Gödel's Ontological Proof of God's Existence

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Possible witnessing of positive properties

Axioms:

- **(1)** Properties necessarily entailed by *positive* properties are also positive:

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

- **(2)** A property's negation is positive iff the property is not positive:

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Theorem 1: Positive properties possibly have a witness:

$$\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$$

Formal proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}{\forall\psi.[(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}}{(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \rightarrow \neg\varphi'(x)]) \rightarrow P(\neg\varphi')}}{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \rightarrow P(\neg\varphi')} \qquad \dfrac{\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}{P(\neg\varphi') \leftrightarrow \neg P(\varphi')}}{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \rightarrow \neg P(\varphi')}}{P(\varphi') \rightarrow \Diamond\exists x.\varphi'(x)}}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}$$

2 Possible existence of a God

Axioms:

- **(3)** Being God is a positive property:

$$P(G)$$

Theorem 2: It is possible that a God exists:

$$\Diamond\exists x.G(x)$$

Formal proof:

$$\dfrac{P(G) \qquad \dfrac{\dfrac{}{\forall\varphi.[P(\varphi)\to\Diamond\exists x.\varphi(x)]}\,\text{Th. 1}}{P(G)\to\Diamond\exists x.G(x)}}{\Diamond\exists x.G(x)}$$

3 Essentiality of being God

Definitions:

- A property is *essential* for an individual if and only if it holds for that inidividual and necessarily entails every other property that holds for that individual: $\varphi\ ess\ x \leftrightarrow \varphi(x)\wedge\forall\psi.(\psi(x)\to\Box\forall x.(\varphi(x)\to\psi(x)))$

Axioms:

- **(4)** Positive properties are necessarily positive:

$$\forall\varphi.[P(\varphi)\to\Box\ P(\varphi)]$$

Theorem 3: If an individual is a God, then being God is an essential property for that individual:

$$\forall y.[G(y)\to G\ ess\ y]$$

Formal proof:

Let the following derivation with the open assumption $G(x)$ be Π_1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(\psi)^1 \qquad \dfrac{\dfrac{}{\forall\varphi.(\neg P(\varphi)\to P(\neg\varphi))}\,\text{Ax. 2}}{\neg P(\psi)\to P(\neg\psi)}\,\forall\text{E}}{P(\neg\psi)}\,\to\text{E} \qquad \dfrac{\dfrac{G(x)}{\forall\varphi.(P(\varphi)\to\varphi(x))}\,\text{Definition of G}}{P(\varphi)\to\varphi(x)}\,\forall\text{E}}{\neg\psi(x)}\,\to\text{E} \qquad \psi(x)^2}{\bot}\,\to\text{E}}{P(\psi)}\,\text{RAA, 1}}{\psi(x)\to P(\psi)}\,\to\text{I, 2}}{}$$

Let the following derivation with the open assumption $G(x)$ be Π_2:

$$\dfrac{\dfrac{\psi(x)^1 \qquad \begin{array}{c}\Pi_1\\ \hdashline \psi(x)\to P(\psi)\end{array}}{P(\psi)}\,{\to}\text{E} \qquad \dfrac{\dfrac{}{\forall\psi.(P(\psi)\to\Box P(\psi))}\,\text{Ax. 4}}{P(\psi)\to\Box P(\psi)}\,\forall\text{E}}{\dfrac{\Box P(\psi)}{\psi(x)\to\Box P(\psi)}\,{\to}\text{I, 1}}\,{\to}\text{E}$$

Let the following derivation without open assumptions be Π_3:

$$\dfrac{\dfrac{\dfrac{\dfrac{P(\psi)^1 \qquad \dfrac{\begin{array}{c}G(x)^2\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ \forall\varphi.(P(\varphi)\to\varphi(x))\end{array}\ \text{Definition of G}}{P(\psi)\to\psi(x)}\,\forall\text{E}}{\psi(x)}\,{\to}\text{E}}{G(x)\to\psi(x)}\,{\to}\text{I, 2}}{\forall x.(G(x)\to\psi(x))}\,\forall\text{I}}{P(\psi)\to\forall x.(G(x)\to\psi(x))}\,{\to}\text{I, 1}$$

Let the following derivation with the open assumption $G(x)$ be Π_4:

$$\dfrac{\dfrac{\dfrac{\psi(x)^1 \qquad \begin{array}{c}\Pi_2\\ \hdashline \psi(x)\to\Box P(\psi)\end{array}}{\Box P(\psi)}\,{\to}\text{E} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^2}{P(\psi)}\,\Box\text{E} \qquad \begin{array}{c}\Pi_3\\ \hdashline P(\psi)\to\forall x.(G(x)\to\psi(x))\end{array}}{\forall x.(G(x)\to\psi(x))}\,{\to}\text{E}}{\Box\forall x.(G(x)\to\psi(x))}\,\text{Necessitation}}{\Box P(\psi)\to\Box\forall x.(G(x)\to\psi(x))}\,{\to}\text{I, 2}}{\Box\forall x.(G(x)\to\psi(x)}\,{\to}\text{E}}{\psi(x)\to\Box\forall x.(G(x)\to\psi(x))}\,{\to}\text{I, 1}$$

The use of the necessitation rule is valid, because the only open assumption $\Box P(\psi)$ is boxed.

We construct a derivation of theorem 3 with a subderivation $\Pi_4[G(x)^1]$, which means that the open assumption $G(x)$ in Π_4 is discharged with the rule labeled 1.

$$\dfrac{\dfrac{\begin{array}{c}\dfrac{G(x)^1 \qquad \dfrac{\begin{array}{c}\Pi_4[G(x)^1]\\ \hdashline \psi(x)\to\Box\forall x.(G(x)\to\psi(x))\end{array}}{\forall\psi.(\psi(x)\to\Box\forall x.(G(x)\to\psi(x)))}\,\forall\text{I}}{G(x)\&\forall\psi.(\psi(x)\to\Box\forall x.(G(x)\to\psi(x)))}\,\&\text{I}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ \text{Definition of ess}\\ G\ ess\ x\end{array}}{G(x)\to G\ ess\ x}\,{\to}\text{I, 1}}{\forall y.[G(y)\to G\ ess\ y]}$$

Note: the formal proof above uses the necessitation rule of the basic modal logic **K** and Axiom 4, which corresponds to a restricted form of necessitation.

4 Necessity of God's existence

Definitions:

- An individual is a *God* if and only if he possesses all positive properties:

$$G(x) \leftrightarrow \forall \varphi.[P(\varphi) \rightarrow \varphi(x)]$$

- An individual *necessarily exists* if and only if all its essential properties are necessarily witnessed:

$$E(x) \leftrightarrow \forall \varphi.[\varphi \ ess \ x \rightarrow \Box \exists x.\varphi(x)]$$

Axioms:

- **(5)** Necessary existence is a positive property:

$$P(E)$$

Theorem A: If there is a God, then there necessarily exists a God:

$$\exists z.G(z) \rightarrow \Box \exists x.G(x)$$

Formal proof:

$$\dfrac{\overline{\exists z.G(z)}\ 1}{G(g)}$$

$$\dfrac{\dfrac{\dfrac{G(g) \qquad \dfrac{\forall y.[G(y) \rightarrow G \ ess \ y]\ \text{Th. 3}}{G(g) \rightarrow G \ ess \ g}}{G \ ess \ g} \qquad \dfrac{\dfrac{\dfrac{P(E) \qquad \dfrac{\dfrac{G(g)}{\forall \varphi.[P(\varphi) \rightarrow \varphi(g)]}}{P(E) \rightarrow E(g)}}{E(g)}}{\forall \varphi.[\varphi \ ess \ g \rightarrow \Box \exists x.\varphi(x)]}}{G \ ess \ g \rightarrow \Box \exists x.G(x)}}{\Box \exists x.G(x)}}{\exists z.G(z) \rightarrow \Box \exists x.G(x)}\ 1$$

Note: Theorem A could be proved more quickly using the necessitation rule. Interestingly, the proof above shows that, by using the given axioms and definitions of god and necessary existence, theorem A can be derived even without the necessitation rule.

5 Necessary existence of a God

Theorem 4: The existence of a God is necessary:

$$\Box\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{\dfrac{}{\forall\varphi.[\Diamond\ldots\Diamond\Box\varphi \leftrightarrow \Box\varphi]}\ \mathbf{S5}}{\Diamond\Box\exists x.G(x) \leftrightarrow \Box\exists x.G(x)} \qquad \dfrac{\dfrac{}{\Diamond\exists x.G(x)}\ \text{Th. 2} \qquad \dfrac{}{\exists z.G(z) \rightarrow \Box\exists x.G(x)}\ \text{Th. A}}{\Diamond\Box\exists x.G(x)}}{\Box\exists x.G(x)}$$

Note: The proof above relies on a theorem of the modal logic **S5**, which is a quite strong modal logic. It would be interesting to try to derive theorem 4 with weaker modal axioms.

6 God's existence

Axioms:

- **(M)** What is necessary is the case:

$$\forall\varphi.[\Box\varphi \rightarrow \varphi]$$

Theorem: There exists a God:

$$\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{}{\Box\exists x.G(x)}\ \text{Th. 4} \qquad \dfrac{\forall\varphi.[\Box\varphi \rightarrow \varphi]}{\Box\exists x.G(x) \rightarrow \exists x.G(x)}}{\exists x.G(x)}$$