The Ontological Modal Collapse as a Collapse of the Square of Opposition

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Abstract. The *modal collapse* that afflicts Gödel's modal ontological argument for God's existence is discussed from the perspective of the modal square of opposition.

Mathematics Subject Classification (2010). Prim. 03A02; Sec. 68T02 .

Keywords. Modal Logics, Higher-Order Logics, Ontological Argument.

1. Introduction

Attempts to prove the existence (or non-existence) of God by means of abstract, ontological arguments are an old tradition in western philosophy, with contributions by several prominent philosophers, including St. Anselm of Canterbury, Descartes and Leibniz. Kurt Gödel studied and further improved this argument, bringing it to a mathematically more precise form, as a chain of axioms, lemmas and theorems in a modal logic [23, 30], shown in Fig. 1.

Gödel defines God as a being who possesses all *positive* properties and states a few reasonable (but debatable) axioms that such properties should satisfy. The overall idea of Gödel's proof is in the tradition of Anselm's argument, who defined God as some entity of which nothing greater can be conceived. Anselm argued that existence in the actual world would make such an assumed being even greater; hence, by definition, God must exist. However, for Anselm existence was treated as a predicate and the possibility of God's existence was assumed as granted. These issues were criticized by Kant and Leibniz, respectively, and succesfully addressed by Gödel.

Nevertheless, Gödel's work still leaves room for criticism. In particular, his axioms are so strong that they entail a *modal collapse* [**?**, 31]: everything that is the case is so necessarily. There has been an impressive body of recent and ongoing work (cf. [31, 19, 3, 2, **?**, 18] and the references therein) proposing solutions for the modal collapse. The goal of this short note is to discuss the modal collapse from the point of view of the modal square of opposition.

A1 Either a property or its negation is positive, but not both:

$$\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \equiv \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\, P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi \; ess \; x \equiv \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G \; ess \; x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \equiv \forall\varphi[\varphi \; ess \; x \rightarrow \Box\exists y\varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

L1 If a god-like being exists, then necessarily a god-like being exists:

$$\exists x G(x) \rightarrow \Box\exists y G(y)$$

L2 If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists x G(x) \rightarrow \Box\exists y G(y)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists x G(x)$$

FIGURE 1. Scott's version of Gödel's ontological argument [30].

2. A Collapse of the Modal Square

A crucial step of most ontological arguments is the claim that if God's existence is possible, then it is necessary. This is Lemma **L2** in Gödel's proof. In the modal square of opposition (Fig. 2), this is an unusual situation in which the **I** corner must imply and entail the **A** corner, in the particular case when ϕ is $\exists x G(x)$. Gödel's proof shows that his axioms are strong enough to invert

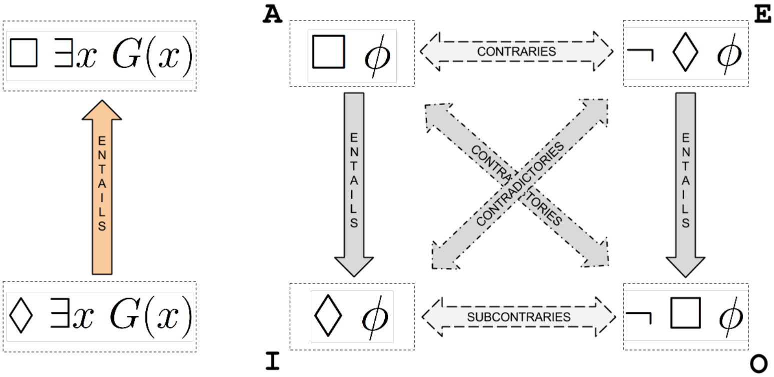

FIGURE 2. Modal Square of Opposition.

MC	Everything that is the case is so necessarily: $\forall\phi[\phi \to \Box\phi]$
MC'	Everything that is possible is necessary: $\forall\phi[\Diamond\phi \to \Box\phi]$
T	Everything that is necessary is the case: $\forall\phi[\Box\phi \to \phi]$
ExImp	(Modal Existential Import): $\Diamond\top$
AI	Everything that is necessary is possible: $\forall\phi[\Box\phi \to \Diamond\phi]$
MC"	Modalities collapse completely: $\forall\phi[\phi \leftrightarrow \Diamond\phi \leftrightarrow \Box\phi]$

FIGURE 3. Modal Collapse

the direction of entailment for the sentence at issue. The question, however, is whether the axioms are not too strong, also allowing the inverted entailment for arbitrary ϕ. That is essentially the question asked by Sobel [**?**]; and his proof of the modal collapse (**MC**) provides an affirmative answer. It is possible to show that this form of the modal collapse entails (in modal logic **K**) a collapse of the modal square (**MC'**), causing the subcontraries to entail (and even imply) their respective contraries. Normally, as shown in Fig. 2, in the modal square of opposition only the other direction of entailment holds: the contraries entail their subcontraries, assuming the *modal existential import* **ExImp**.

Moreover, in any modal logic where the axiom **T** holds (i.e. where the accessibility relation is reflexive), a total collapse of the modalities (**MC"**) occurs. Interestingly, under this stronger form of modal collapse, the contraries entail their subcontraries even without the existential import.

Although Gödel's axioms lead to modal collapse, there are several variants (e.g. [**?**, **?**]) that are known to be immune to the modal collapse. This

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

A:MC The modal collapse happens for any positive properties applied to any god-like being:

$$\forall\varphi\forall x[(P(\varphi) \wedge G_A(x)) \rightarrow \mathit{collapse}(\varphi(x))]$$

A:MC1 The modal collapse does *not* happen for positive properties applied to arbitrary individuals (*counter-satisfiable*):

$$\forall\varphi\forall x[P(\varphi) \rightarrow \mathit{collapse}(\varphi(x))]$$

A:MC2 The modal collapse does *not* happen for an arbitrary properties applied to a god-like being (*counter-satisfiable*):

$$\forall\varphi\forall x[G_A(x) \rightarrow \mathit{collapse}(\varphi(x))]$$

FIGURE 4. Restricted Collapse for Anderson's Emendation [**?**]

means there must be at least one proposition ϕ such that $\phi \rightarrow \Box\phi$ (from now on abbreviated $\mathit{collapse}(\phi)$) is not valid under the axioms and definitions used by the variant. But if the variant is sufficiently similar to Gödel's argument, following the path deriving Lemmas **L1** and **L2**, then $\mathit{collapse}(\exists x G(x))$ must be valid. Therefore, one may wonder how strong is their immunity to the modal collapse: is there any other proposition ϕ for which $\mathit{collapse}(\phi)$ is also valid?

For Anderson's emendation [**?**], for example, a form of the modal collapse (**A:MC**), restricted to positive properties applied to god-like beings, can be derived. The proof, under the modal logic **K**, depends only on Anderson's alternative definition of god-like being (**A:D1**). This class of propositions for which the collapse occurs is tight: weaker restrictions (**A:MC1** and **A:MC2**), which could lead to larger classes, are counter-satisfiable.

ToDo: write a similar paragraph about Bjordal's alternative??

Independently of the variant of the ontological argument under consideration, the following can be said about the class of collapsing propositions:

1. Valid propositions are collapsing: if ϕ is valid, then $\mathit{collapse}(\phi)$ is valid.
2. The class of collapsing propositions is closed under logical equivalence: if $\mathit{collapse}(\phi)$ is valid and $\phi \leftrightarrow \phi'$ is valid, then $\mathit{collapse}(\phi')$ is valid.
3. The class of collapsing propositions is not generally closed under equi-validity: even if $\mathit{collapse}(\phi)$ is valid and ϕ and ϕ' are equi-valid, $\mathit{collapse}(\phi')$ may not be valid.
4. The class of collapsing propositions is not generally closed under implication: even if $\mathit{collapse}(\phi)$ is valid and $\phi \rightarrow \phi'$ is valid, $\mathit{collapse}(\phi')$ may not be valid.

3. Conclusions

Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (cf. [31, 3, 2, 19, 1, 18]).

In theoretical philosophy, formal logical confrontations with such ontological arguments had been so far (mainly) limited to paper and pen. Up to now, the use of computers was prevented, because the logics of the available theorem proving systems were not expressive enough to formalize the abstract concepts adequately. Gödel's proof uses, for example, a complex higher-order modal logic (HOML) to handle concepts such as *possibility* and *necessity* and to support quantification over individuals and properties.

controversies, care with parameters

ToDo: Leibniz calculemus, Rushby, Zalta [27, 28]

The technique enabling this analysis is the embedding of quantified modal logics into higher-order logics [10, 9, 6], for which automated theorem provers exist [**?**, **?**, **?**, **?**]. This technique has already been succesfully employed in the verification and reconstruction of Gödel's proof [**?**], and a detailed mathematical description of the technique is available in [**?**].

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