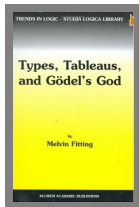


Gödel's Proof of God's Existence

Christoph Benz Müller and Bruno Woltzenlogel Paleo

Square of Opposition
Vatican, May 6, 2014



$$\frac{\text{Axiom 3} \quad \frac{\overline{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}}{P(G) \rightarrow \Diamond \exists x. G(x)} \forall_E}{\Diamond \exists x. G(x)} \rightarrow_E$$

A gift to **Priest Edvaldo** in Piracicaba, Brazil

First time mechanization and automation of

- (variants of) a modern ontological argument
- (variants of) higher-order modal logic

Work context/history:

- **Proposal:** exploit classical higher-order logic (HOL) as universal meta-logic — cf. previous talks at UNILog
 - for object-level reasoning (in embedded non-classical logics)
 - for meta-level reasoning (about embedded non-classical logics)
- **Proof of concept:** demonstrate practical relevance of the approach by an interesting and relevant application
- **Experiments:** systematic study of Gödel's argument
- **Relation to Square of Opposition:** should be easy to analyze variants of the Square within our approach

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

What we did:

A: Pen and paper: detailed natural deduction proof

B: Formalization: in classical higher-order logic (HOL)

Automation: theorem provers LEO-II(E) and SATALLAX

Consistency: model finder NITPICK (NITROX)

C: Step-by-step verification: proof assistant Coq

D: Automation & verification: proof assistant ISABELLE

Did we get any new results?

Yes — let's discuss this later!



Germany

- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

Austria

- Die Presse
- Wiener Zeitung
- ORF
- ...

Italy

- Repubblica
- L'Espresso
- ...

India

- DNA India
- Delhi Daily News
- India Today
- ...

US

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- ...

International

- Spiegel International
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SCIENCE NEWS

HOME / SCIENCE NEWS / RESEARCHERS SAY THEY USED MACBOOK TO PROVE GOEDEL'S GOD THEOREM

Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | [1 comments](#)

Do you really need a MacBook to obtain the results?

No

Did Apple send us some money?

No

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Rich history on ontological arguments (pros and cons)

... Anselm v. C.
Gaunilo Th. Aquinas Descartes
Spinoza Leibniz Hume
Kant Hegel Frege Hartshorne
Malcolm Lewis Plantinga
Gödel ...

Anselm's notion of God:

"God is that, than which nothing greater can be conceived."

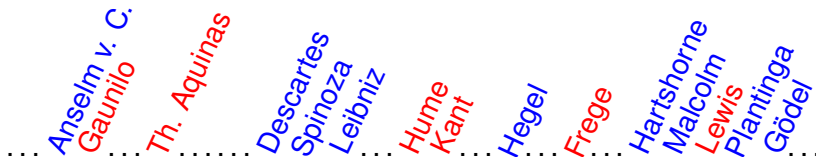
Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"(Necessarily) God exists."

Rich history on ontological arguments (pros and cons)



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Different Interests in Ontological Arguments:

- **Philosophical:** Boundaries of Metaphysics & Epistemology
 - We talk about a metaphysical concept (God),
 - but we want to draw a conclusion for the real world.
- **Theistic:** Successful argument should convince atheists
- **Ours:** Can computers (theorem provers) be used ...
 - ...to formalize the definitions, axioms and theorems?
 - ...to verify the arguments step-by-step?
 - ...to fully automate (sub-)arguments?

Towards: *“Computer-assisted Theoretical Philosophy”*

(cf. Leibniz dictum — Calculemus!)

Gödel's Manuscript: 1930's, 1941, 1946-1955, 1970

Ontologische Beweise

Feb 10, 1970

$P(\varphi)$ φ is positive ($\varphi \in P$)

At 1 $P(\varphi) \cdot P(\psi) \supset P(\varphi \cdot \psi)$ At 2 $P(\varphi) \cdot \neg P(\neg \varphi)$

P1 $G(x) \equiv (\varphi) [P(\varphi) \supset \varphi(x)]$ (Good)

P2 $\varphi \text{ Ess } x \equiv (\psi) [\psi(x) \supset N(\psi) \cdot (\varphi \supset \psi)]$ (Essence of x)

$P \supset Nq = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$
 $\neg P(\varphi) \supset N \neg P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess } x$

Df. $E(x) \equiv (\varphi) [\varphi \text{ Ess } x \supset N \exists x \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists x) G(y)$

hence $(\exists x) G(x) \supset N(\exists x) G(y)$

" $M(\exists x) G(x) \supset M N(\exists x) G(y)$

" $\supset N(\exists x) G(y)$

$M = possibility$

any two sentences of x are nec. equivalent

exclusive or * and for any number of humanoids

$M(\exists x) G(x)$ means ^{the system of} all pos. props. is compatible
 This is true because of:

At 4: $P(\varphi) \cdot \varphi \supset N \psi \supset P(\psi)$ which impl

~~then~~ $\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. props. were inconsistent it would mean that the same prop. A (which is positive) would be $x \neq x$

Positive means positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. It also means "attribution" as opposed to "privation" (or containing privation). This interprets $x \neq x$ as

$\exists x \varphi$ privation: $(x) N \neg \varphi(x)$ - otherwise $\varphi(x) \supset N x \neq x$

hence $x \neq x$ positive prop. $x=x$ neg. property At 4

or the equiv. of pos. prop. $\neg \neg x$

x i.e. the normal form in terms of elem. prop. contains a member without negation.

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall\phi[P(\neg\phi) \equiv \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive:
 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \supset \psi(x)]) \supset P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall\phi[P(\phi) \supset \Diamond\exists x\phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \equiv \forall\phi[P(\phi) \supset \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond\exists xG(x)$

Axiom A4 Positive properties are necessarily positive: $\forall\phi[P(\phi) \supset \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \text{ ess. } x \equiv \phi(x) \wedge \forall\psi(\psi(x) \supset \Box\forall y(\phi(y) \supset \psi(y)))$

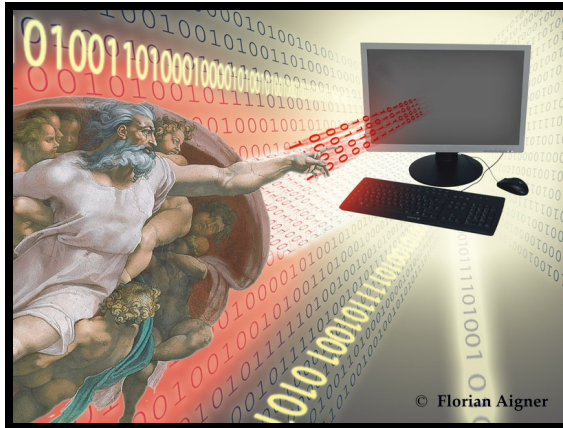
Thm. T2 Being God-like is an essence of any God-like being: $\forall x[G(x) \supset G \text{ ess. } x]$

Def. D3 *Necessary existence* of an individ. is the necessary exemplification of all its essences: $NE(x) \equiv \forall\phi[\phi \text{ ess. } x \supset \Box\exists y\phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box\exists xG(x)$

- Embedding of QML in HOL and Proof Automation (myself)
- Proof Overview (Bruno)
- Experiments and Results (Bruno)
- Conclusion and Outlook (Bruno)



Embedding of **QML** in **HOL** and Proof Automation

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

Then use existing HOL theorem provers for reasoning in QML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[BenzmüllerRaths, LPAR, 2013]

$$\text{QML} \quad \varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \supset \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\varphi \mid \exists x\varphi \mid \forall P\varphi$$

- Kripke style semantics (possible world semantics)

$$\text{HOL} \quad s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$$

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exist

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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HOL $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

QML in **HOL**: **QML** formulas φ are mapped to **HOL** predicates $\varphi_{t \rightarrow o}$

\neg	=	$\lambda\varphi_{t \rightarrow o} \lambda s_t \neg\varphi s$	Ax
\wedge	=	$\lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\varphi s \wedge \psi s)$	
\supset	=	$\lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\neg\varphi s \vee \psi s)$	
\Box	=	$\lambda\varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg r s u \vee \varphi u)$	
\Diamond	=	$\lambda\varphi_{t \rightarrow o} \lambda s_t \exists u_t (r s u \wedge \varphi u)$	
\forall	=	$\lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu h d s$	
\exists	=	$\lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_\mu h d s$	
\forall	=	$\lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu H d s$	
valid	=	$\lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$	

The equations in **Ax** are given as axioms to the **HOL** provers!

(Remark: Note that we are here dealing with constant domain quantification)

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Example:

QML formula

QML formula in **HOL**

expansion, $\beta\eta$ -conversion

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expansion, $\beta\eta$ -conversion

$\Diamond \exists x G(x)$

valid $(\Diamond \exists x G(x))_{t \rightarrow o}$

$\forall w_t (\Diamond \exists x G(x))_{t \rightarrow o} w$

$\forall w_t \exists u_t (rwu \wedge (\exists x G(x))_{t \rightarrow o} u)$

$\forall w_t \exists u_t (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in **QML**,

\rightarrow we instead prove that valid $\varphi_{t \rightarrow o}$ can be derived from **Ax** in **HOL**.

This can be done with interactive or automated **HOL** theorem provers.

Soundness and Completeness:

wrt. Henkin semantics

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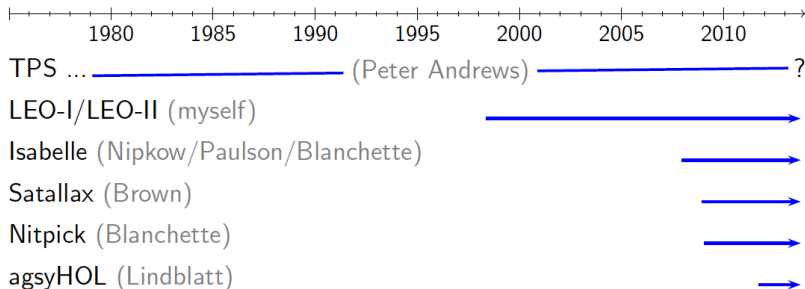
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Automated Theorem Provers and Model Finders for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
 - can be called remotely via SystemOnTPTP at Miami
 - they significantly gained in strength over the last years
 - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— **HOL-P** becomes a **Universal Reasoner** —



Proof Overview

Experiments and Results

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C1: $\Diamond \exists z. G(z)$

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$$\frac{\mathbf{C1:} \Diamond \exists z. G(z) \quad \mathbf{L2:} \Diamond \exists z. G(z) \supset \Box \exists x. G(x)}{\mathbf{T3:} \Box \exists x. G(x)}$$

L2: $\Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

C1: $\Diamond \exists z. G(z)$

L2: $\Diamond \exists z. G(z) \supset \Box \exists x. G(x)$

T3: $\Box \exists x. G(x)$

$$\begin{array}{c}
 \text{S5} \\
 \hline
 \forall \bar{x}. [\Diamond \bar{\Box x} \supset \bar{\Box x}] \\
 \hline
 \text{L2: } \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \hline
 \text{C1: } \Diamond \exists z. G(z) \quad \text{L2: } \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \hline
 \text{T3: } \Box \exists x. G(x)
 \end{array}$$

$$\begin{array}{c}
 \diamond \exists z. G(z) \supset \diamond \Box \exists x. G(x) \qquad \text{S5} \quad \overline{\forall \xi. [\diamond \Box \xi \supset \Box \xi]} \\
 \hline
 \text{L2: } \diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \\
 \text{C1: } \diamond \exists z. G(z) \qquad \text{L2: } \diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \hline
 \text{T3: } \Box \exists x. G(x)
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$$\begin{array}{c}
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 \hline
 \Diamond \exists z.G(z) \supset \Diamond \Box \exists x.G(x)
 \end{array}
 \qquad
 \begin{array}{c}
 \text{S5} \\
 \hline
 \forall \xi. [\Diamond \Box \xi \supset \Box \xi]
 \end{array}$$

$$\text{L2: } \Diamond \exists z.G(z) \supset \Box \exists x.G(x)$$

$$\begin{array}{c}
 \text{C1: } \Diamond \exists z.G(z) \qquad \text{L2: } \Diamond \exists z.G(z) \supset \Box \exists x.G(x) \\
 \hline
 \text{T3: } \Box \exists x.G(x)
 \end{array}$$

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$$

$$\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)$$

$$\frac{\Box \exists z. G(z) \supset \Box \Box \exists x. G(x)}{\Box \exists z. G(z) \supset \Box \exists x. G(x)}$$

S5

$$\forall \xi. [\Diamond \Box \xi \supset \Box \xi]$$

$$\mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)$$

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$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y] \qquad P(E) \\
 \hline
 \mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x) \\
 \hline
 \Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x) \qquad \mathbf{S5} \quad \forall \xi. [\Diamond \Box \xi \supset \Box \xi] \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \\
 \mathbf{C1:} \ \Diamond \exists z. G(z) \qquad \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
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 \end{array}$$

$$\mathbf{D1: } G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$$

$$\mathbf{D3: } E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \mathbf{T2: } \forall y. [G(y) \supset G \text{ ess. } y] \qquad \mathbf{A5} \\
 \hline
 \mathbf{L1: } \exists z. G(z) \supset \Box \exists x. G(x) \qquad \overline{P(E)} \\
 \hline
 \Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x) \qquad \mathbf{S5} \\
 \hline
 \mathbf{L2: } \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \qquad \overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]} \\
 \hline
 \mathbf{C1: } \Diamond \exists z. G(z) \qquad \mathbf{L2: } \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \hline
 \mathbf{T3: } \Box \exists x. G(x)
 \end{array}$$

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y] \qquad \mathbf{A5} \\
 \hline
 \mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x) \qquad \overline{P(E)} \\
 \hline
 \Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x) \qquad \mathbf{S5} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \qquad \overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]} \\
 \hline
 \mathbf{C1:} \ \Diamond \exists z. G(z) \qquad \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
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 \end{array}$$

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \supset \varphi(x)]$$

$$\mathbf{D2:} \ \varphi \text{ ess. } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \supset \Box \forall x. (\varphi(x) \supset \psi(x)))$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \frac{\overline{\forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)]} \quad \overline{\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]}}{\mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y]} \quad \frac{\mathbf{A5}}{P(E)} \\
 \frac{\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)}{\diamond \exists z. G(z) \supset \diamond \Box \exists x. G(x)} \quad \frac{\mathbf{S5}}{\overline{\forall \xi. [\diamond \Box \xi \supset \Box \xi]}} \\
 \mathbf{L2:} \ \diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \frac{\mathbf{C1:} \ \diamond \exists z. G(z) \quad \mathbf{L2:} \ \diamond \exists z. G(z) \supset \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
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$$\mathbf{C1:} \ \Diamond \exists z. G(z)$$

$$\begin{array}{c}
 \frac{\overline{\forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)]} \quad \overline{\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]}}{\mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y]} \quad \frac{\mathbf{A5}}{P(E)} \\
 \frac{\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)}{\Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x)} \quad \frac{\mathbf{S5}}{\overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]}} \\
 \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
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$$P(G)$$

$$\mathbf{C1:} \ \Diamond \exists z. G(z)$$

$$\frac{\overline{\forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)]}}{\mathbf{A1b}}$$

$$\frac{\overline{\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]}}{\mathbf{A4}}$$

$$\frac{\overline{P(E)}}{\mathbf{A5}}$$

$$\mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y]$$

$$\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)$$

$$\Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x)$$

$$\frac{\overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]}}{\mathbf{S5}}$$

$$\mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)$$

$$\mathbf{C1:} \ \Diamond \exists z. G(z)$$

$$\mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)$$

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$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess. } x \supset \Box \exists y. \varphi(y)]$$

$$\frac{\mathbf{A3}}{\overline{P(G)}}$$

$$\mathbf{C1:} \ \Diamond \exists z. G(z)$$

$$\frac{\mathbf{A1b}}{\overline{\forall \varphi. [\neg P(\varphi) \supset \overline{P(\neg \varphi)}]}}$$

$$\frac{\mathbf{A4}}{\overline{\forall \varphi. [P(\varphi) \rightarrow \Box \overline{P(\varphi)}]}}$$

$$\frac{\mathbf{A5}}{\overline{P(E)}}$$

$$\mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y]$$

$$\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)$$

$$\Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x)$$

$$\frac{\mathbf{S5}}{\overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]}}$$

$$\mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)$$

$$\mathbf{C1:} \ \Diamond \exists z. G(z)$$

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$$\begin{array}{c}
 \frac{\mathbf{A3} \quad \overline{P(\overline{G})}}{\mathbf{T1:} \ \forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)]} \\
 \hline
 \mathbf{C1:} \ \Diamond \exists z. G(z) \\
 \\
 \frac{\frac{\mathbf{A1b} \quad \overline{\forall \varphi. [\neg P(\varphi) \supset P(\neg \varphi)]}}{\mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y]} \quad \frac{\mathbf{A4} \quad \overline{\forall \varphi. [P(\varphi) \rightarrow \Box \overline{P(\varphi)}]}}{\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)} \quad \frac{\mathbf{A5} \quad \overline{P(E)}}{\mathbf{S5} \quad \overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]}} \\
 \hline
 \frac{\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x) \quad \mathbf{S5} \quad \overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]}}{\mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)} \\
 \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

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$$\begin{array}{c}
 \frac{\frac{\mathbf{A3}}{\overline{P(\bar{G})}} \quad \frac{\frac{\mathbf{A2}}{\overline{\forall \varphi. \forall \psi. [(P(\bar{\varphi}) \wedge \Box \forall x. [\varphi(x) \supset \psi(x)]) \supset P(\bar{\psi})]} \quad \frac{\mathbf{A1a}}{\overline{\forall \varphi. [P(\bar{\neg \varphi}) \supset \neg P(\bar{\varphi})]}}}{\mathbf{T1:} \ \forall \varphi. [P(\varphi) \supset \Diamond \exists x. \varphi(x)]} \\
 \hline
 \mathbf{C1:} \ \Diamond \exists z. G(z) \\
 \\
 \frac{\frac{\mathbf{A1b}}{\overline{\forall \varphi. [\neg P(\bar{\varphi}) \supset P(\bar{\neg \varphi})]}} \quad \frac{\mathbf{A4}}{\overline{\forall \varphi. [P(\bar{\varphi}) \rightarrow \Box P(\bar{\varphi})]}} \quad \frac{\mathbf{A5}}{\overline{P(\bar{E})}}}{\mathbf{T2:} \ \forall y. [G(y) \supset G \text{ ess. } y]} \\
 \hline
 \frac{\mathbf{L1:} \ \exists z. G(z) \supset \Box \exists x. G(x)}{\Diamond \exists z. G(z) \supset \Diamond \Box \exists x. G(x)} \\
 \hline
 \frac{\mathbf{S5}}{\overline{\forall \xi. [\Diamond \Box \xi \supset \Box \xi]}} \\
 \hline
 \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x) \\
 \\
 \frac{\mathbf{C1:} \ \Diamond \exists z. G(z) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \supset \Box \exists x. G(x)}{\mathbf{T3:} \ \Box \exists x. G(x)}
 \end{array}$$

$$\mathbf{D1:} \ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

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$$\begin{array}{c}
 \begin{array}{c}
 \mathbf{A3} \\
 \hline \hline \overline{P(\bar{G})}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A2} \\
 \hline \hline \overline{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \supset \psi(x)]) \supset P(\psi)]}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A1a} \\
 \hline \hline \overline{\forall\varphi.[P(\neg\varphi) \supset \neg P(\varphi)]}
 \end{array}
 \\
 \hline \hline
 \mathbf{T1:} \ \forall\varphi.[P(\varphi) \supset \Diamond\exists x.\varphi(x)]
 \\
 \hline \hline
 \mathbf{C1:} \ \Diamond\exists z.G(z)
 \\
 \\
 \begin{array}{c}
 \mathbf{A1b} \\
 \hline \hline \overline{\forall\varphi.[\neg P(\varphi) \supset P(\neg\varphi)]}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A4} \\
 \hline \hline \overline{\forall\varphi.[P(\varphi) \rightarrow \Box P(\varphi)]}
 \end{array}
 \quad
 \begin{array}{c}
 \mathbf{A5} \\
 \hline \hline \overline{P(E)}
 \end{array}
 \\
 \hline \hline
 \mathbf{T2:} \ \forall y.[G(y) \supset G \text{ ess. } y]
 \\
 \hline \hline
 \mathbf{L1:} \ \exists z.G(z) \supset \Box\exists x.G(x)
 \\
 \hline \hline
 \Diamond\exists z.G(z) \supset \Diamond\Box\exists x.G(x)
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 \begin{array}{c}
 \mathbf{S5} \\
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 \end{array}
 \\
 \hline \hline
 \mathbf{L2:} \ \Diamond\exists z.G(z) \supset \Box\exists x.G(x)
 \\
 \\
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 \\
 \hline \hline
 \mathbf{T3:} \ \Box\exists x.G(x)
 \end{array}$$

$$\frac{A \vee B \quad \begin{array}{c} \overline{A} \\ \vdots \\ C \end{array} \quad \begin{array}{c} \overline{B} \\ \vdots \\ C \end{array}}{C} \vee_E$$

$$\frac{A \quad B}{A \wedge B} \wedge_I$$

$$\frac{\begin{array}{c} \overline{A}^n \\ \vdots \\ B \end{array}}{A \supset B} \supset_I^n$$

$$\frac{A}{A \vee B} \vee_{I_1}$$

$$\frac{A \wedge B}{A} \wedge_{E_1}$$

$$\frac{B}{A \supset B} \supset_I$$

$$\frac{B}{A \vee B} \vee_{I_2}$$

$$\frac{A \wedge B}{B} \wedge_{E_2}$$

$$\frac{A \quad A \supset B}{B} \supset_E$$

$$\frac{A[\alpha]}{\forall x.A[x]} \forall_I$$

$$\frac{\forall x.A[x]}{A[t]} \forall_E$$

$$\frac{A[t]}{\exists x.A[x]} \exists_I$$

$$\frac{\exists x.A[x]}{A[\beta]} \exists_E$$

$$\neg A \equiv A \supset \perp$$

$$\frac{\neg\neg A}{A} \neg\neg_E$$

$$\frac{\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Box A} \Box_I$$

$$\frac{\Box A}{t : \boxed{\begin{array}{c} A \\ \vdots \end{array}}} \Box_E$$

$$\frac{t : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Diamond A} \Diamond_I$$

$$\frac{\Diamond A}{\beta : \boxed{\begin{array}{c} A \\ \vdots \end{array}}} \Diamond_E$$

$$\Diamond A \equiv \neg \Box \neg A$$

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Natural Deduction Proofs

T2 (Partial)

$$\begin{array}{c}
 \frac{\psi(x)^6 \quad \frac{\psi(x) \rightarrow \Box P(\psi)}{\Box P(\psi)} \Pi_2}{\Box P(\psi)} \rightarrow E \\
 \frac{\Box P(\psi) \quad \frac{\frac{\Box P(\psi)^7 \quad \frac{P(\psi) \rightarrow \forall x.(G(x) \rightarrow \psi(x))}{\forall x.(G(x) \rightarrow \psi(x))} \Pi_3}{\Box P(\psi) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow E}{\Box \forall x.(G(x) \rightarrow \psi(x))} \Box I \\
 \frac{\Box \forall x.(G(x) \rightarrow \psi(x))}{\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow I^6 \\
 \frac{\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))}{\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow I^7
 \end{array}$$

- Formal encoding(s) of the axioms, definitions, and theorems in Scott's proof script
- Calls to the HOL reasoners mentioned before
- Interactive proofs in proof assistants Isabelle and Coq

The source files of these experiments are available at:

<https://github.com/FormalTheology/GoedelGod/>

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Addresses criticisms: modal logic S5 is too strong

$$\forall P. [\Diamond \Box P \supset \Box P]$$

If something is possibly necessary, then it is necessary.

S5 usually considered adequate

(But KB is sufficient! — shown by HOL ATPs)

- Axioms and definitions are consistent
- Logic K is sufficient for proving T1, C and T2.
- For proving the final theorem T3, logic KB is sufficient
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Fundamental criticism against Gödel's argument.

Everything that is the case is so necessarily.

Follows from T2, T3 and D2 (as shown by HOL ATPs).

There are no contingent “truths”.

Everything is determined.

There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, ...

- Gödel's axioms imply the *modal collapse*: $\phi \supset \Box\phi$
- For proving T1, only the \supset -direction of A1 is needed. Some proposals try to avoid modal collapse by replacing the \supset -direction of A1. However, the \subset -direction of A1 is required for proving T2.

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- Gödel's axioms imply a 'flawless god', that is, an entity that can only have 'positive' properties.
- Another implication of Gödel's axioms is monotheism.
- All of the above findings hold for both
 - constant domain semantics and
 - varying domain semantics (for the domain of individuals).



Conclusions

Achievements:

- Verification of Gödel's ontological argument with HOL provers
 - exact parameters known: constant domain quantification, Henkin Semantics
 - experiments with different parameters could be performed
- Gained some novel results and insights
- Major step towards **Computer-assisted Theoretical Philosophy**
 - see also Ed Zalta's *Computational Metaphysics* project at Stanford University
 - see also John Rushby's recent verification of Anselm's proof in PVS
 - remember Leibniz' dictum — *Calculemus!*
- Interesting bridge between CS, Philosophy and Theology

Ongoing and future work

- Formalize and verify literature on ontological arguments
 - ... in particular the criticisms and proposed improvements
- Own contributions — supported by theorem provers

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- Interesting bridge between CS, Philosophy and Theology

Ongoing and future work

- Formalize and verify literature on ontological arguments
 - ... in particular the criticisms and proposed improvements
- Own contributions — supported by theorem provers