mathrm{T}}$ CON(T), where 'CON(T)' is the formalized statement that asserts the consistency of T. Theorem 1 tells us that G cannot be decided by T but that it is true if T is consistent. The theorem does not tell us that G is absolutely undecidable. Indeed, Gödel frequently emphasizes how sentences that are undecidable in some theories are in fact decided in certain natural extensions of those theories (e.g., by ascending to higher types). T can be, for example, primitive recursive arithmetic (PRA), Peano arithmetic (PA), Zermelo-Fraenkel set theory (ZF), and so on. The theorems tell us that such formal theories T cannot be finitely axiomatizable, consistent and complete. Theorem 2 suggests, generally speaking, that if there is a consistency proof for a theory T then it will be necessary to look for $\vdash_{\mathrm{T}'}$ CON(T), where T is a proper subsystem of T′.

A very likely candidate for C is PRA. PA is arguably less suitable, given the distinctions described in the first paragraph of this section. In whatever manner we construe C, however, it follows from Theorem 1 that if C is consistent then the Gödel sentence for C cannot be decided by C even though it is true. It follows from Theorem 2 that if C is consistent then C cannot prove CON(C). Given the way we characterized C above, it follows that deciding the Gödel sentence for C or proving CON(C) must require objects or concepts that *cannot* be completely represented in space-time as finitary, concrete, real and immediately intuitable. In other words, deciding the Gödel sentence for C or proving that CON(C) must require appeal to the meanings of sign configurations, to objects or concepts that are in some sense infinitary, ideal or abstract and not immediately intuitable.

Either there are such 'abstract' entities or not. Suppose that there are no such entities or that we can have no insight into such entities. (There are a variety of philosophical views on which such a claim would be made.

I return to this point in Section 9 below.) Then it would follow, within the perspective of C, that we must stop or 'become static' with respect to deciding the Gödel sentence for C or obtaining a proof of CON(C). That is, we could not decide some clearly posed mathematical problem. (I mean we could not decide it unless the decision were to be made arbitrarily or on non-mathematical grounds. I return to this point below in order to ward off possible objections.) However, for some T (e.g., PA) we in fact do have proofs of CON(T) and decisions of related problems. Contradiction. Therefore, there are such entities and we must have some insight into them.

It is worth noting that we can obtain a reasonably good understanding of the sense in which the objects or concepts used in the decisions or consistency proofs must be abstract, infinitary and not completely captured in immediate intuition. In the case of PA, for example, the consistency proof requires the use of transfinite induction on ordinals $< \epsilon_0$, or it requires primitive recursive functionals of finite type. Perhaps the level of abstraction involved here is not very substantial. It is certainly more substantial, however, in the case of consistency proofs for real analysis or in proofs of the consistency of ZF $+$ the continuum hypothesis (CH) or of ZF $+ \neg$CH.

It needs to be emphasized again that C cannot be just any formal theory. We cannot keep extending C with new axioms that allow us to decide sentences that were previously undecidable and still expect to have concrete mathematics, for then we might as well have started with something like ZF in the first place. As a purely formal theory ZF is of course concrete in the same sense as PRA but the difference is that ZF certainly could not codify concrete, finitary mathematics. To hold that it could is to completely subvert the philosophical basis of Hilbertian proof theory.

The kinds of remarks I have made here on the incompleteness theorems are expressed in many places in Gödelś writings. For example, Gödel (Gödel 1972, 271–272) says that

> P. Bernays has pointed out on several occasions that, in view of the fact that the consistency of a formal system cannot be proved by any deduction procedures available in the system itself, it is necessary to go beyond the framework of finitary mathematics in Hilbert's sense in order to prove the consistency of classical mathematics or even of classical number theory. Since finitary mathematics is defined as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory ... [What Hilbert means by "Anschauung" is substantially Kant's space-time intuition confined, however, to configurations of a finite number of discrete objects.] By abstract concepts, in this context, are meant concepts which are essentially of the second or higher level, i.e., which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or *thought contents* (e.g., proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from

a reflection upon the combinatorial (space-time) properties of the symbols representing them, but rather from a reflection upon the *meanings* involved.

The idea that it is necessary to reflect upon meaning is given a central role in the 1961 paper (Gödel *1961/?) in which Gödel invokes Husserl's ideas. The 1961 paper clearly contains the idea that reflection on meaning (or intuition of concepts) is of a 'higher level' than reflection on the combinatorial properties of concrete symbols.

It is well-known that these ideas are also part of Gödel's anti-mechanist views on computation and cognition. Consider, for example, the following remarks in an early paper:

The generalized undecidability results do not establish any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics. ... Turing's analysis of mechanically computable functions is independent of the question whether there exist finite *non-mechanical* procedures ... such as involve the use of abstract terms on the basis of their meaning. (Gödel 1934, Vol. I, 370)

The idea that (finite) humans intuit abstract concepts or meanings is a central theme in Gödel's comments on the difference between minds and machines.

In his discussion of Husserl's views in his 1961 paper Gödel distinguishes 'leftward', empiricist views of mathematics from 'rightward', rationalist views. Hilbert had attempted a particular kind of reconciliation of these views. The incompleteness theorems, however, show that Hilbert's attempt to reconcile these views is unworkable. Since the theorems suggest that we must reflect on meaning, or on abstract concepts, we can instead try to obtain a workable combination of the two types of view through

cultivating (deepening) knowledge of the abstract concepts which themselves lead to the setting up of those mechanical systems, and further, according to the same procedures [for clarifying meaning], seeking to gain insights about the solvability, and the actual methods for the solving of all meaningful mathematical propositions.(Gödel *1961/?, 383)

Gödel says that there "exists today the beginnings of a science which claims to possess a systematic method for such clarification of meaning, and that is the phenomenology founded by Husserl". Continuing, he says

Here clarification of meaning consists in concentrating more intensely on the concepts in question by directing our attention in a certain way, namely, onto our own acts in the use of those concepts, onto our own powers in carrying out those acts, etc. In so doing, one must keep clearly in mind that this phenomenology is not a science in the same sense as the other sciences. Rather it is [or in any case should be] a procedure or technique that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other, hitherto unknown, basic concepts. (Gödel *1961/?, 383)

Gödel argues that the phenomenological approach cannot be dismissed on *a priori* grounds. Empiricists, in particular, should be the last to suppose there is an *a priori* argument against the phenomenological approach since *a priori* arguments about such matters are not available to them and would merely be dogmatic.

Gödel makes a number of remarks on the nature of concepts at various places in his writings. They are mentioned, for example, in his general views on logic. Gödel often speaks of logic in a manner that calls to mind ideas in Leibniz, Bolzano, Husserl and other philosophers who associated logic with the idea of science as rational inquiry in a very broad sense. Logic, Gödel says, is about the most general abstract (and 'formal') concepts and the relationships of these concepts to one another (see, e.g., Gödel *1953/59, 354). Sense perception is about particular objects and their properties and relations. In some passages, Gödel likens the conceptual content of sentences to Frege's notion of sense (Sinn) (Gödel *1953/59, 350). Concepts are abstract, objective intensional entities (Gödel 1944, 1958/72, *1961/?). They are not merely subjective and, unlike conscious states, they do not themselves have temporal phases. Concepts are not sets, although one might conjecture that every set is the extension of some concept. Concepts, it seems, are objects *sui generis*. One could make a list of things with which they are not to be identified or to which they are not to be reduced. Gödel rejects those forms of reductionism that ignore or seek to eliminate the intuition of concepts. He mentions specifically Hilbert's formalism, mechanism, nominalism, conventionalism, positivism, empiricism, psychologism and Aristotelian realism (Aristotelian universals) (Gödel *1951, *1953/59, *1961/?). In Section 9 of this paper I briefly consider some of the reasons for rejecting each of these positions. Hao Wang recorded a remark from a conversation with Gödel that puts the point succinctly, albeit rather roughly (1996, 167):

> Some reductionism is right: reduce to concepts and truths, but not to sense perceptions ... Platonic ideas [what Husserl calls "essences" and Gödel calls "concepts"] are what things are to be reduced to. Phenomenology makes them clear.

2. THE PLACE OF CONCEPTS IN HUSSERLIAN PHENOMENOLOGY: INTENTIONALITY AND OBJECTS OF EXPERIENCE

Gödel, as we have seen, says that in phenomenology "clarification of meaning consists in concentrating more intensely on the concepts in question by directing our attention in a certain way, namely, onto our own acts in the use of those concepts, onto our own powers in carrying out those acts, etc." Concepts are abstract, objective intensional entities. We now consider in more detail the place of such concepts in Husserlian phenomenology. This will give us a basis for understanding the idea that we do or can intuit concepts. A key ingredient in the description of the intuition of concepts is the fact that human cognition exhibits *intentionality*. This fact is often overlooked in recent work in the philosophy of logic and the philosophy of mathematics but it is probably not possible to understand the idea of intuiting concepts unless one starts with it. Without this insight the idea of intuiting concepts may in fact be absurd.

To say that cognition exhibits intentionality means that our cognitive acts are always *about* or directed toward something. Consciousness is, for many types of acts, always consciousness *of*. One can see this clearly in acts of believing, knowing, willing, desiring, remembering, hoping, imagining, and so on. For example, an act of belief is always directed toward a particular object or state of affairs. Try to deny that cognition exhibits intentionality and you are faced with an absurdity. Imagine for a moment what it would be like for a belief to not be about anything in particular.

Intentionality means directedness. *For a cognitive act to be directed in a particular way means that it is not directed in other ways.* Cognitive acts are perspectival and we cannot take all possible perspectives on an object or domain. We do not experience everything all at once. Perhaps only a god could do that, but we cannot. Our beliefs and other cognitive acts at any given time, in other words, always involve certain *categories* of objects. The mind always categorizes in this way. It cannot help doing so, for our experience would have to be very different otherwise. This means that we are always *interpreting* the world is some way or other. As I have been saying, categories or concepts are always at work in our experience, even in everyday sense experience. It is a basic feature of intentionality that directedness is always directedness by way of concepts, even if we do not always reflect on these concepts. In fact, we may not reflect on them at all. There is no such thing, however, as directedness that is not directedness by way of some concept. The standard form of intentionality can be characterized as follows:

$$\begin{array}{ccccc} & & & \overbrace{\qquad\qquad\qquad}^{\text{meaning or intention}} & \\ \underbrace{\text{A}}_{\text{subject}} & \underbrace{\text{believes}}_{\text{act type}} & \text{that} & \text{S} & \text{is directed toward (refers to)} \\ & & & \downarrow & \\ & & & \text{[object, state of affairs]} & \end{array}$$

'S' expresses a sense or meaning and it is by virtue of this meaning that we are directed toward a particular object or state of affairs. We 'bracket' the existence of the object or state of affairs since it may not always exist. There can be directedness toward an object by virtue of the meaning of a cognitive act even if, as sometimes happens, the object does not exist. In order to impute existence to the object we must have evidence for existence. Evidence takes us beyond the merely possible to the actual. When there is evidence for an object the object stabilizes and takes on a certain identity or invariance in the field of our experience.

Suppose S has the form Pa. As a matter of terminology, I will say that 'P' expresses a *concept*, and that 'a' is an expression for an object. Pa is merely the form of an expression that has a content or meaning. The following are examples of informal expressions of mathematical concepts by virtue of which the mind would be directed in different ways: "x is a triangle", "x is a natural number", "x is a function", "x is a mechanically computable function", "x is a set", "x is an infinite set", "x is a measurable cardinal", "x is a weakly compact cardinal". (Some of these are compound concepts.) These differences disappear if we have the pure form Px. Consider how your mind is directed in this latter case.

In a strictly defined formal language we could assign a Gödel number to the expression 'Px'. 'Px' could, for example, be characterized as a certain bit string. As such it would have a particular length and algorithmic complexity. This is to be contrasted with the meaning or concept expressed, for example, by "x is a set". Meanings or concepts do not have a particular length. We cannot measure the complexity of a concept in terms of the shortest program that would generate the bit string we take to express the concept.

It should be noted that in the context of the intentionality scheme presented above concepts or categories are intensional entities. Concepts are not, for example, to be confused with sets and their identity conditions are not determined extensionally We will say that a concept applies to or is true of an object or it fails to apply to or to be true of an object. Similarly, we can say that an object falls under or fails to fall under a concept. Applicability is a relation that should in general be distinguished from the set-theoretic relation of membership '$\in$'.

It is also important to note that, quite generally, meanings (or concepts) are different from mathematical objects (e.g., numbers, sets, functions). A concept or meaning can itself become an object of consciousness when we are directed toward it but to be directed toward the meaning of a mathematical expression is not the same thing as being directed toward a particular mathematical object. It should not be assumed that intuiting the one is exactly the same thing as intuiting the other.

3. SOME EXAMPLES OF HOW CONCEPTS FUNCTION IN OUR EXPERIENCE

Let us consider an example of how concepts are already present in our everyday perceptual experience. Suppose that I am standing some distance from a wall on which I perceive a round dark dot. As I approach it I see that there are light colored surfaces in its interior. Moving closer I can discern some markings in its interior. Continuing in this manner it occurs to me that it is a mandala. I could then make many further determinations about it, noting various designs used in it, and so on. According to the thesis of intentionality, we always experience things by way of concepts. In reflecting on this example, therefore, it is clear that a progression of concepts must be involved in gaining knowledge in ordinary sense perception. Given the description of the experience, one might think of this sequence as beginning with the concept "x is a dark round dot on a wall" and proceeding through "x has light colored surfaces in its interior", "x has markings in its interior", "x is a mandala", "x contains thirty three concentric circles", and so on. These concepts are already present in the (prereflective) experience but they are of course not themselves the objects of the experience. They are individuated and become objects only in reflection on the experience. Once we reflect on the experience we can describe many aspects of the experience in terms of the concepts involved and their relations to one another. The concepts involved at earlier stages, for example, are successively filled in by and harmonized with the later concepts, where the content of the earlier concepts is obviously retained in a certain way at later stages. What was quite indeterminate becomes more and more determinate in the process. The concepts in the progression must be related to one another in certain ways in order for there to be a continuous, harmonious development of the perception. The earlier concepts provide a basis for the later concepts. They are not rejected but are supplemented and mutually adjusted to more determinate concepts. The concepts, taken together, form a consistent set, but some may be more determinate than others.

The determinacy/indeterminacy feature of concepts in our experience simply reflects our epistemic situation. A horizon of possibilities about what may fall under the concept is determined by the concept, against the background of its context, but this often leaves many things indeterminate. The horizon of a concept Px may be defined, for our purposes here, in terms of those concepts that would in future possible experience be consistent with our present experience regarding Px. The possibilities (and probabilities) with respect to Px are always to be considered in the context of and against the background of our previous experience. The concepts in the example all have their own horizons but clearly the range of possible experience of the object x is narrowed at later phases in the sequence. The horizon of possible experience associated with "x is a dark round dot on a wall" leaves open many possibilities that are closed off by the subsequent determinations. Among other things, this means that some questions about x that could not be decided at earlier stages of the experience are *decidable* at later stages as a function of the new concepts that emerged in the course of the experience.

One might imagine that this little slice of my history resulted from my active interest in the object and in coming to know more about it. On the other hand, I may not have approached it in any conscious and systematic way but I just happened to notice these things. In natural science one takes an active, conscious, systematic approach in obtaining knowledge about natural objects.

Of course the development is not always harmonious. There can be various frustrations, annulments, and so on. For example, I might perceive an object x as a snake at a given stage of my experience and perceive it at a later stage as a coiled rope. In this case we do not have a harmonious progression and filling in of the earlier experience but a misperception. There can be no x in our experience such that x is a snake and x is a coiled rope. These concepts are inconsistent with one another. At later stages I perceive the object under the concept "x is a coiled rope".

Examples of the sort we have considered show how concepts are already at work in our ordinary, engaged prereflective experience and how in reflection we can become aware of these concepts and make them the objects of our experience or intuition. Of course in ordinary everyday experience we do not intuit concepts and it is not the case that we must intuit concepts in order for our everyday experience to be what it is. Instead, concepts are implicit in our everyday experience and make it possible. They are very much in the background of experience. Since they exist in experience in this manner, however, they can be brought into the foreground and they can become explicit in our experience through reflection. We can in reflection become conscious of the concept itself and its relations to other concepts. The fact that we can reflect on concepts does not at all entail that we will immediately have a perfect grasp of the concept and its relations to other concepts. On the contrary, our grasp of the concepts that are at work in our experience is often quite hazy and imperfect. In reflection we can clarify our grasp of a concept and its relations to other concepts. We can try to perfect our intuition of the concept itself.

Some degree of reflection is required if there is to be anything more to human knowledge and awareness than simple forms of everyday perception. Indeed, all of the ‘higher’, more theoretical forms of experience, like those found in logic and mathematics, will require some degree of reflection. These higher, more theoretical forms of experience are founded on sense experience and need not exist. Perhaps there were times in the history of human beings when they did not exist. To the extent that they do exist, however, they show that there must be some intuition of concepts. At a certain point, for example, one comes to know that some things are not natural numbers and some things are. This presupposes some grasp of the concept of natural number. Reflection is a condition for the possibility of sciences like mathematics and logic. All higher forms of awareness involved in science and culture are in this sense built up from reflection on everyday, prereflective, passive forms of awareness. The existence of sciences like mathematics and logic indicates a more active and systematic cognitive achievement.

Concepts have the same basic function in these more theoretical forms of awareness that they have in everyday awareness. Of course in pure mathematics, unlike everyday perception, the sensory contraints on experience fall away. There are, however, still logical, conceptual or meaning-theoretic constraints on the experience. The concepts involved simply direct us toward different kinds of objects in the two cases. Following Husserl, we might also argue that the concepts of logic and mathematics are exact in that they involve idealizations, while the concepts involved in everyday perception are inexact or ‘morphological’ since idealization is not involved in this case (see, e.g., Husserl 1913b, §74). The points, lines, planes, circles and spheres of Euclidean geometry, for example, are idealizations of the shapes of objects given to us in everyday sense perception. It is possible to give a detailed account of this cognitive activity of idealization but there is no space to do so here (see, e.g., Tieszen 2003).

Now certainly we can distinguish particular numbers from the concept of number, particular sets from the concept of set, particular functions from the concept of function, and so on. Imagine a cognitive life, for example, in which particular natural numbers bore no more relation to one another than they did to any other particulars. There could be no systematicity for these objects. Each natural number would be utterly unique and singular and could bear no more relation to another natural number than to, say, a chair. Of course this flatly contradicts our experience with natural numbers and it contradicts the fact that we have a science of these objects.

The concept of natural number is operative in our experience with natural numbers. We also know, for example, from our experience with natural numbers (as this is shown in mathematical practice) that some concepts are not consistent with “x is a natural number” or that some are simply from the wrong categories to be applicable to the objects that fall under this concept. An example of the latter type would be a concept like “x has a color”. Some of the concepts that are consistent with “x is a natural number” are “x is even”, “x is prime”, “x is perfect”, and so on. Each of these concepts in turn has its own horizon. The fact that the concepts have horizons and are not grasped with perfect clarity is shown by the fact that there are open problems involving combinations of these concepts. There are often various refinements and adjustments over time in our intuition of concepts.

The same points could be made about sets. However incomplete and unfinished our thinking about sets may be it nonetheless displays the kind of systematicity and organization that indicates the presence of concepts. Our early experience with the concept “x is a set”, for example, appears quite indeterminate by later standards. The early experience has been filled in with many concepts with various branchings, refinements, and adjustments. Against this historical background we might single out the concepts associated with Zermelo-Fraenkel set theory and expect new concepts to emerge that consistently unfold the existing concepts and add further determinations that may allow us to decide questions about these objects that were not decidable at earlier stages. The situation here is analogous to the way that decisions may come about through the progression of concepts involved in the perception of the mandala. Gödel indeed says that the idea of deciding undecidable mathematical propositions by extending mathematical theories with new axioms or by ascending to higher types is best described as developing or unfolding our intuition of the concepts (meanings) that appear in these propositions (Gödel 1958/72, 1947/64, *1961/?). These decisions will be non-arbitrary, as they are in the example above of sense perception. They will be forced or constrained in certain ways. Misperceptions are also possible here.

As another example one might consider the concept of proof. The intuitive concept of proof is clarified by the incompleteness theorems themselves because they show that provability in any given formal system cannot fully capture the intuitive concept (Wang 1974, 83). Purely formal proof is always ‘relative’ to a given formal system. The incompleteness theorems show us that number-theoretic provability is not the same thing as number-theoretic truth, whereas it was not clear at earlier points whether they could be equated or not.

Our examples show how concepts can become objects in reflection. There is, in these cases, intuition of concepts. If we take intentionality seriously then we must say that *in being directed toward a concept or category of concrete particulars we are not (primarily) directed toward the particulars themselves.* It is the concept that we grasp. The key idea to remember here is that we are speaking of what the mind can be directed toward. Intuition is to be understood only in these terms. It is a basic fact of consciousness that we can be directed toward many different kinds of things including, as our examples are meant to show, concepts. If different categories of concepts will be applicable depending upon whether we are directed toward concrete particulars or concepts then, according to the thesis of intentionality, we must be directed toward different things. The properties and relations of concepts are indeed different from the properties and relations of concrete particulars.

An immediate consequence of our remarks on directedness is that directedness toward concepts is not the same thing as directedness toward sets. The latter are governed by extensional identity conditions and the former are not. We can of course also be directed toward sets in our thinking, as is the case when we use the concept "x is a set" or some specification of this concept.

4. SOME RELATIONS OF CONCEPTS AND THE SPACE OF CONCEPTS

In following out the consequences of the thesis of intentionality and the earlier examples we have argued that there is directedness toward concepts. We can also be directed toward the relations of concepts to one another. Not all relations are relations of concrete, sensible particulars. Certain relations are naturally construed as being relations of concepts. Some of these relations can be characterized to a first approximation by using the notation of intensional logic for possibility, '$\Diamond$', and necessity, '$\Box$', where these are relativized to what can occur in our experience.

The inconsistency of concepts P and Q, Inc(P, Q), can be characterized as $\neg\Diamond(\exists x)(Px \wedge Qx)$.
Similarly, Con(P, Q): $\Diamond(\exists x)(Px \wedge Qx)$ (consistency of concepts P, Q),
Imp(P, Q): $\Box(\forall x)(Px \rightarrow Qx)$ (concept P implies concept Q),
Equiv(P, Q): $\Box(\forall x)(Px \leftrightarrow Qx)$ (equivalence of concepts P and Q),
Ind(P, Q) (independence of concepts P and Q), and so on.

Consider again some of the examples above. If the x in our experience is a dark round dot on a wall then is it possible that x is a mandala? Certainly these concepts are consistent with one another but the second concept is not implied by the first. On the other hand, the concepts “x is a snake” and “x is a coiled rope” are not consistent with one another. Note that this is not a purely formal contradiction, in the sense of $Px \wedge \neg Px$. One must know the origin of the symbols Px and Qx. It is necessary to know what the symbols represent. To shift to some mathematical concepts: if x is a natural number is it possible that x is prime? Again, these concepts are consistent with one another. If x is a natural number is it possible that x not have a successor? Here we find an inconsistency given the standard concept of the natural numbers. Here is another example of inconsistent concepts: x is a triangle and x has four sides. It is worth noting that we can often know that concepts are inconsistent with one another without knowing everything about the concepts. We need not know *exactly* what their applications are in order to know that they are inconsistent.

Critics of realism will sometimes ask the supposedly embarrassing question of where logical or mathematical concepts and objects are. They are supposed to be ‘out there’ in some platonic heaven or third realm. Since the position I am outlining is not a form of naive ontological realism it is easy to respond to this worry. The view I am describing holds that mathematical concepts and at least some mathematical objects are given to us in our experience as transcendent and as existing ‘in themselves’. They acquire this meaning of being in our experience. From my point of view the question of where concepts are is based on a category mistake. Concepts are obviously not in some place as physical objects are in a place. Concepts, however, can be thought of as having a ‘place’ in the network or system of logical and mathematical objects, concepts and propositions even though we may not have a clear grasp of the network. They are in the (conceptual) space of logic and mathematics. This is, in a sense, a ‘world’ but it is a cognitive world which, being based on the intentionality of consciousness, is not at all inaccessible to us even though much of it may be implicit at any given stage. These concepts and propositions imply one another, are (in)consistent with one another, independent of one another, equivalent, and so on. To know their ‘place’ is to know their relations to other concepts. To know the place of a concept Px is to know what it implies, is implied by, is (in)consistent with, etc. It seems that one could come to know concepts and their places as well as one knows about physical objects and their places. Perhaps only a few have an overview of this mathematical and logical space. To learn mathematics or logic is to have it to some extent if only implicitly.

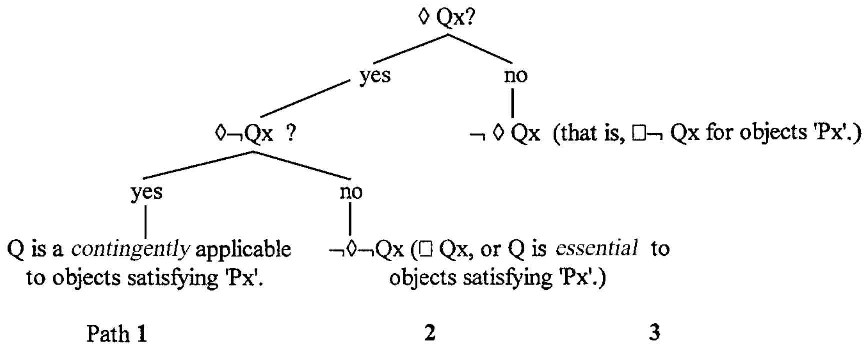
5. MEANING CLARIFICATION

In the discussion of intentionality above it was said that we are directed toward objects by way of the 'meanings' associated with our cognitive acts. In reflection on experience we can explore and unfold the meanings of our concepts. Thus, we are in effect describing a kind of meaning clarification. One might seek to clarify, for example, the concept "x is a natural number" or the concept "x is set", along with related concepts. We can take a more or less active role in this meaning clarification. Meaning clarification often takes place in these sciences as a matter of course, albeit not always in a reflective and systematic manner. We may cease to make progress if, under some reductionistic scheme, the meaning of mathematics is lost or distorted. This is what lies behind Gödel's remarks on meaning clarification in phenomenology.

Gödel and Husserl have made remarks about the phenomenological method that have raised great expectations in some philosophers and logicians about applying the method to obtain solutions to open mathematical problems. Gödel no doubt had some hopes of using Husserlian ideas to make more progress in mathematics but I think one has to be cautious about what can be achieved. No one has a method for solving deep open problems and one ought not to expect phenomenology to have one either. What the phenomenological view does foster, among other things, is recognition of the existence and importance of meaning and informal concepts in mathematical practice. As I see it, it holds that pure mathematics is an autonomous science with its own concepts and that it is possible to clarify such concepts in a variety of ways (including but not limited to the technique of formalization). It is a view about the nature of mathematics that can be contrasted with other philosophical views. It describes mathematical practice in terms of unfolding and clarifying our grasp of the meaning of mathematical concepts through activities like abstraction, generalization, specification, analogizing, idealization, formalization, axiomatization, and free variation in imagination. Some of the activities to which it would appeal in the analysis of *mathematical* cognition are different from the activities to which it would appeal in the analysis of concepts involved in everyday perception. This phenomenological view must have seemed especially appealing to Gödel because it stood in such stark contrast to the ax-wielding logical positivists of his time who held, among other things, that mathematical statements were contentless tautologies. So, in a general sense, it is a view that does give more hope for progress in mathematics than many of the other reductionistic or dismissive 'leftward' views of mathematics that were and still are around. Fortunately, the philosophical attitude toward mathematics does not seem quite as grim now as it did in the heyday of positivism.

What one can do, therefore, is to give examples of informal concepts that have been clarified through the kinds of activities mentioned above. Gödel, Kreisel, Wang and Troelstra have given examples of this type (e.g., the concept of a mechanically computable function, of choice sequence, of continuity, of proof, of points as parts of the continuum, and so on). One could do more of this sort of thing with an emphasis on specific ideas in phenomenology.

It might be useful to pause in particular over Husserl's idea of free variation in imagination since it is something he specifically recommends. This is a way in which meaning clarification might be made more active and reflective, given the context of the concepts involved and the appropriate background and expertise. Frege (Frege 1892), for example, has said that comprehensive knowledge of a reference requires us to be able to say immediately whether any given sense belongs to the reference. Similarly, we could say that comprehensive knowledge of a mathematical object requires us to be able to say immediately whether any given concept belongs to the object. One way to try to improve our mathematical knowledge therefore would be to (1) start with an actual or imagined arbitrary instance of the type of mathematical object under consideration; (2) vary the instance as many ways as possible, and freely or arbitrarily, in imagination, i.e., try to imagine it under as many different concepts as possible; and (3) as 2 proceeds determine under which concepts, if any, the object could no longer be experienced as an instance of the same type of phenomenon. Schematically, for an arbitrary x that could be imagined or perceived to be a P, determine whether x can also be Q, R, S, T, ... and so on: If Px, then



Compared to some of the restrictive or constraining conditions of reductionist methodologies, this way of looking at things might help to free the mind. It could help to open up the mind, based on the use of imagination in the service of reason. One generates a bunch of concepts and then looks back through them at what was given. The general formulation just mentioned is of course quite open-ended and non-deterministic but the process is significantly more determinate once we contextualize it against the background and expertise found in our actual practice in various spheres.

One need not have a great wealth of experience in mathematics, for example, to notice that a set or a natural number cannot be imagined to be just anything. Our imagination here is not absolutely free (compare with Gödel's remarks on free creation and constraints, Gödel *1951, 314, and elsewhere). It is bounded or limited in certain respects. This is not a matter of convention. Social or linguistic conventions are susceptible to variation and could be imagined differently. This is the nature of conventions. The limits under which we may vary our thinking of a mathematical object, however, are determined by something that is invariant: the concept of the object.

It should also be noted that there are two types of variation or change that we may observe in this process: essential change and alteration or 'accidental' change. In the cases in which we determine that $\Box$Qx given the concept Px we are dealing with the kind of information found in axioms or theorems, the denial of which would amount to essential change in the objects, while in the other case we arrive at "accidental" properties or relations concerning the objects. In the latter case we can vary the concepts under which an object of a given type is thought without changing the object, while in the former case we cannot do this. We see this necessity, for example, in thinking of the procedure in connection with the axioms of Peano arithmetic.

Meaning clarification is premised on the fact that there is much that is implicit in the intuition of a concept that could be made explicit. The idea is that we extend our understanding of the possibilities involving the phenomenon and that necessary or essential properties will appear through the multiplicity of variations. By generating a multiplicity of variations in imagination one consciously and purposely creates a background against which necessities may emerge. It is only the stability of the essential concepts that allows us to vary the other concepts in the first place. Any variation presupposes that something is constant which makes the variation possible. Imagining possibilities in this way might help us to arrive at a new, higher state of consciousness concerning the phenomenon. Through the process of free variation one might be able to uncover a "rule" governing the horizon of the concept which would not have been seen prior to the process of variation. The rule, even if only partially understood or not fully determinate, would suggest further possible insights. In some cases we would be forced outside of a given concept. But we are not necessarily forced into a void. We may have stumbled onto a new concept that can be consistently developed.

As noted above, Gödel says that a non-reductionistic clarification of meaning, or this kind of informal rigor, can be expected to play a role in the non-mechanical decidability of open mathematical problems. In his paper on Cantor's continuum problem (Gödel 1947/64, *1951), for example, Gödel says that a proof of the undecidability of generalized continuum hypothesis (GCH) from the present axioms of set theory does not solve the problem of GCH. Its undecidability from the axioms being assumed today only means that these axioms do not contain a complete description of the well-determined reality in which GCH must be either true or false. By a proof of undecidability a problem loses its meaning only if the system of axioms under consideration is interpreted as a hypothetico-deductive system, i.e., if the meanings of the primitive terms are left undetermined. If the meanings of the primitive terms are not left undetermined then we can develop and sharpen our intuition of the concepts involved.

On the basis of his own work on (un)decidability, Gödel also notes that sometimes we might be expected to ascend to higher levels of abstraction and meaning clarification to decide problems at lower levels (see Gödel 1947/1964, *1951). It can be shown that the axioms involved in extensions of transfinite set theory, for example, have consequences outside the domain of very great transfinite numbers, which is their immediate subject matter. Each of the axioms of infinity, under the assumption of consistency, can be shown to increase the number of decidable propositions even in the field of Diophantine equations.

6. THE PHENOMENOLOGICAL ONTOLOGY OF CONCEPTS

I considered some simple examples in Section 2 in which there is directedness toward concrete sensible particulars in space-time in order to show how there is a natural shift in some of our cognitive activities to directedness toward concepts that are applicable to concrete particulars. There can also be a shift of directedness toward concepts that are *not* directly applicable to concrete, sensible particulars. It is in the nature of pure mathematical thinking that it is not directed toward concrete, sensible particulars in space-time. Even in applications of mathematics we come to interpret the natural world in a special way: we mathematize it. The mathematical concepts and structures used in applications shape and organize the way we see the natural world into more idealized and abstract forms.

In mathematical thinking there is of course directedness, but it is now directedness toward 'abstract' or ideal objects like numbers, sets and functions. Pure mathematical concepts (e.g., "x is a triangle", "x is a natural number", "x is prime") have abstract or ideal objects as instances. Husserl distinguishes, for example, 'real' individuals and universals from 'ideal' individuals and universals (Husserl 1970, Investigation II, §2). The directedness toward ideal objects as instances appears at higher, founded levels of awareness of the type we see in mathematics. Our awareness moves from sensory individuals up to the more theoretical, conceptual forms of directedness. Of course mathematical concepts are not themselves typically the objects of mathematical thinking. The objects that fall under such concepts are what mathematics is about. One could, however, always reflect on and analyze mathematical concepts. This may usually be more of a philosophical, logical or meaning-theoretic task.

In all of this I am speaking about what the mind is directed toward and even if there is always a more or less vague background of sensory awareness in all of our cognitive activities it is not the case that the mind is always directed toward objects of sense experience. (At least mine is not.) If the mind were directed only toward objects of sense experience there would be no mathematics and logic.

I wish to take mathematics and logic (in a broad sense) as sciences that are given. They are or can be part of our conscious life, and they provide data that may be substituted for S in the intentionality schema described above. Once we accept the idea that consciousness exhibits intentionality we can see why we must think of mathematical concepts and objects as abstract or ideal. Indeed, the putative 'ontology' of concepts should be arrived at by way of the application of the notion of intentionality to what is given in mathematics and logic. One consequence is that in being directed toward a concept that applies to particulars we are not directed toward the particulars themselves. The concepts or relations that apply to concrete particulars are different from the concepts or relations that apply to concepts. Guided by the idea of directedness, we see that there are many types of concepts and relations that do not apply to concepts themselves.

The phenomenological ontology of concepts results from the simple observation that certain categories of concepts and relations are not applicable to concepts. For example, categories involving spatial relations are not applicable to concepts. Concepts do not have spatial properties and do not stand in spatial relations. Concepts do not have temporal properties and do not stand in temporal relations. By way of contrast, the mental process of conceptualizing does have temporal duration. Concepts also do not stand in causal relations to one another or to anything else (e.g., to us). Because they do not have temporal properties they are distinct from subjective phases of consciousness. Phases of consciousness do have temporal duration and vary through time. Concepts are in this sense objective. They are invariants in the stream of consciousness. They are independent of phases of consciousness in the sense that they are not variable, 'real' elements in the stream of consciousness, but they are not mind-independent in the sense that they are not in the stream of consciousness at all. They are in the stream of consciousness as 'abstract' or 'ideal' and invariant. Thus, they are not mind-independent in a manner that precludes our being directed toward them (or by them). Of course some concepts do apply to concepts and concepts do stand in certain kinds of relations to one another. Not all relations are empirical, e.g., spatial, temporal or causal. Some relations are meaning-theoretic, logical or formal.

The form of rationalism that starts with the notion of intentionality and takes mathematics and logic as given yields a kind of phenomenological realism about mathematics and logic. It can be contrasted with classical metaphysical realism because it is a realism that starts from inside the domain of our conscious life and, given the intentionality of human reason, finds that the mind is directed, *prima facie*, toward different kinds of things in its different activities. Concepts are among the things toward which it may be directed in reflection. It is not an untenable realism that places concepts outside of all possible experience. The concepts it recognizes are only those that are objects of possible experience.

To speak of *phenomenological* ontology is to hold that all we have to go on are the invariants in our experience, even if we are under some massive illusion about these for many years. We can distinguish real invariants from apparent invariants relative to our experience thus far, but we are never in a position to do this once and for all. This is just the nature of phenomenology and is part of its way of avoiding a naive and insupportable metaphysical realism.

7. CONCEPTS ARE NOT SUBJECTIVE IDEAS

We have been saying that concepts are not subjective. They should be distinguished from the individual changing ideas of each subject. Since subjects are supposed to be directed toward objects by way of concepts it might be objected that consciousness could not be directed by abstract, objective entities but only by individual subjective ideas. This objection gives expression to a very basic misunderstanding of my view of concepts or senses. Thus, it is important to respond to it.

First, in Fregean/Husserlian fashion, we can say that many different subjective ideas may be associated with the same abstract concept. 'Ideas' in this sense occur at particular times in particular subjects. They are unique to each subject. One person's idea is not that of another. The same concept need not always be connected with the same idea, even in the same person. The concept, however, can be shared by many different people at different times and places. Indeed, we see how a common store of concepts is invariant from generation to generation, especially in the sciences of mathematics and logic. A good start on an account of this phenomenon can be found, for example, in Husserl's essay on the origin of geometry (Husserl 1939). Thus, various subjective ideas may be associated with the same, invariant concept. The relationship is another instance of the variation/invariance structure of consciousness. Epistemically speaking, concepts and subjective ideas depend on and emerge in relation to one another. The existence of a concept is required for the existence of a subjective idea, and the existence of a subjective idea is required for knowledge of a concept.

It is true that if there were no subjective ideas then a subject's acts would not be directed. A subjective idea must, in a sense, be present if there is to be directedness toward an object. The point, however, is to see that directedness would not be possible without concepts. The presence of a subjective idea is a necessary condition for directedness but it is not sufficient. The reason it is not sufficient is that without concepts there would be no invariance across different subjective experiences within the same subject, to say nothing of invariance across different subjects at different times and places. Directedness toward objects (= identities in experience) requires such invariance. Otherwise there would be nothing that these diverse ideas or experiences would have in common. The concept or sense is precisely that which diverse ideas can have in common. Thus, the directedness of consciousness requires the notion of concept or sense. Moreover, as we have been saying, it is not possible to deny that consciousness is directed.

Frege liked to illustrate some of these ideas with his telescope analogy (Frege 1892). Consider the observation of the moon though a telescope. The moon is the object or reference (invariant), mediated by the real image projected by the glass in the interior of the telescope, and by the retinal image of the observer. The real image is like the sense (another invariant), and the retinal image is like the subjective idea or experience. The optical image in the telescope is one-sided and dependent upon the standpoint of the observation but it is still objective and invariant inasmuch as it can be used by different observers or even the same observer on different occasions. Each different observer, however, would have his or her own retinal image on account of the diverse shapes of the observers' eyes, different physiological features, etc.

8. CONCEPTUAL INTUITION AND THE ALLEGED PROBLEM OF 'EPISTEMIC CONTACT'

On the view of conceptual or rational intuition that we have been developing, concepts are not abstract 'things-in-themselves' that are forever cut off from consciousness. Rather, they are given as invariants in our experience toward which we can be directed. Directedness is not a spatial, temporal or causal relation. As our examples show, the absence of a causal relation makes not the slightest difference to the fact that we can be directed toward concepts. Its absence does not somehow block cognitive directedness toward such objects. The real mystery is why anyone would insist that if directedness is not a causal relation then it is an illusion. This insistence is probably just a misapplication of the desire to be 'scientific' (i.e., empirical). One need not accept the assumption, however, that all science is empirical science. One can perfectly well remain scientific without such misplaced zeal. Directedness is not a causal relation but it is a fundamental and undeniable feature of consciousness. To hold that we cannot be directed toward such different objects is to lose track of the facts of conscious life. Once we understand intuition in terms of intentionality and directedness, and not in terms of causal relations, it is possible to develop an account of the intuition of concepts. Such an account is a natural consequence of adopting the intentional stance on our experience, as opposed to the physical stance or the design stance (see, e.g., Dennett 1971).

Of course in being directed toward ordinary sensory objects causal relations do play a role but even here it can be argued that causal attributions are parasitic on invariance in the field of consciousness. Since we can misperceive and be under illusions about objects, the 'object' to which we take ourselves to be causally related will be that which stabilizes and becomes invariant in our experience. If I perceive a house at one point and then see at a later stage that it is actually only a house façade (as in a movie prop) then at the later stage I should not be able to say that I was causally related to a house at any point in the experience. What I take myself to be causally related to will be a function of what remains invariant in my experience. In a phenomenological account of knowledge we can speak of what motivates knowledge in various domains without having to suppose that motivation is always the same thing as a causal relation to a sensory object.

The more fundamental structure, even in sense perception, is therefore directedness toward invariants. What this means is that our directedness model applies uniformly across the domains of mathematics and sense perception and that we simply particularize it for each of these domains. Virtually every empiricist account of knowledge and intuition I know of starts with sense perception and never gets beyond it. Some philosophers who have developed these accounts admit outright that their models do not apply to mathematics and that they do not know how to account for mathematics. Since empiricist accounts do not apply uniformly across domains we are presented with dilemmas about how we could have mathematical knowledge. The *locus classicus* of this in recent times is Benacerraf's paper "Mathematical Truth" (Benacerraf 1973). It should be obvious that I challenge Benacerraf's claim that our best account of knowledge will be some kind of causal account. Benacerraf's paper ignores the fact that in "knowledge" we have an example *par excellence* of intentionality. If we have knowledge anywhere it is in the sciences of mathematics and logic. This is arguably our most secure and highest grade of knowledge. Why downplay it and insist on empirical knowledge as the paradigm? The way to account for mathematical knowledge is in terms of directedness toward invariants. The fact that intentionality in mathematics has not been investigated has led to a huge gap in current thinking about mathematical knowledge. This contributes to the feeling that Benacerraf's dilemma is genuine.

9. SOME POSITIONS THAT IGNORE OR SEEK TO ELIMINATE THE INTUITION OF ABSTRACT CONCEPTS

Gödel mentions a number of philosophical positions that either ignore or seek to eliminate the intuition of abstract concepts. It is useful to consider these positions in order to obtain a sharper view of the place of concepts in mathematics.

Hilbert's formalism would provide a way to ignore or eliminate the intuition of abstract concepts. If Hilbert's original plan for proof theory had succeeded it would be possible to appeal only to the intuition of concrete, finite sign configurations and to finitary, purely formal or mechanical rules for manipulating these sign configurations in order to establish the consistency and completeness of mathematical theories. Gödel's incompleteness theorems, however, show that Hilbert's original plan for proof theory will not succeed. Mathematics cannot be completely represented in finitely axiomatizable purely formal systems if it is consistent. The intuition of abstract concepts cannot be ignored or eliminated because the incompleteness theorems show that the appeal to only concrete, finite particulars and mechanical operations on these will not suffice for obtaining decisions about formally undecidable sentences, for obtaining consistency proofs for formal systems, or for the general development of mathematics. There must be some informal meaning that has not been captured in the finitary formal system. This notion of meaning, and the kind of informal rigor called for in dealing with it, goes far beyond the narrow 'contentual' meaning that Hilbert assigned to finitist, concrete mathematics. What will be required is the intuition of abstract concepts and meaning clarification. The finitary restriction of Hilbert's program is replaced with the realization that there is an inexhaustibility to many mathematical concepts. These concepts exceed or transcend our intuition at any given stage. The intuition of concepts is partial. All we can do is to work in a more or less systematic and reflective manner to unfold more and more of the meaning of these concepts.

Given the technical relationships that exist between finitary formal systems and the mathematical concept of mechanical computation, Gödel applies many of these same remarks to mechanism (see, e.g., Gödel 1934, *193?, 1972a). The intuition of abstract concepts could be ignored or eliminated if mathematics could be completely mechanized. The incompleteness theorems, however, cast doubt on the complete mechanization of mathematics. The intuition of abstract concepts cannot be completely mechanized. What cannot be captured in a machine is the reflection on abstract informal meanings, where these meanings transcend our grasp but are nonetheless still partially available to us. The finitary restrictions on machines are contrasted with the fact that there is an inexhaustibility to many mathematical concepts. It is this kind of view that underlies the late note on a "philosophical error in Turing's work" (1972a) that Gödel appended to his 1972 paper. Gödel says that Turing presents an argument at one point to show that mental procedures cannot go beyond mechanical procedures. The problem is that there may be finite, non-mechanical procedures that make use of the meaning of terms. Turing does not recognize that

> *mind*, *in its use*, *is not static*, *but constantly developing*, i.e., that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding. (Vol. II, 306)

> Therefore, although at each stage the number and precision of the abstract terms at our disposal may be *finite*, both (and therefore, also Turing's number of *distinguishable states of mind*) may *converge toward infinity* in the course of the application of the procedure. (Vol. II, 306)

Human minds are not machines because human minds intuit abstract concepts (see also Tieszen 1994). Moreover, intuition of concepts in my description is to be understood in terms of the intentionality of consciousness and intentionality is just what machines lack.

Nominalism is the view that only concrete particulars exist. Universals do not exist or, in the language I have used, abstract concepts or objects do not exist. Nominalism is usually part of an empiricist epistemology and ontology. Nominalists would certainly seek to eliminate the idea of intuiting abstract concepts. According to the arguments in this paper, however, we can say that concepts must exist in our experience in order for it to be the way that it is. Nominalism is either a naive metaphysical view that ignores the nature of our experience or if it does not ignore the nature of our experience then it is untenable. It is not the case that only concrete particulars exist in our experience (see Sections 1–3). Gödel notes that Hilbert's formalism may be viewed as a kind of nominalism. In his 1953/59 papers he also shows how the incompleteness theorems can be applied in particular to Carnap's positivism, which can be viewed as a combination of nominalism and conventionalism.

One of Gödel's main objections (*1951, *1953/59) to conventionalism is that it portrays mathematics as our own free creation or invention in a way that does not square with the facts. As we said above, not just anything falls under a given concept. Not just anything could be, for example, a natural number. We are constrained or forced in certain ways by the meanings of concepts and we cannot change these meanings at will. There may be a certain amount of freedom in unfolding a concept, depending on how indeterminate the concept is. Conventionalism, however, misrepresents this. It treats mathematics and logic as 'man made' and variable at will. Conventions can be varied and imagined differently. Concepts, however, are not variable in this manner and are not man made (see also Tieszen 1998b).

Gödel's remarks about positivism are related to his remarks about conventionalism and nominalism. A central problem with positivism lies in the way that it distinguishes statements of logic and mathematics from empirical statements. The latter, being based on sense experience, are supposed to have content and to be meaningful. The statements of logic and mathematics, on the other hand, are supposed to be tautological or void of content. They do not have their own content or meaning nor do they refer to distinctively mathematical objects and facts. They are true 'by definition', where this is a matter of linguistic convention. They are not considered to be true on the basis of the abstract concepts they contain. The positivist view thus leaves no room for intuition of abstract concepts and meaning clarification in mathematics. Apart from its conventionalism, Gödel thinks that positivism does not square with facts about how, in light of the incompleteness theorems, we will need to decide undecidable but meaningful mathematical statements and obtain consistency proofs. It does not do justice to mathematics.

Psychologism is a species of empiricism. According to psychologism, the objects logic and mathematics are mental objects and the truths of logic and mathematics are truths about our factual psychological makeup. They are empirical, *a posteriori* invariants that could be different from how they now are. As is the case with many of the views that Gödel rejects, this introduces an untenable relativism into mathematics and logic. We can see why it would be set aside. The properties of the mental objects and invariants found by empirical science are not the properties of concepts.

The general 'leftward' worldview (see Gödel *1961/?) that underlies nominalism, conventionalism, positivism, and psychologism is empiricism. It may also be seen as underwriting Hilbertian formalism and mechanism. Empiricism is the view that all knowledge is derived from sense experience. Thus, there cannot be anything like an experience of abstract concepts. Gödel thinks that attempts to reconcile mathematics and logic with empiricism have failed and the incompleteness theorems and related results can be used to show how they have failed. What is needed is a more balanced view that includes elements of rationalism that accord with our experience but that excludes earlier excesses or mistakes associated with naive rationalism. I think that the view I have outlined above indicates some of the major empiricist sins of omission. Chief among these is its blindness about certain facts of experience, including the fact that consciousness exhibits intentionality and that the mind can be directed in many different ways. The mind is not locked onto sense experience and only sense experience, even if some elements of sense experience are always in the background of consciousness. In particular, concepts make our experience possible and the mind can be directed toward these concepts.

Finally, Aristotelian realism is not necessarily a view that eliminates intuition of abstract concepts. The problem is that it is just too limited to do justice to mathematics and logic. It allows only those concepts or universals that have physical, concrete instances. No other concepts or universals exist. This is clearly not acceptable if we are to do justice to pure mathematics and pure logic.

In light of more recent work on theories of concepts one could add that concepts are not mereological sums, sets of actual or possible particulars, tropes, and so on. It might be possible to devise a formal theory of concepts (see, e.g., Wang's suggestions about this in Wang 1987, 309–311) but, as we have seen, Gödel makes various remarks about concepts and meaning clarification that suggest it is necessary to develop insights into the content as well as the form of concepts.

About the process of the intuition Gödel says, in summary, that our intuitions of concepts are (1) constrained or 'forced' in certain respects (e.g., not just anything falls under the concept 'natural number'); (2) they are fallible; (3) more or less clear and distinct, precise; and (4) for many mathematical concepts they are inexhaustible (*1951, 1947/64, and citations in Wang).

10. CONCLUSION

Why would anyone think that there is or could be an intuition of abstract concepts? I have now provided some of the elements of an answer to this question. It is an answer that preserves some of the rationalistic, 'rightward' features (Gödel *1961/?) that have always been associated with logic and mathematics while avoiding the metaphysical excesses of naive rationalism and naive realism. Our experience in mathematics is not random or arbitrary. It displays some degree of systematicity, owing to the concepts at work in experience. We should be able to uncover additional relations among concepts. Viewed aright, the idea that we can intuit concepts should hardly be the great mystery its detractors have made it out to be. This of course is not to say that more work is not needed.

ACKNOWLEDGEMENTS

Parts of this paper were presented to the Spring 2000 Working Group on the History and Philosophy of Logic at UC-Berkeley and to the Spring 2000 Logic Lunch group at Stanford. Thanks for comments are due to the audience members and especially to Solomon Feferman, Grisha Mints, Paolo Mancosu, Tom Ryckman, Joel Friedman, Richard Zach, Ed Zalta, Johan van Bentham and John Etchemendy. I also thank the *Synthese* referees.

NOTES

[1] These arguments have been discussed by a number of writers. See especially Wang 1974, 1987, 1996; Parsons 1995; Tragesser 1977; and Tieszen 1992, 1994, 1998a, 1998b, 2000.

REFERENCES

Benacerraf, P.: 1973, 'Mathematical Truth', *Journal of Philosophy* **70**, 661–679.

Dennett, D.: 1971, 'Intentional Systems', reprinted in D. Dennett, *Brainstorms*, Bradford Books, Montgomery, Vermont, 1978, pp. 3–22.

Feferman, S. et al. (eds): 1986/1990/1995, *Kurt Gödel: Collected Works*, Vols. I, II, III, Oxford University Press, Oxford.

Føllesdal, D.: 1995, 'Gödel and Husserl', in J. Hintikka (ed.), *Essays on the Development of the Foundations of Mathematics*, Kluwer, Dordrecht, pp. 427–446.

Frege, G. 1892, 'Über Sinn und Bedeutung', translated as 'On Sense and Reference', in P. Geach and M. Black (eds.), *Translations from the Philosophical Writings of Gottlob Frege*, Blackwell, Oxford, 1970, pp. 56–78.

Gödel, K.: 1934, 'On Undecidable Propositions of Formal Mathematical Systems', in *Feferman et al.*, *1986*, 346–371.

Gödel, K.: *193?, 'Undecidable Diophantine Propositions', in *Feferman et al.*, *1995*, 164–175.

Gödel, K.: 1944, 'Russell's Mathematical Logic', in *Feferman et al.*, *1990*, 119–143.

Gödel, K.: 1946, 'Remarks Before the Princeton Bicentennial Conference on Problems in Mathematics', in *Feferman et al.*, *1990*, 150–153.

Gödel, K.: 1947/64, 'What is Cantor's Continuum Problem?', in *Feferman et al.*, *1990*, 176-187, 254–270.

Gödel, K.: *1951, 'Some Basic Theorems on the Foundations of Mathematics and Their Implications', in *Feferman et al*., 1995, 304-323.

Gödel, K.: *1953/9, III and V: 'Is Mathematics Syntax of Language?', in *Feferman et al.*, *1995*, 334–363.

Gödel, K.: *1961/?, 'The Modern Development of the Foundations of Mathematics in the Light of Philosophy', in *Feferman et al.*, *1995*, 374–387.

Gödel, K.: 1972, 'On an Extension of Finitary Mathematics Which Has Not Yet Been Used', in *Feferman et al.*, *1990*, 271–280.

Gödel, K.: 1972a, 'Some Remarks on the Undecidability Results', in *Feferman et al.*, *1990*, 305–306.

Husserl, E.: 1913a, *Logische Untersuchungen*, 2nd ed. Max Niemeyer, Halle. Reprinted in Husserl 1984. English translation in Husserl 1970.

Husserl, E.: 1913b, *Ideen zu einer reinen Phänomenologie und phänomenologischen Philosophie*, Max Niemeyer., Halle. Reprinted in Husserl 1950. English translation in Husserl 1982.

Husserl, E.: 1939, 'Der Ursprung der Geometrie als intentional-historisches Problem', translated as 'The Origin of Geometry' in E. Husserl, *The Crisis of the European Sciences and Transcendental Phenomenology*, Northwestern University Press, Evanston, IL, 1970, Appendix VI.

Husserl, E.: 1950, *Husserliana III*, Nijhoff, The Hague.

Husserl, E.: 1970, *Logical Investigations*, Vols. I, II. English translation of *Husserl 1913b and 1921*, Routledge and Kegan Paul, London.

Husserl, E.: 1982, *Ideas Pertaining to a Pure Phenomenology and to a Phenomenological Philosophy*, Nijhoff, The Hague.

Husserl, E.: 1984, *Husserliana XIX*, Nijhoff, The Hague.

Parsons, C.: 1995, 'Platonism and Mathematical Intuition in Kurt Gödel's Thought', *The Bulletin of Symbolic Logic* **1**, 44–74.

Tieszen, R.: 1992, 'Kurt Gödel and Phenomenology', *Philosophy of Science* **59**, 176–194.

Tieszen, R.: 1994, 'Mathematical Realism and Gödel's Incompleteness Theorems', *Philosophia Mathematica* **3**, 177–201.
Tieszen, R.: 1998a, 'Gödel's Philosophical Remarks on Logic and Mathematics: Critical Notice of Kurt Gödel: Collected Works, Vols. I, II, III', *Mind* **107**, 219–232.
Tieszen, R.: 1998b, 'Gödel's Path from the Incompleteness Theorems (1931) to Phenomenology (1961)', *Bulletin of Symbolic Logic* **4**, 181–203.
Tieszen, R.: 2000, 'Gödel and Quine on Meaning and Mathematics', in G. Sher and R. Tieszen (eds.), *Between Logic and Intuition: Essays in Honor of Charles Parsons*, Cambridge University Press, Cambridge.
Tieszen, R.: 2003 (forthcoming), 'Husserl's Logic', in D. Gabbay and J. Woods (eds.), *Handbook of the History and Philosophy of Logic*, Springer, Berlin.
Tragesser, R.: 1977, *Phenomenology and Logic*, Cornell University Press, Ithaca, New York.
Wang, H.: 1974, *From Mathematics to Philosophy*, Humanities Press, New York.
Wang, H.: 1987, *Reflections on Kurt Gödel*, MIT Press, Cambridge, MA.
Wang, H.: 1996, *A Logical Journey: From Gödel to Philosophy*, MIT Press, Cambridge, MA.

Department of Philosophy
San Jose State University
San Jose, CA 95192-0096
USA
E-mail: RichardTieszen@aol.com