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"PARFAITS MIROIRS DE L'UNIVERS": A 'VIRTUAL' INTERPRETATION OF LEIBNIZIAN METAPHYSICS

> Mir scheint daher, daß in der Leibniz-Forschung mit recht wiederholt der Versuch von Leibniz kritisiert worden ist, das mathematische Konvergenzmodell auch auf Begriffe zu übertragen ([Schneider], 291, quoted in [Mates], 112)
>
> Mea Principia talia sunt, ut vix a se invicem divelli possint. Qui unum bene novit, omnia novit (Leibniz to des Bosses, 7 November 1710, GII,412) (My principles are such that one could hardly disjoin them. Who knows one well, knows them all.)

0. INTRODUCTION

In "Virtual Modality" (in press) – a long technical article – I developed an adequate metalogical semantics for modal extensions of a recursively axiomatisable first-order theory T, and called the motivations for this semantic framework '*Leibnizian*". The purpose of this somewhat less formal essay will be to offer a partial justification my invocation of one of the more venerable names in the history of western philosophy.

To this end, let T be such an incomplete axiomatisable theory, and **C** the completion of T's Lindenbaum algebra – the (Dedekind–MacNeill) completion of the the boolean algebra of formulas of T modulo T-provable equivalence.

In the present context, **C** may be concretely realised as the *Stone algebra*, or boolean algebra of regular open sets in T's *Stone space*, where the latter is construed as the (metatheoretically defined) space of Henkin interpretations of T. Detailed information about Stone algebras and Stone spaces (so named for their twentieth-century discoverer and investigator, Marshall Stone), as well as the notions of boolean-valued 'randomness' which underlie these results, may be found in (Bell) and (Bell and Machover), (Jech) and other articles, textbooks and monographs cited in (Boos 1998).

Synthese **136:** 281–304, 2003.

Briefly, the semantic framework set out in "Virtual Modality" may be characterised as follows.

1. *Modal 'intensions' or 'intensional objects'* in this framework are topologically '*random*' or '*virtual*' (**C**-valued) 1-*types* over T (maximal consistent sets of *properties* in the language of T) (I will assimilate such '*types*' in the sequel to Leibniz' "*complete individual notions*").
2. *Modal 'truth'* **u** is a canonically defined 'random' (**C**-valued) Henkin interpretation of T (**u** will also provide a metalogical gloss of Leibniz' elusive notion of "*metaphysical necessity*" for T).
3. *Modal valuations of the 'names' or constants of* T are 'random' (**C**-valued) valuations of the constants of T in the 'virtual' model **u** (and therefore particular instances of 1's 'modal intensions').
4. Valuations of modal '*necessity*' are '*random*' (**C**-*valued*) *theoretical extensions* **N** of T in the language of T. Such **N** may be also construed as *'virtual' 'accessibility'-relations* between the **C**-valued Henkin interpretations **w** of T. Finally,
5. '*Leibnizian' valuations of modal necessity*' are particular 'random' (**C**-*valued*) extensions of T which are **C**-valued intersections of the vacuously 'true' theory **u** for T, and one other, '*actual*' random Henkin model **w** of T. I will construe such **w** as virtual sources of "*sufficient reason*" and "*hypothetical necessity*" for T below.

The principal theorem of "Virtual Modality" was a *completeness-result*, in which I derived the existence of 'countermodels' **N** for unprovable modal assertions *s* from the bridge between virtual theories and accessibility-relations mentioned above in 4. The existence of simpler 'Leibnizian' countermodels **N** for *s* then followed as a corollary.

I will not try to provide an extended review of these results here, but sketch instead some of their potential implications, and construe the structures they employ as metalogical ectypes of metaphysical archetypes in Leibniz' "*principes*".

Particular metamathematical glosses, for example, emerge for

6. Leibniz' appeals to "*complete individual notions*";
7. his '*predicate-in-notion*'-principle;
8. the '*principle of the identity of indiscernibles*';
9. his 'holist' '*mirroring principle*';
10. the elusive distinction(s) he tried to draw between "*metaphysical*" and *"hypothetical" necessity*;
11. his "*grand principe . . . de la raison suffisante*".
12. his many attempts to characterise "*perfection*";
13. his "*principle of plenitude*"; and finally

14. his persistent invocations of "*virtual identity*", and many allusions to continually (and continuously) graduated forms of monadic "*confusion*".

I will provide more detailed metalogical accounts of 6–9 in Section 1 below, turn to 10 and 11 in Section 2, consider 12 and 13 in Section 3, and assimilate 14's boolean-valued 'virtuality' to Leibnizian "confusion" in Section 4.

I will also offer detailed textual rationales for these reconstructions and assimilations, and make a final effort to draw together these conjectural interrelations between my metalogical miniatures and Leibniz' original *principes* in the essay's conclusion.

All of these metamathematical glosses, it should be mentioned, will also be theory-*relative*, in a dual sense: they are formulated in tacit *metatheoretic* venues U, in which one can define canonical *spaces of interpretations* for *particular theories* T – the *Stone spaces* mentioned above. I will therefore conclude this introductory section with a brief preliminary rationale (with apologies to Wittgenstein) for my introduction of such *(meta)logische Räume*.

In a very terse but systematic and (to me) suggestive note, reprinted as (Couturat, 1–3), Leibniz tried to outline a '*numerical*' analogue of '*truth*'(s) of 'reason' and of 'fact' (cf. also other, kindred texts such as FdC187ff.), and I will assimilate these explicitly analogical arguments – arrayed in parallel columns – to a proto-analysis of an unspecified theory's Stone space.

In the original fragment, of course (also translated in (AG 98-101), as "The Source of Contingent Truths"), Leibniz did not talk about any formal background-theory T for his reflections, much less observe that such a theory might be *incomplete*. He was, however, aware that "contingency" *required* some sort of incompleteness or underdetermination of the 'merely logical' *characteristica(e)* he had elsewhere posited.

In any event, Leibniz' still-inchoate analogies in their original form led him to distinguish between

15. *finite* (binary?) decimals, which he assimilated (in running entries in two adjacent columns) assertions that are *decidable* (provable or refutable), in ways one might now call '*finitary*' (from finitely many premises); and
16. *infinite* (binary?) decimals, which he aligned with assertions that could *not* be finitarily decided, and would, in that sense, be *contingent*.

It is now quite standard to observe

17. that Stone spaces of incomplete theories T are homeomorphic to (Cantor-) spaces of *infinite binary (rational as well as irrational) decimal expansions*; and
18. that such spaces, in turn, are homeomorphic to '*random*' spaces of *countably many independent coin-tosses*;.

In ways I will try to elaborate further below, my introduction of Stone spaces of theories and their free-variable counterparts may therefore be defensible attempts to elaborate an incomplete but deeply original Leibnizian insight into the *semantics* of the *characteristicae magnae* he so cherished.

1. 'MIROIRS', 'NOTIONS' ET 'PREDICATS'

In this section, I will begin to sketch the interrelations promised above between the metalogically 'random' formal semantics of 1–5, and Leibniz'

1. ontology of '*complete individual notions*'; his
2. '*predicate-in-notion*'-principle; his
3. '*principle of the identity of indiscernibles*'; and, finally, his
4. (holist, 'flower-in-the-crannied-wall'-, or) '*mirroring*'-principle.
   I will also begin to examine
5. the graduated states of '*confusion*', '*potentiality*' and relational *underdetermination*. Leibniz attributed to monadic '*perception*'.

Consider first the following well-known passages, from sections 8, 9 and 13 of the text we call the "Discourse on Metaphysics" (1686); from Primary Truths (1686); and from Principles of Nature and of Grace (1714). I have cross-referenced them here for convenience with the translations of [AG]:

6. **Complete individual notions and the predicate-in-notion-principle**

> ("Discourse on Metaphysics" (1686), section 8, GIV,432-433, AG40-1) ... puisque les actions et passions appartiennent proprement aux substances individuelles (*actiones sunt suppositorum*), il seroit necessaire d'expliquer ce que c'est qu'une telle substance. Il est bien vray, que lorsque plusieurs prédicats s'attribuent à un même sujet, et que ce sujet ne s'attribue à aucun autre, on l'appelle substance individuelle; mais cela n'est pas assez, et une telle explication n'est que nominale ....
> ... lors qu'une proposition n'est pas indentique, c'est à dire lorsque le predicat n'est pas compris expressement dans le sujet, it faut qu'il y

soit compris virtuellement, et c'est que les philosophes appellent *inesse* .... en sorte que celuy qui entendroit parfaitement la notion du sujet, jugeroit aussi que le predicat lui appartient.
...la nature d'une substance individuelle ou d'un estre complet, est d'avoir une notion si accomplie qu'elle soit suffisante à comprendre et à en faire deduire tous les predicats du sujet à qui cette notion est attribuée.

7. **Indiscernibility**
("Primae Veritates" (1686), C519, AG32) Sequitur etiam hinc, non dari posse in natura duas res singulares solo numero differentes.
("Discourse on Metaphysics" (1686), section 9, GIV434, AG42)
Il s'ensuit de cela plusieurs paradoxes considerables, comme entre autres qu'il n'est pas vray que deux substances se ressemblent entierement, et soyent differentes solo numero ....

8. **Mirroring and *Confusion***
("Discourse on Metaphysics" (1686), section 9, GIV434, AG42)
De plus toute substance est comme un monde entier et comme un miroir de Dieu ou bien de tout l'univers, qu'elle exprime chacune à sa façon, à peu pres comme une même ville es diversement representée selon les differentes situations de celuy qui la regarde .... Car il exprime quoyque confusement tout ce qui arrive dans l'univers, passé, present ou avenir, ce qui a quelque ressemblance à une perception ou une connoissance infinie. ...
("Principles of Nature and of Grace" (1714), section 3, GVI,599, AG207)
...[i]l s'ensuit que chaque Monade est un miroir vivant, ou doué d'action interne, representatif de l'univers, suivant son point de vue, et aussi reglé que l'univers lui-même.

9. **'Absolute' and 'Metaphysical' Necessity**
("Discourse on Metaphysics" (1686), section 13, GIV436-7, AG45)
...il semble que par là la difference des verités contingentes et necessaires sera détruite, ...et que la fatalité absolue regnera .... A quoy je réponds ...que la connexion ou consécution est de deux sortes, l'une est absolument necessaire, dont le contraire implique contradiction ...; l'autre n'est necessaire qu'ex hypothesi, et pour ainsi dire par accident, et elle est contingente en elle même .... Et cette connexion est fondée ...sur la suite de l'univers.

10. **Compossibility and Graduated 'Perfection'**
(Letter to Bourguet, 1714, GIII573, L662)
Vous y ajoutés ces paroles: Si l'on regarde l'univers comme une collection, on ne peut pas dire qu'il puisse y en avoir plusieurs. Cela seroit

vray, si l'univers etoit la collection de tous les possibles; mais cela n'est point, parce que tous les possibles ne sont point compossibles; et l'Univers actuel est la collection de tous les possibles existans [sic], c'est à dire de ceux qui forme le plus riche composé. Et comme il y a de differentes combinaisons des possibles, les unes meilleurs que les autres, il y en a plusieurs Univers possibles, chaque collection de compossibles en faisant un.

11. **"Chaque Ame connoit l'infini, connoit tout, mais confusement, comme en me promenant sur le rivage de la mer . . . "**
('. . . through a glass, darkly . . . .')
("Principles of Nature and of Grace" (1714), section 13, GVI604, AG211)
On pourroit connoitre la beauté de l'univers dans chaque Ame, si l'on pouvoit deplier tous ses replis, que ne se developpent sensiblement qu'avec le temps. Mais comme chaque perception distincte de l'Ame comprend une infinité de perceptions confuses, qui enveloppent tout l'univers, l'Ame même ne connoit dont elle a perception, qu'autant qu'elle en a des perceptions distinctes et revelées; et elle a de la perfection, en mesure de ses perceptions distinctes. Chaque Ame connoit l'infini, connoit tout, mais confusement, comme en me promenant sur le rivage de la mer, et entendant le grand bruit qu'elle fait, j'entends les bruits particuliers de chaque vague, dont le bruit total est composé, mais sans les discerner; nos perceptions confuses sont le resultat des impressions que tout l'univers fait sur nous. Il en est de même dans chaque Monade. Dieu seul a une connoissance de tout, car il en est la source. On a fort bien dit, qu'il est comme centre partout; mais sa circomference n'est nulle part, tout luy étant present immediatement, sans aucun eloignement de ce Centre.

On my metalogical reading of these middle- and late-Leibnizian passages, they reflected Leibniz' decision to *identify* (at least "*virtuellement*"),

12. "complete individual notions", "monads" and/or "metaphysical points" with
13. *maximally consistent classes of theoretical predicates that might 'correctly' be attributed to them*, in the sense of a proto-metatheoretical *characteristica*.

I have therefore proposed to assimilate these classes to

14. metalogical *types* – 1-types, more precisely - which appear naturally in the semantics of first-order theories (cf., e.g., Bell and Machover, 205), that is, to
15. *maximally consistent sets* **m** *of formulas in a single free variable in the language of* T.

This assimilation also fits precisely into a wider pattern of "*Stone duality*", which may be sketched as follows.

If a theory T is given as above, let S(T) be the set of all maximally consistent set of formulae in the language of T (or equivalently, using Henkin constants to emulate free variables, Henkin interpretations of T).

One may then say of a *property or 'predicate'* $s(x_1)$ in T that

16. **m** *verifies s*, or *s holds in* **m**, iff [s] is *in* **m** – is a set-theoretic member of **m**; or equivalently, iff
17. the sentence obtained by substituting the first Henkin constant for the variable x_1 in s holds, in Tarski's sense, in the Henkin model which corresponds to **m**.

One can also define the *Stone topology* for S(T). Each formula s in the language of T corresponds to a *basic open* (and closed) set [s]: [s] consists of the '*notions*' **m** *which verify s* – equivalently, are *such that the predicate s is in the notion* **m**).

Let us quickly check that this assimilation bears out analogues of Leibniz' original claims.

It does, first, closely parallel Leibniz' attempts to anchor *substantiality* in some sort of metatheoretic *completeness*, as in 1 above, *via* a conception of such 'completeness' as a systematic canvass of 'all' the predicates one might attribute to a given 'substance'.

Leibniz, moreover, took pains at several points to express his admiration for Plato, and hypothetically 'complete' canvasses of predicates had already appeared in the **Parmenides** as glosses of the forms.

Integration of time-evolution and dynamics of infinitesimally-generated 'well-founded properties' *into* the formalism of T, finally, can accommodate quite readily the "New System"-account of substantiality Leibniz developed in the 1690s (cf., e.g., Rutherford (1995a, 145–158), and Rutherford (1995b, 124–132)).

Such 'complete' canvasses also surfaced later in the metaphysics of Kant, most conspicuously in the "completeness" or *Vollständigkeit der Bedingungen* that characterised 'transcendent' *Vernunftideen*. Kant also argued that the *Bereich der Erscheinungen* is "complete", and alleged to 'deduce' this in some elusive 'transcendental' sense, but struggled for much of the rest of his career to discern such 'constitutive' completeness from 'merely regulative' counterparts.

Be that as it may – the metalogical reconstruction of Leibniz' *predicate-in-notion-principle*, in the framework sketched above, is extensionally correct. All one need do is construe "*in*" as "*is a member of*", as in 10 above.

The *principle of the identity of indiscernibles* is also immediate, in the framework of the identification sketched above in 6 through 9. 'Complete notions', identified as types in L(T), are simply sets of predicates in L(T). By completeness and the axiom of extensionality, they can therefore be discerned in a set-metatheory for T iff the negation of a predicate 'in' one is 'in' the other.

The *mirroring principle*, finally, holds in the following qualified, theory-relative sense. A type or 'complete notion' t trivially includes many predicates of the form "s and $x = x$", where s is a sentence in the language L(T) of T. The set of such sentences s forms a *complete theory* in L(T), and this theory is in fact the reduct to L(T) of any complete Henkin theory m which realises t.

It would seem reasonable, therefore, to argue that 'complete' knowledge of the *type t* would 'express' everything that could be said *in the original language of* T about the 'world' determined by **m**.

In a number of passages, Leibniz also defended, or seemed to defend, a stronger '*compossibilist*' formulation of the mirroring principle, loosely associated with a relational property of complete notions he called "*compossibility*" (cf. e.g., the remarks in (GIII573), (L662), the letter to Bourguet quoted above).

"Compossibility" is a somewhat vexed notion, historically and metalogically. There is an obvious if somewhat simplistic sense in which every Henkin structure or complete Henkin extension of T may be regarded as a collection of 'compossibles' – its so-called elementary diagram – in ways that seem to fit the passage in the letter to Bourguet.

Other passages, however, suggest a need to search for stronger or more carefully differentiated glosses. According to one of these, defended by Mates, Rescher and others,

18. every complete notion **t** would express its binary, tertiary. ... *(inter)relations* with every other complete individual notion $\mathbf{t}^*$; and
19. a (maximal consistent) *aggregate* of such complete notions (and therefore of the interrelations between them) would consitute a '*world*'.

Recall again that in "Virtual Modality", the metalogical counterparts **t** of Leibniz' *complete notions* are "*one-types over* T" – maximally consistent collections of predicates in the original language L(T) of T.

In this context, a rough but natural counterpart of 19's '*aggregate*' can be elicited from the 'Leibnizian' structure **w** mentioned in 0.5 – namely the **C**-valued collection of *one-types* (in the *larger* language L(H) of the *Henkin closure* H *of* T) that are *realised in* **w**.

In the same context, one could also sketch metalinguistically stratified partial gloss of 18. For each random **C**-valued 1-type **t** *over a Henkin-closed extension* H *of* T uniquely defines a random Henkin structure **w**, essentially as in the interpretation of the 'mirroring principle' above. And each *pair* (**u**, **w**), in turn, would then determine a '*Leibnizian' interpretation of* T-*expressible modal 'relations' between virtual 1-types*.

Some attempt at metalinguistic disambiguation (here between 'type over T' and 'type over H') may be needed to 'save' the informal claim made in 18. For straightforward metalogical interpretations of it would be untenable.

Such *disambiguations*, however, typically give rise to *metatheoretic hierarchies*, and problematic attempts to '*collapse*' such hierarchies seem to underlie 'vexed' aspects of many classical metapysical thought-experiments – Leibniz' 'compossibility', perhaps, among them.

Similar patterns of *metalinguistic* and *metatheoretic ascent* also emerge, for example, when one tries to 'collapse' hierarchies which disambiguate notions of '*definability*' (cf., e.g., section 2 of Boos (1998)). Both 'definability' and type-formation, of course, are also forms of *(meta)logical individuation* – pointwise definability in one case, and 'complete' specification in the other.

More generally, I argued in Boos (1998) that 'all' disambiguations of significant *semantic* or '*alethic*' notions give rise to such ascents. To the extent that one can generalise or philosophise about such notions, they seem to be *relational*, or perhaps '*liminal*': expressible only in ways that generate (and demand) new thresholds of reinterpretation.

In this sense, they may 'all' be spiritual descendents of Parmenidean and Aristotlelian '*tritos-anthropos*'-arguments, or ancient skeptical modes of 'lapse into infinites'.

Be that as it may, I wish to argue more concretely here – in support of the particular claims made in Section 1 above – that the metalogical reconstructions in "Virtual Modality" *do* in fact yield a kind of *modal ontology*.

(In the light of Willard van Orman Quine's opposition to modal ontologies of any sort, it may also be slightly ironic that I make this assertion in the sense of his well-known dictum that "*to be is to be the value of a bound variable*".)

For in "Virtual Modality" 's formal semantics for modal extensions of T, outlined above, the entities one metatheoretically (and 'virtually') *quantifies* over *are*, in fact, the 1-types I have assimilated to Leibniz' "*complete individual notions*".

In the next section, I will examine the tenability of "Virtual Modality" 's semantics as an analogical reconstruction for Leibniz' "*principle of sufficient reason*", and his somewhat elusive distinction between "*metaphysical*" and "*hypothetical*" *necessity*.

2. 'NECESSITÉ(S)' ET 'RAISON(S) SUFFISANTE(S)'

It may be relevant, first, that Leibniz explicitly introduced his second "*great principle*", in the **Monadology** (GVI, 612), as a semantic refinement of its *first* (and, implicitly, weaker) counterpart – the Aristotelian principle of "(non)contradiction" or *consistency*. For *consistency* of a first-order theory T is a *precondition* for *nontriviality* of T's Stone space in one sense (the space is nonempty), and *incompleteness* of T is so in another (the space is larger than $\{0, 1\}$).

Indeed, one may gloss assorted semantic paradoxes and results of Skolem, Gödel, Chaitin and others as refinements of

1. Leibniz' attempts to discern consistency (or interpretability) from 'truth'; and
2. his remarks about the elusiveness and liminality of these and other criterial, 'alethic notions', viewed 'from within'.

I will return to such glosses at the end of this section, and in the essay's conclusion. For now, I wish to argue that Leibniz' "second great principle" – that of 'sufficient reason' – fulfilled the following offices in his system.

It first

3. postulated forms of *meta*mathematical *coherence*, in the form of initial and boundary-conditions, for the illimitably complex "*raisons particulières*"of his "monads" graduated sentience and experience.

In the process, it also

4. re-*secularized* – or at least re-*mathematised* – Descartes' appeals to the veracity of 'god' as the source of a quasi-stoic 'criterion of truth', including mathematical truth.

And finally, the principle of sufficient reason *anticipated*, in my view – or at least provided some carefully-considered prototypes for – Kant's claims (in very different language)

5. that the 'forms' and 'categories' of '*Erfahrung*" are *uniquely "constitutive"* of 'our' experience,

and

6. that the latter therefore has some sort of 'transcendentally' *identifiable* as well as *determinate* structure.

In the process, it also preempted the views expressed in two of Albert Einstein's well-known quasi-eschatological *dicta*:

7. that the most *in*explicable *datum* about 'the universe' is that 'it' *is* (he firmly believed) *explicable*; and
8. that "*der Alte*" – Einstein's "*old man*", who is *"raffiniert" aber nicht "boshaft" - nicht "würfelt"*
   (a wry but thoroughly 'Leibnizian' remark).
   From the 1670s on, Leibniz *also* believed quite clearly
9. that *both* of his "two great principles" - of consistency and sufficent reason – and
10. most of the ancillary '*principes*, he adduced to clarify and elaborate them, were '*logical*', in some deep sense, rather than *theo*logical; and
11. that they were integrally related in some way to an as-yet-unrealised '*characteristica universalis*'.

It is no accident, I believe, that twentieth-century *metalogicians* have derived enormous benefit from Thoralf Skolem's and Kurt Gödel's insights into the *limitations* of such just '*characteristicae*', elaborated as precisely expressible schemes for *coding* and '*arithmetising*' finitary proof-systems of the sort introduced and annotated by Frege, Peano, Russell and others (reviewed, e.g., in Boos (1998)).

It would seem appropriate, therefore, look for explicit *Rückbeziehungen* between Skolem and Gödel's ground-breaking metalogical ideas about the limitations of metalogical ideas, and the insights and expressions of hope Leibniz cast in the form of his allegedly constitutive '*principes*'.

Finally, it also seems to me appropriate historically, as well as methodologically, to adduce Stone's metatheoretically definable *spaces of interpretations*, and look for interrelations *not* in explicitly *modal* formalisms – for these (I believe) were *derivative* in Leibniz metaphysics – but in the *not*-so-universal *characteristicae* and theory-relative 'ontologies' provided by axiomatic number- and set-metatheories.

For it is precisely *within* these rather vacuously abstract metatheoretic contexts that one might try to *impose* more concrete formal constraints, which might serve, in turn – in various alternative ways – as conjectural counterparts for Leibniz' '*principles*'.

Indeed, one of Leibniz' greatest virtues as a metaphysician, I believe, was his amiable willingness to (try to) *identify* – as such '*principes*' – the assumptions he knew he had to *beg* – a practice most of his predecessors, contemporaries and successors (including some who notoriously claimed

to present their more 'systematic' conclusions *more geometrico*) honored only in the breach.

In any case, the "*principle of sufficient reason*"– in my reconstruction, at least – is an intrinsically *meta*theoretic existence-claim. It posits the *existence* of a unique structure divinely-'intended' *structure* or *ratio* for T's 'experience(s)'.

In effect, I have assimilated such a *ratio* to an element **w** of a 'virtual' space St(T) of *interpretations* for a given theory T. Such a model – Leibniz observed – could only be specified '*completely*' by an what he often called an '*analysis infinitorum*' – which we human beings, in our finitude, can only adumbrate with the aid of mathematical 'analysis' and indefinite extrapolation.

Indeed, in "Virtual Modality", I have not only

12. assimilated Leibniz' '*analyses infinitorum*' to Bolzano–Weierstrass-like canvasses which converge to element **w** of St(T); but also
13. observed that our '*finitude*' may also induce us to demand '*absolutely*' '*sharp*' interpretations of experiential evidence, and of modal assertions about such evidence, when only '*relatively*' ('*hypothetically*'?) '*sharp*' counterparts of such interpretations are to be had.

Arguments cognate to these apply, finally, to one of Leibniz' most notoriously vexed and elusive distinctions – the one he struggled to draw between '*absolute*' and *'hypothetical' necessity*.

In his study of the '*dual*' distinction, I would argue – the distinction between '*absolute*' and '*hypothetical*' *possibility* – Leibniz effectively outlined an inchoate but well-thought-through attempt to discern

14. the existence of *an interpretation for* T – *a* (Henkin) structure which models a given (incomplete) theory T (any structure at all which models T) – as a guarantor of *'absolute' or 'metaphysical' possibility*; and
15. the existence of *an 'intended' interpretation* **w** for T ('discerned' or individuated, if at all, only by (a) '*god*').

The latter – the fixed structure of **w**, including its internal 'time-evolution' – would also determine the boundary conditions, as it were, for what might be called *actuality*, regarded as a kind of '*hypothetical possibility*'.

The language and substance of Leibniz' arguments was greatly complicated, I believe, by the fact that he *also* believed the divinely 'intended' structure **w** would also lock into place a kind of **w**-*internal 'hypothetical' 'necessity'* – which one could identify with some appropriate sort

of '*Laplacean determinism*': the notion that time-evolution of **w**-internal events is subject to some sort of universal (differential) equation.

Much metaphysical and eschatological *Unfug*, of course – brilliantly parodied by Voltaire – resulted from Leibniz' attempts to reconcile his temperamental meliorism with this then-paradigmatically-'scientific', 'determinist' view.

In any event, I believe that Leibniz tried to base an inchoate but well-grounded modal semantics for 'universal' but expressible theories T in which 'contingencies' (undecidabilities) arise on some kind of *superposition of two* determining structures – **u** and **w** – for such T.

These are, once again:

16. a comparably 'universal' but rather vacuous notion **u** of '*truth*' for T; and
17. a *second* structure **w** for T, which provides a '*sufficient reason*' – that is,
18. a 'complete' and particular account (*logos*, *ratio*, *raison*) for 'all' of our (past, present and potential future) 'experience(s)'.

The first structure would impose a kind of initial condition of "absolute necessity", given by 'merely logical' inference (and in the tension between "*absolute*" and '*merely*", one might see the source of Wittgenstein's "*Sinnlosigkeit*").

The second structure **w** *would* "virtually" decide – 'analyse' – the boundary- conditions of "metaphysical necessity" – the illimitably nuanced complexities and interrelations of monadic 'perceptions'. Such a structure 'must' exist, Leibniz believed, or postulated, even though we finite monadic minds might 'perceive' its patterns and superpositions only in "*confused*" ways.

3. 'PERFECTION' AND 'PLENITUDE'

The last section's metalogical reconstructions of Leibnizian 'actuality' and 'sufficient reason' provide complementary interpretations of his notion(s) of *perfection* and *plenitude*, if one construes the latter as stipulations imposed on 'actual' worlds or interpretations **w** for T, and types t over T.

Consider first the following middle- and late-Leinizian characterisations of (moral and metaphysical) "**perfection**".

1. ("Discourse on Metaphysics" 6, GIV431, AG9)
   . . . Dieu a choisi celuy [le monde] qui est plus parfait, c'est à dire celuy

qui est en même temps le plus simple en hypotheses et le plus riche en phenomenes, ...

2. ("De rerum originatione radicali", GVII304, AG151)
   Et ut possibilitas est principium Essentiae, ita perfectio seu Essentiae gradus (per quem plurima sunt compossibilia) principium existentiae.
3. ("Monadology" 50, GVI615, AG219)
   Et une Creature est plus parfait qu'une autre en ce qu' on trouve en celle ce qui sert à rendre raison a priori de ce qui se passe dans l'autre, et c'est par là qu'on dit, qu'elle agit sur l'autre.
4. (Letter to Wolff, LW161, AG230)
   Perfectio, de qua quaeris, est gradus realitatis positivae, vel quod eodem redit, intelligibilitatis affirmativae, ut illud sit perfectius, in quo plura reperiuntur notatu.
5. (Letter to Wolff, LW170, AG233)
   Plus observabilitatis est in re, est plures in ea proprietates universales, plus harmoniae; ergo idem est perfectionem quaerere in essentia, et quaerere in proprietatibus quae ex essentia fluunt.
6. (Letter to Wolff, LW171, AG233)
   Consensus quaeritur in varietate, hic placet eo magis, quo facilius observatur, et in hoc consistit sensus perfectionis. Perfectio autem in re ipsa est tanto major, quanto major est consensus in majore varietate, sive a nobis observatur vel non.
7. (Letter to Wolff, LW172, AG233-4)
   Perfectio est harmonia rerum, vel observabilitas universalium, seu consensus vel identitas in varietate, posses etiam dicere gradum considerabilitatis.

Leibniz also used "*perfectum*" from time to time as a near-synonym for "*completum*" (cf., e.g., "...in perfecta notione cujusque substantiae individualis continentur omnia ejus praedicata tam necessaria quam contingentia ...", GVII311)).

In passages such as the following, Leibniz also interwove claims about "*perfection*" with similar-sounding but metalogically distinct ontological assertions (often implicit) about "*plenitude*".

8. (Letter to Malebranche (1679), GI331, L211)
   Quicquid agit, quatenus agit, liberum est. Il faut dire aussi que Dieu fait le plus de choses qu'il peut, et ce qui l'oblige à chercher des loix simples, c'est à fin de trouver place pour tout autant de choses qu'il est possible de placer ensemble.
9. ("De rerum originatione radicali" (1697), GVII303, AG150)
   Ut autem paulo distinctius explicemus, quomodo ex veritatibus aeternis sive essentialibus vel metaphysicis oriantur veritates tempor-

> ales, contingentes sive physicae, primum agnoscere debemus eo ipso, quod aliquid potius existere quam nihil, aliquam in rebus possibilibus seu in ipsa possibilitate vel essentia esse egentiam existentiae, vel (ut sic dicam) praetensionem ad existendum et, ut verbo complectar, essentiam per se tendere ad existentiam. Unde porro sequitur, omnia possibilia, seu essentiam vel realitatem possibilem exprimentia, pari jure ad essentiam tendere pro quantitate essentiae seu realitatis, vel pro gradu perfectionis quem involvunt; est enim perfectio nihil aliud quam essentiae quantitas.
>
> Hinc vero manifestissime intelligitur ex infinitis possibilium combinationibus seriebusque possibilibus existere eam, per quam plurimum essentiae seu possibilitatis perducitur ad existendum.

Moderately complex interrelations can be traced between metalogical analogues of these notions, and I will review several of these below.

Preliminary attempts to disambiguate metalogical counterparts of Leibnizian "*perfectio*" might distinguish *'perfection' of a theory* form *'perfection' of its (Henkin-) interpretations*, seen as particular extensions of T.

Considering theories first, and abusing for a moment the distinction between theories and their axiomatisations, one might, for example, introduce an early-modern descendent of Occam's razor, and define *'perfection' of* T as

10. *independence* of T's *axiomatisation* (cf. 3.1 above); or, more sharply, as
11. *recursive axiomatisability* of T.

On the simpler reconstruction in 10, a given *axiomatisation* of a theory T would be more '*perfect*' – more *streamlined*, one might say – if it is *minimally redundant*.

Given appropriate metatheoretic premises, it is not hard to establish that every first-order theory may be 'perfected' in this relatively straightforward way. But the axiomatisations that witness this 'perfection' would not in general be unique, and such 'perfection' would not change the consequences of T.

The *second* candidate for *theory-'perfection'* – in 11 – might seem to offer a more promising realisation for Occam's basic idea. For it would require that T have an epistemically accessible and communicable axiomatisation – if not finite, then at least expressible, in 'principle', in a prescriptive or algorithmic form. As the Germans put it, such a condition might require that 'god's design for T be "*nachvollziehar*".

It would follow however from Gödel's results that 11's accessibility-, communicability- and *Nachvollziehbarkeits*-condition *conflicts*, in general, with another metalogical reconstruction of 'perfection' Leibniz evidently valued, namely

12. *completeness* of T (cf. 2, 5 and 6 above).

Indeed, Gödel's arguments suggest a kind of *complementarity*-relation may exist between the *'epistemic' desiderata* in 11 and their '*ontological*' counterparts in 12.

For a theory is '*epistemically*' more '*perfect*' if it '*abstracts*' elegantly from more structures in its language, considered as sources of 'particular', merely 'empirical' data. But it is '*ontologically*' more '*perfect*' if it accounts – metaphorically speaking – for every sparrow that falls. And Gödel's results pointed to an inherent tension between these two conditions

Historical antecedents of metalogical completeness-notions had already appeared in Parmenidean infinite-canvass-reconstructions of the Platonic forms, mentioned earlier, as well as in Leibniz' own *notiones completae*. And metalogical descendents of these notions also lie at the heart of proofs of the completeness-theorem, and of the Stone-duality-properties briefly invoked in Section 1. *Every* Henkin-interpretaton of T, in particular, is 'ontologically perfect' in the sense of 12.

This points, in turn, to another difficulty with 12 - that '*ontologically*' more 'perfect' extensions of '*epistemically*' more 'perfect' theories are anything but *unique*. There are, for example, continuum-many such extensions of *any* theories that inductively *recapitulate ('encode') their own syntax* (itself a reasonable criterion of 'epistemic perfection'), among them the theories in 11. 'Ontological arguments' focused on the criterion in 12 would therefore have to consider whole *pandaimonia* of such 'godlike' extensions.

In response to these persistent problems, one might shift one's attention to *'perfection' of interpretations* **w** for a given theory T, and try to elicit from more nuanced semantic conditions on such interpretations notions of

13. '*epistemic perfection*' of such **w**, as their 'inductive' accessibility 'from within' (cf. also 4.1 and 4.2 below); and
14. '*ontological perfection*', as their "*richesse*", "*plenitude*" or *realisation of maximally many types over* T (or *notiones completae* for T).

Interestingly enough, these notions – more latitudinarian forms of 11 and 12, which also reflect Leibnizian desiderata of the sort quoted earlier - are *also* incompatible. For the natural metalogical counterpart of 13 – called '*model-theoretic genericity*' of such **w** – is equivalent to the assertion that they *omit – fail* to realise – *every type they possibly can*.

Also interesting, perhaps, is that the metalogical counterpart of the condition in 14 – various forms of which are called "*saturation*" – *does* yield a kind *unicity*-result for such **w** (up to isomorphism) – *when it holds*. But such validity is itself contingent on acceptance of detailed (and defeasible) background-set-theoretic assumptions.

Roughly speaking, then, one way to formulate persistent '*complementarities*' between 11/13 and 12/14 might be to suggest

15. that *if* a 'god' (or demiurge) *were* to devise a **w** that mirrored 'all' the 'actual' variety it could, *we* would be unable to gain an 'inductive' epistemic overview of that 'ontological' variety 'from within'.

This might provide warrants for a (controversial) '*theodicy*', perhaps'; but not a '*theology*'.

Be that as it may, such formulations also suggest that Leibniz may have been '*right*' to postulate some sort of '*plenitude*' (or 'richness', or 'continuity') as a guarantor of relevant forms of metalogical *unicity*. Monotheists who wish to formalise modal or metalogical counterparts of 'ontological arguments' might keep this mind.

But Leibniz may have been '*wrong*', once again, if he believed that the '*existence*' of such 'rich' structures could unproblematically be '*proved*'; or that such 'plenitude' could be anticipated or characterised processively '*from within*'. Indeed, the 'complementarity'- relations just sketched suggest that 'ontological arguments from within' might not even be (expressible, communicable, intelligible) *arguments*.

(By way of technical side-comment: the 'complementarity'-relations between 13 and 14 also suggest that the modal Barcan formula-scheme may reflect an '*epistemic*' contraint on modal quantification. For it holds in the 'Leibnizian' semantics associated with **w** ifand only if **w** satisfies a very strong boolean-valued form of 13.)

Such considerations of conflicting 'epistemic' and 'ontological' alternatives have not exhausted the range of Leibnizian attempts to provide proto-metalogical reconstructions of '*perfection*' (perhaps it is not exhaustible).

Consider once again, for example, the passages in which he considered

15. eam [combinationem possibilium seriemque possibilem], per quam plurimum essentiae seu possibilitatis perducitur ad existentiam (GVII303, AG150);

and the

16. "[c]reature ... plus parfait qu'une autre", in which one finds "ce qui sert à rendre raison a priori de ce qui se passe dans l'autre" (GVII615, AG 219).

If one counts *theories*, for example, among Leibniz' "*creatures*" (his spirit might be willing), one might assimilate

17. the '*essence*' of a class of structures for a given theory T to the theoretical extension of T common to (satisfied in) all of them;
18, '*existence*' of such an 'essence' to metatheoretic existence of a structure which *interprets* it (David Hilbert once embraced such a 'distributive' notion) ....

Returning to 'perfection' of *theories*, one could then stipulate formally that

19. a *theory* U is *'more perfect' than another theory* V, if and only if one can *rendre raison in* U *for* V's *existence* (prove in U that V is consistent, or equivalently for interesting U's, that V has an interpretation).

This formal relation – definable in reasonable set-metatheories – is sometimes called *proof-* or *consistency-theoretic strength*, and informal antecedents of it have also played a role in assorted 'proofs' of the 'existence of god' (consider, for example, Descartes' implicitly metatheoretic hierarchy of 'formal' and 'objective' 'realities' in Meditation III (ATVII, 40–42).

It might therefore have seemed quite reasonable for Leibniz to expect

20. that new adumbrations of 'perfection' would emerge from them in his hoped-for *characteristica magna*; and
21. that these notions would *augment*, or at least be *compatible with*, the proof-theoretic elegance (epistemic 'perfection') he sought to reconcile with them.

In the event, however, another metalogical 'complementarity'-principle applies to 11, 13 and 19. For it is not hard to prove in metatheories of the sort mentioned above – relatively weak subtheories of ZFC, for example – that

22. '*epistemically perfect*' theories in the sense of 11 or 13 *cannot* be 'perfect' in the sense of 19, if such '*perfection*' is *maximality* with respect to proof theoretic strength.

Shifting again to perfection of interpretations, however, one can make another observation which applies to a wide variety of interesting theories T (Peano arithmetic and assorted set-theories among them): that

23. *interpretations* **w** of such T that are '*ontologically perfect*' in the sense of 14 (satisfy appropriate 'saturation'-properties) are '*more perfect*'

(in the sense of 19) than any of their '*epistemically accessible*' *substructures* or *subtheories* (in particular, 'more perfect' than any of their subtheories that are '*recursively axiomatisable*' in various ways that generalise the original sense mentioned above in 11).

To me at least, these lines of argument suggest that such 'complementarity'-properties lie along a persistent unsharp boundary between epistemology and metaphysics, and that the *Unschärfe* of this boundary has interesting implications for attempts to make rigorous certain forms of 'the ontological argument', as well as assorted roughly cognate 'arguments from (or to) design'.

In the concluding section of this essay, I will draw further on the evidence of the modal semantics developed in "Virtual Modality" to suggest that the "*confusion*" Leibniz (unlike Descartes) gracefully attributed to all finitary monadic "knowledge" (including his own) – and tried also intermittently to put to fideist uses - may be systematically ineliminable.

More precisely, I will argue

24. that the structures **w** of 0.5, 2.15 and 2.17 – as Solovay, Chaitin and other investigators have observed in many forms – will almost certainly turn out to be *random* elements of the Stone space St(T) of T – or *random* binary decimals (cf., once again, Leibniz' attempts to model 'contingency' in (Couturat) 1–3, reviewed above in 0.15–0.18); and finally
25. that recurrent randomness of this sort refracts (metalogical counterparts of) Leibnizian 'perfection' and 'plenitude' in potentially interesting ways.
(Vielleicht "würfelt" also 'der Alte' doch?)

4. CONCLUSION

In this concluding section, I will try to sketch some interrelations and analogies between

1. the epistemic '*virtuality*' of "Virtual Modality" 's modal semantics;
2. the incompleteness. plurality of interpretation and '*liminality*' of discursive modalities; and
3. the diffusion, underdetermination and '*confusion*' of Leibniz' "*promenade sur le rivage de la mer*".

In a period in which popularisers of 'chaos theory' have tried to describe sensitivity to initial conditions in terms of 'butterfly effects', many

readers may also understand the sort of holism and semantic realism about "*Fait(s)*" that Leibniz tried to express in the following familiar passages.

4. **Raison(s), (In)definability and analyses infinitorum**
   ("Monadology", sections 33, 35 and 36, GVI612-613, AG217)
   Il y a aussi deux sortes de Verités, celles de Raisonnement et celles de Fait. Les Verités de Raisonnement sont necessaires et leur opposé est impossible, et celles de Fait sont contingentes et leur opposé est possible ....
   Et il y a enfin des idées simples, don't on ne sauroit donner la définition ....
   Mais la raison suffisante se doit aussi trouver dans les verités contingentes ou de fait, c'est à dire dans la suite des choses repandues pars l'univers des Créatures, ou la Résolution en raisons particulières pourroit aller à un detail sans bornes, à cause de la varieté immense des choses de la Nature et de la division des corps à l'infini. Il y a une infinité de figures et de mouvements présents et passés, qui entrent dans la cause éfficiente de mon écriture présente, et il y a une infinité des petites inclinations et dispositions de mon âme présentes et passées, qui entrent dans la cause finale.

In an attractive metaphor, mentioned earlier – an obvious prototype of Fourier-analysis – Leibniz also likened the "*confusion*" of "*detail sans bornes*" and "*la varieté immense des choses de la Nature et ... la division des corps à l'infini*" to our pleasant but somewhat vague auditory perception(s) of "*the waves at the seashore*" in section 13 of "Principles of Nature Grace" (GVI604, AG211).

Less poetically evocative but comparably defensible likeness for such "confusion" (and for Nicholas of Cusa' universally distributed "*centre partout*", cited by Leibniz two sentences further on) might be found in the **C**-valued superpositions of this essay's semantics, its 'mixtures' and its virtual 'intensions'.

Such superpositions *can* be 'localised' and 'sharpened', but only *provisionally*, in partial and processive ways, adequate to 'decide' *data* that emerge in countable simulacra of a given set-theoretic structure **M**.

M cannot localise *itself*, however – cannot judge *its own* adequacy and 'sharpness'. It may only '*think*', for example, that it is uncountable, in ways that dissolve in the yet-'sharper' focal resolution of wider set-metatheoretic contexts.

In "Virtual Modality", accordingly, the 'actual worlds' **w** that determine Leibnizian models and countermodels for modal extensions of T are *not* models of T, but *'virtual' models of* T, or equivalently, *topologically random variables of models of* T. And interpretations of designation (nam-

ing) and predication are *distributive* and *graduated* along similarly random lines.

By this, for example, I mean that the interpretations of 'Sophie Charlotte' and properties of 'Sophie Charlotte' expressible in a theory T that talked about 'Sophie Charlotte' would be 'random', **C**-valued interpretations of these notions in the inherently 'virtual' model **u**.

By results of Robert Solovay, these **C**-valued interpretations of T's constants and atomic predicates 'distribute' uniformly over other **C**-valued models **w** of T, as definable borel images of their values in **u**.

Leibniz' search for higher-order "*principles*" and efforts to discern different *levels* of '*necessity*' might also bear witness to another early insight – into the recurrent but elusive distinction between *theory* and *metatheory*. Indeed, one might draw a thumbnail distinction between his metaphysics and Spinoza's by observing that Leibniz' 'god' '*chooses*' – *meta*theoretically – among alternative interpretations of Spinoza's "*deus sive natura*".

Work of Solovay, Chaitin and others has also suggested that '*randomness*' – which I've assimilated in admittedly rather heuristic ways to Leibniz' "*confusion*" – is *theory-relative*, rather like other, more obviously relational semantic notions, such as 'consistency' (Gödel), and 'truth' (Tarski). In particular, randomness-in-a-theory might be assimilated to what is underdetermined or *imprécisable* in that theory (compare Leibniz' many allusions to what "only [a] god could know").

In (Boos 1994), finally, I observed once again

1. that '*most*' models for interesting first-order theories T are '*random*', in a sense that can be made precise in topological as well as measure-theoretic ways; and
2. that attempts to define models in '*inductive*' ways, '*from within*', lead precisely to such *randomness* (whose topological variant is usually called '*genericity*') in the models they '*induce*' (cf. again 3.13 above, and the remarks that follow).

Let me once again call such modal, alethic and quasi-alethic notions – which lie at the boundary or epistemic horizon of theories for which they may be formulated – "*liminal*" (at least with respect to those theories to which they are 'intended' to apply).

The consistency of Peano number theory, for example, is a theorem of ZF, but *liminal* for Peano number theory itself, as is the consistency of ZF in ZF. What objects or predicates are *determinate* or *expressible* in a given theory T, similarly, is *liminal* for T, but becomes *object*-theoretically *determinate* and *expressible* in a variety of *metatheories* for T – for which analogous questions can be posed in their turn.

To me at least, such considerations suggest that some outlines of a network of interrelations may be discerned between 'liminal' notions of

2. '*modality*';
3. '*counterfactual support*';
4. '*potentiality*';
5. '*dispositionality*';
6. '*probability*';
7. *'distributive' notions of designation and predication*; and
8. *'randomness' of epistemic and theoretical boundaries*;

where *all* of these notions are *relativised to* particular theories T and *expressed in* metatheories U for T.

On my readings of such "*liminality*" – most of them prompted by one or another result in the metalogical textbook- and monograph-literature – one might also draw on such patterns to formulate a few preliminary conjectures.

One is that semantic '*bivalence*' and *sharpness* of measurement and modal individuation may be both

8. *transcendent*, in the sense that 'the' distinction between what is determinate and indeterminate may be indeterminate, as well as undecidable 'from with'; and even
9. *transcendental illusions*, in the sense that thoroughly indeterminate and undecidable interpretations may look incontrovertibly determinate, again, 'from within' ('the' 'universal' structure **u**, for example – a C-valued 'redundant truth- definition' – '*looks*' bivalent in the multivalent context of the random V(**C**)).

Another is that one should perhaps *expect* some sort of '*fuzziness*', or '*randomness*', or *semantic gradualism* to emerge in studies of modality and counterfactuality – as updated and more complex forms, perhaps, of Leibniz' '*continuity*'-principles.

If what is '*factual*', for example, were as illimitably complex as Leibniz thought – not '*chaotically*' *determinate*, but *recurrently random and indeterminate*, with respect to 'our' theories and metatheories as they evolve – might not 'the' *(modal) distinction(s)* between *factual* and *counterfactual* be *comparably graduated and nondeterministic as well*?

Whatever the merits of such conjectures, these late-twentieth-century metamathematical miniatures of Leibniz ideas (heard at the seashore, as it were) prompted me to propose another hypothesis at the end of "Virtual Modality", with which I will also conclude here.

The two anachronistic clauses of this conjecture are:

10. that Leibniz may have been '*wrong*' about the semantic adequacy for modal discourse of pairs of structures {**u**, **w**} of the sort mentioned above; but
11. that Leibniz may have been *'virtually' right* about the semantic adequacy of such pairs, in the sense that
12. Leibniz was *'almost surely' right* about their semantic adequacy – with value 1 in T's Stone algebra **C**.

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