Gödel's Ontological Proof of God's Existence (Draft)

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Introduction

ToDo: Do also Scott's and Gdel's proofs.

2 Natural Deduction

A *derivation* is a directed acyclich graph whose nodes are formulas and whose edges correspond to applications of the inference rules shown in Figures 1 and 3. Parts of a derivation may be surrounded by boxes. A *proof* is a derivation that additionally satisfies the following conditions:

- **eigen-variable conditions:** if ρ is a $\forall_I$ inference eliminating a variable α, then any occurrence of α in the proof should be an ancestor of the occurrence of α eliminated by ρ; if ρ is a $\exists_E$ inference introducing a variable β, then any occurrence of β in the proof should be a descendant of the occurrence of β introduced by ρ.
- **boxed assumption condition:** any assumption should be discharged within the box where it is made.
- **unboxed root condition:** the proof's root should not be inside any box.

Double lines are used to abbreviate tedious propositional reasoning steps in the derivations. Dashed lines are used to refer to a proof shown elsewhere. Dotted lines are used to indicate folding and unfolding of definitions.

Figure 1: The intuitionistic natural deduction calculus **ND**

$$\dfrac{B}{A \to B} \to_I \qquad \dfrac{\begin{array}{c}\overline{A}^{\;n}\\ \vdots \\ B\end{array}}{A \to B} \to_I^n \qquad \dfrac{A \quad A \to B}{B} \to_E$$

$$\dfrac{A \quad B}{A \wedge B} \wedge_I \qquad \dfrac{A \wedge B}{A} \wedge_{E_1} \qquad \dfrac{A \wedge B}{B} \wedge_{E_2}$$

$$\dfrac{A \vee B \quad \begin{array}{c}\overline{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}\\ \vdots \\ C\end{array}}{C} \vee_E \qquad \dfrac{A}{A \vee B} \vee_{I_1} \qquad \dfrac{B}{A \vee B} \vee_{I_2}$$

$$\dfrac{A[\alpha]}{\forall x.A[x]} \forall_I \qquad \dfrac{\forall x.A[x]}{A[t]} \forall_E \qquad \dfrac{A[t]}{\exists x.A[x]} \exists_I \qquad \dfrac{\exists x.A[x]}{A[\beta]} \exists_E$$

ToDo: add intuitionistic rule for negation/bottom

Figure 2: Classical Rules/Axioms

ToDo: add classical rules/axioms that we need

Figure 3: Rules for Modal Operators

$$\dfrac{\boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A} \Box_I \qquad \dfrac{\Box A}{\boxed{\begin{array}{c}A\\ \vdots\end{array}}} \Box_E$$

3 Possibly, God Exists

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall \varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall\varphi.\forall\psi.[(P(\varphi) \land \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

Theorem 1 *Positive properties are possibly exemplified:*

$$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$$

Proof

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom 2}}{\forall\varphi.\forall\psi.[(P(\varphi) \land \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}}{\forall\psi.[(P(\rho) \land \Box\forall x.[\rho(x) \to \psi(x)]) \to P(\psi)]}\,\forall_E}{(P(\rho) \land \Box\forall x.[\rho(x) \to \neg\rho(x)]) \to P(\neg\rho)}\,\forall_E}{(P(\rho) \land \Box\forall x.[\neg\rho(x)]) \to P(\neg\rho)} \qquad \dfrac{\dfrac{\text{Axiom 1}}{\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}}{P(\neg\rho) \leftrightarrow \neg P(\rho)}\,\forall_E}{(P(\rho) \land \Box\forall x.[\neg\rho(x)]) \to \neg P(\rho)}}{P(\rho) \to \Diamond\exists x.\rho(x)}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}\,\forall_I$$

Definition 1 *A* God-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \to \varphi(x)]$$

Axiom 3 *The property of being God-like is positive:*

$$P(G)$$

Corollary 1 *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{P(G) \to \Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\to_E$$

4 Being God is an essence of any God

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi) \to \Box\, P(\varphi)]$$

Definition 2 *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi\ ess\ x \leftrightarrow \varphi(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

Theorem 2 *Being God-like is an essence of any God-like being:*

$$\forall y.[G(y) \to G \ ess \ y]$$

Proof Let the following derivation with the open assumption $G(x)$ be $\Pi_1[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\neg P(\psi)^1$}
\AxiomC{Axiom 1}
\dashedLine
\UnaryInfC{$\forall\varphi.(\neg P(\varphi) \to P(\neg\varphi))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$\neg P(\psi) \to P(\neg\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$P(\neg\psi)$}
\AxiomC{$G(x)$}
\RightLabel{D1}
\dottedLine
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \varphi(x))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\neg\psi) \to \neg\psi(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\neg\psi(x)$}
\AxiomC{$\psi(x)^2$}
\RightLabel{$\to_E$}
\BinaryInfC{$\bot$}
\RightLabel{RAA^1}
\UnaryInfC{$P(\psi)$}
\RightLabel{$\to_I^2$}
\UnaryInfC{$\psi(x) \to P(\psi)$}
\end{prooftree}
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_2[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\psi(x)^3$}
\AxiomC{$\Pi_1[G(x)]$}
\dashedLine
\UnaryInfC{$\psi(x) \to P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$P(\psi)$}
\AxiomC{Axiom 4}
\dashedLine
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \Box P(\varphi))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\psi) \to \Box P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box P(\psi)$}
\RightLabel{$\to_I^3$}
\UnaryInfC{$\psi(x) \to \Box P(\psi)$}
\end{prooftree}
$$

Let the following derivation without open assumptions be Π_3:

$$
\begin{prooftree}
\AxiomC{$P(\psi)^4$}
\AxiomC{$G(x)^5$}
\RightLabel{D1}
\dottedLine
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \varphi(x))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\psi) \to \psi(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\psi(x)$}
\RightLabel{$\to_I^5$}
\UnaryInfC{$G(x) \to \psi(x)$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^4$}
\UnaryInfC{$P(\psi) \to \forall x.(G(x) \to \psi(x))$}
\end{prooftree}
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_4[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\psi(x)^6$}
\AxiomC{Π_2}
\dashedLine
\UnaryInfC{$\psi(x) \to \Box P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box P(\psi)$}
\AxiomC{$\Box P(\psi)^7$}
\RightLabel{$\Box_E$}
\UnaryInfC{$P(\psi)$}
\AxiomC{Π_3}
\dashedLine
\UnaryInfC{$P(\psi) \to \forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_E$}
\BinaryInfC{$\forall x.(G(x) \to \psi(x))$}
\RightLabel{Necessitation}
\UnaryInfC{$\Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^7$}
\UnaryInfC{$\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^6$}
\UnaryInfC{$\psi(x) \to \Box\forall x.(G(x) \to \psi(x))$}
\end{prooftree}
$$

The use of the necessitation rule above is correct, because the only open assumption $\Box P(\psi)$ is boxed. In the derivation of Theorem 2 below, the assumption $G(x)$ in the subderivation $\Pi_4[G(x)^8]$ is discharged by the rule labeled 8.

$$\dfrac{\dfrac{\dfrac{G(x)^8 \qquad \dfrac{\dfrac{\Pi_4[G(x)^8]}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}}{\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\,\forall_I}{G(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\,\wedge_I \;\; \vdots\,\text{D2}}{\dfrac{G\ ess\ x}{G(x) \to G\ ess\ x}\,\to_I^8}}{\forall y.[G(y) \to G\ ess\ y]}\,\forall_I$$

5 If God's existence is possible, it is necessary

Definition 3 Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1 *If there is a God, then necessarily there exists a God:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{}{\exists z.G(z)}\,1}{G(g)}$$

$$\dfrac{\dfrac{G(g) \qquad \dfrac{\dfrac{\text{Theorem 2}}{\forall y.[G(y) \to G\ ess\ y]}}{G(g) \to G\ ess\ g}}{G\ ess\ g} \qquad \dfrac{\dfrac{\dfrac{\text{Axiom 5}}{P(E)} \qquad \dfrac{\dfrac{G(g)}{\forall\varphi.[P(\varphi) \to \varphi(g)]}}{P(E) \to E(g)}}{E(g)}}{\dfrac{\forall\varphi.[\varphi\ ess\ g \to \Box\exists x.\varphi(x)]}{G\ ess\ g \to \Box\exists x.G(x)}}}{\dfrac{\Box\exists x.G(x)}{\exists z.G(z) \to \Box\exists x.G(x)}\,1}$$

6 Necessarily, God exists

ToDo: this is proven in a way that is slightly different from Gödel's 1970.

Theorem 3 *Necessarily, God exists:*

$$\Box\exists x.G(x)$$

Formal proof: Let the following derivation be Π:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\exists x.G(x)}\,1}{\Box\Box\exists x.G(x)}\,\Box_I}{\Box\exists x.G(x)\to\Box\Box\exists x.G(x)}\,\to_I^1}{\neg\Box\Box\exists x.G(x)\to\neg\Box\exists x.G(x)}\,\text{propositional logic}}{\Box(\neg\Box\Box\exists x.G(x)\to\neg\Box\exists x.G(x))}\,\Box_I \qquad \text{Axiom K}}{\Box\neg\Box\Box\exists x.G(x)\to\Box\neg\Box\exists x.G(x)}}{\Box\Diamond\neg\Box\exists x.G(x)\to\Box\neg\Box\exists x.G(x)}\,\Diamond$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\Diamond\neg\Box\exists x.G(x)}\,1 \qquad \dfrac{\Pi}{\Box\Diamond\neg\Box\exists x.G(x)\to\Box\neg\Box\exists x.G(x)}}{\dfrac{\Box\neg\Box\exists x.G(x)}{\neg\Box\exists x.G(x)}\,\Box_E\text{, axiom T}} \qquad \dfrac{}{\neg\Box\exists x.G(x)\to\neg\exists x.G(x)}\,\text{Lemma 1}}{\dfrac{\neg\exists x.G(x)}{\Box\neg\exists x.G(x)}\,\Box_I}\,\to_E}{\neg\Diamond\exists x.G(x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\Diamond\exists x.G(x)} \qquad \dfrac{\text{Corollary 1}}{\Diamond\exists x.G(x)}}{\bot}\,\to_E}{\neg\Box\Diamond\neg\Box\exists x.G(x)}\,\to_I^1 \qquad \dfrac{\dfrac{\text{Axiom B}}{\neg\Box\exists x.G(x)\to\Box\Diamond\neg\Box\exists x.G(x)}}{\neg\Box\Diamond\neg\Box\exists x.G(x)\to\Box\exists x.G(x)}}{\Box\exists x.G(x)}\,\to_E$$

Better proof, that directly proves $\exists x.G(x)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Box\neg\Box\exists x.G(x)}\,1}{\neg\Box\exists x.G(x)}\,\Box_E\text{, axiom T} \qquad \dfrac{\text{Lemma 1}}{\neg\Box\exists x.G(x)\to\neg\exists x.G(x)}}{\dfrac{\neg\exists x.G(x)}{\Box\neg\exists x.G(x)}\,\Box_I}\,\to_E}{\neg\Diamond\exists x.G(x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\Diamond\exists x.G(x)} \qquad \dfrac{\text{Corollary 1}}{\Diamond\exists x.G(x)}}{\bot}\,\to_E}{\neg\Box\neg\Box\exists x.G(x)}\,\to_I^1}{\neg\Box\Diamond\neg\exists x.G(x)} \qquad \dfrac{\dfrac{\text{Axiom B}}{\neg\exists x.G(x)\to\Box\Diamond\neg\exists x.G(x)}}{\neg\Box\Diamond\neg\exists x.G(x)\to\neg\neg\exists x.G(x)}}{\dfrac{\neg\neg\exists x.G(x)}{\exists x.G(x)}\,\neg\neg\text{ E}}\,\to_E$$

Note that the last step is classical and we do not prove the existential statement by providing an object for which the statement holds. This proof makes section 7 superflous and the use of axiom "M" unnecessary.

The system used contains the $\Box_E$-rule with the restriction that we have a $\Box_I$ below it. This is equivalent to modal sytem K that contains axiom K and necessitation rule N. We aslo use axiom B ($A \to \Box\Diamond A$). No other modal axioms are needed.

7 God exists

Axiom 6 (M) *What is necessary is the case:*

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Corollary 2 *There exists a God:*

$$\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\text{Theorem 3}}{\Box\exists x.G(x)} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$

8 Proof system equivalent to system K

We show that the proof system with boxed parts of derivations is equivalent to the system K. The modal system K consists of the axiom K and the necessitation rule N.

Axiom 7 (The transitivity axiom K)

$$\Box(A \to B) \to (\Box A \to \Box B)$$

Axiom 8 (The necessitation rule) *If A is a theorem, then $\Box A$ is a theorem.*

Lemma 2 *The axiom K is derivable in the system.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Box(A \to B)^2}{A \to B}\,\Box_E \qquad \dfrac{\Box A^1}{A}\,\Box_E}{B}\,\to_E}{\Box B}\,\Box_I}{\Box A \to \Box B}\,\to_I^1}{\Box(A \to B) \to (\Box A \to \Box B)}\,\to_I^2$$

Lemma 3 *Assuming the axiom K and the necessitation rule N, the open formula $\Box A$ and the existence of a derivation of B from the open assumption A, then we can derive $\Box B$ without the rules for boxed parts of derivations.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}A^1\\ \vdots\\ B\end{matrix}}{A \to B}\,{\to_I^1}}{\Box(A \to B)}\,\text{Necessitation} \qquad \dfrac{\text{Axiom K}}{\Box(A \to B) \to (\Box A \to \Box B)}}{\Box A \to \Box B}\,{\to_E} \qquad \Box A^1}{\Box B}\,{\to_E}$$