

The Ontological Modal Collapse as a Collapse of the Square of Opposition

Christoph Benz Müller,
Leon Weber and
Bruno Woltzenlogel Paleo

Abstract. The *modal collapse* that afflicts Gödel’s modal ontological argument for God’s existence is discussed from the perspective of the modal square of opposition. Furthermore, a computer-assisted verification of the claims that the emendations by Anderson and by Frode are immune to the modal collapse is presented.

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1. Introduction

Attempts to prove the existence (or non-existence) of God by means of abstract, ontological arguments are an old tradition in western philosophy, with contributions by several prominent philosophers, including St. Anselm of Canterbury, Descartes and Leibniz. Kurt Gödel studied and further improved this argument, bringing it to a mathematically more precise form, as a chain of axioms, lemmas and theorems in a modal logic [23, 30], as in Fig. 1.

Gödel defines God (see Fig. ??) as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable (but debatable) axioms that they should satisfy. The overall idea of Gödel’s proof is in the tradition of Anselm’s argument, who defined God as some entity of which nothing greater can be conceived. Anselm argued that existence in the actual world would make such an assumed being even greater; hence, by definition God must exist. Gödel’s ontological argument is clearly related to this reasoning pattern. However,

A1	Either a property or its negation is positive, but not both:
	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:
	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:
	$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$
D1	A <i>God-like</i> being possesses all positive properties:
	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:
	$P(G)$
C	Possibly, a God-like being exists:
	$\Diamond\exists xG(x)$
A4	Positive properties are necessarily positive:
	$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$
D2	An <i>essence</i> of an individual is a property possessed by it and necessarily implying any of its properties:
	$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:
	$\forall x[G(x) \rightarrow G \text{ ess. } x]$
D3	<i>Necessary existence</i> of an individual is the necessary exemplification of all its essences:
	$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:
	$P(NE)$
T3	Necessarily, a God-like being exists:
	$\Box\exists xG(x)$

FIGURE 1. Scott's version of Gödel's ontological argument [30].

it also tries to fix some fundamental weaknesses in Anselm's work. For example, Gödel explicitly proves that God's existence is possible, which has been a basic assumption for Anselm. Because of this, Anselm's argument has been criticized as incomplete by Leibniz. Leibniz claimed that the assumption should be derivable from the definition of God as a perfect being and from the notion of perfection. Gödel's proof addresses this critique, and it also addresses Kant's objection that existence should not be treated as a predicate.

Nevertheless, Gödel's work still leaves room for criticism. In particular, his axioms are so strong that they entail a *modal collapse* [?, 31]: everything that is the case is so necessarily. There has been an impressive body of recent

MC	Everything that is the case is so necessarily:
	$\forall \phi[\phi \rightarrow \Box \phi]$
MCC	Modalities collapse:
	$\forall \phi[\phi \leftrightarrow \Diamond \phi \leftrightarrow \Box \phi]$

FIGURE 2. Modal Collapse [?, ?].

and ongoing work (cf. [31, 19, 3, 2, ?, 18] and the references therein) proposing solutions for the modal collapse. The goal of this paper is to present a computer-assisted analysis of the solutions proposed by Anderson and by Frode. The technique enabling this analysis is the embedding of quantified modal logics into higher-order logics [10, 9, 6], for which automated theorem provers exist [?, ?, ?, ?]. This technique has already been successfully employed in the verification and reconstruction of Gödel's proof [?], and a detailed mathematical description of the technique is available in [?].

2. The Modal Collapse

A1 Either a property or its negation is positive, but not both:

$$\forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall \phi \forall \psi [(P(\phi) \wedge \Box \forall x [\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall \phi [P(\phi) \rightarrow \Diamond \exists x \phi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \leftrightarrow \forall \phi [P(\phi) \rightarrow \phi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond \exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall \phi [P(\phi) \rightarrow \Box P(\phi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x [G(x) \rightarrow G \text{ ess. } x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall \phi [\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

T3 Necessarily, a God-like being exists:

$$\Box \exists x G(x)$$

FIGURE 3. Anderson's Emendation [?].

3. Anderson's Emendation

A1 Either a property or its negation is positive, but not both:

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A5 Necessary existence is a positive property:

$$P(NE)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists xG(x)$$

FIGURE 4. Frode's Alternative [?].

4. Frode's Alternative

5. Other Solutions

ToDo: Fitting

6. Conclusions

Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (cf. [31, 3, 2, 19, 1, 18]).

In theoretical philosophy, formal logical confrontations with such ontological arguments had been so far (mainly) limited to paper and pen. Up to now, the use of computers was prevented, because the logics of the available theorem proving systems were not expressive enough to formalize the abstract concepts adequately. Gödel's proof uses, for example, a complex higher-order modal logic (HOML) to handle concepts such as *possibility* and *necessity* and to support quantification over individuals and properties.

ToDo: Leibniz calculemus, Rushby, Zalta [27, 28]

References

- [1] R.M. Adams, 'Introductory note to *1970', in *Kurt Gödel: Collected Works Vol. 3: Unpubl. Essays and Letters*, Oxford Univ. Press, (1995).
- [2] A.C. Anderson and M. Gettings, 'Gödel ontological proof revisited', in *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics: Lecture Notes in Logic 6*, 167–172, Springer, (1996).
- [3] C.A. Anderson, 'Some emendations of Gödel's ontological proof', *Faith and Philosophy*, 7(3), (1990).
- [4] P.B. Andrews, 'General models and extensionality', *Journal of Symbolic Logic*, 37(2), 395–397, (1972).
- [5] P.B. Andrews, 'Church's type theory', in *The Stanford Encyclopedia of Philosophy*, ed., E.N. Zalta, spring 2014 edn., (2014).
- [6] C. Benzmüller, 'HOL based universal reasoning', in *Handbook of the 4th World Congress and School on Universal Logic*, ed., J.Y. Beziau et al., pp. 232–233, Rio de Janeiro, Brazil, (2013).
- [7] C. Benzmüller and D. Miller, 'Automation of higher-order logic', in *Handbook of the History of Logic, Volume 9 — Logic and Computation*, Elsevier, (2014). Forthcoming; preliminary version available at <http://christoph-benzmueller.de/papers/B5.pdf>.
- [8] C. Benzmüller, J. Otten, and Th. Rath, 'Implementing and evaluating provers for first-order modal logics', in *Proc. of the 20th European Conference on Artificial Intelligence (ECAI)*, pp. 163–168, (2012).
- [9] C. Benzmüller and L.C. Paulson, 'Exploring properties of normal multimodal logics in simple type theory with LEO-II', in *Festschrift in Honor of Peter B. Andrews on His 70th Birthday*, ed., C. Benzmüller et al., 386–406, College Publications, (2008).

- [10] C. Benzmüller and L.C. Paulson, ‘Quantified multimodal logics in simple type theory’, *Logica Universalis*, **7**(1), 7–20, (2013).
- [11] C. Benzmüller, F. Theiss, L. Paulson, and A. Fietzke, ‘LEO-II - a cooperative automatic theorem prover for higher-order logic’, in *Proc. of IJCAR 2008*, number 5195 in LNAI, pp. 162–170. Springer, (2008).
- [12] C. Benzmüller and B. Woltzenlogel-Paleo, ‘Formalization, mechanization and automation of Gödel’s proof of God’s existence’, *arXiv:1308.4526*, (2013).
- [13] C. Benzmüller and B. Woltzenlogel-Paleo, ‘Gödel’s God in Isabelle/HOL’, *Archive of Formal Proofs*, (2013).
- [14] C. Benzmüller and B. Woltzenlogel-Paleo, ‘Gödel’s God on the computer’, in *Proceedings of the 10th International Workshop on the Implementation of Logics*, EPiC Series. EasyChair, (2013). Invited abstract.
- [15] Y. Bertot and P. Casteran, *Interactive Theorem Proving and Program Development*, Springer, 2004.
- [16] J.C. Blanchette and T. Nipkow, ‘Nitpick: A counterexample generator for higher-order logic based on a relational model finder’, in *Proc. of ITP 2010*, number 6172 in LNCS, pp. 131–146. Springer, (2010).
- [17] C.E. Brown, ‘Satallax: An automated higher-order prover’, in *Proc. of IJCAR 2012*, number 7364 in LNAI, pp. 111 – 117. Springer, (2012).
- [18] R. Corazzon. Contemporary bibliography on ontological arguments: <http://www.ontology.co/biblio/ontological-proof-contemporary-biblio.htm>.
- [19] M. Fitting, *Types, Tableaux and Gödel’s God*, Kluwer, 2002.
- [20] M. Fitting and R.L. Mendelsohn, *First-Order Modal Logic*, volume 277 of *Synthese Library*, Kluwer, 1998.
- [21] D. Gallin, *Intensional and Higher-Order Modal Logic*, North-Holland, 1975.
- [22] P. Garbacz, ‘PROVER9’s simplifications explained away’, *Australasian Journal of Philosophy*, **90**(3), 585–592, (2012).
- [23] K. Gödel, *Appx.A: Notes in Kurt Gödel’s Hand*, 144–145. In [31], 2004.
- [24] L. Henkin, ‘Completeness in the theory of types’, *Journal of Symbolic Logic*, **15**(2), 81–91, (1950).
- [25] R. Muskens, ‘Higher Order Modal Logic’, in *Handbook of Modal Logic*, ed., P Blackburn et al., 621–653, Elsevier, Dordrecht, (2006).
- [26] T. Nipkow, L.C. Paulson, and M. Wenzel, *Isabelle/HOL: A Proof Assistant for Higher-Order Logic*, number 2283 in LNCS, Springer, 2002.
- [27] P.E. Oppenheimer and E.N. Zalta, ‘A computationally-discovered simplification of the ontological argument’, *Australasian Journal of Philosophy*, **89**(2), 333–349, (2011).
- [28] J. Rushby, ‘The ontological argument in PVS’, in *Proc. of CAV Workshop “Fun With Formal Methods”*, St. Petersburg, Russia,, (2013).
- [29] S. Schulz, ‘E – a brainiac theorem prover’, *AI Communications*, **15**(2), 111–126, (2002).
- [30] D. Scott, *Appx.B: Notes in Dana Scott’s Hand*, 145–146. In [31], 2004.
- [31] J.H. Sobel, *Logic and Theism: Arguments for and Against Beliefs in God*, Cambridge U. Press, 2004.
- [32] G. Sutcliffe, ‘The TPTP problem library and associated infrastructure’, *Journal of Automated Reasoning*, **43**(4), 337–362, (2009).

- [33] G. Sutcliffe and C. Benzmüller, ‘Automated reasoning in higher-order logic using the TPTP THF infrastructure.’, *Journal of Formalized Reasoning*, **3**(1), 1–27, (2010).
- [34] B. Woltzenlogel-Paleo and C. Benzmüller, ‘Automated verification and reconstruction of Gödel’s proof of God’s existence’, *OCG J.*, (2013).

Christoph Benzmüller
Department of Mathematics and Computer Science
Arnimallee 7
Room 115
14195 Berlin
Germany
e-mail: c.benzmueller@gmail.com

Leon Weber
ToDo
ToDo
ToDo Berlin
Germany
e-mail: leon.weber@fu-berlin.de

Bruno Woltzenlogel Paleo
Favoritenstraße 9
Room HA0402
1040 Wien
Austria
e-mail: bruno.wp@gmail.com