# A completeness theorem in second order modal logic

by

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In what follows we present a second order formulation of S5 which is shown to be complete relative to a secondary sense of validity corresponding to that relative to which standard second order logic is known to be complete.[1] In our semantical metalanguage we consider various indexed sets of possible worlds and allow that not all objects existing in one indexed world need exist in another. However, as we have therefore confessed in the metalanguage our ontological commitment to all the objects that exist in one world or another, we acknowledge and formalize this confession in our object languages through allowing for quantification over *possibilia.*

Our means for distinguishing the existent from the mere *possibile* is through a distinction between those attributes that entail existence (with respect to each of their argument places), referred to hereafter as *e-attributes,* and those attributes that do not.[2] Accordingly, we understand 'x exists' to mean 'There is some e-attribute which x possesses', thus rendering existence essentially impredicative. An alternative and equivalent route—but which we shall not follow here—is possible through taking existence as primitive in the form of quantification over existing objects and defining e-attributes as those attributes which *necessarily* are possessed only by existing objects.

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[1] Cf. Henkin [9].

[2] For a modal free complete (in a secondary sense) formulation of second order logic with existence attributes, see Cocchiarella [5]. For a more philosophical discussion of this approach toward existence, see Cocchiarella [6] and [7].

The particular axiom set we present in §3 below is applicable under obvious changes to other modal systems besides S5, although, of course, the semantics must be appropriately modified in well-known ways, i.e., through consideration of accessibility relations between the indices of world systems as defined below.[3] E.g., where the S5 modal axioms (Group II) are replaced by those of S4 or any of the systems between S4 and S5, lemma 21 alone of those listed is no longer provable and when added to the axioms the resulting set yields a completeness theorem for each of these modal logics.[4] In regard to second order tense logic, where $\Box$ is definable in terms of the tense operators as meaning "at all times", we need only replace the S5 modal axioms by those of tense logic—within which the S5 axioms are provable—in order to obtain a completeness theorem for second order tense logic.[5] Incidentally, a significant feature of the characterization of the axiom set is that it in no way requires the notion of proper substitution of either a term for an individual variable or of a formula for a predicate variable.[6]

[3] Cf. Kripke [8] and Hintikka [10].

[4] The proof of completeness is similar to that given for Theorem 2 with the supplementation of the following definition of accessibility between the maximally consistent sets (of the respective systems) in K: Where $K, K' \in K$, K is accessible from K' iff for each formula $\varphi$, if $\Box\varphi \in K'$, then $\varphi \in K$. For systems below S5, the statement (I) in the proof of Theorem 2 no longer holds but it plays only a simplifying rather than an essential role as the claims made in each of its specific uses can be justified on other grounds—given certain minor modifications in the construction of $[\eta]_K$ and $Ind_K$, specifically, modifications so as to guarantee that the individual constants used to determine the *possibilia* at any one index determine the same *possibilia* as any other index. In particular, redefining $[\eta]_K$ and $Ind_K$ as $\{\zeta_i : i \in \omega$ and $\zeta_i = \eta \in K$ and $\bigvee \alpha \Box \alpha = \zeta_i \in K^*$, for some individual variable $\alpha\}$ and $\{[\zeta_i]_{K^*} : i \in \omega\}$, respectively, suffices.

[5] Cf. Cocchiarella [2] or the abstract [3] for an axiomatization of tense logic. Cocchiarella [4] indicates complete axiom sets for this and several other temporal interpretations of $\Box$.

[6] It is even possible to formulate the axioms so that the notion of freedom and bondage of a variable is replaced by that of an occurrence *simpliciter* of one expression within another. However, it is not the notion of freedom and bondage which is troublesome in modal logic but rather that of proper substitution, and, accordingly, it is appropriate that the latter not be used to characterize logical axioms since otherwise those characterizations may need to be

## §1. *Terminology*

We take $\rightarrow$, the conditional sign, $\wedge$, the universal quantifier, $\wedge^e$, the universal e-quantifier, and $\Box$ as our only primitive logical constants. All other logical constants are assumed to be defined in the usual way. By a language we understand a set of predicate and operation constants of arbitrary many places. The terms and atomic formulas of a language are assumed to be defined in the usual manner. The set of formulas of a language L is the intersection of all those sets K such that each atomic formula of L is in K and $\Box\varphi$, $(\varphi \rightarrow \psi)$, $\wedge\mu\varphi$, $\overset{e}{\wedge}\mu\varphi$ are in K whenever $\varphi$, $\psi$ are in K and $\mu$ is an individual or a predicate variable. By a *modal free* formula we mean one in which $\Box$ does not occur.

We use '$\alpha$' and '$\beta$' to refer to individual variables, '$\pi$', '$\rho$' and '$\sigma$' to refer to predicate expressions (i.e., predicate variables or predicate constants), '$\varphi$', '$\psi$' and '$\chi$' to refer to formulas, and '$\zeta$' and '$\eta$' to refer to terms. We shall use '$\varphi(\alpha_0, \ldots, \alpha_{n-1})$' in place of '$\varphi$' when we wish to indicate that $\alpha_0, \ldots, \alpha_{n-1}$ are all the distinct individual variables occurring free in $\varphi$.

We say that $\psi$ is a (modal) *generalization* of $\varphi$ if $\psi$ is the result of prefixing $\varphi$ with a string (possibly empty) of expressions, each of which is either $\Box$, $\wedge\mu$ or $\overset{e}{\wedge}\mu$, where $\mu$ is a predicate or individual variable. We take $\varphi[{}^{\alpha}_{\zeta}]$ to be the result of properly substituting $\zeta$ for $\alpha$ in $\varphi$ if $\zeta$ can be so substituted for $\alpha$; otherwise $\varphi[{}^{\alpha}_{\zeta}]$ is $\varphi$ itself. In regard to the notion of proper substitution of a formula $\psi$ for a predicate expression $\pi$ in a formula $\varphi$, we utilize the notation

$$\varphi\left[{}^{\pi(a_0, \ldots, a_{n-1})}_{\psi}\right]$$

in place of Church's notation

$$\check{S}\, {}^{\pi(a_0, \ldots, a_{n-1})}_{\psi}\varphi\Big|$$

(assuming, of course, that Church's definition has been amended so as to accord with the fact that a predicate or individual variable may be bound by $\wedge^e$ as well as by $\wedge$).[7] Where $\sigma$ is a predicate

seriously qualified in further extensions of the notion of formulahood (Cf. Cocchiarella [6], §1).

[7] Cf. Church [1], p. 192 f. Aside from this emendation, the notion of proper sub-

expression of the same number of places as $\pi$, we understand $\varphi\left[\begin{smallmatrix}\pi\\ \sigma\end{smallmatrix}\right]$ to be $\varphi\left[\begin{smallmatrix}\pi(a_0, \ldots, a_{n-1})\\ \sigma(a_0, \ldots, a_{n-1})\end{smallmatrix}\right]$.

For convenience, we shall take '$\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\varphi$' as an abbreviation of '$\Box\bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}[\pi(\alpha_0, \ldots, \alpha_{n-1}) \leftrightarrow \varphi]$', where it is always understood that $\alpha_0, \ldots, \alpha_{n-1}$ are pairwise distinct individual variables. Where $\pi$ and $\sigma$ are both n-place predicate expressions, we shall use '$\pi = \sigma$' instead of '$\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\sigma(\alpha_0, \ldots, \alpha_{n-1})$'.[8] This notation suggests, of course, that attribute identity be construed as necessary co-extensiveness (with respect to all *possibilia*).[9] Propositional identity, then, is to be understood as necessary material equivalence.[10]

## §2. *Semantics*

Where L is a language, we say that $\mathfrak{A}$ is an *L-model* if there are A, B, R such that $\mathfrak{A} = \langle A, B, R\rangle$, A, the set of objects existing in $\mathfrak{A}$, is included in B, the set of *possibilia* of $\mathfrak{A}$, i.e., $A \subseteq B$, B is not empty and R is a function with L as domain and such that for all natural numbers n: (1) for all n-place predicate constants $\pi$ in L, $R(\pi) \subseteq B^n$, and (2) for all n-place operation constants $\delta$ in L, $R(\delta) \in B^{B^n}$, Where $\mathfrak{A} = \langle A, B, R\rangle$, we set $\mathcal{U}_{\mathfrak{A}} = A$ and $\mathcal{D}_{\mathfrak{A}} = B$. In addition, if $\theta$ is a constant in the language of $\mathfrak{A}$, we set $\theta_{\mathfrak{A}} = R(\theta)$.

We shall say that the I-indexed family $\langle\mathfrak{A}_i\rangle_{i\in I}$ of L-models is a *world system* for L if for all $i, j \in I$, (1) $\mathcal{D}_{\mathfrak{A}_i} = \mathcal{D}_{\mathfrak{A}_j}$ and (2) $\bigcup_{k\in I} \mathcal{U}_{\mathfrak{A}_k} \subseteq \mathcal{D}_{\mathfrak{A}_i}$. We speak of the members of I as being the *indices* or *reference points* of the world system $\langle\mathfrak{A}_i\rangle_{i\in I}$.

Where $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a world system, we say that X is an n-ary *attribute* in $\langle\mathfrak{A}_i\rangle_{i\in I}$ if X is a function with I as domain and such that for all $i \in I$, $X_i \subseteq \mathcal{D}^n_{\mathfrak{A}_i}$. In addition, we shall say that Y is an

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stitution (of terms for individual variables and formulas for predicate variables) is the usual one for standard second order logic.

[8] We also use '=' for identity in the metalanguage as well. However, it will be clear in each context which use is intended.

[9] Cf. Montague [11].

[10] We understand properties to be 1-place attributes and propositions to be 0-place attributes (whose extensions are truth values).

n-ary *e-attribute* in $\langle\mathfrak{A}_i\rangle_{i\in I}$ if Y is an n-ary attribute in $\langle\mathfrak{A}_i\rangle_{i\in I}$ such that for all $i \in I$, $Y_i \subseteq \mathcal{U}^n_{\mathfrak{A}_i}$.

We understand $\langle\langle\mathfrak{A}_i\rangle_{i\in I}, \langle F_n\rangle_{n\in\omega}, \langle E_n\rangle_{n\in\omega}\rangle$ to be a *secondary world system* for L if $\langle\mathfrak{A}_i\rangle_{i\in I}$ is a world system for L and for each $n \in \omega$, (1) every member of $F_n$ is an n-ary attribute in $\langle\mathfrak{A}_i\rangle_{i\in I}$, (2) every member of $E_n$ is an n-ary e-attribute in $\langle\mathfrak{A}_i\rangle_{i\in I}$, and (3) $E_n \subseteq F_n$.

By an *assignment* (of values to variables) in a secondary world system such as defined above, we mean a function $\mathfrak{a}$ with the set of predicate and individual variables as domain and such that (1) for each individual variable $\alpha$, $\mathfrak{a}(\alpha) \in \mathcal{D}_{\mathfrak{A}_i}$, for some $i \in I$, and (2) for each $n \in \omega$ and each n-place predicate variable $\pi$, $\mathfrak{a}(\pi) \in F_n$. Where $\mathfrak{a}$ is an assignment and $\mu$ is a predicate or individual variable, we take $\mathfrak{a}(^{\mu}_{x})$ to be that funtion which is exactly like $\mathfrak{a}$ except in its assigning x to $\mu$.

The *extension* of a term or predicate expression is relativized not only to secondary world systems but also to the indices and assignments (of values to variables that may occur free in the term or predicate expression) in such world systems. If $\mathfrak{B} = \langle\langle\mathfrak{A}_i\rangle_{i\in I}, \langle F_n\rangle_{n\in\omega}, \langle E_n\rangle_{n\in\omega}\rangle$ is a secondary world system for L, $\mathfrak{a}$ is an assignment in $\mathfrak{B}$, and $i \in I$, then (1) if $\alpha$ is an individual variable, $\mathrm{ext}(\alpha, \mathfrak{B}, i, \mathfrak{a}) = \mathfrak{a}(\alpha)$, (2) if $\pi$ is a predicate variable, $\mathrm{ext}(\pi, \mathfrak{B}, i, \mathfrak{a}) = \mathfrak{a}(\pi)_i$, (3) if $\pi$ is a predicate constant in L, $\mathrm{ext}(\pi, \mathfrak{B}, i, \mathfrak{a}) = \pi_{\mathfrak{A}_i}$, and (4) if $\delta$ is an n-place operation constant in L and $\zeta_0, \ldots, \zeta_{n-1}$ are terms of L, then $\mathrm{ext}(\delta(\zeta_0, \ldots, \zeta_{n-1}), \mathfrak{B}, i, \mathfrak{a}) = \delta_{\mathfrak{A}_i}(\mathrm{ext}(\zeta_0, \mathfrak{B}, i, \mathfrak{a}), \ldots, \mathrm{ext}(\zeta_{n-1}, \mathfrak{B}, i, \mathfrak{a}))$.

In addition to the extension (at an index) of a *predicate constant* $\pi$ of L we shall want to also speak of the *intension* of $\pi$ in $\mathfrak{B}$ (without relativization to particular indices): $\mathrm{int}(\pi, \mathfrak{B})$ = the function f whose domain is I and which is such that for some assignment $\mathfrak{a}$ in $\mathfrak{B}$ and for each $i \in I$, $f(i) = \mathrm{ext}(\pi, \mathfrak{B}, i, \mathfrak{a}) = \pi_{\mathfrak{A}_i}$.

### §2.1. *Satisfaction and Truth*

If $\mathfrak{B} = \langle\langle\mathfrak{A}_i\rangle_{i\in I}, \langle F_n\rangle_{n\in\omega}, \langle E_n\rangle_{n\in\omega}\rangle$ is a secondary world system for L, $\mathfrak{a}$ is an assignment in $\mathfrak{B}$, and $i \in I$, then:

(1) for all $n \in \omega$, for all n-place predicate expressions $\pi$ of L and all terms $\zeta_0, \ldots, \zeta_{n-1}$ of L, $\mathfrak{a}$ *satisfies* $\pi(\zeta_0, \ldots, \zeta_{n-1})$ *in* $\mathfrak{B}$ *at* i iff $\langle ext(\zeta_0, \mathfrak{B}, i, \mathfrak{a}), \ldots, ext(\zeta_{n-1}, \mathfrak{B}, i, \mathfrak{a})\rangle \in ext(\pi, \mathfrak{B}, i, \mathfrak{a})$;

(2) if $\varphi$, $\psi$ are formulas of L, then $\mathfrak{a}$ *satisfies* $(\varphi \rightarrow \psi)$ *in* $\mathfrak{B}$ *at* i iff either $\mathfrak{a}$ does not satisfy $\varphi$ in $\mathfrak{B}$ at i or $\mathfrak{a}$ satisfies $\psi$ in $\mathfrak{B}$ at i;

(3) if $\varphi$ is a formula of L and $\alpha$ is an individual variable, then

  (a) $\mathfrak{a}$ *satisfies* $\bigwedge\alpha\varphi$ *in* $\mathfrak{B}$ *at* i iff for all $x \in \mathcal{D}_{\mathfrak{A}_i}$, $\mathfrak{a}\binom{\alpha}{x}$ satisfies $\varphi$ in $\mathfrak{B}$ at i, and

  (b) $\mathfrak{a}$ *satisfies* $\overset{e}{\bigwedge}\alpha\varphi$ *in* $\mathfrak{B}$ *at* i iff for all $y \in \mathcal{U}_{\mathfrak{A}_i}$, $\mathfrak{a}\binom{\alpha}{y}$ satisfies $\varphi$ in $\mathfrak{B}$ at i;

(4) if $\varphi$ is a formula of L and $\pi$ is an n-place predicate variable, then

  (c) $\mathfrak{a}$ *satisfies* $\bigwedge\pi\varphi$ *in* $\mathfrak{B}$ *at* i iff for all $X \in F_n$, $\mathfrak{a}\binom{\pi}{X}$ satisfies $\varphi$ in $\mathfrak{B}$ at i, and

  (d) $\mathfrak{a}$ *satisfies* $\overset{e}{\bigwedge}\pi\varphi$ *in* $\mathfrak{B}$ *at* i iff for all $Y \in E_n$, $\mathfrak{a}\binom{\pi}{Y}$ satisfies $\varphi$ in $\mathfrak{B}$ at i;

(5) if $\varphi$ is a formula of L, then $\mathfrak{a}$ *satisfies* $\Box\varphi$ *in* $\mathfrak{B}$ *at* i iff for all $j \in I$, $\mathfrak{a}$ satisfies $\varphi$ in $\mathfrak{B}$ at j.

As usual, a formula $\varphi$ of a language L is understood to be *true in* a secondary world system $\mathfrak{B}$ for L *at an index* i of $\mathfrak{B}$ if $\varphi$ is satisfied in $\mathfrak{B}$ at i by every assignment in $\mathfrak{B}$. If $\varphi$ is true in $\mathfrak{B}$ at every index of $\mathfrak{B}$, we say that $\varphi$ is *valid in* $\mathfrak{B}$.

The following semantical lemma is useful in establishing our completeness result. We omit the proof here as it is a simple inductive argument on the formulas of L.

LEMMA: If L is a language of which $\varphi$ is a formula, $\mathfrak{B}$ is a secondary world system for L, i is a reference point of $\mathfrak{B}$, and $\mathfrak{a}$ is an assignment in $\mathfrak{B}$, then:

(1) if $\zeta$ is an individual constant (0-place operation constant) of L and $\mathfrak{a}$ satisfies $\bigvee\alpha\Box\alpha = \zeta$ in $\mathfrak{B}$ at i, then $\mathfrak{a}$ satisfies $\varphi\left[\frac{\alpha}{\zeta}\right]$ in $\mathfrak{B}$ at i iff $\mathfrak{a}\binom{\alpha}{ext(\zeta, \mathfrak{B}, i, \mathfrak{a})}$ satisfies $\varphi$ in $\mathfrak{B}$ at i; and

(2) if $\pi$ is a predicate constant of L of the same number of places as $\sigma$, then $\mathfrak{a}$ satisfies $\varphi[{}^{\sigma}_{\pi}]$ in $\mathfrak{B}$ at i iff $\mathfrak{a}\left({}^{\sigma}_{\mathrm{int}(\pi,\mathfrak{B})}\right)$ satisfies $\varphi$ in $\mathfrak{B}$ at i.

## §2.2. *Normalcy and Validity*

If $\mathfrak{B}$ is a secondary world system for a language L, then $\mathfrak{B}$ is said to be *normal* if for all formulas $\varphi$ of L, $\bigvee\pi\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\varphi(\alpha_0, \ldots, \alpha_{n-1})$ and $\overset{e}{\bigvee}\pi\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}[\varphi(\alpha_0, \ldots, \alpha_{n-1}) \wedge \overset{e}{\bigvee}\sigma\sigma(\alpha_0, \ldots, \alpha_{n-1})]$ are valid in $\mathfrak{B}$, where $\pi$ is an n-place predicate variable which does not occur in $\varphi$.

We shall say that $\varphi$ is *valid* if for some language L, $\varphi$ is a formula of L and $\varphi$ is valid in every normal secondary world system for L. Where $\Gamma$ is a set of formulas, we shall write '$\Gamma \models \varphi$' to indicate that $\varphi$ is a *consequence* of $\Gamma$, i.e., to indicate that if every member of $\Gamma$ is satisfied at any index in any normal secondary world system by any assignment in that world system, then $\varphi$ is also satisfied in that world system at that index by that assignment.

## §3. *Axioms*

An axiom set which provides a completeness theorem for this notion of validity (and consequence) consists of all (modal) generalizations of formulas of the following forms (*modus ponens* being the only inference rule):

(I) Sentential axioms:

(1) $\varphi \rightarrow (\psi \rightarrow \varphi)$
(2) $[\varphi \rightarrow (\psi \rightarrow \chi)] \rightarrow [(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi)]$
(3) $(\sim\varphi \rightarrow \sim\psi) \rightarrow (\psi \rightarrow \varphi)$

(II) Modal axioms:

(4) $\Box\varphi \rightarrow \varphi$
(5) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
(6) $\sim\Box\varphi \rightarrow \Box\sim\Box\varphi$

(III) Quantificational axioms for *possibilia* and attributes:

(7) $\bigwedge\mu(\varphi \to \psi) \to (\bigwedge\mu\varphi \to \bigwedge\mu\psi)$, where $\mu$ is a predicate or individual variable,

(8) $\varphi \to \bigwedge\mu\varphi$, where $\mu$ is a predicate or individual variable not occurring in $\varphi$,

(9) $\bigvee\pi\pi = \hat{\alpha}_0 \dots \hat{\alpha}_{n-1}\varphi(\alpha_0, \dots, \alpha_{n-1})$, where $\pi$ is an n-place predicate variable which does not occur in $\varphi$,

(10) $\bigvee\alpha\bigwedge\rho[\rho(\alpha) \to \rho(\zeta)]$, where $\alpha$ does not occur in $\zeta$,

(11) $\bigwedge\rho[\rho(\zeta) \to \rho(\eta)] \to (\varphi \to \psi)$, where $\varphi$ is an atomic formula and $\psi$ is obtained from $\varphi$ by replacing an occurrence of $\eta$ by an occurrence of $\zeta$,

(12) $\Diamond\bigwedge\rho[\rho(\alpha) \to \rho(\beta)] \to \Box\bigwedge\rho[\rho(\alpha) \to \rho(\beta)]$

(IV) Existence axioms:

(13) $\overset{e}{\bigwedge}\pi(\varphi \to \psi) \to (\overset{e}{\bigwedge}\pi\varphi \to \overset{e}{\bigwedge}\pi\psi)$

(14) $\bigwedge\pi\varphi \to \overset{e}{\bigwedge}\pi\varphi$

(15) $\overset{e}{\bigwedge}\pi\Box\overset{e}{\bigvee}\sigma\sigma = \hat{\alpha}_0 \dots \hat{\alpha}_{k-1}[\pi(\beta_0, \dots, \beta_{n-1}) \wedge \varphi(\beta_0, \dots, \beta_{n-1})]$, where $\{\alpha_0, \dots, \alpha_{k-1}\} \subseteq \subseteq \{\beta_0, \dots, \beta_{n-1}\}$ and $\pi$, $\sigma$ are distinct predicate variables that do not occur in $\varphi$,

(16) $\overset{e}{\bigvee}\pi\pi = \hat{\alpha}_0 \dots \hat{\alpha}_{n-1}[\overset{e}{\bigvee}\sigma\sigma(\alpha_0, \dots, \alpha_{n-1})]$[11]

(17) Definitional axiom:
$\overset{e}{\bigwedge}\alpha_0 \dots \overset{e}{\bigwedge}\alpha_{n-1}\varphi \leftrightarrow \bigwedge\alpha_0 \dots \bigwedge\alpha_{n-1}[\overset{e}{\bigvee}\pi\pi(\alpha_0, \dots, \alpha_{n-1}) \to \varphi]$

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[11] Axiom 16 can be replaced by $\bigvee \overset{e}{\pi}\pi = \hat{\alpha}_0 \dots \hat{\alpha}_{n-1}[\varphi(\alpha_0, \dots, \alpha_{n-1}) \wedge \bigvee \overset{e}{\delta}\sigma(\alpha_0, \dots, \alpha_{n-1})]$.

We use '$\Gamma$' to refer to sets of formulas. We shall write '$\Gamma \vdash \varphi$' to indicate that $\varphi$ is derivable from $\Gamma$ by means of these axioms with *modus ponens* as the only inference rule. Where $\Gamma$ is empty, we write '$\vdash \varphi$', indicating that $\varphi$ is a theorem of the system. Theorem 1 below is proved in a manner similar to that of Theorem 1 of Cocchiarella [5].

THEOREM 1: If $\Gamma \vdash \varphi$, then $\Gamma \models \varphi$.

## §4. *Some Lemmas*

Since we can easily establish a rewrite law for the present system, let us adopt the convention that $\zeta = \eta$ is an abbreviation for $\Lambda\pi[\pi(\zeta) \rightarrow \pi(\eta)]$, where it is left unspecified as to which 1-place predicate variable $\pi$ is. We observe that although our informal conventions now allow us to use the identity sign to express both individual and attribute identity (as well as identity in the metalanguage), no confusion between the two need arise so long as we keep in mind the additional conventions regarding the use of distinct groups of Greek letters for referring to the different types of expressions. Where proofs are not given, it is understood that they proceed in the usual manner. Because of (A1)—(A3), we have:

LEMMA 1: If $\varphi$ is a tautologous formula, then $\vdash \varphi$.

LEMMA 2: If $\Gamma \vdash \varphi$ and $\Gamma \vdash \varphi \rightarrow \psi$, then $\Gamma \vdash \psi$.

LEMMA 3: If $\mu$ is a predicate or individual variable which does not occur free in $\varphi$, then $\vdash \varphi \rightarrow \Lambda\overset{e}{\mu}\varphi$.

PROOF: By (A8), (A14), (A17) and Lemmas 1 and 2. (Q.E.D.)

LEMMA 4: $\vdash \Lambda\alpha\varphi \rightarrow \Lambda\overset{e}{\alpha}\varphi$.

PROOF: By (A17) and Lemmas 1 and 2. (Q.E.D.)

LEMMA 5: If $\Gamma \vdash \varphi$ and $\mu$ is a predicate or individual variable which does not occur free in any member of $\Gamma$, then $\Gamma \vdash \Lambda\mu\varphi$ and $\Gamma \vdash \Lambda\overset{e}{\mu}\varphi$.

LEMMA 6: If $\psi'$ is obtained from $\psi$ by replacing an occurrence of $\varphi$ by an occurrence of $\varphi'$, and $\vdash \varphi \leftrightarrow \varphi'$, then $\vdash \psi \leftrightarrow \psi'$.

PROOF: By a simple inductive argument using Lemmas 1—3, (A7), (A13) and (A17). (Q.E.D.)

LEMMA 7: If $\zeta$ can be properly substituted for $\alpha$ in $\varphi$, $\Gamma \vdash \varphi[{}^{\alpha}_{\zeta}]$, and $\zeta$ does not occur in any member of $\Gamma$, then $\Gamma \vdash \Lambda\alpha\varphi$.

LEMMA 8: If $\sigma$ can be properly substituted for $\pi$ in $\varphi$, $\Gamma \vdash \varphi[{}^{\pi}_{\sigma}]$, and $\sigma$ does not occur in any member of $\Gamma$, then $\Gamma \vdash \Lambda\pi\varphi$.

LEMMA 9: If $\varphi$ is modal free and $\psi$ is obtained from $\varphi$ by replacing a free occurrence of $\zeta$ by a free occurrence of $\eta$, then $\vdash \zeta = \eta \rightarrow$ $\rightarrow (\varphi \leftrightarrow \psi)$.

PROOF: By a simple inductive argument on modal free formulas, using (A11) in the atomic case and Lemmas 1—3, 5 and (A7), (A8) and (A17) on the inductive hypothesis. (Q.E.D.)

LEMMA 10: If $\psi$ is obtained from $\varphi$ by replacing a free occurrence of $\zeta$ by a free occurrence of $\eta$, then $\vdash \Box\zeta = \eta \rightarrow (\varphi \leftrightarrow \psi)$.

PROOF: Similar to that for Lemma 9, except that the stronger antecedent allows us to cover the additional inductive case for $\Box$ by obvious and well-known consequences of the S5 (or S4) modal axioms. (Q.E.D.)

LEMMA 11: If $\alpha$, $\beta$ are distinct individual variables such that $\beta$ can be properly substituted for $\alpha$ in $\varphi$, then $\vdash \Lambda\alpha\varphi \rightarrow \varphi[{}^{\alpha}_{\beta}]$ and $\vdash \overset{e}{\vee}\alpha\,\alpha = \beta \rightarrow (\overset{e}{\Lambda}\alpha\varphi \rightarrow \varphi[{}^{\alpha}_{\beta}])$.

PROOF:

$\vdash \Box\alpha = \beta \rightarrow (\varphi \rightarrow \varphi[{}^{\alpha}_{\beta}])$ by repeated use of Lemmas 10, 1 and 2,

$\vdash \alpha = \beta \rightarrow (\varphi \rightarrow \varphi[{}^{\alpha}_{\beta}])$ by (A4), (A12) and Lemmas 1 and 2,

$\vdash \Lambda\alpha \sim \varphi[{}^{\alpha}_{\beta}] \rightarrow (\Lambda\alpha\varphi \rightarrow \Lambda\alpha\,\alpha \neq \beta)$

and

$\vdash \overset{e}{\Lambda}\alpha \sim \varphi[{}^{\alpha}_{\beta}] \rightarrow (\overset{e}{\Lambda}\alpha\varphi \rightarrow \overset{e}{\Lambda}\alpha\,\alpha \neq \beta)$

by Lemma 5, (A7), (A17) and Lemmas 1 and 2,

$\vdash \sim\varphi[^{\alpha}_{\beta}] \rightarrow \Lambda\alpha \sim\varphi[^{\alpha}_{\beta}]$ by (A8),
$\vdash \sim\varphi[^{\alpha}_{\beta}] \rightarrow \overset{e}{\Lambda}\alpha \sim\varphi[^{\alpha}_{\beta}]$ by Lemma 3,
$\vdash V\alpha\alpha = \beta \rightarrow (\Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\beta}])$ by Lemmas 1 and 2,
$\vdash V\alpha\alpha = \beta$ by (A10), and therefore
$\vdash \Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\beta}]$ and
$\vdash \overset{e}{V}\alpha\alpha = \beta \rightarrow (\overset{e}{\Lambda}\alpha\varphi \rightarrow \varphi[^{\alpha}_{\beta}])$
by Lemmas 1 and 2.
(Q.E.D.)

LEMMA 12: If $\alpha$, $\beta$ are distinct individual variables, then $\vdash \overset{e}{V}\pi\pi(\beta) \leftrightarrow \overset{e}{V}\alpha\alpha = \beta$.

PROOF: By (A17) and Lemmas 9, 11, and 1 and 2. (Q.E.D.)

LEMMA 13: If $\alpha$, $\beta$ are distinct individual variables, then $\vdash \overset{e}{\Lambda}\beta\ \overset{e}{V}\alpha\alpha = \beta$.

PROOF: By Lemmas 12, 1, 2 and 5. (Q.E.D.)

LEMMA 14: If $\beta$ does not occur in $\zeta$ or in $\Lambda\alpha\varphi$, then $\vdash V\beta\Box\beta = \zeta \rightarrow$ $\rightarrow(\Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\zeta}])$ and $\vdash \overset{e}{V}\beta\Box\beta = \zeta \rightarrow (\overset{e}{\Lambda}\alpha\varphi \rightarrow \varphi[^{\alpha}_{\zeta}])$.

PROOF: $\vdash \Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\beta}]$ by Lemma 11,
$\vdash \Box\beta = \zeta \rightarrow (\Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\zeta}])$ by repeated use of Lemmas 10, 1 and 2, and therefore
$\vdash V\beta\Box\beta = \zeta \rightarrow (\Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\zeta}])$
by Lemmas 5, 6, (A7), (A8) and Lemmas 1 and 2.

Proof of the second part is similar except for the additional use of Lemma 13. (Q.E.D.)

LEMMA 15: If $\varphi$ is modal free and $\beta$ does not occur in $\zeta$ or in $\Lambda\alpha\varphi$, then $\vdash \Lambda\alpha\varphi \rightarrow \varphi[^{\alpha}_{\zeta}]$ and $\vdash \overset{e}{V}\beta\beta = \zeta \rightarrow (\overset{e}{\Lambda}\alpha\varphi \rightarrow \varphi[^{\alpha}_{\beta}])$.

PROOF: Similar to the proof for Lemma 14 except for using Lemma 9 in place of Lemma 10. (Q.E.D.)

LEMMA 16: $\vdash \overset{e}{V}\alpha\varphi \leftrightarrow V\alpha(\overset{e}{V}\pi\pi(\alpha) \wedge \varphi)$

PROOF: By (A17) and Lemmas 1, 2 and 6. (Q.E.D.)

LEMMA 17: If $\alpha$, $\beta$ are distinct individual variables, then $\vdash \bigvee\beta\Box\beta = \alpha$.

PROOF: By (A4), (A12), Lemma 5, (A7) and Lemmas 1 and 2. (Q.E.D.)

LEMMA 18: If $\psi$ is modal free and $\alpha_0, \ldots, \alpha_{n-1}$ are all the distinct individual variables free in $\psi$, then $\vdash \pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\psi \rightarrow (\varphi \leftrightarrow$ $\leftrightarrow \varphi\left[{}^{\pi(\alpha_0, \ldots, \alpha_{n-0})}_{\psi}\right])$.

PROOF: Assume $\psi$ can be properly substituted for $\pi$ in $\varphi$ (with respect to the free occurrences of $\alpha_0, \ldots, \alpha_{n-1}$ in $\psi$), since otherwise Lemma 18 holds trivially. The proof then is straightforward by an inductive argument on the structure of $\varphi$. Where $\varphi$ is atomic, Lemma 18 follows by (A4) and Lemmas 15, 1 and 2. (The fact that $\psi$ is modal free is essential for this application of Lemma 15.) Where $\varphi$ is a conditional or a formula beginning with a universal quantifier, Lemma 18 follows from the inductive hypothesis by Lemma 5, (A7), (A13), (A17), (A8) and Lemmas 1—3. Where $\varphi$ begins with $\Box$, Lemma 18 follows from the inductive hypothesis by *modal generalization* (an obvious derived rule, hereafter referred to as M. G.), (A5) and the obvious fact (of S4 as well as of S5) that iterated occurrences of $\Box$ can be collapsed to a single occurrence. (Q.E.D.)

LEMMA 19: If $\sigma$ is a predicate expression of the same number of places as $\pi$, and $\sigma$ does not occur in $\bigwedge\pi\varphi$, then $\vdash \bigwedge\pi\varphi \rightarrow \varphi[{}^{\pi}_{\sigma}]$ and $\vdash \overset{e}{\bigvee}\pi\pi = \sigma \rightarrow (\overset{e}{\bigwedge}\pi\varphi \rightarrow \varphi[{}^{\pi}_{\sigma}])$.

PROOF: By an argument similar to that for Lemma 11 except for using Lemma 18 in place of Lemma 10, (A9) in place of (A10), and (A13) in place of (A17). (Q.E.D.)

LEMMA 20: If $\psi$ is modal free, $\sigma$ does not occur in $\psi$, and $\alpha_0, \ldots, \alpha_{n-1}$ are all the distinct individual variables occurring free in $\psi$, then $\vdash \bigwedge\pi\varphi \rightarrow \varphi\left[{}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\right]$ and $\vdash \overset{e}{\bigvee}\sigma\sigma = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\psi \rightarrow (\overset{e}{\bigwedge}\pi\varphi \rightarrow$ $\rightarrow \varphi\left[{}^{\pi(\alpha_0, \ldots, \alpha_{n-1})}_{\psi}\right])$.

PROOF: By an argument similar to that for Lemma 14 except for using Lemma 19 in place of Lemma 11, Lemma 18 in place of

Lemma 10, adding a use of (A9) to dispatch the second order analogue of the antecedent of Lemma 14, and using (A15) in place of Lemma 13.[12] (Q.E.D.)

LEMMA 21: (a) $\vdash \bigwedge\alpha\Box\varphi \rightarrow \Box\bigwedge\alpha\varphi$
(b) $\vdash \bigwedge\pi\Box\varphi \rightarrow \Box\bigwedge\pi\varphi$
(c) $\vdash \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \Box\bigwedge\overset{e}{\pi}\varphi$

PROOF: In regard to (c):

$\vdash \overset{e}{\bigvee}\sigma\sigma = \pi \rightarrow (\bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \Box\varphi)$ by Lemma 20,

$\vdash \Box\overset{e}{\bigvee}\sigma\sigma = \pi \rightarrow (\Box \sim \Box\varphi \rightarrow \Box \sim \bigwedge\overset{e}{\pi}\Box\varphi)$ by M. G., (A5), (A6) and Lemmas 1 and 2,

$\vdash \sim\varphi \rightarrow \Box \sim \Box\varphi$ by (A4), (A6) and Lemmas 1 and 2,

$\vdash \Box\,\overset{e}{\bigvee}\sigma\sigma = \pi \rightarrow (\sim \Box \sim \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \varphi)$ by Lemmas 1 and 2,

$\vdash \bigwedge\overset{e}{\pi}\Box\overset{e}{\bigvee}\sigma\sigma = \pi \rightarrow (\sim \Box \sim \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \bigwedge\overset{e}{\pi}\varphi)$ by Lemma 5, (A13) and Lemmas 1—3,

$\vdash \sim \Box \sim \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \bigwedge\overset{e}{\pi}\varphi$ by (A15) and Lemma 2,

$\vdash \Box \sim \Box \sim \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \Box\bigwedge\overset{e}{\pi}\varphi$ by M. G., (A5) and Lemmas 1 and 2,

$\vdash \sim \Box \sim \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \Box\bigwedge\overset{e}{\pi}\varphi$ by (A6) and Lemmas 1 and 2, and therefore

$\vdash \bigwedge\overset{e}{\pi}\Box\varphi \rightarrow \Box\bigwedge\overset{e}{\pi}\varphi$ by (A4) and Lemmas 1 and 2.

The proof for (b) is similar except for using (A7) in place of (A13) and for deletion of all considerations of the formula $\overset{e}{\bigvee}\sigma\sigma = \pi$ and of the use of (A15). The proof for (a) is entirely similar to that for

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[12] This use of (A15) presupposes a lemma to the effect that $\vdash \bigwedge\overset{e}{\pi}\Box\chi \rightarrow$ $\rightarrow \bigwedge\overset{e}{\pi}\chi$. Such a lemma follows trivially from (A4), Lemma 5, (A13) and Lemmas 1 and 2.

(b) except for using Lemmas 14, 17 and 2 in place of Lemma 20. (Q.E.D.)

It is worth pointing out that the first order analogue to part (c) of Lemma 21 does not hold. The proof for (c) relies essentially on (A15) whose first order analogue $\overset{e}{\bigwedge}\alpha\Box\overset{e}{\bigvee}\beta\beta=\alpha$ (i.e., every existent of this world is an existent of all worlds—or, everything which exists necessarily exists) is not a theorem, which is exactly as it should be. A similar observation holds for Lemmas 22 and 23 to follow.

LEMMA 22: (a) $\vdash \bigvee\alpha\Box\varphi \rightarrow \Box\bigvee\alpha\varphi$
(b) $\vdash \bigvee\pi\Box\varphi \rightarrow \Box\bigvee\pi\varphi$
(c) $\vdash \overset{e}{\bigvee}\pi\Box\varphi \rightarrow \Box\overset{e}{\bigvee}\pi\varphi$

PROOF: In regard to (c):

$\vdash \overset{e}{\bigvee}\sigma\sigma=\pi \rightarrow (\overset{e}{\bigwedge}\pi\sim\varphi \rightarrow \sim\varphi)$ by Lemma 20,

$\vdash \Box\overset{e}{\bigvee}\sigma\sigma=\pi \rightarrow (\Box\varphi \rightarrow \Box\overset{e}{\bigvee}\pi\varphi)$ by M. G., and Lemmas 1 and 2,

$\vdash \overset{e}{\bigwedge}\pi\sim\Box\overset{e}{\bigvee}\pi\varphi \rightarrow \overset{e}{\bigwedge}\pi\sim\Box\varphi$ by Lemma 5, (A15), (A13) and Lemmas 1 and 2, and therefore

$\vdash \overset{e}{\bigvee}\pi\Box\varphi \rightarrow \Box\overset{e}{\bigvee}\pi\varphi$ by Lemmas 1—3.

The proof for (b) is similar except for using (A7) in place of (A13) and for deleting all considerations of $\overset{e}{\bigvee}\sigma\sigma=\pi$ and (A15). The proof for (a) is similar to that for (b) except for using Lemmas 14, 17 and 2 in place of Lemma 20. (Q.E.D.)

LEMMA 23: (a) $\vdash \bigvee\alpha\Box\alpha=\zeta \rightarrow \Box\bigvee\alpha\Box\alpha=\zeta$
(b) $\vdash \bigvee\pi\pi=\hat{\alpha}_0\ldots\hat{\alpha}_{n-1}\varphi \rightarrow \Box\bigvee\pi\pi=\hat{\alpha}_0\ldots\hat{\alpha}_{n-1}\varphi$
(c) $\vdash \overset{e}{\bigvee}\pi\pi=\hat{\alpha}_0\ldots\hat{\alpha}_{n-1}\varphi \rightarrow \Box\overset{e}{\bigvee}\pi\pi=\hat{\alpha}_0\ldots\hat{\alpha}_{n-1}\varphi$

PROOF: By Lemma 22 and the well-known fact that the S4 principle $\Box\psi \rightarrow \Box\Box\psi$ is provable in S5. (Q.E.D.)

LEMMA 24: If $\pi$ does not occur in $\varphi$, $\alpha_0, \ldots, \alpha_{n-1}$ are all the individual variables free in $\varphi$, and $i<n$, then $\vdash \overset{e}{\bigvee}\pi\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\varphi \rightarrow \rightarrow \bigwedge\alpha_0 \ldots \bigwedge\alpha_{n-1}[\varphi \rightarrow \overset{e}{\bigvee}\sigma\sigma(\alpha_i)]$.

PROOF: Similar to its modal free analogue in [5]. (Q.E.D.)

LEMMA 25: $\vdash \overset{e}{\bigvee}\pi\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}[\varphi(\alpha_0, \ldots, \alpha_{n-1}) \wedge \overset{e}{\bigvee}\sigma\sigma(\alpha_0, \ldots, \alpha_{n-1})]$.

PROOF: By (A15), (A16), Lemma 20, (A4) and Lemmas 1 and 2. (Q.E.D.)

## §5. *A Completeness Theorem for Second Order S5*

Theorem 2 below constitutes our strong completeness result for countable languages. It, of course, also constitutes a weak completeness result for arbitrary languages. Moreover, since a compactness theorem can be proved in the usual manner through the use of ultraproducts, Theorem 2, together with this compactness theorem yields a strong completeness result for arbitrary languages.

THEOREM 2: If $\Gamma$ is a consistent set of formulas of a countable language, then there exists a normal secondary world system $\mathfrak{B}$ (with countably many *possibilia* and countably many attributes[13]) such that for some reference point i of $\mathfrak{B}$, every member of $\Gamma$ is satisfiable in $\mathfrak{B}$ at i.

PROOF: Suppose that $\Gamma$ is a consistent set of formulas of a countable language L. Let $\zeta_0, \ldots, \zeta_n, \ldots$ ($n \in \omega$) be pairwise distinct individual constants not in L; let, for each natural number k, $\pi^k_0, \ldots, \pi^k_n, \ldots$ ($n \in \omega$) be pairwise distinct k-place predicate constants not in L; and let L$*$ be the language resulting from the addition to L of all these constants. Let $\Sigma_1, \ldots, \Sigma_n, \ldots$ ($n \in \omega$) be an ordering of the set of formulas $\varphi$ of L$*$ for which there are a natural number k, a k-place predicate variable $\sigma$, an individual variable $\alpha$, and a formula $\psi$ of L$*$ such that $\varphi$ is either $\bigvee\alpha\psi$, $\bigvee\sigma\psi$ or $\overset{e}{\bigvee}\sigma\psi$. We define

[13] By an alternative proof than that given here, e.g., one which utilized semantic tableaux, we can also show that $\mathfrak{B}$ need have only countably many reference points.

by recursion the chain $\Gamma_0, \ldots, \Gamma_n, \ldots$ $(n \in \omega)$ as follows: (1) $\Gamma_0 = \Gamma$; (2) if $\Gamma_n \cup \{\Sigma_{n+1}\}$ is not consistent, then $\Gamma_{n+1} = \Gamma_n$; (3) if $\Gamma_n \cup \cup \{\Sigma_{n+1}\}$ is consistent, then:

(a) if $\Sigma_{n+1} = \bigvee\alpha\psi$, for some formula $\psi$ and some individual variable $\alpha$, then $\Gamma_{n+1} = \Gamma_n \cup \{\psi[^{\alpha}_{\zeta_i}], \bigvee\alpha\Box\alpha = \zeta_i\}$, where i is the first natural number such that $\zeta_i$ does not occur in any member of $\Gamma_n$;
(b) if $\Sigma_{n+1} = \bigvee\sigma\psi$, for some formula $\psi$ and some k-place predicate variable $\sigma$, then $\Gamma_{n+1} = \Gamma_n \cup \{\psi[^{\sigma}_{\pi^k_i}]\}$, where i is the first natural number such that $\pi^k_i$ does not occur in any member of $\Gamma_n$;
(c) if $\Sigma_{n+1} = \overset{e}{\bigvee}\sigma\psi$, for some formula $\psi$ and some k-place predicate variable $\sigma$, then $\Gamma_{n+1} = \Gamma_n \cup \cup \{\psi[^{\sigma}_{\pi^k_i}], \overset{e}{\bigvee}\sigma\sigma = \pi^k_i\}$, where i is the first natural number such that $\pi^k_i$ does not occur in any member of $\Gamma_n$.

We observe that for each $n \in \omega$, $\Gamma_n$ is consistent. The proof is by induction. If $\Sigma_{n+1} = \bigvee\alpha\psi$, the consistency of $\Gamma_{n+1}$ follows by the inductive hypothesis and Lemmas 7 and 17. If $\Sigma_{n+1} = \bigvee\sigma\psi$ or $\Sigma_{n+1} = \overset{e}{\bigvee}\sigma\psi$, then the consistency of $\Gamma_{n+1}$ follows by the inductive hypothesis and Lemma 8, with (A14) and (A15) also being used in the second case. In the usual manner, we conclude that $\bigcup_{n\in\omega} \Gamma_n$ is consistent, since otherwise $\Gamma_n$ is not consistent for some $n \in \omega$. Accordingly, by Lindenbaum's lemma, there exists a maximally consistent set $K*$ of formulas of $L*$ such that $\bigcup_{n\in\omega} \Gamma_n \subseteq K*$.

Now let $\mathbf{K}$ be the set of maximally consistent sets K of formulas of $L*$ satisfying the following conditions for all formulas $\varphi$ of $L*$, all individual variables $\alpha$, and, for all $n \in \omega$, all n-place predicate variables $\sigma$:

(1) if $\Box\varphi \in K*$, then $\Box\varphi \in K$,
(2) $\bigwedge\alpha\varphi \in K$ iff for all $k \in \omega$, if $\bigvee\alpha\Box\alpha = \zeta_k \in K$, then $\varphi[^{\alpha}_{\zeta_k}] \in K$,
(3) $\overset{e}{\bigwedge}\alpha\varphi \in K$ iff for all $k \in \omega$, if $\overset{e}{\bigvee}\alpha\Box\alpha = \zeta_k \in K$, then $\varphi[^{\alpha}_{\zeta_k}] \in K$,

(4) $\bigwedge\sigma\varphi \in K$ iff for all $k \in \omega$, $\varphi[^{\sigma}_{\pi^n_k}] \in K$,
(5) $\overset{e}{\bigwedge}\sigma\varphi \in K$ iff for all $k \in \omega$, if $\overset{e}{\bigvee}\sigma\sigma = \pi^n_k \in K$, then $\varphi[^{\sigma}_{\pi^n_k}] \in K$.

(I) In regard to **K**, we show first that for all formulas $\varphi$ of $L*$ and all $K \in \mathbf{K}$, if $\Box\varphi \in K$, then for all $K' \in \mathbf{K}$, $\Box\varphi \in K'$. Suppose $K, K' \in \mathbf{K}$ and by *reductio* that though $\Box\varphi \in K$, $\Box\varphi \notin K'$. Then, by clause (1) for $K'$, $\Box\varphi \notin K*$, i.e., $\sim\Box\varphi \in K*$, from which it follows by (A6) that $\Box\sim\Box\varphi \in K*$. But then, by clause (1) for K, $\Box\sim\sim\Box\varphi \in K$, from which, by (A4), it follows that $\sim\Box\varphi \in K$. This last, however, is impossible since $\Box\varphi \in K$ and K is consistent by assumption.

(II) We next show in regard to **K** that for each formula $\varphi$ of $L*$ and for each $K \in \mathbf{K}$, if $\varphi \in K$ and $\sim\Box\varphi \in K$, then there is a $K' \in \mathbf{K}$ such that $\sim\varphi \in K'$. Suppose $K \in \mathbf{K}$, $\varphi \in K$ and $\sim\Box\varphi \in K$. We reconsider the sequence $\Sigma_1, \ldots, \Sigma_n, \ldots$ $(n \in \omega)$ and define by recursion the following sequence of formulas of $L*$: (1) $\theta_0 = \sim\varphi$; (2) if $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \Sigma_{n+1}) \notin K*$, then $\theta_{n+1} = \theta_n$; (3) if $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \Sigma_{n+1}) \in K*$, then:

(a) if $\Sigma_{n+1} = \bigvee\alpha\psi$, then $\theta_{n+1} = \psi[^{\alpha}_{\zeta_i}]$, where i is the first natural number such that $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \psi[^{\alpha}_{\zeta_i}] \wedge \bigvee\alpha\Box\alpha = \zeta_i) \in K*$[14];
(b) if $\Sigma_{n+1} = \bigvee\sigma\psi$, where $\sigma$ is a k-place predicate variable, then $\theta_{n+1} = \psi[^{\sigma}_{\pi^k_i}]$, where i is the first natural number such that $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \psi[^{\sigma}_{\pi^k_i}]) \in K*$[15];

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[14] That there is at least one such number i is justified as follows: since $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \bigvee\alpha\psi) \in K^*$, then by rewriting $\alpha$ to an individual variable $\beta$ not occurring in $\theta_0, \ldots, \theta_n$ or in $\psi$, we know that $\Diamond\bigvee\beta(\theta_0 \wedge \ldots \wedge \theta_n \wedge \psi[^{\alpha}_{\beta}]) \in K^*$, from which it follows by Lemma 21 and the way $K^*$ was constructed that $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \psi[^{\alpha}_{\zeta_i}]) \wedge \bigvee\alpha\Box\alpha = \zeta_i \in K^*$, for some $i \in \omega$. Therefore, by Lemma 23 and the fact that $\Diamond\chi_1 \wedge \Box\chi_2 \rightarrow \Diamond(\chi_1 \wedge \chi_2)$ is provable in S5 (or in S4), $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \psi[^{\alpha}_{\zeta_i}] \wedge \bigvee\alpha\Box\alpha = \zeta_i) \in K^*$.

[15] Compare footnote 14 for justification of the fact that there is at least one such number i.

(c) if $\Sigma_{n+1} = \overset{e}{\vee}\sigma\psi$, then $\theta_{n+1} = \psi[^{\sigma}_{\pi^k_i}]$, where i is the first natural number such that $\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \wedge \psi[^{\sigma}_{\pi^k_i}] \wedge \overset{e}{\vee}\sigma\sigma = \pi^k_i) \in K*$.[16]

We note that for all $n \in \omega$, $\{\theta_0, \ldots, \theta_n\}$ is consistent. The proof is by induction. We know that $\theta_0$ is consistent since otherwise $\vdash \varphi$ and therefore $\vdash \Box\varphi$, from which it follows that both $\Box\varphi$ and $\sim\Box\varphi$ are in K, which is impossible as $K \in \mathbf{K}$. If $\theta_{n+1} = \theta_n$, then $\{\theta_0, \ldots, \theta_{n+1}\}$ is consistent by the inductive hypothesis. Assume $\theta_{n+1} \neq \theta_n$. If $\Sigma_{n+1} = \vee\alpha\psi$, then $\theta_{n+1} = \psi[^{\alpha}_{\zeta_i}]$, etc., in which case if $\{\theta_0, \ldots, \theta_{n+1}\}$ were not consistent, then $\vdash \theta_0 \wedge \ldots \wedge \theta_n \rightarrow \sim\psi[^{\alpha}_{\zeta_i}]$ and therefore $\sim\Diamond(\theta_0 \wedge \ldots \wedge \theta_n \wedge \psi[^{\alpha}_{\zeta_i}]) \in K*$, which is impossible by definition of $\theta_{n+1}$. If $\Sigma_{n+1} = \vee\sigma\psi$ or $\Sigma_{n+1} = \overset{e}{\vee}\sigma\psi$, then $\theta_{n+1} = = \psi[^{\sigma}_{\pi^k_i}]$, etc., and hence by a similar argument, $\{\theta_0, \ldots, \theta_{n+1}\}$ must be consistent.

Now let $\Xi = \{\theta_n : n \in \omega\}$ and let $M = \Xi \cup \{\psi \in K*$: for some $\chi$, $\psi = \Box\chi\}$. We point out that M is consistent. For if M were not consistent, then for some j, $k \in \omega$, $\vdash \sim(\theta_0 \wedge \ldots \wedge \theta_j \wedge \Box\chi_0 \wedge \ldots \wedge \wedge \Box\chi_{k-1})$, for some $\Box\chi_0, \ldots, \Box\chi_{k-1} \in K*$, in which case $\sim\Diamond(\theta_0 \wedge \wedge \ldots \wedge \theta_j \wedge \Box\chi_0 \wedge \ldots \wedge \Box\chi_{k-1}) \in K*$; but as $\Diamond(\theta_0 \wedge \ldots \wedge \theta_j) \in K*$ and $\Box\Box\chi_0, \ldots, \Box\Box\chi_{k-1} \in K*$, then $\Diamond(\theta_0 \wedge \ldots \wedge \theta_j \wedge \Box\chi_0 \wedge \ldots \wedge \wedge \Box\chi_{k-1}) \in K*$, and thus K* would be inconsistent, which it is not. Accordingly, since M is consistent, there exists a maximally consistent set K' of formulas of L* such that $M \subseteq K'$.

We now show that $K' \in \mathbf{K}$. Clearly, by definition of M, K' satisfies clause (1) for **K**. In addition, by Lemmas 14 and 20, K' satisfies the left-to-right directions of clauses (2)—(5) for **K**.

Suppose $\wedge\alpha\psi \notin K'$ even though for all $k \in \omega$, if $\vee\alpha\Box\alpha = \zeta_k \in K'$, then $\psi[^{\alpha}_{\zeta_k}] \in K'$. Then $\Sigma_k = \vee\alpha \sim\psi \in K'$, for some $k \in \omega$. We note that $\Diamond(\theta_0 \wedge \ldots \wedge \theta_{k-1} \wedge \Sigma_k) \in K*$ since if not, then $\Box(\theta_0 \wedge \ldots \wedge \wedge \theta_{k-1} \rightarrow \sim\Sigma_k) \in K*$, in which case $\Box(\theta_0 \wedge \ldots \wedge \theta_{k-1} \rightarrow \sim\Sigma_k) \in M$, and therefore $\theta_0 \wedge \ldots \wedge \theta_{k-1} \rightarrow \sim\Sigma_k \in K'$; but as $\theta_0, \ldots, \theta_{k-1} \in K'$, then $\sim\Sigma_k \in K'$, i.e., $\wedge\alpha\psi \in K'$, which, by assumption, is impossible. Thus, by definition of $\theta_k$, $\theta_k = \sim\psi[^{\alpha}_{\zeta_i}]$ and $\Diamond(\theta_0 \wedge \ldots \wedge \theta_k \wedge \sim \sim\psi[^{\alpha}_{\zeta_i}] \wedge \vee\alpha\Box\alpha = \zeta_i) \in K*$. Accordingly, $\Diamond\vee\alpha\Box\alpha = \zeta_i \in K*$ and

[16] Compare footnote 14 for justification of the fact that there is at least one such number i.

therefore, by Lemma 23, $\Diamond\Box V\alpha\Box\alpha = \zeta_i \in K*$, from which by (A6) it follows that $\Box V\alpha\Box\alpha = \zeta_i \in K*$. Accordingly, $\Box V\alpha\Box\alpha = = \zeta_i \in M \subseteq K'$ and thus by (A4) and assumption, $\psi[^{\alpha}_{\zeta_i}] \in K'$. But $\theta_k = \sim\psi[^{\alpha}_{\zeta_i}] \in K'$, and therefore $K'$ is not consistent, which is impossible. Thus, $K'$ satisfies clause (2) for K. In addition then, since by Lemma 16 $V\overset{e}{\alpha}\sim\psi$ is equivalent to $V\alpha(V\overset{e}{\pi}\pi(\alpha) \wedge \sim\psi)$, $K'$ also satisfies clause (3) for K. The argument that $K'$ also satisfies clauses (4) and (5) for K is entirely similar to that above (except that for clause (4) it is even simpler). Accordingly, $K' \in \mathbf{K}$ which was to be shown.

Now for each $K \in \mathbf{K}$, for each term $\eta$ of $L*$, and for each $i \in \omega$, let:

(1) $[\eta]_K =_{df} \{\zeta_i : i \in \omega$ and both $\zeta_i = \eta$ and $V\alpha\Box\alpha = = \zeta_i \in K$, for some individual variable $\alpha\}$;

(2) $Ind_K =_{df} \{[\zeta_i]_K : i \in \omega$ and $V\alpha\Box\alpha = \zeta_i \in K$, for some individual variable $\alpha\}$;

(3) $Ind^e_K =_{df} \{[\zeta_i]_K : i \in \omega$ and $V\overset{e}{\alpha}\Box\alpha = \zeta_i \in K$, for some individual variable $\alpha\}$;

(4) $Ext_K =_{df}$ the function R whose domain is $L*$ and which is such that:

   (a) for all $n \in \omega$ and all n-place predicate constants $\pi$ in $L*$, $R(\pi) = \{\langle[\eta_0]_K, \ldots, [\eta_{n-1}]_K\rangle : : \pi(\eta_0, \ldots, \eta_{n-1}) \in K\}$,

   (b) for all $n \in \omega$ and all n-place operation constants $\delta$ in $L*$, $R(\delta) =$ the function f whose domain is $(Ind_K)^n$ and which is such that for all $\eta \in (Ind_K)^n$, $f([\eta_0]_K, \ldots, [\eta_{n-1}]_K) = [\delta(\eta_0, \ldots, \eta_{n-1})]_K$; and

(5) $\mathfrak{A}_K = \langle Ind^e_K, Ind_K, Ext_K\rangle$.

We note that because of (A10) and the way $K*$ was constructed, if $\eta$ is a term of $L*$ and $K \in \mathbf{K}$, then $[\eta]_K$ is not empty. In addition, if $\beta$ is an individual variable and $K, K' \in \mathbf{K}$, then $[\beta]_K = [\beta]_{K'}$. For if $\zeta_i \in [\beta]_K$, then $\zeta_i = \beta$, $V\alpha\Box\alpha = \zeta_i \in K$, and therefore by Lemma 23 and (I) above, $V\alpha\Box\alpha = \zeta_i \in K'$. But, by Lemma 17, $V\alpha\Box\alpha = \beta \in K$, and therefore, by Lemma 14 and (A12), $\Box\zeta_i = = \beta \in K$. Then, by (I) above, $\Box\zeta_i = \beta \in K'$, i.e., by (A4), $\zeta_i = \beta \in K'$, and hence $\zeta_i \in [\beta]_{K'}$. Similarly, if $\zeta_i \in [\beta]_{K'}$, then $\zeta_i \in [\beta]_K$, and therefore $[\beta]_K = [\beta]_{K'}$.

We also observe that by definition if $K \in \mathbf{K}$, then $\mathfrak{A}_K$ is an $L*$-model. In addition, if $K, K' \in \mathbf{K}$, then $\mathcal{D}_{\mathfrak{A}_K} = \mathcal{D}_{\mathfrak{A}_{K'}}$. For if $x \in \mathcal{D}_{\mathfrak{A}_K}$, then $x = [\zeta_i]_K$, for some $i \in \omega$, where $\vee\alpha\Box\alpha = \zeta_i \in K$, and, accordingly, by Lemma 23, (I) above and (A4), $\vee\alpha\Box\alpha = \zeta_i \in K'$; that is $x = [\zeta_i]_K = [\zeta_i]_{K'}$, and therefore $x \in \mathcal{D}_{\mathfrak{A}_{K'}}$. The converse argument is similar. Consequently, $\langle \mathfrak{A}_K \rangle_{K \in \mathbf{K}}$ is a world system with respect to $L*$.

Now let *Int* be that function whose domain is $\{\pi_i^n : n \in \omega$ and $i \in \omega\}$ and which is such that for all $n, i \in \omega$, $Int(\pi_i^n) =$ the function $X$ whose domain is $\mathbf{K}$ and which is such that for all $K \in \mathbf{K}$, $X_K = Ext_K(\pi_i^n)$. In addition, let $F$ and $E$ be those $\omega$-indexed families such that for all $n \in \omega$, $F_n = \{Int(\pi_i^n) : i \in \omega\}$ and $E_n = \{Int(\pi_i^n) : i \in \omega$ and $\overset{e}{\vee}\sigma\sigma = \pi_i^n \in K*$, for some n-place predicate variable $\sigma\}$. We note that if $K \in \mathbf{K}$ and $\overset{e}{\vee}\sigma\sigma = \pi_i^n \in K*$, then, by Lemma 23 and (I) above, $\overset{e}{\vee}\sigma\sigma = \pi_i^n \in K$. Moreover, clearly for each $X \in F_n$, $X$ is an n-ary attribute in $\langle \mathfrak{A}_K \rangle_{K \in \mathbf{K}}$. If, in addition, $X \in E_n$, then $X$ is an n-ary e-attribute in $\langle \mathfrak{A}_K \rangle_{K \in \mathbf{K}}$. For if $Int(\pi_i^n) \in E_n$ and $\langle [\eta_0]_K, \ldots, [\eta_{n-1}]_K \rangle \in Int(\pi_i^n)_K$, for any $K \in \mathbf{K}$, then $\pi_i^n(\eta_0, \ldots, \eta_{n-1}) \in K$, in which case, since $\overset{e}{\vee}\sigma\sigma = \pi_i^n \in K$, $\overset{e}{\vee}\alpha\alpha = \eta_k \in K$, for all $k < n$, by Lemmas 15, 24 and 12; and therefore $[\eta_k]_K \in Ind_K^e$, for all $k < n$. Finally, let $\mathfrak{B} = \langle \langle \mathfrak{A}_K \rangle_{K \in \mathbf{K}}, \langle F_n \rangle_{n \in \omega}, \langle E_n \rangle_{n \in \omega} \rangle$. Then $\mathfrak{B}$, by the foregoing, is a secondary world system with respect to $L*$.

Now let $\mathfrak{a}$ be that function whose domain is the set of individual and predicate variables and which is such that (1) for all individual variables $\alpha$, $\mathfrak{a}(\alpha) = [\alpha]_{K*}$, and (2) for all $n \in \omega$ and all n-place predicate variables $\sigma$, $\mathfrak{a}(\sigma) = Int(\pi_i^n)$, where $i$ is the first number such that for all $K \in \mathbf{K}$, $\sigma = \pi_i^n \in K$. (Note that by (A9), Lemma 23, and the way $K*$ was constructed, if $\sigma$ is an n-place predicate variable, then there is an $i \in \omega$ such that for all $K \in \mathbf{K}$, $\sigma = \pi_i^n \in K$.) Accordingly, by definition, $\mathfrak{a}$ is an assignment in $\mathfrak{B}$.

(III) By a simple inductive argument on the terms of $L*$, it is easily seen that for all $K \in \mathbf{K}$ and for all terms $\eta$ of $L*$, $ext(\eta, \mathfrak{B}, K, \mathfrak{a}) = [\eta]_K$. For if $\eta$ is an individual variable, then $ext(\eta, \mathfrak{B}, K, \mathfrak{a}) = \mathfrak{a}(\eta) = [\eta]_{K*} = [\eta]_K$ for all $K \in \mathbf{K}$. And if $ext(\eta_i, \mathfrak{B}, K, \mathfrak{a}) = [\eta_i]_K$ for all $i < n$, and $\delta$ is an n-place operation constant of $L*$, then $ext(\delta(\eta_0, \ldots, \eta_{n-1}), \mathfrak{B}, K, \mathfrak{a}) = \delta_{\mathfrak{A}_K}(ext(\eta_0, \mathfrak{B}, K, \mathfrak{a}), \ldots ext(\eta_{n-1}, \mathfrak{B}, K, \mathfrak{a})) = \delta_{\mathfrak{A}_K}([\eta_0]_K, \ldots, [\eta_{n-1}]_K) = [\delta(\eta_0, \ldots, \eta_{n-1})]_K$.

(IV) We note that for all $K \in \mathbf{K}$ and for all $j \in \omega$, $\mathfrak{a}$ satisfies $\bigvee\alpha\Box\alpha = \zeta_j$ in $\mathfrak{B}$ at K iff $\bigvee\alpha\Box\alpha = \zeta_j \in K$. For if $\mathfrak{a}$ satisfies $\bigvee\alpha\Box\alpha = \zeta_j$ in $\mathfrak{B}$ at K, then for some $k \in \omega$, $[\zeta_k]_K \in Ind_K$, i.e., $\bigvee\alpha\Box\alpha = \zeta_k \in K$ and $\mathfrak{a}(^{\alpha}_{[\zeta_k]_K})$ satisfies $\Box\alpha = \zeta_j$ in $\mathfrak{B}$ at K; and therefore by the semantical lemma of §2.1, $\mathfrak{a}$ satisfies $\Box\zeta_k = \zeta_j$ in $\mathfrak{B}$ at K; and, accordingly, by (III), $[\zeta_k]_{K'} = [\zeta_j]_{K'}$ for all $K' \in \mathbf{K}$; and therefore, by (II), $\Box\zeta_k = \zeta_j \in K$; hence, by Lemma 10, $\bigvee\alpha\Box\alpha = \zeta_j \in K$. On the other hand, if $\bigvee\alpha\Box\alpha = \zeta_j \in K$, then $[\zeta_j]_K \in Ind_K$; and therefore, since $\mathfrak{a}$ satisfies $\Box\zeta_j = \zeta_j$ in $\mathfrak{B}$ at K, by the semantical lemma of §2.1, $\mathfrak{a}(^{\alpha}_{[\zeta_j]_K})$ satisfies $\Box\alpha = \zeta_j$ in $\mathfrak{B}$ at K, i.e., $\mathfrak{a}$ satisfies $\bigvee\alpha\Box\alpha = \zeta_j$ in $\mathfrak{B}$ at K.

(V) We now show that for all formulas $\varphi$ of $L*$ and for all $K \in \mathbf{K}$, $\mathfrak{a}$ satisfies $\varphi$ in $\mathfrak{B}$ at K iff $\varphi \in K$. Our proof is by strong induction on the set A of natural numbers k such that (V) holds of every formula of $L*$ whose length is k. We assume that every number less than k is in A and show that $k \in A$ by induction on the formulas of $L*$. Suppose $\varphi$ is an atomic formula $\sigma(\eta_0, \ldots, \eta_{n-1})$ of length k, where $\sigma$ is an n-place predicate expression and $\eta_0, \ldots, \eta_{n-1}$ are terms of $L*$. Then $\mathfrak{a}$ satisfies $\sigma(\eta_0, \ldots, \eta_{n-1})$ in $\mathfrak{B}$ at K iff $\langle ext(\eta_0, \mathfrak{B}, K, \mathfrak{a}), \ldots, ext(\eta_{n-1}, \mathfrak{B}, K, \mathfrak{a})\rangle \in ext(\sigma, \mathfrak{B}, K, \mathfrak{a})$, i.e., by (III), iff $\langle [\eta_0]_K, \ldots, [\eta_{n-1}]_K\rangle \in ext(\sigma, \mathfrak{B}, K, \mathfrak{a})$, and thus iff $\sigma(\eta_0, \ldots, \eta_{n-1}) \in K$, since if $\sigma$ is a predicate constant, then $ext(\sigma, \mathfrak{B}, K, \mathfrak{a}) = Ext_K(\sigma)$, and if $\sigma$ is a predicate variable, then $ext(\sigma, \mathfrak{B}, K, \mathfrak{a}) = \mathfrak{a}(\sigma) = Int(\pi^n_i)_K = Ext_K(\pi^n_i)$, for some $i \in \omega$ sucht that $\sigma = \pi^n_i \in K$. If $\varphi$ is a conditional $(\psi \to \chi)$ of length k, then (V) holds for $\varphi$ by the inductive hypothesis and the fact that K is maximally consistent. If $\varphi$ is of the form $\bigwedge\alpha\psi$ and of length k, then: $\mathfrak{a}$ satisfies $\bigwedge\alpha\psi$ in $\mathfrak{B}$ at K only if for all $i \in \omega$, if $[\zeta_i]_K \in Ind_K$, i.e., if $\bigvee\alpha\Box\alpha = \zeta_i \in K$, and hence, by (IV), if $\mathfrak{a}$ satisfies $\bigvee\alpha\Box\alpha = \zeta_i$ in $\mathfrak{B}$ at K, then $\mathfrak{a}(^{\alpha}_{[\zeta_i]_K})$ satisfies $\psi$ in $\mathfrak{B}$ at K, and therefore, by the semantical lemma of §2.1, $\mathfrak{a}$ satisfies $\psi[^{\alpha}_{\zeta_i}]$ in $\mathfrak{B}$ at K, in which case, by the inductive hypothesis, $\psi[^{\alpha}_{\zeta_i}] \in K$. Accordingly, if $\mathfrak{a}$ satisfies $\bigwedge\alpha\psi$ in $\mathfrak{B}$ at K, then, by definition of K, $\bigwedge\alpha\psi \in K$. On the other hand, if $\bigwedge\alpha\psi \in K$, then, by definition of K, for all $i \in \omega$, if $[\zeta_i]_K \in Ind_K$, then $\psi[^{\alpha}_{\zeta_i}] \in K$; and thus, by the inductive hypothesis and the semantical lemma of §2.1, if $\bigwedge\alpha\psi \in K$, then for all $x \in \mathcal{D}_{\mathfrak{B}}$, $\mathfrak{a}(^{\alpha}_{x})$ satisfies $\psi$ in $\mathfrak{B}$ at K, i.e., $\mathfrak{a}$ satisfies $\bigwedge\alpha\psi$ in $\mathfrak{B}$ at K. A similar argument applies if $\varphi$ is

$\Lambda\overset{e}{\alpha}\psi$ instead, except we note, by Lemma 4, that $\vdash \overset{e}{V\alpha}\Box\alpha = \zeta_i \leftrightarrow V\alpha\Box\alpha = \zeta_i \wedge \overset{e}{V\alpha}\Box\alpha = \zeta_i$. If $\varphi$ is $\Lambda\sigma\psi$ or $\Lambda\overset{e}{\sigma}\psi$, then again a similar argument applies, except we note that for all i, $n \in \omega$, $Int(\pi_i^n) = int(\pi_i^n, \mathfrak{B})$ and $Int(\pi_i^n) \in E_n$ iff $\overset{e}{V\sigma}\sigma = \pi_i^n \in K*$, and thus iff $\overset{e}{V\sigma}\sigma = \pi_i^n \in K$. Finally, if $\varphi$ is of the form $\Box\psi$, then $\mathfrak{a}$ satisfies $\Box\psi$ in $\mathfrak{B}$ at K iff for all $K' \in \mathbf{K}$, $\mathfrak{a}$ satisfies $\psi$ in $\mathfrak{B}$ at K'; and therefore, by the inductive hypothesis, iff for all $K' \in \mathbf{K}$, $\psi \in K'$. Suppose then that $\mathfrak{a}$ satisfies $\Box\psi$ in $\mathfrak{B}$ at K but that $\Box\psi \notin K$. Then, since $K \in \mathbf{K}$, $\sim\Box\psi \in K$. However, since $\mathfrak{a}$ satisfies $\Box\psi$ in $\mathfrak{B}$ at K, $\mathfrak{a}$ then satisfies $\psi$ in $\mathfrak{B}$ at K, in which case $\psi \in K$, by the inductive hypothesis. Accordingly, by (II), $\sim\psi \in K'$, for some $K' \in \mathbf{K}$, which is impossible since $\psi \in K'$, for all $K' \in \mathbf{K}$. Suppose on the other hand that $\psi \in K'$, for all $K' \in \mathbf{K}$, but that $\mathfrak{a}$ does not satisfy $\Box\psi$ in $\mathfrak{B}$ at K. Then, by definition, there is a $K' \in \mathbf{K}$ such that $\mathfrak{a}$ does not satisfy $\psi$ in $\mathfrak{B}$ at K'; and therefore, by the inductive hypothesis, $\psi \notin K'$, which is impossible by assumption. Thus we conclude that (V) holds of every formula of $L*$ of length k, and therefore that $k \in A$.

We observe that since every member of $\Gamma$ is in $K*$ which is in $\mathbf{K}$, then by (V), every member of $\Gamma$ is satisfiable in $\mathfrak{B}$ at K*. It remains then only to show that $\mathfrak{B}$ is normal. But for each formula $\varphi$ of $L*$, by (A9), Lemmas 25 and 5 and the fact that every member of $\mathbf{K}$ is maximally consistent, the universal closures of $V\pi\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\varphi(\alpha_0, \ldots, \alpha_{n-1})$ and $\overset{e}{V\pi}\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}[\varphi(\alpha_0, \ldots, \alpha_{n-1}) \wedge \overset{e}{V\sigma}\sigma(\alpha_0, \ldots, \alpha_{n-1})]$ are in every member of $\mathbf{K}$, and thus, by (V) and the fact that these are closed formulas, they are true in $\mathfrak{B}$ at all $K \in \mathbf{K}$, i.e., they are valid in $\mathfrak{B}$. But then, by Lemmas 11 and 20, any generalization of $V\pi\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}\varphi(\alpha_0, \ldots, \alpha_{n-1})$ and $\overset{e}{V\pi}\pi = \hat{\alpha}_0 \ldots \hat{\alpha}_{n-1}[\varphi(\alpha_0, \ldots, \alpha_{n-1}) \wedge \overset{e}{V\sigma}\sigma(\alpha_0, \ldots, \alpha_{n-1})]$ is valid in $\mathfrak{B}$. Therefore, $\mathfrak{B}$ is normal. (Q.E.D.)

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