The Ontological Modal Collapse as a Collapse of the Square of Opposition

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Abstract. The *modal collapse* that afflicts Gödel's modal ontological argument for God's existence is discussed from the perspective of the modal square of opposition. Furthermore, a computer-assisted verification of the claims that the emendations by Anderson and by Frode are immune to the modal collapse is presented.

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1. Introduction

Attempts to prove the existence (or non-existence) of God by means of abstract, ontological arguments are an old tradition in western philosophy, with contributions by several prominent philosophers, including St. Anselm of Canterbury, Descartes and Leibniz. Kurt Gödel studied and further improved this argument, bringing it to a mathematically more precise form, as a chain of axioms, lemmas and theorems in a modal logic [23, 30], as shown in Fig. 1.

Gödel defines God (see Fig. **??**) as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable (but debatable) axioms that they should satisfy. The overall idea of Gödel's proof is in the tradition of Anselm's argument, who defined God as some entity of which nothing greater can be conceived. Anselm argued that existence in the actual world would make such an assumed being even greater; hence, by definition God must exist. Gödel's ontological argument is clearly related to this reasoning pattern. However,

This work was completed with the support of ToDo.

A1 Either a property or its negation is positive, but not both:

$$\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \equiv \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\ P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G\ ess\ x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \equiv \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

L1 If a god-like being exists, then necessarily a god-like being exists:

$$\exists x G(x) \rightarrow \Box\exists y G(y)$$

L2 If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists x G(x) \rightarrow \Box\exists y G(y)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists x G(x)$$

FIGURE 1. Scott's version of Gödel's ontological argument [30].

it also tries to fix some fundamental weaknesses in Anselm's work. For example, Gödel explicitly proves that God's existence is possible, which has been a basic assumption for Anselm. Because of this, Anselm's argument has been criticized as incomplete by Leibniz. Leibniz claimed that the assumption should be derivable from the definition of God as a perfect being and from the notion of perfection. Gödel's proof addresses this critique, and it

MC Everything that is the case is so necessarily:

$$\forall\phi[\phi \rightarrow \Box\phi]$$

MC' Everything that is possible is necessary:

$$\forall\phi[\Diamond\phi \rightarrow \Box\phi]$$

MC" Modalities collapse:

$$\forall\phi[\phi \leftrightarrow \Diamond\phi \leftrightarrow \Box\phi]$$

Figure 2. Modal Collapse [**?**, **?**].

also addresses Kant's objection that existence should not be treated as a predicate.

Nevertheless, Gödel's work still leaves room for criticism. In particular, his axioms are so strong that they entail a *modal collapse* [**?**, 31]: everything that is the case is so necessarily. There has been an impressive body of recent and ongoing work (cf. [31, 19, 3, 2, **?**, 18] and the references therein) proposing solutions for the modal collapse. The goal of this paper is to present a computer-assisted analysis of the solutions proposed by Anderson and by Frode. The technique enabling this analysis is the embedding of quantified modal logics into higher-order logics [10, 9, 6], for which automated theorem provers exist [**?**, **?**, **?**, **?**]. This technique has already been succesfully employed in the verification and reconstruction of Gödel's proof [**?**], and a detailed mathematical description of the technique is available in [**?**].

2. The Modal Collapse

A crucial step of the ontological argument is the claim that if God's existence is possible, then it is necessary. This is Lemma L2 in Gödel's proof. In the modal square of opposition (Fig. **??**), this is an unusual situation in which the **I** corner implies and entails the **A** corner, for the particular case when ϕ is $\exists x G(x)$. Gödel's proof shows that his axioms are strong enough to invert the direction of entailment for the particular ϕ at issue. The question is whether the axioms are then not too strong, also allowing the inverted entailment for arbitrary ϕ. That is essentially the question asked by Sobel [**?**]; and his proof of the modal collapse (MC) answers it affirmatively. The modal collapse is a collapse of the modal square: it shows that, assuming Gödel's axioms, the subcontraries entail the contraries.

ToDo: include figure of Modal Square of Opposition.

ToDo: discuss what the arrows mean (entailment or implication?) in the modal square. This is not as obvious as in the non-modal square, because the deduction theorem does not hold for modal logics.

A:A1 If a property is positive, its negation is not positive:

$$\forall\varphi[P(\varphi) \to \neg P(\neg\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \to \Diamond\exists x\varphi(x)]$$

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

A3' The property of being God-like is positive:

$$P(G_A)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \to \Box\, P(\varphi)]$$

A:D2 An *essence* of an individual is a property that necessarily implies those and only those properties that the individual has necessarily:

$$\varphi\ ess_A\ x \equiv \forall\psi[\Box\psi(x) \leftrightarrow \Box\forall y(\varphi(y) \to \psi(y))]$$

T2' Being God-like is an essence of any God-like being:

$$\forall x[G_A(x) \to G_A\ ess_A\ x]$$

D3' *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE_A(x) \equiv \forall\varphi[\varphi\ ess_A\ x \to \Box\exists y\varphi(y)]$$

A5' Necessary existence is a positive property:

$$P(NE_A)$$

L1' If a god-like being exists, then necessarily a god-like being exists:

$$\exists x G_A(x) \to \Box\exists y G_A(y)$$

L2' If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists x G_A(x) \to \Box\exists y G_A(y)$$

T3' Necessarily, a God-like being exists:

$$\Box\exists x G_A(x)$$

FIGURE 3. Anderson's Emendation [?].

3. Anderson's Emendation

I have meanwhile added another file "Anderson_var_partial.thy". In this version the actualist e-Quantifier is used exclusively in the definition of NE and everywhere else ithe possibilist Quantifiers are used. It is still unclear to me, what is actually meant. Reading Anderson 1990, last sentence on p.

302, it could mean that "Anderson_var_partial.thy" is the correct version; but when reading AndersonGettings, then one could think that "Anderson_var.thy" is meant. But ok we have now both versions. Interestingly in "Anderson_var_partial.thy" A5 is not inferable anymore, and this also holds for "anderson_implies_fuhrmann". Nitpick finds countermodels.

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$$\forall\phi[P(\phi) \to \Diamond\exists x\phi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \leftrightarrow \forall\phi[P(\phi) \to \phi(x)]$$

A3 The property of being God-like is positive:

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D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \leftrightarrow \forall\phi[\phi\ ess\ x \to \Box\exists y\phi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists x G(x)$$

FIGURE 4. Frode's Alternative [**?**].

4. Frode's Alternative

5. Other Solutions

ToDo: Fitting

6. Conclusions

Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (cf. [31, 3, 2, 19, 1, 18]).

In theoretical philosophy, formal logical confrontations with such ontological arguments had been so far (mainly) limited to paper and pen. Up to now, the use of computers was prevented, because the logics of the available theorem proving systems were not expressive enough to formalize the abstract concepts adequately. Gödel's proof uses, for example, a complex higher-order modal logic (HOML) to handle concepts such as *possibility* and *necessity* and to support quantification over individuals and properties.

controversies, care with parameters

ToDo: Leibniz calculemus, Rushby, Zalta [27, 28]

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