

Gödel's Ontological Proof of God's Existence (Draft)

Bruno Woltzenlogel Paleo, Annika Siders

August 18, 2013

“There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not.” - Kurt Gödel

1 Possible witnessing of positive properties

Axioms:

- **(A1)** Properties necessarily entailed by *positive* properties are also positive:

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

- **(A2)** A property's negation is positive iff the property is not positive:

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Lemma 1: Positive properties possibly have a witness:

$$\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$$

Formal proof:

$$\frac{\frac{\frac{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{\forall\psi.[(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \rightarrow \neg\varphi'(x)]) \rightarrow P(\neg\varphi')}}{\frac{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \rightarrow P(\neg\varphi')}{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \rightarrow \neg P(\varphi')}}}{\frac{P(\varphi') \rightarrow \Diamond\exists x.\varphi'(x)}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}$$

2 Possible existence of a God

Axioms:

- (A3) Being God is a positive property:

$$P(G)$$

Lemma 2: It is possible that a God exists:

$$\Diamond \exists x.G(x)$$

Formal proof:

$$\frac{P(G) \quad \frac{\frac{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x.G(x)]}{P(G) \rightarrow \Diamond \exists x.G(x)} \text{Th. 1}}{\Diamond \exists x.G(x)}$$

3 Essentiality of being God

Definitions:

- (D1) A property is *essential* for an individual if and only if it holds for that individual and necessarily entails every other property that holds for that individual: $\varphi \text{ ess } x \leftrightarrow \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$

Axioms:

- (A4) Positive properties are necessarily positive:

$$\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]$$

Lemma 3: If an individual is a God, then being God is an essential property for that individual:

$$\forall y. [G(y) \rightarrow G \text{ ess } y]$$

Formal proof:

Let the following derivation with the open assumption $G(x)$ be Π_1 :

$$\frac{\frac{\frac{\neg P(\psi)^1 \quad \frac{\frac{\forall \varphi. (\neg P(\varphi) \rightarrow P(\neg \varphi))}{\neg P(\psi) \rightarrow P(\neg \psi)} \text{Ax. 2}}{P(\neg \psi)} \text{VE}}{\neg \psi(x)} \text{VE} \quad \frac{\frac{\frac{G(x) \quad \dots \dots \dots}{\forall \varphi. (P(\varphi) \rightarrow \varphi(x))} \text{Definition of G}}{P(\varphi) \rightarrow \varphi(x)} \text{VE}}{\psi(x)^2} \text{VE}}{\psi(x) \rightarrow P(\psi)} \text{RAA, 1} \rightarrow \text{I, 2}$$

Let the following derivation with the open assumption $G(x)$ be Π_2 :

$$\frac{\frac{\psi(x)^1 \quad \frac{\Pi_1 \quad \overline{\psi(x)} \rightarrow \overline{P(\psi)}}{\overline{P(\psi)}} \rightarrow E \quad \frac{\overline{\forall \psi.(P(\psi) \rightarrow \Box P(\psi))} \text{ Ax. 4}}{\overline{P(\psi) \rightarrow \Box P(\psi)}} \forall E}{\Box P(\psi)} \rightarrow E \quad \frac{\Box P(\psi)}{\psi(x) \rightarrow \Box P(\psi)} \rightarrow I, 1$$

Let the following derivation without open assumptions be Π_3 :

$$\frac{\frac{\frac{P(\psi)^1 \quad \frac{\frac{G(x)^2 \quad \dots \text{ Definition of G}}{\forall \varphi.(P(\varphi) \rightarrow \varphi(x))} \forall E}{P(\psi) \rightarrow \psi(x)} \rightarrow E}{\psi(x)} \rightarrow I, 2 \quad \frac{G(x) \rightarrow \psi(x)}{\forall x.(G(x) \rightarrow \psi(x))} \forall I}{P(\psi) \rightarrow \forall x.(G(x) \rightarrow \psi(x))} \rightarrow I, 1$$

Let the following derivation with the open assumption $G(x)$ be Π_4 :

$$\frac{\frac{\psi(x)^1 \quad \frac{\Pi_2 \quad \overline{\psi(x)} \rightarrow \overline{\Box P(\psi)}}{\overline{\Box P(\psi)}} \rightarrow E \quad \frac{\frac{\frac{\Box P(\psi)^2}{P(\psi)} \Box E \quad \frac{\frac{\Pi_3 \quad \overline{P(\psi)} \rightarrow \overline{\forall x.(G(x) \rightarrow \psi(x))}}{\forall x.(G(x) \rightarrow \psi(x))} \rightarrow E}{\Box \forall x.(G(x) \rightarrow \psi(x))} \text{ Necessitation}}{\Box P(\psi) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow E}{\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow I, 1$$

The use of the necessitation rule is valid, because the only open assumption $\Box P(\psi)$ is boxed.

We construct a derivation of theorem 3 with a subderivation $\Pi_4[G(x)^1]$, which means that the open assumption $G(x)$ in Π_4 is discharged with the rule labeled 1.

$$\frac{\frac{G(x)^1 \quad \frac{\frac{\Pi_4[G(x)^1] \quad \overline{\psi(x)} \rightarrow \overline{\Box \forall x.(G(x) \rightarrow \psi(x))}}{\forall \psi.(\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x)))} \forall I}{G(x) \& \forall \psi.(\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x)))} \& I}{\dots \text{ Definition of ess } G \text{ ess } x} \rightarrow I, 1 \quad \frac{G(x) \rightarrow G \text{ ess } x}{\forall y.[G(y) \rightarrow G \text{ ess } y]}$$

4 Necessity of God's existence

Definitions:

- **(D2)** An individual is a *God* if and only if he possesses all positive properties:

$$G(x) \leftrightarrow \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

- **(D3)** An individual *necessarily exists* if and only if all its essential properties are necessarily witnessed:

$$E(x) \leftrightarrow \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists x. \varphi(x)]$$

Axioms:

- **(A5)** Necessary existence is a positive property:

$$P(E)$$

Auxiliary Lemma: If there is a God, then there necessarily exists a God:

$$\exists z. G(z) \rightarrow \Box \exists x. G(x)$$

Formal proof:

$$\frac{\frac{\frac{\overline{\exists z. G(z)}}{G(g)}^1 \quad \frac{\frac{\overline{\forall y. [G(y) \rightarrow G \text{ ess } y]}}{G(g) \rightarrow G \text{ ess } g} \quad \text{Th. 3} \quad \frac{\frac{\overline{P(E)} \quad \frac{\overline{\overline{\overline{G(g)}}} \quad \overline{\forall \varphi. [P(\varphi) \rightarrow \varphi(g)]}}{P(E) \rightarrow E(g)} \quad \frac{E(g)}{\overline{\forall \varphi. [\varphi \text{ ess } g \rightarrow \Box \exists x. \varphi(x)]}}}{G \text{ ess } g \rightarrow \Box \exists x. G(x)} \quad \frac{\Box \exists x. G(x)}{\exists z. G(z) \rightarrow \Box \exists x. G(x)}^1}{G \text{ ess } g}$$

5 Necessary existence of a God

Theorem: The existence of a God is necessary:

$$\Box \exists x. G(x)$$

Formal proof:

$$\frac{\frac{\overline{\forall \varphi. [\Diamond \dots \Diamond \Box \varphi \leftrightarrow \Box \varphi]}}{\Diamond \Box \exists x. G(x) \leftrightarrow \Box \exists x. G(x)} \quad \text{S5} \quad \frac{\overline{\Diamond \exists x. G(x)}}{\Diamond \Box \exists x. G(x)} \quad \text{Th. 2} \quad \frac{\overline{\exists z. G(z) \rightarrow \Box \exists x. G(x)}}{\Diamond \Box \exists x. G(x)} \quad \text{Th. A}}{\Box \exists x. G(x)}$$

6 God's existence

Axioms:

- (M) What is necessary is the case:

$$\forall \varphi. [\Box \varphi \rightarrow \varphi]$$

Corollary: There exists a God:

$$\exists x. G(x)$$

Formal proof:

$$\frac{\frac{\Box \neg \exists x. G(x) \quad \text{Th. 4} \quad \frac{\forall \varphi. [\Box \varphi \rightarrow \varphi]}{\Box \exists x. G(x) \rightarrow \exists x. G(x)}}{\exists x. G(x)}}$$