Formalization, Mechanization and Automation of Gödel's Proof of God's Existence*

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Attempts to prove the existence (or non-existence) of God by means of abstract ontological arguments are an old tradition in philosophy and theology. Gödel's proof [12] is a modern culmination of this tradition, following particularly the footsteps of Leibniz. Gödel defines God as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable but debatable axioms that they should satisfy. Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (e.g. [17, 18, 2, 11, 1, 10]). Our formalization employs the following axioms (A*) and definitions (D*):[3]

Chris: The reference to Scott's version is kind of confusing, we need to verify this sources.

Any property necessarily implied by a positive property is positive.

$$\forall P \forall Q(pos\, P \wedge (\Box \forall x(P\, x \Rightarrow Q\, x)) \Rightarrow pos\, Q) \tag{A1}$$

A property is positive if and only if its negation is not positive.

$$\forall P(pos\, P \Leftrightarrow \neg(pos\, \neg P)) \tag{A2}$$

The property of being God-like is positive.

$$pos\, god \tag{A3}$$

Positive properties are necessarily positive.

$$\forall P(pos\, P \Rightarrow \Box(pos\, P)) \tag{A4}$$

Necessary existence is a positive property.

$$pos\, nec\_exists \tag{A5}$$

x is God-like if and only if x has every positive property.

$$god\, x := \forall P(pos\, P \Rightarrow P\, x) \tag{D1}$$

A property P is an essence of x if and only if P is a property of x and every property Q that x has is necessarily implied by P.

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[3] (A1), (A2), (A5), (D1), (D3) are logically equivalent to, respectively, axioms 5, 2 and 4 and definitions 1 and 3 in Gödel's manuscript [12]. (A3) was introduced by Scott [17] and could be derived from Gödel's axiom 1 and (D1) in a logic with infinitary conjunction. (A4) is a weaker form of Gödel's axiom 3. (D2) has an extra conjunct lacking in Gödel's definition 2; this lack is believed to have been an oversight by Gödel [13].

$$ess\,P\,x := P\,x \wedge \forall Q(Q\,x \Rightarrow \Box\forall y(P\,y \Rightarrow Q\,y)) \tag{D2}$$

x necessarily exists if and only if every essence of x is necessarily exemplified.

$$nec\_exists\,x := \forall P(ess\,P\,x \Rightarrow \Box\exists y(P\,y)) \tag{D3}$$

From these axioms and definitions we then infer:

Positive properties are possibly exemplified.

$$\forall P(pos\,P \Rightarrow \Diamond\exists x(P\,x)) \tag{L1}$$

Possibly God exists.

$$\Diamond\exists x(god\,x) \tag{L2}$$

If x is God-like, then the property of being God-like is an essence of x.

$$\forall x(god\,x \Rightarrow ess\,god\,x) \tag{L3}$$

Necessarily God exists.

$$\Box\exists x(god\,x) \tag{T}$$

The above variant of Gödel's proof has now been constructed for the first-time with an unprecedent degree of detail and formality; cf. [16]. The following has been done (and in this order):

Chris: We need to make sure that this ND proof is based on a sound calculus.

- A detailed natural deduction proof.
- A formalization of the axioms, definitions and theorems in the TPTP THF syntax [19].
- Automatic verification of the consistency of the axioms and definitions with Nitpick [8].
- Automatic demonstration of the theorems with the provers LEO-II [5] and Satallax [9].
- A step-by-step formalization using the Coq proof assistant [6].
- A formalization using the Isabelle proof assistant [15] where the theorems (and some additional lemmata) have been automated with Sledgehammer [7] and Metis [14].

Gödel's proof is challenging to formalize and verify because it requires an expressive logical language with modal operators (*possibily* and *necessarily*) and with quantifiers for individuals and sets of individuals (properties). Our computer-assisted formalizations rely on an embedding of the modal logic S5 into classical higher-order logic with Henkin semantics [4, 3]. The formalization is thus essentially done in classical higher-order logic where quantified S5 is emulated.

This work attests the maturity of contemporary interactive and automated deduction tools for classical higher-order logic and it demonstrates the elegance and practical relevance of the embeddings based approach. Most importantly, our work opens new perspectives for a computer-assisted theoretical philosophy. The critical discussion of the underlying concepts, definitions and axioms remains a human responsibility, but the computer can assist in building and checking rigorously correct logical arguments. In case of logico-philosophical disputes, the computer can check the disputing arguments and partially fulfill Leibniz' dictum: Calculemus — Let us calculate!

Future work includes an extensive study of other formalizations of ontological arguments with our machinery. Variations of the distinctive features of the base logic S5 (e.g. non-rigid symbols, varying domains, etc.) are enabled in these studies due to the flexibility of the embeddings based approach.

References

1. R.M. Adams. Introductory note to *1970. In *Kurt Gödel: Collected Works Vol. 3: Unpublished Essays and Letters*. Oxford University Press, 1995.
2. A.C. Anderson and M. Gettings. Gödel ontological proof revisited. In *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics: Lecture Notes in Logic 6*, pages 167–172. Springer, 1996.
3. C. Benzmüller and L.C. Paulson. Exploring properties of normal multimodal logics in simple type theory with leo-ii. In *Festschrift in Honor of Peter B. Andrews on His 70th Birthday*. College Publications, 2008.
4. C. Benzmüller and L.C. Paulson. Quantified multimodal logics in simple type theory. *Logica Universalis (Special Issue on Multimodal Logics)*, 7(1):7–20, 2013.
5. C. Benzmüller, F. Theiss, L. Paulson, and A. Fietzke. LEO-II - a cooperative automatic theorem prover for higher-order logic. In *Proc. of IJCAR 2008*, volume 5195 of *LNAI*, pages 162–170. Springer, 2008.
6. Y. Bertot and P. Casteran. *Interactive Theorem Proving and Program Development*. Springer Verlag, 2004.
7. J.C. Blanchette, S. Böhme, and L.C. Paulson. Extending Sledgehammer with SMT solvers. *Journal of Automated Reasoning*, 51(1):109–128, 2013.
8. J.C. Blanchette and T. Nipkow. Nitpick: A counterexample generator for higher-order logic based on a relational model finder. In *Proc. of ITP 2010*, number 6172 in LNCS, pages 131–146. Springer, 2010.
9. C.E. Brown. Satallax: An automated higher-order prover. In *Proc. of IJCAR 2012*, number 7364 in LNAI, pages 111 – 117. Springer, 2012.
10. R. Corazzon. Contemporary bibligraphy on the ontological proof (`http://www.ontology.co/biblio/ontological-proof-contemporary-biblio.htm`).
11. M. Fitting. *Types, Tableaux and Gödel's God*. Kluver Academic Press, 2002.
12. K. Gödel. Ontological proof. In *Kurt Gödel: Collected Works Vol. 3: Unpublished Essays and Letters*. Oxford University Press, 1970.
13. A.P. Hazen. On gödel's ontological proof. *Australasian Journal of Philosophy*, 76:361–377, 1998.
14. J. Hurd. First-order proof tactics in higher-order logic theorem provers. In *Design and Application of Strategies/Tactics in Higher Order Logics, number NASA/CP-2003-212448 in NASA Technical Reports*, pages 56–68, 2003.
15. T. Nipkow, L.C. Paulson, and M. Wenzel. *Isabelle/HOL: A Proof Assistant for Higher-Order Logic*. Number 2283 in LNCS. Springer, 2002.
16. B. Woltzenlogel Paleo and C. Benzmüller. Formal theology repository (`https://github.com/FormalTheology/GoedelGod`).
17. D. Scott. Gödel's ontological proof. In *On Being and Saying. Essays for Richard Cartwright*, pages 257–258. MIT Press, 1987.

18. J.H. Sobel. Gödel's ontological proof. In *On Being and Saying. Essays for Richard Cartwright*, pages 241–261. MIT Press, 1987.
19. G. Sutcliffe and C. Benzmüller. Automated reasoning in higher-order logic using the TPTP THF infrastructure. *Journal of Formalized Reasoning*, 3(1):1–27, 2010.