Gödel's Ontological Proof of God's Existence (Draft)

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Possible witnessing of positive properties

Axioms:

- **(A1)** Properties necessarily entailed by *positive* properties are also positive:
$$\forall\varphi.\forall\psi.[(P(\varphi) \land \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

- **(A2)** A property's negation is positive iff the property is not positive:
$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Lemma 1: Positive properties possibly have a witness:

$$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$$

Formal proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall\varphi.\forall\psi.[(P(\varphi) \land \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}{\forall\psi.[(P(\varphi') \land \Box\forall x.[\varphi'(x) \to \psi(x)]) \to P(\psi)]}}{(P(\varphi') \land \Box\forall x.[\varphi'(x) \to \neg\varphi'(x)]) \to P(\neg\varphi')}}{(P(\varphi') \land \Box\forall x.[\neg\varphi'(x)]) \to P(\neg\varphi')} \qquad \dfrac{\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}{P(\neg\varphi') \leftrightarrow \neg P(\varphi')}}{(P(\varphi') \land \Box\forall x.[\neg\varphi'(x)]) \to \neg P(\varphi')}}{P(\varphi') \to \Diamond\exists x.\varphi'(x)}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}$$

2 Possible existence of a God

Axioms:

- **(A3)** Being God is a positive property:

$$P(G)$$

Lemma 2: It is possible that a God exists:

$$\Diamond\exists x.G(x)$$

Formal proof:

$$\begin{prooftree}
\AxiomC{$P(G)$}
\AxiomC{}
\dashedLine
\RightLabel{Th. 1}
\UnaryInfC{$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$}
\UnaryInfC{$P(G) \to \Diamond\exists x.G(x)$}
\BinaryInfC{$\Diamond\exists x.G(x)$}
\end{prooftree}$$

3 Essentiality of being God

Definitions:

- **(D1)** An individual is a *God* if and only if he possesses all positive properties:

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \to \varphi(x)]$$

- **(D2)** A property is *essential* for an individual if and only if it holds for that inidividual and necessarily entails every other property that holds for that individual: $\varphi \ ess \ x \leftrightarrow \varphi(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$

Axioms:

- **(A4)** Positive properties are necessarily positive:

$$\forall\varphi.[P(\varphi) \to \Box\, P(\varphi)]$$

Lemma 3: If an individual is a God, then being God is an essential property for that individual:

$$\forall y.[G(y) \to G \ ess \ y]$$

Formal proof:

Let the following derivation with the open assumption $G(x)$ be Π_1:

$$\begin{prooftree}
\AxiomC{$\neg P(\psi)^1$}
\AxiomC{}
\RightLabel{Ax. 2}
\UnaryInfC{$\forall\varphi.(\neg P(\varphi) \to P(\neg\varphi))$}
\RightLabel{$\forall$E}
\UnaryInfC{$\neg P(\psi) \to P(\neg\psi)$}
\RightLabel{$\to$E}
\BinaryInfC{$P(\neg\psi)$}
\AxiomC{$G(x)$}
\dottedLine
\RightLabel{Definition of G}
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \varphi(x))$}
\RightLabel{$\forall$E}
\UnaryInfC{$P(\varphi) \to \varphi(x)$}
\RightLabel{$\to$ E}
\BinaryInfC{$\neg\psi(x)$}
\AxiomC{$\psi(x)^2$}
\RightLabel{$\to$E}
\BinaryInfC{$\bot$}
\RightLabel{RAA, 1}
\UnaryInfC{$P(\psi)$}
\RightLabel{$\to$ I, 2}
\UnaryInfC{$\psi(x) \to P(\psi)$}
\end{prooftree}$$

Let the following derivation with the open assumption $G(x)$ be Π_2:

$$\dfrac{\dfrac{\dfrac{\psi(x)^1 \qquad \begin{array}{c}\Pi_1\\ \psi(x) \to P(\psi)\end{array}}{P(\psi)}\to\mathrm{E} \qquad \dfrac{\dfrac{}{\forall\psi.(P(\psi) \to \Box P(\psi))}\text{Ax. 4}}{P(\psi) \to \Box P(\psi)}\forall\mathrm{E}}{\Box P(\psi)}\to\mathrm{E}}{\psi(x) \to \Box P(\psi)}\to\mathrm{I},\ 1$$

Let the following derivation without open assumptions be Π_3:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\psi)^1 \qquad \dfrac{\dfrac{G(x)^2}{\forall\varphi.(P(\varphi) \to \varphi(x))}\text{Definition of G}}{P(\psi) \to \psi(x)}\forall\mathrm{E}}{\psi(x)}\to\mathrm{E}}{G(x) \to \psi(x)}\to\mathrm{I},\ 2}{\forall x.(G(x) \to \psi(x))}\forall\mathrm{I}}{P(\psi) \to \forall x.(G(x) \to \psi(x))}\to\mathrm{I},\ 1$$

Let the following derivation with the open assumption $G(x)$ be Π_4:

$$\dfrac{\dfrac{\dfrac{\psi(x)^1 \qquad \begin{array}{c}\Pi_2\\ \psi(x) \to \Box P(\psi)\end{array}}{\Box P(\psi)}\to\mathrm{E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^2}{P(\psi)}\Box\mathrm{E} \qquad \begin{array}{c}\Pi_3\\ P(\psi) \to \forall x.(G(x) \to \psi(x))\end{array}}{\forall x.(G(x) \to \psi(x))}\to\mathrm{E}}{\Box\forall x.(G(x) \to \psi(x))}\text{Necessitation}}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))}\to\mathrm{I},\ 2}{\Box\forall x.(G(x) \to \psi(x)}\to\mathrm{E}}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}\to\mathrm{I},\ 1$$

The use of the necessitation rule is valid, because the only open assumption $\Box P(\psi)$ is boxed.

We construct a derivation of theorem 3 with a subderivation $\Pi_4[G(x)^1]$, which means that the open assumption $G(x)$ in Π_4 is discharged with the rule labeled 1.

$$\dfrac{\dfrac{\dfrac{\dfrac{G(x)^1 \qquad \dfrac{\begin{array}{c}\Pi_4[G(x)^1]\\ \psi(x) \to \Box\forall x.(G(x) \to \psi(x))\end{array}}{\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\forall\mathrm{I}}{G(x)\&\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\&\mathrm{I}}{G\ ess\ x}\text{Definition of ess}}{G(x) \to G\ ess\ x}\to\mathrm{I},\ 1}{\forall y.[G(y) \to G\ ess\ y]}$$

4 Necessity of God's existence

Definitions:

- **(D3)** An individual *necessarily exists* if and only if all its essential properties are necessarily witnessed:

$$E(x) \leftrightarrow \forall \varphi.[\varphi \ ess \ x \to \Box \exists x.\varphi(x)]$$

Axioms:

- **(A5)** Necessary existence is a positive property:

$$P(E)$$

Auxiliary Lemma: If there is a God, then there necessarily exists a God:

$$\exists z.G(z) \to \Box \exists x.G(x)$$

Formal proof:

$$\dfrac{\overline{\exists z.G(z)}^{\,1}}{G(g)}$$

$$\dfrac{\dfrac{G(g) \qquad \dfrac{\forall y.[G(y) \to G \ ess \ y] \ \text{Th. 3}}{G(g) \to G \ ess \ g}}{G \ ess \ g} \qquad \dfrac{\dfrac{P(E) \qquad \dfrac{\dfrac{G(g)}{\forall \varphi.[P(\varphi) \to \varphi(g)]}}{P(E) \to E(g)}}{\dfrac{E(g)}{\forall \varphi.[\varphi \ ess \ g \to \Box \exists x.\varphi(x)]}}}{G \ ess \ g \to \Box \exists x.G(x)}}{\dfrac{\Box \exists x.G(x)}{\exists z.G(z) \to \Box \exists x.G(x)}\ 1}$$

5 Necessary existence of a God

ToDo: This section still needs more details. See Coq formalization for more details.
Theorem: The existence of a God is necessary:

$$\Box \exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{\forall \varphi.[\Diamond \ldots \Diamond \Box \varphi \leftrightarrow \Box \varphi] \ \textbf{S5}}{\Diamond \Box \exists x.G(x) \leftrightarrow \Box \exists x.G(x)} \qquad \dfrac{\Diamond \exists x.G(x) \ \text{Th. 2} \qquad \exists z.G(z) \to \Box \exists x.G(x) \ \text{Th. A}}{\Diamond \Box \exists x.G(x)}}{\Box \exists x.G(x)}$$

6 God's existence

Axioms:

- **(M)** What is necessary is the case:

$$\forall \varphi.[\Box\varphi \to \varphi]$$

Corollary: There exists a God:

$$\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{}{\Box\exists x.G(x)}\text{ Th. 4} \qquad \dfrac{\forall \varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$