# bin

By knork

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## Contents

**theory** *SalamuchaPDF*

**imports** *Main*

**begin**
**declare** [[*smt-timeout* = *300*]]

High timeout for smt, so that there is a high probability that smt terminates. Not using this setting makes pdf creation really annoying.

## 1 Types and Definitions

**typedecl** *a*

things and stuff in the world

**consts** $R$:: $a \Rightarrow a \Rightarrow bool$

$_{m}oves$

**consts** $f$ :: $a \Rightarrow bool$

is in motion

**consts** *partof* :: $a \Rightarrow a \Rightarrow bool$ (**infixr** *M52*)

$_{i}saproperpartof$

**consts** *aspactu* :: $a \Rightarrow {}'a \Rightarrow a \Rightarrow bool$ (-$A$- -)
**consts** *asppot* :: $a \Rightarrow {}'a \Rightarrow a \Rightarrow bool$ (- $P$- -)
**consts** *body*:: $a \Rightarrow bool$ ($C$)
**consts** *duration* :: $'a \Rightarrow a \Rightarrow bool$ (- $F$ -)

not entirely sure if we need a /time type/ here; I donacute¿t think it will change anything

**consts** *finitetime* :: $'a \Rightarrow bool$ ($H$)

**abbreviation** $CC$:: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow 'a\ set$
**where** $CC\ r \equiv \{a.\ \exists\, t.\ ((r\ a\ t) \vee (r\ t\ a))\}$

Salamucha has Cacute¿R

**abbreviation** *irreflexive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
**where** *irreflexive* $r \equiv (\forall\, x.\ \neg\ (r\ x\ x))$

**abbreviation** *transitive* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
**where** *transitive* $r \equiv (\forall\, x\ y\ z.\ ((r\ x\ y) \wedge (r\ y\ z) \longrightarrow (\ r\ x\ z)))$

**abbreviation** *connected* :: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$
**where** *connected* $r \equiv \forall\, x\ y.\ ((x \in (CC\ r) \wedge y \in (CC\ r) \wedge (x \neq y)) \longrightarrow (r\ x\ y \vee r\ y\ x))$

**abbreviation** $K$:: $('a \Rightarrow 'a \Rightarrow bool) \Rightarrow bool$ $(K\text{-})$
**where** $K\ r \equiv ((connected\ r) \wedge (transitive\ r) \wedge (irreflexive\ r))$

## 2 Proof of Lemma T

Sledgehammer can prove the Lemma directly

**lemma** *Tauto*: $((\forall\, x.\ (f\ x \longrightarrow (\exists\, t.\ (R\ t\ x)))) \wedge (K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$ **by** (*metis* (*no-types*, *lifting*) *CollectI*)

Using the steps from Salamucha is actually worse performancewise (and needs smt)

Note that in the second version of Salamuchas notation (the boxed one) there is a consistent typo. The y in the Consequens of almost all formulas should be a v

Note that in step threeb neither threea nor twob is used

(*Warning: Stepseven is an smt proof. Other proof methods fail here [but if need be the proof can be made explicit*)

**lemma** $T$: $((\forall\, x.\ (f\ x \longrightarrow (\exists\, t.\ (R\ t\ x)))) \wedge (K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
**proof** –
**have** *one*: $(\forall\, x.\ ((f\ x\ ) \longrightarrow (\exists\, t.\ (R\ t\ x)))) \longrightarrow (\forall\, x.\ ((\forall\, t.\ (\neg\ R\ t\ x)) \longrightarrow (\neg\ f\ x)))$ **by** *blast*
**have** *twoa*: $(K\ R) \longrightarrow (\forall\, y\ u.\ (R\ y\ u \longrightarrow \neg\ R\ u\ y))$ **by** *blast*
**have** *twob*: $(K\ R) \longrightarrow (\forall\, y\ u.\ ((u \in (CC\ R) \wedge u \neq y \wedge R\ y\ u) \longrightarrow (\neg\ R\ u\ y)\ ))$ **by** *meson*
**have** *threea*: $((K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))) \longrightarrow$

$(\exists v.\ (\forall u.\ ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$ **by** *meson*
**have** *threeb*: $((K\ R) \wedge (\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq y\ ) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v.\ (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (\neg\ R\ u\ v)\ )))$ **by** (*metis* (*mono-tags*, *lifting*))
**have** *threec*: $((K\ R) \wedge (\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq y\ ) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v.\ (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (\neg\ R\ u\ v \wedge R\ v\ u))))$ **by** *meson*
**have** *four*: $((K\ R) \wedge (\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq y\ ) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v.\ ((\forall u.\ ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (\neg\ R\ u\ v))) \wedge (\forall u.\ (\ (u \in (CC\ R) \wedge (u \neq v)) \longrightarrow (R\ v\ u)))\ ))$ **by** *meson*
**have** *five*: $\forall u\ v.\ ((\neg\ (u \in (CC\ R))) \longrightarrow (\neg\ R\ u\ v))$ **by** *simp*
**have** *six*: $(K\ R) \longrightarrow (\forall u\ v.\ (u = v \longrightarrow (\neg\ R\ u\ v)))$ **by** *simp*
**have** *seven*: $((K\ R) \wedge (\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq y\ ) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v.\ ((\forall u.\ ((\neg\ R\ u\ v)\ )) \wedge (\forall u.((u \in (CC\ R) \wedge (u \neq v)) \longrightarrow (R\ v\ u)))))$ **using** *five six four*
**by** (*smt ext*)
**have** *eigth*: $((\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))) \wedge (K\ R) \wedge (\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ (\ (u \in (CC\ R) \wedge u \neq y\ ) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\exists v.\ (((\neg\ f\ v) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))))$ **using** *seven one* **by** *meson*
**then show** *?thesis* **by** *blast*
**qed**

The first two conjuncts of the antecedents of T imply the stronger Thesis T1

**lemma** *TtoT1*:
**assumes** *firsttwoT*: $(K\ R) \wedge (\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x))))$
**shows** $\forall x.\ ((f\ x) \longrightarrow (\exists t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *firsttwoT* **by** *blast*

Are the conjuncts of the antecedent of Thesis T all necessary?

**lemma** *T12*: $((\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))) \ \wedge\ (K\ R)\ )$
$\longrightarrow\ (\exists v.\ (\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
**nitpick**[*verbose*]
**oops**

Nitpick does NOT find a counterexample; perhaps someone with more computing power could run this again*) (*Salamucha gives the following counterexample (p.115f): Let R be the greater-than relation on the positive natural numbers. Let f x mean that x is a positive number (hold trivially). Since ¿ is an ordering relation and there always is a bigger number the antecedents hold. There is however no positive number that is not positive therefore the conditional is false.

**lemma** *T13*: $((\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))) \ \wedge (\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow\ (\exists v.\ (\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$

**nitpick**[*verbose*]
**oops**

**lemma** *T23*: ( $(K\ R) \wedge (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow\ (\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
**nitpick**[*verbose*]
**oops**

nitpick finds a counterexample

# 3 Proof of Thesis T1

fast can prove T1 in 1s

**lemma** *T1auto*:
**assumes** *onea*: $\forall\, x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *oneb*: $\forall\, x.\ ((\exists\ a\ b.\ ((a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b)))\ )) \longrightarrow (\neg\ R\ x\ x))$
**and** *onec*: $\forall\, x.\ ((f\ x) \longrightarrow (\exists\, t.\ (R\ t\ x)))$
**shows** $\forall\, x.\ ((f\ x) \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *onea oneb onec* **by** *fast*

N.B.: Salamucha implies that this proof hold for other definitions of identities as well

Now with Salamuchas more expicit proof

Nitpick confirms consistency

Contrary to what Salamucha thinks, for step two both 11 and 12 are needed, not just 12; see below.

**lemma** *T1*:
**assumes** *11*: $\forall\, x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *12*: $\forall\, x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**and** *13*: $\forall\, x.\ ((f\ x) \longrightarrow (\exists\, t.\ (R\ t\ x)))$
**shows** $\forall\, x.\ ((f\ x) \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
**proof** –

**have** *onea*: $\forall\, x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$ **using** *11* **by** *blast*
**have** *oneb*: $\forall\, x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee (f\ a \vee (\neg\ f\ b))))$ **using** *onea* **by** *auto*
**have** *onec*: $\forall\, x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))$ **using** *oneb* **by** *blast*

**have** *two*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\neg\ (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))\quad)$ **using** *12 onea* **by** *blast*
**have** *threea*: $\forall x.\ ((\neg\ f\ x) \vee (\neg\ R\ x\ x))$ **using** *two onec* **by** *blast*
**have** *threeb*: $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ (\ (R\ t\ x) \wedge (\neg\ R\ x\ x))))$ **using** *13 threea* **by** *auto*
**have** *threec*: $\forall x.\ (f\ x \longrightarrow (\exists\ t.\ (R\ t\ x \wedge t \neq x)))$ **using** *threeb* **by** *fast*
**thus** *?thesis* **by** *simp*
**qed**

12 does not imply two

**lemma** $(\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))) \longrightarrow (\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\neg\ (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))\quad))$
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

Are all assumptions necessary?

**lemma** *T1wo1*:
**assumes** *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**and** *13*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ (R\ t\ x)))$
**shows** $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick doesnacute¿t find a counterexample For a counterexample consider: x R y := x = y f x := True x M y := False 12 then simlifies to: "forall¿x. (False longrightarrow¿ (not¿ R x x))" which holds 13 is trivally true (for t = x) The thesis doesnacute¿t hold.

**lemma** *T1wo2*:
**assumes** *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *13*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ (R\ t\ x)))$
**shows** $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick doesnacute¿t find a counterexample*) (*For an easy counterexample consider x R y := x = y x M y := True f x := exists¿t. (R t x) then both 11 and 13 hold but clearly the thesis is wrong

**lemma** *T1wo3*:
**assumes** *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\ a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *12*: $\forall x.\ ((\exists\ a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**shows** $\forall x.\ ((f\ x) \longrightarrow (\exists\ t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick doesnacute¿t find a counterexample*) (* For a counterexample consider: x R y := False x M y := True f x : = True 11 holds trivially and the thesis is false for 12 we have: "forall¿x. (exists¿a b. ((True and¿ (False or¿ True)) longrightarrow¿ True)" ergo: "forall¿x. exists¿a b. (True)" which is a theorem.

## 4 Irreflexivity of R

first automated

**lemma** *irreflexivityRauto*:
**assumes** *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *12*: $\forall x.\ ((\exists\, a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**and** *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R* **using** *11 12 14* **by** *presburger*

then using the steps in Salamuchas book

Nitpick runs out of time trying to find a model

N.B.: steps until threea are the same as in the proof of T1

**lemma** *irreflexivityR*:
**assumes** *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *12*: $\forall x.\ ((\exists\, a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**and** *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**proof** –

**have** *onea*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x)))$ **using** *11* **by** *blast*
**have** *oneb*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee (f\ a \vee (\neg\ f\ b))))$ **using** *onea* **by** *auto*
**have** *onec*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))$ **using** *oneb* **by** *blast*
**have** *two*: $\forall x.\ ((f\ x \wedge (R\ x\ x)) \longrightarrow (\neg\ (\exists\, a\ b.\ (a\ M\ x \wedge b\ M\ x) \wedge ((\neg\ f\ a) \wedge f\ b) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b)))))$ **using** *12 onea* **by** *blast*
**have** *threea*: $\forall x.\ ((\neg\ f\ x) \vee (\neg\ R\ x\ x))$ **using** *two onec* **by** *blast*
**have** *foura*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (\neg\ R\ y\ y))$ **using** *14 threea* **by** *fastforce*
**have** *fourb*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow ((R\ x\ y) \wedge (\neg\ R\ y\ y)))$ **using** *foura* **by** *simp*
**have** *fourc*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (x \neq y))$ **using** *fourb* **by** *fast*
**thus** *?thesis* **by** *auto*
**qed**

Are the assumption all necessary?

**lemma** *irreflexivityRwo1*:

**assumes** $12$: $\forall x.\ ((\exists a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**and** $14$: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick runs out of time*) (*For a counterexample consider: x R y := x = y; x M y := False; f x := False

**lemma** *irreflexivityRwo2*:
**assumes** $11$: $\forall x.\ ((f\ x) \longrightarrow (\exists a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** $14$: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick runs out of time*) (*For a counterexample consider: x R y := x = y; f x := False; x M y := False

**lemma** *irreflexivityRwo4*:
**assumes** $11$: $\forall x.\ ((f\ x) \longrightarrow (\exists a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** $12$: $\forall x.\ ((\exists a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

Show that the weaker assumption doesnacute¿t work to prove irreflexivity

**lemma** *weaker12*:
**assumes** $11$: $\forall x.\ ((f\ x) \longrightarrow (\exists a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** $w12$: $\forall x.((\exists a\ b.\ (\ (a\ M\ x \wedge b\ M\ x) \wedge \neg\ (f\ a \longleftrightarrow f\ b)\ \ )) \longrightarrow (\neg\ R\ x\ x))$
**and** $14$: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

## 5 The third proof

first automated

**lemma** *thirdproofauto*:
**assumes** $21$: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
**and** $22$: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
**and** $23$: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
**and** $24$: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$

**shows** $\forall x.\ (f\,x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *21 22 23 24* **by** *blast*

Nitpick confirms consistency

**lemma** *thirdproof*:
**assumes** *21*: $\forall\, x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
**and** *22*: $\forall\, x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
**and** *23*: $\forall\, x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
**and** *24*: $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ (R\ t\ x)))$
**shows** $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
**proof** –

**have** *one*: $\forall\, x\ y.\ ((x\ A\ R\ y) \longrightarrow \neg(x\ P\ R\ y))$ **using** *21* **by** *simp*
**have** *two*: $\forall\, x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow ((x\ P\ R\ y) \wedge (y\ A\ R\ x)))$ **using** *22 23* **by** *simp*
**have** *threea*: $\forall\, x.((f\,x \wedge (R\ x\ x)) \longrightarrow ((x\ P\ R\ x) \wedge (x\ A\ R\ x)))$ **using** *two* **by** *simp*
**have** *threeb*: $\forall\, x.((f\,x \wedge (R\,x\,x)) \longrightarrow \neg((x\ A\ R\ x) \longrightarrow \neg(x\ P\ R\ x)))$ **using** *threea* **by** *simp*
**have** *four*: $\forall\, x.\ ((x\ A\ R\ x) \longrightarrow \neg(x\ P\ R\ x))$ **using** *one* **by** *simp*
**have** *five*: $\forall\, x.\ ((f\,x \wedge (R\ x\ x)) \longrightarrow ((x\ A\ R\ x) \longrightarrow \neg\ (x\ P\ R\ x)))$ **using** *four* **by** *simp*
**have** *six*: $\forall\, x.\ (f\,x \longrightarrow \neg\ (R\ x\ x))$ **using** *five threeb* **by** *simp*
**have** *seven*: $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge \neg(R\ x\ x))))$ **using** *24 six* **by** *simp*
**have** *eight*: $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *seven* **by** *fastforce*
**thus** *?thesis* **by** *simp*
**qed**

Are all assumptions necessary?

**lemma** *thirdproofwo1*:
**assumes** *22*: $\forall\, x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
**and** *23*: $\forall\, x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
**and** *24*: $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ (R\ t\ x)))$
**shows** $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

**lemma** *thirdproofwo2*:
**assumes** *21*: $\forall\, x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
**and** *23*: $\forall\, x\ y.\ ((f\,x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
**and** *24*: $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ (R\ t\ x)))$
**shows** $\forall\, x.\ (f\,x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

**lemma** *thirdproofwo3*:
**assumes** *21*: $\forall\, x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$

**and** *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
**and** *24*: $\forall x.\ (f\, x \longrightarrow (\exists\, t.\ (R\ t\ x)))$
**shows** $\forall x.\ (f\, x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

**lemma** *thirdproofwo4*:
**assumes** *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
**and** *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
**and** *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
**shows** $\forall x.\ (f\, x \longrightarrow (\exists\, t.\ ((R\ t\ x) \wedge t \neq x)))$
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

Next we show that also assumptions 21 22 23 and 14 imply irreflexivity

**lemma** *IrreflexivityRv2*:
**assumes** *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
**and** *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
**and** *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
**and** *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*

**using** *21 22 23 14* **by** *meson*

Nitpick confirms consistency

Are the assumptions all necessary?

**lemma** *IrreflexivityRv2wo1*:
**assumes** *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ \ R\ y))$
**and** *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
**and** *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

**lemma** *IrreflexivityRv2wo2*:
**assumes** *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ \ S\ y) \longrightarrow \neg(x\ P\ \ S\ y))$
**and** *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ \ R\ x))$
**and** *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

**lemma** *IrreflexivityRv2wo3*:

**assumes** *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
**and** *22*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
**and** *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

**lemma** *IrreflexivityRv2wo4*:
**assumes** *21*: $\forall x\ y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
**and** *22*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
**and** *23*: $\forall x\ y.\ ((f\ x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
**shows** *irreflexive R*
**nitpick**[*verbose*]
**oops**

Nitpick finds a counterexample

# 6 Arguments for there being a first element

N.B. my local sledgehammer (and try0 etc.) can t prove the following theorem; the only remote prover that finds a proof is vampire but proof reconstruction fails even here. I would be interested if sledgehammer find a proof on a faster machine useful theorems to add are mem$_Collect_eqandperhapsTauto$

**lemma** *TpThenNotC3*:
**assumes** *Tp*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
**and** *c1*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$
**and** *c2*: $K\ R$
**shows** $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists u.\ ((u \in (CC\ R) \wedge u \neq x) \wedge (\neg\ R\ x\ u))))$
**proof** –

**have** *one*: $\forall x\ y.\ ((R\ y\ x) \longrightarrow (f\ y \wedge f\ x))$ **using** *Tp* **by** *fastforce*
**have** *two*: $\forall x\ y.\ ((R\ x\ y \vee R\ y\ x) \longrightarrow (f\ x \wedge f\ y))$ **using** *one Tp* **by** *blast*
**have** *threea*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists t.\ (R\ t\ x \vee R\ x\ t)))$ **by** *auto*
**have** *threeb*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists t.\ (f\ x \wedge\ f\ t)))$ **using** *threea two* **by** *blast*
**have** *threec*: $\forall x.\ (x \in (CC\ R) \longrightarrow f\ x)$ **using** *threeb* **by** *blast*
**have** *threed*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists u.\ (R\ u\ x)))$ **using** *threec c1* **by** *simp*
**have** *threee*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists u.\ ((u \in (CC\ R) \wedge u \neq x) \wedge (\neg\ R\ x\ u))))$
**proof** –
{ **fix** $aa :: a$
**obtain** $aaa :: a \Rightarrow a$ **where**
*ff1*: $\forall a.\ a \notin CC\ R \vee R\ (aaa\ a)\ a$
**by** (*metis* (*lifting*) *threed*)
{ **assume** $\neg\ R\ aa\ aa \wedge aa \in CC\ R$
**then have** $aa \neq aaa\ aa$
**using** *ff1* **by** (*metis* (*lifting*))
**moreover**

{ **assume** $aa \neq aaa\ aa \wedge aa \in CC\ R$
**then have** $(\exists\, a.\ R\ (aaa\ aa)\ a \vee R\ a\ (aaa\ aa)) \wedge aa \neq aaa\ aa$
**using** *ff1* **by** *meson*
**then have** $aaa\ aa \in CC\ R \wedge aa \neq aaa\ aa$
**using** *mem-Collect-eq* **by** *blast*
**then have** $R\ aa\ (aaa\ aa) \vee aa \notin CC\ R \vee (\exists\, a.\ a \in CC\ R \wedge aa \neq a \wedge \neg$
$R\ aa\ a)$
**by** (*metis* (*lifting*))
**moreover**
{ **assume** $R\ aa\ (aaa\ aa)$
**then have** $aa \notin CC\ R \vee (\exists\, a.\ a \in CC\ R \wedge aa \neq a \wedge \neg\ R\ aa\ a)$
**using** *ff1* **by** (*meson c2*) }
**ultimately have** $aa \notin CC\ R \vee (\exists\, a.\ a \in CC\ R \wedge aa \neq a \wedge \neg\ R\ aa\ a)$
**by** *blast* }
**ultimately have** $aa \notin CC\ R \vee (\exists\, a.\ a \in CC\ R \wedge aa \neq a \wedge \neg\ R\ aa\ a)$
**by** *blast* }
**then have** $aa \notin CC\ R \vee (\exists\, a.\ a \in CC\ R \wedge aa \neq a \wedge \neg\ R\ aa\ a)$
**by** (*meson c2*) }
**then show** *?thesis*
**by** (*metis* (*lifting*)) **qed**
**thus** *?thesis* **by** *blast*
**qed**

Nitpick doesnacute¿t find a model. That is however not really of importance since this is (sort of) supposed to be a reductio*)

whether the assumptions are all necessary is irrelevant here (since itacute¿s supposed to be a reductio).

Arguments for Tp (for a reductio)

Automated:

**lemma** *Tpauto*:
**assumes** *c2*: $K\ R$
**and** *NotC3*: $\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
**and** *35*: $\forall\, x.\ ((\exists\, t.\ (t\ R\ x)) \longrightarrow f\ x)$
**shows** $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$ **using** *c2 NotC3 35* **by** *meson*

With Salamuchaacute¿s steps:

slight differences between both notational variants of Salamucha; Probably typos; the more intuitive version is used

**lemma** *Tp*:
**assumes** *c2*: $K\ R$
**and** *NotC3*: $\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
**and** *35*: $\forall\, x.\ ((\exists\, t.\ (t\ R\ x)) \longrightarrow f\ x)$
**shows** $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
**proof** –

**have** *one*: $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (x \in (CC\ R) \wedge y \in (CC\ R)))$ **by** *auto*
**have** *twoa*: $(\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
$\longrightarrow (\forall\, y.\ (y \in (CC\ R) \longrightarrow (\exists\, u.\ (u \in (CC\ R) \wedge u \neq y \wedge \neg(R\ y\ u)))))$ **by** *presburger*
**have** *twob*: $(\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))$
$)))$
$\longrightarrow (\forall\, y.\ (y \in (CC\ R) \longrightarrow (\exists\, u.\ (R\ u\ y))))$ **using** *c2* **by** *meson*
**have** *twoc*: $(\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))$
$)))$
$\longrightarrow (\forall\, y.\ (y \in (CC\ R) \longrightarrow f\ y))$ **using** *35* **by** *meson*
**have** *twod*: $(\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))$
$)))$
$\longrightarrow (\forall\, x\ y.\ (R\ x\ y \longrightarrow (f\ x \wedge f\ y)))$ **using** *twoc* **by** *blast*
**thus** *?thesis* **using** *NotC3* **by** *blast*
**qed**

Are all assumptions necessary? (Kind of an academic question, since this is supposed to be a reductio)

No!

**lemma** *Tpwo1*:
**assumes** *NotC3*: $\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
**and** *35*: $\forall\, x.\ ((\exists\, t.\ (t\ R\ x)) \longrightarrow f\ x)$
**shows** $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$ **using** *NotC3 35* **by** *meson*

A

**lemma** *Tpo2*:
**assumes** *c2*: *K R*
**and** *35*: $\forall\, x.\ ((\exists\, t.\ (t\ R\ x)) \longrightarrow f\ x)$
**shows** $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
**nitpick**[*verbose*]
**oops**

Nitpick doesnacute¿t find a counterexample*) (*For a counterexample consider: let R be a relation on the natural numbers (n ge¿ 0) where: x R y := x = 0 and¿ y = 1 R is transitive irreflexive and connected hence c2 holds. let f x := exists¿t. (t ¡ x) then, if t R x holds then t = 0 and x = 1 and there is a smaller number than 1, namely 0. hence 35 holds. however for x = 0 and y = 1 x R y holds but it is not true that f 0, since by definition there is no smaller natural number

**lemma** *Tpwo3*:
**assumes** *c2*: *K R*
**and** *NotC3*: $\neg\ (\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
**shows** $\forall\, x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
**nitpick**[*verbose*]
**oops**

Nitpick doesnacute¿t find a counterexample*) (*For a (trivial) counterexample consider: let R be the less-than relation on the natural numbers. It is obviously an Ordering Relation. There is also no smalles element. therefore c2 and NotC3 hold. let f x := False then the conclusion is wrong for all x y.

Again for the following proof we have to declare the type of the /time/ elements explicitly; we will just use type a here

For the following lemma sledgehammer proof reconstruction fails, but the results strongly suggest that the set of assumptions are inconsistent. This is however not a problem since the intention of this lemma is to show that Tp should be rejected

N.B. Step seven has a typo in the second notational variant!

N.B. Salamucha mentions that for some definitions of identity (e.g. a Leibnizian) the x noteq¿ y can be omitted in none. He argues that this is however not very helpful and leads to more problems than the apparent simplification solves. I tend to agree.

**lemma** *Unwantedconsequences*:
**assumes** *31*: $\forall x.\ (f\ x \longrightarrow C\ x)$
**and** *32*: $\forall x.\ ((C\ x \wedge f\ x) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$
**and** *33*: $\forall x\ (t_2{::}a).\ (C\ x \longrightarrow ((t_2\ F\ x) \longrightarrow (H\ t_2)))$
**and** *34*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow (t_1 = t_2))$
**and** *c2*: $K\ R$
**and** *Tp*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
**shows** $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y) \wedge (t_1\ F\ x) \wedge (t_2\ F\ y)) \quad \longrightarrow t_1 = t_2)$
**proof** −
  **have** *one*: $\forall x\ y.\ ((R\ x\ y \vee R\ y\ x) \longrightarrow (f\ x \wedge f\ y))$ **using** *Tp* **by** *auto*
  **have** *twoa*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists z.\ (R\ z\ x \vee R\ x\ z)))$ **by** *auto*
  **have** *twob*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists z.\ (f\ z \wedge f\ x)))$ **using** *twoa one* **by** *meson*
  **have** *twoc*: $\forall x.\ (x \in (CC\ R) \longrightarrow f\ x)$ **using** *twob* **by** *simp*
  **have** *three*: $\forall x.\ (x \in (CC\ R) \longrightarrow C\ x)$ **using** *twoc 31* **by** *blast*
  **have** *four*: $\forall x.\ (x \in (CC\ R) \longrightarrow (C\ x \wedge f\ x))$ **using** *three twoc* **by** *simp*
  **have** *five*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$ **using** *four 32* **by** *blast*
  **have** *six*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\forall (t_2{::}a).\ ((t_2\ F\ x) \longrightarrow (H\ t_2))))$ **using** *three 33* **by** *blast*

  **have** *seven*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ ((t_1\ F\ x) \wedge (H\ t_1))))$ **using** *five six* **by** *blast*
  **have** *eight*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y \vee R\ y\ x) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow t_1 = t_2)$ **using** *34* **by** *blast*
  **have** *nine*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y) \wedge (t_1\ F\ x) \wedge (t_2\ F\ y)) \quad \longrightarrow t_1 = t_2)$ **using** *eight c2* **by** *meson*
  **thus** *?thesis* **by** *blast*
**qed**

# 7 The Consequens of Thesis T

Ex Motu implies Monotheism

**lemma** *monotheismauto*:
**assumes** *god*: $(\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
**and** *c2*: $K\ R$
**and** *c3*: $(\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
**shows** $((\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))) \wedge (\neg\ (f\ w) \wedge (\forall\, u.$ $(u \in (CC\ R) \wedge u \neq w) \longrightarrow (R\ w\ u))))$
$\longrightarrow v = w$ **using** *c2 c3 god* **by** (*metis* (*full-types*, *lifting*) *mem-Collect-eq*)

the step /vwin/ is not part of Salamuchaacute¿s outline, but needed for Isabelleacute¿s provers*)

**lemma** *monotheism*:
**assumes** *god*: $(\exists\, v.\ (\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))))$
**and** *c2*: $K\ R$
**and** *c3*: $(\exists\, y.\ (y \in (CC\ R) \wedge (\forall\, u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u)))))$
**shows** $((\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u))) \wedge (\neg\ (f\ w) \wedge (\forall\, u.$ $(u \in (CC\ R) \wedge u \neq w) \longrightarrow (R\ w\ u))))$
$\longrightarrow v = w$
**proof** −
  {**assume** *asm1*: $(\neg\ (f\ v) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u)))$
   **and** *asm2*: $(\neg\ (f\ w) \wedge (\forall\, u.\ (u \in (CC\ R) \wedge u \neq w) \longrightarrow (R\ w\ u)))$
    {**assume** *poly*: $v \neq w$
     **from** *asm1* **have** *v1*: $\forall\, x.\ ((x \in (CC\ R) \wedge (x \neq v)) \longrightarrow (R\ v\ x))$ **by** *auto*
     **from** *asm2* **have** *w1*: $\forall\, x.\ ((x \in (CC\ R) \wedge (x \neq w)) \longrightarrow (R\ w\ x))$ **by** *auto*

     **have** *vwin*: $v \in (CC\ R) \wedge w \in (CC\ R)$
      **proof** −
      **from** *c3* **obtain** $y$ **where** *obty*: $y \in (CC\ R)$ **by** *auto*
      {**assume** $y \neq v$
       **hence** $v \in (CC\ R)$ **using** *v1 obty* **by** *auto*}
      **moreover**
      {**assume** $y = v$
       **hence** $v \in (CC\ R)$ **using** *obty* **by** *simp*}
      **ultimately have** $v \in (CC\ R)$ **by** *fastforce*
      **thus** *?thesis* **using** *w1 obty* **by** *blast* **qed**
     **hence** $(R\ v\ w) \vee (R\ w\ v)$ **using** *c2 poly* **by** *blast*
    **moreover**
    {**assume** $R\ v\ w$
     **hence** $\neg\ (R\ w\ v)$ **using** *c2* **by** *blast*
     **hence** *False* **using** *w1 vwin poly* **by** *auto*}
    **moreover**
    {**assume** $R\ w\ v$
     **hence** $\neg\ (R\ v\ w)$ **using** *c2* **by** *blast*
     **hence** *False* **using** *v1 vwin poly* **by** *auto*}
    **ultimately have** *False* **by** *blast*}
  **hence** $v = w$ **by** *blast*}
**thus** *?thesis* **by** *fast*

**qed**

## 8 The entire proof(s) (as specified on p.131ff)

**lemma** *AC*:
**assumes** *one*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$
**and** *two*: $\forall x\ y\ z.\ (((R\ x\ y) \wedge\ (R\ y\ z)) \longrightarrow (R\ x\ z))$
**and** *three*: $\forall x\ y.\ ((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y)) \longrightarrow ((R\ x\ y) \vee (R\ y\ x)))$

**and** *11*: $\forall x.\ ((f\ x) \longrightarrow (\exists a\ b.\ (a\ M\ x \wedge b\ M\ x)))$
**and** *12*: $\forall x.\ ((\exists a\ b.\ (((a\ M\ x) \wedge (b\ M\ x)) \wedge (((\neg\ f\ a) \wedge (f\ b)) \vee ((\neg\ f\ a) \longrightarrow (\neg\ f\ b))))) \longrightarrow (\neg\ R\ x\ x))$
**and** *14*: $\forall x\ y.(x\ R\ y \longrightarrow f\ y)$

**and** *31*: $\forall x.\ (f\ x \longrightarrow C\ x)$
**and** *32*: $\forall x.\ ((C\ x \wedge f\ x) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$
**and** *33*: $\forall x\ (t_2{::}a).\ (C\ x \longrightarrow ((t_2\ F\ x) \longrightarrow (H\ t_2)))$
**and** *34*: $\forall x\ y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow (t_1 = t_2))$
**and** *35*: $\forall x.\ ((\exists t.\ (t\ R\ x)) \longrightarrow f\ x)$

**and** *A*: $\neg\ (\forall x.\ (x \in \{y.\ (y \in (CC\ R) \wedge (C\ y))\} \longrightarrow (\exists t_1{::}a.\ ((t_1\ F\ x) \wedge (H\ t_1)))))$

**shows** $\exists v.\ (\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u)))$
**proof** −

**from** *11 12 14* **have** *irreflexive R* **using** *irreflexivityR* **by** *blast*
**hence** *c2*: *K R* **using** *one two three* **by** *blast*
**have** *T1*: $\forall x.\ ((f\ x) \longrightarrow (\exists t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *11 12 14 T1auto one* **by** *blast*
**hence** *c1*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$ **by** *blast*
**{assume** *Tp*: $\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y))$
**have** *seven*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ ((t_1\ F\ x) \wedge (H\ t_1))))$ **using** *Tp 31 32 33* **by** *blast*
**hence** *False* **using** *A* **by** *blast***}**
**hence** *NOTTp*: $\neg\ (\forall x\ y.\ ((R\ x\ y) \longrightarrow (f\ x \wedge f\ y)))$ **by** *blast*
**{assume** *NOTC3*: $\neg\ ((\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
**have** *False* **using** *Tpauto 35 NOTTp c2 NOTC3* **by** *blast***}**
**hence** *c3*: $((\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$ **by** *blast*
**show** *?thesis* **using** *c1 c2 c3 Tauto* **by** *blast*
**qed**

**lemma** *BC*:
**assumes** *one*: $\forall x.\ (f\ x \longrightarrow (\exists t.\ (R\ t\ x)))$

**and** *two*: $\forall x\, y\, z.\ (((R\ x\ y) \wedge\ (R\ y\ z)) \longrightarrow (R\ x\ z))$
**and** *three*: $\forall x\, y.\ ((x \in (CC\ R) \wedge y \in (CC\ R) \wedge (x \neq y)) \longrightarrow ((R\ x\ y) \vee (R\ y\ x)))$

**and** *21*: $\forall x\, y\ (S{::}a \Rightarrow a \Rightarrow bool).\ ((x\ A\ S\ y) \longrightarrow \neg(x\ P\ S\ y))$
**and** *22*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (x\ P\ R\ y))$
**and** *23*: $\forall x\, y.\ ((f\, x \wedge (R\ y\ x)) \longrightarrow (y\ A\ R\ x))$
**and** *14*: $\forall x\, y.(x\ R\ y \longrightarrow f\, y)$

**and** *31*: $\forall x.\ (f\, x \longrightarrow C\ x)$
**and** *32*: $\forall x.\ ((C\ x \wedge f\, x) \longrightarrow (\exists (t_1{::}a).\ (t_1\ F\ x)))$
**and** *33*: $\forall x\ (t_2{::}a).\ (C\ x \longrightarrow ((t_2\ F\ x) \longrightarrow (H\ t_2)))$
**and** *34*: $\forall x\, y\ (t_1{::}a)\ (t_2{::}a).\ (((R\ x\ y) \wedge ((t_1\ F\ x) \wedge (t_2\ F\ y))) \longrightarrow (t_1 = t_2))$
**and** *35*: $\forall x.\ ((\exists t.\ (t\ R\ x)) \longrightarrow f\, x)$

**and** *A*: $\neg\ (\forall x.\ (x \in \{y.\ (y \in (CC\ R) \wedge (C\ y))\} \longrightarrow (\exists t_1{::}a.\ ((t_1\ F\ x) \wedge (H\ t_1)))))$
**shows** $\exists v.\ (\neg\ (f\ v) \wedge (\forall u.\ (u \in (CC\ R) \wedge u \neq v) \longrightarrow (R\ v\ u)))$

**proof** –
 **from** *21 22 23 14* **have** *irreflexive R* **using** *IrreflexivityRv2* **by** *blast*
 **hence** *c2*: *K R* **using** *one two three* **by** *blast*
 **have** *T1*: $\forall x.\ ((f\, x) \longrightarrow (\exists t.\ ((R\ t\ x) \wedge t \neq x)))$ **using** *21 22 23 one thirdproof* **by** *blast*
 **hence** *c1*: $\forall x.\ (f\, x \longrightarrow (\exists t.\ (R\ t\ x)))$ **by** *blast*
 **{assume** *Tp*: $\forall x\, y.\ ((R\ x\ y) \longrightarrow (f\, x \wedge f\, y))$
 **have** *seven*: $\forall x.\ (x \in (CC\ R) \longrightarrow (\exists (t_1{::}a).\ ((t_1\ F\ x) \wedge (H\ t_1))))$ **using** *Tp 31 32 33* **by** *blast*
 **hence** *False* **using** *A* **by** *blast***}**
 **hence** *NOTTp*: $\neg\ (\forall x\, y.\ ((R\ x\ y) \longrightarrow (f\, x \wedge f\, y)))$ **by** *blast*
 **{assume** *NOTC3*: $\neg\ ((\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$
 **have** *False* **using** *Tpauto 35 NOTTp c2 NOTC3* **by** *blast***}**
 **hence** *c3*: $((\exists y.\ (y \in (CC\ R) \wedge (\forall u.\ ((u \in (CC\ R) \wedge u \neq y) \longrightarrow (R\ y\ u))))))$ **by** *blast*
 **show** *?thesis* **using** *c1 c2 c3 Tauto* **by** *blast*
**qed**

Nitpick times out while trying to find a model for both proofs. Sledgehammer and remote provers canacute¿t prove false, but consistency is still an open question.

**end**