MODAL ALGEBRAS AND THE ONTOLOGICAL ARGUMENT

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1. Introduction

The logical structure of St. Anselm's Ontological Argument has been subject to critical analyses ever since it was formulated. Some of these are only of historical or purely philosophical interest, while others seem to have a logical content worthwhile for its own sake. The latter include Perzanowski [2], where a rather comprehensive picture of the logical structure of the Ontological Argument is drawn against the background of the lattice of propositional modal logics. Perzanowski's logical reconstruction was based upon, and indeed motivated by, an earlier attempt of Hartshorne [1].

Perzanowski's reconstruction employs a propositional constant α, whose intended reading is "the perfect being exists", and the following two conditions, characterising α:

(A1) $$\alpha \to \Box\alpha,$$

(A2) $$\Diamond\alpha.$$

The Ontological Argument then, or at least the formal part thereof, boils down to the following inference:

$$\frac{\Diamond\alpha, \alpha \to \Box\alpha}{\alpha.}$$

Hartshorne [1] shows that in **S5** the above is an instance of a derivable inference rule, namely:

(AR) $$\frac{\Diamond p, p \to \Box p}{p}$$

Perzanowski [2] shows that some weaker logics are equally good for the purpose, for example the 'Brouwerian' system **B**. It is tempting to say that logics in which AR is derivable are exactly what we are looking for. Alas, not quite so, as the following example shows. For each positive integer n consider the logic $\mathbf{V}_n$ defined relative to **K** by the axiom:

(V_n) $$\Box\bot \vee \Diamond\Box\bot \vee \cdots \vee \Diamond^n\Box\bot.$$

These form a strictly descending chain whose intersection is **K** (see for instance, Perzanowski [3]). We claim that AR is derivable in each $\mathbf{V}_n$. To see it, observe first that V_n is equivalent to:

(V'_n) $$(\Diamond\top \wedge \Box\Diamond\top \wedge \cdots \wedge \Box^n\Diamond\top) \to \bot.$$

Then, assume that the premisses of AR hold for some p. Since $\Diamond p \to \Diamond\top$ is true in all normal logics, we get that $\Diamond\top$ holds. By repeated necessitation we obtain that

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the antecedent of V'_n holds, hence $\bot$ holds as well, and by *ex falsi quodlibet* we get p. Therefore, AR is derivable, but only in virtue of the fact that the assumption $\Diamond p$ is inconsistent with the underlying logic $\mathbf{V}_n$. Although logically impeccable, it seems to be somewhat pathological from the point of view of interpretation. After all, the advocates of the Ontological Argument would presumably believe that its premisses are consistent with the logic they use. One possible way out of this corner goes via the constant α. Indeed, what the argument above also shows is that the logic arising from $\mathbf{V}_n$ by adding α to the language and A2 to the set of axioms is inconsistent. It is then reasonable to require that adding α to the language and A1, A2 to the set of axioms, results in a consistent extension of the logic we take as basic. It seems even more satisfactory to require this extension to be *conservative*.

Definition 1. *A modal logic* Λ *is* Anselmian *iff* $\Lambda \oplus \mathrm{A1} \oplus \mathrm{A2}$ *is a conservative extension of* Λ *and AR is derivable in* Λ*, where* $\Lambda \oplus \mathrm{A1} \oplus \mathrm{A2}$ *stands for the logic in the language augmented by the constant* α *with* A1 *and* A2 *added to the set of axioms.*

In what follows we will look at some Anselmian logics, mostly in the guise of varieties of modal algebras. However, we also assume familiarity with the relational semantics, in particular with the duality between algebras and frames, as we intend to move freely between the two.

2. Preliminaries

Throughout the paper we will be concerned with normal modal logics, i.e., axiomatic extensions of the minimal Kripke logic $\mathbf{K}$ closed under the Necessitation Rule:

$$\text{(NR)} \qquad \frac{p}{\Box p.}$$

We will use boldface and calligraphic capitals, respectively, for logics and corresponding varieties. Thus, for instance, $\mathbf{K}$ will stand for the minimal normal modal logic and $\mathcal{K}$ for the variety of all normal modal algebras.

We assume that modal logics and modal algebras use the same language. This may be viewed as a kind of systematic ambiguity: variables are thought of as propositional variables in the context of modal logic and as first-order individual variables in the context of modal algebras, and the same goes for connectives/functions. One deviation from this is that we will follow the custom of using l and m instead of $\Box$ and $\Diamond$ as well as 1 and 0 instead of $\top$ and $\bot$, in algebraic contexts. We will conform to this custom in the easiest way possible, namely by pretending that $\Box$, $\Diamond$, $\top$ and $\bot$ are respective shorthands for l, m, 1 and 0, so whenever we write something like $\Box\bot$, what we "really mean" is $l0$.

Notice that the decision to focus on axiomatic extensions of $\mathbf{K}$ has an important consequence for algebraic semantics, that can be briefly described as follows. Since a rule of inference, say,

$$\frac{\alpha_1, \dots, \alpha_n}{\beta,}$$

can be rendered algebraically as a quasi-identity

$$\alpha_1 = 1 \,\&\, \dots \,\&\, \alpha_n = 1 \Rightarrow \beta = 1,$$

modal logics with additional rules of inference correspond to quasivarieties of modal algebras, whereas normal modal logics correspond to varieties. To state it more formally, we recall the following well-known result.

Theorem 1. *The lattice of normal modal logics is dually isomorphic to the lattice of subvarieties of* $\mathcal{K}$.

Against this background, we need to distinguish between *derivable* and *admissible* rules of inference. Let Λ be a logic of a consequence operation C. A rule ρ is derivable in Λ if the conclusion of ρ can be inferred from the premisses of ρ using the axioms of Λ and rules of C. A rule ρ is admissible in Λ if $C^\rho(\emptyset) = \Lambda = C(\emptyset)$, where C^ρ is the consequence operation obtained by augmenting C by the rule ρ. In other words, ρ is admissible if we do not gain any new theorems by adding ρ to the rules of inference already in stock. Let $\mathcal{V}_\Lambda$ stand for the variety corresponding to Λ; let $\mathcal{Q}^\rho_\Lambda$ stand for the quasivariety axiomatised by all the axioms (identities) of Λ, plus all the rules (quasi-identities) of C, plus the quasi-identity corresponding to ρ.

Lemma 1. *The rule ρ is admissible in Λ iff the varietal closure of $\mathcal{Q}^\rho_\Lambda$ is $\mathcal{V}_\Lambda$.*

Proof. Suppose ρ is admissible in Λ. Therefore, the identities valid in $\mathcal{Q}^\rho_\Lambda$ and in $\mathcal{V}_\Lambda$ are precisely the same. By definition of $\mathcal{Q}^\rho_\Lambda$, we have $\mathcal{Q}^\rho_\Lambda \subseteq \mathcal{V}_\Lambda$, thus $H(\mathcal{Q}^\rho_\Lambda) \subseteq \mathcal{V}_\Lambda$. If this inclusion were proper, there would be an identity valid in $H(\mathcal{Q}^\rho_\Lambda)$ but not in $\mathcal{V}_\Lambda$. In particular, this identity would be valid throughout $\mathcal{Q}^\rho_\Lambda$ but not in $\mathcal{V}_\Lambda$, which contradicts admissibility of ρ.

Conversely, assume $H(\mathcal{Q}^\rho_\Lambda) = \mathcal{V}_\Lambda$. The conclusion then follows immediately, since $H(\mathcal{Q}^\rho_\Lambda)$ and $\mathcal{Q}^\rho_\Lambda$ have exactly the same valid identities. □

3. A not-quite-Anselmian quasivariety

Let Λ be an Anselmian logic. Then, A1, A2 and AR force α to be $\top$, i.e., $\Lambda \oplus \mathrm{A1} \oplus \mathrm{A2}$ contains $\alpha \leftrightarrow \top$. Thus, any Anselmian logic Λ must satisfy $\top \to \Box\top$ and $\Diamond\top$. The former is true in all normal logics, the latter is the characteristic axiom of the logic $\mathbf{D} = \mathbf{K} \oplus \Diamond\top$. Our first result follows from this.

Theorem 2. *All Anselmian logics contain* $\mathbf{D}$.

Now, let C_A be the consequence operator defined by all the axioms and rules of $\mathbf{D}$ plus the inference rule AR. Consider $C_A(\emptyset)$, and define $\mathbf{D_A}$ to be the smallest normal modal logic equal to $C_A(\emptyset)$. Observe that although AR is among the inference rules of C_A it does not follow that AR is derivable in $\mathbf{D_A}$. Indeed, we will show that $\mathbf{D_A}$ is not Anselmian, and in fact $\mathbf{D_A} = \mathbf{D}$.

An algebraic rendering of C_A is provided by a quasivariety $\mathcal{Q}_A$ of modal algebras, defined relative to $\mathcal{D}$, by the following quasi-identity:

($\star$) $mx = 1 = x \to lx \Rightarrow x = 1$.

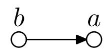
Now, the algebraic counterpart of the logic $\mathbf{D_A}$ is the varietal closure of $\mathcal{Q}_A$, call it $\mathcal{D}_A$. Certainly, $\mathcal{D}_A = H(\mathcal{Q}_A)$.

Theorem 3. *The logic $\mathbf{D_A}$ is Anselmian iff $\mathcal{Q}_A = \mathcal{D}_A$ iff $\mathcal{Q}_A$ is a variety.*

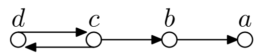
Proof. By a simple restatement of the definition of 'Anselmian' in algebraic terms, we have: the logic $\mathbf{D_A}$ is Anselmian iff the quasi-identity ($\star$) holds throughout the variety $\mathcal{D}_A$. Since $\mathcal{D}_A = H(\mathcal{Q}_A)$, it means that ($\star$) holds in all homomorphic images of algebras from $\mathcal{Q}_A$, which amounts to saying that $\mathcal{Q}_A = H(\mathcal{Q}_A) = \mathcal{D}_A$. □

Theorem 4. *The quasivariety $\mathcal{Q}_A$ is not a variety. Therefore, the logic $\mathbf{D}_\mathbf{A}$ is not Anselmian.*

Proof. Consider the algebras $\mathbf{A}$ and $\mathbf{B}$ whose frame representations $\mathfrak{A} = (W_A, R_A)$ and $\mathfrak{B} = (W_B, R_B)$ are depicted below. We use the standard device of representing reflexive points by "empty" dots and irreflexive points by filled ones, so both these frames are reflexive. Then, $\mathfrak{A}$ is a generated subframe of $\mathfrak{B}$, and thus $\mathbf{A}$ is a homomorphic image of $\mathbf{B}$.



$\mathfrak{A}$



$\mathfrak{B}$

Now, $\mathfrak{A}$ falsifies $(\star)$ for we have: $R_A^{-1}\{a\} = W_A$, hence $\Diamond\{a\} = \top$, and $R_A^{-1}(W_A \setminus \{a\}) = R_A^{-1}\{b\} = \{b\}$, hence $\Diamond(\neg\{a\}) \subseteq \neg\{a\}$, so the antecedent holds; however, $\{a\} \neq A_W$ and thus the consequent fails. To see that $\mathfrak{B}$ satisfies $(\star)$ observe that if X is a proper subset of W_B with $R_B^{-1}(X) = B$, then X contains $\{a, c\}$ or $\{a, d\}$. Thus, $W_B \setminus X \subseteq \{b, d\}$, or $W_B \setminus X \subseteq \{b, c\}$. Since X is proper, it leads to five possible cases, none of which satisfies $R_B^{-1}(W_B \setminus X) \subseteq W_B \setminus X$. $\square$

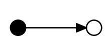
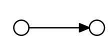
Theorem 5. *The varietal closure of $\mathcal{Q}_A$ is the variety $\mathcal{D}$. Thus, $\mathbf{D}_\mathbf{A} = \mathbf{D}$ and therefore the rule AR is admissible in $\mathbf{D}$.*

Proof. The construction we have just given for a particular case is in fact generally applicable, which shows that each algebra $\mathbf{A}$ in $\mathcal{D}$ is a homomorphic image of an algebra $\mathbf{B}$ in $\mathcal{Q}_A$. We will not give the details here, as the claim will also follow as a corollary to Theorem 9 in the next section. Admissibility of AR then follows by Lemma 1. $\square$

4. Anselmian varieties

Admissibility of AR in $\mathbf{D}$ suggests that there should be Anselmian logics somewhere rather close to $\mathbf{D}$. We will characterise Anselmian logics, or rather the varieties corresponding to them (i.e., *Anselmian varieties*), in two ways: by an exclusion principle, and equationally. Then, we will show that there is no largest Anselmian variety, indeed, the varietal join of all Anselmian varieties is $\mathcal{D}$.

4.1. **Two characterisations.** Certainly, for any variety $\mathcal{V}$ contained in $\mathcal{Q}_A$, its corresponding logic is Anselmian, and conversely, all varieties corresponding to Anselmian logics are contained in $\mathcal{Q}_A$. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be algebras whose Kripke frames are pictured below:

$\mathfrak{E}_1$  $\mathfrak{E}_2$ 

Notice that $\mathbf{E}_2$ is isomorphic to the algebra $\mathbf{A}$ from the proof of Theorem 4.

Theorem 6. *A variety $\mathcal{V}$ is Anselmian iff neither $\mathbf{E}_1$ nor $\mathbf{E}_2$ belong to $\mathcal{V}$.*

Proof. That $\mathbf{E}_2$ falsifies $(\star)$ was shown in the course of proving Theorem 4. The argument for $\mathbf{E}_1$ is analogous. Thus, it remains to be shown that if $\mathcal{V}$ is not Anselmian, it must contain either $\mathbf{E}_1$ or $\mathbf{E}_2$. Let $\mathbf{A} \in \mathcal{V}$ falsify $(\star)$. There is then an element $a \in A \setminus \{1\}$ with $a \leq la$ and $l\neg a = 0$. We have two cases:

Case 1. If $a = la$, then $\{1, a, \neg a, 0\} \subseteq A$ is closed under the operations. The algebra with this universe is easily seen to be isomorphic to $\mathbf{E}_2$. Thus, $\mathbf{E}_2 \subseteq \mathbf{A} \in \mathcal{V}$.
Case 2. If $a < la$, then let θ be the congruence on $\mathbf{A}$ generated by $(la, 1)$. From $a < la$ it follows that $la \leq l^2 a$, so the principal filter $\uparrow\{a\}$ is closed under l, hence a congruence class. In A/θ, we then have $la/\theta = 1$ and $a/\theta < 1/\theta$, therefore $\{1/\theta, a/\theta, \neg a/\theta, 0/\theta\}$ is closed under the operations, and the algebra with this universe is isomorphic to $\mathbf{E}_1$. Thus, $\mathbf{E}_1 \in HS(\mathbf{A}) \subseteq \mathcal{V}$. □

Since $\mathbf{E}_1$ and $\mathbf{E}_2$ are subdirectly irreducible, but not simple, we immediately get the following result, generalising the observation that **S5** is Anselmian.

Corollary 1. *All semisimple varieties of modal algebras are Anselmian.*

We will see shortly that the converse does not hold. Now we state the promised equational characterisation.

Theorem 7. *A subvariety $\mathcal{V}$ of $\mathcal{D}$ is Anselmian if and only if it satisfies the identity*

$$l^{\leq k}((x \to lx) \wedge mx) \leq x \tag{I$_k$}$$

for some $k \in \omega$, where $l^{\leq k} z$ stands for $z \wedge lz \wedge \cdots \wedge l^k z$.

Proof. Let $\mathcal{V} \subseteq \mathcal{D}$ satisfy $l^{\leq k}((x \to lx) \wedge mx) \leq x$ for some $k \in \omega$. Suppose $a \in \mathbf{A} \in \mathcal{V}$ has $ma = 1$ and $a \to la = 1$. Then, $l^{\leq k}((a \to la) \wedge ma) = 1$ and so $a = 1$ as required. For the converse, we will prove the contrapositive. Assume $\mathcal{V}$ falsifies $l^{\leq k}((x \to lx) \wedge mx) \leq x$ for all k, and let $\mathbf{F}(x)$ be the free algebra in $\mathcal{V}$ generated by x. Put $\Theta = \mathbf{Cg}^{\mathbf{F}(x)}(u, 1)$, where $u = (x \to lx) \wedge mx$. Since $(x, 1) \in \Theta$ iff $l^{\leq k} u \leq x$ for some k, we get that $(x, 1) \notin \Theta$. Letting $\mathbf{B} = \mathbf{F}(x)/\Theta$ and $b = x/\Theta$, we obtain that $\mathbf{B} \models mb = 1, b \leq lb, b \neq 1$. Thus $\mathbf{B}$ is not Anselmian, as claimed. □

4.2. **Cyclic varieties.** Let $\mathcal{DB}$ stand for the variety defined relative to $\mathcal{D}$ by the 'Brouwerian' identity

(B) $x \leq lmx$.

It was shown in [2] that B implies ($\star$), and so $\mathcal{DB}$ is Anselmian (as $\mathcal{DB}$ is not semisimple, it also shows that the converse of Corollary 1 fails). It is a good exercise to derive (I$_1$) from the defining identites of $\mathcal{DB}$, but we leave it to the interested reader. Instead, we will generalise the observation from [2], showing that certain weaknings of (B) imply ($\star$) as well. For any positive integer n consider the identity

(B$_n$) $x \leq l(mx \vee m^2 x \vee \cdots \vee m^n x)$

and let $\mathcal{DB}_n$ be the variety defined relative to $\mathcal{D}$ by (B$_n$). Note that $\mathcal{DB}_1 = \mathcal{DB}$ and $\mathcal{DB}_{n+1} \supset \mathcal{DB}_n$ for all positive integers n; these inclusions are proper. Moreover, $\mathcal{DB}_n$ has a natural characterisation in terms of Kripke frames, namely, $\mathcal{DB}_n$ is the logic of all frames $\langle W, R\rangle$ such that R is *serial* (i.e., has no endpoints), and satisfies

(S$_n$) if xRy, then $yR^k x$, for some $k \in \{1, \dots, n\}$,

which amounts to a weak form of symmetry. Such frames are also known as *cyclic.*

Theorem 8. *For every positive integer n, the variety $\mathcal{DB}_n$ is Anselmian. Therefore, if $\mathcal{V}$ is a subvariety of $\mathcal{DB}_n$ for some n, then $\mathcal{V}$ is Anselmian.*

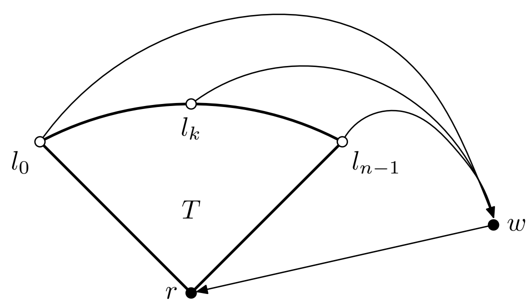
Proof. Take an $a \in \mathbf{A} \in \mathcal{DB}_n$ with $ma = 1$ and $a \leq la$. Thus, $l\neg a = 0$ and $\neg a \geq m\neg a$. By monotonicity of m, we obtain $m\neg a \geq m^2\neg a$. Repeating it n times yields $\neg a \geq m\neg a \geq \cdots \geq m^n \neg a$ and therefore $\neg a \geq m\neg a \vee m^2\neg a \vee \cdots \vee m^n\neg a$.

Then the assumptions above together with B_n yield: $0 = l\neg a \geq l(m\neg a \vee m^2\neg a \vee \cdots \vee m^n\neg a) \geq \neg a$. Thus, $\neg a = 0$, i.e., $a = 1$ as required. The second statement is then obvious. □

Now we are in a position to prove the result of which Theorem 5 is a corollary.

Theorem 9. *The varietal join* $\bigvee_{n\in\omega} \mathcal{DB}_{n+1}$ *is equal to* $\mathcal{D}$.

Proof. We will argue via Kripke frames. It suffices to show that every non-theorem of $\mathbf{D}$ can be falsified on a frame satisfying S_n, for some positive integer n. It is known that every $\varphi \notin \mathbf{D}$ can be falsified on a finite frame $\langle T, S\rangle$, such that the reflexive transitive closure $\leq$ of S makes $\langle T, \leq\rangle$ into a tree, and the relation S itself is intransitive and irreflexive, except on maximal elements of T. Take a $\varphi \notin \mathbf{D}$. Let $\langle T, S\rangle$ be a finite frame of the kind described above and falsifying φ. Let r (*root*) be the smallest element of T under $\leq$, and $l_0, \dots, l_{k-1}$ (*leaves*) be all maximal elements of T under $\leq$. We can assume that φ is falsified at r and for any leaf l, the distance $rS \circ \cdots \circ Sl$ is equal to the *modal depth* (the greatest number of nested occurences of modal operators) of φ. Construct the frame $\langle W, R\rangle$ as follows: put $W = T \cup \{w\}$, for some $w \notin T$; then define $R = S \cup \{(l_i, w) : 0 \leq i \leq k-1\} \cup \{(w, r)\}$. Our $\langle W, R\rangle$ looks like this:

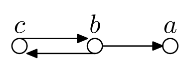


Since T is finite, $\langle W, R\rangle$ satisfies S_n, with $n-1$ equal to modal depth of φ. Moreover, the construction ensures that $\langle W, R\rangle$ still falsifies φ at r. □

Theorem 10. *The intersection of all Anselmian logics is equal to* $\mathbf{D}$. *In particular, there is no smallest Anselmian logic.*

Proof. The first statement follows by Theorem 6 and duality. Namely, the intersection of all Anselmian logics corresponds to the varietal join of all Anselmian varieties, and this contains $\bigvee_{n\in\omega} \mathcal{DB}_{n+1} = \mathcal{D}$. Thus, the join of all Anselmian varieties is exactly $\mathcal{D}$. For the second statement, observe that the smallest Anselmian logic, if it existed, would be equal to the intersection of all Anselmian logics, that is to $\mathbf{D}$. Since $\mathbf{D}$ is not Anselmian the claim follows. □

Let us now show that the converse of Theorem 8 does not hold. Consider the variety generated by the algebra $\mathbf{C}$ of the frame depicted below.



Theorem 11. *The variety* $V(\mathbf{C})$ *is Anselmian, but* $V(\mathbf{C}) \nsubseteq \mathcal{DB}_n$ *for any* n.

Proof. The variety $V(\mathbf{C})$ is clearly not cyclic, so $V(\mathbf{C})$ is not a subvariety of $\mathcal{DB}_n$ for any n. To show that $V(\mathbf{C})$ is Anselmian, it suffices to show that $(\star)$ holds in all subdirectly irreducible members of $V(\mathbf{C})$. But all subdirectly ireducible members of $V(\mathbf{C})$ belong to $HS(\mathbf{C})$, and it is not difficult to see that there are precisely two of these: $\mathbf{C}$ itself, and the algebra of a single reflexive point. Both satisfy ($\star$). $\square$

The example above seems to indicate that at least some amount of cyclicity is essential for being Anselmian. Observe that removing any one of the twin arrows joining c and b results in a frame falsifying ($\star$). We will probe into this a little deeper in the next section.

5. Anselmian frames

We will call a Kripke frame Anselmian, if it satisfies ($\star$). To keep the note self-contained, we will now recall the *standard translation* of modal formulae to monadic second-order formulae, called their *second-order correspondents*. For a modal formula ϕ, its standard translation $st_x(\phi)$ is recursively defined as:

- $st_x(p) = Px$, for a propositional variable p
- $st_x(\phi \wedge \psi) = st_x(\phi) \wedge st_x(\psi)$
- $st_x(\neg\phi) = \neg st_x(\phi)$
- $st_x(\Diamond\phi) = \exists y\colon xRy \mathbin{\&} st_y(\phi)$

where x, y are individual variables, R is a binary predicate constant, and P is a unary predicate variable. Let $\phi = \phi(p_1, \dots, p_k)$. The second-order correspondent of ϕ, is the formula $\forall P_1, \dots, P_k \forall x\colon st_x(\phi)$. Now, consider an inference rule

$$\frac{\alpha_1, \dots, \alpha_n}{\beta} \tag{Q}$$

The second-order correspondent of Q is the formula

$$\forall P_1, \dots, P_k\colon (\forall x\colon st_x(\alpha_1)) \mathbin{\&} \dots \mathbin{\&} (\forall x\colon st_x(\alpha_n)) \Rightarrow \forall x\colon st_x(\beta)$$

The following lemma is (most probably) folklore.

Lemma 2. *A Kripke frame $\mathfrak{F}$ satisfies an inference rule Q if and only if $\mathfrak{F}$ satisfies the second-order correspondent of Q.*

Proof. The usual modal satisfiability conditions are second-order statements about Kripke frames. Standard translation just formalises these as open formulae, and second-order correspondents are obtained by an appropriate universal quantification. $\square$

Lemma 3. *The second-order correspondent of ($\star$) is the formula*

$$\forall P\colon (\forall x \exists y\colon xRy \mathbin{\&} Py) \mathbin{\&} (\forall x\colon Px \Rightarrow (\forall y\colon xRy \Rightarrow Py)) \Rightarrow \forall x\colon Px.$$

Less formally but perhaps more informatively, we can say that $\mathfrak{F} = (W, R)$ is Anselmian iff it satisfies the following: if a subset P of W contains a successor of every point and is successor closed, then $P = W$.

Theorem 12. *Let $\mathfrak{F} = (W, R)$ be Anselmian. If $\mathfrak{F}$ is rooted and $|W| > 1$, then every $w \in W$ is a successor of some w' distinct from w.*

Proof. Take any $w \in W$ and let $P = W \setminus \{w\}$. We will first show that wRu holds for some $u \neq w$. If w is irreflexive, then wRu must hold for some $u \neq w$, because $\mathfrak{F}$ has no endpoints. If w is reflexive and wRu does not hold for any $u \neq w$, then $\mathfrak{F}$ is a disjoint union of $(P, R|_P)$ and $(\{w\}, R|_{\{w\}})$. But then $\mathfrak{F}$ is not rooted, contradicting the assumptions. Thus, we have established that wRu holds for some $u \neq w$. It follows that P contains a successor of every point. Now, since $\mathfrak{F}$ is Anselmian, P

cannot be successor closed, by Lemma 3. Therefore, $w'Rw$ holds for some $w' \in P$ proving the claim. $\square$

6. Loose ends

*** This will get organised some day.

One. Adding reflexivity. We get the same results as for **D**.

Two. Adding transitivity. Whoops! As we have EDPC, all finite si's are splitting. Then by the exclusion principle we get the largest Anselmian variety, namely $\mathcal{V}_1 \cap \mathcal{V}_2$, where $\mathcal{V}_i$ is the splitting companion of $V(\mathbf{E}_i)$, for $i \in \{1, 2\}$. Then, $\mathcal{V}_1 \cap \mathcal{V}_2$ can be finitely axiomatised by Jankov formulas (or by Zakhariaschev subframe formulas). Moreover, it can be easily shown that it has finite model property: a version of standard filtration preserves $\star$.

Three. Adding k-transitivity for some k. Effect as for transitivity, because of EDPC.

Four. Deontic logics. Axioms like $x \leq mlx$ (and generalisations) more natural than $x \leq lmx$ etc. Interesting also because of the Kripke-frame condition $xRy \Rightarrow \exists z \neq x \colon yRz$.

References

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