Gödel's Ontological Proof of God's Existence (Draft)

Bruno Woltzenlogel Paleo, Annika Siders

August 25, 2013

"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Introduction

2 Natural Deduction

ToDo: Show and explain here the rules of the calculus we are using.

ToDo: We should use a calculus for the basic modal logic K. Everything else should be stated as axioms.

ToDo: cite a paper that proves soundness and completeness for this calculus.

3 Possibly, God Exists

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Theorem 1 *Positive properties are possibly exemplified:*

$$\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$$

Proof

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom 2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}}{\forall\psi.[(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \psi(x)]) \to P(\psi)]}\;\forall_E}{(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \neg\varphi'(x)]) \to P(\neg\varphi')}\;\forall_E}{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to P(\neg\varphi')} \qquad \dfrac{\dfrac{\text{Axiom 1}}{\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}}{P(\neg\varphi') \leftrightarrow \neg P(\varphi')}\;\forall_E}{\dfrac{\dfrac{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to \neg P(\varphi')}{P(\varphi') \to \Diamond\exists x.\varphi'(x)}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}\;\forall_I}
$$

Definition 1 *A* God-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \to \varphi(x)]$$

Axiom 3 *The property of being God-like is positive:*

$$P(G)$$

Corollary 1 *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof

$$
\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{P(G) \to \Diamond\exists x.G(x)}\;\forall_E}{\Diamond\exists x.G(x)}\;\to_E
$$

4 Being God is an essence of any God

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi) \to \Box\ P(\varphi)]$$

Definition 2 *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi \ ess\ x \leftrightarrow \varphi(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

ToDo: instead of using the $\Box_E$ rule, we should use the M axiom.

Theorem 2 *Being God-like is an essence of any God-like being:*

$$\forall y.[G(y) \to G\ ess\ y]$$

Proof Let the following derivation with the open assumption $G(x)$ be $\Pi_1[G(x)]$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\neg P(\psi)^1 \quad \dfrac{\dfrac{\text{Axiom 1}}{\forall\varphi.(\neg P(\varphi)\to P(\neg\varphi))}}{\neg P(\psi)\to P(\neg\psi)}\,\forall_E}{P(\neg\psi)}\,\to_E \quad \dfrac{\dfrac{G(x)}{\forall\varphi.(P(\varphi)\to\varphi(x))}\,\text{D1}}{P(\varphi)\to\varphi(x)}\,\forall_E}{\neg\psi(x)}\,\to_E \quad \psi(x)^2}{\bot}\,\to_E}{P(\psi)}\,RAA^1
$$

$$
\dfrac{P(\psi)}{\psi(x)\to P(\psi)}\,\to_I^2
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_2[G(x)]$:

$$
\dfrac{\dfrac{\dfrac{\psi(x)^3 \quad \dfrac{\Pi_1[G(x)]}{\psi(x)\to P(\psi)}}{P(\psi)}\,\to_E \quad \dfrac{\dfrac{\text{Axiom 4}}{\forall\psi.(P(\psi)\to\Box P(\psi))}}{P(\psi)\to\Box P(\psi)}\,\forall_E}{\Box P(\psi)}\,\to_E}{\psi(x)\to\Box P(\psi)}\,\to_I^3
$$

Let the following derivation without open assumptions be Π_3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(\psi)^4 \quad \dfrac{\dfrac{G(x)^5}{\forall\varphi.(P(\varphi)\to\varphi(x))}\,\text{D1}}{P(\psi)\to\psi(x)}\,\forall_E}{\psi(x)}\,\to_E}{G(x)\to\psi(x)}\,\to_I^5}{\forall x.(G(x)\to\psi(x))}\,\forall_I}{P(\psi)\to\forall x.(G(x)\to\psi(x))}\,\to_I^4
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_4[G(x)]$:

$$
\dfrac{\dfrac{\dfrac{\psi(x)^6 \quad \dfrac{\Pi_2}{\psi(x)\to\Box P(\psi)}}{\Box P(\psi)}\,\to_E \quad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\,\Box_E \quad \dfrac{\Pi_3}{P(\psi)\to\forall x.(G(x)\to\psi(x))}}{\forall x.(G(x)\to\psi(x))}\,\to_E}{\Box\forall x.(G(x)\to\psi(x))}\,\text{Necessitation}}{\Box P(\psi)\to\Box\forall x.(G(x)\to\psi(x))}\,\to_I^7}{\Box\forall x.(G(x)\to\psi(x)}\,\to_E}{\psi(x)\to\Box\forall x.(G(x)\to\psi(x))}\,\to_I^6
$$

The use of the necessitation rule above is correct, because the only open assumption $\Box P(\psi)$ is boxed. In the derivation of Theorem 2 below, the assumption $G(x)$ in the subderivation $\Pi_4[G(x)^8]$ is discharged by the rule labeled 8.

$$
\cfrac{\cfrac{\cfrac{\cfrac{G(x)^8 \qquad \cfrac{\cfrac{\Pi_4[G(x)^8]}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}}{\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\,\forall_I}{G(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\,\land_I}{G\ ess\ x}\,\mathrm{D2}}{G(x) \to G\ ess\ x}\,\to_I^8}{\forall y.[G(y) \to G\ ess\ y]}
$$

5 If God's existence is possible, it is necessary

Definition 3 Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi\ ess\ x \to \Box\exists x.\varphi(x)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1 *If there is a God, then necessarily there exists a God:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof

$$
\cfrac{\cfrac{}{\exists z.G(z)}\,1}{G(g)}
$$

$$
\cfrac{\cfrac{\cfrac{G(g) \qquad \cfrac{\cfrac{\text{Theorem 2}}{\forall y.[G(y) \to G\ ess\ y]}}{G(g) \to G\ ess\ g}}{G\ ess\ g} \qquad \cfrac{\cfrac{\cfrac{\text{Axiom 5}}{P(E)} \qquad \cfrac{\cfrac{G(g)}{\forall\varphi.[P(\varphi) \to \varphi(g)]}}{P(E) \to E(g)}}{E(g)}}{\cfrac{\forall\varphi.[\varphi\ ess\ g \to \Box\exists x.\varphi(x)]}{G\ ess\ g \to \Box\exists x.G(x)}}}{\Box\exists x.G(x)}}{\exists z.G(z) \to \Box\exists x.G(x)}\,1
$$

6 Necessarily, God exists

ToDo: This section still needs more details. See Coq formalization for more details.

ToDo: this is proven in a way that is slightly different from Gödel's 1970.

Theorem 3 *Necessarily, God exists:*

$$\Box\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\dfrac{\textbf{S5}}{\forall\varphi.[\Diamond\ldots\Diamond\Box\varphi \leftrightarrow \Box\varphi]}}{\Diamond\Box\exists x.G(x) \leftrightarrow \Box\exists x.G(x)} \qquad \dfrac{\dfrac{\text{Corollary 1}}{\Diamond\exists x.G(x)} \qquad \dfrac{\text{Lemma 1}}{\exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\Box\exists x.G(x)}}{\Box\exists x.G(x)}$$

7 God exists

Axiom 6 (M) *What is necessary is the case:*

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Corollary 2 *There exists a God:*

$$\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\text{Theorem 3}}{\Box\exists x.G(x)} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$