Formalization, Mechanization and Automation of Gödel's Proof of God's Existence*

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Attempts to prove the existence (or non-existence) of God by means of abstract ontological arguments are an old tradition in philosophy and theology. Gödel's proof [12, 13] is a modern culmination of this tradition, following particularly the footsteps of Leibniz. Gödel defines God as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable (but debatable) axioms that they should satisfy. Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof (cf. [19, 2, 11, 1, 10]).

Dana Scott's version of Gödel's proof [18] employs the following axioms (**A**), definitions (**D**), corollaries (**C**) and theorems (**T**), and it proceeds in the following order:[3]

A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$
D1	A *God-like* being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists x G(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box\, P(\phi)]$
D2	An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G\ ess.\ x]$
D3	*Necessary existence* of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists x G(x)$

The above version of Gödel's proof has now been analysed for the first-time with an unprecedent degree of detail and formality; cf. [17]. The following has been done (and in this order):

- A detailed natural deduction proof.
- A formalization of the axioms, definitions and theorems in the TPTP THF syntax [20].
- Automatic verification of the consistency of the axioms and definitions with Nitpick [8].
- Automatic demonstration of the theorems with the provers LEO-II [5] and Satallax [9].
- A step-by-step formalization using the Coq proof assistant [6].
- A formalization using the Isabelle proof assistant [16] where the theorems (and some additional lemmata) have been automated with Sledgehammer [7] and Metis [15].

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[3] A1, A2, A5, D1, D3 are logically equivalent to, respectively, axioms 5, 2 and 4 and definitions 1 and 3 in Gödel's notes [12, 13]. A3 was introduced by Scott [18] and could be derived from Gödel's axiom 1 and D1 in a logic with infinitary conjunction. A4 is a weaker formof Gödel's axiom 3. D2 has an extra conjunct $\phi(x)$ lacking in Gödel's definition 2; this is believed to have been an oversight by Gödel [14].

Gödel's proof is challenging to formalize and verify because it requires an expressive logical language with modal operators (*possibly* and *necessarily*) and with quantifiers for individuals and properties. Our computer-assisted formalizations rely on an embedding of the modal logic into classical higher-order logic with Henkin semantics [4, 3]. The formalization is thus essentially done in classical higher-order logic where quantified modal logic is emulated.

In our ongoing computer-assisted study of Gödel's proof we have obtained the following results:

- The basic modal logic K is sufficient for proving T1, C and T2.
- Modal logic S5 is not needed for proving T3; the logic KB is sufficient.
- Without the first conjunct $\phi(x)$ in D2 the set of axioms and definitions would be inconsistent.
- For proving theorem T1, only the left to right direction of axiom A1 is needed. However, the backward direction of A1 is required for proving T2.

This work attests the maturity of contemporary interactive and automated deduction tools for classical higher-order logic and demonstrates the elegance and practical relevance of the embeddings-based approach. Most importantly, our work opens new perspectives for a computer-assisted theoretical philosophy. The critical discussion of the underlying concepts, definitions and axioms remains a human responsibility, but the computer can assist in building and checking rigorously correct logical arguments. In case of logico-philosophical disputes, the computer can check the disputing arguments and partially fulfill Leibniz' dictum: Calculemus — Let us calculate!

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