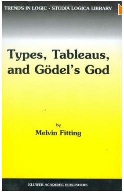
Formalization, Mechanization and Automation of Gödel's Proof of God's Existence

Christoph Benzmüller and **Bruno Woltzenlogel Paleo**

November 1, 2013



$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{P(G) \rightarrow \Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\rightarrow_E$$

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Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

Drucken | Versenden | Merken

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Germany
- Telepolis & Heise
- Spiegel Online
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- Die Welt
- Berliner Morgenpost
- Hamburger Abendpost
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Austria
- myscience.at
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Def: Ontological Argument/Proof

* deductive argument
* for the existence of god
* starting from premises, which are justified by pure reasoning, i.e. they do not depend on observation in the world.

Existence of God: different types of arguments/proofs

- a posteriori (use experience/observation in the world)
  - teleological
  - cosmological
  - moral
  - ...
- a priori (based on pure reasoning, independent)
  - ontological argument
    - definitional
    - modal
    - ...
  - other a priori arguments

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Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)

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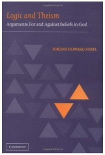

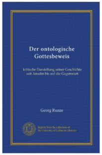

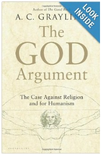

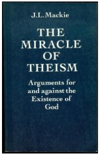

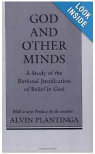

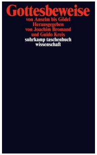

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Rich history on ontological arguments (pros and cons)

... Anselm v. C. ... Th. Aquinas ...... Descartes Spinoza Leibniz ... Hume Kant ... Hegel ... Frege ... Hartshorne Malcolm Lewis Plantinga Gödel ...

Anselm's notion of God:

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"(Necessarily) God exists."

Introduction

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Different Interests in Ontological Arguments:

- Philosophical: Boundaries of Metaphysics & Epistemology
  - We talk about a metaphysical concept (God),
  - but we want to draw a conclusion for the real world.
  - Necessary Existence (NE): metaphysical NE vs. logical NE vs. modal NE
- Theistic: Successful argument should convince atheists.
- Our: Can computers (theorem provers) be used
  - to formalize the definitions and axioms?
  - to verify the arguments step-by-step?
  - to fully automate (sub-)arguments?

"Computer-assisted Theoretical Philosophy"

Main challenge: No theorem provers for *Higher-order Modal Logic*

Our idea: Exploit an embedding in *Higher-order Classical Logic*

[BenzmüllerPaulson, Logica Universalis, 2013]

What we did (rough outline for remaining presentation!):

A: pen and paper: detailed natural deduction proof

B: formalization: axioms, defs, thms in TPTP THF

C: consistency: automatic verification with NITPICK

D: proof automation: theorems provers Leo-II and SATALLAX

E: step-by-step verification: proof assistant COQ

F: automation & verification: proof assistant ISABELLE

Did we get new results? Yes — let's discuss later!

ToDo: Show Goedel's Manuscript

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D2: } \varphi\ ess\ x \equiv \varphi(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

$$\textbf{D3: } E(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{}{P(G)}\,\textbf{A3} \qquad \dfrac{\dfrac{}{\forall\varphi.\forall\psi.[(P(\varphi) \land \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}\,\textbf{A2} \qquad \dfrac{}{\forall\varphi.[P(\neg\varphi) \to \neg P(\varphi)]}\,\textbf{A1a}}{\textbf{T1: } \forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}}{\textbf{C1: } \Diamond\exists x.G(x)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\forall\varphi.[\neg P(\varphi) \to P(\neg\varphi)]}\,\textbf{A1b} \qquad \dfrac{}{\forall\varphi.[P(\varphi) \to \Box\, P(\varphi)]}\,\textbf{A4}}{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y]} \qquad \dfrac{}{P(E)}\,\textbf{A5}}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}\,\textbf{S5}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists x.G(x) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: }}$$

Natural Deduction Calculus

$$\dfrac{A \vee B \quad \begin{matrix}\overline{A}\\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overline{B}\\ \vdots \\ C\end{matrix}}{C}\ \vee_E \qquad \dfrac{A \quad B}{A \wedge B}\ \wedge_I \qquad \dfrac{\begin{matrix}\overline{A}^{\,n}\\ \vdots \\ B\end{matrix}}{A \rightarrow B}\ \rightarrow_I^n$$

$$\dfrac{A}{A \vee B}\ \vee_{I_1} \qquad \dfrac{A \wedge B}{A}\ \wedge_{E_1} \qquad \dfrac{B}{A \rightarrow B}\ \rightarrow_I$$

$$\dfrac{B}{A \vee B}\ \vee_{I_2} \qquad \dfrac{A \wedge B}{B}\ \wedge_{E_2} \qquad \dfrac{A \quad A \rightarrow B}{B}\ \rightarrow_E$$

$$\dfrac{A[\alpha]}{\forall x.A[x]}\ \forall_I \qquad \dfrac{\forall x.A[x]}{A[t]}\ \forall_E \qquad \dfrac{A[t]}{\exists x.A[x]}\ \exists_I \qquad \dfrac{\exists x.A[x]}{A[\beta]}\ \exists_E$$

$$\neg A \equiv A \rightarrow \bot \qquad \dfrac{\neg\neg A}{A}\ \neg\neg_E$$

$$
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  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\mathbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi)\wedge\Box\forall x.[\varphi(x)\rightarrow\psi(x)])\rightarrow P(\psi)]}
        }{\forall\psi.[(P(\rho)\wedge\Box\forall x.[\rho(x)\rightarrow\psi(x)])\rightarrow P(\psi)]}\,\forall_E
      }{(P(\rho)\wedge\Box\forall x.[\rho(x)\rightarrow\neg\rho(x)])\rightarrow P(\neg\rho)}\,\forall_E
    }{(P(\rho)\wedge\Box\forall x.[\neg\rho(x)])\rightarrow P(\neg\rho)}
    \qquad
    \dfrac{
      \dfrac{\mathbf{A1a}}{\forall\varphi.[P(\neg\varphi)\rightarrow\neg P(\varphi)]}
    }{P(\neg\rho)\rightarrow\neg P(\rho)}\,\forall_E
  }{
    \dfrac{
      \dfrac{(P(\rho)\wedge\Box\forall x.[\neg\rho(x)])\rightarrow\neg P(\rho)}{P(\rho)\rightarrow\Diamond\exists x.\rho(x)}
    }{\forall\varphi.[P(\varphi)\rightarrow\Diamond\exists x.\varphi(x)]}\,\forall_I
  }
}{}
$$

$$
\dfrac{
  \dfrac{\mathbf{A3}}{P(G)}
  \qquad
  \dfrac{
    \dfrac{\mathbf{T1}}{\forall\varphi.[P(\varphi)\rightarrow\Diamond\exists x.\varphi(x)]}
  }{P(G)\rightarrow\Diamond\exists x.G(x)}\,\forall_E
}{\Diamond\exists x.G(x)}\,\rightarrow_E
$$

Natural Deduction Calculus
Rules for Modalities

$$\frac{\alpha : \boxed{\begin{array}{c}\vdots \\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \frac{\Box A}{t : \boxed{\begin{array}{c}A \\ \vdots\end{array}}}\ \Box_E$$

$$\frac{t : \boxed{\begin{array}{c}\vdots \\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \frac{\Diamond A}{\beta : \boxed{\begin{array}{c}A \\ \vdots\end{array}}}\ \Diamond_E$$

$$\Diamond A \equiv \neg A$$

$$
\dfrac{\dfrac{\dfrac{\psi(x)^6 \qquad \dfrac{\Pi_2}{\psi(x)\rightarrow\Box P(\psi)}}{\Box P(\psi)}\rightarrow_E \qquad \dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\Box_E \qquad \dfrac{\Pi_3}{P(\psi)\rightarrow\forall x.(G(x)\rightarrow\psi(x))}}{\forall x.(G(x)\rightarrow\psi(x))}\rightarrow_E}{\Box\forall x.(G(x)\rightarrow\psi(x))}\Box_I}{\Box P(\psi)\rightarrow\Box\forall x.(G(x)\rightarrow\psi(x))}\rightarrow_I^7}{\Box\forall x.(G(x)\rightarrow\psi(x))}\rightarrow_E}{\psi(x)\rightarrow\Box\forall x.(G(x)\rightarrow\psi(x))}\rightarrow_I^6
$$

Embedding, Application of THF Provers

todo

- Goal: verification of the natural deduction proof
  - Step-by-step formalization
  - Almost no automation (intentionally!)
- Interesting facts to note:
  - Embedding is transparent to the user
  - Embedding gives labeled calculus for free

todo

Summary of Results

Conclusion

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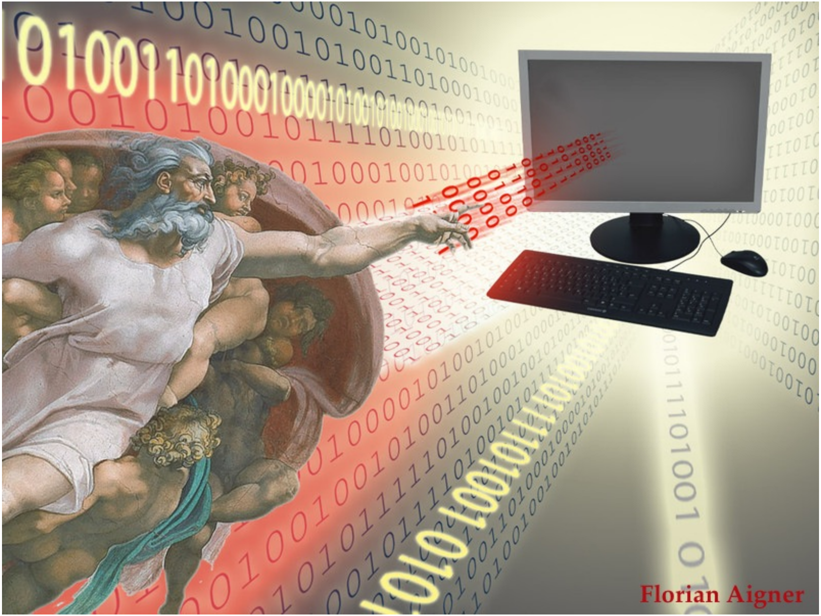
Florian Aigner