Formalization, Mechanization and Automation of Gödel's Proof of God's Existence*

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Attempts to prove the existence (or non-existence) of God by means of abstract ontological arguments are an old tradition in philosophy and theology. Gödel's proof [12] is a modern culmination of this tradition, following particularly the footsteps of Leibniz. Gödel defines God as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable but debatable axioms that they should satisfy. Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (e.g. [17, 18, 2, 11, 1, 10]). Our formalization employs the following axioms (A*) and definitions (D*):[3]

Chris: The reference to Scott's version is kind of confusing, we need to verify this sources.

Any property necessarily implied by a positive property is positive.

$$\forall P \forall Q(pos\, P \wedge (\Box \forall x(P\, x \Rightarrow Q\, x)) \Rightarrow pos\, Q) \tag{A1}$$

A property is positive if and only if its negation is not positive.

$$\forall P(pos\, P \Leftrightarrow \neg(pos\, \neg P)) \tag{A2}$$

The property of being God-like is positive.

$$pos\, god \tag{A3}$$

Positive properties are necessarily positive.

$$\forall P(pos\, P \Rightarrow \Box(pos\, P)) \tag{A4}$$

Necessary existence is a positive property.

$$pos\, nec\_exists \tag{A5}$$

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* This work has been supported by the German Research Foundation under grant BE2501/9-1. Bruno ????

[3] (A1), (A2), (A5), (D1), (D3) are logically equivalent to, respectively, axioms 5, 2 and 4 and definitions 1 and 3 in Gödel's manuscript [12]. (A3) was introduced by Scott [17] and could be derived from Gödel's axiom 1 and (D1) in a logic with infinitary conjunction. (A4) is a weaker form of Gödel's axiom 3. (D2) has an extra conjunct lacking in Gödel's definition 2; this lack is believed to have been an oversight by Gödel [13].

x is God-like if and only if x has every positive property.

$$god\,x := \forall P(pos\,P \Rightarrow P\,x) \tag{D1}$$

A property P is an essence of x if and only if P is a property of x and every property Q that x has is necessarily implied by P.

$$ess\,P\,x := P\,x \wedge \forall Q(Q\,x \Rightarrow \Box\forall y(P\,y \Rightarrow Q\,y)) \tag{D2}$$

x necessarily exists if and only if every essence of x is necessarily exemplified.

$$nec\_exists\,x := \forall P(ess\,P\,x \Rightarrow \Box\exists y(P\,y)) \tag{D3}$$

From these axioms and definitions we then infer:

Positive properties are possibly exemplified.

$$\forall P(pos\,P \Rightarrow \Diamond\exists x(P\,x)) \tag{L1}$$

Possibly God exists.

$$\Diamond\exists x(god\,x) \tag{L2}$$

If x is God-like, then the property of being God-like is an essence of x.

$$\forall x(god\,x \Rightarrow ess\,god\,x) \tag{L3}$$

Necessarily God exists.

$$\Box\exists x(god\,x) \tag{T}$$

The above variant of Gödel's proof has now been constructed for the first-time with an unprecedent degree of detail and formality; cf. [16]. The following has been done (and in this order):

Chris: We need to make sure that this ND proof is based on a sound calculus.

- A detailed natural deduction proof.
- A formalization of the axioms, definitions and theorems in the TPTP THF syntax [19].
- Automatic verification of the consistency of the axioms and definitions with Nitpick [8].
- Automatic demonstration of the theorems with the provers LEO-II [5] and Satallax [9].
- A step-by-step formalization using the Coq proof assistant [6].
- A formalization using the Isabelle proof assistant [15] where the theorems (and some additional lemmata) have been automated with Sledgehammer [7] and Metis [14].

Gödel's proof is challenging to formalize and verify because it requires an expressive logical language with modal operators (*possibily* and *necessarily*) and with quantifiers for individuals and sets of individuals (properties). Our computer-assisted formalizations rely on an embedding of the modal logic S5 into classical higher-order logic with Henkin semantics [4, 3]. The formalization is thus essentially done in classical higher-order logic where quantified S5 is emulated.

This work attests the maturity of contemporary interactive and automated deduction tools for classical higher-order logic and it demonstrates the elegance and practical relevance of the embeddings based approach. Most importantly, our work opens new perspectives for a computer-assisted theoretical philosophy. The critical discussion of the underlying concepts, definitions and axioms remains a human responsibility, but the computer can assist in building and checking rigorously correct logical arguments. In case of logico-philosophical disputes, the computer can check the disputing arguments and partially fulfill Leibniz' dictum: Calculemus — Let us calculate!

Future work includes an extensive study of other formalizations of ontological arguments with our machinery. Variations of the distinctive features of the base logic S5 (e.g. non-rigid symbols, varying domains, etc.) are enabled in these studies due to the flexibility of the embeddings based approach.

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