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Magari and others on Gödel's ontological proof

Petr Hájek
Institute of Computer Science, Academy of Sciences
182 07 Prague, Czech Republic
e-mail: hajek@uivt.cas.cz

1 Introduction

This paper is a continuation of my paper [H] and concentrates almost exclusively to mathematical properties of logical systems underlying Gödel's ontological proof [G] and its variant by Anderson [A], with special care paid to Magari's criticism [M]. Since [H] is written in German, we shall try to summarize its content in such a way that knowledge of [H] will be not obligatory for reading the present paper (even it remains advantageous). Here we describe the development related to [G] in rough way, in Section 1 (Preliminaries) we tell more formal details. [G] uses variables for individuals, variables for properties, one predicate P (positive) applicable to properties, modalities $\Box$ and $\Diamond$ and equality among individuals. There are 5 axioms and some definitions; in particular, x is *godlike*, notation $G(x)$, if x has all positive properties; a property X is an *essence* of x, notation $XEss.x$, if $X(x)$ and each Y such that $Y(x)$ necessarily includes X; x *necessarily exists* if each essence X of x is necessarily instantiated, i.e. $\Box(\exists y)X(y)$. There are some few lemmas and Theorem 3 saying $\Box(\exists x)G(x)$ - necessarily there is a godlike individual. (See Sect.1,4 for more details). Gödel's text resembles sacral texts; it has no introduction, motivations, statement of underling logic, and therefore needs some "hermeneutic" interpretation. In particular, it has become clear that following questions are relevant: (1) what *comprehension axiom(s)* does one have to assume, i.e. which formulas define properties? (2) what *equality axioms* are to be assumed? And (3), what *modal axioms* are to be assumed? Sobel [S] and Polívka (cf.[H]) observed that assuming full comprehension leads to *collapse of modalities*, i.e. $(\forall X)(\forall x)(X(x) \equiv \Box X(x))$ become provable, which means that the system trivializes. Magari [M] claims that the first three Gödel's axioms $(A1) - (A3)$ already imply the main theorem; we shall analyze his claim in Section 2 and show that his proof implicitly uses a too strong equality axiom (without which the claim is not true, as shown in [H]). Anderson [A] presented an emended system admitting full comprehension and not suffering by collapse of modalities; I proved in [H] that for Anderson's system Magari's claim becomes true; the analogs of Gödel's $A1 - A3$ prove necessary existence of a godlike individual (in the modified sense), but this does not mean that the system trivializes because Anderson's system (of his $A1 - A3$ plus full comprehension) can be presented as a conservative extension of Gödel's system $A1 - A5$ with a certain "cautious" comprehension axiom. Briefly, Anderson's system admits some "non-convertible" properties; Gödel's system results when we drop all non-convertible properties. We give more details in Section 1 (Preliminaries.) Section 2 analyzes Magari's claim; Section 3 presents a completeness theorem for the underlying logic (which shows that no hidden axioms remain to be formulated). Up to that point, the modal logic is $S5$; in Section 4 we show that a weaker logic $KD45$, known as *logic of belief* (in contradistinction to $S5$, called *logic of knowledge* in *AI*) is sufficient to the proof of the main theorem both in Gödel's and in Anderson's system but is not sufficient to prove that Anderson's system includes Gödel's system via restriction to convertible properties. Section 5, the only place discussing extra-mathematical things, contemplates on Magari's philosophical standpoint, as apparent from [M].

2 Preliminaries

The propositional skelet of the ontological proof, as presented by [B], is simple: one has to establish (q standing for $(\exists x)G(x)$)

(1) $\Diamond q$ (as intuitively accepted),

(2) $q \to \Box q$ (Anselm's principle).

Then $\Diamond q \to \Diamond\Box q \to \Box q$ (e.g. by S5).

Gödel's axioms are

(A1) $P(X) \equiv \neg P(\neg X)$
(A2) $P(X)$ & $\Box(\forall x)(X(x) \to Y(x)) \to P(Y)$
(A3) $P(G)$
(A4) $P(X) \to \Box P(X)$.
(A5) $P(NE)$

(here $G(x) \equiv (\forall Y)(P(Y) \to Y(x))$, $XEss.x \equiv X(x)$ & $(\forall Y)(Y(x) \to \Box(\forall z)(X(z) \to Y(z))$, $NE(x) \equiv (\forall Y)(YEss.x \to \Box(\exists z)Y(z))$. We quickly go throug Gödel's proof.

Lemma: (i) $P(X) \to \Diamond(\exists x)X(x)$; (ii) $G(x) \to GEss.x$

Proof.: (i) If $\Box(\forall x)\neg X(x)$ then $\Box(\forall x)(X(x) \to \neg X(x))$, thus $P(X) \to P(\neg X)$, contradiction. (ii) One easily shows $G(x) \to (\forall Y)(P(Y) \equiv Y(x))$; now if $G(x)$ and $Y(x)$ then $P(Y)$, hence $\Box P(Y)$, hence $\Box(\forall z)(G(z) \to Y(z))$. This is $GEss.x$.

Theorem. $\Box(\exists x)G(x)$

Proof. Lemma (i) gives $\Diamond(\exists x)G(x)$; we prove $G(x) \to \Box(\exists x)G(x)$. Indeed, assume $G(x)$; since $P(NE)$ and $GEss.x$ we get $\Box(\exists z)G(z)$ by the definition of NE. The theorem follows by Hartshorne's observation above.

Remark. (i) $G(x) \to \Box G(x)$, thus $(\exists x)\Box G(x)$.
(ii) $G(x)$ & $Y(x). \to \Box Y(x)$.

Proof. (i) Let $Z(y)$ be $y = x$; then $G(x) \to \Box(\exists y)(G(y) \to Z(y))$, thus $G(a)$ & $\Box(\exists y)G(y) \to \Box(\exists y)(G(y)$ & $y = a) \to \Box G(a)$.
(ii) $G(a)$ & $Y(a) \to \Box G(a)$ & $\Box(\forall z)(G(z) \to Y(z)) \to \Box Y(a)$.

The *comprehension axiom* for a formula $\varphi(x)$ (and possibly other free variables, but not Y) is the axiom

$$(\exists Y)\Box(\forall x)(Y(x) \equiv \varphi(x)).$$

The *full comprehension scheme* (C_{full}) is the scheme of *all* comprehension axioms. We implicitly used three comprehension axioms above, e.g. to know that $G, NE, \Phi_x(y) \equiv y = x$, exists as properties. As mentioned above, Gödel's system with (C_{full}) suffers by the collapse of modalities; the following is the *cautious comprehension scheme* (C_{caut}):

$$(\forall x)(G(x) \to (\Box\varphi(x) \vee \Box\neg\varphi(x))) \to$$
$$\to (\exists Y)\Box(\forall z)(Y(z) \equiv \varphi(z)).$$

$(GO)_{caut}$ will denote the axiom system (A1)-(A5), comprehension axioms for G, NE, I_x (see below) and the schema (C_{caut}).

$$(\exists G)\Box(\forall x)(G(x) \equiv (\forall Y)(P(Y) \to Y(x))$$

$$(\exists NE)\Box(\forall x)(NE(x) \equiv (\forall Y)(YEss.x \to \Box(\exists z)Y(z))$$

$$(\forall x)(\exists I)\Box(\forall y)(I(y) \equiv y = x).$$

$(GO)_{caut}$ is consistent and does not suffer by collapse of modalities (see next section).

Andersen's variant: (A1) is weakened to
(AA1) $P(X) \to \neg P(\neg X)$,
(AA2) is identical with (A2). The definition of a godlike object is changed to

$$H(x) \equiv (\forall Y)(P(Y) \equiv \Box Y(x))$$

(thus x is godlike in new sense iff positive properties are exactly properties necessarily applying to x).
(AA3) is $P(H)$ (godlikeness is positive).
Anderson's axioms (AA4), (AA5) are irrelevant:

Theorem: $(AA1) - (AA3) \vdash \Box(\exists x)H(x)$

Proof. $\Diamond(\exists x)H(x)$ is provable as above; we prove $H(x) \to \Box H(x)$. Indeed, if $H(x)$ and $P(H)$ then H necessarily applies to x, by the definition of H.

Here we have used just the comprehension axiom for H. $(AO)_{full}$ will be (AA1)-(AA3) plus (C_{full}). $(AO)_{full}$ is consistent and does have collapse of modalities, by [A] (and by obvious models, see below). More than that: define in $(AO)_{full}$ a property X to be *convertible* if

$$(\forall x)(H(x) \to (\Box X(x) \vee \Box\neg X(x))).$$

Note that e.g. H is convertible. ("Convertible" since going from X to $\neg X$ we convert from positive to negative; nonconvertible classes are neither positive non negative. Also "convertible" should resemble "constructible"; the reason for this will be immediately clear to a reader knowing properties of the Gödel's constructible inner model of set theory).

Theorem: ([H]) Restricting properties to convertible properties, interpreting G as H and letting everything else absolute we get an interpretation $\#$ of $(GO)_{caut}$ in $(A5)_{full}$. This interpretation is faithful. i.e. for each closed formula φ of $(GO)_{caut}$,

$$(GO)_{caut} \vdash \varphi \text{ iff } (AO)_{full} \vdash \varphi^{\#}$$

This is proved by constructing an interpretation $\flat$ of $(AO)_{full}$ in $(GO)_{caut}$ such that the composition $\# * \flat$ is identical.

3 The status of Magari's claim

In [M], Magari presents a passionate criticism of Gödel's proof (and proofs of God in general - see Sect.5). Here we shall analyze his claim, mentioned above, that Gödel's axioms (A1)-(A3) are enough to prove $\Box(\exists x)G(x)$. His proof is semantical: he describes a class of models of the underlying language and then shows that each model of this class which satisfies (A1)-(A3) satisfies also $\Box(\exists x)G(x)$. The difficulty is that the system is not complete with respect to this class; we shall show which additional axiom, true in all Magari models, suffices to make his claim valid, then we refer to [H] for another semantics which is enough to show that without any additional axiom Magari's claim is false. (But note again the remarkable fact explained in the preceding section that for Anderson's variant Magari's claim does hold.)

Define for a moment: a *Magari model* (see [M]) is a tuple $\langle W, (M_w)_{w\in W}, (B_w)_{w\in W}, (P_w)_{w\in W}\rangle$ where W is a set of possible worlds and for each $w \in W$: $M_w \neq \emptyset$ is a set of individuals, B_w is a boolean algebra of subsets of M_w, $P_w \subseteq B_w$ is a system of positive sets. Each object variable is interpreted as a mapping f_x such that $f_x(w) \in M_w$ for all w, similarly $f_X(w) \in B_w$ for call w; $w \Vdash X(x)$ iff $f_x(w) \in F_X(w)$, $w \Vdash P(X)$ iff $f_X(w) \in P_w$. One defines $w \Vdash X = Y$ iff $w \Vdash (\forall x)(X(x) \equiv Y(x))$, i.e. $f_X(w) = f_Y(w)$. But observe that under this definition the following axiom becomes true in each

Magari model (pointwise equality axiom for P):
(PEP) $\qquad P(X)\ \&\ X = Y. \rightarrow P(Y).$

Theorem (Magari revised) (A1)-(A3), in presence of (PEP), (and some comprehension) prove $\Box(\exists x)G(x)$.

Proof. We first show $P(X) \rightarrow (\exists x)X(x)$. Assume $P(X)$ and $(\forall x)\neg X(x)$. Then for all Y, $Y = X \cup Y$ (the existence of a property being (necessarily) the union of two gives properties is the comprehension axiom needed and valid in all Magari models); we have $\Box(X \subseteq X \cup Y)$ and therefore $P(X \cup Y)$ by (A2). Using PEP (!) we get $P(Y)$, thus $(\forall Y)P(Y)$ which contradicts (A1).

In particular, by (A3) we have $P(G)$, hence $(\exists x)G(x)$, which gives $\Box(\exists x)G(x)$ by necessitation.

Note that similar semantics is considered by Czermak [C]. Now we recall the semantics of [H]; we just speak on models.

A *model* is a tuple $K = \langle W, M, Prop, \mathcal{P}\rangle$ where $W \neq \emptyset$ is a set of possible worlds, $M \neq \emptyset$ is a set of individuals (common for all possible worlds), *Prop* is a non-empty set of mappings $F : W \times M \rightarrow \{0,1\}$ and $\mathcal{P} : W \times Prop \rightarrow \{0,1\}$. Individual variables x are interpreted by elements m_x of M, property variables X by elements F_X of *Prop*; $w \Vdash X(x)$ iff $F_X(w, m_x) = 1$. $w \Vdash P(X)$ iff $\mathcal{P}(w, F_x) = 1$; $w \Vdash x = y$ iff $m_x = m_y$. The rest is obvious (in particular, $w \Vdash \Box\varphi$ iff for all $w' \in W$, $w' \Vdash \varphi$). Note that here we also have a "hidden" axiom, namely $(x = y \rightarrow \Box(x = y))$ (objects are fixed); this could be removed but we prove completeness (in the next section and it is some simplification to have just one universe of individuals.

Fact. Let $K = \langle W, M, Prop, \mathcal{P}\rangle$ be a model and let $g \in M$. (1) Assume that *Prop* consists of all $F : M \times W \rightarrow \{0,1\}$ such that F is constant as $W \times \{g\}$ and that $\mathcal{P}(w, F) = 1$ iff (independently of w) for all $v \in W$, $F(v, g) = 1$. Then K is a model of $(GO)_{cau}$.

(2) Assume that *Prop* consists of *all* mappings $F : M \times W \rightarrow \{0,1\}$; let $\mathcal{P}$ be as above. Then K is a model of $(AO)_{full}$.

(3) Let $W_0 \in W$ let *Prop* be as in (2) and let $\mathcal{P}(w, F) = 1$ iff $F(W_0) = 1$ (independently of w). Then $K \Vdash (A1)-(A3)$ and $\Diamond(\exists x)G(x)$ and $\Diamond\neg(\exists x)G(x)$. This refutes Magari's claim in its original form. (All these examples an from [H].)

4 A completeness theorem

We prove, in a standard manner, a completeness theorem giving completeness of systems like $(AO)_{full}$, $(GO)_{cant}$ as a particular case.

The system. *Variables:* object variables $x, y, \ldots$; property variables $X, Y, \ldots$. *Atomic formulas:* $X(x)$ (this can be written as $x \in X$), $P(X)$, $x = y$, $X = Y$.

Axioms: Two-sorted *predicate calculus* (examples of axioms: $(\forall x)\varphi(x) \to \varphi(y)$, $(\forall X)\varphi(X) \to \varphi(Y)$ - under the usual substitutibility conditions); *equality axioms* for object (reflexivity, symmetry, transitivity; $\varphi(x)$ & $x = y. \to \varphi(y)$; definition of *equality for properties:* $X = Y \equiv (\forall x)(X(x) \equiv Y(x))$ (extensionality); *equality axiom* for P: $P(X)$ & $\Box(X = Y). \to P(Y)$. *Modal axioms* (S5):
$\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$, $\Box\varphi \to \varphi$,
$\psi \equiv \Box\psi$ for ψ boxed (i.e. resulting from formulas of the form $\Box\chi$ using connectives and quantifiers).
Necessary equality for objects: $x = y \to \Box(x = y)$.

Global and local theories. A *global theory* Q is given by some special axioms; φ is *provable* in Q if there is a proof of φ from logical axioms above and special axioms of Q using modus ponens and necessitation as deduction rules. Similarly for *local theories* T, but the only deduction rules is modus ponens. $Q \vdash \varphi$ means global provability, $T \vdash^{o} \varphi$ local provability. $(AO)_{full}$ and $(G5)_{caut}$ are examples of global theories, i.e. we are interested in their global consequences. $Cn(Q)$ is the set of all global consequences of Q.

Models are structures as above, i.e. $K = \langle W, M, Prop, \mathcal{P}\rangle$ with $Prop$ being a set of mappings $X : (W \times M) \to \{0,1\}, \mathcal{P} : (W \times Prop) \to \{0,1\}$.

Completeness theorem. If Q is a global theory α a closed formula and $Q \nvdash \neg\alpha$ then there is a model $K = \langle W, M, Prop, \mathcal{P}\rangle$ and a possible world $n \in W$ such that $n \Vdash \neg\alpha$. The rest of the section contains a proof of the completeness, together with some auxiliary lemmas.

Notation. Q and α are fixed; T_{-1} is $Cn(Q) \cup \{\alpha\}$. For each $n = 0, 1, \ldots$, $Const_n$ is a set of object and property constants (infinitely many of either kind), for $m \neq n$ $Const_m$ and $Const_n$ are disjoint.

Lemma. (1) $\vdash \Box(\forall x)\varphi \equiv (\forall x)\Box\varphi$
(2) $\vdash (\exists x)\Box\varphi \to \Box(\exists x)\varphi$

Proof. $\vdash \Box(\forall x)\varphi \to \varphi$, $\vdash \Box(\Box(\forall x)\varphi \to \varphi)$, $\vdash \Box\Box(\forall x)\varphi \to \Box\varphi$, $\vdash \Box(\forall x)\varphi \to \Box\varphi$, $\vdash (\forall x)(\Box(\forall x)\varphi \to \Box\varphi)$, $\vdash \Box(\forall x)\varphi \to (\forall x)\Box\varphi$.
The rest is similar.

Notation. Q and α are fixed; T_{-1} is $Cn(Q) \cup \{\alpha\}$ each $n = 0, 1, \ldots$, $Cont_n$ is a set of object and property contents (infinitely many of either kind), for $m \neq n$ $Const_m$ and $Const_n$ are disjoint. $Const_\infty = \bigcup_n Const_n$; $Const_{<n} = \bigcup_{i<n} Const_i$. $\{\varphi_n\}$ is an enumeration of all formulas of $L(Const_\infty)$ (the language enriched by all the constants) such that for each $n, \varphi_n \in L(Const_{<n})$, $\varphi_0 = \perp$. We assume that each closed formula φ occurs infinitely many times in the enumeration; n is a *starter* if φ_n occurs first time, otherwise n is a *repeater* and $t(n)$ is the biggest $m < n$ such that φ_n is the same formula as φ_m.

Construction. T_0 is a complete Henkin extension of T_{-1} using $Const_0$ as Henkin constants; i.e. for each closed formula $(\exists x)\varphi(x)$ of $L(Const_0)$ there is a constant $c \in Const_0$ such that $T_0 \vdash^0 (\exists x)\varphi(x) \to \varphi(c)$, and similarly for $(\exists X)\varphi(X)$.

For each T, $B(T) = \{\Box\psi \mid \psi \text{ closed}, T \vdash \Box\psi\}$. Note that if $T \supseteq Cn(Q)$ is complete and ψ is closed then either $(\Box\psi) \in B(T)$ or $(\Box\Diamond\neg\psi) \in B(T)$ and $B(T) \vdash^0 \Diamond\neg\psi$.

We construct complete Henkin local theories T_n; for each n we shall have $T_n \supseteq Cn(Q) \cup \{\alpha\} \cup B(T_{n-1})$; if n is a repeater we shall have $T_n \supseteq T_{t(n)}$. The definition is as follows:
Let T_{n-1} be defined $(n > 0)$. We define an anxiliary theory T':
Case 1. n is a starter and $(\Box\neg\varphi_n) \in B(T_{n-1})$; put $T' = T_{n-1}$
Case 2. n is a starter and $(\Box\Diamond\varphi_n) \in B(Y_{n-1})$; put $T' = Cn(Q) \cup B(T_{n-1}) \cup \{\varphi_n\}$.
Case 3. n is a repeater: put $T' = Cn(Q) \cup B(T_{n-1}) \cup T_{t(n)}$.
Claim 1. T' is consistent (as a local theory). (Proof below). Non let Tn be a complete Henkin local theory using $Const_n$ as witnessing constants. We finish the construction.

For each closed φ, let $w_\varphi = \bigcup\{Tn \mid \varphi_n = \varphi\}$.
Claim 2. w_φ is a complete Henkin local theory in $L(Const_\infty)$; $B(n_\varphi)$ is the

some set B for all φ 's . (Obvious.)

Definition of the model. For each object constant c, let $[c] = \{d \mid (\Box(c = d)) \in B\}$; let M be the set of all $[c]$. W is the set of all w_φ. For each property constant C, let F_C be the mapping on $W \times M$ such that $F_C(w_\varphi, [c]) = 1$ iff $w_\varphi \vdash^0 C(c)$ (clearly, the definition does not depend on the choice of a representative c of $[c]$). *Prop* is the set of all mapping F_C; $\mathcal{P}(w_\varphi, F_C) = 1$ iff $w_\varphi \vdash^0 P(C)$. This completes the definition of $\langle W, M, Prop, \mathcal{P}\rangle$.

Claim 3. $(F_C = F_D \ \& \ \mathcal{P}(w_\varphi, C) = 1) \rightarrow \mathcal{P}(w_\varphi, D) = 1$.

Claim 4. For each closed φ and each $w \in W$, $w \Vdash \varphi$ iff $w \vdash^0 \varphi$. In particular, $w_\perp \Vdash \alpha$, which completes the proof of completeness. Since soundness of our semantics is evident we have the usual.

Corollary. $Q \vdash \alpha$ iff α is true in each world of each model.

Remark. $(GO)_{caut}$ and $(AO)_{full}$ are examples of global theories, thus we have completeness for them. Note that both theories prove $(\forall Y)(P(Y) \rightarrow \Box P(Y))$, thus they prove $(\forall Y)(P(Y) \equiv \Box P(Y))$ and we may confine us to models $\langle W, M, Prop, \mathcal{P}\rangle$ where $\mathcal{P}$ does not depend on possible worlds, i.e. $\mathcal{P} : Prop \rightarrow \{0, 1\}$.

Appendix to Section 3: Proofs of claims.

Proof of claim 1. Case 2: Let $Cn(Q) \cup B(T_{n-1}) \cup \{\varphi_n\}$ be inconsistent (as a local theory); then $Cn(Q) \cup B(T_{n-1}) \vdash^0 \neg\varphi_n$, thus for some $(\Box\psi) \in B(T_{n-1})$, $Cn(Q) \vdash^0 \Box\psi \rightarrow \neg\varphi_n$, thus $Q \vdash \Box\psi \rightarrow \varphi_n$, $Q \vdash \Box\psi \rightarrow \Box\neg\varphi_n$, thus $T_{n-1} \vdash \Box\neg\varphi_n$, a contradiction (with the assumption of Case 2).

Case 3. Assume $Cn(Q) \cup B(T_{n-1}) \cup T_{t(n)}$ inconsistent, thus $Cn(Q) \cup B(T_{n-1}) \cup T_{t(n)} \vdash^0 \perp$. Assume axioms of $T_{t(n)}$ to be closed. Then there is a $\chi \in T_{t(n)}$ and a $(\Box\psi) \in B(T_{n-1})$ such that $Cn(Q) \cup \{\Box\psi\} \vdash^0 \neg\chi$, thus $Q \vdash \Box\psi \rightarrow \neg\chi$, $Q \vdash \Box\psi \rightarrow \Box\neg\chi$, $T_{n-1} \vdash^0 \Box\neg\chi$, but $T_{t(n)} \vdash^0 \psi$, $T_{t(n)} \vdash^0 \Diamond\psi$, $(\Diamond\psi) \in B_{t(n)} \subseteq T_{n-1}$, thus T_{n-1} is contradictory.

Proof of claim 3. To prove claim 3 it is enough to show that if $F_C = F_D$ then $\Box(\forall x)(C(x) \equiv D(x)) \in B$. Indeed, if $F_C = F_D$ then for all object constants c and all w_φ, $w_\varphi \vdash^0 C(c) \equiv D(c)$; since w_φ is Henkin, we get $w_\varphi \vdash (\forall x)(C(x) \equiv D(x))$. And if the formula $\Box(\forall x)(C(x) \equiv D(x))$ were not in B then for the n which is the starter for this formula we would get $(\exists x)(C(x) \not\equiv D(x))$, a contradiction.

Proof of claim 4. We prove $w \Vdash \varphi$ iff $w \vdash^0 \varphi$ by induction on the complexity of the closed formula φ. (For non-closed formulas one works with satisfaction by a given evaluation e of object and property variables.) The claim is obvious for atomic formulas $c = d$, $C(c)$, $P(C)$; the induction step for connectives and for quantifiers is also obvious (using completeness and existence of Henkin constants. Consider $\Box\varphi, \varphi$ being closed:
$(\forall w)(w \Vdash \Box\varphi)$ iff $(\forall w)(w \Vdash \varphi)$ iff $(\forall w)(w \vdash^0 \varphi)$ iff $(\forall w)(w \vdash^0 \Box\varphi)$;
in the last equivalence, $\Leftarrow$ is obvious and $\Rightarrow$ uses the fact that if $(\exists w)(w \vdash^0 \Diamond\neg\varphi)$ then $(\exists w')(w' \vdash^0 \neg\varphi)$: take n to be the starter for $\neg\varphi$, then $w_n \vdash^0 \neg\varphi$ since $(\Box\varphi) \notin B(T_{n-1})$ (because $(\Box\Diamond\neg\varphi) \in B$). This completes the proof.

5 Weakening the modal logic.

Let us now delete the axiom $\Box\varphi \rightarrow \varphi$, i.e. let us weaken the modal logic from S5 (logic of knowledge) to KD45 (logic of belief) and let us see what happens with our theories. (Cf. e.g. [V] for information on KD45.) $(AO)^-_{full}$ and $(GO)^-_{caut}$ denote the theories with the same special axioms as $(A5)_{full}$ and $(GO)_{caut}$ respectively but with the weakened logic.

Observation. $(GO)^-_{caut} \vdash \Box(\exists x)G(x)$ and $(AO)^-_{full} \vdash \Box(\exists x)H(x)$, by the same proofs as in Section 1. (Just check that the deleted modal axiom is not used.)

This means that with this modal logic both system prove that it is *believed* that a godlike individual exists. We consider some models of both systems and exhibit some unprovabilities.

Models for $(GO)^-_{caut}$: Consider models of the form

$$K_1 = \langle W, M, Prop, \mathcal{P}, W_0, g\rangle$$

where W, M are as above, $\emptyset \neq W_0 \subseteq W$, $g \in M$, *Prop* consists of all mappings $X : (W \times M) \rightarrow \{0, 1\}$ constant on $W_0 \times \{g\}$; $\mathcal{P} : Prop \rightarrow \{0, 1\}$, $\mathcal{P}(X) = 1$ iff X equals 1 on $W_0 \times \{g\}$; $w \Vdash \Box\varphi$ iff for all $w' \in W_0$, $w'' \Vdash \varphi$. Then K_1 is a model of $(GO)^-_{caut}, K \Vdash \Box G(g)$, but for $w \in W - W_0$, $w \Vdash (\forall x)\neg G(x)$.

Corollary 1. $(GO)^-_{caut} \not\vdash (\exists x)G(x)$.

Models for $(A5)^-_{full}$: If we modify the model above just letting *Prop* to be the set of all $X : (W \times M) \rightarrow \{0, 1\}$ but let the definition of $\mathcal{P}$ unchanged

then we get a model K_2 of $(AO)^-_{full}$ in which for each w, $w \Vdash H(g)$ (since the truth-value of $H(g)$ is world-independent in this model). But we may get other models of $(AO)^-_{full}$, not satisfying $(\forall Y)(P(Y) \equiv \Box P(Y))$, as follows: $K = \langle W, M, Prop, \mathcal{P}, W_0, g\rangle$, $Prop$ consists of all $X : (W \times M) \to \{0,1\}$, $\mathcal{P} : (W \times Prop) \to \{0,1\}$ and the following holds:
$W \in W_0$ implies $\mathcal{P}(w, X) = 1$ iff x has constantly value 1 on $W_0 \times \{g\}$,
$w \in W - W_0$ implies $\mathcal{P}(w, X) = 0$.
Note that necessity is defined using $W_0 : w \Vdash \Box\varphi$ iff $(\forall w' \in W_0)(w' \Vdash \varphi)$.

In this model we have $K \Vdash \Box H(g)$, (i.e. for each $w \in W_0$, $w \Vdash H(g)$), but for $w \in W - W_0$ we have $w \Vdash (\forall x)\neg H(x)$.

Corollary 2. $(AO)^-_{full} \nvdash (\exists x)H(x)$.

Corollary 3. $(AO)^-_{full} \nvdash G^\#(x) \equiv H(x)$.

Indeed, in $K_2 w \Vdash H(g)$ for each w, but if $w \in W - W_0$ then $w \Vdash \neg G^\#(g)$ (note that $\#$ defines K_1 in K_2). Thus $\#$ is not an interpretation of $(GO)^-_{caut}$ in $(AO)^-_{full}$. On the positive side, we have the following:

Observation. (1) $(AO)^-_{full} \vdash \Box(G^\#(x) \equiv H(x))$
(2) $(AO)^-_{full} + (\forall Y)(P(Y) \equiv \Box P(Y)) \vdash (\exists x)H(x)$

Proof. (1) follows from the fact that $KD45 \vdash \Box(\Box\varphi \to \varphi)$.
(2) Let g be such that $\Box H(g)$, thus
$\Box((\forall Y)(P(Y) \equiv \Box Y(g))$,
$(\forall Y)(\Box P(Y) \equiv \Box Y(g))$,
$(\forall Y)(P(Y) \equiv Y(g))$.
which gives $H(g)$.

Summarizing, the removal of the axiom $\Box\varphi \to \varphi$, i.e. replacement of S5 by KD45 does not affect proofs of the main theorem in either system, but the existence of a godlike object becomes unprovable and the nice relation between both systems is lost. A deeper investigation of the weakened systems (including some completeness theorem) could shed more light to them.

6 Logica, teofilia e morte.

Professor Magari starts his paper [M] by saying "Teofili hanno spesso fornito ingegnosi argomenti... esistono anche teofobi (io lo sono di tutto cuore)", thus "theophils have collected ingenious arguments [for the existence of God]; there are also theophobs (as me, by my whole hearth)". I cannot share Magari's "theophoby" (being a Christian); but, as I remarked in [H], I agree with him saying "non è più facile ammettere gli assiomi che ammettere direttamente il teorema" (thus "it is not easier to accept the axioms [of Gödel] than to accept directly the theorem). Quotations from [Küng] will not be repeated here; but let me state explicitly that I regret very much that I have postponed serious discussion with Magari - until it was too late.

Quoting from the end of [M], I think that both "teofili" and "teofobi" may agree with him saying "occore in ogni caso stare molto in guardia contro tutto ciò che può essere sugerito dal desiderro di credere"(in any case one has to be very sensitive against everything possibly suggested with the desire to believe) - just because one's faith is not and should not be a result of somebody else's demand. The immediate continuation of the last quotation documents, in my opinion, Magari's tolerance and preference of objectivity against his own feelings: "Debo ammettere, con una certa riluttanza, che analogamente va trattato il desiderio di non credere" (I have to admit, with a certain reluctance, that the desire not to believe has to be treated analogously). The end not quotation reads "...che però mio sembra assai più raro" (but the latter seems to me rather more rare); I comment that during the communist system in my country the official atheist propaganda and "desiderio di non credere" was very strong and frequent. But this does not concern the main topic of discussion.

Professor Magari is dead and we remain grateful for his contribution to logic and various other domains of mathematics.

7 References

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