

# Formalization, Mechanization and Automation of Gödel's Proof of God's Existence\*

Christoph Benz Müller<sup>1</sup> and Bruno Woltzenlogel Paleo<sup>2</sup>

<sup>1</sup> Dahlem Center for Intelligent Systems, Freie Universität Berlin, Germany  
c.benzmueller@gmail.com

<sup>2</sup> Theory and Logic Group, Vienna University of Technology, Austria  
bruno@logic.at

Attempts to prove the existence (or non-existence) of God by means of abstract ontological arguments are an old tradition in philosophy and theology. Gödel's proof [12, 13] is a modern culmination of this tradition, following particularly the footsteps of Leibniz. Gödel defines God as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable (but debatable) axioms that they should satisfy. Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (cf. [19, 2, 11, 1, 10]).

Dana Scott's version of Gödel's proof [18] employs the following axioms (**A**), definitions (**D**), corollaries (**C**) and theorems (**T**), and it proceeds in the following order:<sup>3</sup>

- |  |   |
|--|---|
| <b>A1</b> Either a property or its negation is positive, but not both:   | $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$   |
| <b>A2</b> A property necessarily implied<br>by a positive property is positive:  | $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$                         |
| <b>T1</b> Positive properties are possibly exemplified:  | $\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$   |
| <b>D1</b> A <i>God-like</i> being possesses all positive properties:   | $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$   |
| <b>A3</b> The property of being God-like is positive:  | $P(G)$  |
| <b>C</b> Possibly, God exists:   | $\Diamond\exists xG(x)$   |
| <b>A4</b> Positive properties are necessarily positive:  | $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$   |
| <b>D2</b> An <i>essence</i> of an individual is<br>a property possessed by it and<br>necessarily implying any of its properties: | $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$ |
| <b>T2</b> Being God-like is an essence of any God-like being:  | $\forall x[G(x) \rightarrow G \text{ ess. } x]$   |
| <b>D3</b> <i>Necessary existence</i> of an individual is<br>the necessary exemplification of all its essences:                   | $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$  |
| <b>A5</b> Necessary existence is a positive property:  | $P(NE)$   |
| <b>T3</b> Necessarily, God exists:   | $\Box\exists xG(x)$   |

Scott's version of Gödel's proof has now been analysed for the first-time with an unprecedented degree of detail and formality with the help of theorem provers; cf. [17]. The following has been done (and in this order):

- A detailed natural deduction proof.
- A formalization of the axioms, definitions and theorems in the TPTP THF syntax [20].
- Automatic verification of the consistency of the axioms and definitions with Nitpick [8].
- Automatic demonstration of the theorems with the provers LEO-II [5] and Satallax [9].
- A step-by-step formalization using the Coq proof assistant [6].
- A formalization using the Isabelle proof assistant [16], where the theorems (and some additional lemmata) have been automated with Sledgehammer [7] and Metis [15].

\* This work has been supported by the German Research Foundation under grant BE2501/9-1.

<sup>3</sup> A1, A2, A5, D1, D3 are logically equivalent to, respectively, axioms 2, 5 and 4 and definitions 1 and 3 in Gödel's notes [12, 13]. A3 was introduced by Scott [18] and could be derived from Gödel's axiom 1 and D1 in a logic with infinitary conjunction. A4 is a weaker form of Gödel's axiom 3. D2 has an extra conjunct  $\phi(x)$  lacking in Gödel's definition 2; this is believed to have been an oversight by Gödel [14].

Gödel's proof is challenging to formalize and verify because it requires an expressive logical language with modal operators (*possibly* and *necessarily*) and with quantifiers for individuals and properties. Our computer-assisted formalizations rely on an embedding of the modal logic into classical higher-order logic with Henkin semantics [4, 3]. The formalization is thus essentially done in classical higher-order logic where quantified modal logic is emulated.

In our ongoing computer-assisted study of Gödel's proof we have obtained the following results:

- The basic modal logic K is sufficient for proving T1, C and T2.
- Modal logic S5 is not needed for proving T3; the logic KB is sufficient.
- Without the first conjunct  $\phi(x)$  in D2 the set of axioms and definitions would be inconsistent.
- For proving theorem T1, only the left to right direction of axiom A1 is needed. However, the backward direction of A1 is required for proving T2.

This work attests the maturity of contemporary interactive and automated deduction tools for classical higher-order logic and demonstrates the elegance and practical relevance of the embeddings-based approach. Most importantly, our work opens new perspectives for a computer-assisted theoretical philosophy. The critical discussion of the underlying concepts, definitions and axioms remains a human responsibility, but the computer can assist in building and checking rigorously correct logical arguments. In case of logico-philosophical disputes, the computer can check the disputing arguments and partially fulfill Leibniz' dictum: *Calculemus* — Let us calculate!

## References

1. R.M. Adams. Introductory note to \*1970. In *Kurt Gödel: Collected Works Vol. 3: Unpublished Essays and Letters*. Oxford University Press, 1995.
2. A.C. Anderson and M. Gettings. Gödel ontological proof revisited. In *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics: Lecture Notes in Logic 6*. Springer, 1996.
3. C. Benz Müller and L.C. Paulson. Exploring properties of normal multimodal logics in simple type theory with LEO-II. In *Festschrift in Honor of Peter B. Andrews on His 70th Birthday*, pages 386–406. College Publications.
4. C. Benz Müller and L.C. Paulson. Quantified multimodal logics in simple type theory. *Logica Universalis (Special Issue on Multimodal Logics)*, 7(1):7–20, 2013.
5. C. Benz Müller, F. Theiss, L. Paulson, and A. Fietzke. LEO-II - a cooperative automatic theorem prover for higher-order logic. In *Proc. of IJCAR 2008*, volume 5195 of *LNAI*, pages 162–170. Springer, 2008.
6. Y. Bertot and P. Casteran. *Interactive Theorem Proving and Program Development*. Springer, 2004.
7. J.C. Blanchette, S. Böhme, and L.C. Paulson. Extending Sledgehammer with SMT solvers. *Journal of Automated Reasoning*, 51(1):109–128, 2013.
8. J.C. Blanchette and T. Nipkow. Nitpick: A counterexample generator for higher-order logic based on a relational model finder. In *Proc. of ITP 2010*, number 6172 in *LNCS*, pages 131–146. Springer, 2010.
9. C.E. Brown. Satallax: An automated higher-order prover. In *Proc. of IJCAR 2012*, number 7364 in *LNAI*, pages 111 – 117. Springer, 2012.
10. R. Corazzon. Contemporary bibliography on the ontological proof (<http://www.ontology.co/biblio/ontological-proof-contemporary-biblio.htm>).
11. M. Fitting. *Types, Tableaux and Gödel's God*. Kluwer Academic Press, 2002.
12. K. Gödel. Ontological proof. In *Kurt Gödel: Collected Works Vol. 3: Unpublished Essays and Letters*. Oxford University Press, 1970.
13. K. Gödel. Appendix A. Notes in Kurt Gödel's Hand, pages 144–145. In [19], 2004.
14. A.P. Hazen. On gödel's ontological proof. *Australasian Journal of Philosophy*, 76:361–377, 1998.
15. J. Hurd. First-order proof tactics in higher-order logic theorem provers. In *Design and Application of Strategies/Tactics in Higher Order Logics, NASA Tech. Rep. NASA/CP-2003-212448*, 2003.
16. T. Nipkow, L.C. Paulson, and M. Wenzel. Isabelle/HOL: A Proof Assistant for Higher-Order Logic. Number 2283 in *LNCS*. Springer, 2002.
17. B. Woltzenlogel Paleo and C. Benz Müller. Formal theology repository (<https://github.com/FormalTheology/GoedelGod>).
18. D. Scott. Appendix B. Notes in Dana Scott's Hand, pages 145–146. In [19], 2004.
19. J.H. Sobel. *Logic and Theism: Arguments for and Against Beliefs in God*. Cambridge U. Press, 2004.
20. G. Sutcliffe and C. Benz Müller. Automated reasoning in higher-order logic using the TPTP THF infrastructure. *Journal of Formalized Reasoning*, 3(1):1–27, 2010.