# Formalization, Mechanization and Automation of Gödel's Proof of God's Existence

Christoph Benzmüller and Bruno Woltzenlogel Paleo

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**Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis**

*Von Tobias Hürter*

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

**Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.**

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany
- Telepolis & Heise
- Spiegel Online
- FAZ
- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
- ...

Austria
- myscience.at
- Wiener Zeitung
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Italy
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- ...

**Def: Ontological Argument/Proof**

* deductive argument
* for the existence of god
* starting from premises, which are justified by pure reasoning, i.e. they do not depend on observation in the world.

Existence of God: different types of arguments/proofs

— a posteriori (use experience/observation in the world)
—— teleological
—— cosmological
—— moral
—— ...
— a priori (based on pure reasoning, independent)
—— ontological argument
——— definitional
——— modal
——— ...
—— other a priori arguments

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*Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis*

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)

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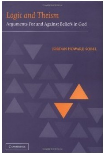


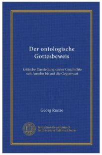


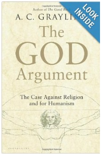


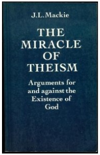


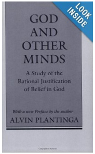


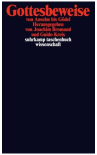


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Rich history on ontological arguments (pros and cons)

… Anselm v. C. … Th. Aquinas …… Spinoza Leibniz … Hume Kant … Hegel … Frege … Hartshorne Malcolm Lewis Plantinga Gödel …

Anselm's notion of God:

*"God is that, than which nothing greater can be conceived."*

Gödel's notion of God:

*"A God-like being possesses all 'positive' properties."*

To show by logical reasoning:

*"(Necessarily) God exists."*

# Introduction

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Different Interests in Ontological Arguments:

- Philosophical: Boundaries of Mataphysics & Epistemology
  - We talk about a metaphysical concept (God),
  - but we want to draw a conclusion for the real world.

- Theistic: Successful argument should convince atheists.

- Our: Can computers (theorem provers) be used
  - to formalize the definitions and axioms?
  - to verify the arguments step-by-step?
  - to fully automate (sub-)arguments?

*"Computer-assisted Theoretical Philosophy"*

Main Challenge: No theorem provers for *Second-order Modal Logic*

Our Idea: Exploit an embedding in *Classical Higher-order Logic*

[BenzmüllerPaulson, Logica Universalis, 2013]

What we did (rough outline for remaining presentation!):

A: pen and paper: detailed natural deduction proof

B: formalization: axioms, defs, thms in TPTP THF

C: consistency: automatic verification with NITPICK

D: proof automation: theorems provers Leo-II and SATALLAX

E: step-by-step verification: proof assistant COQ

F: automation & verification: proof assistant ISABELLE

Conclusion

Did we get anything new? . . . Probably yes — let's discuss afterwards!

# Gödel's Manuscript

ToDo: Show Goedel's Manuscript

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \rightarrow \Box\forall x.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3:}\ E(x) \equiv \forall\varphi.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{}{P(G)}\,\textbf{A3} \qquad \dfrac{\dfrac{}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}\,\textbf{A2} \qquad \dfrac{}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}\,\textbf{A1a}}{\textbf{T1:}\ \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1:}\ \Diamond\exists x.G(x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]}\,\textbf{A1b} \qquad \dfrac{}{\forall\varphi.[P(\varphi) \rightarrow \Box\ P(\varphi)]}\,\textbf{A4}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{}{P(E)}\,\textbf{A5}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}\,\textbf{S5}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}}{}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists x.G(x) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}}$$

$$\dfrac{A \lor B \qquad \begin{array}{c}\overline{A}\\ \vdots\\ C\end{array} \qquad \begin{array}{c}\overline{B}\\ \vdots\\ C\end{array}}{C}\ \lor_E \qquad\qquad \dfrac{A \quad B}{A \land B}\ \land_I \qquad\qquad \dfrac{\begin{array}{c}\overline{A}^{\;n}\\ \vdots\\ B\end{array}}{A \to B}\ \to_I^n$$

$$\dfrac{A}{A \lor B}\ \lor_{I_1} \qquad\qquad \dfrac{A \land B}{A}\ \land_{E_1} \qquad\qquad \dfrac{B}{A \to B}\ \to_I$$

$$\dfrac{B}{A \lor B}\ \lor_{I_2} \qquad\qquad \dfrac{A \land B}{B}\ \land_{E_2} \qquad\qquad \dfrac{A \quad A \to B}{B}\ \to_E$$

$$\dfrac{A[\alpha]}{\forall x.A[x]}\ \forall_I \qquad \dfrac{\forall x.A[x]}{A[t]}\ \forall_E \qquad\qquad \dfrac{A[t]}{\exists x.A[x]}\ \exists_I \qquad \dfrac{\exists x.A[x]}{A[\beta]}\ \exists_E$$

$$\neg A \equiv A \to \bot \qquad\qquad \dfrac{\neg\neg A}{A}\ \neg\neg_E$$

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\dfrac{\dfrac{\mathbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}}{\forall\psi.[(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}\ \forall_E
}{(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \neg\rho(x)]) \rightarrow P(\neg\rho)}\ \forall_E
}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \rightarrow P(\neg\rho)}
\qquad
\dfrac{\dfrac{\mathbf{A1a}}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{P(\neg\rho) \rightarrow \neg P(\rho)}\ \forall_E
}{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \rightarrow \neg P(\rho)}
}{
\dfrac{P(\rho) \rightarrow \Diamond\exists x.\rho(x)}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}\ \forall_I
}
$$

$$
\dfrac{
\dfrac{\mathbf{A3}}{P(G)}
\qquad
\dfrac{\dfrac{\mathbf{T1}}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{P(G) \rightarrow \Diamond\exists x.G(x)}\ \forall_E
}{\Diamond\exists x.G(x)}\ \rightarrow_E
$$

$$\dfrac{\alpha : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\;\Box_I \qquad\qquad \dfrac{\Box A}{t : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\;\Box_E$$

$$\dfrac{t : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\;\Diamond_I \qquad\qquad \dfrac{\Diamond A}{\beta : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\;\Diamond_E$$

$$\Diamond A \equiv \neg A$$

# Natural Deduction Proofs
## Part of T2

$$
\dfrac{
\dfrac{\psi(x)^6 \qquad \dfrac{\Pi_2}{\psi(x) \to \Box P(\psi)}}{\Box P(\psi)}\;{\to_E}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\;\Box_E \qquad \dfrac{\Pi_3}{P(\psi) \to \forall x.(G(x) \to \psi(x))}}{\forall x.(G(x) \to \psi(x))}\;{\to_E}}{\Box\forall x.(G(x) \to \psi(x))}\;\Box_I}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))}\;{\to_I^7}
}{\dfrac{\Box\forall x.(G(x) \to \psi(x))}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}\;{\to_I^6}}\;{\to_E}
$$

# Embedding, Application of THF Provers

todo

# Coq Proof

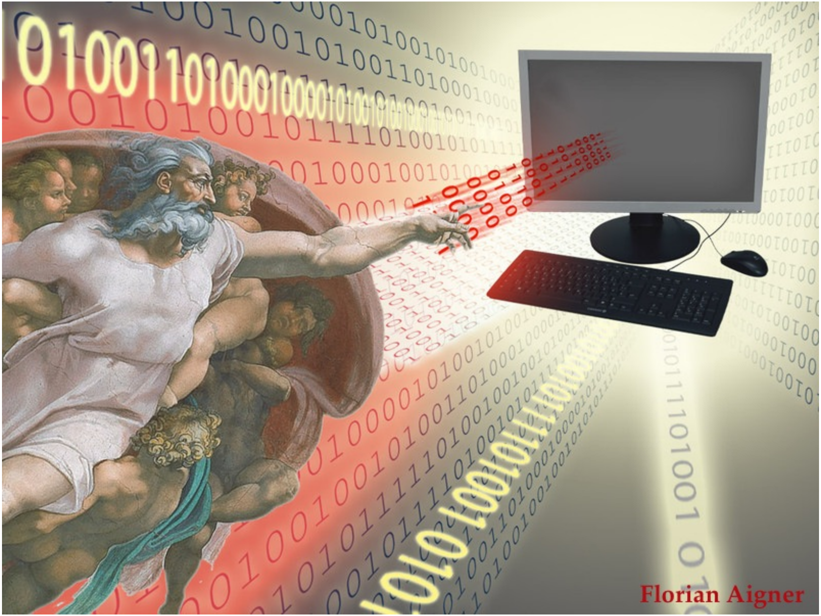
todo

# Isabelle Proof

todo

# Conclusion

todo


Florian Aigner