

Gödel's God in Isabelle/HOL

Christoph Benz Müller and Bruno Woltzenlogel Paleo

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A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$
D1	A <i>God-like</i> being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists xG(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$
D2	An <i>essence</i> of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G \text{ ess. } x]$
D3	<i>Necessary existence</i> of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists xG(x)$

1 Introduction

A formalization and (partial) automation of Dana Scott's version [11] of Goedel's ontological argument [8] in quantified modal logic KB (QML KB) is presented. QML KB is in turn modeled as a fragment of classical higher-order logic (HOL). Thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benz Müller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (poperties).

The formalization presented here has been carried out and verified in the Isabelle/HOL proof assistant; for more information on this system see the textbook by Nipkow, Paulson, and Wenzel [10]. More recent tutorials on Isabelle can be found at: <http://isabelle.in.tum.de>.

Some further notes:

1. This LaTeX text document has been produced automatically from the Isabelle source code document at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Isabelle/GoedelGodSession> with the Isabelle build tool.
2. The formalization presented here is related to the THF [13] and Coq [4] formalizations at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/>.

3. All reasoning gaps in Scott's proof script have been automated with Sledgehammer [5], performing remote calls to the higher-order automated theorem prover LEO-II [3]. These calls suggest the Metis [9] calls as given below. The Metis proofs are verified in Isabelle/HOL.
4. For consistency checking, the model finder Nitpick [6] has been employed.

2 An Embedding of QML KB in HOL

The types i for possible worlds (or states) and μ for individuals are introduced.

typeddecl i — the type for possible worlds
typeddecl μ — the type for individuals

Possible worlds are connected by an accessibility relation r .

consts $r :: i \Rightarrow i \Rightarrow \text{bool}$ (**infixr** r 70) — accessibility relation r

The B axiom (symmetry) for relation r is stated. B is needed only for proving theorem T3.

axiomatization where $\text{sym}: x\ r\ y \longrightarrow y\ r\ x$

QML formulas are identified with certain HOL terms of type $i \Rightarrow \text{bool}$. This type will be abbreviated in the remainder as σ .

type-synonym $\sigma = (i \Rightarrow \text{bool})$

The classical connectives $\neg, \wedge, \rightarrow$, and \forall (over individuals and over sets of individuals) and \exists (over individuals) are lifted to type σ . The lifted connectives are $m\neg, m\wedge, m\Rightarrow, \forall, \Pi$, and \exists . Further connectives could be introduced analogously. Definitions could be used instead of abbreviations.

abbreviation $mnot :: \sigma \Rightarrow \sigma$ ($m\neg$) **where** $m\neg\ \varphi \equiv (\lambda w. \neg\ \varphi\ w)$
abbreviation $mand :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\wedge$ 79) **where** $m\wedge\ \varphi\ \psi \equiv (\lambda w. \varphi\ w \wedge \psi\ w)$
abbreviation $mimplies :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\Rightarrow$ 74) **where** $m\Rightarrow\ \varphi\ \psi \equiv (\lambda w. \varphi\ w \longrightarrow \psi\ w)$
abbreviation $mforall\text{-ind} :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$ (\forall) **where** $\forall\ \Phi \equiv (\lambda w. \forall x. \Phi\ x\ w)$
abbreviation $mforall\text{-indset} :: ((\mu \Rightarrow \sigma) \Rightarrow \sigma) \Rightarrow \sigma$ (Π) **where** $\Pi\ P \equiv (\lambda w. \forall x. P\ x\ w)$
abbreviation $mexists\text{-ind} :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$ (\exists) **where** $\exists\ \Phi \equiv (\lambda w. \exists x. \Phi\ x\ w)$
abbreviation $mbox :: \sigma \Rightarrow \sigma$ (\Box) **where** $\Box\ \varphi \equiv (\lambda w. \forall v. \neg\ w\ r\ v \vee \varphi\ v)$
abbreviation $mdia :: \sigma \Rightarrow \sigma$ (\Diamond) **where** $\Diamond\ \varphi \equiv (\lambda w. \exists v. w\ r\ v \wedge \varphi\ v)$

For grounding lifted formulas, the meta-predicate *valid* is introduced.

abbreviation $valid :: \sigma \Rightarrow \text{bool}$ ($[-]$) **where** $[p] \equiv \forall w. p\ w$

3 Gödel's Ontological Argument

Constant symbol P (Gödel's 'Positive') is declared.

consts $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms $A1(a/b): \forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$ (Either a property or its negation is positive, but not both.) and $A2: \forall \phi \forall \psi [(P(\phi) \wedge \Box \forall x [\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ (A property necessarily implied by a positive property is positive).

axiomatization where

A1a: $[\Pi (\lambda \Phi. P (\lambda x. m \neg (\Phi x)) m \Rightarrow m \neg (P \Phi))]$ **and**
A1b: $[\Pi (\lambda \Phi. m \neg (P \Phi) m \Rightarrow P (\lambda x. m \neg (\Phi x)))]$ **and**
A2: $[\Pi (\lambda \Phi. \Pi (\lambda \psi. (P \Phi m \wedge \Box (\forall (\lambda x. \Phi x m \Rightarrow \psi x))) m \Rightarrow P \psi))]$

We prove theorem *T1*: $\forall \varphi [P(\varphi) \rightarrow \Diamond \exists x \varphi(x)]$ (Positive properties are possibly exemplified). *T1* is proved directly by Sledgehammer with command *sledgehammer [provers = remote-leo2]*. This successful attempt then suggests to instead try the Metis call in the line below. This Metis call generates a proof object that is verified in Isabelle/HOL's kernel.

theorem *T1*: $[\Pi (\lambda \Phi. P \Phi m \Rightarrow \Diamond (\exists \Phi))]$
sledgehammer *[provers = remote-leo2]*
by (*metis A1a A2*)

Next, the symbol *G* for ‘God-like’ is introduced and defined as $G(x) \leftrightarrow \forall \phi [P(\phi) \rightarrow \phi(x)]$ (A God-like being possesses all positive properties).

definition *G* :: $\mu \Rightarrow \sigma$ **where** $G = (\lambda x. \Pi (\lambda \Phi. P \Phi m \Rightarrow \Phi x))$

Axiom *A3* is added: $P(G)$ (The property of being God-like is positive). Sledgehammer and Metis then prove corollary *C*: $\Diamond \exists x G(x)$ (Possibly, God exists).

axiomatization where *A3*: $[P G]$

corollary *C*: $[\Diamond (\exists G)]$
sledgehammer *[provers = remote-leo2]* **by** (*metis A3 T1*)

Axiom *A4* is added: $\forall \phi [P(\phi) \rightarrow \Box P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where *A4*: $[\Pi (\lambda \Phi. P \Phi m \Rightarrow \Box (P \Phi))]$

Symbol *ess* for ‘Essence’ is introduced and defined as $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$ (An *essence* of an individual is a property possessed by it and necessarily implying any of its properties).

definition *ess* :: $(\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$ (**infixr** *ess* 85) **where**
 $\Phi \text{ ess } x = \Phi x m \wedge \Pi (\lambda \psi. \psi x m \Rightarrow \Box (\forall (\lambda y. \Phi y m \Rightarrow \psi y)))$

Next, Sledgehammer and Metis prove theorem *T2*: $\forall x [G(x) \rightarrow G \text{ ess. } x]$ (Being God-like is an essence of any God-like being).

theorem *T2*: $[\forall (\lambda x. G x m \Rightarrow G \text{ ess } x)]$
sledgehammer *[provers = remote-leo2]* **by** (*metis A1b A4 G-def ess-def*)

Symbol *NE*, for ‘Necessary Existence’, is introduced and defined as $NE(x) \leftrightarrow \forall \phi [\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)]$ (Necessary existence of an individual is the necessary exemplification of all its essences).

definition *NE* :: $\mu \Rightarrow \sigma$ **where** $NE = (\lambda x. \Pi (\lambda \Phi. \Phi \text{ ess } x m \Rightarrow \Box (\exists \Phi)))$

Moreover, axiom *A5* is added: $P(NE)$ (Necessary existence is a positive property).

axiomatization where *A5*: $[P NE]$

Finally, Sledgehammer and Metis prove the main theorem *T3*: $\Box \exists x G(x)$ (Necessarily, God exists).

theorem *T3*: $[\Box (\exists G)]$

sledgehammer [*provers = remote-leo2*] **by** (*metis A5 C T2 sym G-def NE-def*)

corollary *C2*: $[\exists G]$

sledgehammer [*provers = remote-leo2*] **by** (*metis T1 T3 G-def sym*)

The consistency of the entire theory is checked with Nitpick.

lemma *True nitpick* [*satisfy, user-axioms, expect = genuine*] **oops**

It has been criticized that Gödel's ontological argument implies what is called the modal collapse. The prover Satallax [7] can indeed show this, but verification with Metis still fails.

lemma *MC*: $[p \Rightarrow (\Box p)]$

using *T2 T3 ess-def sym* **sledgehammer** [*provers = remote-satallax*] **oops**

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4 Further results on Gödel's God.

abbreviation *mequals* :: $\mu \Rightarrow \mu \Rightarrow \sigma$ (**infixr** *m=* 90) **where** $x \text{ m= } y \equiv \Pi (\lambda \varphi. (\varphi \ x \Rightarrow \varphi \ y))$

Gödel's God is flawless, that is, he has no negative properties.

theorem *Flawless*: $[\Pi (\lambda \varphi. \forall (\lambda x. (G \ x \Rightarrow (m \neg (P \ \varphi) \Rightarrow m \neg (\varphi \ x)))))]$

sledgehammer [*provers = remote-leo2*] **by** (*metis A1b G-def*)

Moreover, it can be shown that any two God-like beings are (Leibniz-)equal, that is there is there only one God-like being.

theorem *Monotheism*: $[\forall (\lambda x. \forall (\lambda y. (G(x) \Rightarrow (G(y) \Rightarrow (x \text{ m= } y)))))]$

sledgehammer [*provers = remote-leo2*] **by** (*metis C sym T2 ess-def*)

Add-on: We briefly show that Leibniz equality denotes equality.

lemma *test2*: $x = y \Rightarrow [x \text{ m= } y]$

sledgehammer [*provers = remote-leo2*]

by *metis*

lemma *test1*: $[x \text{ m= } y] \Rightarrow x = y$

sledgehammer [*provers = remote-satallax*] **oops**

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