bin

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Contents

theory *SalamuchaPDF*

imports *Main*

begin
declare [[*smt-timeout* = *300*]]

High timeout for smt, so that there is a high probability that smt terminates. Not using this setting makes pdf creation really annoying.

1 Types and Definitions

typedecl *a*

things and stuff in the world

consts R:: $a \Rightarrow a \Rightarrow bool$

$_moves$

consts f :: $a \Rightarrow bool$

is in motion

consts *partof* :: $a \Rightarrow a \Rightarrow bool$ (**infixr** *M52*)

$_isaproperpartof$

consts *aspactu* :: $a \Rightarrow {}'a \Rightarrow a \Rightarrow bool$ (-A_- -)
consts *asppot* :: $a \Rightarrow {}'a \Rightarrow a \Rightarrow bool$ (- P_- -)
consts *body*:: $a \Rightarrow bool$ (C)
consts *duration* :: $'a \Rightarrow a \Rightarrow bool$ (- F -)

not entirely sure if we need a /time type/ here; I donacute¿t think it will change anything

consts *finitetime* :: $'a \Rightarrow bool$ (H)

abbreviation CC:: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set$
where $CC\ r \equiv \{a.\ \exists\, t.\ ((r\ a\ t) \vee (r\ t\ a))\}$

Salamucha has Cacute¿R

abbreviation *irreflexive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where *irreflexive* $r \equiv (\forall\, x.\ \neg\ (r\ x\ x))$

abbreviation *transitive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where *transitive* $r \equiv (\forall\, x\ y\ z.\ ((r\ x\ y) \wedge (r\ y\ z) \longrightarrow (\ r\ x\ z)))$

abbreviation *connected* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
where *connected* $r \equiv \forall\, x\ y.\ ((x \in (CC\ r) \wedge y \in (CC\ r) \wedge (x \neq y)) \longrightarrow (r\ x\ y \vee r\ y\ x))$

abbreviation K:: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ (K-)
where $K\ r \equiv ((connected\ r) \wedge (transitive\ r) \wedge (irreflexive\ r))$

2 Proof of Lemma T

Sledgehammer can prove the Lemma directly

lemma *Tauto*: $((\forall\, x.\ (f\, x \longrightarrow (\exists\, t.\ (R\ t\ x)))) \wedge (K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$ **by** (*metis* (*no-types*, *lifting*) *CollectI*)

Using the steps from Salamucha is actually worse performancewise (and needs smt)

Note that in the second version of Salamuchas notation (the boxed one) there is a consistent typo. The y in the Consequens of almost all formulas should be a v

Note that in step threeb neither threea nor twob is used

(*Warning: Stepseven is an smt proof. Other proof methods fail here [but if need be the proof can be made explicit*)

lemma T: $((\forall\, x.\ (f\, x \longrightarrow (\exists\, t.\ (R\ t\ x)))) \wedge (K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists\, v.\ (\neg\ (f\, v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
proof –
have *one*: $(\forall\, x.\ ((f\, x\ ) \longrightarrow (\exists\, t.\ (R\ t\ x)))) \longrightarrow (\forall\, x.\ ((\forall\, t.\ (\neg\ R\ t\ x)) \longrightarrow (\neg\ f\ x)))$ **by** *blast*
have *twoa*: $(K\ R) \longrightarrow (\forall\, y\ u.\ (R\ y\ u \longrightarrow \neg\ R\ u\ y))$ **by** *blast*
have *twob*: $(K\ R) \longrightarrow (\forall\, y\ u.\ ((u \in (CC\ R) \wedge u \neq y \wedge R\ y\ u) \longrightarrow (\neg\ R\ u\ y)\ ))$ **by** *meson*
have *threea*: $((K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))) \longrightarrow$

$(\exists v. (\forall u. ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$ **by** *meson*

have *threeb*: $((K\ R) \wedge (\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v. (\forall u. ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (\neg\ R\ u\ v))))$ **by** (*metis* (*mono-tags*, *lifting*))

have *threec*: $((K\ R) \wedge (\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v. (\forall u. ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (\neg\ R\ u\ v \wedge R\ v\ u))))$ **by** *meson*

have *four*: $((K\ R) \wedge (\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v. ((\forall u. ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (\neg\ R\ u\ v))) \wedge (\forall u. ((u \in (CC\ R) \wedge (u \neq v)) \longrightarrow (R\ v\ u)))))$ **by** *meson*

have *five*: $\forall u\ v. ((\neg\ (u \in (CC\ R))) \longrightarrow (\neg\ R\ u\ v))$ **by** *simp*

have *six*: $(K\ R) \longrightarrow (\forall u\ v. (u = v \longrightarrow (\neg\ R\ u\ v)))$ **by** *simp*

have *seven*: $((K\ R) \wedge (\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v. ((\forall u. ((\neg\ R\ u\ v))) \wedge (\forall u.((u \in (CC\ R) \wedge (u \neq v)) \longrightarrow (R\ v\ u)))))$
using *five six four*
by (*smt ext*)

have *eigth*: $((\forall x. (f\ x \longrightarrow (\exists t. (R\ t\ x)))) \wedge (K\ R) \wedge (\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v. (((\neg\ f\ v) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))))$ **using** *seven one* **by** *meson*

then show *?thesis* **by** *blast*

qed

The first two conjuncts of the antecedents of T imply the stronger Thesis T1

lemma *TtoT1*:
assumes *firsttwoT*: $(K\ R) \wedge (\forall x. (f\ x \longrightarrow (\exists t. (R\ t\ x))))$
shows $\forall x. ((f\ x) \longrightarrow (\exists t. ((R\ t\ x) \wedge t \neq x)))$ **using** *firsttwoT* **by** *blast*

Are the conjuncts of the antecedent of Thesis T all necessary?

lemma *T12*: $((\forall x. (f\ x \longrightarrow (\exists t. (R\ t\ x)))) \ \wedge\ (K\ R)\ )$
$\longrightarrow\ (\exists v. (\neg\ (f\ v) \wedge (\forall u. (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
nitpick[*verbose*]
oops

Nitpick does NOT find a counterexample; perhaps someone with more computing power could run this again*) (*Salamucha gives the following counterexample (p.115f): Let R be the greater-than relation on the positive natural numbers. Let f x mean that x is a positive number (hold trivially). Since ¿ is an ordering relation and there always is a bigger number the antecedents hold. There is however no positive number that is not positive therefore the conditional is false.

lemma *T13*: $((\forall x. (f\ x \longrightarrow (\exists t. (R\ t\ x)))) \ \wedge (\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow\ (\exists v. (\neg\ (f\ v) \wedge (\forall u. (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$

nitpick[*verbose*]
oops

lemma *T23*: $(\ (K \ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow \ (\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
nitpick[*verbose*]
oops

nitpick finds a counterexample

3 Proof of Thesis T1

fast can prove T1 in 1s

lemma *T1auto*:
assumes *onea*: $\forall\, x.\ ((f\, x) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *oneb*: $\forall\, x.\ ((\exists\, a\ b.\ ((a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
and *onec*: $\forall\, x.\ ((f\, x) \longrightarrow (\exists\, t.\ (R\ t\ x)))$
shows $\forall\, x.\ ((f\, x) \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *onea oneb onec* **by** *fast*

N.B.: Salamucha implies that this proof hold for other definitions of identities as well

Now with Salamuchas more expicit proof

Nitpick confirms consistency

Contrary to what Salamucha thinks, for step two both 11 and 12 are needed, not just 12; see below.

lemma *T1*:
assumes *11*: $\forall\, x.\ ((f\, x) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *12*: $\forall\, x.\ ((\exists\, a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
and *13*: $\forall\, x.\ ((f\, x) \longrightarrow (\exists\, t.\ (R\ t\ x)))$
shows $\forall\, x.\ ((f\, x) \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
proof –

have *onea*: $\forall\, x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x)))$ **using** *11* **by** *blast*
have *oneb*: $\forall\, x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee (f\ a \vee (\neg\ f\ b))))$ **using** *onea* **by** *auto*
have *onec*: $\forall\, x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))$ **using** *oneb* **by** *blast*

have *two*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\neg\ (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))\ \ )$ **using** *12 onea* **by** *blast*
have *threea*: $\forall x.\ ((\neg\ f\ x) \vee (\neg\ R\ x\ x))$ **using** *two onec* **by** *blast*
have *threeb*: $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ (\ (R\ t\ x) \wedge (\neg\ R\ x\ x))))$ **using** *13 threea* **by** *auto*
have *threec*: $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ (R\ t\ x \wedge t \neq x)))$ **using** *threeb* **by** *fast*
thus *?thesis* **by** *simp*
qed

12 does not imply two

lemma $(\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))) \longrightarrow (\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\neg\ (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))\ \ ))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

Are all assumptions necessary?

lemma *T1wo1*:
assumes *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
and *13*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ (R\ t\ x)))$
shows $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *T1wo2*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *13*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ (R\ t\ x)))$
shows $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *T1wo3*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
shows $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

4 Irreflexivity of R

first automated

lemma *irreflexivityRauto*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
shows *irreflexive R* **using** *11 12 14* **by** *presburger*

then using the steps in Salamuchas book

Nitpick runs out of time trying to find a model

N.B.: steps until threea are the same as in the proof of T1

lemma *irreflexivityR*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
shows *irreflexive R*
proof –

have *onea*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$ **using** *11* **by** *blast*
have *oneb*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee (f\ a \vee (\neg\ f\ b))))$ **using** *onea* **by** *auto*
have *onec*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))$ **using** *oneb* **by** *blast*
have *two*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\neg\ (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))\quad)$ **using** *12 onea* **by** *blast*
have *threea*: $\forall x.\ ((\neg\ f\ x) \vee (\neg\ R\ x\ x))$ **using** *two onec* **by** *blast*
have *foura*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (\neg\ R\ y\ y))$ **using** *14 threea* **by** *fastforce*
have *fourb*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow ((R\ x\ y) \wedge (\neg\ R\ y\ y)))$ **using** *foura* **by** *simp*
have *fourc*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (x \neq y))$ **using** *fourb* **by** *fast*
thus *?thesis* **by** *auto*
qed

Are the assumption all necessary?

lemma *irreflexivityRwo1*:
assumes *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
shows *irreflexive R*
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *irreflexivityRwo2*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
shows *irreflexive R*

nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *irreflexivityRwo4*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
shows *irreflexive R*
nitpick[*verbose*]
oops

Nitpick finds a counterexample

Show that the weaker assumption doesnacute¿t work to prove irreflexivity

lemma *weaker12*:
assumes *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
and *w12*: $\forall x.((\exists\ a\ b.\ (\ (a\ M\ x \wedge b\ M\ x) \wedge \neg\ (f\ a \longleftrightarrow f\ b)\ \ )) \longrightarrow (\neg\ R\ x\ x))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
shows *irreflexive R*
nitpick[*verbose*]
oops

Nitpick finds a counterexample

5 The third proof

first automated

lemma *thirdproofauto*:
assumes *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
and *22*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
and *23*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
and *24*: $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ (R\ t\ x)))$
shows $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *21 22 23 24* **by** *blast*

Nitpick confirms consistency (see commented call below).

lemma *thirdproof*:
assumes *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
and *22*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
and *23*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
and *24*: $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ (R\ t\ x)))$
shows $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
proof –

have *one*: $\forall x\ y.\ ((x\ A\ \ R\ y) \longrightarrow \neg(x\ P\ \ R\ y))$ **using** *21* **by** *simp*
have *two*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow ((x\ P\ \ R\ y) \wedge (y\ A\ \ R\ x)))$ **using** *22 23* **by** *simp*

have *threea*: $\forall x.((f\,x \wedge (R\ x\ x)) \longrightarrow ((x\ P\ \ R\ x) \wedge (x\ A\ \ R\ x)))$ **using** *two* **by** *simp*
have *threeb*: $\forall x.((f\,x \wedge (R\ x\ x)) \longrightarrow \neg((x\ A\ \ R\ x) \longrightarrow \neg(x\ P\ \ R\ x)))$ **using** *threea* **by** *simp*
have *four*: $\forall x.\ ((x\ A\ \ R\ x) \longrightarrow \neg(x\ P\ \ R\ x))$ **using** *one* **by** *simp*
have *five*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow ((x\ A\ \ R\ x) \longrightarrow \neg\ (x\ P\ \ R\ x)))$ **using** *four* **by** *simp*
have *six*: $\forall x.\ (f\,x \longrightarrow \neg\ (R\ x\ x))$ **using** *five threeb* **by** *simp*
have *seven*: $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ ((R\ t\ x) \wedge \neg(R\ x\ x))))$ **using** *24 six* **by** *simp*
have *eight*: $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *seven* **by** *fastforce*
thus *?thesis* **by** *simp*
qed

Are all assumptions necessary?

lemma *thirdproofwo1*:
assumes *22*: $\forall x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
and *23*: $\forall x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
and *24*: $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ (R\ t\ x)))$
shows $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *thirdproofwo2*:
assumes *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
and *23*: $\forall x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
and *24*: $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ (R\ t\ x)))$
shows $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *thirdproofwo3*:
assumes *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
and *22*: $\forall x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
and *24*: $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ (R\ t\ x)))$
shows $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *thirdproofwo4*:
assumes *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
and *22*: $\forall x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
and *23*: $\forall x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
shows $\forall x.\ (f\,x \longrightarrow (\exists\,t.\ ((R\ t\ x) \wedge t \neq x)))$
nitpick[*verbose*]
oops

Nitpick finds a counterexample

Next we show that also assumptions 21 22 23 and 14 imply irreflexivity

lemma *IrreflexivityRv2*:
assumes *21*: $\forall x\, y\, (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
and *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
and *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
and *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\, y)$
shows *irreflexive R*

using *21 22 23 14* **by** *meson*

Nitpick confirms consistency

Are the assumptions all necessary?

lemma *IrreflexivityRv2wo1*:
assumes *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
and *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
and *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\, y)$
shows *irreflexive R*
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *IrreflexivityRv2wo2*:
assumes *21*: $\forall x\, y\, (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
and *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
and *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\, y)$
shows *irreflexive R*
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *IrreflexivityRv2wo3*:
assumes *21*: $\forall x\, y\, (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
and *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
and *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\, y)$
shows *irreflexive R*
nitpick[*verbose*]
oops

Nitpick finds a counterexample

lemma *IrreflexivityRv2wo4*:
assumes *21*: $\forall x\, y\, (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
and *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
and *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
shows *irreflexive R*
nitpick[*verbose*]

oops

Nitpick finds a counterexample

6 Arguments for there being a first element

N.B. my local sledgehammer (and try0 etc.) can t prove the following theorem; the only remote prover that finds a proof is vampire but proof reconstruction fails even here. I would be interested if sledgehammer find a proof on a faster machine useful theorems to add are mem$_Collect_eqandperhapsTauto$

lemma *TpThenNotC3*:
assumes *Tp*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
and *c1*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$
and *c2*: $K\ R$
shows $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists u.\ ((u \in (CC\ R) \wedge u \neq x) \wedge (\neg\ R\ x\ u))))$
proof −

have *one*: $\forall x\ y.\ ((R\ y\ x) \longrightarrow (f\ y \wedge f\ x))$ **using** *Tp* **by** *fastforce*
have *two*: $\forall x\ y.\ ((R\ x\ y \vee R\ y\ x) \longrightarrow (f\ x \wedge f\ y))$ **using** *one Tp* **by** *blast*
have *threea*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists t.\ (R\ t\ x \vee R\ x\ t)))$ **by** *auto*
have *threeb*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists t.\ (f\ x \wedge\ f\ t)))$ **using** *threea two* **by** *blast*
have *threec*: $\forall x.\ (x \in (CC\ R) \longrightarrow f\ x)$ **using** *threeb* **by** *blast*
have *threed*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists u.\ (R\ u\ x)))$ **using** *threec c1* **by** *simp*
have *threee*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists u.\ ((u \in (CC\ R) \wedge u \neq x) \wedge (\neg\ R\ x\ u))))$
proof −
{ fix $aa :: a$
obtain $aaa :: a \Rightarrow a$ **where**
ff1: $\forall a.\ a \notin CC\ R \vee R\ (aaa\ a)\ a$
by (*metis* (*lifting*) *threed*)
{ assume $\neg\ R\ aa\ aa \wedge aa \in CC\ R$
then have $aa \neq aaa\ aa$
using *ff1* **by** (*metis* (*lifting*))
moreover
{ assume $aa \neq aaa\ aa \wedge aa \in CC\ R$
then have $(\exists a.\ R\ (aaa\ aa)\ a \vee R\ a\ (aaa\ aa)) \wedge aa \neq aaa\ aa$
using *ff1* **by** *meson*
then have $aaa\ aa \in CC\ R \wedge aa \neq aaa\ aa$
using *mem-Collect-eq* **by** *blast*
then have $R\ aa\ (aaa\ aa) \vee aa \notin CC\ R \vee (\exists a.\ a \in CC\ R \wedge aa \neq a \wedge \neg$
$R\ aa\ a)$
by (*metis* (*lifting*))
moreover
{ assume $R\ aa\ (aaa\ aa)$
then have $aa \notin CC\ R \vee (\exists a.\ a \in CC\ R \wedge aa \neq a \wedge \neg\ R\ aa\ a)$
using *ff1* **by** (*meson c2*) **}**
ultimately have $aa \notin CC\ R \vee (\exists a.\ a \in CC\ R \wedge aa \neq a \wedge \neg\ R\ aa\ a)$
by *blast* **}**

ultimately have *aa* $\notin$ *CC R* $\vee$ ($\exists$ *a*. *a* $\in$ *CC R* $\wedge$ *aa* $\neq$ *a* $\wedge$ $\neg$ *R aa a*)
by *blast* **}**
then have *aa* $\notin$ *CC R* $\vee$ ($\exists$ *a*. *a* $\in$ *CC R* $\wedge$ *aa* $\neq$ *a* $\wedge$ $\neg$ *R aa a*)
by (*meson c2*) **}**
then show *?thesis*
by (*metis* (*lifting*)) **qed**
thus *?thesis* **by** *blast*
qed

Nitpick doesnacute¿t find a model. That is however not really of importance since this is (sort of) supposed to be a reductio*)

whether the assumptions are all necessary is irrelevant here (since itacute¿s supposed to be a reductio).

Arguments for Tp (for a reductio)

Automated:

lemma *Tpauto*:
assumes *c2*: *K R*
and *NotC3*: $\neg$ ($\exists$ *y*. (*y* $\in$ (*CC R*) $\wedge$ ($\forall$ *u*. ((*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$ *y*) $\longrightarrow$ (*R y u*)))))
and *35*: $\forall$ *x*. (($\exists$ *t*. (*t R x*)) $\longrightarrow$ *f x*)
shows $\forall$ *x y*. ((*R x y*) $\longrightarrow$ (*f x* $\wedge$ *f y*)) **using** *c2 NotC3 35* **by** *meson*

With Salamuchaacute¿s steps:

slight differences between both notational variants of Salamucha; Probably typos; the more intuitive version is used

lemma *Tp*:
assumes *c2*: *K R*
and *NotC3*: $\neg$ ($\exists$ *y*. (*y* $\in$ (*CC R*) $\wedge$ ($\forall$ *u*. ((*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$ *y*) $\longrightarrow$ (*R y u*)))))
and *35*: $\forall$ *x*. (($\exists$ *t*. (*t R x*)) $\longrightarrow$ *f x*)
shows $\forall$ *x y*. ((*R x y*) $\longrightarrow$ (*f x* $\wedge$ *f y*))
proof –

have *one*: $\forall$ *x y*. ((*R x y*) $\longrightarrow$ (*x* $\in$ (*CC R*) $\wedge$ *y* $\in$ (*CC R*))) **by** *auto*
have *twoa*: ($\neg$ ($\exists$ *y*. (*y* $\in$ (*CC R*) $\wedge$ ($\forall$ *u*. ((*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$*y*) $\longrightarrow$ (*R y u*))))))
$\longrightarrow$ ($\forall$ *y*. (*y* $\in$ (*CC R*) $\longrightarrow$ ($\exists$ *u*. (*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$ *y* $\wedge$ $\neg$(*R y u*))))) **by** *presburger*
have *twob*: ($\neg$ ($\exists$ *y*. (*y* $\in$ (*CC R*) $\wedge$ ($\forall$ *u*. ((*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$ *y*) $\longrightarrow$ (*R y u*)))
)))
$\longrightarrow$ ($\forall$ *y*. (*y* $\in$ (*CC R*) $\longrightarrow$ ($\exists$ *u*. (*R u y*)))) **using** *c2* **by** *meson*
have *twoc*: ($\neg$ ($\exists$ *y*. (*y* $\in$ (*CC R*) $\wedge$ ($\forall$ *u*. ((*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$ *y*) $\longrightarrow$ (*R y u*)))
)))
$\longrightarrow$ ($\forall$ *y*. (*y* $\in$ (*CC R*) $\longrightarrow$ *f y*)) **using** *35* **by** *meson*
have *twod*: ($\neg$ ($\exists$ *y*. (*y* $\in$ (*CC R*) $\wedge$ ($\forall$ *u*. ((*u* $\in$ (*CC R*) $\wedge$ *u* $\neq$ *y*) $\longrightarrow$ (*R y u*)))
)))
$\longrightarrow$ ($\forall$ *x y*. (*R x y* $\longrightarrow$ (*f x* $\wedge$ *f y*))) **using** *twoc* **by** *blast*
thus *?thesis* **using** *NotC3* **by** *blast*

qed

Are all assumptions necessary? (Kind of an academic question, since this is supposed to be a reductio)

No!

lemma *Tpwo1*:
assumes *NotC3*: $\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
and *35*: $\forall\, x.\ ((\exists\, t.\ (t\ R\ x)) \longrightarrow f\ x)$
shows $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$ **using** *NotC3 35* **by** *meson*

A

lemma *Tpo2*:
assumes *c2*: $K\ R$
and *35*: $\forall\, x.\ ((\exists\, t.\ (t\ R\ x)) \longrightarrow f\ x)$
shows $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
nitpick[*verbose*]
oops

Nitpick doesnacute¿t find a counterexample*) (*For a counterexample consider: let R be a relation on the natural numbers (n ge¿ 0) where: x R y := x = 0 and¿ y = 1 R is transitive irreflexive and connected hence c2 holds. let f x := exists¿t. (t ¡ x) then, if t R x holds then t = 0 and x = 1 and there is a smaller number than 1, namely 0. hence 35 holds. however for x = 0 and y = 1 x R y holds but it is not true that f 0, since by definition there is no smaller natural number

lemma *Tpwo3*:
assumes *c2*: $K\ R$
and *NotC3*: $\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
shows $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
nitpick[*verbose*]
oops

Nitpick doesnacute¿t find a counterexample*) (*For a (trivial) counterexample consider: let R be the less-than relation on the natural numbers. It is obviously an Ordering Relation. There is also no smalles element. therefore c2 and NotC3 hold. let f x := False then the conclusion is wrong for all x y.

Again for the following proof we have to declare the type of the /time/ elements explicitly; we will just use type a here

For the following lemma sledgehammer proof reconstruction fails, but the results strongly suggest that the set of assumptions are inconsistent. This is however not a problem since the intention of this lemma is to show that Tp should be rejected

N.B. Step seven has a typo in the second notational variant!

N.B. Salamucha mentions that for some definitions of identity (e.g. a Leibnizian) the x noteq¿ y can be omitted in none. He argues that this is however not very helpful and leads to more problems than the apparent simplification solves. I tend to agree.

lemma *Unwantedconsequences*:
assumes *31*: $\forall x.\ (f\ x \longrightarrow C\ x)$
and *32*: $\forall x.\ ((C\ x \wedge f\ x) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$
and *33*: $\forall x\ (t_2{::}a).\ (C\ x \longrightarrow ((t_2\ F\ x) \longrightarrow (H\ t_2)))$
and *34*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow (t_1 = t_2))$
and *c2*: $K\ R$
and *Tp*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
shows $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y) \wedge (t_1\ F\ x) \wedge (t_2\ F\ y)) \longrightarrow t_1 = t_2)$
proof −
  have *one*: $\forall x\ y.\ ((R\ x\ y \vee R\ y\ x) \longrightarrow (f\ x \wedge f\ y))$ **using** *Tp* **by** *auto*
  have *twoa*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists z.\ (R\ z\ x \vee R\ x\ z)))$ **by** *auto*
  have *twob*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists z.\ (f\ z \wedge f\ x)))$ **using** *twoa one* **by** *meson*
  have *twoc*: $\forall x.\ (x \in (CC\ R) \longrightarrow f\ x)$ **using** *twob* **by** *simp*
  have *three*: $\forall x.\ (x \in (CC\ R) \longrightarrow C\ x)$ **using** *twoc 31* **by** *blast*
  have *four*: $\forall x.\ (x \in (CC\ R) \longrightarrow (C\ x \wedge f\ x))$ **using** *three twoc* **by** *simp*
  have *five*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$ **using** *four 32* **by** *blast*
  have *six*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\forall (t_2{::}a).\ ((t_2\ F\ x) \longrightarrow (H\ t_2))))$ **using** *three 33* **by** *blast*

  have *seven*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ ((t_1\ F\ x) \wedge (H\ t_1))))$ **using** *five six* **by** *blast*
  have *eight*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y \vee R\ y\ x) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow t_1 = t_2)$ **using** *34* **by** *blast*
  have *nine*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y) \wedge (t_1\ F\ x) \wedge (t_2\ F\ y)) \longrightarrow t_1 = t_2)$ **using** *eight c2* **by** *meson*
  thus *?thesis* **by** *blast*
qed

7 The Consequens of Thesis T

Ex Motu implies Monotheism

lemma *monotheismauto*:
assumes *god*: $(\exists v.\ (\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
and *c2*: $K\ R$
and *c3*: $(\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
shows $((\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))) \wedge (\neg\ (f\ w) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq w) \longrightarrow (R\ w\ u))))$
$\longrightarrow v = w$ **using** *c2 c3 god* **by** (*metis* (*full-types*, *lifting*) *mem-Collect-eq*)

the step /vwin/ is not part of Salamuchaacute¿s outline, but needed for Isabelleacute¿s provers*)

lemma *monotheism*:

```
assumes god: (∃ v. (¬ (f v) ∧ (∀ u. (u ∈ (CC R) ∧ u ≠ v) ⟶ (R v u))))
and c2: K R
and c3: (∃ y. (y ∈ (CC R) ∧ (∀ u. ((u ∈ (CC R) ∧ u ≠ y) ⟶ (R y u)))))
shows ((¬ (f v) ∧ (∀ u. (u ∈ (CC R) ∧ u ≠ v) ⟶ (R v u))) ∧ (¬ (f w) ∧ (∀ u.
(u ∈ (CC R) ∧ u ≠ w) ⟶ (R w u))))
⟶ v = w
proof −
  {assume asm1: (¬ (f v) ∧ (∀ u. (u ∈ (CC R) ∧ u ≠ v) ⟶ (R v u)))
   and asm2: (¬ (f w) ∧ (∀ u. (u ∈ (CC R) ∧ u ≠ w) ⟶ (R w u)))
    {assume poly: v ≠ w
     from asm1 have v1: ∀ x. ((x ∈ (CC R) ∧ (x ≠ v)) ⟶ (R v x)) by auto
     from asm2 have w1: ∀ x. ((x ∈ (CC R) ∧ (x ≠ w)) ⟶ (R w x)) by auto

     have vwin: v ∈ (CC R) ∧ w ∈ (CC R)
       proof −
       from c3 obtain y where obty: y ∈ (CC R) by auto
       {assume y ≠ v
         hence v ∈ (CC R) using v1 obty by auto}
       moreover
       {assume y = v
         hence v ∈ (CC R) using obty by simp}
       ultimately have v ∈ (CC R) by fastforce
       thus ?thesis using w1 obty by blast qed
     hence (R v w) ∨ (R w v) using c2 poly by blast
    moreover
    {assume R v w
     hence ¬ (R w v) using c2 by blast
     hence False using w1 vwin poly by auto}
    moreover
    {assume R w v
     hence ¬ (R v w) using c2 by blast
     hence False using v1 vwin poly by auto}
    ultimately have False by blast}
  hence v = w by blast}
thus ?thesis by fast
qed
```

8 The entire proof(s) (as specified on p.131ff)

Salamucha offers several different ways to combine sets of assumptions to get the conclusion. Only those that have been formalized in the paper (and are not just natural language assumptions) are proven here. Even those two possible combinations however rely on an additional assumption "A" [that Salamucha claims follows from two other assumptions that are only stated in natural language].

In the (apparently somewhat sloppy) translation A is stated as: "An infinite and ordered set of moving bodies and bodies that move is not in motion for the limited period of time [sic].”

The best fit for this seems to be the formula ”A” below. makes the argument valid, uses the same concepts and fits neatly in the dialectic the previous reductio arguments provide.

lemma *AC*:
assumes *one*: $\forall x. (f\ x \longrightarrow (\exists t. (R\ t\ x)))$
and *two*: $\forall x\ y\ z. (((R\ x\ y) \wedge (R\ y\ z)) \longrightarrow (R\ x\ z))$
and *three*: $\forall x\ y. ((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y)) \longrightarrow ((R\ x\ y) \vee (R\ y\ x)))$

and *11*: $\forall x. ((f\ x) \longrightarrow (\exists a\ b. (a\ M\ x \wedge b\ M\ x)))$
and *12*: $\forall x. ((\exists a\ b. (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg f\ a) \wedge (f\ b)) \vee ((\neg f\ a) \longrightarrow (\neg f\ b))))) \longrightarrow (\neg R\ x\ x))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$

and *31*: $\forall x. (f\ x \longrightarrow C\ x)$
and *32*: $\forall x. ((C\ x \wedge f\ x) \longrightarrow (\exists (t_1::a). (t_1\ F\ x)))$
and *33*: $\forall x\ (t_2::a). (C\ x \longrightarrow ((t_2\ F\ x) \longrightarrow (H\ t_2)))$
and *34*: $\forall x\ y\ (t_1::a)\ (t_2::a). (((R\ x\ y) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow (t_1 = t_2))$
and *35*: $\forall x. ((\exists t. (t\ R\ x)) \longrightarrow f\ x)$

and *A*: $\neg (\forall x. (x \in \{y. (y \in (CC\ R) \wedge (C\ y))\} \longrightarrow (\exists t_1::a. ((t_1\ F\ x) \wedge (H\ t_1)))))$

shows $\exists v. (\neg (f\ v) \wedge (\forall u. (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u)))$
proof –

from *11 12 14* **have** *irreflexive R* **using** *irreflexivityR* **by** *blast*
hence *c2*: *K R* **using** *one two three* **by** *blast*
have *T1*: $\forall x. ((f\ x) \longrightarrow (\exists t. ((R\ t\ x) \wedge t \neq x)))$ **using** *11 12 14 T1auto one* **by** *blast*
hence *c1*: $\forall x. (f\ x \longrightarrow (\exists t. (R\ t\ x)))$ **by** *blast*
{assume *Tp*: $\forall x\ y. ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
have *seven*: $\forall x. (x \in (CC\ R) \longrightarrow (\exists (t_1::a). ((t_1\ F\ x) \wedge (H\ t_1))))$ **using** *Tp 31 32 33* **by** *blast*
hence *False* **using** *A* **by** *blast***}**
hence *NOTTp*: $\neg (\forall x\ y. ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y)))$ **by** *blast*
{assume *NOTC3*: $\neg ((\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
have *False* **using** *Tpauto 35 NOTTp c2 NOTC3* **by** *blast***}**
hence *c3*: $((\exists y. (y \in (CC\ R) \wedge (\forall u. ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$ **by** *blast*
show *?thesis* **using** *c1 c2 c3 Tauto* **by** *blast*
qed

lemma *BC*:

assumes *one*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$
and *two*: $\forall x\ y\ z.\ (((R\ x\ y) \wedge\ (R\ y\ z)) \longrightarrow (R\ x\ z))$
and *three*: $\forall x\ y.\ ((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y)) \longrightarrow ((R\ x\ y) \vee (R\ y\ x)))$

and *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
and *22*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
and *23*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
and *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$

and *31*: $\forall x.\ (f\ x \longrightarrow C\ x)$
and *32*: $\forall x.\ ((C\ x \wedge f\ x) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$
and *33*: $\forall x\ (t_2{::}a).\ (C\ x \longrightarrow ((t_2\ F\ x) \longrightarrow (H\ t_2)))$
and *34*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow (t_1 = t_2))$
and *35*: $\forall x.\ ((\exists t.\ (t\ R\ x)) \longrightarrow f\ x)$

and *A*: $\neg\ (\forall x.\ (x \in \{y.\ (y \in (CC\ R) \wedge (C\ y))\} \longrightarrow (\exists t_1{::}a.\ ((t_1\ F\ x) \wedge (H\ t_1)))))$
shows $\exists v.\ (\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u)))$

proof –
from *21 22 23 14* **have** *irreflexive R* **using** *IrreflexivityRv2* **by** *blast*
hence *c2*: *K R* **using** *one two three* **by** *blast*
have *T1*: $\forall x.\ ((f\ x) \longrightarrow (\exists t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *21 22 23 one thirdproof* **by** *blast*
hence *c1*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$ **by** *blast*
{assume *Tp*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
have *seven*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ ((t_1\ F\ x) \wedge (H\ t_1))))$ **using** *Tp 31 32 33* **by** *blast*
hence *False* **using** *A* **by** *blast***}**
hence *NOTTp*: $\neg\ (\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y)))$ **by** *blast*
{assume *NOTC3*: $\neg\ ((\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
have *False* **using** *Tpauto 35 NOTTp c2 NOTC3* **by** *blast***}**
hence *c3*: $((\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$ **by** *blast*
show *?thesis* **using** *c1 c2 c3 Tauto* **by** *blast*
qed

Nitpick times out while trying to find a model for both proofs. Sledgehammer and remote provers canacute¿t prove false, but consistency is still an open question.

end