A Variant of Gödel's Ontological Proof in a Natural Deduction Calculus (Draft)

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [. . .] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Introduction

Ontological arguments for the existence of God can be traced back at least to St. Anselm (1033-1109). His argument considered a maximally conceivable being, who must exist, because if it did not have the property of existence, then we could conceive of a greater being that, apart from the other properties, also has the property of existence.

A major critique of this argument is that we do not know whether the concept maximal conceivable being in fact designates anything or if it is inconsistent, like a round square [2][p.134]. As Bertrand Russell pointed out, the definition of maximal allows us to define properties, like having boots, which the maximal being then also must have [**?**].

Kant argued against the ontological argument on the basis that existence is not an analytic property [3]. This means that existence cannot be contained in the definition of a concept, because it is generally synthetic. All that we can say is that if God exists, then God necessarily exists [**?**].

St. Anselm's argument was elaborated further by Descartes and Leibniz. Leibniz identified that establishing the possible existence of God is a critical missing step in St. Anselm's argument. To fill this gap, he argued that the properties of God, the perfections, are compatible. This implies that it is possible to have all perfections at once and therefore the existence of a maximal being with all these properties is possible.

Gödel studied Leibniz's work [**?**] and brought his ontological argument to a modern form using a modal logic with higher-order quantification over properties. In this setting, he gave precise axioms describing the notion of *positive* property and defined God as a being having all positive properties.

Gödel's notion of positive property and Leibniz's notion of perfection are not exactly the same [**?**]. On a technical level, the main distinction seems to be that Gödel's positive properties are not just atomic properties, like Leibniz's perfections, but can also consist of complex combinations of atomic properties [2][p.139.] (TODO: Check this claim). In particular, one of Gödel's axioms states that the conjunction of any set of positive properties is positive. And from this axiom, it follows immediately that the property of being God-like is positive. While this step is intuitively and informally clear, it is not easily formalizable in a typical logical calculus, because it requires inferring that being god-like is a (possibly infinite) conjuntion of positive properties, while it has only been defined as the property holding for individuals who have all positive properties. This interplay between universal quantification (in the definition of a God-like being) and conjunction (in Gödel's mentioned axiom) is a technical difficulty that probably explains why, starting with Scott [**?**], to whom Gödel confided his manuscript, this axiom of Gödel's has been replaced by another one that simply assumes the positivity of the property of being God-like.

The main criticism against Gödel's ontological argument is the *modal collapse*, an undesirable consequence of the stipulated axioms. Most recent works [**?**] on the ontological argument have focused on proposing modifications of the argument that would not entail a modal collapse. This is discussed in more detail in Section **??**.

The first contribution of this paper is a detailed formalization of Scott's version [**?**] of Gödel's ontological argument [**?**] (as shown in Section **??**) in a natural deduction calculus (as defined in Section 2). A natural deduction style [**?** **?**] was chosen mainly for three reasons. Firstly, presentations of Göedel's proof are typically either informal or formalized in other styles of calculi (e.g. Fitting's tableaux [2] or Sobel's sentential modal calculus [5]). Therefore, a formalization in natural deduction is a valuable complement to the existing presentations. Secondly, it makes the ontological proof accessible to people who are familiarized with a natural deduction style. Finally, as natural deduction is the style used by proof assistants such as Coq [**?**] and Isabelle [**?**], the natural deduction formalizations have been easily checked step-by-step in Coq [**?**].

The main contribution of the paper, however, is the presentation of new proofs (also in natural deduction style) of the lemmas, theorems and corollaries in Gödel's manuscript. In contrast to Scott's proofs [**?**], the proofs presented here are simpler and shorter, as discussed in Section **??**.

2 Natural Deduction

The language of higher-order modal logic used here is inspired by that of Church's simple type theory [**?**].

Definition 1. Simple types *are given by the following grammar:*

$$\theta, \tau \quad ::= \quad \mu \;\mid\; o \;\mid\; \theta \to \tau$$

where μ is the atomic type for individuals, o is the atomic type for propositions and $\theta \to \tau$ is the type for functions taking an argument of type θ and returning something of type τ. '$\to$' is assumed to be right associative.

Definition 2. Terms and formulas *are given by the following grammar:*

$$\begin{aligned} s,t \quad ::= \quad & p_\tau \ \mid\ X_\tau \ \mid\ (\lambda X_\theta . s_\tau)_{\theta\to\tau} \ \mid\ (s_{\theta\to\tau}\, t_\theta)_\tau \ \mid \\ & \bot_o \ \mid \to_{o\to o\to o} \mid\ \wedge_{o\to o\to o} \ \mid\ \vee_{o\to o\to o} \ \mid \\ & \forall_{(\tau\to o)\to o} \ \mid\ \exists_{(\tau\to o)\to o} \ \mid\ \Box_{o\to o} \ \mid\ \Diamond_{o\to o} \end{aligned}$$

where p_τ and X_τ range over, respectively, constants and variables of type τ. Parenthesis conventions, infix notation for propositional connectives and binding notation for quantifiers are assumed. Furthermore, subscript types are omitted when they are clear from the context. Negation ($\neg_{o\to o}$) and equivalence ($\leftrightarrow_{o\to o\to o}$) are defined by $\neg A \equiv A \to \bot$ and $(A \leftrightarrow B) \equiv (A \to B) \wedge (B \to A)$.

The natural deduction calculus used here has standard rules for propositional connectives and quantifiers, as shown in Figures 1 and 3. The extension to classical logic is achieved by adding a rule for double negation elimination, shown in Figure 2. Finally, modal operators are handled by special rules that insert or remove formulas from boxes, as shown in Figure 4. Apart from the use of labels and the dual rules for '$\Diamond$', these rules are essentially the modal rules from [**?**]. Beta-reduction is implicit; all rules are assumed modulo beta-reduction. A *derivation* is a directed acyclic graph whose nodes are formulas and whose edges correspond to applications of the inference rules. A *proof* of a formula F is a derivation without open assumptions and having F as root not inside any box.

Double lines are used to abbreviate tedious propositional reasoning steps in the derivations. Dashed lines are used to refer to an axiom or theorem with proof shown elsewhere. Dotted lines are used to indicate folding and unfolding of definitions. Furthermore, as it is inconvenient to draw boxes around large derivations in LaTeX, formulas inside boxes are labeled with the names of the boxes surrounding them. Therefore, the boxes can be omitted without loss of information.

Fig. 1. propositional rules

$$\frac{\bot}{A}\ \bot_E \qquad \frac{B}{A \to B}\ \to_I \qquad \frac{\begin{array}{c}\overline{A}^{\,n}\\ \vdots \\ B\end{array}}{A \to B}\ \to_I^n \qquad \frac{A \quad A \to B}{B}\ \to_E$$

$$\frac{A \quad B}{A \wedge B}\ \wedge_I \qquad \frac{A \wedge B}{A}\ \wedge_{E_1} \qquad \frac{A \wedge B}{B}\ \wedge_{E_2}$$

The calculus having only the rules shown in Figures 1, 2 and 3 is named **ND**. The calculus with the additional rules shown in Figure 4 is called $\mathbf{ND_K}$.

Fig. 2. double negation elimination

$$\frac{\neg\neg A}{A}\ \neg\neg_E$$

Fig. 3. quantifier rules

$$\frac{A[\alpha]}{\forall x_\tau . A[x]}\ \forall_I \qquad \frac{\forall x_\tau . A[x]}{A[t]}\ \forall_E \qquad \frac{A[t]}{\exists x_\tau . A[x]}\ \exists_I \qquad \frac{\exists x_\tau . A[x]}{A[\beta]}\ \exists_E$$

eigen-variable conditions:
if ρ is a $\forall_I$ inference eliminating a variable α, then any occurrence of α in the proof should be an ancestor of the occurrence of α eliminated by ρ;
if ρ is a $\exists_E$ inference introducing a variable β, then any occurrence of β in the proof should be a descendant of the occurrence of β introduced by ρ.

Fig. 4. Rules for Modal Operators

$$\frac{\omega : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Box A}\ \Box_I \qquad \begin{array}{c} \dfrac{\Box A}{w : \boxed{\begin{array}{c} A \\ \vdots \end{array}}}\ \Box_E \end{array}$$

eigen-box condition:
ω must be a fresh name for a box.

boxed assumption condition:
assumptions should be discharged within the box where they are made.

2.1 Suitability for Rigid Higher-Order Modal Logic K

Adding the modal rules results in a calculus that is suitable for the basic normal modal logic K. In other words, $\mathbf{ND_K}$ is sound and complete relative to **ND** extended with axiom K ($\Box(A \to B) \to (\Box A \to \Box B)$) and the necessitation rule (which establishes that $\Box A$ is a theorem if A is a theorem).

The straightforward combinations of the quantifier rules of **ND** either with the modal rules of $\mathbf{ND_K}$ or with axiom K and the necessitation rule are suitable for a higher-order modal logic where constants and variables are *rigid*. From the point of view of a *possible worlds* semantics, rigidity means that their interpretation is independent of the world in which they are being interpreted. Rigidity is silently assumed by most works investigating the ontological argument, and is explicitly assumed here. Nevertheless, it should be noted that its adequacy has already been contested [2].

Theorem 1. $\mathbf{ND_K}$ *is complete, relative to* **ND** *extended with axiom K and the necessitation rule.*

Proof. The necessitation rule can be immediately simulated with the $\square_I$ rule. Axiom K can be derived in $\mathbf{ND_K}$ as shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\square(A\to B)^2}{\omega : A\to B}\,\square_E \qquad \dfrac{\square A^1}{\omega : A}\,\square_E}{\omega : B}\,\to_E}{\square B}\,\square_I}{\square A \to \square B}\,\to_I^1}{\square(A\to B)\to(\square A\to\square B)}\,\to_I^2$$

Theorem 2. $\mathbf{ND_K}$ *is sound, relative to* **ND** *extended with axiom K and the necessitation rule.*

Proof. It is necessary to show that $\mathbf{ND_K}$ proofs of the following form can be translated to proofs in **ND** extended with the axiom K and the necessitation rule.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\square A_1}{\omega : A_1}\,\square_E\\ \vdots\end{array} \qquad \dots \qquad \begin{array}{c}\dfrac{\square A_n}{\omega : A_n}\,\square_E\\ \vdots\end{array}}{\omega : B}}{\square B}\,\square_I$$

A translation to **ND** extended with axiom K and necessitation is shown below for the case when $n = 1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A_1}\,1\\ \vdots\\ B\end{array}}{A_1\to B}\,\to_I^1}{\square(A_1\to B)}\,\text{necessitation} \qquad \overset{\text{Axiom K}}{\overline{\square(A_1\to B)\to(\square A_1\to\square B)}}}{\square A_1\to\square B}\,\to_E \qquad \square A_1}{\square B}\,\to_E$$

For $n > 1$, the translation is a straightforward generalization:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{}{A_1}\,1\\ \vdots\end{array} \quad \dots \quad \begin{array}{c}\dfrac{}{A_n}\,\text{n}\\ \vdots\end{array}}{B}}{\overline{A_1\to\dots\to A_n\to B}}\,\to_I^*}{\square(A_1\to\dots\to A_n\to B)}\,\text{nec.} \qquad \overset{\text{Axiom K, iterated}}{\overline{\overline{\square(A_1\to\dots\to A_n\to B)\to(\square A_1\to\dots\to\square A_n\to\square B)}}}}{\square A_1\to\dots\to\square A_n\to\square B}\,\to_E$$

$$\dfrac{\dfrac{}{\Box A_1 \to \ldots \to \Box A_n \to \Box B} \qquad \Box A_1 \quad \ldots \quad \Box A_n}{\Box B} \to_E$$

3 Scott's Proof in Natural Deduction

The acceptance of the correctness of the ontological argument by Gödel's work boils down to the intuitive correctness of the axioms and definitions and the belief in the soundness of the deductive system. The formal argument of Gödel is based on Leibniz proof, which in turn is based on Descartes proof. These proofs have two parts; a proof that if God's existence is possible, then it is necessary and a proof that God's existence is in fact possible.

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall \varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

4 New Proof

4.1 Possibly, God Exists

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall \varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

Theorem 1. *Positive properties are possibly exemplified:*

$$\forall \varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom 2}}{\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}}{\forall \psi.[(P(\rho) \wedge \Box\forall x.[\rho(x) \to \psi(x)]) \to P(\psi)]}\forall_E}{(P(\rho) \wedge \Box\forall x.[\rho(x) \to \neg\rho(x)]) \to P(\neg\rho)}\forall_E}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \to P(\neg\rho)} \qquad \dfrac{\dfrac{\text{Axiom 1}}{\forall \varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}}{P(\neg\rho) \leftrightarrow \neg P(\rho)}\forall_E}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \to \neg P(\rho)}}{P(\rho) \to \Diamond\exists x.\rho(x)}}{\forall \varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}\forall_I$$

Definition 3. *A* God-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \to \varphi(x)]$$

Axiom 3 *The property of being God-like is positive:*

$$P(G)$$

Corollary 1. *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 3}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{P(G) \to \Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\to_E$$

4.2 Being God is an essence of any God

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi) \to \Box\, P(\varphi)]$$

Definition 4. *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi \ ess\ x \leftrightarrow \varphi(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

Theorem 2. *Being God-like is an essence of any God-like being:*

$$\forall y.[G(y) \to G\ ess\ y]$$

Proof. Let the following derivation with the open assumption $G(x)$ be $\Pi_1[G(x)]$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(\psi)^1 \qquad \dfrac{\dfrac{\text{Axiom 1}}{\forall\varphi.(\neg P(\varphi) \to P(\neg\varphi))}}{\neg P(\psi) \to P(\neg\psi)}\,\forall_E}{P(\neg\psi)}\,\to_E \qquad \dfrac{\dfrac{G(x)}{\forall\varphi.(P(\varphi) \to \varphi(x))}\,\text{D3}}{P(\neg\psi) \to \neg\psi(x)}\,\forall_E}{\neg\psi(x)}\,\to_E \qquad \psi(x)^2}{\bot}\,\to_E}{\neg\neg P(\psi)}\,\to_I^1}{P(\psi)}\,\neg\neg_E}{\psi(x) \to P(\psi)}\,\to_I^2$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_2[G(x)]$:

$$
\dfrac{\dfrac{\psi(x)^3 \qquad \begin{array}{c}\Pi_1[G(x)]\\ \vdots \\ \psi(x) \to P(\psi)\end{array}}{P(\psi)}\,{\to_E} \qquad \dfrac{\begin{array}{c}\text{Axiom 4}\\ \vdots \\ \forall\varphi.(P(\varphi) \to \Box P(\varphi))\end{array}}{P(\psi) \to \Box P(\psi)}\,{\forall_E}}{\dfrac{\Box P(\psi)}{\psi(x) \to \Box P(\psi)}\,{\to_I^3}}\,{\to_E}
$$

Let the following derivation without open assumptions be Π_3:

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(\psi)^4 \qquad \dfrac{\begin{array}{c}G(x)^5\\ \vdots\ \text{D3} \\ \forall\varphi.(P(\varphi) \to \varphi(x))\end{array}}{P(\psi) \to \psi(x)}\,{\forall_E}}{\psi(x)}\,{\to_E}}{G(x) \to \psi(x)}\,{\to_I^5}}{\forall x.(G(x) \to \psi(x))}\,{\forall_I}}{P(\psi) \to \forall x.(G(x) \to \psi(x))}\,{\to_I^4}
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_4[G(x)]$:

$$
\dfrac{\dfrac{\psi(x)^6 \qquad \begin{array}{c}\Pi_2\\ \vdots \\ \psi(x) \to \Box P(\psi)\end{array}}{\Box P(\psi)}\,{\to_E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\,{\Box_E} \qquad \begin{array}{c}\Pi_3\\ \vdots \\ P(\psi) \to \forall x.(G(x) \to \psi(x))\end{array}}{\forall x.(G(x) \to \psi(x))}\,{\to_E}}{\Box\forall x.(G(x) \to \psi(x))}\,{\Box_I}}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))}\,{\to_I^7}}{\dfrac{\Box\forall x.(G(x) \to \psi(x))}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}\,{\to_I^6}}\,{\to_E}
$$

The use of the necessitation rule above is correct, because the only open assumption $\Box P(\psi)$ is boxed. In the derivation of Theorem 4 below, the assumption $G(x)$ in the subderivation $\Pi_4[G(x)^8]$ is discharged by the rule labeled 8.

$$
\dfrac{\dfrac{\begin{array}{c}\dfrac{G(x)^8 \qquad \dfrac{\begin{array}{c}\Pi_4[G(x)^8]\\ \vdots \\ \psi(x) \to \Box\forall x.(G(x) \to \psi(x))\end{array}}{\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\,{\forall_I}}{G(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\,{\wedge_I}\\ \vdots\ \text{D4} \\ G\ ess\ x\end{array}}{G(x) \to G\ ess\ x}\,{\to_I^8}}{\forall y.[G(y) \to G\ ess\ y]}\,{\forall_I}
$$

4.3 If God's existence is possible, it is necessary

Definition 5. Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1. *If there is a God, then necessarily there exists a God:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{}{\exists z.G(z)}\;1}{G(g)}\;\exists_E$$

$$\dfrac{\dfrac{\dfrac{}{G(g)} \qquad \dfrac{\dfrac{\text{Theorem 4}}{\forall y.[G(y) \to G \ ess \ y]}}{G(g) \to G \ ess \ g}\;\forall_E}{G \ ess \ g}\;\to_E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom 5}}{P(E)} \qquad \dfrac{\dfrac{\dfrac{}{G(g)}}{\forall\varphi.[P(\varphi) \to \varphi(g)]}\;\text{D3}}{P(E) \to E(g)}\;\forall_E}{E(g)}\;\to_E}{\forall\varphi.[\varphi \ ess \ g \to \Box\exists x.\varphi(x)]}\;\text{D5}}{G \ ess \ g \to \Box\exists x.G(x)}\;\forall_E}{\dfrac{\Box\exists x.G(x)}{\exists z.G(z) \to \Box\exists x.G(x)}\;\to_I^1}\;\to_E$$

4.4 God exists

ToDo: this is proven in a way that is slightly different from Gödel's 1970.

Theorem 3. *God exists:*

$$\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{\dfrac{}{\Box\neg\Box\exists x.G(x)}\;1}{\neg\Box\exists x.G(x)}\;\Box_E,\text{ axiom T} \qquad \dfrac{\text{Lemma 1}}{\neg\Box\exists x.G(x) \to \neg\exists x.G(x)}}{\dfrac{\dfrac{\neg\exists x.G(x)}{\Box\neg\exists x.G(x)}\;\Box_I}{\neg\Diamond\exists x.G(x)}}\;\to_E$$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg\Diamond\exists x.G(x)} \qquad \dfrac{\text{Corollary 1}}{\Diamond\exists x.G(x)}}{\bot}\;\to_E}{\dfrac{\neg\Box\neg\Box\exists x.G(x)}{\neg\Box\Diamond\neg\exists x.G(x)}}\;\to_I^1 \qquad \dfrac{\dfrac{\text{Axiom B}}{\neg\exists x.G(x) \to \Box\Diamond\neg\exists x.G(x)}}{\neg\Box\Diamond\neg\exists x.G(x) \to \neg\neg\exists x.G(x)}}{\dfrac{\neg\neg\exists x.G(x)}{\exists x.G(x)}\;\neg\neg\text{ E}}\;\to_E$$

Note that the last step is classical and we do not prove the existential statement by providing an object for which the statement holds. This proof makes section 4.6 superflous and the use of axiom "M" unnecessary.

The system used contains the $\Box_E$-rule with the restriction that we have a $\Box_I$ below it. This is equivalent to modal sytem K that contains axiom K and necessitation rule N. We aslo use axiom B ($A \to \Box\Diamond A$). No other modal axioms are needed.

4.5 Necessarily, God exists

We can also prove that god exists necessarily in our system by simply introducing box on a theorem by rule N in modal sytem K. So we get:

Corollary 2. *Necessarily, God exists:*

$$\Box\exists x.G(x)$$

4.6 God exists

This section is superfluous.

Axiom 6 (M) *What is necessary is the case:*

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Corollary 3. *There exists a God:*

$$\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{\text{Theorem 2}}{\Box\exists x.G(x)} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$

5 Modal collapse

A major criticism against Gödel's proof is that its axioms lead to the so-called *modal collapse* [4]: it is possible to prove that everything that is the case is so necessarily, and hence actuality, possibility and necessity coincide [5][Ch. 4, section 6, theorems 9 and 10]. As this is an undesirable consequence, many solutions to the problem of the modal collapse have been proposed.

Anderson's solution [**?**] modifies the definitions of God-like being and essence, and eliminates half of an axiom. This not only avoids the modal collapse, but also makes two of Gödel's five axioms derivable from the others [**?**] under some implicit additional assumptions [**?**]. Another solution involving more substantial modifications is that of Bjørdal [**? ?**].

On another track, Fitting has argued that greater care has to be taken with the semantics of higher-order modal logics. Quantified variables may be rigid or flexible; and properties may be treated as intensional or extensional. Making the right choices may prevent the modal collapse [2][Sections 11.9 and 11.10].

Anderson [**?**][p. 292] and Sobel [5][p. 133] also discuss the idea that the notion of property over which quantification is allowed might be too general and restrictions might be appropriate.

It is beyond the scope of this paper to analyze these solutions in detail or propose new solutions. The purpose of this section is simply to show natural deduction derivations of the modal collapse, thus confirming that it holds for the axioms used in the previous sections.

Theorem 4. *For all constant fomulas (without free variables), A, we have:*

$$A \rightarrow \Box A$$

Note that in intuitionistic predicate logic we have $\forall y.[B \rightarrow C] \leftrightarrow [\exists y.B \rightarrow C]$ if y is not free in C.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\text{Theorem 4}}{\forall y.[G(y) \rightarrow G\ ess\ y]}}{\forall y.[G(y) \rightarrow G(y) \wedge \forall \psi.(\psi(y) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x)))]}\text{D2}}{\forall y.[G(y) \rightarrow \forall \psi.(\psi(y) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x)))]}\text{Prop. logic}}{\forall y.[G(y) \rightarrow (A(y) \rightarrow \Box \forall x.(G(x) \rightarrow A(x)))]}\text{Second order quantifier elimination and logical steps}}{\forall y.[G(y) \rightarrow (A \rightarrow \Box \forall x.(G(x) \rightarrow A))]}A\text{ is constant}}{\exists y.G(y) \rightarrow (A \rightarrow \Box \forall x.(G(x) \rightarrow A))}\text{intuitionistic predicate logic}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\exists y.G(y) \rightarrow (A \rightarrow \Box \forall x.(G(x) \rightarrow A)) \qquad \dfrac{\text{Theorem 5}}{\exists y.G(y)}}{A \rightarrow \Box \forall x.(G(x) \rightarrow A)}\rightarrow_E}{A \rightarrow \Box(\exists x.G(x) \rightarrow A)}\text{intuitionistic predicate logic} \qquad \dfrac{}{A}1}{\Box(\exists x.G(x) \rightarrow A)}\rightarrow_E}{\exists x.G(x) \rightarrow A}\Box_E \qquad \dfrac{\text{Theorem 5}}{\exists x.G(x)}\rightarrow_E}{\dfrac{A}{\Box A}\Box_I}\rightarrow_E}{A \rightarrow \Box A}\rightarrow_I^1$$

6 Conclusions

The proofs of theorem T1 and corollary C1 ($\Diamond \exists x.G(x)$) do not rely on equality and use axiom A2 ($\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box \forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$) only once. Corollary C2 ($\exists x.G(x)$), which is usually regarded as a trivial corollary of main theorem T3 ($\Box \exists x.G(x)$) using the modal logic axiom T, is here derived directly from lemma L1 and corollary C1, not relying on T. The main theorem T3 becomes derivable from C2 by a single application of the necessitation rule.

Furthermore, all proofs are done in the modal logic K, except for the proof of corollary C2, which requires one use of the axiom B of modal logic KB.

ToDo: Sobel Anderson on KB: Sobel page 152

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