Gödel's God in Isabelle/HOL

Christoph Benzmüller and Bruno Woltzenlogel Paleo

October 29, 2013

1 Introduction

A formalization and (partial) automation of Dana Scott's version [10] of Goedel's ontological argument [7] in quantified modal logic KB (QML KB) is presented. QML KB is in turn modeled as a fragment of classical higher-order logic (HOL). Thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benzmüller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (poperties).

The formalization presented here has been carried and formally verified in the Isabelle/HOL proof assistant; for more information on this system see the textbook by Nipkow, Paulson, and Wenzel [9]. More recent tutorials on Isabelle can be found at the Isabelle homepage: http://isabelle.in.tum.de.

Some further notes:

1. This LaTeX text document has been produced automatically from the Isabelle source code document at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Isabelle/GoedelGodSession with the Isabelle build tool.

2. The formalization presented here is related to the THF [12] and Coq [4] formalizations available at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/.

3. All reasoning gaps in Scott's proof script have been automated with Sledgehammer [5] performing remote calls to the higher-order automated theorem prover LEO-II [3]. These calls then suggested respective Metis [8] calls as given below. The Metis proofs are then verified in Isabelle/HOL.

4. For consistency checking, the Nitpick model finder [6] has been employed.

2 An Embedding of QML KB in HOL

The types i for possible worlds (or states) and mu for individuals are introduced.

typedecl i — the type for possible worlds
typedecl mu — the type for indiviuals

Possible worlds are connected by an accessibility relation .

consts $r :: i \Rightarrow i \Rightarrow bool$ (**infixr** r *70*) — accessibility relation r

The B axiom (symmetry) for relation r is stated. B is needed only for proving theorem T3.

axiomatization where *sym*: $x\ r\ y \longrightarrow y\ r\ x$

QML formulas are identified with certain HOL terms of type $i \Rightarrow bool$. This type will be abbreviated in the remainder as σ

type-synonym $\sigma = (i \Rightarrow bool)$

The classical connectives $\neg, \wedge, \Rightarrow$, and $\forall$ (for individuals and over sets of individuals) and $\exists$ (over individuals) are lifted to type σ. Further connectives could be introduced analogously. *definition* could be used instead of *abbreviation*; the latter are always fully expanded/rewritten, which is fine here, where the focus has been on proof automation, but which would lead to overly complex proof tasks in a purely interactive session.

abbreviation *mnot* $:: \sigma \Rightarrow \sigma$ $(m\neg)$ **where** $m\neg\ \varphi \equiv (\lambda w.\ \neg\ \varphi\ w)$
abbreviation *mand* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\wedge$ *79*) **where** $\varphi\ m\wedge\ \psi \equiv (\lambda w.\ \varphi\ w \wedge \psi\ w)$
abbreviation *mimplies* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\Rightarrow$ *74*) **where** $\varphi\ m\Rightarrow\ \psi \equiv (\lambda w.\ \varphi\ w \longrightarrow \psi\ w)$
abbreviation *mforall-ind* $:: (mu \Rightarrow \sigma) \Rightarrow \sigma$ $(\forall i)$ **where** $\forall i\ \Phi \equiv (\lambda w.\ \forall x.\ \Phi\ x\ w)$
abbreviation *mexists-ind* $:: (mu \Rightarrow \sigma) \Rightarrow \sigma$ $(\exists i)$ **where** $\exists i\ \Phi \equiv (\lambda w.\ \exists x.\ \Phi\ x\ w)$
abbreviation *mforall-indset* $:: ((mu \Rightarrow \sigma) \Rightarrow \sigma) \Rightarrow \sigma$ $(\forall p)$ **where** $\forall p\ P \equiv (\lambda w.\ \forall x.\ P\ x\ w)$
abbreviation *mbox* $:: \sigma \Rightarrow \sigma$ $(\Box)$ **where** $\Box\ \varphi \equiv (\lambda w.\ \forall v.\ \neg\ w\ r\ v \vee \varphi\ v)$
abbreviation *mdia* $:: \sigma \Rightarrow \sigma$ $(\Diamond)$ **where** $\Diamond\ \varphi \equiv (\lambda w.\ \exists v.\ w\ r\ v \wedge \varphi\ v)$

For the grounding of lifted formulas the meta-predicate *valid* is introduced.

abbreviation *valid* $:: \sigma \Rightarrow bool$ ([-]) **where** $[p] \equiv \forall w.\ p\ w$

The model finder Nitpick confirms that the axioms and definitions above are consistent. Unfortunately, the respective command syntax for Nitpick is not very intuitive.

lemma *True* **nitpick** [*satisfy*, *user-axioms*, *expect* = *genuine*] **oops**

Constant symbol P (Gödel's "Positive") is introduced.

consts $P :: (mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms *A1(a/b)*: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$ (Either a property or its negation is positive, but not both.) and *A2*: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ (A property necessarily implied by a positive property is positive.).

axiomatization where
A1a: $[\forall p\ (\lambda\Phi.\ P\ (\lambda x.\ m\neg\ (\Phi\ x))\ m\Rightarrow\ m\neg\ (P\ \Phi))]$ **and**
A1b: $[\forall p\ (\lambda\Phi.\ m\neg\ (P\ \Phi)\ m\Rightarrow\ P\ (\lambda x.\ m\neg\ (\Phi\ x)))]$ **and**
A2: $[\forall p\ (\lambda\Phi.\ \forall p\ (\lambda\psi.\ (P\ \Phi\ m\wedge\ \Box\ (\forall i\ (\lambda X.\ \Phi\ X\ m\Rightarrow\ \psi\ X)))\ m\Rightarrow\ P\ \psi))]$

We prove theorem T1: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$ (Positive properties are possibly exemplified). T1 is proved directly by Sledghammer with command *sledgehammer* [*provers* = *remote-leo2 remote-satallax*]. This successful attempt then suggest to instead try the Metis call in the line below. This Metis call generates a proof object that is verified in Isabelle/HOL's kernel.

theorem *T1*: $[\forall p\ (\lambda\Phi.\ P\ \Phi\ m\Rightarrow\ \Diamond\ (\exists i\ \Phi))]$
sledgehammer [*provers* = *remote-leo2*]
using *A2 A1a* **by** *metis*

Next, the symbol G, for "God-like", is introduced and defined as $G(x) \leftrightarrow \forall\phi[P(\phi) \to \phi(x)]$ (A God-like being possesses all positive properties:).

definition $G :: mu \Rightarrow \sigma$ **where** $G = (\lambda x.\ \forall p\ (\lambda\Phi.\ P\ \Phi\ m{\Rightarrow}\ \Phi\ x))$

Axiom *A3* is added: $P(G)$ (The property of being God-like is positive.). Sledgehammer and Metis then prove corollary C: $\Diamond\exists x G(x)$ (Possibly, God exists.).

axiomatization where *A3*: $[P\ G]$

corollary C: $[\Diamond\ (\exists\ i\ G)]$
sledgehammer [*provers = remote-leo2*]
using *A3 T1* **by** *metis*

We add axiom *A4*: $\forall\phi[P(\phi) \to \Box\ P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where *A4*: $[\forall p\ (\lambda\Phi.\ P\ \Phi\ m{\Rightarrow}\ \Box\ (P\ \Phi))]$

Symbol *ess*, for "Essence", is introduced and defined as $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \to \Box\forall y(\phi(y) \to \psi(y)))$ (An *essence* of an individual is a property possessed by it and necessarily implying any of its properties.).

definition $ess :: (mu \Rightarrow \sigma) \Rightarrow mu \Rightarrow \sigma$ (**infixr** *ess 85*) **where**
$\Phi\ ess\ x = \Phi\ x\ m{\wedge}\ \forall p\ (\lambda\psi.\ \psi\ x\ m{\Rightarrow}\ \Box\ (\forall i\ (\lambda y.\ \Phi\ y\ m{\Rightarrow}\ \psi\ y)))$

Next, Sledgehammer and Metis prove theorem *T2*: $\forall x[G(x) \to G\ ess.\ x]$ (Being God-like is an essence of any God-like being).

theorem *T2*: $[\forall i\ (\lambda x.\ G\ x\ m{\Rightarrow}\ G\ ess\ x)]$
sledgehammer [*provers = remote-leo2*]
by (*metis* (*lifting*) *A1b A4 G-def ess-def*)

Symbol *NE*, for "Necessary Existence", is introduced and defined as $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \to \Box\exists y\phi(y)]$ (Necessary existence of an individual is the necessary exemplification of all its essences.).

definition $NE :: mu \Rightarrow \sigma$ **where** $NE = (\lambda x.\ \forall p\ (\lambda\Phi.\ \Phi\ ess\ x\ m{\Rightarrow}\ \Box\ (\exists\ i\ \Phi)))$

Moreover, axiom *A5* is added: $P(NE)$ (Necessary existence is a positive property.).

axiomatization where *A5*: $[P\ NE]$

Finally, Sledgehammer and Metis prove the main theorem *T3*: $\Box\exists x G(x)$ (Necessarily, God exists).

theorem *T3*: $[\Box\ (\exists\ i\ G)]$
sledgehammer [*provers = remote-leo2*]
using *A5 C T2 sym G-def NE-def* **by** *metis*

corollary *T4*: $[\exists\ i\ G]$
sledgehammer [*provers = remote-leo2*]
using *T1 T3 sym G-def* **by** *metis*

Finally, the consistency of the entire theory is checked with Nitpick.

lemma *True* **nitpick** [*satisfy*, *user-axioms*, *expect = genuine*] **oops**

Acknowledgments: Nik Sultana, Jasmin Blanchette and Larry Paulson provided very important help wrt consistency checking in Isabelle. Jasmin Blanchette instructed us on how to produce latex documents from Isabelle sources, and he pointed to many useful tricks in Isabelle.

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