GodProof comment

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The necessitation rule or box-introduction requires that all open assumptions be boxed. This can be compared to the universal introduction rule, which requires that the eigenvariable of the rule must not occur free in any open assumption. Both conditions are seemingly violated in your proof of theorem 3.

Below we will give a derivation, which does not violate these conditions. The proof uses both axiom 4 and necessitation.

Let the following derivation with the open assumption $G(x)$ be Π_1:

$$\dfrac{\dfrac{\dfrac{\psi(x)^2 \qquad \dfrac{\dfrac{\neg P(\psi)^1 \qquad \dfrac{\dfrac{}{\forall\varphi(\neg P(\varphi) \supset P(\neg\varphi))}{\scriptstyle Ax.2.}}{\neg P(\psi) \supset P(\neg\psi)}{\scriptstyle \forall E}}{P(\neg\psi)}{\scriptstyle \supset E} \qquad \dfrac{\dfrac{G(x)}{\forall\varphi(P(\varphi) \supset \varphi(x))}{\scriptstyle \text{Definition of G}}}{P(\neg\psi) \supset \neg\psi(x)}{\scriptstyle \forall E}}{\neg\psi(x)}{\scriptstyle \supset E}}{\bot}{\scriptstyle \supset E}}{P(\psi)}{\scriptstyle RAA,1}}{\psi(x) \supset P(\psi)}{\scriptstyle \supset I,2}$$

Let the following derivation with the open assumption $G(x)$ be Π_2:

$$\dfrac{\dfrac{\dfrac{\psi(x)^1 \qquad \begin{matrix}\Pi_1\\ \vdots\\ \psi(x) \supset P(\psi)\end{matrix}}{P(\psi)} \qquad \dfrac{}{P(\psi) \supset \Box P(\psi)}{\scriptstyle Ax.4}}{\Box P(\psi)}{\scriptstyle \supset E}}{\psi(x) \supset \Box P(\psi)}{\scriptstyle \supset I,1}$$

Let the following derivation without open assumptions be Π_3:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\psi)^1 \qquad \dfrac{\dfrac{G(x)^2}{\forall\varphi(P(\varphi)\supset\varphi(x))}\;\text{Definition of G}}{P(\psi)\supset\psi(x)}}{\psi(x)}}{G(x)\supset\psi(x)}\;{\scriptstyle \supset I,2}}{\forall x(G(x)\supset\psi(x))}\;{\scriptstyle \forall I}}{P(\psi)\supset\forall x(G(x)\supset\psi(x))}\;{\scriptstyle \supset I,1}$$

Let the following derivation with the open assumption $G(x)$ be Π_4:

$$\dfrac{\dfrac{\dfrac{\psi(x)^1 \qquad \begin{matrix}\Pi_2\\ \vdots\\ \psi(x)\supset\Box P(\psi)\end{matrix}}{\Box P(\psi)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^2}{P(\psi)}\;{\scriptstyle \Box E} \qquad \begin{matrix}\Pi_3\\ \vdots\\ P(\psi)\supset\forall x(G(x)\supset\psi(x))\end{matrix}}{\forall x(G(x)\supset\psi(x))}}{\Box\forall x(G(x)\supset\psi(x))}\;{\scriptstyle \textit{nessecitation}}}{\Box P(\psi)\supset\Box\forall x(G(x)\supset\psi(x))}\;{\scriptstyle \supset I,2}}{\Box\forall x(G(x)\supset\psi(x))}}{\psi(x)\supset\Box\forall x(G(x)\supset\psi(x))}\;{\scriptstyle \supset I,1}$$

The use of the necessitation rule is valid, because the only open assumption $\Box P(\psi)$ is boxed.

We construct a derivation of theorem 3 with a subderivation $\Pi_4[G(x)^1]$, which means that the open assumption $G(x)$ in Π_4 is discharged with the rule labeled 1.

$$\dfrac{\dfrac{\dfrac{G(x)^1 \qquad \dfrac{\begin{matrix}\Pi_4[G(x)^1]\\ \vdots\\ \psi(x)\supset\Box\forall x(G(x)\supset\psi(x))\end{matrix}}{\forall\psi(\psi(x)\supset\Box\forall x(G(x)\supset\psi(x)))}\;{\scriptstyle \forall I}}{G(x)\&\forall\psi(\psi(x)\supset\Box\forall x(G(x)\supset\psi(x)))}}{G \text{ ess } x}\;\text{Definition of ess}}{G(x)\supset G \text{ ess } x}\;{\scriptstyle \supset I,1}$$

The use of the universal introduction rule is valid, because the only open assumption is $G(x)$, which does not contain the property ψ as a free variable.