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Richard Tieszen

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To the memory of Hao Wang (1921–1995)

Preface

In his book *A Logical Journey: From Gödel to Philosophy*, Hao Wang writes that "In his philosophy Gödel tried to combine and go beyond the main contributions of his three heroes: Plato, Leibniz, and Husserl" (Wang 1996, p. 289). Elaborating somewhat, he says that

> Before 1959 Gödel had studied Plato, Leibniz, and Kant with care: his sympathies were with Plato and Leibniz. Yet he felt he needed to take Kant's critique of Leibniz seriously and find a way to meet Kant's objections to rationalism . . . It seems likely that, in the process of working on his Carnap paper in the 1950s, Gödel had realized that his realism about the conceptual world called for a more solid foundation than he then possessed. At this juncture [1959] it was not surprising for him to turn to Husserl's phenomenology, which promises a general framework for justifying certain fundamental beliefs that Gödel shared. (Wang 1996, p. 164)

In the mid 1980s I began a series of discussions with Wang about Gödel's philosophical and technical work. Our exchanges continued for several years. I mention this fact because the remarks just quoted already figured into our discussions, and because the structure of the present book is shaped by the general framework outlined in these passages. What I want to do in this book is to take Wang's report about Gödel's three philosophical heroes seriously and to set out some ideas about how to combine and to go beyond the contributions of Plato, Leibniz, and Husserl, based on Gödel's philosophical and technical writings, his comments to Hao Wang, and various items from the Gödel *Nachlass*. Kant also figures into the mix in an important way because, as Wang notes, Gödel wanted to be able to respond to Kant's objections to rationalism. Kant's transcendental philosophy lies in the background of Gödel's interest in Husserl's transcendental phenomenology. By 1961 or thereabouts, Gödel says that it is Husserl's (transcendental) phenomenology that for the first time does justice to the core of Kant's philosophy (Gödel *1961/?, p. 387). Gödel also discusses Kant's views in a number of his earlier writings on the foundations of mathematics and logic, usually to object to the restriction of the notion of intuition to sensory intuition, or to the fact that it is not possible to recognize the objectivity of abstract entities of mathematics or logic in Kant's system. Gödel's papers on relativity theory, time, and idealism are of course also directly related to Kant. As Wang says, Gödel wanted to use Plato, Leibniz, and Husserl in a positive way and to use Kant in a "mixed way" (Wang 1996, p. 327).

I think there can be little doubt that in his later philosophical writing, and perhaps even as early as 1925, Gödel favored some form of platonic rationalism. A central point of this book is to develop a defensible version of Gödelian platonic rationalism, based on Gödel's own philosophical views, and especially his views about using Husserl's transcendental phenomenology to provide a general framework for supporting a type of platonic realism and Leibnizian rationalism about logic and mathematics. In the first chapter of the book I indicate how particular elements in the philosophy of Plato and Leibniz are combined in the work of Husserl that was attractive to Gödel. The combination that results is supported in its general form in chapter 1 by a host of quotations and other textual material from Gödel, Wang, and Husserl. I do not claim that the line of argument I develop to defend a type of Gödelian platonic rationalism reflects all of the elements of Gödel's own view. There are materials in the *Nachlass* that have not yet been transcribed from Gödel's Gabelsberger shorthand and these might shed more light on exactly what Gödel's views were (see Dawson and Dawson 2005). Many of Gödel's remarks about combining and going beyond Plato, Leibniz, and Husserl are pointers. They indicate a path, even if they are short on details. Based on my discussions with Wang, I believe that in this book I am headed in the direction toward which Gödel pointed but I also note some possible lines of divergence along the way. The reason I do not follow up on the alternative paths that I sometimes indicate is always the same: I want to develop a *defensible* version of platonic rationalism that is nonetheless informed by a close reading of Gödel's work on the foundations of mathematics and logic. If the position that results turns out to diverge from particular claims to which Gödel was clearly committed then so be it. I am content to let the arguments stand for themselves and to pursue the path for which the stronger arguments can be made. It is for this reason that my book is entitled *After Gödel*.

It should perhaps be emphasized that it is not the point of *After Gödel* to show that Gödel himself managed to develop Husserl's ideas or to synthesize into a whole the ideas in Plato, Leibniz, Kant, and Husserl in which he was interested. Rather, the point is that I want to develop these ideas on the basis of close readings and detailed analyses and discussions of all of Gödel's central texts on the philosophy and foundations of mathematics, showing various important links between these texts along the way and, as already mentioned, supplementing the treatment with citations from the Gödel *Nachlass* and Hao Wang's books on Gödel. I wish not only to establish a basis in the texts for what I do later in the book but, in so doing, to also inform those readers who are not familiar with Gödel's relevant writings or the supplementary materials. It seemed to me to be a priority in writing the book to start with Gödel's own texts, many of which have been published only in recent years, since the 1990s, and to then develop a position in the philosophy of mathematics and logic on this basis, *after* Gödel. There is no other book that does these things.

Among the other novel or noteworthy features of the book that might be mentioned are the following: Gödel's incompleteness theorems are discussed in connection with his interest in Husserl's phenomenology, with the idea of showing how this connection can be systematically linked to Gödel's critique of Hilbert, Carnap, and Logical Positivism. This material is, in turn, related to Gödel's interest in the Husserlian and Leibnizian ideas of philosophy as rigorous science. All of these points are woven together with Gödel's views on minds and machines, and with a discussion of his views about extrinsic justification of axioms and the relation of such views to holism about mathematics. One of the highlights of the book perhaps lies in how it brings together "monads" with a type of platonism, and argues for a solution to the plight of the platonist on these grounds. The argument against a mechanistic view of reason in chapter 7 is novel in its use of genetic analysis to undermine the identification of human minds with machines.

I would also like to make it clear from the outset that my focus throughout the book will not be on platonic rationalism in general, but only on platonic rationalism about mathematics and logic. Attempting to defend platonic rationalism in areas outside of mathematics and logic would necessitate consideration of many additional issues and could also distract us from some of the unique features of Gödel's thought. Gödel associates his platonic rationalism with his own profound technical work in the foundations of mathematics and logic, especially the incompleteness theorems. This is a new development in the history of platonism and rationalism, and I want to keep it in the foreground in this book.

It will be seen that I steer clear of Gödel's remarks on religion, angels, demons, ghosts, and the like. The previously unpublished papers included in Vol. III of the *Collected Works*, along with many letters that are of scientific significance in Volumes IV and V, the previously published papers in Volumes I and II, and many items in the *Nachlass*, are filled with sober, interesting and deep philosophical arguments. They include no mention of religion, angels, or ghosts. Gödel's short formalized ontological argument for the existence of God, published in Volume III of the *Collected Works* (Gödel *1970), is in a sense an exception but it is of interest as an exercise in logic and rational theology. In any case, I do not discuss it in this book. I think there is a defensible core of ideas in Gödel's philosophical work on mathematics and logic, and it is this core that will emerge in the chapters below. Many of Gödel's comments on religion, immortality of the soul, and similar matters have been widely quoted. There are also comments and certain activities of Gödel's that are just paranoid or bizarre. Readers who are interested in this more "scandalous" material will have to look elsewhere. I think it has little to do with finding a defensible form of platonic rationalism about mathematics and logic.

A final general point that I would like to make in this short preface concerns the recent history of philosophy. That Gödel, who is one of the greatest logicians of all time, should have found Husserl to be the most interesting philosopher since Leibniz, and not thinkers such as Carnap, Wittgenstein, Schlick, and others, speaks volumes about the division that occurred between so-called analytic and Continental philosophy. In spite of the fact that this division has been institutionalized in various professional organizations, university curricula, university hiring practices, and myriad other ways, there are many respects in which it is artificial and should be resisted. I hope that upcoming generations of thinkers will look back with puzzlement on its episodes of rancor, exclusion, and small-mindedness.

I am happy to acknowledge a number of individuals and institutions for their help in making this book possible in its present form. In one way or another I have been thinking about the material in the book since I was graduate student. The impetus for this came especially from Hao Wang. Charles Parsons, whose breadth and depth of knowledge in the philosophy of mathematics is exemplary, also offered encouragement at an early stage. Several visits to the Gödel *Nachlass* in the mid 1980s, when it was first made available to scholars, revealed a remarkable wealth of material and were a source of inspiration. I regret that the Head Librarian at the Institute for Advanced Study at the time never responded to my requests for permission to quote from items in the *Nachlass*. In the intervening years I produced many notes and research manuscripts on Gödel, some of which I published. It was only a few years ago that I sat down and began to finally write up the material in a book-length treatment. At various points over the years, however, I benefitted from comments of a number of people on my papers or presentations on Gödel. I thank Aldo Antonelli, Paul Benacerraf, Andreas Blass, Charles Chihara, Steven Crowell, Zlatan Damnjanovic, John Etchemendy, Sol Feferman, Joel Friedman, Michele Friend, Pieranna Garavaso, Jaakko Hintikka, Thomas Hofweber, Jerrold Katz, Juliette Kennedy, Lila Luce, Paolo Mancosu, John McCarthy, Grisha Mints, J.N. Mohanty, Izchak Miller, Mike Resnik, Gian-Carlo Rota, Tom Ryckman, Wilfried Sieg, David Stump, Amie Thomasson, Robert Tragesser, Johan van Bentham, Albert Visser, Richard Zach, and Ed Zalta. To Dagfinn Føllesdal I owe a special debt of gratitude for reading and commenting on the entire manuscript of this book.

Parts of the manuscript have been presented in the past few years at various meetings and conferences, starting with the colloquium *Kurt Gödel: les textes*, held at the Université Lille, France, May 2006, organized by Mark van Atten and Pierre Cassou-Noguès. A section of chapter 7 was presented at this meeting, and exchanges with the organizers and with Gabriella Crocco were especially helpful. The conference *Phenomenology and Mathematics*, held at the University of Tampere, Finland, May 2007, and organized by Mirja Hartimo, Leila Haaparanta, Sara Heinämaa, and Juliette Kennedy, provided a wonderful opportunity to try out some of the material in chapter 4. Material in chapter 1 was presented at the annual California Phenomenology Circle meeting, San Luis Obispo, April 2008, with David Smith, Dallas Willard, and Jeff Yoshimi offering thoughtful comments. The *Semana de la Fenomenología*, held at Universidad de Puerto Rico, October 2008, and organized by Guillermo Rosado-Haddock, was also very helpful in providing time to discuss issues raised in chapter 1 with Rosado-Haddock, Claire Ortiz Hill, and Jairo da Silva. Further opportunity to discuss this material was afforded at a lecture I gave at Stanford University in April 2009, thanks to Sol Feferman and Grisha Mints.

I was invited by Gerard Jorland at the Ecole des hautes études en sciences sociales (EHESS), Paris, to present a series of three lectures on Gödel in January of 2009. This month-long visit was especially fruitful since I was able to present several parts of the book manuscript for discussion. While in Paris on this occasion I also gave talks on parts of the manuscript in the Séminaire de philosophie et mathématiques, Ecole normale supérieure, at the invitation of Giuseppe Longo, and in the Ecole normale supérieure

Philosophy Department/Husserl Archive at the invitation of Jocelyn Benoist. Exchanges on all of these occasions were very beneficial, and I must thank Jean Petitot in particular for attending many of the lectures in Paris and offering excellent comments and observations. I also presented parts of chapters 1 and 3 in two invited lectures in Göttingen in November 2009, with one lecture at the Lichtenberg-Kolleg and one at Georg August Universität. For comments on the Göttingen talks I am especially grateful to Felix Mühlhölzer, Ulrich Mayer, Juliette Floyd, Aki Kanamori, Christian Beyer, and Kevin Mulligan. I thank members of the audiences at all of these meetings for their questions and comments.

Special thanks are due to Mark van Atten for his insightful comments on my work in recent years and also for providing me with a copy of notes from his research sessions in the Gödel *Nachlass*. John and Cheryl Dawson also deserve special thanks for their comments and assistance with materials in the *Nachlass*.

I am grateful to the Dean of my college at San José State University, Karl Toepfer, for a course reduction grant in Spring 2008 that allowed me to eliminate a course from my heavy teaching load in order to work on the book. Professor Rita Manning graciously continued her service as Chair of my department during the 2008–2009 academic year, sparing me that duty so that I could work on the book manuscript. The members of my Fall 2008 Intermediate Logic class, almost all of whom were in our Master's degree program, were good sports for taking on large parts of the manuscript. Their comments and objections were interesting, often trenchant, and much appreciated. As a National Endowment for the Humanities Fellow in 2006–2007 I was able to take a one-year leave from my teaching duties to work on parts of the book. I am very grateful to the National Endowment for the Humanities for this support.

Peter Momtchiloff of Oxford University Press deserves thanks for his efforts in seeing the book through the publication process. I also thank two anonymous readers of the Press for their comments on the manuscript. I am grateful to Barbara Ball for her careful work in copy-editing the book.

Without the love and support of my wife, Nancy, I am not sure that *After Gödel* would have been completed.

This could have easily been a much longer book. It is, in some ways, an introduction. I am keenly aware of the fact that various points could be elaborated, arguments developed in more detail, and so on. As a finite "monad" (see below) with many obligations, I felt it was best to bring the project to completion in its present form. It is with pleasant thoughts of my discussions with Hao Wang that I dedicate this work to his memory.

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I

Setting the Stage

> Mathematical logic should be used by more non-positivistic philosophers. The positivists have a tendency to represent their philosophy as a consequence of logic—to give it scientific dignity. Other philosophers think that positivism is identical with mathematical logic, which they consequently avoid.
>
> (Kurt Gödel, as reported by Hao Wang, Wang 1996, p. 174)

The logician who conducted and recorded the most extensive philosophical discussions with Kurt Gödel during Gödel's later years was Hao Wang. We know from the work of Wang and others that Gödel's favorite philosophers were Plato, Leibniz, and Husserl. The work of Kant also figures into Gödel's philosophical thinking in a prominent manner. Let me quote some passages from Wang that are, I think, important for indicating, if only very generally, how ideas in the work of Plato, Leibniz, Kant, and Husserl were related in Gödel's thinking. I start with the full quotation of the passage that was cited in part in the preface:

> Before 1959 Gödel had studied Plato, Leibniz, and Kant with care: his sympathies were with Plato and Leibniz. Yet he felt he needed to take Kant's critique of Leibniz seriously and find a way to meet Kant's objections to rationalism. He was not satisfied with Kant's dualism or with his restriction of intuition to sense intuition, which ruled out the possibility of intellectual or categorial intuition. It seems likely that, in the process of working on his Carnap paper in the 1950s, Gödel had realized that his realism about the conceptual world called for a more solid foundation than he then possessed. At this juncture it was not surprising for him to turn to Husserl's phenomenology, which promises a general framework for justifying certain fundamental beliefs that Gödel shared: realism about the conceptual world, the analogy of concepts and mathematical objects to physical objects, the possibility and importance of categorial intuition or immediate conceptual knowledge, and the one-sidedness of what Husserl calls "the naive or natural standpoint." (Wang 1996, p. 164)

Wang says that

> In his philosophy Gödel tried to combine and go beyond the main contributions of his three heroes: Plato, Leibniz and Husserl. Leibniz had defined the ideal by giving a preliminary formulation of monadology. Husserl had supplied the method for attaining this ideal. Plato had proposed, in his rudimentary objectivism in mathematics, an approach that could serve as

foundation for Husserl's method and, at the same time, make plausible for Gödel the crucial belief that we are indeed capable of perceiving the primitive concepts of metaphysics clearly enough to set up the axioms. (Wang 1996, p. 289)

Gödel uses Plato, Leibniz, and Husserl in a positive way, Kant and Hegel in a mixed way, and positivism and Wittgenstein negatively. (Wang 1996, p. 327)

Husserl is the most recent philosopher on Gödel's list of favorites, and it is to Gödel's interest in Husserl that I will especially turn in this book. Reports of Gödel's interest in Husserl have surfaced in many places over the years. Gian-Carlo Rota has written that Gödel believed Husserl to be the greatest philosopher since Leibniz (Kac, Rota, and Schwartz 1986, p. 177). Heinz Pagels has written that "During his later years he [Gödel] continued to pursue foundational questions and his vision of philosophy as an exact science. He became engaged in the philosophy of Edmund Husserl, an outlook that maintained that there is a first philosophy that could be grasped by introspective intuition into the transcendental structure of consciousness—the very ground of being" (Pagels 1988, p. 293). As part of his description, Pagels mentions how Gödel thought it meaningful to ask questions about the truth of axioms, and to consider their philosophical foundations, and he then mentions Gödel's view on mathematical intuition. Georg Kreisel has also noted Gödel's interest in Husserl in his article on Gödel in the *Biographical Memoirs of Fellows of the Royal Society* (Kreisel 1980, pp. 218–219). Hao Wang has remarked, in connection with Gödel's views in "What is Cantor's Continuum Problem?," that "presumably Husserl's elaborate analysis of our perception of a physical object can…be viewed as supporting G[ödel]'s conclusion" (Wang 1987, p. 303) about the objective existence of mathematical objects and about mathematical intuition. He comments in another place that "perhaps Husserl's considerations of *Wesensschau* can be borrowed to support G[ödel]'s belief in the objective existence of mathematical objects" (Wang 1987, p. 304). Also, Charles Parsons has conjectured that it is Husserl's conception of intuition that is the model in Gödel's "What is Cantor's Continuum Problem?" (Parsons 1983a, p. 24).

In this chapter I present an overview of central themes in Gödel's study of Husserl's phenomenology, culled from the books of Hao Wang (Wang 1974, 1987, 1996), my discussions in the 1980s with Hao Wang about Gödel's philosophical interests, and some items from the Gödel *Nachlass*.[1] Many of the items from the *Nachlass* that I will cite are not widely known. In this chapter, unlike later chapters, I will quote extensively from some texts of Wang, Gödel, and Husserl. I do this in order to inform the reader of various ideas and claims but also to indicate the basis in the texts for the

[1] See also Tieszen 2008. For some general background on Gödel see the biography of Dawson 1997. Solomon Feferman's "Gödel Life and Work" (Feferman 1986), John Dawson's "A Gödel Chronology" (Dawson 1986), and Juliette Kennedy's encyclopedia entry (Kennedy 2007) provide good overviews. Historical background in logic and the foundations of mathematics leading up to Gödel's work can be found in Grattan-Guinness 2000. For some interesting and useful discussion of Gödel on platonism and mathematical intuition see Parsons 1995a, and on Gödel and idealism see Parsons 2010.

position I develop in the book. It is not my intention to be exhaustive in describing Gödel's study of Husserl, or his views on Plato, Leibniz, and Kant. There are entries in the philosophical notebooks in Gödel's *Nachlass* that will probably be of interest in this connection but they are still awaiting transcription from the Gabelsberger shorthand used by Gödel. What I will do is to sketch how some of the central ideas in the work of Plato, Leibniz, Kant, and Husserl coalesce in Gödel's later philosophical remarks.

Gödel (1906–1978) is known to have studied philosophy seriously from the early 1940s until the end of his life. He was exposed to the work of Kant fairly early in his studies, at around the age of 16, according to the questionnaire designed by Burke Grandjean that Gödel answered in 1974 (reproduced in Wang 1987, pp. 16–21). We know that he continued to think seriously about Kant's work off and on over many years. Karl Menger (Menger 1994) indicates that Gödel had a serious interest in Leibniz already in the early 1930s, and Gödel indicates in the Grandjean questionnaire that he studied Leibniz intensively from 1943 to 1946. Gödel's work on Leibniz thus antedated his study of Husserl. Gödel was perhaps first acquainted with the philosophy of Plato through the lectures of Heinrich Gomperz on the history of philosophy at the University of Vienna, but it is not clear that he studied Plato in detail. Gödel turned to Husserl's philosophy in 1959 and he continued to study Husserl's work through the 1970s. As Wang notes, Gödel's library includes all of Husserl's major writings, many marked with underlinings and marginal comments and accompanied by inserted pages written mostly in Gabelsberger shorthand. "The Modern Development of Mathematics in the Light of Philosophy" (Gödel ⋆1961/?) is the only text we have thus far in which Gödel explicitly discusses Husserl's philosophy at any length. It is a very interesting text for the manner in which it connects certain ideas in Husserl's transcendental phenomenology to various central theses in Gödel's philosophical views on logic and mathematics. I will discuss this paper in more detail in chapter 3.

The reasons for emphasizing Gödel's interest in Husserl's philosophy are numerous and they will become apparent as the argument of the book is developed. Husserl (1859–1938) was in a position to assess, appropriate, and qualify ideas in the work of Plato, Leibniz, and Kant, and I will note many of the developments in this direction in the pages that follow. The central point, however, is that many of Husserl's ideas lead to more defensible positions since they are informed both by philosophical arguments and by results in mathematics and logic that appeared after Kant.[2]

Husserl's views in foundations, it should be noted, were not informed by Gödel's incompleteness theorems. Gödel's results on incompleteness were published in 1931 when Husserl was in his seventies and was not focused on logic and foundations of mathematics. It has in fact been argued that Husserl's views on "definite" formal systems and their "ontological correlates," definite manifolds, are incompatible with Gödel's incompleteness theorems. I think this matter is complicated by a number of

[2] See also the work of Tragesser on Husserl and mathematics and logic, Tragesser 1977 and 1984.

factors. It is not clear that Husserl thought all formal systems and manifolds had to be definite. He sometimes says, for example, that it is manifolds in the "pregnant sense" that are definite.[3] In any case, in the writings and notes of Gödel that are currently available, there are no reflections on Husserl's specific texts on formal systems and manifolds. Husserl, of course, says in the *Logical Investigations* and other later writings that his idea of pure logic is influenced in part by Leibniz, and he embraces the vision of *mathesis universalis*, which he thinks of as a science of "idealities." Gödel would no doubt find this part of Husserl's view to his liking. The following themes in Husserl's work, which overlap to some extent with ideas in Plato, Leibniz, or Kant, were clearly of interest to Gödel:

1. the idea that philosophy can be a rigorous, universal, a priori science (which is related especially to Gödel's interest in Leibniz and rationalism)[4],
2. transcendental idealism, and the use of the phenomenological method (*epochē*), to develop a new kind of monadology, a monadology that would be aided by phenomenology, but would be combined with
3. a type of platonism that recognizes the objectivity of ideal or abstract objects and concepts of mathematics, logic, and philosophy, and
4. that acknowledges and seeks to cultivate categorial or eidetic intuition of such objects
5. in order to clarify the meaning of primitive concepts of logic and mathematics
6. to ideally be used, in connection with Gödel's technical results, in helping to decide open problems in the foundations of mathematics and logic, especially higher set theory, and also in providing a foundation for the sciences and for philosophy itself.

As we will see, Gödel opposes logical positivism, naturalism, conventionalism, nominalism, and empiricism about logic, mathematics, and philosophy. He argues at length against Carnap's view of mathematics as syntax of language, and against certain aspects of Hilbert's formalism about the foundations of mathematics. He argues against the mechanist conception of the human mind as a Turing machine. We will see below how Gödel connects this last point directly with his study of Husserl's phenomenology. Gödel also argues against subjectivism, psychologism, and Aristotelian realism about the concepts and objects of logic and mathematics.

[3] See, for example, Lohmar 1989. There is a growing literature on Husserl's ideas about "definite" formal systems and definite manifolds. See, for example, da Silva 2000 and 2002, Tieszen 2004, and for more detailed discussion and references Centrone 2010.

[4] Gödel even speaks in some places of the project of finding the basic concepts of metaphysics and of axiomatizing metaphysics. The idea of axiomatizing metaphysics is, fortunately, not completely moribund in our times. See the interesting work of Ed Zalta (*Principia Metaphysica*, URL = http://mally.stanford.edu/publications.html, and Zalta 2002).

§ 1. Gödel on Leibniz, Kant, and Husserl

I would now like to present and discuss some of Gödel's remarks on Leibniz, Kant, and Husserl that will be taken up in more detail in later chapters of this book.

Let us start with a few of Gödel's comments about Leibniz. In remarks related to his discussions with Gödel in the 1970s, Wang (Wang 1996, p. 166) says that "Gödel 's own main aim in philosophy was to develop metaphysics—specifically, something like the monadology of Leibniz transformed into exact theory—with the help of phenomenology." Gödel told Wang (Wang 1996, pp. 55, 288, 309) that he considered Leibniz's monadology close to his own philosophy. We know that there are notes on Leibniz in Gabelsberger shorthand in the Gödel *Nachlass* but what we do not know is exactly which parts of Leibniz's monadology Gödel would or would not have accepted. Was he prepared, for example, to accept Leibniz's view that there are many different kinds of monads? It is worth noting that Gödel read and evidently appreciated the essay *Eine neue Monadologie* (Mahnke 1917) by one of Husserl's students, Dietrich Mahnke (van Atten and Kennedy 2003, p. 457). Mahnke obtained his doctoral degree with Husserl in 1922, writing a thesis entitled *Leibnizens Synthese von Universalmathematik und Individualmetaphysik*. This thesis was published in Husserl's *Jahrbuch für Philosophie und phänomenologische Forschung* in 1925.

Although Gödel was quite interested in some updated version of a monadology that used the methodology of transcendental phenomenology, he thought his own work in the foundations of mathematics (the incompleteness theorems in particular) showed that a mechanist view of reason or of the mind of the kind found in Leibniz's work on logic would have to be abandoned. In his 1939 lectures on logic at Notre Dame, which are in the *Nachlass* but are not yet published, he says about Leibniz's Program that the rules of logic can be applied in a purely mechanical way and therefore it is possible

> to construct a machine that would do the following thing: The supposed machine is to have a crank and whenever you turn the crank once around the machine would write down a tautology of the calculus of predicates and it would write down every existing tautology . . . if you turn the crank sufficiently often. So the machine would really replace thinking completely as far as deriving formulas of the calculus of predicates is concerned. It would be a thinking machine in the literal sense of the word. For the calculus of propositions you can do even more. You could construct a machine in the form of a typewriter such that if you type down a formula of the calculus of propositions then the machine would ring a bell [if the formula is a tautology] and if it is not it would not. You could do the same thing for the calculus of monadic predicates.
>
> (cited in Sieg 2006, pp. 197–198; see also Cassou-Noguès 2009)

Gödel then says that "it is impossible to construct a machine which would do the same thing for the whole calculus of predicates."

> So here already one can prove that Leibniz's program of the "calculemus" cannot be carried through, i.e., one knows that the human mind will never be able to be replaced by a machine already for this comparatively simple question to decide whether a formula is a tautology or not.

In another note in the *Nachlass* (see van Atten and Kennedy 2003, p. 433) he says that "The universal characteristic claimed by Leibniz (1677) does not exist. Any systematic procedure for solving problems of all kinds would have to be nonmechanical." Gödel amended the first sentence of this note to read: "The universal characteristic claimed by Leibniz (1677) if interpreted as a formal system does not exist." For Gödel, however, this did not mean abandoning a rationalistic optimism about solving open problems in mathematics and logic. At the end of his 1944 paper on Russell he says that

> It seems reasonable to suspect that it is this incomplete understanding of the foundations which is responsible for the fact that mathematical logic has up to now remained so far behind the high expectations of Peano and others who (in accordance with Leibniz's claims) had hoped that it would facilitate theoretical mathematics to the same extent as the decimal system of numbers has facilitated numerical computations. For how can one expect to solve problems systematically by mere analysis of the concepts occurring if our analysis so far does not even suffice to set up the axioms? But there is no need to give up hope. Leibniz did not in his writings about the *Characteristica universalis* speak of a utopian project; if we are to believe his words he had developed this calculus of reasoning to a large extent, but was waiting for its publication till the seed could fall on fertile ground. He went even so far as to estimate the time which would be necessary for his calculus to be developed by a few select scientists to such an extent "that humanity would have a new kind of an instrument increasing the powers of reason more than any optical instrument has ever aided the power of vision."
>
> (Gödel 1944, pp. 140–141; see also Parsons 1990)

In fact, Gödel retained a rationalistic optimism about mathematical problem-solving on the basis of analyses of concepts but later he shifted the philosophical foundation for his view from Leibniz and Hilbert to Husserl. The optimism in the later writings is not based on a mechanist conception of reason but rather on a conception of reason that allows for the possibility of finding systematic and finite but non-mechanical methods for the decision of mathematical questions on the basis of clarification of the intuition of the abstract meanings of the terms involved in the problems. The appeals here to the grasp or intuition of meaning, and to the fact that this meaning is "abstract" (connecting meaning with a kind of platonism), are based on Gödel's view of the philosophical consequences of his incompleteness theorems, and they all mirror elements in Husserl's philosophy that were of interest to Gödel.

In addition to what was said about Kant above, I would also like to note here the following points concerning Kant. We know that Gödel was interested in aspects of Kant's transcendental idealism. Gödel connected his own idealistic views on time and relativity theory directly to Kant (Gödel 1949, 1949a, *1946/9), and in his later unpublished 1961 paper on the foundations of mathematics (Gödel *1961/?, p. 387) he speaks about how we can come to a better understanding of some of Kant's important insights on the basis of Husserl's phenomenology. From 1954 to 1959 he corresponded with Gotthard Günther at some length about transcendental philosophy. In a letter written to Günther of 30 June 1954, Gödel says

The reflection on the subject treated in idealistic philosophy…the distinction of levels of reflection, etc., seem to me very interesting and important. I consider it entirely possible, that this is "the" way to the correct metaphysics. However, I cannot go along with the denial of the objective meaning of thought that is connected with it, [although] it is really entirely independent of it. I do not believe that any Kantian or positivistic argument or the antinomies of set theory or quantum mechanics has proved that the concept of objective being (no matter whether for things or abstract entities) is senseless or contradictory. When I say that one can (or should) develop a theory of classes as objectively existing entities, I do indeed mean by that existence in the sense of ontological metaphysics, by which, however, I do not want to say that abstract entities are present in nature. They seem rather to form a second plane of reality, which confronts us just as objectively and independently of our thinking as nature.

(Gödel 1954–1959, in Gödel 2003a, Vol. IV, pp. 502, 504; see also Parsons 2003)

The complaint about Kant in this passage, reflecting Gödel's platonism or objectivism about mathematics, is a recurring theme in Gödel's philosophical writing.

Let us consider for a moment what Kant says about platonism in the *Critique of Pure Reason*. Kant writes (*CPR* A3/B7 – A6/B10) that once we are outside the circle of sense experience we can be sure of not being contradicted by sense experience. The charm of extending our knowledge is so great that nothing short of encountering a direct contradiction can suffice to arrest us in our course. Contradiction can perhaps be avoided if we are careful with the fabrications that occur when we leave behind sense experience, although we are nonetheless still dealing with fabrications. Mathematics, Kant says, gives us a shining example of how far, independently of sense experience, we can progress in a priori knowledge. Misled by such a proof of the power of reason, however, the demand for the extension of knowledge recognizes no limits.

The light dove, cleaving the air in her free flight, and feeling its resistance, might imagine that its flight would be easier still in empty space. It was thus that Plato left the world of the senses, as setting too narrow limits to the understanding, and ventured out beyond it on the wings of the ideas, in the empty space of the pure understanding.

Kant says that Plato did not observe that with all his efforts he made no advance. It is a common fate of human reason to complete its speculative structures as speedily as possible and only afterwards enquire whether the foundations are reliable. Platonic realism, in a word, is unfounded. It is just this kind of claim in Kant's philosophy that Gödel wants to overcome.

Now note, by way of contrast, what Gödel says about Husserl's transcendental idealism in a draft letter of 1972 to Gian-Carlo Rota:

I believe that his [Husserl's] transcendental phenomenology, carried through, would be nothing less than Kant's critique of pure reason transformed into an exact science, except for the fact that [in footnote: Kant's subjectivism and negativism for the most part would be eliminated] the result (of the "critique") would be far more favorable for human reason.

(van Atten and Kennedy 2003, p. 446)

The Kantian critique of reason was clearly too restrictive by Gödel's sights. Husserl agrees that it is too restrictive, as we will see below in a number of quotations in which Husserl portrays the phenomenological method as a way to develop and defend a new kind of rationalism that avoids the excesses of older forms of rationalism but also avoids any kind of mysticism. In a passage of a 1935 letter written to Levy-Bruhl, for example, Husserl says that phenomenology is a "method by which I want to establish, against mysticism and irrationalism, a kind of super-rationalism which transcends the old rationalism as inadequate and yet vindicates its inmost objectives." This letter is quoted in Herbert Speigelberg's *The Phenomenological Movement* (see Spiegelberg 1960, p. 84). Gödel owned a copy of Spiegelberg's book and it is likely that he would have read this passage (see also van Atten and Kennedy 2003, p. 450). These kinds of remarks are to be found in various places in Husserl's writings and they are very much in the spirit of Gödel's outlook.

Let us now turn to some of Gödel's comments about Husserl. Among Wang's own comments on his discussions with Gödel about Husserl are the following:

> For Gödel, the appeal of Husserlian phenomenology was, I think, that it developed the transcendental method in a way that accommodated his own beliefs in intellectual intuition and the reality of concepts. (Wang 1996, p. 165)

> In the 1960s he recommended to some logicians that they should study the sixth investigation in the *Logical Investigations* for its treatment of *categorial intuition*. In his discussions with me in the 1970s he repeatedly urged me to study Husserl's later work. (Wang 1996, p. 164)

> Gödel told me that the most important of Husserl's published works are *Ideas* and *Cartesian Meditations*: "The latter is closest to real phenomenology—investigating how we arrive at the idea of the self." According to Gödel, Husserl just provides a program to be carried out; his *Logical Investigations* is a better example of the execution of this program than is his later work, but it has no correct technique because it still adopts the "natural attitude." (Wang 1996, p. 164)

> I once asked Gödel about Husserl's *Formal and Transcendental Logic*, because I thought it might be more accessible to me than some of the other books. Gödel said that "it is only programmatic: it is suggested that formal logic is objective and transcendental logic is subjective, but the transcendental part—which is meant to give justifications—is rudimentary." (Wang 1996, p. 164)

Wang also recorded in his notes certain direct comments of Gödel on Husserl. I reproduce a few of these here, in order to refer to them in the analysis that follows in this chapter and in later chapters:

> Husserl's is a very important method as an entrance into philosophy, so as to finally arrive at some metaphysics. Transcendental phenomenology with *epochē* as its methodology is the investigation (without knowledge of scientific facts) of the cognitive process, so as to find out what really appears to be—to find the objective concepts. (Wang 1996, p. 166)

> Husserl used Kant's terminology to reach, for now, the foundations and, afterwards, used Leibniz to get the world picture. Husserl reached the end, arrived at the science of metaphysics. [Wang notes that this last sentence is different from what Gödel said on other occasions.] (Wang 1996, p. 166)

> Some reductionism is right: reduce to concepts and truths, but not to sense perceptions. Really it should be the other way around: Platonic ideas [Wang includes: what Husserl calls "essences" and Gödel calls "concepts"] are what things are to be reduced to. Phenomenology makes them [the ideas] clear. (Wang 1996, p. 167)

> Leibniz believed in the ideal of seeing the primitive concepts clearly and distinctly. When Husserl affirmed our ability to "intuit essences" he had in mind something like what Leibniz believed. (Wang 1996, p. 168)

Among other things, these comments of Gödel and Wang indicate that it is Husserl's *transcendental* phenomenology, with its *epochē* (= phenomenological reduction), that is of most interest to Gödel. Gödel mentions the *epochē* explicitly in one of the comments cited above. Here is another comment that Gödel makes about the *epochē*:

> Introspection is an important component of thinking; today it has a bad reputation. Introspective psychology is completely overlooked today. *Epochē* concerns how introspection should be used, for example, to detach oneself from influences of external stimuli (such as fashions of the day). (Wang 1996, p. 169)

I discussed Gödel's references to introspection with Wang, since I was worried about various objections that have been raised against introspection as a source of knowledge. It is my impression that when Gödel spoke of introspection in connection with the *epochē* that what he had in mind was just the kind of turning of regard that Husserl in various writings takes to be characteristic of transcendental, eidetic phenomenology. Without going into details about the *epochē* here (see chapter 4) I will only note for now that with the *epochē* we are supposed to suspend or "bracket" the "natural attitude," that is, the ordinary assumption of the existence of the world around us (see, e.g. Husserl 1907, 1913, 1923–24a, b). The suspension applies also to the sciences, including psychology, that assume the existence of the objects they study. The point of such a suspension is to shift attention away from the *objects* and facts in any domain to *consciousness of the objects* and facts. This shift from focusing on objects to the consciousness of objects seems, at least loosely, like introspection, but it is necessary to be very careful about this. It is important to distinguish what Husserl has in mind from (empirical) introspectionist psychology. There is a tradition of thinking of introspection as "inner sense," analogous in some ways to "outer sense." Outer sense, that is, the deliverances of the five senses that put us in touch with things external to us, gives us particulars. Similarly, inner sense is supposed to give us particulars about our own mental lives. It reports about private or subjective individual acts, act-contents, feelings, images, and so on. It yields the kind of information that allows us to distinguish one human subject from another. Acts of introspection are not directed toward mind-independent abstract objects. Introspecting is thus quite different from engaging in the *Wesensanalyse* of consciousness. Phenomenology, as an eidetic science, is supposed to be a priori in nature (see, e.g. Husserl 1911, 1913). It would issue in an a priori, material or regional ontology. It is supposed to be concerned with universal features of consciousness. These features should be deliverances of reason. This would all be quite distinct from introspection, at least on standard conceptions of introspection. Essence analysis is not about what is individual, private, or subjective. It does not, by its own nature, seek what is particular, what makes one human subject different from another. Essence analysis involves a kind of abstraction. The actual contents of particular beliefs, the feelings and images with which they are associated in different subjects, and so on—all of that would be data for introspection, and introspection would presumably be corrigible, just as what is given in outer sense is corrigible.

In the *Logical Investigations* and other writings Husserl says that there will no doubt be difficulties in phenomenological analysis due to the seemingly unnatural direction of intuition and thought required by phenomenology (see the "Introduction" to the six Logical Investigations). Instead of becoming lost in the performance of acts built intrinsically on one another and instead of naively positing the existence of objects, we must practice phenomenological reflection. We must, that is, make these acts themselves and their meaning-content our objects. This is a direction of thought that runs counter to deeply ingrained habits, as Gödel notes in the next passage quoted below. Among other things, the *epochē* involves a shift to analysis of the *meanings* by virtue of which we are directed toward objects in the world (see chapter 5 below). This is supposed to allow us to focus on our experience itself, on the constitution of the meaning of being, without the prejudices or presuppositions that may be built into the natural attitude or the existing sciences. As Gödel says, it should allow us to detach ourselves from external influences, including fashions of the day.

One of the central features of consciousness that we find after engaging the *epochē* is intentionality. Gödel refers to this in connection with psychology, perhaps because he is thinking of Husserl's introductions to transcendental phenomenology by way of phenomenological psychology, but the point remains the same in transcendental phenomenology:

> One fundamental discovery of introspection marks the true beginning of psychology. This discovery is that the basic form of consciousness distinguishes between an intentional object and our being pointed (directed) toward it in some way (willing, feeling, cognizing). There are various kinds of intentional object. There is nothing analogous in physics. This discovery marks the first division of phenomena between the psychological and the physical. Introspection calls for learning how to direct attention in an unnatural way. (Wang 1996, p. 169)

§ 2. Husserl on Plato, Leibniz, and Kant

It is very interesting to see how several themes concerning the work of Plato, Leibniz, and Kant mentioned above come together in Husserl's own writings, especially in works such as the "London Lectures" (Husserl 1922), *Erste Philosophie* (Husserl 1923–24a, b), the drafts of the *Encyclopedia Britannica* article (Husserl 1927–28), *Cartesian Meditations* (Husserl 1931), and others.

Husserl was already beginning to connect his phenomenology with ideas in Leibniz's monadology around 1917, and this continued in his writings throughout the 1920s and early 1930s. In the *Cartesian Meditations*, for example, Husserl says that

> The [transcendental] ego, taken in full concreteness [vs as mere identical pole, as substrate of habitualities], we propose to call by the Leibnizian name: monad... Since the monadically concrete ego includes also the whole of actual and potential conscious life, it is clear that the problem of *explicating this monadic ego phenomenologically* (the problem of his constitution for himself) must include *all constitutional problems without exception*. Consequently the phenomenology of *self-constitution* coincides with *phenomenology as a whole* (Husserl 1931, § 33).

The remark here about explicating the monadic ego phenomenologically should be compared with Gödel's remark to Günther, cited above, that the reflection on the subject treated in idealistic philosophy might be the way to the correct metaphysics, except that the denial of the objective meaning of thought connected with idealism must be resisted. It should also be noted that Husserl restricts his attention to the transcendental ego as monad. We know that Leibniz has a range of different kinds of monads, but Husserl's focus is much narrower. It is on the kinds of "monads" that we are. There are suggestions in some of Gödel's writings that he was prepared to go Leibniz's way on this matter but, for a variety of reasons, I am not going to pursue this line of inquiry. I am not sure about what could count as evidence, on phenomenological grounds, for the existence of certain types of non-human Leibnizian monads. In any case, there will already be plenty to say about the kinds of monads that we are. Monads, in Husserl's sense, are not mysterious. Monads, in the sense that will be in play in the argument below, are just you and me after the suspension of what Husserl calls the "natural atttitude" by the phenomenological reduction (*epochē*). The phenomenological reduction does not, for example, make us into disembodied spirits. We do not have evidence for such things. As we will see below (chapter 7), monads in Husserl's sense are also not the kinds of beings whose essence we can expect to understand completely through the one-sidedness of the natural sciences.

Elsewhere in Husserl's *Cartesian Meditations* we find this:

> Phenomenological transcendental idealism has presented itself as a *monadology*, which, despite all our deliberate suggestions of Leibniz's metaphysics, draws its content purely from phenomenological explication of the transcendental experience laid open by transcendental reduction, accordingly from the most originary evidence, wherein all conceivable evidences must be grounded... Actually, therefore, phenomenological explication is nothing like "metaphysical construction."... (Husserl 1931, § 62)

In claiming in this passage that phenomenological explication is nothing like "metaphysical construction," Husserl is saying, among other things, that phenomenology is not engaged in the naive metaphysics of earlier philosophical projects. He is expressing a kind of skepticism about earlier metaphysical schemes albeit, as we will see in chapter 3, not the extreme eliminativist skepticism of the logical positivists. In language that Gödel uses in his 1961 text (Gödel *1961/?), phenomenology seeks to avoid "the death-defying leap into a new metaphysics" that would only amount to another dubious metaphysical scheme. We will see below how Husserl wishes to distinguish naive metaphysics from phenomenological ontology.

In a long, interesting passage from a draft of the *Encyclopedia Britannica* entry that contains language quite similar to some of Gödel's remarks on Husserl we are told that

> Remarkable consequences arise when one weighs the significance of transcendental phenomenology. In its systematic development, it brings to realization the Leibnizian idea of a universal ontology as the systematic unity of all conceivable a priori sciences, but on a new foundation which overcomes "dogmatism" through the use of the transcendental phenomenological method. Phenomenology as the science of all conceivable transcendental phenomena and especially the synthetic total structures in which alone they are concretely possible—those of the transcendental single subjects [monads] bound to communities of subjects [monads] is *eo ipso* the a priori science of all conceivable beings [*Seienden*]. But [it is the science], then, not merely of the totality of objectively existing beings taken in an attitude of natural positivity, but rather of the being as such in full concretion, which produces its sense of being and its validity through the correlative intentional constitution. It also deals with the being of transcendental subjectivity itself, whose nature is to be demonstrably constituted transcendentally in and for itself. Accordingly, a phenomenology properly carried through is the truly universal ontology, as over against the only illusorily all-embracing ontology in positivity—and precisely for this reason it overcomes the dogmatic one-sidedness and hence the unintelligibility of the latter, while at the same time it comprises within itself the truly legitimate content [of an ontology of positivity] as grounded originally in intentional constitution. (Husserl 1927–28, p. 175)

From the notes for the "London Lectures" we have

> Transcendental phenomenological subjectivity or monadologism as [is a] necessary consequence of the transcendental phenomenological attitude. The knowledge that any objectivity is only what it is through intentional meaning or significance shows that there is only one possibility for an absolute and concrete being: the being of a concretely full transcendental subjectivity. *It* is the only genuine "Substance." The *ego* is what it is from its own fundamental meaning. The *ego* is in so far as it constitutes itself for itself as being. All other being is merely relative to the ego and is encompassed within the regulated intentionality of subjectivity. (Husserl 1922, p. 72)

Apart from the references to Leibniz, universal science (ontology), and transcendental phenomenological method in these passages, it is important to note the language about how the monad produces the meaning (sense) of being and of validity through "intentional constitution." There are many other passages in Husserl's later writings that contain similar ideas but I will spare the reader some of the longer quotations (interested readers might want to consider, for example, the passages from the draft of the *Encyclopedia Britannica* article on pp. 150–152 of 1927–28). The idea that monads constitute the meaning of being of the objects toward which they are (intentionally) directed by their mental acts plays a very important role in the position I will develop below.

Finally, in a formulation that brings together ideas in Leibniz, Plato, and transcendental philosophy, Husserl says

> Thus, as Phenomenology is developed, the Leibnizian foreshadowing of a Universal Ontology, the unification of all conceivable a priori sciences, is improved, and realized upon the new and non-dogmatic basis of phenomenological method. For Phenomenology as the science of all concrete Phenomena proper to Subjectivity and Intersubjectivity, is *eo ipso* an a priori science of all possible existence and existences. Phenomenology is universal in its scope, because there is no a priori which does not depend upon its intentional constitution, and derives from this its power of engendering habits in the consciousness that knows it, so that the establishment of any a priori must reveal the subjective process by which it is established.
>
> ...Once the a priori disciplines, such as the mathematical sciences, are incorporated within Phenomenology, they cannot thereafter be beset by "paradoxes" or disputes concerning principles: and those sciences which have become a priori independently of Phenomenology can only hope to set their methods and premises beyond criticism by founding themselves upon it. For their very claim to be positive, dogmatic sciences, bears witness to their dependency, as branches merely, of that universal, eidetic ontology which is Phenomenology.
>
> ...The endless task, this exposition of the *Universum* of the a priori, by referring all objectivities to their transcendental "origin," may be considered as one function in the construction of a universal science of Fact, where every department, including the positive, will be settled on its a priori.
>
> ...Thus the ancient conception of Philosophy as the Universal Science, Philosophy in the Platonic, Philosophy in the Cartesian, sense, that shall embrace all knowledge, is once more justly restored. (Husserl 1927–28, pp. 191–194)

Hence, in a remark that could perhaps have been made by Gödel, Husserl says that

> The ideal of the future is essentially that of phenomenologically based ("philosophical") sciences, in unitary relation to an absolute theory of monads.

After these comments on Plato, Leibniz, and phenomenology, I would also like to take note of some of Husserl's critical comments about Kant that would have resonated with Gödel. We saw above that Gödel recommended the Sixth Investigation of Husserl's *Logical Investigations* to various logicians in the 1970s. I will discuss some specific elements of the Sixth Investigation in chapter 6 but let us consider for now what Husserl says about Kant in this text. We should note, first of all, that the idea that human consciousness exhibits intentionality is, at best, only implicit in Kant's philosophy, while it is front and center in Husserl's work. This means that the ideas about intentionality, meaning, and constitution that are so central to Husserl's transcendental philosophy are not present in Kant's thinking at all. Husserl does argue for a general Kantian kind of distinction between thinking and intuiting, or signification and intuition. It seems to me that Gödel also wants to preserve a distinction of this type. Knowledge, including mathematical knowledge, requires intuition, not mere conception. Departing substantially from Kant, however, Husserl argues that if we take the intentionality of human consciousness seriously then we must recognize both sensory and categorial (rational) intuition. There can be mere thinking or signification concerning sensory objects, and there can also be intuition of sensory objects. Analogously, there can be mere thinking or signification concerning categorial (abstract or ideal) objects, and there can also be intuition of categorial objects. Intuition of categorial objects does not have to mean intuition of noumenal objects, as Kant might have us believe. As we will see in chapters 4 and 6, Husserl sets aside the notion of noumenal objects and develops a genetic analysis of the intuition of abstract objects in terms of invariants in our experience. Viewed in terms of genetic epistemological analysis, the thinking and intuition in the case of categorial objects, which are objects such as natural numbers, sets, propositions, and the like, is not the most basic kind of *founding* thinking or intuition, but is a *founded* kind of thinking and intuition. It is a thinking about and, where possible, an intuiting of ideal or abstract objects. Husserl sometimes calls the intuition of ideal objects, especially in connection with intuition of essences, "ideation."[5] In the *Logical Investigations* he distinguishes "real" from "ideal" objects. "Real" objects are objects that are either temporal in nature (such as "inner" mental processes), or temporal and spatial (such as "outer" physical objects), while ideal objects such as numbers, sets, and propositions are neither temporal nor spatial. Regarding intuition, Husserl holds that there can be adequate and inadequate intuitions and, in fact, that there are degrees of (in)adequacy. We also need to recognize a difference between individual and universal intuition. Husserl objects to the "baseless" view according to which individual intuition is usually conceived in a narrow way as sensory intuition exclusively. On his alternative view the distinction between individual and universal intuition also has an application with respect to ideal objects. There are both individual and universal ideal objects.

In Investigation VI of his *Logical Investigations* Husserl says that Kant fails to draw any of these distinctions clearly in his theory of knowledge.

In Kant's thought categorial (logical) functions play a great role, but he fails to achieve our fundamental extension of the concepts of perception and intuition over the categorial realm, and this because he fails to appreciate the deep difference between intuition and signification, their possible separation, and their commixture. And so he does not complete his analysis of the difference between the inadequate and the adequate adaptation of meaning to intuition. He therefore also fails to distinguish between concepts, as the universal meanings of words, and concepts as species of *authentic* universal presentation, and between both of these and concepts as universal objects, as the intentional correlates of universal presentations. Kant drops from the outset into the channel of a metaphysical epistemology in that he attempts a critical "saving" of

[5] In the *Logical Investigations* and other works around this time Husserl speaks of "categorial intuition" in connection with logic and objects of logic and mathematics, but in later works he mostly speaks of "eidetic intuition" (intuition of essences). Both can be viewed as types of rational intuition, with the latter focused on essences in particular. I do not propose in this book to go into details about the shift from the language of categorial intuition to the language of essence intuition. I am interested in both, as species of rational intuition, and the context should make it apparent to which I am referring.

mathematics, natural science and metaphysics, before he has subjected knowledge as such, the whole sphere of acts in which pre-logical objectivation and logical thought are performed, to a clarifying critique and analysis of essence, and before he has traced back the primitive logical concepts and laws to their phenomenological sources. It was ominous that Kant (to whom we nonetheless feel ourselves quite close) should have thought he had done justice to the domain of pure logic in the narrowest sense, by saying that it fell under the principle of contradiction. Not only did he never see how little the laws of logic are all analytic propositions in the sense laid down by his own definition, but he failed to see how little his dragging in of an evident principle for analytic propositions really helped to clear up the achievements of analytic thinking.

All of the main obscurities of the Kantian critique of reason depend ultimately on the fact that Kant never made clear to himself the peculiar character of pure Ideation, the adequate survey of conceptual essences, and the laws of universal validity rooted in those essences. He accordingly lacked the phenomenologically correct concept of the a priori. For this reason he could never rise to adopting the only possible aim of a strictly scientific critique of reason: the investigation of the pure, essential laws which govern acts as *intentional* experiences, in all their modes of sense-giving objectivation, and their fulfilling constitution of "true being." (Husserl 1973, p. 65)

Husserl elaborates on his critique of Kant's view of logic on p. 100 of *Formal and Transcendental Logic* (Husserl 1929). Here he points out how Kant failed to ask transcendental questions about logic itself. Kant asks about conditions for the possibility of the kind of cognition involved in natural science and in our everyday sensory experience of the natural world, but he does not ask about conditions for the possibility of the kind of cognition involved in logic. Husserl thinks that pure logic, like pure mathematics, is concerned with ideal objects and states of affairs. As he says in p. 100,

Pure logic has as its thematic sphere ideal formations. But they would have had to be clearly seen, and definitely apprehended, as such ideal objectivities, before transcendental questions about them and about pure logic could have been asked. The eighteenth century and the age that followed were so strongly actuated by empiricism (or better, by anti-platonism) that nothing was remoter from them than recognition of ideal formations as being objectivities—in the manner of the good and never-relinquishable sense whose legitimacy we have established in detail.

Nothing else hindered a clear insight into the sense, into the proper questions and methods of genuine transcendental philosophy so much as did this anti-platonism, which was so influential that it actuated all parties, and even the thinking of a Kant, struggling to free himself from empiricism.

For the succeeding age this meant, however, that *those investigations in the psychology of cognition, or rather those transcendental phenomenological investigations, that are the thing actually needed for a full and, therefore, two-sided logic were never seriously undertaken*. But that was because no one ventured, or had the courage to venture, to take the *ideality of the formations with which logic is concerned* as the characteristic of a separate, self-contained, *"world" of ideal Objects* and, in so doing, to come face to face with the painful question of how subjectivity can in itself bring forth, purely from sources appertaining to its own spontaneity, formations that can be rightly accounted as ideal *Objects* in an ideal "world."

> For only then was one faced with the unintelligibility of *how ideal objectivities* that originate purely in our own subjectivities of judgment and cognition, that are there originaliter in our field of consciousness purely as formations produced by our own spontaneity, *acquire the being-sense of "Objects,"* existing in themselves over against the adventitiousness of the acts and the subjects. How does this sense "come about," how does it originate in us ourselves? And where else could we get it, if not from our own sense-constituting performance?

Note the formulation of the problem of the relation of the subjectivity of consciousness to the objectivity and ideality of logic in these last two quoted paragraphs. Husserl asks how human subjectivity (the monad) can bring forth formations from sources of its own spontaneity (reason), formations that can be considered as ideal objects in an ideal world. He asks how the objects of cognition in logic can acquire their sense or meaning as ideal and existing in themselves, over against the subjective acts in which they are known. How does this sense or meaning *originate* in us? These questions will play a central role in the developments in chapters 4, 5, and 6 below.

Husserl thus says that

> Accordingly the *transcendental problem* that *Objective logic*...must raise concerning its field of ideal objectivities takes a position *parallel to the transcendental problems of the sciences of realities*, the problems that must be raised concerning the regions of realities to which those sciences pertain, and, in particular, the transcendental problems concerning Nature, which were treated by Hume and Kant. It seems then, that the immediate consequence of bringing out the world of ideas and, in particular (thanks to the effectuation of impulses received from Leibniz, Bolzano, and Lotze), the world of ideas with which pure logic is concerned, should have been an *immediate extension* of transcendental problems to this sphere. (Husserl 1929, p. 100)

§ 3. A new combination of philosophical views

With the material described above in mind, the general picture of Gödel's turn to Husserl's transcendental phenomenology that we obtain thus far is roughly as follows:

The transcendental ego in its full concreteness is a "monad" ("substance"). It constitutes the meaning of being of the world through its intentionality. In the case of mathematics, logic, and the other a priori sciences, including phenomenology itself, it constitutes the meaning of the being of its objects (essences, categorial objects) in a rationally motivated way as ideal or abstract and non-mental. Evidence and objectivity in these domains is acquired on the basis of categorial or eidetic intuition. This suggests a kind of platonism with its emphasis on non-mental and mind-independent ideal objects (in the sense of "mind-independence$_2$" discussed in chapters 4 and 6), with its rationalism, and its robust sense of objectivity. I call this kind of platonism *constituted platonism* and I discuss it in much more detail in chapters 4 and 6. Constituted platonism is unlike traditional mathematical platonism since traditional platonists have not been transcendental (phenomenological) idealists. In the case of logic and mathematics, it is on the basis of constituted platonism that we need to understand phenomenological ontology. Phenomenological ontology, as we will see below, is quite different from naive metaphysics.

Plato certainly did not speak of the constitution of the meaning of being by "monads," and he is engaged, by Husserl's sights, in "naive" (pre-critical, in the sense of transcendental philosophy) metaphysics. This is also true of other traditional mathematical platonists. As we saw above, Husserl says that through transcendental phenomenology "...the ancient conception of Philosophy as the Universal Science, Philosophy in the Platonic sense, that shall embrace all knowledge, is once more justly restored." In his "London Lectures" (Husserl 1922, p. 73) he says that

> Phenomenology realizes (thought of as developed) the original and genuine idea of logic. For originally (in the Platonic dialectic) logic was to be the science of rendering clear the significance, result and legitimacy of possible knowledge and was thereby to make possible genuine wisdom and a universal philosophy.

One can think of the Platonic dialectic as a method that is supposed to help us cultivate an awareness of the ideal forms. This is how we can come to know about ideal or abstract objects, thereby attaining genuine wisdom and a universal philosophy. In this sense, a rudimentary methodology for cultivating our knowledge of ideal or abstract objects is already found in Plato, and one can then read Husserl and Gödel as attempting to clarify, develop, and improve upon such an epistemology. In Husserl's case, however, Plato's doctrine of recollection is not part of the picture. At the same time, the Socratic statement that the unexamined life is not worth living, when coupled with the dialectical method, can be seen as a precursor in some ways of Husserl's idea of suspending the "natural attitude" in order to neutralize various prejudices (prejudgments) so that we might see essences or ideal forms more clearly. I will discuss the idea of the suspension of the natural attitude in some detail in chapter 4.

Leibniz is not in an obvious way a platonist about mathematical objects or facts, although it might be suggested that Leibniz has the means at his disposal to preserve at least some of the features of mathematical platonism. Mark van Atten has noted (in conversation), for example, that if natural numbers are ideas in the mind of God, of the central monad, then they are not our own subjective mental entities. They are mind-independent entities, in the sense that they are independent of any particular human mind. Mind-independence is one of the hallmarks of platonism. It is possible that Gödel wanted to adopt such a view. For Gödel, however, the view would evidently have to be extended to mathematical objects undreamed of in Leibniz's philosophy, such as huge transfinite sets. Assuming such an extension were possible, there would still be many questions about such a position. I want to steer clear of explicit consideration of it in this book for several reasons. One would need to give a defensible account of how we humans could come to know about ideas in the mind of God, to say nothing of the fact that we now have to introduce God into the picture at the very beginning of the investigation. This is one of the best ways I know of to attract dozens of philosophical objections. To be frank, I have serious doubts about whether Gödelian platonic rationalism can be defended on these grounds.

Leibniz, in any case, is clearly a rationalist who is interested in philosophy as a rational (not empirical) universal science. He is interested in deciding mathematical and other problems by human reason, through the analysis of concepts, although in his writings on logic he appears to think of decidability in a mechanical way. For Leibniz, as for other classical rationalists, concepts of reason, including those of logic and mathematics, are exact and our grasp of such concepts either is, or can be made, clear and distinct, whereas empirical knowledge lacks, in various degrees, just these features. Leibniz holds that the science of possibilities and necessities precedes sciences of actualities. Leibniz is a monadologist but his monadology is not brought into line with the methods of transcendental phenomenology, and in this respect it remains, by Husserl's sights, naive. In the quotation from the *Encyclopedia Britannica* draft above, however, Husserl says that the systematic development of transcendental phenomenology brings to realization the Leibnizian idea of a universal ontology as the systematic unity of all conceivable a priori science on a foundation that overcomes dogmatism and one-sidedness through the use of the transcendental method. Phenomenology is the science of all conceivable beings, taken not in the attitude of naive positivity, but rather as understood though correlative intentional constitution.

Kant is not a monadologist, although his idea of the transcendental unity of apperception foreshadows Husserl's transcendental ego. Husserl, as we saw, refers to the transcendental ego in its full concreteness as a monad. Kant, like Leibniz and Plato, does not put the intentionality of human consciousness at the center of his philosophy. Kant is also not a platonist about mathematical objects or facts, and he mounts a critique of classical rationalism (including Leibniz). For Kant, knowledge is restricted to sensory intuition and the two forms of sensory intuition, space and time. Kant, unlike Husserl, distinguishes phenomena from noumena (which is what Wang calls Kant's dualism in the first passage quoted above), and is able to develop the transcendental method far enough to show how *empirical realism* is compatible with transcendental idealism (see chapter 4), but in his work there is no question of showing how a kind of platonism or mathematical objectivism is compatible with transcendental idealism. Neither Husserl nor Gödel wants to reject the Kantian claim that intuition is required for knowledge. A distinction between mere conception and intuition is to be retained, but now we need to recognize not only sensory intuition but also categorial (or "eidetic") intuition.

Thus, in transcendental phenomenology, the transcendental ego in its full concreteness as a monad can now be combined with a kind of (constituted) platonism about logic and mathematics (unlike in Leibniz and Kant), and with the idea of universal science (as in Leibniz and Plato) in a way that keeps Kantian transcendental method or idealism in broad outlines but extends it to mathematics, logic, and philosophy itself, avoiding Kant's dualism, his restrictions on intuition, his overwrought critique of rationalism, and his skepticism about ideal or abstract objects (concepts). Elements in the work of Plato, Leibniz, Kant, and Husserl come together in one picture in which the monad (as a concrete transcendental ego), in a community of monads, constitutes the meaning of being of its objects in mathematics and logic as ideal or abstract and non-mental, and acquires evidence in these domains on the basis of categorial intuition or *Wesensanalyse*.

One can, in principle, substitute for "monad" in the singular in this picture the plural "monads," or transcendental egos. The constitution of the meaning of being of one objective world, Husserl says, requires the community of monads, a single universe of compossible ("harmonious") monads. Intersubjectivity is required for the constitution of the meaning of being of one objective world. Each monad (transcendental ego), to extend the analogy, would presumably be "windowless," but would mirror all of the others if there is to be constitution of one objective world. Of course the issues of intersubjectivity and of the layers of constitution involved in the analysis of the meaning of being of the objective world have been analyzed in great detail by Husserl and others in the phenomenological movement.

One caveat that should be entered, as indicated above, is that it is not clear how much of Leibniz's original monadology, with all of its attendant ideas, Husserl and Gödel wanted to preserve. We can see that certain elements of Leibniz's monadology are at least loosely echoed in Husserl's thinking. Recall also that Gödel says that he wants something like Leibniz's monadology transformed into an exact theory with the help of phenomenology.

§ 4. Developments to come

In subsequent chapters we will see how many of the general themes discussed in this chapter either are, or can be, related to Gödel's *philosophical* thinking about his technical results in logic and mathematics, and to the incompleteness theorems in particular. Chapter 2 opens with a brief non-technical presentation of Gödel's incompleteness theorems, a note on his speed-up results, and some related matters. It then presents central philosophical ideas involved in Hilbert's program, ideas to which Gödel reacts in his own philosophical writing on the incompleteness theorems. Gödel thought that Hilbert's finitistic formalism was inadequate as a foundation of mathematics. According to the second incompleteness theorem, it would not be possible to provide a finitistically acceptable consistency proof, even of finitist mathematics. One might widen one's view of constructivism in order to find consistency proofs for finitist mathematics, subsystems of arithmetic, full classical arithmetic, subsystems of real analysis, and so on. While Gödel himself contributed some interesting and important work to constructive foundations and intuitionism (Gödel 1933e, 1933f, *1938a, *1941, 1958, 1972), he nonetheless thought that we could not do justice to classical set theory, especially higher set theory, on the grounds of intuitionism or constructivism. In his paper on Cantor's continuum problem (Gödel 1947/64), to be discussed in later chapters, he is at pains to show that higher set theory is a meaningful endeavor in which, in particular, the continuum hypothesis has a meaning that, on the basis of further clarification, should lead us to a solution.

There has been some debate about whether Gödel was already a platonic rationalist about mathematics and logic early in his career. He says in at least one place, the Grandjean questionnaire (see Wang 1987, pp. 16–21), that he already held such a view in 1925. If he did not really hold such a view early in his career then there is some reason to think that his own work on set theory, starting in the late 1930s, led him in this direction. In his later philosophical work he frequently speaks in favor of a kind of platonic rationalism about mathematics and logic, although he does appear to acknowledge that the ascent from finitism to full set theory involves us in choices about degrees of idealization and evidence. Chapter 2 lays out some of the elements involved in the ascent from finitistic formalism through intuitionism (or constructivism) to platonic rationalism, based on Gödel's technical work and his own philosophical interpretation of that work.

In the later part of his career Gödel thought that finitistic formalism, intuitionism, and other forms of constructivism were inadequate as foundations of mathematics and logic. He also thought that the views of the logical positivists, whose Vienna Circle meetings he had attended, were inadequate. He argued that Carnap's view of mathematics as syntax of language, in particular, was seriously flawed. This is another place where Gödel brings his incompleteness theorems to bear on philosophical matters. Chapter 3 contains a presentation and discussion of the arguments against Carnap's view that Gödel developed in the six drafts of the paper he wrote for *The Library of Living Philosophers* volume on Carnap (Schilpp 1963) but never published. In chapter 3 I follow up in some detail Hao Wang's remark, cited at the beginning of this chapter, about Gödel's shift from his Carnap paper to the study of Husserl's philosophy. Wang, recall, says that

> It seems likely that, in the process of working on his Carnap paper in the 1950s, Gödel had realized that his realism about the conceptual world called for a more solid foundation than he then possessed. At this juncture it was not surprising for him to turn to Husserl's phenomenology, which promises a general framework for justifying certain fundamental beliefs that Gödel shared: realism about the conceptual world, the analogy of concepts and mathematical objects to physical objects, the possibility and importance of categorial intuition or immediate conceptual knowledge, and the one-sidedness of what Husserl call "the naive or natural standpoint."

In chapter 3, I juxtapose Gödel's arguments against Carnap's view with what he says in his 1961 text (Gödel *1961/?) about the modern development of the foundations of mathematics, Husserl, and the empiricist *Zeitgeist*. This will take us deeper into what Gödel thought he might derive from Husserl's transcendental phenomenology in connection with the themes in Plato, Leibniz, and Kant discussed in this chapter. As we will see, Gödel also comments briefly, but revealingly, on the relationship of Husserl's work to Kant in the 1961 text.

Chapter 4 builds on the first three chapters, focusing in particular on the kind of platonic rationalism that can be developed on the basis of ideas in Husserl's transcendental phenomenology. It starts with a discussion of how Kant thought he could reconcile his transcendental idealism with empirical realism. The idea here is to set up some of the background for Husserl's view of the possibility of reconciling transcendental phenomenological idealism with *mathematical* and *logical* realism (= platonism). This leads to a discussion of the new kind of platonic rationalism that was referred to earlier in this chapter as constituted platonism. This is a platonic rationalism that depends on the phenomenological *epochē*, as we will see. Constituted platonism, it can be argued, gives us a better basis than naive metaphysical platonism for holding that human reason can know about ideal or abstract transcendent objects and concepts in mathematics and logic. Very little of the discussion of Gödel's platonism in the secondary literature has been informed by Gödel's own philosophical interests. Chapter 4 should help to address this lacuna and, in any case, to develop a version of Gödelian platonism that is arguably more defensible than traditional forms of platonism.

Chapter 5 takes up the epistemology of constituted platonism in more detail. It opens with an overview of deficiencies that have come to light in reductionistic treatments of human consciousness. These deficiencies, it is argued, have marred our understanding of human reason. I present and discuss some of the important general features of reason that have been emphasized in the rationalist tradition in philosophy. Husserl's work adds to this tradition the view that a central feature of reason, as of many other forms of consciousness, is intentionality. Acts of reason exhibit intentionality. Relevant aspects of the Husserlian theory of intentionality are then described. It is argued that it is crucial to understand the epistemology of constituted platonism on the basis of the intentionality of human reason. The chapter also introduces what I call the "intentional difference principle." This principle will figure directly into tracking the kinds of directedness of human consciousness involved in our different kinds of cognitive acts. It is on the basis of such a principle, along with the notion of rationally motivated meaning constitution, that we can address the question of how human subjects can know about abstract or ideal objects and states of affairs in mathematics and logic.

The analysis of human reason is continued in the first part of chapter 6, where it is argued that acts of reason in mathematics and logic, viewed genetically, are *founded* on lower-level cognitive activities, in particular on sense perception. The chapter opens with a discussion of founding and founded types of conscious directedness and meaning constitution. A kind of abstraction is already involved in ordinary sense perception. Cognitive acts in science that are founded on sense perception involve various other types of abstraction. Once this part of the chapter is complete I can turn to the question of how knowledge of transcendent abstract objects and truths of mathematics and logic is possible. Gödel's views on the use of the transcendental method in philosophy are coupled with the idea that human reason in mathematics and logic exhibits intentionality. This makes it possible to analyze knowledge in mathematics and logic in terms of the distinction between the mere intentions by virtue of which we are directed toward objects and the fulfillment of those intentions (= intuition). This will be an integral part of constituted platonism. It is in this chapter that I discuss in more detail the notion of categorial intuition or *Wesensschau*. Analyses of intuition in philosophy and mathematics have proceeded in many different directions, and not always with happy results. As we will see, there is a distinctive account of intuition in Husserl's philosophy, an account that I think can be developed and defended. Acts of reason in mathematical and logical practice themselves display a distinction between mere conception and intuition, expectation and realization, problem and solution. In acts of reason in mathematics and logic we find conjecture but also proof. At the end of the chapter I take up the manner in which rational or "eidetic" intuition in mathematics and logic is analogous to sensory intuition, and dispel some misunderstandings about the analogy.

In chapter 7, I return to themes in Gödel's thinking about minds and machines. In a note in his *Nachlass*, cited in chapter 7, Gödel suggests that a phenomenological investigation of the processes of reasoning might be used to show that the human mind contains an element totally different from a finite combinatorial mechanism. This suggestion is explored in some detail in chapter 7 on the basis of ideas in earlier chapters of the book and in connection with Gödel's remarks on the implications of the incompleteness theorems for minds and machines. In his Gibbs Lecture of 1951 (Gödel *1951) Gödel formulates a central implication of his incompleteness theorems, which he says is "of great philosophical interest," as a disjunction: either the human mind infinitely surpasses the powers of any finite machine or else there exist absolutely unsolvable Diophantine problems. In a note in the *Nachlass* Gödel says that he hopes to show that the first disjunct is true. In his later work he suggests, in effect, that we replace a mechanist conception of reason with a conception that allows for the possibility of finding systematic and finite but non-mechanical methods for the decision of mathematical questions on the basis of clarification of the intuition of the abstract meanings of the terms (concepts) involved in the questions. This conception of reason, along with constituted platonism and an analysis of the genesis of modern mechanism, is used in chapter 7 to bolster the claim that human minds are not simply computational mechanisms.

The book concludes with a consideration of two important topics concerning the type of platonic rationalism presented in earlier chapters. Gödel's critique of Carnap is featured in chapter 3, but in chapter 8 I discuss in more detail the kind of holistic empiricism about mathematics and logic with which W.V. Quine supplanted Carnap's positivism. Quine's view has been influential with philosophers. It seems to me that Gödel would have viewed it as yet another example of the "leftward" empiricist *Zeitgeist* that he finds troublesome. In the first part of this chapter I want to explain why I think Gödel would see the matter this way. If the empiricist *Zeitgeist* that has been with us since the Renaissance is problematic when applied to mathematics and logic, it is no less true that traditional rationalist views have had their own problems. I close the book with a number of suggestions about how the rationalistic platonism I have developed in connection with mathematical and logical practice can avoid the excesses of traditional forms of rationalism. I cannot say that Gödel was as incisive on the excesses of traditional rationalism as he was in so much of his other technical and philosophical work. This fact opens up his views to a number of objections. I would like to cordon off some elements of Gödelian platonic rationalism about mathematics and logic that I think can be defended. As for the rest, I am content to let Gödel's critics have their way.

2

Incompleteness, Consistency, and the Ascent to Platonic Rationalism

In this chapter I provide a brief overview of Gödel's basic results on incompleteness and undecidability, consistency, finite axiomatizability, independence, and speed-up techniques, along with some of his ideas about set theory and metamathematics, but without entering into technical details.[1] Tarski's theorem on the indefinability of arithmetical truth in arithmetical language is also mentioned. Most of the technical results for which Gödel is famous were produced in the 1930s and early 1940s. The incompleteness theorems and other results I discuss are purely mathematical in character and are unassailable as such on scientific grounds. Gödel, in fact, made a point of formulating the incompleteness theorems so that they would be acceptable to philosophers, logicians, and mathematicians of different foundational persuasions. It is possible to study Gödel's technical results and to understand them without knowledge of philosophical doctrines or viewpoints. In order to see the philosophical significance of Gödel's work, however, we need to understand the results in connection with the philosophical and foundational background against which they emerged. We cannot just consider the purely technical presentation of the results. It is quite possible to read many papers and even book-length treatments that are devoted to presenting Gödel's results with great mathematical rigor and yet to come away with virtually no grasp of the philosophical context and significance of the results.

The main point of the present chapter is to discuss central philosophical ideas associated with Gödel's results. I will outline some of the ideas in Gödel's philosophical thinking about his technical results that lead to a kind of platonic rationalism about

[1] There are many presentations of the incompleteness theorems in particular. Gödel's original papers (Gödel 1931, 1933o, 1934, 193?) are certainly still worth consulting. In addition, the articles Smoryński 1977 and Kleene 1986, the book chapter van Dalen 2004 (Chapter 7) and the books Smullyan 1992, Boolos, Burgess, and Jeffrey 2007, Smith 2007, and the older book by Kleene 1952, are among the presentations that could be recommended. The popularization by Nagel and Newman 1958 is of interest, especially in connection with the correspondence between Nagel and Gödel that accompanied its production (see "Correspondence with Ernest Nagel," *KG:CW*, 2003a, Vol. V, 145–154).
On some of the ways in which Gödel's incompleteness theorems have been misunderstood and abused see Franzén 2005.

mathematics and logic. The kind of platonic rationalism that I develop in subsequent chapters and that I think can be defended is ultimately to be interpreted in terms of the themes in chapter 1, that is, in terms of a transcendental philosophy that includes constituted platonism and the intentionality of human reason, and according to which there can be meaning clarification of basic abstract concepts of logic and mathematics that depends on categorial intuition or *Wesensanschauung* (Husserl). The solution of at least some open problems in mathematics and logic is then held to depend on systematic and finite but non-mechanical methods of human reason. It is Husserl and Leibniz who emphasize the exactness of concepts of mathematics and logic, and the possibility of clarification of our grasp of these concepts, and Husserl (after 1900) and Plato who think of the concepts and objects of mathematics as ideal and abstract. Husserl, building on Kant's transcendental philosophy, would interpret all of this in the context of transcendental phenomenology and, unlike Kant, he recognizes the possibility of a kind of rational intuition. The notion of reason embedded in Husserl's phenomenology, however, leads Gödel to think, in contrast with Leibniz, that decidability on the basis of human reason (of the monad) is not to be identified with purely mechanical or computational decidability.

§ 1. Incompleteness and consistency

Let us start with a general statement of Gödel's incompleteness theorems. For formal theories T that contain enough mathematics to make Gödel numbering possible, the *first incompleteness theorem* says that if T is consistent then there is a sentence G_T, the Gödel sentence for T, such that $\nvdash_T G_T$ and $\nvdash_T \neg G_T$. This theorem will apply to formal systems in which it is possible to meet some standard conditions on formal provability and to do enough arithmetic to set up the Gödel numbering. T can be, for example, primitive recursive arithmetic (PRA), Peano arithmetic (PA), the system of Russell and Whitehead's *Principia Mathematica* (which was Gödel's original target), Zermelo-Fraenkel set theory (ZF), and so on. The idea of Gödel numbering is to encode syntactic objects such as terms, formulas, and proofs as numbers or sequences of numbers in such a way as to be able to formally represent key syntactical notions concerning formal systems within the systems. In the case of a formal system such as Peano Arithmetic, for example, the encoding will allow us to represent metamathematical statements about PA in PA. The Gödel sentence, thanks to the encoding, will itself be an arithmetical sentence. (In some writings Gödel refers to these undecidable sentences as "sentences of Goldbach type," by which he also means Π_1 sentences.) We can obtain such a Gödel sentence in virtually any formal system that allows us to do a minimal amount of arithmetic, and one can be quite precise about exactly how much arithmetic is needed for the encoding to be possible. The Gödel sentence G_T for a formal theory T is the sentence that, in effect, says of itself that it is not provable in T: it says "This sentence is not provable (in T)." To be a bit more specific, it is a formula that says that a sentence with a particular Gödel number is not provable in T but it is itself the formula that has this Gödel number.[2] Thus, we can obtain the kind of self-reference that we find in certain sentences that lead to paradoxes, such as the Liar Paradox. In the Liar Paradox we have a sentence that says of itself that it is false and if we then ask whether this sentence is true or false we land in a paradox (contradiction). The Gödel sentence, however, is not of the form "This sentence is false." Gödel in effect substitutes "unprovable" for "false," but the sentence is then expressible in the formal system T, with the consequence that the sentence does not generate a paradox but only the result that a consistent formal system T could not prove such a sentence.[3] Along with the Gödel numbering, this is a really ingenious part of Gödel's proof.

Suppose T has the feature that only true formulas are provable in it. We would normally intend our formal systems to have such a "soundness" feature. If G_T were provable in T then, given what it says, it would be false. By our supposition, therefore, it is unprovable and hence true. Thus, again by our supposition, $\neg G_T$ must also be unprovable. G_T is therefore formally undecidable in T. In the Introduction to his famous 1931 paper on incompleteness Gödel gives an informal soundness argument such as this, but in the body of the paper he replaces the soundness supposition by, as he says, "a purely formal and much weaker one," namely that T is ω-consistent. In subsequent work by Rosser it was shown how this hypothesis can be replaced by that of simple consistency. It will not be necessary, however, to enter into all of the details about refinements of and variations on Gödel's own original proof of the first incompleteness theorem.

A Gödel sentence can be constructed for any formal system that meets the standard formal provability conditions and that allows of Gödel numbering. Since G_T is a true formula that T cannot prove (if T is consistent), we might consider extending T by adding G_T to it (or a sentence from which G_T follows) as an axiom. We will then, however, be able to construct a new Gödel sentence for the extended formal system. This sentence will be true but undecidable with respect to the extended system. The extended systems here will obviously not be conservative extensions since it is possible to prove sentences in them that are not provable in the previous systems. We could thus construct infinitely many true but unprovable formulas in this manner. If we are, for example, attempting to axiomatize arithmetic with our formal system PA, then it is

[2] The undecidable Gödel sentences, it is sometimes pointed out, are not themselves examples of interesting or "non-artificial" mathematical statements. Examples of such mathematical statements, however, have been found since the time of Gödel's results. The first example of such a statement that is undecidable in PA is due to J. Paris and L. Harrington (Paris and Harrington 1977). This statement, the Paris-Harrington theorem, is also of interest because it refers only to natural numbers, but its proof requires the use of infinite *sets* of natural numbers, which would be an example of Gödel's idea of having to ascend to stronger, more abstract principles in order to solve number-theoretic problems.

[3] Wittgenstein (Wittgenstein 1976) evidently misunderstood Gödel's result at just this point, thinking that Gödel had simply arrived at a logical paradox. For some of Gödel's remarks on Wittgenstein, which are generally quite negative, see Wang 1996, p. 179. A number of papers have, of course, been written on Wittgenstein's remarks on the incompleteness theorems.

clear that a finite axiomatization will not be possible, for every time we add the true but unprovable arithmetic sentence (Gödel sentence) as an axiom to our existing axiomatization there will be another true but unprovable sentence in the stronger system. Since these sentences will all be true but unprovable arithmetic sentences, it is clear that we will not be able to capture all of the arithmetic truths in PA. In general, the incompleteness theorems thus point to the impossibility of exhausting mathematics in formal systems. Gödel discusses the inexhaustibility of mathematics in some detail in his Gibbs Lecture of 1951(Gödel *1951) and I will return to this in a moment.

The first incompleteness theorem does not tell us that G_T is absolutely undecidable. It is only undecidable relative to the formal system T. Gödel frequently emphasizes how sentences that are undecidable in some formal theories are in fact decided in certain natural extensions of those theories (e.g. by ascending to higher types). As we will see later, Gödel was very concerned with questions about relative and absolute provability and definability, with relative and absolute undecidability, and with questions about how mechanical decidability might differ from decidability on the basis of reason. In his later work he wished to deny that there are absolutely unsolvable Diophantine problems. He also thought that the continuum problem in set theory should be decidable on the basis of extending Zermelo-Fraenkel set theory with new evident axioms. In fact, Gödel makes a number of remarks about how it should, in principle, be possible to decide any clearly formulated yes or no question in mathematics. I will come back to this issue in subsequent chapters.

The *second incompleteness theorem* says that if T is consistent then $\nvdash_T$ CON(T), where "CON(T)" is a particular formalized statement that asserts the consistency of T. The second theorem suggests, generally speaking, that if there is a consistency proof for a theory T then it will be necessary to look for $\vdash_{T'}$ CON(T), where T is a proper subsystem of T'. One easy way to see why the second incompleteness theorem is true, without going into all of the details of formalizing the first incompleteness theorem, is that, given the first theorem, CON(T) is equivalent to G_T. If G_T is true, then given what it says (i.e. "I am not provable in T"), it follows that some sentence is not provable in T. This means that T is consistent, for if it were not then every sentence would be provable in T. In the other direction, if CON(T) is true then so is G_T, on the basis of the first theorem. Since G_T is not provable then neither is CON(T).

On the basis of the incompleteness theorems it is natural to think that the notion of arithmetical truth could not be arithmetically definable. The truth predicate for number theory could not be definable in number theory, on pain of contradiction. Although Gödel in effect had this result, Tarski (Tarski 1933) is credited with showing that the set of Gödel numbers of the truths of arithmetic is not the extension of any arithmetical formula. This is referred to as Tarski's indefinability theorem. If we refer to the language of arithmetic as L, then L can be extended to a new language L' by adding to L a predicate that expresses arithmetic truth, "true in L." To be more explicit, it is the predicate "Gödel number of a true sentence in L." The set of Gödel numbers of truths of L' will not be definable in L', but we can add to L' a new predicate "true in L," thus obtaining a new language L". Again, truth for L" will not be definable in L". In this manner of adding new truth predicates we obtain a Tarskian hierarchy of increasingly richer languages L, L', L",...In general, to define truth in a formal language of a particular kind we have to step outside that language to a richer language. Later in this chapter we will see some of Gödel's remarks about how the use of a "highly transfinite concept of objective mathematical truth" helped him to obtain his incompleteness results.

In his most extensive discussion of the philosophical implications of the incompleteness theorems in the Gibbs Lecture (Gödel *1951), Gödel highlights what we referred to above as the inexhaustibility of mathematics in formal systems. He says that the incompleteability of mathematics is shown in a general way by the incompleteness theorems. In this lecture Gödel states the first incompleteness theorem in the following form: Whatever well-defined system of axioms and rules of inference may be chosen there always exist Diophantine problems (which he specifies) that are undecidable by these axioms and rules, provided only that no false propositions of this type are derivable. For a system of axioms and rules to be well-defined means that it is possible to actually write down the axioms in a precise formalism or, if their number is infinite, it must be possible to give a finite procedure for writing them down one after the other. The rules of inference must be such that, given any premises, it is possible to either write down a conclusion or to determine that there exists no immediate conclusion by the rule of inference under consideration. The requirement that the system of axioms and rules be well-defined is equivalent to the requirement that it should be possible to build a finite machine, in the sense of a Turing machine (TM), that will write down all of the consequences of the axioms one after another. The first theorem is thus equivalent to the fact that there exists no finite mechanical procedure (TM) for the systematic decision of all Diophantine problems of the type specified.

The second theorem is stated as follows: For any well-defined system of axioms and rules, the proposition stating their consistency (or rather the equivalent number-theoretical proposition) is undemonstrable from these axioms and rules, provided these axioms and rules are consistent and suffice to derive a certain portion of the finitisitic arithmetic of integers. Gödel says it is this theorem that makes the incompleteability of mathematics particularly evident. The theorem shows that it is impossible for someone to set up a well-defined system of rules and axioms, and consistently make the following assertion about it: I perceive (with mathematical certitude) that all of these axioms and rules are correct and, moreover, I believe that they contain all of mathematics. Anyone who makes such an assertion contradicts himself. If the axioms are perceived to be correct, then they are perceived with the same certitude to be consistent. One then has a mathematical insight not derivable from the axioms.

In the Gibbs Lecture (Gödel *1951), Gödel also illustrates his point about inexhaustibility in connection with the effort to axiomatize set theory. Instead of finding a finite set of axioms, as in elementary geometry, one is faced with an infinite series of axioms that can be extended further and further, but without the possibility of generating all of these axioms by a finite rule. (See how Gödel relates this inexhaustibility to the analogy between sensory intuition and rational intuition in one of the drafts of his paper on Carnap, discussed in chapter 3 below.) In the process of discussing this matter, Gödel points out that, as we ascend to axioms for sets of higher level in the set-theoretic hierarchy, we find that the axioms have consequences even for the theory of integers, that is, the axioms entail solutions of certain Diophantine problems which are undecidable on the basis of preceding axioms. This means that, in principle, such set-theoretic axioms could be used to solve number-theoretic problems (as mentioned in footnote 2).

§ 2. Speed-up results

We know from the incompleteness results that sentences that are not provable in particular (consistent) formal theories are provable in extensions of those theories. In a paper "On the Length of Proofs" published in 1936 (Gödel 1936a), Gödel pointed out a speed-up result that holds when one moves from a weaker formal system to a stronger one. The result is formulated for "logics" of different orders. A higher-order logic S_{i+1} will prove formulas that a logic S_i will not prove. If we consider formulas that can be proved in both systems, however, then the result says that if the length of a proof is defined to be the number of lines in the proof, then the proof of a given formula in S_{i+1} will be much shorter than the shortest proof in S_i. To be more precise, it says that for each function ϕ that is computable in S_i there exist infinitely many formulas *f* such that if *k* is the length of the shortest proof of *f* in S_i and *l* is the length of the shortest proof of *f* in S_{i+1}, then $k > \phi(l)$. As an example, Gödel notes that if we set $\phi(n) = 10^6 n$ then there are infinitely many formulas whose shortest proof in S_i is more than 10^6 times as long as their shortest proof in S_{i+1}. Results analogous to this, where the length of a proof is taken to be its Gödel number (equivalently, the number of symbols) instead of the number of the lines in the proof, were obtained later. (There are some important technical differences related to the matter of how one defines the length of a proof.)

Speed-up results of this type are philosophically interesting because they indicate the benefits of ascending to more powerful formal systems. In terms of the interpretations of the formal systems involved and the related capacities of human reason, they indicate the benefits of ascending to higher-level conceptions in our thinking, and they suggest that it is not a good idea to insist that only the reasoning expressed in first-order logic is legitimate. George Boolos (Boolos 1987) gives an example of the premises and conclusion of an argument written in the language of first-order predicate logic with identity and function symbols that is logically valid and short, in the sense that the premises and conclusion contain only about 60 symbols. Here is the argument, where s is a 1-place and f a 2-place function sign:

1. $\forall n\, fn1 = s1$
2. $\forall x\, f1sx = ssf1x$
3. $\forall n \forall x\, fsnsx = fnfsnx$
4. $D1$
5. $\forall x(Dx \rightarrow Dsx)$

$\therefore$

6. $Dfssss1ssss1$

There is a derivation of the conclusion from the premises of this argument in any standard axiomatic formulation of second-order logic whose every symbol can easily be written down but, as Boolos puts it, "it is beyond the bounds of physical possibility that any actual or conceivable creature or device should ever write down all the symbols of a complete derivation in a standard system of first-order logic…there are far too many symbols in any such derivation for this to be possible," even though there is such a derivation in first-order logic. It is just that no such derivation could possibly be written down in this universe. Boolos shows that on one of the standard formulations of first-order logic, the number of symbols in the derivation would be at least the value of an exponential stack

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$$

with 65,536 '2's in all. The conclusion, in other words, just cannot be feasibly derived in first-order logic, but there is a short and simple argument demonstrating its validity in any standard system of second-order logic. Boolos says that, although this fact "cannot by itself be regarded as an overwhelming consideration for the view that first-order logic ought never to have been accorded canonical status as *Logic*, it certainly is one further consideration" in favor of such a view. An additional observation of Boolos is especially relevant for my purposes: the fact that we recognize the validity of the inference in question "would seem to provide as strong a proof as could be asked for that no standard first-order logical system could be taken to be a satisfactory idealization of the psychological processes or mechanisms whereby we recognize (first-order) logical consequences." Boolos says that cognitive scientists ought to be suspicious of the view that logic as it appears in many logic texts adequately represents the whole of the science of valid inference.

This conclusion of Boolos seems to me to point to recognition of the kinds of cognitive processes involved in human reason that Gödel and Husserl want to investigate in transcendental phenomenology. Recognition of the validity of the inference in question cannot be adequately analyzed in terms of the cognitive acts required by the first-order derivation. The cognitive processes involved could not be adequately represented in this manner. Rather, there is an ascent to a more abstract, conceptual level at which we understand the validity of the inference. A capacity for greater abstraction must be involved. Reason or conception allows us to see or understand the validity of inferences we could never see to be valid if the limits of reason coincided with weaker systems such as first-order logic, that is, systems with more restrictions, in which we cannot even carry out a formal proof, given the number of symbols involved. I will have more to say about such a view of reason later in this chapter and in subsequent chapters.

§ 3. Gödel, Turing, and the generalized incompleteness theorems

Gödel had a very high regard for Alan Turing's work on mechanical computability. He says in various places that it is thanks to Turing's work that a precise, unquestionably adequate, and indeed "absolute" definition of the general concept of formal system can be given, with the consequence that the existence of undecidable arithmetical sentences and the unprovability of the consistency of a system in the same system can be proved for *every* consistent formal system containing a certain amount of finitary number theory (see Gödel 1934, Postscriptum of 3 June 1964). Turing's work gives an analysis of the concept of mechanical procedure (algorithm, computation procedure, or finite combinatorial procedure). The concept is shown to be equivalent to that of a Turing machine. A formal system can be defined to be any mechanical procedure for producing formulas, which are the provable formulas. Well-defined or effectively given formal systems can be viewed as Turing machines, and Turing machines can be viewed as effectively given formal systems. When we speak of formal systems below in connection with Gödel's theorems, we can replace the notion of formal system with "Turing machine." The essence of a formal system, Gödel says, is that, in it, reasoning is completely replaced by mechanical operations on formulas.

Gödel says that the definition of mechanical procedures in terms of general recursiveness or Turing computability is very important because "with this concept one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen" (see Gödel 1946, p. 150). By way of contrast, it has been possible to define demonstrability (provability) or definability only relative to a given language. The incompleteness theorems then show that for each individual language the notion of demonstrability or definability obtained is not the one looked for. In the case of the concept of computability, however, "it is not necessary to distinguish orders, and the diagonal procedure does not lead outside of the defined notion." Gödel goes on to make some remarks in this Princeton lecture about whether there might be concepts of demonstrability and definability that are also "absolute" in this sense.

It is worth noting that some of Gödel's ideas about "perceiving" concepts are developed in connection with his view of Turing's analysis of the concept of mechanical procedure. Wang tells us that "related to [Gödel's] conception of concepts is that part of his views that he attributes to Husserl" (Wang 1987, p. 161). In Wang's notes in *From Mathematics to Philosophy* of his discussions with Gödel we are told that concepts are, in effect, abstract objects. They are supposed to exist independently of our perceptions of them, and we simply have better or worse perceptions of them, much as we have a clearer perception of an animal when it is nearer to us than when it is farther away (see Husserl 1913, §§ 66–69). According to Wang, Gödel thought of the idea of "seeing" a concept as very closely related to understanding the meaning of an expression. Much like Husserl, he claimed that it was a prejudice of our time, a kind of naturalistic prejudice, to suppose that we do not "see" or "intuit" concepts (see Tieszen 1998, 2002, 2005a). On Husserl's view it is supposed to be possible to clarify our intuition of concepts. In *Ideas I*, for example, Husserl discusses the "method of clarification," and the "nearness" and "remoteness" of the data of intuition (Husserl 1913, §§ 67–70 and 1952b, chapter 4; also Tieszen 1992). On Husserl's view, reflection on concepts, on ideal content, is itself something that can be filled in intuition, just as certain other types of object-directed acts might be filled in intuition. Thus, a concept can itself become the object of an act, and one can then set about determining what kinds of properties are true with respect to the concept (see chapter 5, § 4, and chapter 6). Gödel thought that we have an example of such meaning clarification in the concept of a mechanical procedure, which he believed was finally brought into the correct perspective with the work of Turing. He told Wang that there were more similarities than differences between sense perceptions and perceptions of concepts, and that the analog of perceiving physical objects from different perspectives is the perception of different logically equivalent concepts. This last remark refers to one of the hallmarks of intensionality, and intentionality, and it also brings out the connection with the idea of intuiting essences, for we have evidently arrived at the "essence" of the concept of mechanical procedure by way of the many different definitions of this concept that were subsequently proved to be logically equivalent. This is a particular example, about which more will be said below, of what intuiting the essence of something must be like.

Gödel says that the adequacy of the definition of "formal system" and of "mechanical procedure" due to Turing, however, has nothing whatsoever to do with the question whether there exist finite non-mechanical but systematic procedures not equivalent to any algorithm. In a comment that will be important in some of our arguments below, he says that the incompleteness theorems do not establish any bounds for the powers of human reason, but rather only for the potentialities of pure formalism in mathematics (Gödel 1934, 3 June 1964 Postscriptum). An alternative, non-mechanistic conception of reason is linked to Gödel's study of Husserl: the phenomenological view of the monad or of human reason leaves open the possibility of finding systematic and finite but non-mechanical methods for the decision of meaningful mathematical questions on the basis of the clarification of the intuition of the abstract concepts involved in the questions.

§ 4. Some philosophical ideas related to Hilbert's program

In order to understand Gödel's philosophical views on the incompleteness theorems, especially in relation to his views on platonism and rationalism, it is necessary to understand, among other things, the philosophical ideas involved in Hilbert's conception of proof theory or metamathematics (especially Hilbert 1922, 1923, 1926, 1928). Gödel was working within this program when he discovered the incompleteness theorems. He was following Hilbert in attempting to establish the consistency and completeness of axiomatic formal systems for mathematics, and had already made some progress in this direction in his doctoral thesis, where he established a completeness theorem for first-order logic (Gödel 1929).

Hilbert had hoped to show, in response to the foundational crisis brought on by the discovery of the paradoxes of set theory, that the Peano axioms for arithmetic, and indeed the axiomatic formalization(s) of all of higher mathematics, could be proved consistent using only concrete, finitist means. This would put mathematics on a firm foundation, and spare us the kind of shock that was caused by Russell's paradox and other paradoxes of set theory. The hope was that "Cantor's paradise" itself could be secured in this way. The idea was to axiomatize mathematical theories (e.g. ZF set theory) and then completely formalize them. The formalization should be very precise. We specify an alphabet of signs from which the expressions of the formal system are to be composed, we present an inductive definition of the expressions of the system, and we lay down a finite set of axioms or suitable axiom schemata and a finite sets of rules of inference from which theorems are to be derived from the axioms. The entire formal system is then supposed to be seen as a system of concrete, finite sign configurations and manipulations on sign configurations according to the rules of inference, which are simply rules for mechanically generating new bits of syntax from existing bits. Now the sign configurations and the rules for manipulating them are all given, at least as tokens, in straightforward sensory perception. If we focus on only such a concrete, "real" formal system in place of the original (informal) mathematics that led to it, we are concerned with what Hilbert called "metamathematics." Metamathematics will be concerned only with syntax and syntactical properties. If we think of the epistemology here in terms of the intentionality of human consciousness then what the mind is directed toward in this case is finite concrete sign configurations (as tokens) given, at least in principle, in straightforward sensory perception. A "proof," for example, is now just a finite sequence of finite sign configurations called "sentences," and is relative to the formal system specified.

In order to have a secure foundation for mathematics we are supposed to prove that such axiomatic formal systems are consistent on the basis of concrete, finitist consistency proofs. Such proofs would depend only on a finite number of discrete objects—sign configurations—that would be immediately intuitable in space-time, and on the combinatorial properties of these objects. This constructive, finitist part of mathematics would, in Hilbert's view, contain only "real" or "contentual" sentences. The aim would be to formalize mathematics and to show that the formalism for higher parts of mathematics could be proved consistent using only the real, concrete, contentual part of mathematics. The finitisitically meaningful sentences, which are the "real" sentences, correspond roughly to computations or combinatorial manipulations, and sentences that are more complicated than this are considered "ideal." As such, they have no meaning but they can be manipulated in certain ways, just as i is not a real number but can be manipulated algebraically using the fact that $i^2 = -1$. Hilbert's idea was that just as i leads to no new algebraic identities, the use of ideal sentences and reasoning about them would not allow the derivation of any new real sentences. That is, no new sentences could be derived that were not already derivable finitistically. Hilbert's program is, in this sense, a conservation program: the formalizations which included ideal elements in mathematics would be shown to be conservative extensions of the real part of mathematics, thus effectively eliminating the dependence on the ideal elements. The ideal elements would merely serve to shorten proofs, or to simplify a system of reasoning or to make it more perspicuous.

It was part of Hilbert's formalism that there should be no need to consider the meaning of the sign configurations involved in the formalizations. Only syntactical properties and relations should figure into the consistency proof. Moreover, any allegedly dubious type of "intuition" would be eliminated, since only the intuition or perception of finite sign configurations remains. Hilbert puts it the following way:

> as a condition for the use of logical inferences...something must already be given to our faculty of representation, certain extralogical concrete objects that are intuitively present as immediate experience prior to all thought. If logical inference is to be reliable it must be possible to survey these objects completely in all their parts, and the fact that they occur, that they differ from one another, and that they follow each other, or are concatenated, is immediately intuitively given, together with the objects, as something that neither can be reduced to anything else nor requires reduction...And in mathematics, in particular, what we consider is the concrete signs themselves, whose shape, according to the conception we have adopted, is immediately clear and recognizable. (Hilbert 1926)

The concern for reliability in Hilbert's conception of proof theory precluded any appeal to the concepts or meanings associated with the concrete signs. The concrete signs should be given immediately to our faculty of representation "prior to all thought." Accordingly, the foundation of mathematics, the basis of security or reliability, lies in sense perception of finite sign configurations and not in conception. If it is the nature of human *reason* to trade primarily in a (certain type of) conception that is distinct from sensation, then the foundation of mathematics, for Hilbert, could not lie primarily in human reason. In his 1961 text on Husserl (Gödel *1961/?), Gödel points out that in his view of foundations, Hilbert in fact tried to combine empiricist *and* rationalist elements, but that the incompleteness theorems show that Hilbert's combination will not work. We will come back to this in the next chapter.

§ 5. A note on Carnap on logical syntax

It is worth noting now that in his description of logical syntax in *The Logical Syntax of Language* (Carnap 1934a), Carnap was clearly influenced by Hilbert. Carnap says that a theory, rule, definition, or the like, is to be called "formal" when no reference is made in it either to the meaning of the symbols or to the sense of the expressions, but simply and solely to the kinds and order of the symbols from which the expressions are constructed (Carnap 1934a, § 1). The logic of science, Carnap says, is nothing other than the syntax of the language of science. Describing the attitude of the Vienna Circle towards metaphysics in particular, he says that the sentences of metaphysics are pseudo-sentences which on logical analysis are proved to be either empty phrases or phrases that violate the rules of syntax. Of the so-called philosophical problems, the only questions that have any meaning are those of the logic of science. To share this view is to substitute logical syntax for philosophy. Carnap holds that as soon as logic is formulated in an exact manner it turns out to be nothing other than the syntax either of a particular language or of languages in general (Carnap 1934a, § 62).

Gödel says in one of the drafts of his Carnap paper (Gödel *1953/59–III, p. 341, footnote 19) that if Carnap's syntactical program is to achieve its purpose then what must be meant by "syntax" is equivalent to Hilbert's "finitism" in the following sense: it consists of concepts and of reasoning that refer to finite combinations of symbols that are contained within the limits of "what is directly given in sensual intuition" (quoting Hilbert). The part of our knowledge thus defined is equivalent to recursive number theory, except that it could be argued that combinatorial objects with an exorbitant number of elements or operations to be performed must not occur in finitary considerations. The reason is that these would be theoretical constructions which are as far from the "immediately given" as the infinite, and they perhaps even go far beyond everything contained in space-time reality. So the ideas of such objects or operations could not be known to be meaningful or consistent unless we trust in some mathematical intuition of things that are completely inaccessible to sense experience.

According to the *Logical Syntax of Language*, however, questions about anything not formally representable, such as questions about the conceptual content of certain sentences or the perceptual content of certain expressions, do not belong to logic at all but rather to empirical psychology (Carnap 1934a, § 71). Carnap says that all questions in the field of logic can be formally expressed and are then resolved into syntactical questions. A special logic of meaning is superfluous. "Non-formal logic", Carnap says, is a *contradictio in adjecto*. Logic is syntax (see also Carnap 1934a, §§72–73, § 86). As we will see in the next chapter and elsewhere, Gödel is quite critical of these ideas.

§ 6. Hilbert and finitist consistency proofs

Returning to Hilbert's program for the foundations of mathematics, we can see that it fits quite nicely with elements of empiricism and anti-platonism because the consistency of axiomatic formalized parts of mathematics is to be proved using *only* a special theory that I will call *C*, for "concrete mathematics." (This is also what Hilbert calls "finitist" mathematics.) C could not be just any theory. What would make C special is the fact that it would allegedly possess the kinds of properties needed to ensure reliability or security in empiricist or anti-platonistic terms. It should be finitary and not infinitary, for the worry is that we do not understand the infinite very well, and in our thinking about the infinite we may be led into inconsistencies and paradoxes. It should be possible to understand C as a theory involving only concrete and not abstract entities, for one might suppose that abstract entities are mysterious and that we should avoid postulating them whenever possible. The concrete entities in this case are finite sign configurations understood in the first instance, if they are to be perceived with the senses, as sign tokens. In the language of chapter 1, C should be concerned with what is real and not with what is ideal (in the sense of the ideal or abstract objects thought by platonists to exist). All of the parts of mathematics in which we might have thought that ideal or abstract elements (such as abstract "meanings" of semantics) were present would thus be shown to be secure. The sentences and proofs of C should be surveyable in immediate intuition. Immediate intuition is a non-mysterious form of sensory perception. C should not be a creature of pure conception or pure reason, for the worry is that we may be led by pure reason into antinomies and hopeless confusion. C is the part of our mathematical thinking that is contentual and meaningful. The contrast is with parts of mathematical thinking that we may regard as purely formal and "meaningless" in the sense that we need not consider their purported references to abstract concepts or infinitary objects.

A very likely candidate for C is PRA (see, e.g. Tait 1981). PA is arguably less suitable, given the concern for security and issues about finiteness, concreteness, availability of signs to immediate sensory perception, and related matters. In whatever manner we construe C, however, it follows from the first incompleteness theorem that if C is consistent then the Gödel sentence for C cannot be decided by C even though it is true. It follows from the second theorem that if C is consistent then C cannot prove CON(C). Given the way we characterized C above, it is therefore quite natural to draw the conclusion that deciding the Gödel sentence for C or proving CON(C) must require objects or concepts that *cannot be completely represented in space-time as finitary, concrete, real, and immediately intuitable*. In other words, if we adopt the distinctions in

Hilbert's philosophy then *deciding the Gödel sentence for C or proving that CON(C) must require appeal to the meanings of sign configurations, to objects or concepts that are in some sense infinitary, ideal, or abstract, and not immediately intuitable* in Hilbert's sense.

Gödel, in fact, notes such implications in various places in his writings. For example, in his so-called *Dialectica* paper, Gödel (Gödel 1972, pp. 271–72) says that

P. Bernays has pointed out on several occasions that, in view of the fact that the consistency of a formal system cannot be proved by any deduction procedures available in the system itself, it is necessary to go beyond the framework of finitary mathematics in Hilbert's sense in order to prove the consistency of classical mathematics or even of classical number theory. Since finitary mathematics is defined as the mathematics of *concrete intuition*, this seems to imply that *abstract concepts* are needed for the proof of consistency of number theory... [What Hilbert means by "Anschauung" is substantially Kant's space-time intuition confined, however, to configurations of a finite number of discrete objects.] By abstract concepts, in this context, are meant concepts which are essentially of the second or higher level, i.e., which do not have as their content properties or relations of *concrete objects* (such as combinations of symbols), but rather of *thought structures* or thought contents (e.g., proofs, meaningful propositions, and so on), where in the proofs of propositions about these mental objects insights are needed which are not derived from a reflection upon the combinatorial (space-time) properties of the symbols representing them, but rather from a reflection upon the *meanings* involved.

It was part of Hilbert's program, and also of Carnap's in *Logical Syntax of Language*, that there should be no consideration of the meaning of the finite sign configurations in the formalizations. Gödel's idea in this passage that it is necessary to reflect on meaning plays a central role in a number of his philosophical papers to be discussed below (for example, Gödel *1961/?, *1953/59), as does the idea that reflection on meaning or on concepts is of a "higher level" than reflection on the combinatorial properties of concrete symbols.

Gödel was not the only one to draw such conclusions from his incompleteness theorems. Emil Post, writing about Gödel's incompleteness theorems around 1941, evidently had something similar in mind when he remarked that

It is to the writer's continuing amazement that ten years after Gödel's remarkable achievement current views on the nature of mathematics are thereby affected only to the point of seeing the need of many formal systems, instead of a universal one. Rather has it seemed to us to be inevitable that these developments will result in a reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth. Postulational thinking will then remain as but one phase of mathematical thinking.

Gödel, of course, thinks that intuition must still play a role in mathematical knowledge. If intuition, as opposed to mere conception, is going to play a role in our knowledge in mathematics and logic, however, then it can no longer be a matter simply of immediate, prereflective sensory perception of sign tokens. It must be a kind of "rational intuition," such as Husserl's categorial intuition or *Wesensanschauung*. These remarks also suggest that the notion of rigor need not be limited only to formal rigor. There can be a kind of informal rigor in mathematical thinking (see also Kreisel 1967).

Were there no intuition of such abstract objects or concepts, were we able to proceed only from within C, then we would evidently have to stop or to "become static" with respect to deciding the Gödel sentence for C or obtaining a proof of CON (C). That is, we could not decide some clearly posed mathematical problem. What I mean by this is that we could not decide it unless the decision were to be made arbitrarily or on non-mathematical grounds, which would arguably violate conditions on the use of reason in mathematics and logic. Now for some T (e.g. PRA, PA), we in fact do have proofs of CON(T) and decisions of related problems. This suggests that there are such entities and that there must be some conscious directedness toward them. It would be natural to claim that the human mind (monad), in its use, is not destined to remain static in these circumstances but is constantly developing its understanding of the abstract objects or concepts of mathematics and logic.

It needs to be emphasized again that C cannot be just any formal theory. We cannot keep extending C with new axioms that allow us to decide sentences that were previously undecidable and still expect to have concrete mathematics, for then we might as well have started with something like ZF in the first place. As a purely formal theory, ZF is of course concrete in the same sense as PRA, but the difference is that ZF certainly could not codify concrete, finitary mathematics. To hold that it could is to completely subvert the philosophical basis of Hilbertian proof theory.

When we consider the incompleteness theorems against the philosophical background from which they emerged we can therefore see how it is natural to begin an ascent to platonism and rationalism about mathematics and logic, for it appears that we must move from the concrete, finite, and immediately intuitable in sense perception to the abstract, infinite, "ideal," and mediately intuitable, where a grasp of meaning again plays a role, if we are to obtain consistency proofs and to decide undecidable sentences. One might say that platonic rationalism fits the incompleteness theorems and also the speed-up results better than various forms of empiricist anti-platonism. It provides, as it were, a better explanation of the incompleteness phenomena. There is of course much more to say about this, and the claim will be taken up in more detail in later chapters. We can, however, already add some elements to the argument here.

It is possible to obtain a reasonably good understanding of the sense in which the objects or concepts used in the decisions of Gödel sentences or in consistency proofs must be abstract, infinitary, and not completely captured in immediate intuition. In the case of PA, for example, a consistency proof in the style of Gentzen requires the use of transfinite induction on ordinals $< \epsilon_0$, or in the style of Gödel's own *Dialectica* interpretation, it requires primitive recursive functionals of finite type. Thus, some kind of epistemic capacity that takes us beyond C seems to be required. There seems to be an ascent to concepts of a higher level, that is, concepts which do not have as their content properties or relations of concrete objects such as combinations of symbols.

In other words, there is prima facie a capacity for conception and knowledge involved here that has been associated with reason, and not with mere sense perception. It does not seem to be a matter of mysticism, religious belief, or whimsy. The kind of reflection involved in this knowledge, as Gödel says, is not reflection on the combinatorial space-time properties of symbols. Instead, it involves the concept of transfinite ordinals $< \epsilon_0$, or the concept of primitive recursive functionals of finite type. These concepts seem meaningful to us. In the passage we quoted above, Gödel speaks of reflection on the "thought contents" or "thought structures" involved, and he refers to these as "mental objects," as opposed to concrete sense objects. In any careful analysis of human cognitive capacities, acts of reflection should be distinguished from acts of sense perception. On the conception of reason to be discussed in more detail in chapters 5 and 6, we will hold that acts of reflection are a function of reason. In acts of reflection we are no longer simply engaged in prereflective immediate intuition of sensory objects. Acts of abstraction, to be discussed in more detail in chapter 6, are also a function of reason (or conception), and in the case at hand there is an abstraction in which the combinatorial space-time properties of symbols are seen as secondary to the primary function of reason in mathematics. Generalization, idealization, and even imaginative variation figure into the use of reason. According to the phenomenology of human consciousness there are indeed such kinds of cognitive activities, and it appears from Gödel's work that we should be prepared to recognize them if we are not to stop with respect to solving certain kinds of mathematical and metamathematical problems. We need an account of human reason, evidence, justification, and proof that takes us beyond strict finitism and concrete mathematics, and hence beyond the empiricism and anti-platonism that is arguably involved in such a restriction. In a letter written to Leon Rappaport (Gödel 2003a, Vol. V, p. 176) Gödel thus says that

> it does not follow from my theorems that there are no *convincing* consistency proofs for the usual mathematical formalisms, notwithstanding that such proofs must use modes of reasoning not contained in those formalisms. What is practically certain is that there are, for the classical formalisms, no conclusive *combinatorial* consistency proofs (such as Hilbert expected to give), i.e., no consistency proofs that use only concepts referring to finite combinations of symbols and not referring to any infinite totality of such combinations.

§ 7. Beyond intuitionistic restrictions

Intuitionism and some other forms of constructivism already recognize cognitive capacities and forms of evidence that transcend concrete, finitistic mathematics. Intuitionism is a kind of idealism (in the sense of anti-realism) or "conceptualism" about mathematics (see also chapter 4, §2). Intuitionists hold that there are no unexperienced or unexperienceable objects or truths in mathematics or logic. To put a somewhat finer point on it, relative to the discussion in chapter 4, there are no non-mental objects in mathematics and logic. To be is to be intuited, or to be intuitable, according to the intuitionistic view of intuition which, of course, differs in particular from Hilbert's conception of intuition. Hilbert's formalism depends on what Kant would have called "outer" intuition. Intuition for Hilbert is sensory intuition of sign tokens given in space and time, or perhaps it is intuition of sign types associated with such tokens. Brouwerian intuitionism, on the other hand, depends on what Kant would call the form of "inner" intuition, namely, the temporality of inner mental processes, as indicated in many of the papers in Brouwer 1975. Forms of constructivism that differ from intuitionism are not always motivated by philosophical idealism, but they nonetheless involve restrictions of various types in order to avoid what is viewed as an unwarranted epistemology or ontology. Although intuitionism and other forms of constructivism are fascinating and deserve serious study and further development, Gödel would evidently argue that we need to countenance rational activities in which we rise above intuitionism and these other forms of constructivism if we are to do justice to existing parts of mathematical practice, especially higher set theory (see especially Gödel 1964). One could say that Gödel thinks that rational justification is not confined to either Hilbertian or Brouwerian notions of intuition. We need to recognize concepts that have a content different from the content of the concepts of intuitionism and other constructivist programs. Other forms of conception and meaning conferral must play a role in actual mathematical and logical practice. From the perspective of intuitionism or a constructivism that is more liberal than concrete finitism, for example, we would evidently still have to stop with respect to deciding a clearly posed mathematical problem such as the question whether the continuum hypothesis (CH) is consistent with the axioms of ZF. This is evidently a clearly posed mathematical problem that has a mathematical solution. Before the solution was found there might have been skeptics who thought the problem was inherently vague or ill-formed, unsolvable, meaningless, and so on. In order to obtain the proof of the consistency of CH with the existing axioms of ZF it was necessary to allow certain forms of impredicativity and to take the classical ordinals as given. It was necessary to go beyond constructive mathematics, and evidently one thereby moves beyond standard forms of idealism to a platonist position. In recognizing rational scientific activities that transcend those allowed by constructivism or standard mathematical idealism we obtain a result that would not have been forthcoming otherwise. The classical ordinals cannot be identified with, or completely represented by, concrete, finite sign configurations, nor can they be mind-dependent in the sense that human beings can complete the construction of these objects individually in time on the basis of step-by-step mental operations. They cannot be intuited in either Hilbert's or Brouwer's sense.

What this means is that we are countenancing in this case the existence of objects that cannot be completely given in intuition of the type recognized by Hilbert or Brouwer. In his paper on Cantor's continuum problem, Gödel takes note of this fact and maneuvers around finitism and constructivism by saying that the negative attitude toward Cantor's set theory, and toward classical mathematics, of which it is a natural generalization, is by no means a necessary outcome of a closer examination of their foundations, but only the result of a certain philosophical conception of the nature of mathematics, which admits mathematical objects only to the extent to which they are interpretable as our own constructions or, at least, can be completely given in mathematical intuition. For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation of Cantor's set theory in its whole original extent and meaning, namely, axiomatics of set theory interpreted in the way sketched. (Gödel 1964, p. 258)

The sketch of the interpretation Gödel mentions in the Cantor paper refers to an iterative concept of set. Gödel recognizes that the individual objects of higher set theory—transfinite sets—cannot be completely given to human beings (finite monads) in mathematical intuition. Thus, he appeals to a kind of platonic rationalism, referring to those who consider mathematical objects to exist independently of our constructions, and says that it is not necessary that we be able to have an intuition of transfinite sets individually, provided that the concepts involved in higher set theory are sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them. Gödel thus shifts the issue to the question of the clarity of the basic concepts involved in Cantorian set theory, and here he reverts to his call for meaning clarification. We need to clarify our intuition of the abstract concepts by virtue of which we are directed toward the objects of set theory.[4] Transfinite sets (as extensions) cannot be completely given to us in mathematical intuition, but the abstract concepts by virtue of which we are directed toward such objects can be given to us in reflection in a kind of rational intuition, and it is on this basis that there can be a foundation for Cantorian set theory. Gödel of course thinks that Husserlian phenomenology can at least help to provide such a foundation.

At this level we are thus supposing that there are objects—transfinite sets—that in some sense transcend the mind. The objects here exist independently of our constructions but on the Husserlian view it would still be possible to be directed toward such objects by way of the concepts involved in our thinking in set theory. This is just a particular example of the intentionality of human consciousness and the idea, as we will see in later chapters, would then be to reflect carefully on such concepts. According to the constituted platonism we will discuss in more detail later, we would also hold that the objects here—transfinite sets—are meant in our constituting activities non-arbitrarily as existing independently of our constructions. They are meant as

[4] Compare this with Georg Kreisel's remark (Kreisel 1969, p. 97) that the notions or concepts of set theory are difficult to analyze, not that they are dubious. Kreisel goes on to say that "... probably the first step is... *to recognize the objectivity of the basic notions* (subset, power set)... and then, if possible, give a phenomenological analysis of these notions." I think it likely that these comments of Kreisel were influenced by Gödel.

mind-independent in the very acts of consciousness in which we are engaged in the practice of higher set theory. If there are objects that are not concrete and not completely intuitable or graspable according to finitist, nominalist, or idealist conceptions of mathematics, then we have stepped over to the platonist's position. Other kinds of cognitive activities exhibiting intentionality, such as abstraction, higher-level reflection and meaning clarification, idealization, and imaginative variation must be involved. The mind-independence of the objects involved is further highlighted by the fact that it is impredicative set theory that is required in the proof of the consistency of ZF + CH, and the impredicativity involved evidently forces us outside of constructivist bounds. The only philosophical position that does not impose restrictions on reflection, abstraction, generalization, idealization, imaginative variation, and proof of the kind found in constructivism, predicativism, and related views, would be a rationalist epistemology of some type that is compatible with mathematical or logical platonism. Platonic rationalism would not place the same limits on the imagination of possibilities or on the determination of necessities relative to these possibilities. We presumably do not want to impose restrictions on these kinds of cognitive activities in science if, for example, we are to have any hope of meaningfully confirming or rejecting beliefs such as "CH is consistent with ZFC," "¬CH is consistent with ZFC," or even beliefs such as CH itself, $2^{\aleph_0} = \aleph_1$.

There is, of course, a difference between the statements (i) "CH is consistent with ZFC" and (ii) CH itself. The latter is a conjecture, but it is widely held that the former is a fact. We have a proof that establishes that CH is consistent with ZFC. Now how could we argue that CH is inherently vague or even meaningless if it is the same CH we are talking about in (i) and (ii)? Could it possibly be vague or meaningless in the one case but not the other? If CH were inherently vague could we have a mathematical proof of the statement that "CH is consistent with ZFC"? How could we regard this statement as having a truth value? There seems to be an incongruity between holding that CH itself is inherently vague, but that the statement in which it appears, "CH is consistent with ZFC," is not inherently vague.

One can point to some clearly posed mathematical problems that certainly cannot be decided using the resources of C but that can be decided in a meaningful, rational, and principled way if we recognize forms of reflection, abstraction, idealization, imagination, and proof that take us beyond C. I think that clearly posed mathematical problems in fact are not, and ought not to be, decided arbitrarily or on non-mathematical grounds. They ought to be decided on the basis of reason. The existing consistency proofs for PA, for example, are not arbitrary or non-mathematical, nor are the consistency proofs of ZFC + CH and ZFC + ¬CH. In his 1961 paper Gödel says that the problems that led to the focus on concrete mathematics were exaggerated by empiricist skeptics. In particular, the paradoxes of set theory were used as a pretext for a skeptical upheaval. On Gödel's analysis (as in Gödel *1961/?), the paradoxes were not the result of failing to adhere to the limitation of "leftward" empiricist or materialist strictures. Instead, they resulted from generalizations or abstractions in set theory that are unfounded even on rationalist grounds, such as dividing the totality of all existing things into two categories instead of viewing sets as obtained from iterated applications of the operation "set of" to integers or other well-defined objects (Gödel 1964, pp. 258–59). The view would evidently be that it is possible to skirt the paradoxes and to arrive at non-arbitrary, rationally motivated, and yet non-mechanical decisions in higher set theory.

Gödel is prepared, it seems, to hold that, as we ascend from a finitist approach in foundations to a platonic rationalism about set theory, we engage in greater and greater idealizations and adhere to less stringent standards of evidence. With such an ascent we have to be willing to tolerate less clarity in parts of mathematics (see the discussion on pp. 211–220 of Wang 1996). One might, for example, distinguish the following positions on these grounds: strict finitism, finitism, predicativism, intuitionism, classical mathematics, and classical mathematics that includes full set theory. Finer classifications would, of course, be possible (see Gödel's own classification in *1953/59, p. 346, footnote 32, which is also discussed below in chapter 3). At the same time, Gödel says that "nothing remains if one drives to the ultimate intuition or to what is completely evident" (Wang 1996, p. 212). To require that everything be completely evident would be to destroy science altogether, and would serve no positive purpose. "If you give up idealization then mathematics disappears. Consequently, it is a subjective matter where you want to stop on the ladder of idealization" (Wang 1996, p. 217). Similarly, Gödel says that "without idealizations nothing remains: there would be no mathematics at all, except the part about small numbers. It is arbitrary to stop anywhere along the path of more and more idealization" (Wang 1996, p. 217). Brouwer, Heyting, and other philosophical constructivists would, of course, demur. These comments of Gödel, however, can be read as reflecting his conception of the scope of reason in mathematics and logic, a conception that is not shared by intuitionists and other philosophical constructivists. It is in the very nature of the employment of reason in mathematics to idealize and abstract. It does not follow, by Gödel's sights, that there is no evidence at all in higher set theory, or that there is no meaning or order or systematicity. On the contrary, all of these things are present. There is scientific practice at this level of mathematics, rigor, proof, a host of results, and a stability that has been built up over years of investigation. These are arguably all signs of reason at work.

Wang tells us that Gödel even conjectured at one time that if set theory is inconsistent, then elementary number theory is already inconsistent (Wang 1996, p. 216). This is clearly related to his claim that it is arbitrary to stop anywhere along the path of more and more idealization in mathematics. The point is evidently that the idealizations involved at each stage are rationally motivated, and so closely bound up with one another that a serious weakness at one point would bring down the whole of this magnificent development of human reason that we call mathematics.

If you acquire some training in mathematics and pick up a book on classical number theory or higher set theory and study it, then the book will no doubt appear to you to have some meaning, even if it takes a while for you to come to a better understanding of it. It will not appear to you as utterly meaningless syntax. As you read, your thoughts will be directed in certain ways and not others. There is some kind of meaning here that is given in mathematical practice in these areas. Among other things, it appears that the objects about which you are thinking and conversing in this context are not meant as spatial objects, temporal objects, causal objects, and so on. No spatial, temporal, or causal language is used in describing the properties and relations of the objects of number theory or set theory in mathematical practice. This is how the meaning of being of these objects is constituted (see chapters 4 and 6 below). Entire categories of predicates are not applied to these objects, while other categories of predicates are applied. Husserl would then have us ask the following question: *what are the conditions for the possibility of the constitution of this kind of meaning*? The objects are meant or intended in certain ways and not others in mathematical practice. The meaning is there for us. What are the conditions for its possibility? How could this meaning be constituted? There can, of course, also be meaning constitution in constructive mathematics, but in the present context I want to highlight the non-arbitrary, meaningful, and rational nature of mathematical practice in parts of mathematics that are non-constructive.

Gödel's view is evidently that even if you come to understand a book in higher set theory quite well, not everything about sets will be clear and distinct. Given the concepts involved, the established results should in principle be clear and distinct even if it takes some hard work to come to an understanding of them, while our understanding of the conjectures and open problems, relative to the established results, will not be clear and distinct. His response is to call for meaning clarification and to seek a method of meaning clarification, not to give up on the possibilities of reason and science. It is to hold that we can expect to obtain more clarity about the basic concepts of set theory, so that we might solve open problems in this domain of thought, such as the continuum problem.

§ 8. A view of the incompleteness phenomena from the perspective of platonic rationalism

One might imagine that those who are inclined toward forms of empiricism and anti-platonism would start from the bottom up, as it were, in how they understand mathematics and logic. One would start with "outer" sense perception, or generalizations founded on sense perception, or theoretical hypotheses of natural sciences that depend on sensory observation, or with the "inner" perception of an empirical introspectionist psychology, and then view mathematics and logic through these lenses. Similarly, one might imagine that those who are inclined toward platonic rationalism would start from the top down, so to speak, by viewing mathematics and logic as productions of (pure) reason or conception, in which there is conscious directedness toward abstract or ideal objects and truths that are mind-independent in certain ways (Tieszen 1994a). Then there would be applications of the exact concepts of mathematics and logic to the world of sense experience, with various kinds of accommodations and adjustments, by way of the different and additional conceptions involved in the natural sciences.

Now, if we start with some form of platonic rationalism, then the incompleteness of formal systems (or TMs) looks fairly natural. Incompleteness is the incompleteness of finitary, consistent formal systems (TMs), and if mathematical meanings, concepts, or objects did not transcend such formal systems (TMs), then platonic rationalism would just collapse into mechanistic formalism. There would be nothing about mathematics that was not captured in some formal system, or by some machine. This claim appeals not only to the fact that platonic rationalism acknowledges the existence of abstract objects or concepts but also to the fact that it acknowledges, in some more or less robust sense, the existence and mind-transcendence of abstract objects or concepts that are transfinite. In his 1931 paper Gödel already remarks that incompleteness is inherent in all formal systems of mathematics because the formation of ever higher types can be continued into the transfinite, while in any formal system at most denumerably many of these types are available (Gödel 1931, p. 181, footnote 48a). Kleene (Kleene 1986) has noted that it is implicit in this remark that the adjunction of higher types to a formal system permits one to define the notion of truth for that system, and then to show that all of its provable sentences are true, and hence to decide the proposition shown to be undecidable in the system. It is natural to read this in terms of a platonic rationalism.

Gödel, in fact, says that the "heuristic principle" of his construction of undecidable arithmetical propositions in formal systems of mathematics is a form of platonic rationalism about the concept of mathematical truth of just the sort that we have been discussing. The heuristic principle is the "highly transfinite concept of objective mathematical truth," as opposed to the concept of (concrete, relative) formal provability with which mathematical truth was generally confused before the work of Gödel and Tarski (see Wang 1974, p. 9). This transfinite concept eventually leads to finitarily provable results. The argument is that applications of a platonic rationalism about mathematical truth give us specific scientific results in mathematics and metamathematics. In particular, platonist/rationalist assumptions might help us to arrive at results such as the incompleteness theorems, various kinds of relative consistency proofs, and decisions of various mathematical problems.

As Gödel (Wang 1974, p. 9) notes, how could one think of expressing metamathematics in mathematical systems if the latter are supposed to consist of meaningless symbols which acquire meaning only through metamathematics? The basic idea of arithmetizing syntax is essential for the proof of the incompleteness theorems, and the undecidable sentences that Gödel constructs are arithmetic sentences. For the platonist, the sentences of a mathematical system are taken to have meaning or to be true independently of, or in a way that transcends, syntax manipulation or metamathematics. But then there is certainly no bind in thinking of expressing the syntax of the system *in* the system. Similarly, how would it be possible to give a consistency proof for the continuum hypothesis by means of Gödel's inner model technique if consistency proofs have to be finitary? From the viewpoint of mechanistic formalism, an interpretation of set theory in terms of Gödel's model seems preposterous since it is an "interpretation" in terms of something which itself has no meaning. The fact that such an interpretation, as well as any non-finitary consistency proof, yields a finitary relative consistency proof escapes notice otherwise. Gödel points out that failing to apply a platonic rationalism would make it nearly impossible to discover the consistency proof for the continuum hypothesis, because the ramified hierarchy, which had been invented for constructive purposes, had to be used in an entirely non-constructive way. A similar remark applies to the concept of mathematical truth, where mechanistic formalists considered formal provability to be an analysis of the concept of mathematical truth, and were therefore not in a position to distinguish the two. The concept of "mathematical truth" also had to be used in a non-constructive or non-finitary way.

With these remarks we are also on grounds of mathematical practice that favor a form of platonic rationalism over traditional intuitionism. Traditional intuitionism and platonic rationalism differ on the nature of the "abstract" objects and meanings that we have discussed, on the question of their mind-independence, on the nature of mathematical intuition, on the kinds of mathematical sentences that are meaningful, and so on. Now one could observe, in parallel with some of the remarks we just made, that an interpretation of set theory in terms of Gödel's inner model would also seem preposterous to a traditional intuitionist since it would be an "interpretation" in terms of something which itself had no intuitionistic meaning. For traditional intuitionism, what has meaning is what is based on construction or intuitionistic intuitability, and the axioms of ZF and ZFC do not fall within this domain. In addition, as Gödel points out, the ramified hierarchy had to be used in a non-constructive way. The platonic rationalist, however, would say that it is a highly transfinite concept of objective mathematical truth that naturally underwrites the discovery of these relative consistency results. Why would an intuitionistic conception of mathematical truth not suffice? The answer is straightforward. Traditional intuitionists will take intuitionistic provability to be an analysis of the concept of mathematical truth and are not in a position to distinguish the two. Recall, however, that Gödel was working with *PM*, Bernays/Gödel set theory (*BG*), *ZF*, and related systems. Now to think that there could be a truth or a meaningful proposition of *PM, BG, or ZF* that would not be provable in *PM, BG* or *ZF* would certainly require a non-intuitionistic conception of truth or meaning (see Solovay 1990 for a characterization of *BG* and its relation to *ZF*). It is true that intuitionists are anti-formalists and might have expected the incompleteness theorems, but in the context of systems that are this comprehensive, it is a transcendent (and highly transfinite) concept of objective mathematical truth, not an intuitionistic conception of truth or meaning, that would help to provide the kind of clarity needed to find a proof of incompleteness. Moreover, it would not be possible to think of expressing the metamathematics of systems like *PM*, *BG*, or *ZF* in *PM*, *BG*, or *ZF* if these systems are supposed to consist of meaningless symbols which acquire meaning only on the basis of intuitionistic intuition or construction. At the time, this was perhaps even more important, because finitism was not always clearly distinguished from intuitionism, and there were some complicating factors in Brouwerian intuitionism resulting from solipsistic denials of objective truth.

I should add that I think it is not clear how modified versions of Hilbert's program would deal with these points about the incompleteness theorems and consistency proofs. For example, how would it be possible to think of expressing the metamathematics of ZF or systems of real analysis in these systems if these systems are supposed to consist of meaningless symbols which acquire meaning only on the basis of our grasp of the inductive build-up of objects? We must evidently have some insight into meaning or truth independently of our grasp of such a build-up. It is not clear how this could be explained without adopting a form of platonic rationalism of the type we have been discussing. I want to emphasize, however, that these critical remarks about extended forms of Hilbert's program do not seem to me to undermine at all the value of the results that have been obtained by proof theorists. Results on the ordinal strength of formal theories, along with the many other kinds of results that have been obtained by proof theorists are very enlightening and valuable. The point is that the philosophy underlying Hilbertian proof theory, even in its extended form, can lead to interesting and important ideas and results, but that it does not, and cannot, give us the whole picture of the nature of mathematics and logic.

In light of these comments on platonic rationalism, one could read the first incompleteness theorem for PA as follows. The first incompleteness theorem suggests that the abstract concept of objective arithmetic truth transcends our intuition (or constructive abilities) at any given stage, in the sense that we know we can always constitute additional instances of this concept at various times that we have not yet intuited or constructed, in the form of the specific Gödel sentences G_{PA}, $G_{PA'}$, $G_{PA''}$, ... The concept of arithmetic truth then appears to be known as an identity (or "universal") through these differences which "transcends" the construction of the specific instances at any given stage in time. This identity (or "universal") is "outside of" or "independent of" each particular intuition (construction). This is how mathematical platonism is often characterized; that is, as claiming that there are universals or invariants that transcend the mind or our intuition. Apart from problems about solipsism, it is not clear how *traditional* intuitionism can accommodate such a platonic concept of arithmetic truth. Such a concept at least seems meaningful to us, in light of the first theorem, but what is not based on intuition or construction is meaningless for the traditional intuitionist. Traditional intuitionism holds that nothing lies outside of our intuitions or constructions which, being mental in nature, occur in time. There is no intuition-transcendent non-mental mathematical truth. The mind constructs mathematical objects and truths.

In connection with mathematical practice as a whole, I think there is a problem with traditional intuitionism at precisely this point. By way of contrast, suppose that one tried to couple a rationalist account of our mathematical mental acts and processes with constituted platonism about the objects of our mathematical intentions (see chapter 4). Such a view would be compatible with this reading of the first incompleteness theorem, but it is not traditional intuitionism. It also seems quite unlikely that extended versions of Hilbert's program could accommodate such a view of arithmetic truth.

Gödel holds that even the completeness proof for predicate calculus depends on the application of platonic rationalism (Wang 1974, p. 8). He says that the inability to find the completeness proof, although in 1922 Skolem was very close, came from a lack of the required platonic attitude toward metamathematics and non-finitary reasoning. The prevailing philosophical *Zeitgeist* (see chapters 3 and 7) made it difficult to conceive the possibility that mathematical truth might be different from finitistic formal provability. The claim is that when we are too firmly in the grip of mechanistic or finitist formalism, or other types of anti-platonism, we develop a kind of blindness about these matters.

On all these points we see that dividends result from being cautious about certain forms of (eliminative) reductionism in mathematics. It makes sense, for example, to hold that "mathematical truth" may not be reducible to "formal provability," or even to "intuitionistic provability." That is, it should not be assumed from the outset that these expressions have the same meaning, as though they were synonymous, but we should instead see what happens as they are further clarified. Thus, there are some grounds for a reverse skepticism about philosophical views that tend to be eliminativist about basic concepts of logic and mathematics. Apart from formalism and traditional intuitionism, the positions of positivism, conventionalism, fictionalism, naturalism, and empiricism tend toward eliminative reductionism, and therefore might blind us to interesting or important results in mathematics. They might actually hinder some types of *mathematical* progress at the expense of some philosophical theory. We will have much more to say about these positions later in the book.

Would it not be easier to ignore platonic rationalism about mathematics and logic if the incompleteness theorems had not been, or could not be, proved? It seems that the answer is "yes." But we can now see how it might be argued that the incompleteness results and consistency proofs on the one hand, and platonic rationalism about meaning and mathematics on the other, tend to reinforce one another. A form of platonic rationalism helps to explain incompleteness, and incompleteness, along with the various consistency proofs, lends support to a form of platonic rationalism. Platonic rationalism about mathematics and the incompleteness theorems appear to be natural companions.

Other very common phenomena in mathematics are also explained by the kind of platonic rationalism we have been discussing. For example, it explains the fact that, in actually doing mathematics, we frequently experience mathematical statements as meaningful, as having semantic content (not just as meaningless sign-configurations); the fact that we take ourselves to have evidence or proof in mathematics, based on an understanding of this meaning (e.g. in set theory), even when we are not working in a rigorous formal system, or in a finitistically acceptable proof theory; the fact that we seem to employ routinely a kind of informal rigor in mathematical thinking, and so on. These are phenomena one would predict if one were a platonic rationalist about meaning and mathematics of the sort described above, but not if one were a strict formalist. For a strict formalist or mechanist these phenomena are anomalous and need to be explained away or reduced.[5]

§ 9. Some early expressions of Gödel's platonic rationalism

The first published expression of Gödel's mathematical and logical platonism appears in the 1940s in "Russell's mathematical logic" (Gödel 1944; also Parsons 1990). In this paper Gödel remarks on the platonistic attitude that Russell adopted in some of his work, and he develops some analogies between mathematics and natural science that are suggested by Russell. In particular, Gödel notes how Russell compares axioms of mathematics and logic with laws of nature, and mathematical evidence with sense perception. Axioms need not be evident in themselves. Their justification could lie in the fact that they make it possible for the analog of sense perceptions, that is, arithmetical facts, to be deduced. Gödel thinks this view has been largely justified by subsequent developments, and that in the future we will find it even more convincing. He writes that

> It has turned out that (under the assumption that modern mathematics is consistent) the solution of certain arithmetical problems requires the use of assumptions essentially transcending arithmetic, i.e., the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception. (Gödel 1944, p. 121)

The discussion of platonism is deepened when Gödel comes to Russell's approach to the paradoxes. Gödel notes that one of Russell's formulations of the vicious circle principle (VCP) (i.e. no totality can contain members definable only in terms of this totality) makes impredicative definitions impossible, and forces a kind of constructivity. If classical mathematics violates the VCP, however, then this is reason to take the VCP

[5] For some broader reflections on phenomenology and incompleteness, see Bailly and Longo, 2008.

to be false rather than classical mathematics. One of Gödel's most widely quoted philosophical comments is made in this connection:

> Classes and concepts may, however, also be conceived as real objects, namely classes as "pluralities of things" or as structures consisting of a plurality of things and concepts as the properties and relations of things existing independently of our definitions and constructions.
>
> (Gödel 1944, p. 128)

Gödel adds that "the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence." They are just as necessary in obtaining a satisfactory system of mathematics as physical bodies are in obtaining a satisfactory theory of sense perceptions.

The 1947 version of "What is Cantor's Continuum Problem?" also contains expressions of Gödel's platonic rationalism, albeit in a less robust form than the 1964 version of the paper. In the 1947 version Gödel says that, even if CH should turn out to be provably undemonstrable on the basis of the current axioms of set theory, it does not follow that the matter of its truth or falsity is settled. Only someone who denies that the concepts and axioms of classical set theory have any meaning, such as the intuitionist, could be satisfied with such a solution, but not someone who believes that the concepts and axioms describe some well-determined reality:

> For in this reality Cantor's conjecture must be either true of false, and its undecidability from the axioms known today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to point out ways in which a decision of a question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained. (Gödel 1947, p. 181, Vol. II)

The 1964 version of the Cantor paper, which appeared after Gödel had begun to study Husserl's work, elaborates on a kind of platonic rationalism, especially in the "Supplement to the second edition." I will come back to this at the beginning of chapter 4. Chapters 4, 5, and 6 go into much more detail about the kind of platonism and the kind of rationalism about mathematics and logic that can be developed and defended on the basis of the ideas in Husserl's philosophy that were of interest to Gödel. First, however, we will consider in more detail Gödel's arguments in the 1950s against Carnap, and his own account in the early sixties of his turn to Husserl's philosophy, which includes some additional views he held about Hilbert's program and about the relation of Kant to Husserl.

3

Gödel's Path from Hilbert and Carnap to Husserl

Proceeding in roughly historical fashion, I focus in this chapter on Gödel's philosophical writings from the 1950s and early 1960s on Carnap, Hilbert, and Husserl. The editor of the *Library of Living Philosophers* series, Paul A. Schlipp, wrote to Gödel in May of 1953 to invite him to contribute a paper to a volume on the philosophy of Rudolp Carnap. After drafting six versions of the paper and engaging in several rounds of correspondence, Gödel wrote to Schilpp in February of 1959 to say that he was not satisfied with the paper and had therefore decided not to publish it.[1] Hao Wang, writing about Gödel's decision to abandon the Carnap paper and to study Husserl's work in 1959, says that

> It seems to me that the two decisions may have been related. He had, he once told me, proved conclusively in this [Carnap] essay that mathematics is *not* syntax of language but said little about what mathematics *is*. At the time he probably felt that Husserl's work promised to yield convincing reasons for his own beliefs about what mathematics is
>
> (Wang 1996, p. 163; see also the first quotation in chapter 1 above).

Continuing in this vein, Wang says

> It is, therefore, not surprising that, when he commented on various philosophers during his discussions with me, he had more to say about the views of Husserl than about the positivists or empiricists. Indeed, his own criticisms of the empiricists tend to be similar to Husserl's.

I will now follow up on these comments of Wang and consider Gödel's critique of Carnap in the versions of his Carnap paper "Is Mathematics Syntax of Language?"[2] (Gödel *1953/59) in order to show how these are linked to Gödel's discussion of Hilbert and of Husserl's philosophy in "The Modern Development of Mathematics in

[1] *Kurt Gödel: Collected Works, Vol. III* contains versions III and V of this paper, "Is Mathematics Syntax of Language?," and Rodriguez-Consuegra 1995 contains versions II and VI. I have reviewed Rodriguez-Consuegra 1995 in Tieszen 1997. The Schilpp volume (Schilpp 1963) eventually appeared without Gödel's paper.

[2] It should be noted that some of Carnap's early work, especially his thesis on space and *Die logische Aufbau der Welt*, was directly influenced by Husserl's philosophy. See, e.g. Rosado-Haddock 2008 and various essays in Awodey and Klein 2004. The idea that mathematics is nothing more than syntax of language, however, is antithetical to Husserl's views, especially after the "Prolegomena to Pure Logic" of the *Logical Investigations*.

the Light of Philosophy" (Gödel ⋆1961/?). Investigation of the philosophical links between these texts can help us to develop a much better understanding of what Gödel thought mathematics is and of how Husserlian transcendental phenomenology, appropriating certain ideas of Plato, Leibniz, and Kant, might be of aid in articulating and defending his view. This chapter takes us deeper into Gödel's philosophical thinking about mathematics and logic and also provides some of the basic background for the chapters below.

§ 1. Godel's arguments against Carnap's view of mathematics as syntax of language

In drafts of ⋆1953/59 Gödel says that around 1930, Carnap, Hahn, and Schlick, largely under the influence of Wittgenstein, developed a conception of the nature of mathematics that can be characterized as a combination of nominalism and conventionalism. Gödel says that its main objective, according to Hahn and Schlick, was to reconcile strict empiricism with the a priori certainty of mathematics. This objective amounts to an effort to find a workable combination of what Gödel calls "leftward" and "rightward" philosophical views in the 1961 text to be discussed below.

In turning to Carnap, Gödel says that Carnap's program in *Logical Syntax of Language* (Carnap 1934a) and related works in the 1930s aims to establish three basic philosophical points: (i) mathematical intuition (which is later associated by Gödel with Husserl's categorial intuition or *Wesensschau*), for all scientifically relevant purposes, can be replaced by conventions about the use of symbols. Mathematical intuition of abstract objects is, of course, not acknowledged as a source of knowledge by proponents of the syntactical view. The original purpose and chief interest of the syntactical interpretation refer to the question as to whether it can replace the belief in the correctness of mathematical intuition; (ii) mathematics, unlike other sciences, does not describe any existing mathematical objects or facts. Rather, mathematical propositions, because they are nothing but consequences of conventions about the use of symbols and are therefore compatible with all possible experience, are void of content; and (iii) the conception of mathematics as a system of linguistic conventions makes the a priori validity of mathematics compatible with strict empiricism (Gödel ⋆1953/59-V, p. 356). Gödel says that his incompleteness theorems and some of his other mathematical results "tend to bring the falsehood of these assertions to light" (Gödel ⋆1953/59-V, p. 356).

According to the logical positivism of Carnap there is a strict distinction between (analytic) truths of mathematics/logic and (synthetic) empirical truths. It is allegedly possible to reconcile the a priori nature of mathematics and logic with empirical science by holding that the truths of mathematics and logic are based solely on linguistic (syntactical) conventions, while the truths of empirical science depend on verification in the world of sensory experience. The verificationist theory of meaning applies to sentences of empirical theories but not to sentences of mathematics and logic. Carnap says that the meaning of a sentence about reality (= empirical reality) lies in its method of verification. If there is no possible method of verification in sense experience then sentences alleged to be about something or other are in fact meaningless. In works such as "The Elimination of Metaphysics Through the Logical Analysis of Language" (Carnap 1931), Carnap says that meaningful statements are in fact of the following kinds: first, there are statements that are true solely by virtue of their form. These are, following Wittgenstein, the tautologies. These say nothing about reality. The formulas of logic and mathematics are of this kind. They are not factual statements but serve for the transformations of factual statements. There are also negations of such statements, "contradictions," which are false by virtue of their form. The decision about the truth or falsity of all other statements lies in protocol sentences. Protocol sentences are true or false empirical statements and belong to the domain of empirical science. Any statement one desires to construct that does not fall within these categories becomes automatically meaningless (Carnap 1931). Carnap says in various places that sentences of empirical theories thus have content, but sentences of mathematics and logic are without content or object (see, e.g. Carnap 1935; see also Gödel *1953/59, p. 335). Sentences of mathematics are true solely on the basis of syntactical (linguistic) conventions (conventionalism), and the formal systems that embody these conventions are given in sense experience (i.e. empirically) as systems of symbols (nominalism).

In *The Logical Syntax of Language* Carnap indicates that his view of logical syntax is influenced by Hilbert. Carnap says that a theory, rule, definition or the like is to be called "formal" when no reference is made in it to the meanings of symbols. The only reference is to the kinds and order of the symbols from which the expressions are constructed (Carnap 1934a, § 1). Carnap extends this idea into a philosophical position in its own right, based on ideas that are not found in Hilbert's writings but that were afoot in the Vienna Circle. The view in *LSL* is that it depends entirely on the formal structure of a particular language and of the sentences involved whether a sentence should count as "analytic" or not. Analytic sentences, including the sentences of mathematics and logic, are not about anything (Carnap 1934a, § 2). Gödel's own view of analyticity is quite different from this, as we will see below. Carnap is claiming that empirical sentences are *about* something but sentences of logic and mathematics are not. In terms of Husserl's philosophy and Gödel's remarks on meaning and intentionality, this is a claim about intentionality in mathematics and logic. Analytic sentences, it is claimed, are without content or object. All of this applies, in particular, to sentences of mathematics that seem to be about numbers of various kinds, sets, functions, groups, spaces, and so on.[3]

[3] From the point of view of mathematical practice, Carnap's view that mathematical propositions are not about anything seems quite implausible. This is perhaps one reason why Gödel told Wang (Wang 1996, p. 174) that "Carnap's work on the nature of mathematics was remote from actual mathematics; he later came closer to actual science in his book on probability."

It is very interesting to note, in light of his interest in Husserl's idea of philosophy as rigorous science (Husserl 1911), that Gödel should apply his incompleteness theorems to refute Carnap's view of mathematics. It can be argued that Gödel's incompleteness theorems can themselves be seen as examples of philosophy become rigorous science. They establish in a scientifically rigorous way, for example, that we cannot identify proof in formal systems with mathematical truth, and they thereby lead to a clarification of the meaning of the concepts of "formal proof" and "mathematical truth." As we saw in the quotation from Hao Wang at the beginning of this chapter, Gödel thinks the incompleteness theorems can be employed to disprove a philosophical claim, the claim that mathematics is syntax of language.

Gödel's idea of applying his second incompleteness theorem to refute Carnap's view is clever: in order for the truths of mathematics to be based solely on linguistic (syntactical) conventions, the syntactical conventions must be consistent. For if they are not consistent then all statements will follow from them, including all factual (empirical) statements. A rule about the truth of sentences can be called syntactical only if it does not imply the truth or falsehood of any "factual" sentence, that is, one whose truth depends on extralinguistic facts. This requirement follows from the concept of a convention about the use of symbols, but also from the fact that it is the lack of content of mathematics upon which its a priori nature, in spite of strict empiricism, is supposed to depend.

Therefore, a consistency proof for the syntactical conventions would be required in order to uphold Carnap's view. Now we apply the second incompleteness theorem. This theorem should be applicable because the logical positivist will want to retain classical mathematics or at least the mathematics required for physics and natural science. According to the second theorem, no formal system in which it is possible to do elementary arithmetic will, if it is consistent, contain the resources required to prove its own consistency. A consistency proof for any such sets of syntactical conventions will require objects, concepts, or methods that are not part of the systems under consideration. Gödel considers two possibilities for a such a consistency proof: either it will be mathematical in nature or empirical and inductive in nature.

Suppose the consistency proof is mathematical in nature. In order for mathematics to be syntax of language in Carnap's sense it will have to be required that "language" will mean some symbolism that can actually be exhibited and used in the empirical world. In particular, its sentences will have to consist of a finite number of symbols, since sentences of infinite length do not exist in, and cannot be produced in, the empirical world. (The latter kinds of sentences, were they to exist, would presumably have to be purely mathematical objects.) Similarly, the "rules of syntax" will have to be finitary and cannot contain phrases such as "If there exists an infinite set of expressions with a

In a similar vein, in some of his remarks about informal rigor in mathematics and logic, Kreisel says that positivism and pragmatism do not respect the facts of actual intellectual experience (Kreisel 1967, pp. 141, 178).

certain property…", for the simple reason that such phrases could not be finitarily meaningful. Not only must the rules of syntax be finitary, but in the derivation of axioms from them and in the proof of their consistency only finitary syntactical concepts can be used, that is, only concepts referring to finite combinations of symbols. Now the second incompleteness theorem, as we just noted, tells us that no formal system in which it is possible to do elementary arithmetic will, if it is consistent, contain the resources required to prove its own consistency. According to this theorem, therefore, what will be required for the consistency proof will not be finitary, completely given empirically, and so on. Rather, it will be non-finitary, and will not be completely given empirically. If we want a *mathematical* consistency proof then the proof will involve "abstract" and infinitary objects, concepts, or methods, into which we have some insight. If we are to have a consistency proof at all then we will need some mathematical principles into whose content we have insight (intuition), whose meaning we understand. Otherwise there is no hope for a consistency proof. In this case, distinctive mathematical content, the abstract, and the non-finitary, are all back in the philosophical picture, as we already noted in chapter 2 in connection with Hilbert. Mathematical intuition pertaining to these features of mathematical reality is also back in the picture if (rational) intuition, and not mere conception, is required for knowledge and objectivity.

The abstract concepts that Gödel mentions in version III of the Carnap paper are "proof" and "function," where these are understood in their original contentual meaning. (See also Gödel *1951, p. 318, footnote 27, where Gödel also includes "set" among his examples.) Gödel says that, in its original contentual meaning, a proof is not a sequence of expressions satisfying certain formal conditions but rather a sequence of thoughts that convinces a sound mind. A function in its original contentual sense is not an expression of the formalism, but rather an understandable and precise rule associating mathematical objects with mathematical objects. Gödel's example here of a transfinite (i.e. non-constructive) concept is "there is," if this expression means existence of an object irrespective of actual producibility. "Infinite set" is said to be an abstract or transfinite concept, whether it refers to potential or actual infinity. The abstract and transfinite concepts together are said to form the class of "non-finitary" concepts.

Gödel says that in order for mathematical intuition and the assumption of mathematical objects or facts to be dispensed with by means of syntax it will have to be required that the use of "abstract" and "transfinite" concepts of mathematics that can be understood or used only as a consequence of mathematical intuition or of assumptions about their properties can be replaced by considerations about finite combinations of symbols. However,

> If, instead, in the formulation of the syntactical rules some of the very same abstract or transfinite concepts are being used—or in the consistency proof, some of the axioms usually assumed about them—then the whole program completely changes its meaning and is turned into its downright

opposite: instead of clarifying the meanings of the non-finitary mathematical terms by explaining them in term of syntactical rules, non-finitary terms are (used) in order to formulate the syntactical rules; and instead of justifying the mathematical axioms by reducing them to syntactical rules, these axioms (or at least some of them) are necessary in order to justify the syntactical rules (as consistent). (Gödel *1953/59, pp. 341–342)

The conventionalism and nominalism of the logical positivist thus "changes its meaning and turns into its downright opposite." In this case, as Gödel indicates, we are also led to a conception of meaning clarification that is very different from Carnap's conception. Gödel discusses an alternative view of meaning clarification due to Husserl in more detail in the 1961 text, as we will see below.

On the other hand, suppose the consistency "proof" is empirical in nature. In this case the claim to consistency is based on the fact that the conventions have thus far (in our use of them) not been found to lead to inconsistency. The evidence for consistency is based on past experience, that is, it is inductive evidence. In this case we have to rely on empirical facts to sustain syntactical conventionalism about mathematical truths, that is, to support the claim that the syntactical conventions are consistent in order to prevent all statements, including factual statements, from following from the conventions. This reliance on empirical evidence or empirical facts to maintain syntactical conventionalism about mathematical truths again violates the claim that the latter truths should be based solely on syntactical (linguistic) conventions, come what may in the empirical world. Furthermore, the empirical assertions used to support the consistency claim in this case would have content, so that content will again be required, albeit empirical (as opposed to mathematical) content. In short, one would have to appeal to sentences with content in order to be able to hold to strict conventionalism about mathematical statements that are supposed to have no content. Under this alternative, mathematical statements completely lose their a priori character, their character as linguistic conventions, and their voidness of content. Thus, we can again not hold to strict linguistic conventionalism about mathematics.

In sum, it is not possible without a consistency proof to be a conventionalist/nominalist about mathematics in the manner of the logical positivists upon whom Gödel is focusing, but what is needed for the consistency claim, whether it is mathematical or empirical in nature, undermines the conventionalism and nominalism of the logical positivists.[4] The particular attempt of the logical positivists to transform philosophy into rigorous science thus fails. This does not imply, however, the failure of all possible attempts to develop philosophy as rigorous science. Gödel is interested in the alternative that can be found in Husserl's work.

[4] There have been several attempts to defend Carnap against Gödel's argument. See, e.g. Awodey and Carus, 2003. In my view, there are a number of problems with these attempts but I will not consider them here.

§ 2. The resulting characterization of mathematics

What could *not* be true about mathematics if Gödel's critique of Carnap is correct? What *is* mathematics? In the drafts of the Carnap paper Gödel offers the following answers to these questions. Mathematics cannot be interpreted to be syntax of language. Furthermore, it is not possible to hold that mathematical sentences have no content. One can hold that they have no content only if "content" is taken from the beginning in a sense acceptable only to empiricists. In examining the syntactical view, Gödel thinks we are led to the conclusion that there are mathematical objects and facts that are as objective and independent of our conventions and constructions as physical or psychological objects and facts, although they are, of course, objects and facts of an entirely different nature. The conclusion reached by Gödel is that the scheme of the syntactical program to replace mathematical intuition by rules for the use of symbols fails because this replacement destroys any reason for expecting consistency, which is vital for both pure and applied mathematics, and because a consistency proof will require a mathematical intuition of the same power as is needed for discerning the truth of the mathematical axioms or a knowledge of empirical facts involving equivalent mathematical content.

Gödel thinks that this formulation of the unfeasibility of the syntactical program is especially well suited for dealing with the question whether mathematics is void of content. He argues against Carnap that, in fact, it is only laws of nature *together* with mathematics or logic that have consequences verifiable by sense experience. It is therefore arbitrary to place all content in the laws of nature. Mathematics adds something to the laws of nature. The laws of mathematics are not contained in the laws of nature. They involve concepts that are totally different from those occurring in the laws of nature. Mathematics does not add to physical laws any new properties of physical reality, but rather properties of the concepts referring to physical reality, in particular to concepts referring to combinations of things.

The view of Carnap is that mathematical propositions are compatible with all possible experience and are void of content. I think it is quite possible to agree that propositions of pure mathematics are compatible with all possible *sense experience*, because propositions of pure mathematics are separate from, and unaffected by, sense experience, but it does not follow that they are compatible with all possible *mathematical experience*. My experience as I work, for example, in hyperbolic or fractal or a four-dimensional geometry is not identical with my experience as I work in Euclidean geometry. This is not at all a mystery. The natural explanation for it is that I am working with different contents or with different concepts in these different geometries. Propositions of pure geometry are void of sensory or empirical content, but they are not at all void of mathematical content.

Gödel says that mathematics appears to have content or meaning but if this were only an illusion then it would have to be possible to build up mathematics without making use of this content. That mathematics does have content appears in the fact that however we build it up we always need certain undefined terms and certain axioms about them. The question of the existence of a content and of the necessity of axioms here refers to mathematics as a system of sentences knowable or posited to be true, not to mathematics as a hypothetico-deductive system. Gödel says that some body of unconditional mathematical truth must be acknowledged, because even if mathematics is interpreted as a hypothetico-deductive system, the propositions that state that the axioms imply the theorems must be unconditionally true. The field of unconditional mathematical truth is of course delimited very differently according to different foundational views. Gödel (Gödel *1953/59, p. 346, footnote 32) distinguishes the following eight standpoints: 1. Classical mathematics in the broad sense including set theory; 2. Classical mathematics in the strict sense; 3. Semi-intuitionism; 4. Intuitionism; 5. Constructivism; 6. Finitism; 7. Restricted finitism; 8. Implicationism. Implicationism is the view discussed also elsewhere by Gödel according to which it is not the theorems of classical mathematics that are mathematical truths but rather only the conditionals expressing the formal derivability of the theorems from certain axioms. Gödel argues that whichever standpoint is adopted it will still be true that mathematics has content. This holds even for the most restricted standpoint, implicationism. Here the mathematical theorems also express objective facts. The reason is that the singular combinatorial facts concerned consist of relations between the primitive terms of combinatorics, such as "pair," "equality," "iteration," and these cannot be eliminated by basing the use of these terms on conventions. This is true because in order to apply and know of the consistency of these conventions one needs an intuition or an empirical knowledge of facts involving the same or isomorphic concepts. On the other hand, Gödel says that to interpret the combinatorial facts to be nothing but physics of symbols would be unjustifiable and would constitute a return to Mill's conception of mathematics. We might note that, in its original form, Mill's conception of mathematics has virtually no followers.

Gödel argues that for the axioms involving undefined terms there exists no other rational foundation except that they (or propositions implying them) can be directly perceived to be true owing to the meaning of the terms in them or by an intuition of the objects falling under them, or that they are assumed like physical hypotheses on the grounds of inductive arguments, for example their success in applications (see also chapter 8, § 1). Gödel says that by "success" within pure mathematics of some non-evident axiom he means that many of its consequences can be verified on the basis of the evident axioms, the proofs however being more difficult, and that moreover such an axiom would solve important problems not solvable without it (Gödel *1953/59, p. 347, footnote 33). He notes that the second incompleteness theorem implies that the consistency of such a non-evident axiom would be indemonstrable on the basis of the evident axioms, provided it solves any problem of Goldbach type.

It seems, Gödel says, that at least some axioms can be perceived to be true on the basis of the meaning of terms or by an intuition of the objects falling under them. The examples given here are *modus ponens* and complete induction (Gödel *1953/59-III,

p. 347, footnote 34). Expanding a bit on his conception of intuition or the perception of the truth of axioms, he says that he thinks it is arbitrary to consider the proposition "This is red" to be an immediate datum but not to consider the proposition expressing *modus ponens* to be an immediate datum. The only difference lies in the fact that in the first case a relation between an undefined concept and an individual object is perceived, while in the second case a relation between undefined concepts is perceived and the concepts involved are of a different kind from the concept "red." Gödel says that as an axiom *modus ponens* can be considered to be an analytic proposition, even though it does not follow from definitions. It is true "owing to the meaning of undefined terms." Gödel says that complete induction would seem to be an axiom (or consequence of axioms) that is analytic in this sense. One could say that on the basis of the non-constructive standpoint this has even been proved by Dedekind and Frege.

Gödel argues that not only does mathematics have content but its content is unlimited in the sense that outside of every axiomatic system that formalizes mathematical truth there exist propositions expressing new and independent mathematical facts that cannot be reduced to symbolic conventions on the basis of the axioms of this system. If the system contains arithmetic, for example, then the proposition stating the consistency of the system is of this type.

In this context we are told that the neglect of the conceptual content of sentences, which Gödel here associates with Frege's notion of "sense" (*Sinn*) as something objective and non-psychological, is also responsible for the incorrect view that the conclusion in a logical inference contains no information beyond that contained in the premises (Gödel *1953/59, p. 350, footnote 40). Gödel suggests instead that the conclusion represents the extra-logical content of the premises in a conceptually different form. That the conclusion is implied by the premises is itself an objective fact concerning the primitive terms of logic occurring in, and specific to, these terms. In chapter 5, I will return to the idea of associating conceptual content with the notion of "sense" (*Sinn*), especially as this is understood in terms of Husserl's views on "noematic *Sinne*."

The reasoning that leads to the view that no mathematical facts exist is said to be nothing but a *petitio principii*, because "fact" is from the beginning identified with "empirical fact," that is, "synthetic fact concerning sensations." Mathematics is void of content in this sense, but this does not help in any argument against platonists since they agree that mathematics has no content of this kind. For platonists, as Gödel sees it, the content of mathematics consists in the relation between concepts or other abstract objects which subsist (or exist) independently of our sensations, although they are perceived in a different kind of experience and although, in conjunction with certain accepted laws of nature, they even have consequences verifiable by sense perception.

One of the drafts of the Carnap paper contains a passage that is very important for understanding how Gödel thinks of the analogy between sense perception and rational intuition (Gödel *1953/59-III, p. 353, footnote 43). The view here is very similar to some of Husserl's ideas about the analogy between sensory and rational intuition, as we will see in chapter 6. It can in fact be explicated in terms of Husserl's ideas. What Gödel says is that the "inexhaustibility" of mathematics, which we discussed in chapter 2 in connection with the Gibbs Lecture (Gödel *1951), makes the similarity between reason and the senses even closer, because it shows that there exists a practically unlimited number of independent "perceptions" also of reason. The inexhaustibility of mathematics appears not only through foundational investigations (i.e. the incompleteness theorems) but also in the actual development of mathematics in, for example, the unlimited series of axioms of infinity in set theory which are analytic (and evident) in the sense that they only explicate the content of the general concept of set. Here Gödel is speaking about what he later (Gödel 1964) calls the intrinsic necessity of axioms, as distinct from extrinsic justification of axioms (see chapter 8, §1). That such a series may involve a great (perhaps even infinite) number of actually realizable independent rational perceptions is seen in the fact that the axioms involved are not evident from the beginning, but only become so in the course of the development of mathematics. In order to understand the first transfinite axiom of infinity, for example, one must first have developed set theory to a considerable extent. Gödel says that, moreover, if every number-theoretic question of Goldbach type (i.e. expressed in a Π_1 sentence, such as Gödel sentences) is decidable by mathematical proof then there must exist an infinite set of independent evident axioms, that is, a set m of evident axioms that are not derivable from any finite set of axioms. Even if solutions are desired only for all those problems of Goldbach type that are simple enough to be formulated in a few pages, there must exist a great number of evident axioms or evident axioms of great complication, as distinct from the few simple axioms on which present-day mathematics is built. Gödel says it can be proved that, in order to solve all problems of Goldbach type of a certain degree of complication k, one needs a system of axioms whose degree of complication up to minor correction is $\geq k$.

Gödel remarks that the analogy just cited comes pretty close to the true state of affairs except that this additional "sense"—reason—is not counted as a sense because its objects are quite different from those of all the other senses. With sense perception we know particular objects and their properties and relations, while with mathematical reason we perceive the most general (namely "formal") concepts and their relations which are separated from space-time reality, insofar as space-time reality is completely determined by the totality of particularities without any reference to the formal concepts. Mathematical concepts can be introduced by rules for handling symbols on the basis of empirical facts concerning physical symbols, but in reality the situation is just the opposite: the rules for the use of symbols are so chosen that they express properties of previously conceived mathematical concepts or objects.

To summarize three of the basic points about what mathematics *is*: (i) we cannot replace mathematical (or what Husserl calls categorial or "eidetic") intuition with conventions about the use of symbols, and (ii) it could not be the case that mathematics does not describe any existing objects or facts. Mathematical propositions could not be empty tautologies that are void of content. The truths of mathematics could not be based solely on linguistic conventions. Rather, true mathematical propositions and true empirical propositions would both be about objects or facts, and there are analogies in our knowledge of such objects, even if the objects are of different types, and (iii) Gödel thinks that the logical positivist's way of trying to establish the compatibility of the a priori and the a posteriori, the mathematical and the empirical, the analytic and the synthetic, will not work.

This last point is also at the center of Quine's famous criticism of the logical positivists, and it is interesting to compare Gödel's critique with Quine's. Gödel, I think, does not adopt Quine's pragmatic holism about the distinctions (see also chapter 8). Instead, he says that there is one ingredient of Carnap's incorrect theory of mathematical truth that is correct and discloses the true nature of mathematics, namely, it is correct that a mathematical proposition says nothing about physical or psychical reality existing in space and time, because it is already *true owing to the meaning of the terms occurring in it, irrespective of the world of real things*. This idea that mathematical propositions are true owing to the meanings of their terms is not like Quine's view of the distinction. The views of Gödel and Quine on meaning or content, analyticity, intuition, and platonism are quite different from one another. What is wrong with Carnap's view, according to Gödel, is that the meaning of the terms (that is, the concepts they denote) is asserted to be something man-made and consisting merely of linguistic conventions. Gödel's alternative view is that "these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe. Therefore, a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says something about relations of concepts." (Gödel *1951, p. 320) Gödel thus contrasts his own view of analyticity with Carnap's: "analytic" does not mean "true owing to our definitions," but rather it means "true owing to the nature of the concepts" occurring in mathematical statements (Gödel *1951, p. 321).[5] In the Gibbs Lecture Gödel says that this later concept of analyticity is so far from meaning "void of content" that it is possible that an analytic proposition might be undecidable, or decidable only with a certain probability. This is because our knowledge of the world of abstract concepts may be as limited and incomplete as that of the world of physical objects. In chapter 8, I will discuss the contrast between the views of Gödel and Quine in more detail.

It is of interest to note that Gödel says (Gödel *1951, p. 321) that our knowledge of the world of concepts is in certain cases not only incomplete but also indistinct. This latter situation, he says, can be seen in the paradoxes of set theory. The paradoxes are

[5] I will not enter into the complications of trying to decide whether Gödel's notion of analyticity is different from Husserl's notion. In writings such as *Formal and Transcendental Logic*, Husserl thinks of analyticity as purely formal, although not in the sense of the formalists. There are, of course, many different notions of analyticity in the philosophical literature. My strategy is to argue that directedness in contentual mathematical thinking requires a notion of meaning and of relations of meanings to one another. See chapter 5, § 10 and § 12.

sometimes alleged to disprove platonism, but Gödel says this is not justified. He argues that our visual perceptions sometimes contradict our tactile perceptions, as in the case of a rod immersed in water. Nobody will conclude from this, however, that the outer world does not exist. So the set-theoretic paradoxes are supposed to be like certain illusions of the senses. Just as the existence of the one kind of illusion does not show that there is not a mind-independent world of physical objects, so is it not shown by the other illusion that there is not a mind-independent world of abstract concepts or objects.[6]

Before closing this section, I want to note that in his drafts of *1953/59 Gödel is responding to a number of Carnap's views on meaning clarification, the methodology of logical analysis, and metaphysics. Carnap says that modern logic has made it possible to give a new and sharper answer to the question of the validity and justification of metaphysics. Research in applied logic or theory of knowledge aims at clarifying the cognitive content of scientific statements and, thereby, the meanings of the terms that occur in such statements. It is on this basis that metaphysics can be eliminated (Carnap 1931). What is left for philosophy if metaphysics is to be eliminated? What remains, Carnap says, is a method, the method of logical analysis. This method, in its negative use, is supposed to allow us to eliminate meaningless expressions, as we saw above. In its positive use, it is supposed to clarify meaningful concepts and propositions, and to lay logical foundations for factual science and for mathematics. It is only logical analysis in this sense of laying out the syntax of the language of science that can count as "scientific philosophy" (Carnap 1931). In *LSL* Carnap says that philosophy is to be replaced by the logic of science, and the logic of science is the logical syntax of the language of science. We already noted above how, on the basis of the application of the second incompleteness theorem, Gödel suggests a reversal of Carnap's view: "instead of clarifying the meanings of the non-finitary mathematical terms by explaining them in terms of syntactical rules, non-finitary terms are (used) in order to formulate the syntactical rules; and instead of justifying the mathematical axioms by reducing them to syntactical rules, these axioms (or at least some of them) are necessary in order to justify the syntactical rules (as consistent)" (Gödel *1953/59, pp.341–342).[7] In the text to which we will turn in § 4 (Gödel, *1961/?, p. 383) we will see clearly, if only in

[6] Gödel likens the antinomies to illusions of the senses in various places in his writings. Illusions arise in cases where we have not seen concepts clearly enough, and they lose their grip on us once we have achieved more clarity in our perception of concepts. See, for example, the passages in Wang 1974 on perceiving concepts clearly, pp. 81–6. Also, Gödel 1964, p. 268, where he says that "The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics." I return to this in chapter 6, § 4.

[7] Compare this with the remark cited in chapter 1 above: "Some reductionism is right: reduce to concepts and truths, but not to sense perceptions. Really it should be the other way around: Platonic ideas… are what things are to be reduced to. Phenomenology makes them [the ideas] clear." The "reductionism" Gödel mentions in this quote would presumably be subsumed under what Husserl calls the "eidetic reduction" and the phenomenological reduction (*epochē*). Empiricist reductionism, in effect, is contrasted with phenomenological reduction.

outline, the alternative methodology of meaning clarification and the alternative conception of "scientific philosophy" that Gödel thinks we find in Husserl's work.

I would argue that these comments of Gödel on meaning clarification do not entail that strict formalization and formal systems are not important for meaning clarification. The considered position, I think, would be that both careful developments of syntax and formal systems, and cultivation of intuition of abstract concepts can be involved in meaning clarification. A strict formalism that ignores or attempts to eliminate or explain away mathematical intuition of abstract mind-independent concepts or objects would not by itself be sufficient. A kind of dialectical relationship between formalism and rational intuition must be involved. As an example, one might think of adding to a formal system the Gödel sentence for the formal system that we can see to be true if the formal system is consistent, even though this awareness could not be based on operating in the formal system itself. For Gödel this is an awareness based on the intuition of the abstract concepts that underlie the formal system in the first place. Generally speaking, one would work back and forth between abstract concepts and formalization in setting up and developing formalized mathematics (see also Kreisel 1967). As we noted earlier, strict formalizations of mathematics are relatively new in the history of mathematics.

Finally, Gödel is aware of the fact that in later work (e.g. Carnap 1950) Carnap does not object, in scientific semantics, to associating mathematical objects to formulae as their denotation. Carnap thinks the *philosophical* question about the objective existence of mathematical objects, however, does not refer to this "internal" existence, but is concerned with whether the objects introduced by the axioms "really exist." An answer to the latter question is asserted by Carnap to have no "cognitive content," that is, the question is now considered to be meaningless, while formerly Carnap answered it negatively. Although Gödel does not address this later position of Carnap, I think there is no doubt that he would reject it. It is a highly relativistic view in which "to be" is to be quantified over in a particular linguistic framework that could very well differ from numerous other linguistic frameworks. One remark I would like to make about this at the moment is that this kind of freewheeling appeal to the relativity of existence claims to linguistic frameworks does not conform to the manner in which existence claims are actually understood by mathematicians in the history and practice of mathematics. There are, of course, also many arguments in the work of Frege, Husserl, and others against relativism in mathematics, linguistic, or otherwise, but I will not consider these here (see Tieszen 2004).

§ 3. Some general arguments on platonism and free creation in mathematics

One way of holding, against platonism, that mathematics is our own free creation or free invention is to construe "free creation" in terms of linguistic conventions in the style of logical positivism. Gödel thinks that he has offered powerful arguments against this kind of anti-platonism in the drafts of his Carnap paper. There might be some room for free creation in mathematics, but Gödel says in a number of places that there is no such room in the case of the primitive concepts. His arguments are developed most extensively in the Gibbs Lecture of 1951 (Gödel *1951). They are worth considering here since they antedate Gödel's turn to Husserl and the ideas in his 1961 paper that we will consider in a moment. Psychologism and Aristotelian realism are also rejected in the 1951 Gibbs Lecture, as we will see below.

In the Gibbs Lecture Gödel formulates the following implication of his incompleteness theorems:

> Either mathematics is incompleteable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable Diophantine problems. (Gödel *1951, p. 310)

I will discuss this formulation in more detail later, especially in chapter 7. In his later work it is clear that Gödel does not accept the view that there exist absolutely unsolvable Diophantine problems, but in the Gibbs Lecture he argues that the assertion that there are absolutely unsolvable Diophantine problems seems to disprove the view that mathematics is only our own creation, and to imply instead a platonist view according to which mathematical objects and facts exist objectively and independently of our mental acts and decisions. He wants to show that the "empirical interpretation of mathematics" according to which mathematical facts are special kinds of physical or psychical facts is too absurd to be seriously maintained. The empirical interpretation of mathematics is described as the view that there is no essential difference between mathematical objects and physical or psychical objects, and no essential difference in the way we know about mathematics and laws of nature. The true situation, he says, is that if the objectivity of mathematics is assumed, then the objects of mathematics must be totally different from sensory objects because (1) mathematical propositions assert nothing about the actualities of the space-time world. This is particularly clear in applied propositions such as "Either is has rained or it has not rained yesterday"; (2) mathematical objects are known precisely, and general laws about them can be recognized with certainty, that is, by deductive, not inductive inference; (3) they can be known (in principle) without using the senses, that is, by means of reason alone, since they do not concern actualities about which the senses (inner sense included) inform us, but rather they concern possibilities and impossibilities.

Gödel first argues that if there are absolutely unsolvable Diophantine problems then we must accept platonism on the grounds that mathematics could not be entirely our own creation, for a creator would necessarily know all properties of his creations. Our creations cannot have any properties other than those we have given to them. If this is true then the hypothesis that there exist absolutely unsolvable Diophantine problems seems to imply that mathematical objects and facts (or at least something in them) exist objectively and independently of our mental acts and decisions. If a creator knows all the properties of his creation then presumably there could not be *absolutely* undecidable mathematical problems. Now one could object that a constructor need not know every property of what he constructs. We build machines, for example, and cannot predict their behavior in every detail. This would presumably show, by analogy, that mathematics could be our own creation, and that we do not know all the properties of our creations, so that there could be undecidable mathematical problems. The issue at stake, however, is presumably whether mathematical problems could be *absolutely* undecidable. In any case, Gödel says that this objection is very poor. We do not build machines out of nothing. We build them out of some material. If the situation were analogous in mathematics then this "material" or basis for our creations would be something objective and would force some platonistic viewpoint on us even if certain other ingredients of mathematics were our own creation. Gödel is arguing that the kind of objection involved in the analogy with the machine will not work because the creation in such a case involves some "material" out of which the creation is built, and that this material or basis is not itself created. It is something objective and would force some platonistic viewpoint on us.

The argument that our capacity for free creation bottoms out at some point, so to speak, has a long history in philosophy. It is typically involved in positions that combine some kind of realism with a hylomorphism. Already in Aristotelian realism, for example, it is always "matter" that has some form or other. It is not the case that all is form. When we come to Kant's transcendental philosophy, we are told that the senses deliver the "material" that is structured by the forms of intuition and the categories. We do not create what the senses deliver. In our experience the manifold of sensibility constrains the two forms of intuition and the categories of the understanding. We simply do not create everything in our experience. This figures into Kant's claim that he is an empirical realist (and a transcendental idealist).

In Husserl's philosophy there is a similar hylomorphism. Husserl (e.g. in Husserl 1913) uses the Greek term "hylé" to refer to the sensory "matter" that constrains our sense perception. We do not create this. Rather, it is given. It is associated with the passive and immediate part of our experience. He argues that analogous constraining "material" is found in categorial or eidetic intuition, although in the case of pure categorial intuition it is of course not sensory data. This figures directly into what I call constituted platonism, to be discussed in more detail in chapters 4 and 6. The claim is that we finite monads do not constitute everything in our experience. Some "material" is required for the constitution of the consciousness of abstract objects (invariants). This is the "material" from which we abstract, on which we reflect, that we idealize, from which we generalize, on which we perform variations, and so on. The consciousness of abstract or ideal objects has a genesis. Husserl argues that there is a genealogy of this kind of consciousness, with levels of sedimentation, that we can trace. Genetic phenomenology would be completely pointless, however, if we constitute *everything* in our experience. In this case the notion of an "origin" collapses. An idealism according to which we constitute absolutely everything in our experience is nonsense. For the kinds of beings that we are, at any rate, the process bottoms out. In this sense, genetic phenomenology as a kind of transcendental idealism is a natural companion of (constituted) realism. I return to this claim in chapters 4 and 6.

In considering yet another objection to the claim that the existence of absolutely undecidable mathematical problems implies some kind of platonism, Gödel makes the noteworthy observation that if mathematics describes an objective world, on the analogy with physics, then there is no reason why inductive methods should not be applied in mathematics as they are in physics. In the case of mathematics we would, on the basis of such methods, try to determine what is probably true of abstract mathematical objects. A probabilistic view of mathematics that is accompanied by mathematical platonism, one might argue, is a lot less threatening to mathematics than it is in the case where it is not accompanied by mathematical platonism. If we subtract the mathematical platonism from the appeal to a probabilistic epistemology, then we just have an empiricist view according to which mathematical propositions are not about anything, or on which the distinction between being true on the basis of the relation of concepts and being true on the basis of sense experience disappears or is blurred, or on which mathematical truth itself becomes probabilisitic, subject to change, relativistic, and so on. If we retain a platonic rationalism then none of these things need to be true, and yet we can still suppose that the use of inductive methods in mathematics might have heuristic significance. If platonic rationalism is not rejected, then not as much would be at stake in recognizing the possible heuristic significance of such methods in trying to determine what the fixed mathematical truths are. Gödel says that this shows that the philosophical implications of the second disjunct thus do not lie entirely on the side of rationalistic or idealistic philosophy, but that in one respect they favor the empiricist viewpoint. That is, the situation in mathematics is not so very different from that in the natural sciences in terms of how we might come to know about the respective worlds in each case.

Gödel has argued that in considering the second alternative in the disjunction in his Gibbs Lecture we are led to conceptual realism (platonism). He then goes on to argue that conceptual realism is supported by modern developments in the foundations of mathematics irrespective of which alternative in his disjunction holds. Three short arguments are presented for this claim.

The first argument starts with the premise that if mathematics were our own free creation then it is possible that we could still be ignorant of the objects we created, but only through lack of a clear realization of what we have created. This ignorance would, at least in principle, disappear as soon as we attained perfect clarity. Modern developments in the foundations of mathematics, however, have accomplished an insurmountable degree of exactness, and yet this has not helped very much with the solution of open mathematical problems. This points us in the direction of conceptual platonism, with its claim that mathematics is not our own free creation.

The second argument is that the activity of the mathematician shows very little of the freedom that a creator should enjoy. Even if the axioms about, say, the integers were a free invention, it has to be admitted that after a mathematician has imagined the first few properties of his objects he would come to an end of his creative ability, and would not be in a position to create the validity of his theorems at will. If anything like creation exists in mathematics, then what a theorem does is exactly to restrict or constrain the freedom of creation. Reason and conception in mathematics and logic can certainly be constrained, even if the constraint is not due to sensory perception. Reason is clearly constrained, for example, relative to the capacity for imagination because, among other things, it seeks not just pure possibilities but necessities relative to possibilities. Gödel's conclusion is to say that what restricts freedom of creation in mathematics must be independent of the creation. As we saw, this is an argument that realists sometimes use to show that there must be something outside of our own consciousness, or outside of our own cognitive constructions. We do not have a completely free will in this case. There is something that restricts us or that resists us. This is a hallmark of objectivity, of something outside of the subject.

Although he does not mention it in these sections of the Gibbs Lecture, the argument here is related to the claim that there can be genuine surprises in mathematics and logic, or that we might obtain results in mathematics that are entirely unexpected. The idea is that these phenomena are signs of objectivity. They provide evidence for a platonic rationalism. How could we be surprised or reach entirely unexpected results if mathematics were our own free creation? On the other hand, these phenomena are quite compatible with the view that there are mind-independent objects or truths.

Gödel's third argument is that if mathematical objects are our own creations, then evidently integers and sets of integers would have to be two different creations, the first of which does not necessitate the second. In order to prove certain propositions about the integers, however, the concept of set of integers is required. Thus, in order to find out what properties we have given to certain objects of our imagination we must first create certain other objects. Gödel finds this to be a very strange situation. In other words, we have given properties to certain objects (since they are *our* creations), but then, in order to find out what these properties are, we are required to create certain other objects. One might already ask how we can create properties that are, in effect, hidden from us. We supposedly create the properties, and yet they are hidden from us and we can only access them if we create certain other kinds of objects that allow those properties to be revealed.

At the end of the 1951 Gibbs Lecture, Gödel says that psychologism and Aristotelian realism also cannot account for the nature of mathematics. The essence of the psychologistic view, he says, is that the object of mathematics is nothing but the psychological laws by which thoughts, convictions, and so on, occur in us, in the same sense as the object of another part of psychology consists of the laws by which emotions occur in us. The chief objection to this view, Gödel says, is that if it were correct then we would have no mathematical knowledge whatsoever:

> We would not know, for example, that 2 + 2 = 4, but only that our mind is so constituted as to hold this to be true, and there would then be no reason whatsoever why, by some other train of thought, we should not arrive at the opposite conclusion with the same degree of certainty. Hence, whoever assumes that there is some domain, however small, of *mathematical* propositions which we *know* to be true, cannot accept this view. (Gödel *1951, p. 322).

This kind of objection to psychologism, along with many others, is of course discussed at length in the "Prolegomena to Pure Logic" in Husserl's *Logical Investigations* (see also Tieszen 2004), and also in Frege's critique of psychologism (e.g. in Frege 1884, 1893, 1918).

Aristotelian realism is described briefly as the view that concepts are parts or aspects of (material) things whereas, according to platonism, there are two separate worlds, the world of things and the world of concepts. Gödel does not develop detailed critiques of psychologism and Aristotelian realism in the Gibbs Lecture but he does say that he thinks that, after sufficient clarification of the concepts in question, it will be possible to conduct these philosophical discussions with mathematical rigor, and that the result will be that platonism is the only tenable position.

In these writings from the fifties, Gödel thus expresses a range of arguments against positivism, conventionalism, nominalism, naturalism, empiricism, and so on, and in favor of a platonic rationalism about mathematics and logic. Of course there are even earlier expressions of such a position in his papers on Russell and on Cantor's continuum hypothesis, as we already saw in chapter 2. Surrounded by efforts to naturalize mathematics in one way or another, Gödel must have been quite happy, starting in 1959, to read passages such as the following in Husserl's *Ideas I*:

> When it is actually natural science that speaks, we listen gladly and as disciples. But it is not always natural science that speaks when natural scientists are speaking; and it assuredly is *not* when they are talking about "philosophy of Nature" and "epistemology as a natural science." And, above all, it is not natural science that speaks when they try to make us believe that general truisms such as all axioms express (propositions such as "a + 1 = 1 + a," "a judgment cannot be colored," "of only two qualitatively different tones, one is lower and the other higher," "a perception is, *in itself*, a perception of something") are indeed expressions of experiential matters of fact; whereas we know with *full insight* that propositions such as those give explicative expression to data of eidetic intuition. But this very situation makes it clear to us that the "positivists" sometimes confuse the cardinal differences among kinds of intuition and sometimes indeed see them in contrast but, bound by their prejudices, *will* to accept only a single one of them as valid or even as existent. (Husserl 1913, § 20)

Let us now turn to the text in which Gödel writes about the alternative available to us in Husserl's philosophy.

§ 4. The modern development of the foundations of mathematics in the light of philosophy (⋆1961/?)

In the text of ⋆1961/? Carnap is not mentioned by name. Gödel does mention positivism as a philosophy that falls on the "left" side of his schema of possible philosophical worldviews but he focuses mostly on the "leftward" foundational program of Hilbert. Gödel notes some relationships between the views of Hilbert and Carnap in several places. In one of the drafts on Carnap (Gödel ⋆1953/59 – III, footnote 19, p. 341), for example, he says that he thinks that if the syntactical program of Carnap is to serve its purpose then what must be understand by "syntax" is equivalent to Hilbert's finitism, in the sense that it consists of those concepts and forms of reasoning, referring to finite combination of symbols, which are contained within the limits of "that which is directly given in sensual intuition." In the Gibbs Lecture (Gödel ⋆1951, p. 315) he says that the formalistic foundation of mathematics would be a special elaboration of Carnap's syntactical view and that, on the other hand, it turns out that the feasibility of the nominalistic program of the logical positivists implies the feasibility of the formalistic program. In the 1961 text to which I now turn, Husserl's transcendental philosophy is set up as an alternative to both positivism and Hilbert's foundational view (see also Føllesdal's comments on this text in Føllesdal 1995a and 1995b). Gödel links Husserl to Kant at the end of the 1961 text, as we will see, but he also mentions how some modifications of Kant's view will be required.

Gödel sets up a general schema of possible philosophical worldviews in ⋆1961/? according to their degree and manner of affinity to metaphysics. We obtain a division into two groups, with skepticism, materialism, and positivism on one side and spiritualism, idealism, and theology on the other. If one thinks of philosophical doctrines as arranged along a line from left to right in this manner, then empiricism belongs on the left side and a priorism belongs on the right. Pessimism belongs on the left side and optimism in principle toward the right, for empiricist skepticism is a kind of pessimism with regard to knowledge. Materialism is inclined to regard the world as an unordered and therefore meaningless heap of atoms. On the other hand, idealism and theology see meaning, purpose, and reason in everything. Additional examples of theories on the right side would include theories of objective moral values and objective aesthetic values, whereas the interpretation of ethics and aesthetics on the basis of custom, upbringing, and so on would fall on the left.

Gödel says that the development of philosophy since the Renaissance has, on the whole, gone from right to left. This development has also made itself felt in mathematics. Mathematics, as an a priori science, always has an inclination toward the right, and has long withstood the *Zeitgeist* that has ruled since the Renaissance. The empiricist conception of mathematics, such as that set forth by John Stuart Mill in the nineteenth century, did not find much support. Indeed, mathematics evolved into ever higher abstractions, away from matter and into ever greater clarity in its foundations. The foundations of the infinitesimal calculus and the complex numbers, for example, were improved. Mathematics thus moved away from empiricist skepticism. At the beginning of the twentieth century, however, the antinomies appeared in set theory. Gödel says that the significance of the antinomies was exaggerated by skeptics and empiricists, and that the antinomies were employed as a pretext for a leftward upheaval. He says, in response, that the contradictions did not appear in the heart of mathematics but rather near its outer boundary toward philosophy. Moreover, the antinomies in question have been resolved in a manner that is satisfactory to those who understand set theory. These kinds of points are of no use, however, against the prevailing *Zeitgeist*. Many mathematicians came to deny that mathematics, as it had developed previously, represented a system of truths. They acknowledged this for a part of mathematics only, and retained the rest in at best a hypothetical sense.

Now Carnap's view of mathematics as syntax of language is clearly on the left in Gödel's schema, and the arguments of *1953/59 are meant to refute it. Gödel says that the "nihilistic" consequences of the spirit of the times also led to a reaction in mathematics itself. Thus came into being "that curious hermaphroditic thing that Hilbert's formalism represents." It sought to do justice both to the *Zeitgeist* and to the nature of mathematics. In conformity with the ideas prevailing in recent philosophy, it acknowledged that the truth of the axioms from which mathematics starts cannot be justified or recognized in any way, and therefore that the drawing of consequences from them has meaning only in a hypothetical sense. The drawing of consequences itself, to further satisfy the spirit of the time, is construed as a mere game with symbols according to rules, where this is likewise not thought of as supported by insight or intuition.[8] In accordance with the earlier "rightward" philosophy of mathematics and the mathematician's instinct, however, it is held that a proof of a proposition must provide a secure grounding for the propositions and that every precisely formulated yes-or-no question in mathematics must have a clear-cut answer. One aims to prove, that is, for the inherently unfounded rules of the game with symbols that of two sentences A and $\neg$ A, exactly one can always be derived. Such a system is consistent if not both can be derived, and if one can be derived then the mathematical question expressed by A can be unambiguously answered. In order to justify the assertions of consistency and completeness, a certain part of mathematics must be acknowledged to be true in the sense of the old rightward philosophy. The part in question, however, is much less opposed to the spirit of the time than the high abstractions of set theory. It is the part that refers only to concrete and finite objects in space, namely the combinations of symbols. This is Hilbert's finitism, as described in chapter 2.

In Hilbert's program we thus see an interesting mixture of rationalist and empiricist elements. The two rightward, rationalistic elements, namely, the belief in the decidability of clearly formulated mathematical problems and the demand that proofs

[8] As noted above, in various writings going back to the 1930s, Gödel distinguishes the purely formalistic and relative concept of proof from the "abstract" concept of proof as "that which provides evidence." See also chapter 6, §2 below.

provide a secure grounding for propositions, are translated into the context of empirically given formal systems. Empirically given formal systems are in accord with the spirit of the time. The next step in the development comes with Gödel's incompleteness theorems: it turns out that it is impossible to rescue the older rightward aspects of mathematics in such a way as to be in accord with the spirit of the time. Even for elementary arithmetic it is impossible to find a system of axioms and formal rules from which, for every number-theoretic proposition A, either A or ¬ A would always be derivable. Moreover, for reasonably comprehensive axioms of mathematics, it is impossible to provide a proof of consistency merely by reflecting on the concrete combinations of symbols, without introducing more abstract elements.[9] The problem is that we cannot sustain Hilbertian optimism about deciding clearly posed mathematical problems nor can we hold that a proof should provide secure grounding for a proposition once these ideas are translated into a purely formal setting. The first incompleteness theorem shows that for any formal system in which arithmetic is possible, there is always an undecidable formula, and the second theorem shows that proofs in these formal systems cannot provide secure grounding of sentences unless the systems are known to be consistent. To know that the systems are consistent, however, we have to step outside the systems. To preserve the two rationalistic elements we have to depend on mathematical intuition of abstract concepts or objects, which is different from the kind of mathematical intuition that Hilbert recognizes. We have to depend on a kind of meaning clarification that does not just reduce to random development of systems of syntax. Hilbert's combination of materialism and aspects of classical mathematics thus proves to be impossible. This means that the combination of rationalist and empiricist elements involved in Hilbert's program is unworkable. It is not possible to be a finitisitic formalist and to hold that every clearly stated mathematical proposition is decidable. Nor is it possible to hold that proofs provide a secure grounding for mathematical propositions on the basis of the purely formal conception of proof involved in formal systems.

Gödel says that only two possibilities remain. Either we must give up the older rightward aspects of mathematics or attempt to uphold them in contradiction to the spirit of the time. The first alternative suits the *Zeitgeist* and is therefore usually the one adopted. One has to thereby give up on the two features of mathematics just mentioned that would otherwise be very desirable, namely, to safeguard for mathematics the certainty of its knowledge based on the fact that proofs are supposed to provide *evidence*, but also to uphold the optimistic belief that for clear questions posed by reason it is possible for reason to find clear answers.[10] One would give up on these features,

[9] See Gödel's remark in Gödel 1972, Vol. II, p. 271–3, also cited above in chapter 2.

[10] Wang 1974, pp. 324–5, reports that Gödel thought Hilbert was correct in rejecting the view that there exist number theoretic questions undecidable for the human mind, because if it were true it would mean that human reason is utterly irrational in asking questions it cannot answer while asserting emphatically that only reason can answer them. Human reason would then be very imperfect and, in some sense, even inconsistent, which contradicts the fact that parts of mathematics that have been systematically and completely developed

not because any mathematical results compel us to do so, but because this is the only way to remain in agreement with the prevailing philosophy. Gödel grants that great advances have been made on the basis of the leftward spirit in philosophy, and he thinks there have been excesses and wrong directions taken in the preceding rightward philosophies. The correct attitude is that the truth lies in the middle of these philosophies, or consists in a combination of the leftward and rightward views, but not in the manner of Hilbert's conception. Hilbert's combination, like Carnap's, was too primitive, and tended too strongly in the leftward direction. We must look elsewhere for a workable combination. If we want to preserve elements of the earlier rightward view of mathematics then we must suppose that the certainty of mathematics is not to be secured by proving certain properties by a projection onto material systems (i.e. the mechanical manipulation of physical symbols), but rather by cultivating and deepening our knowledge of the abstract concepts that lead to setting up these mechanical systems in the first place. Furthermore, it is to be secured by seeking, according to the same procedures, to gain insights into the solvability of all meaningful mathematical problems.

How is it possible to extend our knowledge of these abstract concepts? How can we make our grasp of these concepts precise, and gain a comprehensive and secure insight into the fundamental relations that hold among them, that is, into the axioms that hold for them? We cannot do this by trying to give explicit definitions for concepts and proofs for axioms, since in that case one needs other undefinable abstract concepts and axioms holding for them. Otherwise one would have nothing from which one could define or prove. Therefore, the procedure must consist to a large extent in a clarification of meaning that does not consist in giving definitions. We thus see here, as in other places in his writing, that Gödel is speaking about the need to reflect on meaning, especially of the primitive terms. What is required is a reflection on meaning or on concepts that is of a "higher level" than reflection on the combinatorial properties of concrete symbols. This is the kind of ascent, suggested in chapter 2 and discussed in more detail in chapters 5 and 6, that is a function of reason. The notion of meaning clarification here is thus quite unlike Carnap's notion, and it involves, on the basis of the second incompleteness theorem, just the kind of reversal of the sensory and the abstract, or the sensory and the categorial, that we noted above in the *1953/59 paper: instead of clarifying the meanings of non-finitary mathematical concepts by explaining them in terms of syntactical rules, non-finitary mathematical concepts are used to formulate the syntactical rules, and instead of justifying mathematical axioms by reducing them to syntactical rules at least some of these mathematical axioms are necessary in order to justify the syntactical rules, in the sense that they are required to show that the syntactical rules are consistent.

(such as the theory of 1st and 2nd degree Diophantine equations) show an amazing degree of beauty and perfection.

In looking for a workable combination of the two directions, Gödel turns to the philosophy of Husserl. He says that there exists today the beginning of a science that claims to possess a systematic method for such a clarification of meaning, and that is the phenomenology founded by Husserl. The conception of scientific philosophy here, however, is quite different from Carnap's view that only the method of "logical analysis," that is, laying out the syntax of the language of science, can count as scientific philosophy. Think again of Husserl's phenomenological version of the Leibnizian idea of universal a priori scientific philosophy, as noted in chapter 1, and of how clarification of meaning would be understood in this context. Clarification of meaning, Gödel now says, "consists in focusing more sharply on the concepts concerned by directing our attention in a certain way, namely onto our own acts in the use of these concepts, onto our powers in carrying out our acts, and so on." Phenomenology is not supposed to be a science in the same sense as other sciences, as Husserl himself points out in the quotations presented in chapter 1. Rather, it is supposed to be a procedure that should produce in us a new state of consciousness in which we describe in detail the basic concepts we use in our thought, or grasp other basic concepts hitherto unknown to us.[11] Gödel says that he sees no reason to reject such a procedure at the outset as hopeless. Empiricists in particular have no reason to do so since that would mean that their empiricism is actually a kind of dogmatic apriorism.

One can in fact present reasons in favor of the phenomenological approach. Gödel's brief gloss on this in the 1961 paper is that if one considers the development of a child one sees that it proceeds in two directions. On the one hand, the child experiments with objects in the external world and with its own sensory and motor organs. On the other hand, it comes to a better and better understanding of language and of the concepts on which language rests. Concerning this second direction we can say that the child passes through states of consciousness of various heights. A higher state is attained, for example, when the child first learns the use of words and, similarly, when for the first time it understands a logical inference. We can view the whole development of empirical science as a systematic and conscious extension of what the child does when it develops in the first direction. The success of this procedure is astonishing and far greater than one might expect a priori. After all it leads to the remarkable technological development of recent times. Gödel thinks that this makes it seem quite

[11] There are references elsewhere in Gödel's thinking to meaning clarification and phenomenology. In Wang 1974, p. 189, for example, Wang says "With regard to setting up the axioms of set theory (including the search for new axioms), we can distinguish two questions, viz., (1) what, roughly speaking, the principles are by which we introduce the axioms, (2) what their precise meaning is and why we accept such principles. The second question is incomparably more difficult. It is my impression that Gödel proposes to answer it by phenomenological investigations." See also the opening of chapter 4 below.

As noted earlier in this chapter, Kreisel (Kreisel 1969, p. 97) says that notions of set theory are difficult notions but not dubious: "probably the first step is . . . *to recognize the objectivity of the basic notions* (subset, power set) . . . and then, if possible, to give a phenomenological analysis of these notions."

possible that a systematic and conscious advance in the second, rationalistic direction might also far exceed the expectations one might have a priori.[12]

There are examples that show how considerable further development in the second direction occurs even without the application of a systematic and conscious procedure, a development that transcends "common sense." Gödel's example here is that in the systematic establishment of axioms of mathematics, new axioms that do not follow by formal logic alone from those previously established again and again become *evident*. His own incompleteness theorems could be used to show this, in the sense that we can augment a given formal system with its Gödel sentence, and then repeat this process indefinitely. He also has in mind the addition of more and more axioms of infinity in set theory. Gödel says that the incompleteness theorems, which are often viewed as negative results, do not exclude the possibility that every clearly posed mathematical yes-or-no question is solvable in this way, for it is just this becoming evident of more and more new axioms on the basis of the meaning of the primitive concepts that a machine cannot emulate. I discuss this last point in much more detail in chapter 7.

In the text of *1961/?, Gödel goes on to say that there is an intuitive grasp of ever newer axioms that are logically independent of the earlier ones, and that this is necessary for the solvability of mathematical problems. (See again Gödel's remarks above on the analogy between sense perception and reason, where he speaks of a series of independent rational perceptions.) He says that this agrees in spirit if not in letter with the Kantian conception of mathematics in the following sense: "Kant asserts that in the derivation of geometrical theorems we always need new geometrical intuitions, and that a purely logical derivation from a finite number of axioms is impossible." Given what we now know about elementary geometry this is demonstrably false. But if we replace the term "geometrical" by "mathematical" or "set-theoretical" then it becomes a demonstrably true proposition. What Gödel does not mention here, although we have seen it in his other comments, is that in this case we also need a conception of rational intuition, such as categorial intuition, that goes beyond Kant's (two forms of) sensory intuition.

At the end of the 1961 text Gödel explicitly describes how he sees the relationship of Husserl's transcendental phenomenology to the philosophy of Kant. He says that many of Kant's assertions are false if understood literally, but in a broader sense contain deep truths. The whole phenomenological method, according to Gödel, goes back in its central idea to Kant. What Husserl did was to formulate it more precisely, make it fully conscious, and actually carry it out for particular domains. It is because in the last analysis Kantian philosophy rests on the idea of phenomenology, albeit not in an

[12] In the letter to Leon Rappaport cited earlier (Gödel 2003b, Vol. V, pp. 176–7), Gödel says that the developments in this direction, however, have been stunted: "Our knowledge of the abstract mathematical entities themselves (as opposed to the *formalisms* corresponding to them) is in a deplorable state. This is not surprising in view of the fact that the prevailing bias even denies their existence." I quote and discuss this letter in more detail in chapter 7.

entirely clear way, that Kant has had such an enormous influence over the entire development of philosophy. "Quite divergent directions have developed out of Kant's thought," however, due to "the lack of clarity and literal incorrectness of many of [his] formulations." None of these developments have really done justice to the core of Kant's thought. It is Husserl's phenomenology that for the first time meets this requirement. It avoids both the death-defying leaps of idealism into a new metaphysics as well as the positivistic rejection of all metaphysics. The *1961/? text concludes with a rhetorical question: "if the misunderstood Kant has already led to so much that is interesting in philosophy, and also indirectly in science, how much more can we expect" from Kant correctly understood by way of Husserl?

§ 5. A workable combination of leftward and rightward views

In this 1961 text Gödel asserts that it is in Husserl's work we might find a workable combination of leftward and rightward elements. Carnap's view of mathematics as syntax of language is on the left side of Gödel 's schema and is to be abandoned, while Hilbert's attempt to combine leftward and rightward elements is not feasible. The 1953/59 drafts indicate the problems with Carnap's view, and in the 1961 text and elsewhere the problems with Hilbert's view are indicated. As indicated in chapter 1, Kant is also arguably too far to the left. In the 1961 text Gödel suggests some modifications of Kant's view on the basis of Husserl's philosophy, and the modifications I mentioned in chapter 1 would also presumably be involved. Kant is not a platonist about mathematical objects or facts, and he mounts a critique of classical rationalism, including Leibniz. For Kant, knowledge is restricted to sensory intuition and the two forms of sensory intuition, space and time. Kant distinguishes phenomena from noumena and is able to develop the transcendental method far enough to show how *empirical realism* is compatible with transcendental idealism (see chapter 4), but Gödel wants to extend the transcendental method to accommodate a kind of platonism that recognizes categorial intuition and the reality of abstract concepts. Categorial intuition or *Wesensanschauung* is supposed to make it possible to grasp the primitive concepts clearly and distinctly, which amounts to a way of updating Leibniz's rationalism in a manner compatible with a kind of platonism but which, on account of this very platonistic rationalism, leads to a conception of decidability other than purely mechanical decidability.

As we have said, Gödel is concerned to preserve two rightward elements about mathematics: (i) optimism about the decidability of mathematical problems by human reason and (ii) the idea that a secure grounding of mathematical propositions can be provided by the abstract concept of proof, that is, the concept of proof as that which provides evidence. On the left we have Carnap's conventionalism/nominalism and Hilbert's formalistic finitism. Both of these are to be rejected, but on the right we also want to avoid a "death-defying leap of idealism" into a dubious metaphysics. We also want to avoid an unfounded and naive rationalism. How can we avoid a dubious metaphysics and skirt Kant's objections to rationalism? Gödel evidently hopes that we can avoid the one-sidedness or prejudices of the positive sciences and yet steer clear of questionable metaphysical views by developing a scientific philosophy that employs the phenomenological *epochē* and still requires that knowledge in mathematics and logic depends not only on mere conception but also on intuition, only now it is categorial intuition or *Wesensanschauung* that we must cultivate.

Skepticism about metaphysics was certainly in the air in the early twentieth century. The views of Carnap and the other logical positivists were one manifestation of this fact. The phenomenological *epochē* can also be seen as an expression of skepticism about metaphysical claims, but it does not issue in the radical reductionistic empiricism and conventionalism of the logical positivists. Husserl's views on the *epochē* and on evidence arguably require that we give up claims to knowledge about some ultimate reality behind appearances (phenomena). This does not require, however, that we give up claims to objectivity or that we abandon ontology altogether. *Phenomenological* ontology is still a possibility. In the next chapter I will begin to unfold the meaning of this claim by considering whether it might be possible to avoid a death-defying leap into a dubious traditional platonist metaphysics about mathematics and logic. This will lead us to a new form of mathematical and logical platonism, constituted platonism. In chapters 5, 6, and 8, I will discuss the rationalist epistemology that is correlated with constituted platonism.

4

A New Kind of Platonism

The combination of "leftward" and "rightward" views that Gödel wants in the foundations of mathematics and logic will involve a kind of platonic rationalism. We have already seen a number of expressions of Gödel's platonic rationalism in previous chapters. Some of Gödel's most widely known views of this type are to be found in the 1964 version of "What is Cantor's Continuum Problem?" He says here that by a proof of undecidability, such as the undecidability of the continuum hypothesis (CH) from the current axioms of set theory, a question loses its meaning only if the system of axioms under consideration is interpreted as a hypothetico-deductive system, that is, if the primitive meanings of the terms are left undetermined. Gödel (Gödel 1964) argues that in set theory the question whether CH is true or not retains its meaning if the primitive terms are taken to have the kind of meaning associated with an iterative concept of set. The objects of transfinite set theory conceived in this manner, however, "clearly do not belong to the physical world, and even their indirect connection with physical experience is very loose (owing primarily to the fact that the set-theoretical concepts play only a minor role in the physical theories of today)."

> But despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics. That new mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis are perfectly possible was pointed out earlier.
> (Gödel 1964, p. 268)

Gödel now has more to say about his own view of mathematical intuition:

> It should be noted that mathematical intuition need not be conceived as a faculty giving an *immediate* knowledge of the objects concerned. Rather, it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents

qualitatively different from sensations or mere combinations of sensations, i.e., the idea of the object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality. (Gödel 1964, p. 268)

In the next sentence of the passage Gödel says that the question of the objective existence of the objects of mathematical intuition is nonetheless an exact replica of the question of the objective existence of the outer world. We have already seen some of the ideas in these passages in earlier chapters, and I will consider the topic of mathematical intuition briefly in this chapter and in more detail in chapter 6.

These quotations are from the famous supplement to the 1964 version of the Cantor paper. In a draft of this supplement in the Gödel *Nachlass* there is an additional sentence after Gödel's comment that "it by no means follows... that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are purely subjective, as Kant asserted." It shows that Gödel had at least some of Husserl's ideas in mind when he wrote the supplement. He says that "Perhaps a further development of phenomenology will, some day, make it possible to decide questions regarding the soundness of primitive terms and their axioms in a completely convincing manner" (*Nachlass* series 4, folder 101, item 040311).[1]

Many of the complaints about Gödel's philosophical views have centered around the kind of platonic rationalism about mathematics and logic expressed in these and other passages of Gödel's work (see, e.g. Chihara's early 1973 and 1982). A major issue that has been raised for Gödel, as it has for all platonists, is this: how is it possible for the human mind or for human reason to know about abstract transcendent objects, including large transfinite sets, that do not belong to the physical world? As Husserl puts it in various places in his writings, how could human consciousness, on the basis of its subjective mental acts and processes, come to know about ideal and mind-independent objects of mathematics and logic? How can we bridge the gap between

[1] Why did Gödel omit this sentence in the published version? Perhaps it was due to his caution in expressing his philosophical views in publication. See Feferman's "Conviction and Caution" (Feferman 1984). At the time that Gödel would have written these words there was a profound split between so-called "analytic" philosophy and the "Continental" tradition with which Husserl's phenomenology was associated. Many aspects of this split are still with us. See, e.g. the books Dummett 1994 and M. Friedman 1999 and 2000. Føllesdal's early paper, Føllesdal 1972, is still helpful. Husserl's eidetic phenomenology was, and still is, often misinterpreted as primarily a form of existentialism because it was, and is, read or understood through the filter of philosophers such as Heidegger, Sartre, and their successors. Sartre, for example, famously asserts that "existence precedes essence," while Husserl holds that to know the essence of a thing is to know the conditions for the possibility of its existence. As we saw in connection with Husserl's comments on Leibniz in chapter 1, the science of necessities and possibilities is said to precede the sciences of actualities.

subjects and objects in this case? Husserl's great contemporary, Gottlob Frege, was wrestling with the same questions about what he called "thoughts" (Frege 1918). Frege did not get very far in answering the question how we could know about the "third realm" of abstract thoughts, but Husserl wrote at great length on this problem, as we shall see below. For the platonist, natural numbers, sets, functions, propositions (or "thoughts") and the like are not supposed to be material or mental entities, are not located in space or time, are not objects to which we could be causally related, and so on. It is therefore a puzzle how we could know about such things.[2] I agree that it is a puzzle, and an unsolvable one, if the only way we can know about anything is through sense perception or introspection. Acts of sense perception ("outer sense") are not directed toward mind-independent abstract or ideal objects, and acts of introspection ("inner sense") are also not directed toward mind-independent abstract or ideal objects. I think that restricting all knowledge to what is derivable from sense perception or psychological introspection, however, flies in the face of basic facts about the intentionality of different modes of human consciousness. If we recognize that intentionality is involved not only in sense perception and introspection but also in the modes of *conception* associated with reason in mathematics and logic, then perhaps the appearance of a puzzle here will evaporate. In chapter 1, I cited some texts and comments in which Gödel indicated that he was interested in a kind of platonism about the abstract concepts and objects of mathematics and logic according to which we know about such mind-independent objects on the basis of categorial intuition or *Wesensschau*. Categorial intuition is a kind of rational intuition. Gödel, taking his lead from Husserl, was interested in developing elements of Kantian transcendental method in such a way as to accommodate categorial intuition and the reality of abstract concepts. How is it possible to do this?

In Husserl's *The Idea of Phenomenology* (Husserl 1907), which Gödel calls a "momentous lecture" (*Nachlass* item 050120.1), we are told that the way to solve the old, vexing philosophical problem of how we can be related to transcendent objects is through the phenomenological reduction (*epochē*). We already noted some of Gödel's remarks about the *epochē* in chapter 1. In the present chapter I will describe the *epochē* in more detail, and in this and the following two chapters I will examine Husserl's claim about solving the problem of transcendent abstract objects, with an eye to developing a defensible version of Gödelian platonic rationalism. The defensible version of platonism I have in mind is what I called constituted platonism in chapter 1 (see also Tieszen 2010). According to constituted platonism, the transcendental ego as monad constitutes in a rationally motivated manner in classical mathematical practice the meaning of being of the objects toward which it is directed as ideal or abstract and

[2] There is, of course, a massive literature on this puzzle. I am certainly not going to address all the attempts at solving it. Rather, I present a position—constituted platonism—that differs from the other positions that have been developed in response to the problem. One attempt that I have discussed in some detail elsewhere is due to Penelope Maddy. See Tieszen 1994b.

non-mental. It constitutes such objects or concepts as transcendent or mind-independent. It is not possible to understand the constitution of meaning of being of such objects and, thus, constituted platonism, without seeing that human consciousness exhibits intentionality, and in chapters 5 and 6, I will discuss the intentionality of human reason in some detail. Constituted platonism is what emerges if we develop the transcendental method in light of the intentionality of human consciousness in a way that allows us to accommodate the rational intuition of abstract concepts or objects. The way to bridge the gap between human subjectivity and mathematical objectivity is to fill in the account of the kinds of founded intentional acts and processes that make the constitution of the meaning of being of mathematical and logical objects possible.

It can be argued that the position I call "constituted platonism" is not really new or groundbreaking, and that it can already be found in Husserl. It will be apparent below that I think a good case can be made for the claim that it is in fact already present in the later Husserl's work.[3] I have heard disagreement in various quarters, however, about whether Husserl actually held such a position. What I want to do in this and subsequent chapters, irrespective of whether Husserl himself held the position or not, is to clarify and lay out the view in some detail, especially against the background in Kantian philosophy, and to discuss its connection with mathematics and the incompleteness theorems (not just logic), in order to establish its importance for making sense of, and defending, Gödelian platonic rationalism. This has not been done anywhere in the literature. Constituted platonism, I think, is likely to strike many readers of the mainstream literature in the philosophy of mathematics and philosophy of logic as a novel position.

§ 1. Some background in the philosophy of Kant

Since the transcendental method in Kant and Husserl is associated with transcendental idealism, the question arises as to how mathematical realism (platonism) could be compatible with transcendental phenomenological idealism. There are forms of (naive) mathematical realism and (naive) idealism that are flatly inconsistent with one another. The division between realism and idealism in philosophy has of course a long history. For my purposes in this book, and especially on the basis of the claims in chapter 1, it is in Kant's transcendental philosophy that we find the most important approach to the realism/idealism debate prior to Husserl. What makes Kant's position so interesting is how he tries to go between the horns of the realism/idealism dilemma.

[3] At least one other Husserl scholar, Jean Petitot, appears to agree with me on this matter. While I was lecturing in Paris on constituted platonism Petitot brought his paper Petitot 1995 to my attention. See also Petitot 2009. He uses the expression "transcendental platonism." It is my impression that he has not explored the position in any detail in relation to Gödel's philosophical writings in particular. I have not yet been able to compare the details of our views.

As we indicated, Gödel was quite interested in Kant's transcendental idealism but it is not at all obvious how this could be compatible with his inclination toward platonic rationalism about mathematics and logic. In his papers in the 1940s on Kant, relativity theory and time (Gödel 1949a, 1949b, *1946/9) we see how Gödel is already thinking about how objectivity could be compatible with transcendental idealism. One approach that he seems to be suggesting at this stage is to hold that knowledge of things-in-themselves in Kant's sense is in fact not ruled out, perhaps not even on Kant's own view. At one point he says, for example, that Kant's view of the unknowability (by theoretical reason) of things-in-themselves should be modified if we want to establish agreement between Kant's doctrines and modern physics: "It should be assumed that it is possible for scientific knowledge, at least partially and step by step, to go beyond the appearances and approach the world of things" (Gödel *1946/49-B2, p. 244). Gödel even presses the claim that the possibility of knowledge of things-in-themselves is not as strictly opposed by Kant himself as it is by many of his followers (Gödel *1946/49-B2, p. 245): Kant "evidently did not want to say that nothing whatsoever can be asserted about the things-in-themselves. For Kant himself asserted, e.g., that they exist, affect our sensibility, and do not exist in time and space, but that the ideas of time and space are completely adequate to their relationship to our sensibility." These are remarks, as indicated, from the late 1940s. There are three reasons why I am not going to pursue the idea that we might be able to know about things-in-themselves: (i) although it perhaps reflects Gödel's efforts at this stage of his career to avoid a flat contradiction between a Kantian idealism and realism, it seems to me to strain credulity as an interpretation of Kant; (ii) it is put forward prior to Gödel's reading of Husserl and, in any case; (iii) I do not see how such a view can possibly be defended.

The first quotation cited in the last paragraph suggests a weaker view according to which it is possible to obtain higher degrees of objectivity in our scientific knowledge, and in this sense to approach the world of things partially and step by step. This view, provided it does not involve the claim that we can at some point in time go beyond appearances, strikes me as defensible and as consistent with Husserl's transcendental phenomenological idealism. I think that we can arrive at a defensible view on the basis of some ideas of Husserl that extend and modify Kant's efforts in the *Critique of Pure Reason* (*CPR*) to reconcile empirical realism with transcendental idealism.

Kant's views on the idealism/realism issue are all concerned with natural science and the world of sensation. His analysis is focused on the issue of whether there are mind-independent physical or material objects, not on whether there are mind-independent abstract or ideal objects. Kant was not what we nowadays think of as a mathematical platonist, and he was in fact critical of platonism in general, as we saw in the passages from the *Critique of Pure Reason* cited in chapter 1. Nonetheless, in order to discern an important pattern in Kant's thinking on this issue, let us briefly consider how it is supposed to be possible to be both a transcendental idealist *and* an empirical realist.

In order to provide a brief overview of Kant's transcendental philosophy, and his transcendental idealism, we can recall his "Copernican turn" in philosophy (*CPR*

Bxvi – Bxix). Kant says that attempts to extend our knowledge of objects by a priori means on the basis of concepts have ended in failure because it was assumed that all our knowledge must conform to objects of sense experience. We might have more success in metaphysics, however, if we abandon the assumption that all our knowledge must conform to objects, and instead suppose that objects must conform to our knowledge. If we are trying to determine how there could be a priori knowledge, then this latter supposition will be more fruitful insofar as it allows us to determine something about objects, prior to their being given, on the basis of concepts. Kant says that if intuition must conform to objects, then he does not see how we could know anything about objects a priori, but if the object (of the senses) must conform to our faculty of intuition then there is no difficulty in entertaining such a possibility. Experience is a species of knowledge that involves "understanding" in Kant's sense, and understanding has rules that we must presuppose as being in us prior to objects being given to us, and therefore as being a priori. Kant says these rules find expression in a priori concepts to which all objects of experience necessarily conform, and with which they must agree. *Transcendental* philosophy is concerned with determining what in our experience or knowledge of objects is due to us human subjects a priori. This does not, however, preclude the possibility that (sensory) objects are given to us in our experience as *transcendent,* not of course in the sense of being noumenal, but only in the sense of being phenomenal, and yet also not merely subjective. As regards objects which are thought solely through reason, and indeed as necessary, but which can never be given in sense experience, this new method of philosophy will help us to see that we can know a priori of things only what we ourselves have put into them.

Kant is also famous for distinguishing phenomena from noumena. This is what Hao Wang refers to as Kant's dualism in the passages we quoted from Wang in chapter 1, and we will see below that Husserl sets aside the notion of the noumenal. In Kant, the distinction is used to separate appearances of objects from objects as things-in-themselves that conceivably lie behind the appearances. Kant holds that we cannot know anything about the noumenal realm behind appearances, if there is such a realm. At one point Kant says that, although we cannot know about the noumenal world, we can think without contradiction that there is such a world. Knowledge, however, is limited to appearances. What is "transcendental," he says, is what is concerned with possible experience. Appearances are ordered and structured by the twelve categories (basic concepts) of the understanding and by the two forms of intuition, space and time. Kant's view of knowledge is hylomorphic, in the sense that the "matter" that we receive from our five senses is formed or structured by the two forms of intuition, space and time. Space is said to be the form of outer intuition, and time is said to be the form of inner intuition. Additional structuring or forming of the manifold of sensibility is a function of the categories of understanding, such as the category of substance and accident and the category of cause and effect. The category of cause and effect applies to the manifold of sensibility, according to Kant, but on a view such as Husserl's it would of course not have application in the case of abstract or ideal objects. Kant holds that our sensory experience has the form or structure that it has thanks to the fact that the categories of understanding and forms of intuition are applied to the manifold of sensibility delivered by the five senses. Concepts and (sensory) intuitions (or forms of sensory intuition) are both required for knowledge. Kant says that concepts without intuitions are empty, and that intuitions without concepts are blind. Knowledge is a product of conception and (sensory) intuition. When we speak of objects, Kant says, we must always be careful to distinguish two senses of the term "object": object in the sense of appearance or phenomenon; and object in the sense of noumenon. It is not possible to investigate knowledge of objects in the latter sense. In theoretical philosophy, knowledge is always knowledge with respect to phenomena, to possible experience.

Principles whose application is confined entirely within the limits of possible (sensory) experience are said by Kant to be transcendental, and those that profess to pass beyond these limits are called transcendent (in the noumenal sense). Kant's empirical realism, however, requires him to argue that, even after we restrict ourselves to the domain of possible experience we have experience of non-mental *objects* and not just subjective phenomena. Restricting ourselves to phenomena, to the domain of possible experience, there are appearances of objects in space and appearances of objects in time. The objects that appear in space and time are physical or material objects, not mental phenomena. Kant says that the objects of "outer intuition" are always subject to the forms of both space and time. We also have appearances pertaining to inner intuition: that is, appearances of mental phenomena such as thoughts, beliefs, desires, and the like. These are not subject to spatial form, but they are subject to time. Time is, in fact, the form of inner intuition.

With this brief overview of Kant in mind we can now state the manner in which Kant is both a transcendental idealist and an empirical realist. Kant holds that we do not have access to objects as noumenal. Thus, we cannot speak of knowing about mind-independent physical objects in the sense of noumenal objects. This does not, however, mean that we cannot speak about knowing objects. Indeed, we know about objects as phenomenal, but this does not carry the strong sense of mind-independence that characterizes the noumenal. Kant is thus not a realist about objects as noumenal. At the same time, he holds that we do know about objects that are mind-independent in the sense that they are not subjective and not "immanent" (in the sense of being our own mental phenomena). Physical objects are given as mind-independent in the sense that they are not our own ideas. Indeed, they appear to us as being in space. They are given in outer intuition. Thus, they are, in this sense, mind-independent and even "transcendent," but not in the strong sense of being noumenal. Some objects *appear* to us as external to us. Thus, Kant can say that he is an empirical realist about physical objects as they appear. They are mind-independent, but not in the sense of being noumenal. At the same time, objects as phenomenal (but not noumenal) are mind-dependent in the sense that they are functions of the forms of understanding and the forms of intuition applied to the "matter" provided by the senses. The form of experience is due to us, but the matter that is formed is not due to us. It is given in receptivity. It is in this manner that Kant can say he is a transcendental idealist and an empirical realist. The subtlety of Kant's position is lost on certain kinds of empiricists and also on certain kinds of rationalists, but it seems to me that it should be taken seriously by anyone who wishes to understand the philosophical views of the later Gödel and the later Husserl.

Let us now come back to the platonist in mathematics or logic. Kant does not say that he is a transcendental idealist and a *mathematical* or *logical* realist. Intuition for Kant, unlike Gödel, is restricted to sensory intuition. Knowledge for Kant depends on the application of concepts to the manifold of sensibility. Arithmetic and geometry are restricted to the *forms* of inner and outer intuition respectively, and these are always forms of sensory intuition. Is Kant a platonist about mathematics or logic in the sense that he holds that there are mind-independent *abstract* objects such as propositions, concepts, essences, numbers, sets, functions, triangles, and the like? Does he have a view according to which we could *know* about such abstract objects? In order for knowledge of such objects to be possible, the objects would need to be given in either inner or outer intuition, or their existence would need to be deduced by a transcendental argument. He does not hold, like Husserl or Gödel, that we can intuit such objects in rational intuition. Kant is certainly not a platonist about mathematics or logic in the manner of, say, Frege, Gödel, or Husserl (in the "Prolegomena to Pure Logic"). As soon as we start asking these kinds of questions we are faced with a host of issues that Kant does not address, as Husserl points out in the passages quoted in chapter 1. Kant felt no need to address these issues. After all, "Plato left the world of the senses, as setting too narrow limits to the understanding, and ventured out beyond it on the wings of the ideas, in the empty space of the pure understanding."

For Kant there would thus not have been a question of trying to reconcile mathematical platonism and transcendental idealism. By 1907 or so, however, it is quite possible to read Husserl as attempting to do such a thing. I would say that it is thanks to Kant that we can consider the possibility of reconciling realism with transcendental idealism at all, and it is thanks to Husserl that we can consider the possibility of reconciling mathematical realism and transcendental phenomenological idealism. One can read Husserl, after 1907 or so, as holding that, conditioned appropriately by one another, there is a form of mathematical platonism that is compatible with transcendental phenomenological idealism. This is perhaps related to Kreisel's observation that Gödel did not view realism and idealism as conflicting philosophies but, rather, "he was ready to treat them more like different branches of the subject, the former concentrating on the things considered, the latter on the processes of acquiring knowledge about these objects or about the processes" (Kreisel 1980, p. 209). This view reported by Kreisel might be read as reflecting the Husserlian idea that if we analyze the nature of our conscious directedness it is quite typical that there are, on the one hand, various constituting acts and mental processes and, on the other hand, the constituted objects and states of affairs. This is just the nature of human consciousness itself, given that it exhibits intentionality. It is then just a matter of analyzing the kind of intentionality involved in the domain of investigation, along with the correlative elements involved in the meaning constitution. Unlike Kant, Husserl has the notion of the intentionality of consciousness and also the correlative idea of the constitution of the meaning of being that is based on his analysis of intentionality.

In order to proceed toward my goal of characterizing a new kind of mathematical and logical platonism, let me now provide a brief opening characterization of mathematical platonism and, on the other hand, of a standard anti-platonism about mathematics.

§ 2. Standard simple formulations of platonism, anti-platonism, and idealism

Mathematical Platonism: There are mind-independent abstract (or "ideal") mathematical objects or truths.

Notice that I have formulated platonism here specifically for mathematical objects or truths. By "mathematical" I just mean the kinds of objects or truths that practicing mathematicians typically take themselves to be thinking about. This includes geometric objects, natural numbers, real numbers, complex or imaginary numbers, functions, groups, sets, or categories, and truths about these objects. Some platonists, such as Gödel, hold that objects such as concepts, meanings, propositions, properties, or essences, also exist. Mathematicians themselves (unlike some logicians) do not typically take themselves to be talking directly about such things in their theories. Logicians who are platonists are more likely to talk about such things. I do not want to make too much of the difference at the moment but I will note that at least part of what is involved here is that concepts, propositions, properties, essences, and the like are usually thought of as overtly *intensional* objects, whereas this is not typical in the case of standard mathematical objects in classical mathematics.

Wang tells us that Gödel associates sets with extensions and mathematics, and concepts with intensions and logic (Wang 1996, p. 273). Logic is concerned with intensional objects (concepts) while mathematics is concerned with extensional objects. This distinction is reflected in Gödel's view that the intensional paradoxes should be distinguished from the semantic and extensional paradoxes. Thus, the intensional paradox associated with the concept of all concepts not applying to themselves is not, according to Gödel, a trivial variant of the extensional paradox associated with the set of all sets not belonging to themselves. This view emerged in Gödel's work some time after he wrote his paper on Russell's logic (Gödel 1944). Gödel thought that we have a fairly satisfactory set theory that resolves the extensional paradoxes, but that the intensional paradoxes had not yet been resolved. I will not go into the many further details about Gödel's views on logic and intensions here, except to note that Gödel thought that the proper understanding of logic called for the development of concept theory. Gödel also makes the interesting observation that paradoxes of either type constitute evidence for objectivism because they reveal a constraint imposed by objective reality on our freedom to form sets and concepts, and because we have been able to resolve the extensional paradoxes by perceiving the concept of set more clearly. Many more details about Gödel's views on logic, concept theory, and the intensional paradoxes can be found in Wang 1996, pp. 264–280. (See also Crocco 2006.)

The general distinction between intensional and extensional objects is an old one in philosophy. There are different views of the distinction but let us consider a typical way of drawing the distinction, due to Frege (Frege 1892 and 1892–95) and also found, at least in part, in Husserl (Husserl 1901, 1913), in connection with the meaning of expressions. The idea is that certain categories of expressions express intensions and that many of these expressions, but not all, have extensions. The extension of an expression is determined by its intension. For example, the expression "the evening star" determines a particular extension, as does the expression "the morning star." It turns out that these expressions have the same extension even though they differ in meaning. A standard way to distinguish intensional from extensional objects is on the basis of their identity conditions. Sets are good examples of extensional objects. According to the axiom of extensionality, a set A is identical to a set B if and only if A and B have the same members. Sameness of membership alone determines identity. Now expressions such as "creature with a kidney" and "creature with a heart" have the same extension. They refer to the same set but they express different intensions. They do not have the same meaning. Intensional identity is something different from extensional identity. Many examples of this type can be given. One can also find expressions that express intensions but lack extensions; for example "the present king of France"or "round square." Universals, essences, concepts, properties, and propositions are typically understood as intensional objects. Fregean senses (*Sinne*) and Husserlian ideal "meanings" also belong here. I will come back to some points about intensions and extensions below in the discussion of intentionality, concepts, and reason.

Among the platonists who recognize the existence of intensional objects, some give priority to intensional objects and claim that extensional objects are derivable from, or dependent on, intensional objects, while others do not give priority to intensional objects.[4] It is possible, it appears, to be a platonist who seeks to reduce intensions to extensions or to not recognize the existence of intensions over and above extensions.

[4] Bealer 1982 is an example of a work that gives priority to intensional objects, i.e. properties, relations, and propositions.

We should note, in any case, that one can evidently be a platonist about extensional objects, intensional objects, or both.

The distinction between intensions and extensions is very difficult to ignore in connection with questions about mathematical *knowledge*. Some principles that are accepted in the forms of reasoning codified in classical, extensional logic (e.g. substitutivity *salva veritate* and existential generalization) appear to fail in the presence of assertions about knowledge, belief, and other cognitive states.[5] One can therefore ask whether such systems really do justice to human reason. In any case, I will argue that it is essential to recognize the distinction between intensions and extensions in considering the nature of human reason and the role of reason in knowledge acquisition.

Given our initial formulation of mathematical platonism, a standard formulation of anti-platonism about mathematics and logic is very easy to come by. Simply negate the formulation of platonism above:

Mathematical Anti-Platonism: It is not the case that there are mind-independent abstract (or "ideal") mathematical or logical objects or truths.

This is the flat denial that there are such objects. In the old debate between realism and idealism, idealists might claim that there are mathematical or logical objects, and that they are even "abstract" in some sense, but they are all mind-dependent and not eternal or atemporal. Mathematics, for example, is about mental entities that are "abstract" in the sense that they cannot be given in sense perception, and yet these mental entities are precisely mind-dependent. Thus, let us formulate mathematical idealism, as a form of anti-platonism, as follows.

Mathematical Idealism (as a form of Anti-Platonism): Mathematical or logical objects (which may be "abstract" in some sense but not eternal or atemporal) are mind-dependent.

On these formulations, mathematical platonism and standard mathematical idealism (which is distinct from transcendental phenomenological idealism) are incompatible. They are incompatible at a level of generality that spares us the need to consider any further details.

If we had captured the essential features of mathematical platonism and transcendental phenomenological idealism in these formulations then we would have an answer to our question whether they are compatible and I could conclude this chapter. Needless to say, I think we have barely scratched the surface. Therefore, let us consider mathematical and logical platonism in somewhat more detail.

[5] Alonzo Church's logics of sense and denotation, especially Alternative 0, are of special interest in this connection (Church 1951b, 1973, 1974). Along with Frege, Russell (at a certain stage), and Gödel, Church is another well-known logician who argued in favor of a platonist view. See Church 1951a.

§ 3. Mathematical or logical platonism

In this section I will indicate a host of general properties that mathematical or logical objects are supposed to possess, according to platonists. Platonists could very well agree that mathematical or logical objects possess these properties and yet disagree about *which* mind-independent abstract or ideal objects exist. As Gödel says at one point, there are four hundred ways to be a platonist (Wang 1996, p. 212). Among the types of mathematical or logical objects about which one might be a platonist are geometric objects of different kinds, natural numbers, real numbers, complex or imaginary numbers, sets of different kinds, functions of different kinds, groups, categories, concepts, essences, or properties. One might be a reductionist or eliminativist about some of the items on this list. For example, one might adopt a platonist view about natural numbers but not about real numbers or imaginary numbers. One popular strategy has been to recognize the existence only of sets and then to define some of the other objects on the list in terms of sets.

In this chapter and in subsequent chapters I will focus on platonic rationalism about natural numbers, sets, and concepts (as intensional entities), not on the basis of some principled position, but only because these are the kinds of objects that Gödel discusses most frequently.

The mind-independent abstract (or ideal) objects that are thought to exist by platonists are usually taken to have the following properties. As the formulation obviously indicates, they are *mind-independent*. This means several things. First, they are not themselves mental entities. They are not the subjective ideas, images, or thoughts of human beings. They are not immanent to human consciousness but they are supposed to *transcend* human consciousness. They are not internal to human consciousness but are in some sense external to it. They are supposed to exist whether there are minds in the universe or not. They would exist even if there were no minds or had never been any minds. The properties of "being expressed" or "being thought of" are not essential to mathematical or logical objects. Such objects are external to human consciousness, but not in the manner of sensory, physical, or material objects. This is what it means to say they are *abstract*. (Note that I'm using the term "abstract" as it is often used in the recent literature on platonism in the philosophy of mathematics, not in the sense of Husserls' theory of parts and wholes in which non-independent parts ("moments") of a whole are said to be "abstract." See below.) To say they are abstract is to say that they are not located in physical space, not involved in causal relations, as material objects presumably are, and not the kinds of objects that can be sensed with one or more of our five senses. "Concrete" objects, however, would have all of these latter properties.

On might ask whether (at least some) geometric or topological objects are located in physical space and can be sensed with one or more of our five senses. Consider the example of a physical knot, tied in a rope or some other material. We in fact sense such a knot and it is obviously present in physical space. The view of the platonist, especially on the kind of platonic rationalism to be discussed below, would be that the knot given to us in this way is not the knot that is meant or intended in topology or knot theory, but is rather a particular concrete representation of such an object. We can speak of knots quite readily without knowing a thing about knot theory or topology. To speak of knot *theory* already presupposes that various categorial activities are at work in our experience (see also chapter 6). Sensory perception would, of course, be involved in handling a physical knot and manipulating it but the acts that are involved in making determinations about it as a mathematical object are already categorial. It is in such categorial acts that a knot is determined to have certain properties and not others, to stand in certain relations to other objects, and so on. The knot, as an object of topology, is not taken to have a color, to be made of a particular material, and so on. It does not share in the inexact ("morphological") character of sensory objects. This is the kind of view that Husserl adopts about the ideal objects of geometry in his essay "The Origin of Geometry" and in many other texts. We should therefore say that it is possible to have particular concrete representations or perhaps proxies in the space of everyday sensory perception for some kinds of ideal geometric and topological objects, but not that such objects are themselves given in sense perception as physical.

Not only are mathematical or logical objects not located in physical space, according to the platonist, but they are also not individuated in time. Unlike objects in physical space or even the objects of "inner sense" (i.e. mental processes, thoughts, images, etc.), they do not have a temporal extension. They are not, as Plato says in works such as *The Republic* (Plato 1969), subject to generation and decay. They are, on the contrary, unchanging. Some platonists say they are "eternal" or "timeless." As I will note below, Husserl has interesting things to say about the relationship of abstract objects to time.

Starting in 1900, in the *Logical Investigations*, Husserl draws a sharp distinction between *real* and *ideal* objects. Although this language is not widely used in the recent literature on mathematical or logical realism, it will be useful to note its relationship to some of the current terminology. The first thing to note is that the "ideal" in this distinction does not refer to "ideas" in a subjective sense. It does not refer to mental entities. It is rather a platonic use of "idealism" that is operative in this case and not, in spite of the potentially confusing language, the use involved in distinguishing idealism from realism. The real/ideal distinction can be drawn in terms of the temporality of objects. Real objects are objects in time. They have temporal duration. This applies to the objects of "inner sense," that is, thoughts, mental processes, and the like, but also to objects of "outer sense," that is, objects in space and in external time. Ideal objects are not in time in the same sense. They do not come into being and pass away.

Much of what I have been saying about abstract objects carries over directly to the ideal objects that Husserl recognizes in the *Logical Investigations*. Mathematical or logical objects, as ideal in Husserl's sense, are not, as I indicated, moments of real objects. Non-independent parts of real objects are just real parts even though they can be thought of as "abstract" in the sense that we can speak and think of them in isolation from the wholes of which they are parts (see Investigation III of the *Logical*

Investigations). This does not mean, however, that they can *exist* in isolation from the wholes of which they are parts. If Husserl is to be a platonist then mathematical or logical objects, as ideal, could not depend for their existence on underlying real wholes. They must exist independently of real objects. Otherwise, Husserl's view would be closer to an Aristotelian realism. There are many remarks in the *Logical Investigations*, especially Investigations II and VI, and elsewhere in Husserl's later writings, that indicate that he is not in this sense an Aristotelian realist about mathematical or logical objects. For example, § 52 of Investigation VI is entitled "Universal objects and their self-constitution in universal intuitions." In this section he contrasts the kind of abstraction involved in setting into relief a non-independent moment of a sensible object with "ideational abstraction" in which an "idea" or "universal," not a non-independent moment, is brought to consciousness.

The platonism here requires that founded constituting activities different from those of abstracting non-independent parts of real objects must be involved in mathematical and logical cognition. In mathematical and logical practice the objects toward which we are directed in our thinking are not meant as moments of real objects. Of which "real" object, for example, would a large transfinite set be a part? The claim is that the objects of mathematics and logic could not be meant in this manner without changing their properties, and that this would contradict the way mathematics and logic are actually given and practiced. We are to start with how mathematics is given and practiced and then ask how this is possible. What kinds of mental acts and processes make it possible? What kind of constituting intentionality must be involved?

I briefly mention one other feature of the real/ideal distinction that is not always salient in the distinction between the concrete and the abstract. The real/ideal distinction in the case of mathematics and logic embodies the difference between the inexact and the exact, or the imperfect and the perfect. This feature of Husserl's distinction has a distinctively platonic pedigree that is often not discussed in modern versions of mathematical or logical realism. Plato's forms were supposed to be perfect in relation to their imperfect or inexact instantiations in the material world. In relation to logic and mathematics, the idea is that logical or mathematical objects are exact and "perfect" in a way that instantiations of, expressions for, or thoughts about, such objects cannot be. In Euclidean geometry, for example, the lines, triangles, circles, and so on, are supposed to be perfect or exact in a way in which drawings of circles and the like, which we can perceive visually, could never be. A globe, which we hold in our hands, could never be exact and perfect in the way that a sphere in Euclidean space is conceived to be perfect or exact. The instantiations in the "real" world can only approximate the ideal.

Modern transfinite set theory is of special interest in connection with mathematical platonism for a number of reasons. It compels philosophers to confront a distinctive and relatively new set of epistemological and ontological issues about mathematical platonism. These are issues, by the way, which either emerged after Husserl's time or to which Husserl himself did not devote much attention. Gödel, however, was keenly aware of them. Modern set theory forces the mind-independence issue in a striking way. Human minds (monads) are finite and have finite capacities. Objects such as natural numbers can be considered finite objects. Even if the human mind cannot actually grasp or form very large natural numbers we can *idealize* the notion of *finite capacity* to cover the grasp or formation of each natural number, thus imagining that there could in principle be a grasp of each natural number as an individual. In modern set theory, however, we are faced with existence statements about huge transfinite sets. Suppose, for example, that we consider some of the existence axioms in Zermelo-Fraenkel set theory with the axiom of choice—in particular, the axioms of infinity, power set, and replacement. These latter three axioms allow us to show rather quickly that very large transfinite sets exist. Not only will denumerably infinite sets exist but also non-denumerably infinite sets will exist, and then power sets of non-denumerably infinite sets, and so on. There is a significant disanalogy with the case of natural numbers: we cannot idealize the *finite* mind or *finite* capacities in such a way as to cover the complete grasp or formation of such transfinite objects. Transfinite sets transcend the (idealized) possibility of being known on the basis of acquaintance with all of their members. A much more substantial idealization has to be involved if there is to be acquaintance with all the members of a transfinite set. Perhaps one could imagine an idealized monad who could grasp such objects, but it would be quite different from the type of monad of which we are examples. If we return to our simple formulation of mathematical platonism then, in connection with set theory, we should ask whether there are mind-independent abstract *infinite* objects. In particular, are there *actual, complete infinite sets*?

Many additional details come into focus once the question of platonism about set theory emerges. There are, of course, the traditional worries about the axiom of choice. Furthermore, with the replacement axiom we also have impredicative specification of sets. Should we therefore hold as part of our mathematical realism that impredicatively specified transfinite sets exist or not? Should we recognize only the existence of predicatively specified sets and restrict ourselves to a predicative set theory?

In his later work Gödel was evidently a platonist about full impredicative set theory with the axiom of choice. Indeed, he is prepared to adopt a platonism that goes beyond the existential commitments of a theory such as ZFC, arguing for the need for new axioms to express more of what already exists in the universe of abstract, mind-independent transfinite sets. Gödel suggests that the search for new axioms depends on sharpening or clarifying our intuition of the concepts or meanings concerning this existing realm of objects or truths, and he says in several places, as we have seen, that phenomenology might help with this task (see the *Nachless* item omitted from Gödel 1964 that was cited above; and Wang 1974, p. 189).

Now let me make some comments about transcendental phenomenological idealism.

§ 4. Transcendental phenomenological idealism

What has been called transcendental phenomenological idealism emerges in the writings of Husserl in which he introduces the phenomenological reduction or *epochē*, starting aound 1907. As noted above, Husserl says in *The Idea of Phenomenology* (Husserl 1907) that the way to solve the problem of how we can be related to transcendent objects is through the phenomenological reduction. As we saw in chapter 1, Gödel says that "transcendental phenomenology with *epochē* as its methodology is the investigation... of the cognitive process, so as to find out what really appears to be—to find the objective concepts."

As indicated by Hao Wang's comments in chapter 1, it appears that Gödel was first drawn to phenomenology for methodological reasons. He was searching for a new method for thinking about the foundations of mathematics and for metaphysics in general, and he thought that phenomenology might offer such a method with its phenomenological reduction or *epochē*. Gödel evidently believed that the method of "bracketing," especially in connection with Husserl's conception of the "eidetic reduction," would enable a person to perceive "concepts" more clearly, or to arrive at essential characteristics of concepts (Wang 1987, p. 193). It would allow for this in part through its suspension of what Husserl called the "naïve" or "natural standpoint," but also because of the sharp distinction between sciences of fact and sciences of essence that one finds in Husserl's conception of the eidetic reduction. The failure to make the latter distinction was, as Husserl saw it, the source of various empiricist or positivistic misunderstandings of sciences of essence such as mathematics, logic and phenomenology itself.

The "natural standpoint" is the naive, prereflective perspective on reality that takes for granted or presupposes the givenness of objects. It is involved in ordinary, everyday experience but also in the natural sciences. The idea of suspending or bracketing the natural standpoint amounts to not making this assumption. It amounts to shifting attention from objects to the *experience of objects*. It is to attend to the presentation of objects instead of to what is presented, and it is to allow that there might not be an object even though our awareness is directed as if there were an object. This involves a kind of reflection in which we do not engage in everyday experience or in the sciences. The natural scientist, for example, is engaged in just straightforwardly studying her objects, their properties, their relations to other objects, and so on. She would no longer be engaged in her science if she shifted from the study of her objects to the study of *the consciousness of the objects*. Similarly, in their everyday research, mathematicians are concerned with their objects—numbers, sets, functions, spaces, and the like—but not with the study of the consciousness of the objects. To shift to the study of the consciousness or experience of objects would be to change the subject. It is the task of Husserlian phenomenology to study the essential features involved in the experience of objects in these various domains of human consciousness.

To say that the natural attitude is "naive" is not at all to dismiss everyday experience or the natural sciences, but it is rather to indicate that the attitude involves certain limitations. I will go into this in much more detail in chapter 7, but we can already note one major kind of limitation. If the natural attitude presupposes that consciousness is in epistemic contact with objects then it is not in a position to account for the possibility of that contact. To account for it would involve a kind of philosophical or phenomenological reflection that would shift us away from the natural attitude and therefore also away from the natural sciences, since the natural sciences depend on the natural attitude.

In the natural attitude in everyday experience it is assumed that the objects toward which we are directed in our conscious acts exist. So the idea of suspending the natural attitude amounts to not making this assumption, allowing that there might not be an object, even though there is a thought or intention directed toward an object, and then understanding the question of knowledge about whether an object exists or not in terms of fulfillments of such thoughts or intentions. This has many consequences which I believe are relevant to Gödel's views on mathematical intuition but I mention here only the obvious consequence that one ought not assume (as one would from the natural standpoint) that the object toward which an act is directed at some stage of experience is necessarily one to which the subject is causally related at that stage. I will return to this point in the next two chapters.

In addition to the idea of suspending the natural standpoint, Husserl's philosophy also includes a sharp distinction between sciences of fact and sciences of essence that was appealing to Gödel. This distinction emerges most forcefully in Husserl's philosophy in works such as *Ideas I* (Husserl 1913) and "Philosophy as a Rigorous Science" (Husserl 1911), where the view that there are no sciences of essence is seen as the curse of positivism, empiricism, and naturalism. Gödel thought that if we could shed empiricist or positivistic confusions about concepts, then we could perhaps make great progress in mathematics and logic, and even metaphysics. As Gian-Carlo Rota has put it, the phenomenological reduction (including the eidetic reduction), with its bracketing of the physical world, is supposed to have the effect of bringing out the experiential reality of "ideal" phenomena. Only when a phenomenon is taken seriously and studied at its own level, Rota says, can it reveal its own properties within its own eidetic domain (Kac, Rota and Schwartz 1986, pp. 169–170). On Husserl's view, some "real" invariants are given to us in a straightforward manner in ordinary sense perception, while other "real" invariants pertaining to sense perception emerge though our various scientific investigations. Husserl emphasizes what he calls "free variation in imagination" as a way of bringing out the experiential reality of essences, as "ideal" invariants. Free variation in imagination is a capacity of reason, not of sense perception or introspection. The kind of synthesis involved in obtaining an invariant in this case is active, not passive (see also Tieszen 2005b).

What the reduction is supposed to show us is how to restrict ourselves to the sphere of appearances, to what is immanent and absolute, but without presupposing naturalism. How does it do this? In *Ideas I, Cartesian Meditations* and several other works, the *epochē* is motivated by way of some comparisons with Descartes' method of doubt. This Cartesian approach to explicating the phenomenological reduction can be contrasted with other paths to the reduction in later writings, such as the path indicated in the *Crisis of the European Sciences and Transcendental Phenomenology* (Husserl 1936). In *Ideas I*, Husserl notes that the reduction is not the same thing as the Cartesian method of doubt, but the Cartesian method, even though it was intended for different purposes, can get us into the neighborhood of what he wishes to obtain provided it is understood in terms of the transcendental turn in philosophy. The *epochē*, for example, plays no role in establishing substance dualism, but is used instead to make us aware in a non-reductionistic manner of mental phenomena as mental phenomena. There is a sense in which the *epochē* is supposed to actually prevent us from engaging in naive and unfounded metaphysics, as we noted in the last chapter. As Husserl indicates in the *Cartesian Meditations*, Descartes did not make the transcendental turn (§ 10). Leibniz also does not make the transcendental turn. As Husserl says in *Formal and Transcendental Logic* (§ 100), Kant does make the transcendental turn but he neglects to carry it out with respect to the ideal objects of logic and mathematics. In the Introduction to *Formal and Transcendental Logic* Husserl says that mathematics and logic are positive sciences that require a foundation in transcendental phenomenology (see also Husserl's lectures on logic of 1906/07 in Husserl 1984b). What the modern sciences lack is a true logic, that is, a transcendental logic, that investigates the cognition behind science and thereby makes science understandable in all its activities. This logic does not intend to be a mere pure and formal logic, a *mathesis universalis*, for while *mathesis* may be a science of logical idealities it is still only a "positive" science. Transcendental phenomenology should bring to light the system of transcendental principles that gives to the sciences the possible sense of genuine science. The positive sciences are completely in the dark about the true sense of their fundamental principles. Transcendental phenomenology, with its analyses of intentionality, genesis, and constitution, is supposed to make it understandable how the positive sciences can bring about only a relative, one-sided rationality.

What can be accomplished with the phenomenological reduction, which makes apparent Husserl's transcendental phenomenological idealism, is this: as we attempt to doubt everything we notice that in fact not everything is doubtful (see Husserl 1907, Lecture II; 1913, § 31), If I think that everything is doubtful, then while I am thinking that everything is doubtful it is indubitable that I am so thinking. In every case of a definite doubt, it is indubitable that I am having this doubt. The same is, in fact, true of every instance of cognition. If I am perceiving or judging, for example, then whether these activities are veridical or not, whether they have objects that exist or not, it is nonetheless clear that I am *perceiving* this or that, or *judging* this or that. The awareness that I am perceiving or judging implies that I have the capacity to *reflect* on my cognitive activities. In this reflection something is given to me that I cannot doubt. It is given, Husserl says, "absolutely" and with certainty. I cannot doubt that I am experiencing something if I am experiencing something, but I can doubt whether that which I am experiencing in fact exists. In this manner we are able to find a way to focus on what appears to us, just as it appears. If we are conscious we cannot doubt that something or other appears to us in our cognitive activities but, of course we can very well doubt that what appears in the appearing is actually the case. In this manner, we can affect a "suspension" or "bracketing" of the (natural) world and everything in it. This means that we also bracket the natural, psychophysical ego or self, the self as the object of natural science. After the reduction we refer to the ego that is directed toward objects as the "transcendental ego" which, in its full concreteness, is said by Husserl to be a "monad."

The method is thus to restrict ourselves to what is "immanent," to disengage from the natural attitude in which we naively and without reflection take ourselves to be experiencing transcendent objects. In the phenomenological attitude, obtained by the reduction, we shift, on the basis of reflection, to the immanent. Husserl then goes on to say that the immanent is absolute, while the transcendent is not. What is transcendent is always relative to consciousness.

Many passages in *Ideas I* (Husserl 1913) express the new transcendental idealism that results from taking the *epochē* seriously. In § 46, for example, Husserl argues that any physical thing that is given "in person" can be non-existent but that no mental process which is given "in person" can be non-existent. The non-existence of the world is conceivable, but the existence of what is immanent—the absolute being of mental processes—would in no respect be altered thereby. In fact, there is a distinct manner in which mental processes would always be presupposed in any effort to doubt the existence of various phenomena. Consider the case, which is certainly possible, in which a perception is corrected by a subsequent perception. Now imagine that this process of correction continues to occur. In § 49 of *Ideas I*, Husserl says that it is conceivable, due to such conflicts, that experience might dissolve into illusion not only in detail but globally. In this case no natural world would be constituted in our experience. There would be no experience of a natural world but in all of this there would still be consciousness. Consciousness, according to such a thought experiment, would indeed be necessarily modified by the "annihilation" of the world of physical things but its own existence would not be touched. The absolute being of the mental processes would in no way be altered thereby. Thus, in § 47 Husserl says that "no limits check us in the process of conceiving of the destruction of the Objectivity of something physical—as the correlate of experimental consciousness." Whatever things are, they are as experienceable things. It is experience alone that prescribes their meaning (or sense). We must not let ourselves be deceived by speaking of the thing naively as something that transcends consciousness and exists in itself, apart from any possible relation to consciousness. "*An object existing in itself is never one with which consciousness or the ego pertaining to consciousness has nothing to do*" (§ 47). In § 49 Husserl says that the whole spatiotemporal world and each of its constituents, according to its sense, has intentional being. It is being posited by consciousness in its experiences. Each constituent of the world, of essential necessity, can be determined and intuited only as something identical through motivated multiplicities of appearances. It is something invariant for consciousness through a manifold of appearances. Beyond that it is nothing. We cannot somehow get a peek at the object behind all the possible appearances of the object, the object as it supposedly "really" is. Any such peek would just be another appearance of the object. Indeed, we can just drop the idea of a noumenal world. The genuine concept of transcendence can only be derived from the contents of our experience itself.

This sphere of absolute consciousness that remains as a residuum after the conceivable annihilation of the world is what provides the subject matter for pure phenomenology. From this point of view, Husserl says, we think of all reality as existent by virtue of a *meaning-bestowing* consciousness (monad) which, for its part, exists absolutely and not by virtue of another sense-bestowal. Consciousness *constitutes* the sense of objectivity. Although this is a form of idealism it is not, Husserl says, a Berkeleyan subjective idealism. Rather, it is transcendental phenomenological idealism. It recognizes that not everything is constituted as a mental phenomenon and it also recognizes the role of the overlapping horizons of different egos (monads) in the constitution of a common, objective world.

In Part II of *Formal and Transcendental Logic (FTL)*, in the context of his investigations of logic, Husserl says similar things. Transcendental phenomenological idealism is represented in *FTL* as the view that it is only in our own experience that things are "there" for us, given as what they are, with the whole content and mode of being that experience attributes to these things. In § 94 of this work Husserl says that "nothing exists for me otherwise than by virtue of the actual and potential performance of my own consciousness." Whatever I encounter as an existing object is something that has received its whole sense of being from my intentionality. Illusion also receives its sense from me. Experience teaches me that the "object" could be an illusion. Objects can be thought of as intentional "poles of identity" (invariants) through the manifold activities of consciousness. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent object that had any other sense than that of an intentional unity or invariant making its appearance in the subjectivity of consciousness. Thus, if what is experienced has the sense of "transcendent being" then it is experience itself that constitutes this sense. If an experience is "imperfect" in the sense that an object is given only partially, then it is only experience that teaches me this.

One of the most interesting features of this transcendental phenomenological idealism is that it does not deny that there is objectivity or objective truth, but rather it makes of objectivity a problem that is to be grasped from what is absolutely given. It enjoins us to investigate how consciousness constitutes the meaning of objectivity. We must now engage in constitutional analysis. We must do this, furthermore, for any kind of objectivity. It is not the case that some objects are supposed to escape the phenomenological reduction. Thus, not only are we supposed to analyze the constitution of the sense (meaning) of the being of objects of ordinary founding acts of sensory perception, but we are also supposed to analyze the constitution of the meaning of the being of objects of founded forms of consciousness that depend on acts of abstraction of different types, acts of generalization, reflection, idealization, and imaginative variation. In particular, we are supposed to analyze the constitution of the meaning of being of categorial objects, of ideal objects, and of mathematical objects in particular.

If we start with the ordinary physical objects (invariants) given to us in founding acts of sensory perception, then we see that they are given to us only partially and as transcendent, as objects that are in space and external (world) time. They are not given to us as subjective or as mental entities. We do not need to hold, as we noted, that everything is a mental phenomenon or a subjective idea. We can recognize that physical objects transcend mental phenomena (as do mathematical objects) only now we say they are constituted by consciousness in this manner. That is, the meaning of the being of physical objects is constituted by consciousness in such a manner that physical objects are not mental entities. They are not meant as mental entities. They are constituted as external objects, as objects that are in space and in external time. We are led, in this sense, to a kind of realism about physical objects. This is different, however, from a naive realism. It is, rather, a phenomenological or "constituted" realism that has its origins in transcendental subjectivity (the monad) itself. Thus, starting with physical objects we can say that it is only *naive* forms of realism about the natural world that take physical objects to somehow exist in themselves, totally independently of consciousness. If we are operating from the position of transcendental phenomenological idealism then, for the reasons discussed above, we cannot be naive realists. We also cannot be crude empiricists, naïve naturalists, or positivists.

Taking our lead from Husserl's comments in *The Idea of Phenomenology*, *Ideas I*, *Formal and Transcendental Logic*, and elsewhere, we can say that ideal objects are also constituted as such by consciousness, by the monad. Let us apply Husserl's words from these texts to mathematical or logical objects in particular: whatever things are, mathematical objects or logical included, they are as experienceable things. It is experience alone that prescribes their sense. The genuine concept of the transcendence of mathematical or logical objects can only be derived from the contents of mathematical or logical experience itself. Nothing exists for me otherwise than by the actual and potential performance of my own consciousness. Whatever is given as an existing object in mathematics or logic is something that has received its whole sense of being from my intentionality. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent mathematical or logical object that had any other sense than that of an intentional unity (invariant) making its appearance in the subjectivity of consciousness.

We need to explicitly note the new twist here: If what is experienced has the meaning "transcendent being" then it is experience itself that constitutes this meaning. If what is experienced in mathematics and logic has the sense of being "ideal," "non-mental," "acausal," "unchanging," "non-spatial," (possibly "partially given") and "non-physical" then it must be experience itself that, in a non-arbitrary manner, constitutes this sense. If mathematical or logical objects are considered to be objects that existed before we became aware of them, and that would exist even if there were no human subjects, then it must be the case that this sense of mathematical or logical objects is constituted in classical mathematical practice in a motivated and non-arbitrary manner. If transfinite sets are considered to be objects that could not be our own constructions, in the sense of constructive mathematics, and cannot be given to us completely in intuition, then it must be mathematical experience itself that constitutes this meaning. If we consider all of the general features of mathematical or logical platonism that we outlined at the beginning of § 2 of this chapter, then we can now say that mathematical or logical objects possess these features, except that we must add the crucial qualification that they are constituted non-arbitrarily as having this meaning by virtue of the intentionality of the consciousness of the transcendental subject (monad).

One feature that we must now modify, according to Husserl, concerns the temporality of mathematical or logical objects. Since we are within the sphere of possible experience for transcendental subjects or monads we are within the sphere of temporality. This means that mathematical or logical objects are also objects that must be in time, only now we will say that they exist at all times. Thus, instead of saying that mathematical or logical objects are atemporal or eternal or timeless—somehow outside of time (and all possible experience) altogether—Husserl says in *Experience and Judgment* that they are omnitemporal (Husserl 1939, § 64). As transcendental phenomenological idealists, we cannot speak about the existence of objects that are somehow outside of all possible appearance or outside of all possible consciousness, and hence outside of all possible time.

We thus arrive at a wholly unique kind of "platonism" about mathematics or logic, that I call "*constituted platonism*." This is, as it were, a platonism embedded within transcendental idealism. In a remarkable new twist in the age-old debate about platonism, we look to the transcendental ego (monad) as the source or origin of platonism about logic and mathematics, where logic and mathematics are built up non-arbitrarily on the basis of reason through acts of abstraction, idealization, reflection, variation, and so on. Just as the "realism" about physical objects is not a naive realism, so this unique kind of platonism about mathematical objects is not a naive platonism.

§ 5. Mind-independence and mind-dependence in formulations of mathematical platonism

Since realism and idealism are viewpoints expressed in terms of mind-independence or mind-dependence, I would now like to single out these characteristics in order to arrive at an explicit formulation of how a form of mathematical or logical platonism might be compatible with transcendental phenomenological idealism. As Husserl says in *The Idea of Phenomenology*, the only way to solve the problem of how we can be related to transcendent (or mind-independent) objects is from within the phenomenological reduction. Once we restrict ourselves to the sphere of appearances, to what is immanent, on the basis of the *epochē*, we see that consciousness exhibits intentionality. "Monads" exhibit intentionality. We find that transcendental subjects are directed by the contents (or what Husserl calls "noemata") of their acts in some domains toward objects that "transcend" these very subjects. In the language of *Ideas I*, we find the noetic-noematic-hyletic structure at work in our experience of ordinary sensory objects. In the case of the founded pure categorial or ideal objects, this same structure will be present, without the category of causality and without the constraints of sensory hylé, but not without grammatical, formal, meaning-theoretic, and other structural and sedimented constraints. The abstractions we arrive at on the basis of various processes of abstraction will themselves act as constraints on further abstractions, reflections, generalizations, and variations. In other words, within the sphere of the immanent and absolute that we obtain after the reduction, we can draw a new distinction between the immanent and transcendent, that is, we can distinguish *what appears as immanent* from *what appears as transcendent*. We can distinguish *what appears as mind-dependent* from *what appears as mind-independent*. Of course the terms "immanent" and "transcendent," or "mind-dependent" and "mind-independent" in this context will have a sense different from their sense prior to the reduction. Similarly, if we view the reduction as depending on a distinction between appearance and reality (in the naïve sense of the natural attitude) and then restricting us to the sphere of appearances, then we find that within the sphere of appearances we can still distinguish appearance from "reality." We can still distinguish knowledge from illusion. This will be true in ordinary sensory experience but also in the case of our experience in mathematics and logic.

How does this work? We can start with an example involving ordinary sensory experience. Suppose that at a certain stage of your experience you perceive a snake lying in a garden. At a later stage, however, you perceive that it is not really a snake lying in a garden but a coiled rope. Now what usually happens in situations such as this is that our experience settles down so that we do not have a continuous series of misperceptions of this sort. Instead, there is typically a more or less harmonious course of experience involving transcendent objects. This opens up the possibility of making an appearance/reality distinction after the *epochē*. Looking back on the experience, we can say that there was merely an appearance of a snake at the earlier stage in the perception and that what we have "in reality" is a coiled rope. We can distinguish knowledge from illusion, where knowledge depends on the evidence provided by intuition. The intention by virtue of which we are directed at the later stage toward a coiled rope, Husserl says, is (partially) fulfilled in intuition, whereas the intention by virtue of which we were directed toward a snake is shown to be empty. The concept or intention "x is a snake," as it figures into the course of our experience, is inconsistent with the concept or intention "x is a coiled rope." No "x" in our experience can be both. This is not a formal inconsistency. Rather, it is a kind of material or contentual a priori inconsistency that is at work in our experience. As Husserl says in some of his writing, the noema (or meaning) by virtue of which we were directed toward a snake at the earlier stage of our experience "explodes" in light of subsequent intuitive experience.

Husserl thus distinguishes empty from fulfilled intentions and holds that we have *knowledge* when our intentions are (partially) fulfilled, are fulfillable, or are frustrated. Intuition is defined in terms of fulfillment of intentions. Intuition is the source of evidence. It is the source of objectivity. We regard frustration as a particular kind of fulfillment, namely the fulfillment of the negation of the original intention P, so that we have $\neg$ P. The original intention is seen to be illusory. Intuition can sometimes be extended or cultivated to a point at which we can *decide* for a given intention P whether P or $\neg$ P. The decision is made by finite beings (monads) and, although it is not arbitrary, it need not be purely mechanical in nature. There is perceptual intuition when some invariant object of sense perception or some generalization regarding sense perception is not only conjectured or postulated, not only merely intended, but is made *present* to consciousness. That which is absent in the mere intention or conjecture is made present. We "see" it, it is revealed. In some kinds of cases it is revealed only partially, indistinctly, unclearly. In cases of the frustration of an intention, an altogether new and even unexpected or surprising invariant may thrust itself upon us. For example, I might experience a policeman on the roadside ahead of me at a certain point, but then realize as I pass that it is really only a cardboard cutout that resembles a policeman.

On this account, intuition is understood in terms of the fulfillment of intentions. It is important not to associate other meanings with the term "intuition" here, especially since this term has been used and abused in so many ways over the years. The argument is that if intuition were not present in our perceptual experience then our perceptual experience would appear to us as nothing more than a set of empty perceptual hypotheses, conjectures, expectations, or problems. As a matter of fact, however, it does not appear this way. Expectations in perception are realized or frustrated, problems of perception are sometimes solved one way or another, perceptual conjectures and hypotheses are confirmed or refuted. There are not only empty intentions in perception but, where there is knowledge, there are also intuitions.

Contrary to standard forms of idealism, we cannot simply say that "to be is to be perceived," because in a case such as the snake/coiled rope example, the subsequent intuitive experience shows that there was no snake. It is not the case that I was really perceiving a snake at the earlier stage. It only appeared that I was. Intuition makes the difference. It is responsible for objectivity. What appears to be mind-independent, given the evidence thus far, is the coiled rope.

The perception of the coiled rope could in principle be overturned in future experience. That is, there might be evidence (experience) in the future that would indicate that it is also not a coiled rope. It is again intuition that would show us this. Its being a coiled rope is *not absolute* even if the coiled rope is given as existent and mind-independent in accordance with all of our evidence thus far. Our evidence that the coiled rope is mind-independent is in this sense presumptive. What is given in intuition in this kind of case can be overturned by subsequent intuition. The fact that we cannot specify some exact point at which we can be assured that no further corrections will occur should not count as an objection to this view. Rather, it is just a recognition of how life really is. It would be artificial to specify such a point.

What this shows is that from within the *epochē* everything is indeed understood as appearance or phenomenon, and that appearances are corrected or verified only by further appearances. Within the sphere of appearance, however, we can still distinguish the "real," the transcendent, and the mind-independent from the "merely apparent," the immanent, and the mind-dependent, on the basis of what stabilizes or becomes invariant in our experience, on the basis of what has a history and a sedimentation that permits of further elaboration, development, constraints, surprises, and so on. We can still distinguish knowledge from illusion. This is a key idea of transcendental phenomenology, and it holds for both sensory experience and mathematical experience. There are illusions and corrections and refinements in mathematical experience, just as there are in sensory experience.

One of Gödel's favorite examples of illusion in mathematics, as we saw earlier, is generated by the naive comprehension axiom in set theory. I will return to this in chapter 6. To the objection that there is an appearance/reality distinction for perceptual objects, but not for mathematical objects, we only need to point out that it is quite possible to be mistaken in our conceptions of mathematical objects and mathematical generalizations. At later stages of mathematical practice we might realize that what appeared to us to be the case in mathematics is actually not the case.

We cannot somehow get outside of appearances to an appearance-independent thing-in-itself. We can just abandon the idea of a noumenal world. The "existent" will be that for which we have evidence across places, times, and persons. This will not hold for the "merely apparent." Rational justification depends on evidence. Imagine a form of experience in which nothing ever stabilizes or becomes invariant. This would be a form of experience that is without reason. It would be experience in which there is no order and no rational connection among things. We are nonetheless not entitled to say that what is stable or invariant is the final, absolute reality. We cannot have a realism that recognizes an appearance-independent absolute reality. At best, the notion of "absolute reality" might be preserved as an infinite ideal. Thus, transcendental phenomenology recognizes an appearance/reality distinction after the reduction that allows for a kind of realism, only it is not naive or absolute realism. It is also not a naive idealism for the same reason: it makes an appearance/reality distinction after the *epochē*.

These considerations show us that there are weak and strong senses of "appearance-independence" or "mind-independence." There could not be mind-independent objects in the strong or absolute sense of lying outside of all possible experience (or appearance). We simply cannot say anything about the possibility of such radically independent things-in-themselves. On the other hand, there are objects that are mind-independent in a weaker sense, according to which objects that are not meant as mental entities are nonetheless invariants in a manifold of appearances. We could be mistaken about objects in our experience, so that we could at some later stage come to see that we had been under an illusion, that we had mere appearances at an earlier stage. To say that there are weak and strong senses of "mind-independence" or "mind-dependence" will then affect the formulations of mathematical platonism and mathematical idealism. In transcendental phenomenology we must set aside the strong (or naive) sense of mind-independence. The weaker sense, however, will allow us to preserve important insights of realism.

§ 6. Compatibility or incompatibility?

To summarize a key point in the discussion thus far we can say that we need to index our conceptions of the mind-dependent and mind-independent, of the immanent and transcendent, and of appearance and reality.

With the phenomenological reduction we turn to phenomena, to appearances, to the immanent. So we first distinguish between appearances (or the immanent), and the naïve view of appearance-independent reality (or of the transcendent). On the one side of this distinction we have appearances, the immanent and mind-dependent, and on the other side of the distinction we have appearance-independent reality or the transcendent as mind-independent. Now, restricting ourselves to the sphere of phenomena, to the immanent and absolute, we find that consciousness exhibits intentionality. Transcendental subjects (monads) are directed toward objects that transcend subjects. We find the noesis-noema-object structure, minus sensory hylé in the case of mathematical or logical objects. Intentionality in pure mathematics is not constrained by sensory hylé, but there are still grammatical, formal, meaning-theoretic, and other structural and sedimented constraints (such as theorems) on it that are reflected in practice in mathematics and logic. One of the marks of objectivity in both sensory and mathematical experience is that we find our awareness to be constrained in certain ways. It is not possible to will objects or states of affairs in either sensory or mathematical experience to be just anything we want them to be. We find all of these moments of experience after the *epochē*. Within the sphere of appearances we can then draw a distinction between the immanent and the transcendent. Here we introduce a new distinction between the immanent and transcendent, the mind-dependent and mind-independent, between appearance and reality. Some things *appear to us* as immanent, some as transcendent.

We can depict the situation in the following diagram, which I will formulate for the mind-dependent/mind-independent distinction, since the issue of mathematical or logical platonism is typically described in these terms. Mathematical or logical objects, indexed according to our comments, are mind-dependent$_1$ mind-independent$_1$

□ ◊

/ \

mind-dependent$_2$ mind-independent$_2$

\# *

Figure 4.1 Positions on realism and idealism

It is inconsistent to say that abstract objects are mind-dependent$_1$ and mind-independent$_1$. Formulated in this way, mathematical platonism and mathematical idealism are incompatible. Most of the debate about realism and idealism, including recent debate, seems to take place at this level. For example, Brouwer might be at the position "□," while Frege might be at the position marked by "◊." It is also inconsistent to say that abstract objects are mind-dependent$_2$ and mind-independent$_2$. Formulated in this way, mathematical platonism and mathematical idealism are still incompatible. It is not inconsistent, however, to say that abstract or ideal mathematical objects are mind-dependent$_1$ and mind-independent$_2$. Indeed, mind-independence$_2$ falls under mind-dependence$_1$. What this means is that mathematical or logical platonism, in this sense, is compatible with transcendental phenomenological idealism. Platonism in this sense, which we can call "*constituted platonism*" or "*constituted realism*," is concerned with non-arbitrarily or rationally motivated constituted mind-independence in our experience in mathematical practice.

What we are now to investigate is the *constitution of the meaning* of mind-independence from within the *epochē*. We need to investigate the rationally motivated *constitution of the sense of the existence of ideal mind-independent mathematical objects*.

As we look back from this viewpoint, we can say that the standard positions of mathematical platonism and mathematical idealism that we set out in our initial formulations in § 2 are too simple. They are ambiguous. If we make the distinctions just indicated, then the assertion that abstract objects are mind-independent$_1$ is naive (or pre-critical) mathematical or logical platonism and is untenable. The assertion that mathematical or logical objects are mind-dependent$_1$, with no further qualification, is naive (or pre-critical) idealism and is untenable. The third position that we outlined combines a transcendental phenomenological idealism and a mathematical or logical platonism in which neither the platonism nor idealism is any longer naive. We have adhered to the *epochē* and left naive metaphysics behind. It also follows that transcendental phenomenological idealism is not compatible with naive mathematical or logical platonism.

Once we make these distinctions, then existence claims, whether in ordinary perception or in the case of mathematics, will have to be understood accordingly. There are of course important disanalogies between sensory objects and mathematical objects, but in either case the existence of mind-independent objects will now have to be understood in the sense of mind-independence$_2$. If these phenomenological considerations are correct, then what other sense could we legitimately give to existence claims?

It has already been intimated how constituted platonism, unlike naive metaphysical platonism, does not cut off the possibility of knowledge of mathematical objects. Knowledge involves intentionality, and our mathematical intentions can be (partially) fulfilled (in intuition), frustrated, or neither fulfilled nor frustrated. Mathematical knowledge is to be spelled out in terms of the intuition of the ideal or abstract objects toward which we are directed by our mathematical intentions, where the objects are to be thought of as (founded) invariants in mathematical or logical experience. What we are describing here is a position about mathematical or logical *experience* in mathematical practice. *Constituted platonism is concerned with the kind of directedness (intentionality) involved in thinking and problem-solving in the practice of mathematics and logic.* Moreover, at the fundamental level, what we are analyzing here is the *genesis* of platonism. I will discuss this point in more detail in chapter 6.

Note how different this is, for example, from a position that starts with neuroscience, with its natural attitude, and then asks how the brain could be related to abstract objects. How could brains be causally related to abstract objects? From my point of view, it is not at all surprising why this question cannot be answered. There are reasons for wanting to suspend or bracket natural sciences of the mind such as neuroscience. The point is not to avoid neuroscience in particular or natural sciences of cognition in general. We, of course, need such important sciences. The reason is rather that the natural sciences presuppose the "natural attitude." They presuppose that we know about existing natural objects, and they are therefore not in a position to account for this knowledge. Such sciences are directed toward their objects but abstract away from analyses of *experience of objects*. There are also other, related reasons. We want to avoid a reductionistic, eliminativist, and one-sided *philosophy of mind* that forgets about consciousness and intentionality. I will discuss this in more detail in chapters 5 and 7.

On the kind of realism described above, objects can be mind-dependent$_1$ and mind-independent$_2$. Before concluding this section I would like to note that such a realism bears more than a passing resemblance to Hilary Putnam's "internal realism" (see, e.g. Putnam 1981, 1987). It would be worthwhile to compare the views in some detail. I am arguing, for example, that the notions of mind-dependence$_1$ and mind-independence$_2$ can be applied in the case of *mathematical* objects or states of affairs. Does Putnam apply his internal realism to the question of mathematical realism? Does he have anything like the idea of constituted mathematical platonism? There are certainly developments in the phenomenological analyses that would not be found in Putnam's internal realism, but there might also be many points on which the views in fact reinforce one another.

§ 7. A conclusion and an introduction

What I have tried to do in this chapter is to show how, at a *general* level, a form of mathematical and logical platonism can be compatible with transcendental phenomenological idealism, and in so doing to indicate the difference between traditional naive platonism about mathematics and logic and constituted platonism. According to constituted platonism, mathematical or logical objects can be mind-dependent$_1$ and mind-independent$_2$. Critics of Gödel do not make such a distinction. They seem to place him at the position marked by "◊" in our diagram. To be frank, if Gödel is in this position then I do not see how his platonic rationalism can be defended. His position will be subject to all of the problems associated with naive metaphysical platonism and naive rationalism. On the other hand, given his serious study of Husserl's philosophy and all of his comments we have noted above, there is good reason to think that Gödel embraced constituted platonism, or was at least moving toward it. I think that this is, in any case, a more defensible form of Gödelian platonism. We have not succumbed to the leftward positions of Hilbert, Carnap, and many other recent philosophies of mathematics, but there is an important sense in which we have embraced a kind of skepticism (thanks to the *epochē*), and avoided the death-defying leap into a new metaphysics. On the basis of the phenomenological *epochē* we are not entitled to hold that either the human mind or abstract objects are noumenal objects or things-in-themselves. We cannot claim that they are somehow the *real* things behind and apart from phenomena. It is not possible to claim that the mind is some disembodied spirit. We must say farewell to naive metaphysics. On the other hand, we do not fall into the trap of reducing everything to naturalism or empiricism. We are not condemned to the natural attitude. We can investigate the intentionality of the kind of *thinking* involved in the practice of mathematics and logic. We can investigate the genesis of mathematical and logical platonism. Phenomenological ontology remains a possibility.

If it is possible to see how a form of platonism is compatible with transcendental phenomenological idealism then we have not yet settled the question of which kinds of mathematical or logical objects and states of affairs can be admitted in constituted platonism. This is now a question of meaning constitution. What kinds of mathematical or logical objects, for example geometric objects, natural numbers, real numbers, complex or imaginary numbers, functions of different types, sets of different types, concepts, and the like, does and can the mind constitute as existing in a rationally motivated and constrained manner in mathematical and logical practice?

Here things are more complicated. This is where the real work of constitutional analysis in the case of mathematics and logic must begin. There are many questions about the constitution of mathematical objects, including objects such as transfinite sets, that need to be considered. To mention just one question of this type, for example, it might be asked whether it is possible to constitute generalities about mathematical objects even if we cannot constitute such objects individually. Could there even be a kind of objectivity in mathematics without objects? There are many questions that could be asked here. One needs to enter into issues about constituted platonism in the case of different kinds of mathematical and logical objects.

In this chapter I have described in general terms the kind of mathematical and logical platonism that arises if we are to develop the transcendental method in such a way as to accommodate the reality of abstract objects or concepts and categorial or rational intuition. I have argued that it is constituted platonism that results from the use of the phenomenological method. In the next two chapters I will take up some important ideas about the rationalism of Gödel's platonic rationalism. This will take us more deeply into ideas about knowledge and rational intuition in mathematics and logic.

5

Consciousness, Reason, and Intentionality

In the last chapter I considered a new kind of platonism that emerges if one takes Gödel's references to Husserlian eidetic transcendental phenomenology seriously. In this chapter I want to focus on some important ideas about the rationalism of Gödelian platonic rationalism. The chapter opens with a brief overview of deficiencies that have come to light in "leftward" treatments of consciousness and the mind and, hence, of reason. We can avoid these deficiencies if we are not reductionistic from the outset about consciousness and the mind. The human mind is capable of sensing, imagining, remembering, willing, hoping, desiring, abstracting, reasoning, and a host of other cognitive activities. It is important to recognize the range of cognitive activities of which we are capable and to investigate their various features, for otherwise it is easy to develop a kind of myopia about the nature of cognition.

In the middle sections of the chapter I provide some background about our capacity for acts of reason by presenting some of the important general features of reason that have been emphasized in the rationalist tradition of philosophy. These will be in play in subsequent chapters, and it is worthwhile to lay them out in order to provide contrast with alternative views that seem to me to have ignored or forgotten about them. First, I discuss the kind of conception that is associated with reason, and then move on to the types of universality and generality that are made possible by conception. Next, I sketch the relationship of reason to objectivity, and provide a few comments about reasoning and logic. A number of rationalists have argued in favor of the idea of rational intuition and, in preparation for the next chapter, I briefly offer a few further observations on this topic. Rationalistic optimism plays an important role in Gödel's thinking. This feature of reason thus deserves a few comments in my sketch of general features of reason. A brief interlude on holism and reason follows.

What Husserlian phenomenology adds to the rationalist tradition is the claim that human reason, like many other forms of human consciousness, exhibits intentionality. The analysis of intentionality that is part of transcendental phenomenology, on which Gödel remarks in one of the quotations in chapter 1, is discussed in several sections toward the end of the chapter. An important principle involving the intentionality of human conscious, which I call the "intentional difference principle," is introduced in § 12 of the chapter.

Many of these points, along with several of the main ideas in earlier chapters, are brought together in the final section of the chapter in order to fill out some of the links between Plato, Leibniz, Kant, and Husserl that were of interest to Gödel in providing a picture of how we are conscious of mathematical reality.

Husserlian phenomenology also adds another important line of thinking to the rationalist tradition in philosophy in holding that there are founded and founding forms of consciousness and that, genetically speaking, acts of reason are founded on sense perception. I take this up at the beginning of the next chapter. Appeals to the intentional difference principle and the founding/founded structure of consciousness will then play a crucial role in the account in the next chapter of the role of reason in our knowledge of mind-independent$_2$ abstract objects in mathematics and logic.

§ 1. Mathematical knowledge and a scandal in the philosophy of mind

I want to open this chapter by pointing out some of the serious limitations and biases that I think have been at work in discussions of mathematical epistemology. First, much of the literature on mathematical knowledge in the past seventy years or so has either just accepted without reflection the prevailing views in the philosophy of mind or ignored the philosophy of mind outright. It is a serious problem, especially now, that grand claims about mathematical epistemology are often ill-informed about work in the philosophy of mind. It is only in recent times that the *mind* itself and its cognitive activities have been "rediscovered." Instead, in much of recent analytic philosophy we have had one "leftward" reductionistic scheme after another concerning human consciousness and the mind.[1] The twentieth century saw a succession of efforts to develop a natural science of the human mind. The "natural" sciences involved were of different types, but they of course all presupposed the "natural attitude." Neuroscience played the central role in identity theory, while behaviorists focused instead on trying to develop a science at the level of observable human behavior, dispositions to behave, and operant conditioning. Since the time that identity theory emerged there have been various forms of neuroscientific reductionism. From a different direction, linguistics was being linked by some thinkers with the effort to develop a natural science of the mind. Functionalism then emerged in response to problems with behaviorism and

[1] Some recent work in the philosophy of mind, in which consciousness is back in the picture, looks to be more promising. It is interesting that some thinkers who work in the analytic tradition of philosophy have been returning to William James' wonderful book, *The Principles of Psychology*, as an entry point into the phenomenology of consciousness. See, e.g. Flanagan 1992, especially chapters 8 and 9. It is known that Husserl read and was influenced by the careful descriptive work of James on first-person experience. See especially the chapters in *The Principles of Psychology* on the stream of consciousness (chapter IX), attention (XI), conception (XII), discrimination and comparison (XIII), the perception of time (XV), memory (XVI), and some parts of chapter XXVIII on necessary truth and the effects of experience.

identity theory. Computational or Turing machine functionalism was the main contender. It was at this stage that computer science entered into the effort to develop a natural science of the mind. This approach itself splintered into "symbolic," serial models of minds, parallel distributed processing (connectionist) models of minds, or various hybrids of such models. At an even later stage such models were criticized for their lack of biological realism. Evolutionary biology, it was argued, should figure into any science of the mind. Perhaps we are, for example, "Darwin machines" of some kind.

Computational functionalism is especially relevant to Gödel's views on minds, machines, and reason. It held that we should understand mental states functionally, in terms of their causal relations. The best way to understand mental states as functional states in the brain, according to the computational view, is along the lines of computational states of a computer. Computer programs provide, in effect, a functional organization of hardware that causes the hardware to produce a desired result. The causes and effects do not themselves have any mental content but consist purely of physical sequences. Being in a mental state (e.g. having a belief) consists entirely of fitting into certain kinds of causal relations. There is not some kind of extra mental content in addition to the causal relations. There is just a physical input which is processed through a sequence of cause-effect relations in the system, and which issues in a physical output. This is how we were supposed to understand "knowing," "believing," and so on. Computational functionalism would thus allow us to be good physicalists and also to set aside worries about the "mysteriousness" of the mental. On this view, various kinds of "systems" (composed of different materials, such as brain tissue or silicon chips) could have mental states, provided they had states with the right causal relations. For us humans, mental states are computational states that happen to be implemented in brains.

Every one of these efforts to develop a natural science of the mind in the twentieth century, however, was faced with the same problem: leaving out or failing to do justice to first-person conscious experience. This "problem of consciousness" has been invariant through all of these positions, as well as a number of other positions, and at present it is just as troublesome for natural science, with its presupposition of the natural attitude, as it has ever been. From the point of view of Husserl's phenomenology it is obvious why the problem of consciousness, from a first-person perspective, has persisted throughout all of the efforts to develop a natural science of the mind: the very methods required in order for the natural sciences to be possible abstract away from subjectivity, consciousness, and the features of experience itself. I will explain this in more detail in chapter 7. Meanwhile, consider what happens to the concept of human reason, as a form of consciousness, if you are a behaviorist, a computational functionalist, a connectionist, or a neuroscientific reductionist about the mind. To put it bluntly, it either disappears or is seriously distorted. Each of these positions has tended toward a kind of eliminativism about consciousness and intentionality.[2]

Answering the question of what he thought the philosophical implications of his incompleteness theorems were in the letter to Leon Rappaport mentioned earlier, Gödel says that

> My theorems only show that the *mechanization* of mathematics, i.e., the elimination of the *mind* and of *abstract* entities, is impossible, if one wants to have a satisfactory foundation and system of mathematics. (Gödel 2003b, Vol. V, p. 176)

Gödel speaks here of the elimination of mind and of abstract entities. This is the "leftward" situation we have gotten ourselves into. It is quite interesting that Gödel thinks his incompleteness theorems show, philosophically, that the elimination of the mind and of abstract entities is impossible if we want a satisfactory foundation of mathematics. We have already discussed some of the reasons why he would think that this is a philosophical implication of his theorems, and there will be more to come below. If we can speak only of brains or behavior but not minds, then of course there can be no question of the mind being directed toward abstract objects in certain domains of consciousness. If we abstract away from consciousness, then there is little point in investigating how the kind of consciousness involved in reason and conception differs from the kind of consciousness involved in sense perception, imagination, introspection, and so on. Rather, there are just brains that are stimulated by sensory data, or there are syntax manipulation devices instantiated in silicon chips that process electrical signals, or there is just observable behavior.

It is not at all surprising that the elimination of the mind and elimination of abstract entities should come together as a package. If everything is a matter of an overheated scientism that recognizes only an empiricist interpretation of natural science then of course there can be no conscious minds and no abstract objects. But what if we do recognize the existence of human subjectivity, consciousness, and features of experience itself? The outcome might then be quite different, especially if it situates a view of consciousness of abstract objects in a broadly scientific setting. The scientific setting here, however, would not consist only of natural science but would have to include logic, pure mathematics, and some other productions of reason where we do not suppose from the outset that these later scientific undertakings are to be understood in terms of a strict empiricist interpretation of natural science. Instead of trying to interpret reason and its productions in terms of empiricist principles and methods, we can interpret empiricist principles and methods in terms of reason and its productions. It would be foolish to deny that natural science requires sense perception (or technological

[2] It does not follow that the positions have nothing valuable to offer to the study of the mind. The main problem is with the eliminativism and the truncation of conceptual schemes that can be used to shed light on the nature of the mind and experience. There is in fact some very interesting work that, for example, brings phenomenology together with neuroscience. See, for example, Varela 1999, and some of the other essays in Petitot *et al.* 1999.

extensions of it) but it can still be claimed that there is much in natural science that cannot be accounted for on the basis of empiricist principles alone. Since logic, pure mathematics, and some other operations of reason are surely subsumed under the category "science," we can bar the non-scientific ideas that are at the base of the worries.

One could argue that scientific theory in fact depends on the consciousness of abstract or ideal objects. It seems to me that something like this is true, and while I cannot consider all of the details in this chapter, I will provide a few indications below about the manner in which I think it is true. Scientific thinking is not just sense perception but is instead built up from sense perception on the basis of cognitive activities of abstraction, idealization, reflection, variation, formalization, axiomatization, and other "higher-order" activities of reason that involve conscious directedness toward abstract objects (see chapter 7). These cognitive activities are indispensable to science, but they cannot be construed as being directed toward objects of sensory experience. What I have in mind here is a kind of indispensability argument.

§ 2. Rationalism and empiricism

Mathematics and logic are fields of endeavor that have appeared to many philosophers to display the very essence of human reason. There are a number of general features of reason that have been highlighted by rationalists in philosophy, and it will be useful to review them here. We can start with an elementary point about the long-standing division between rationalism and empiricism. Rationalism and empiricism, as traditionally construed, are views about knowledge. The empiricist, on a classical formulation, holds that all knowledge is derived from sense experience. Ideas or concepts that cannot be traced back to sense experience are not epistemically legitimate. This is what underlies a number of the positions that we have mentioned, such as the position of Mill, Carnap and, in part, Hilbert. Rationalism can come in weaker or stronger forms. A weaker form of rationalism just negates what the empiricist asserts: not all knowledge is derived from sense experience. A rationalist of this type allows that the empiricist is partially correct but that she just does not see the whole picture. Some knowledge is derived from reason. There have also been strong forms of rationalism according to which no knowledge at all is derived from the senses. Anything worthy of the term "knowledge" is derived from reason, thanks to its rigor, certainty, exactness, ideality, or objectivity. Some platonic rationalists have even held that the entire sensory or material world is an illusion. It is not necessary to take this path, however, and I have no plans to do so in this book.

Humans have a capacity for sensation, but they also evidently have a capacity for reason. Empiricists, who are on the "left" side of Gödel's schema, have been skeptical about the alleged deliverances of reason. They will point to the metaphysical systems of some of the great rationalists, such as Plato, Descartes, Spinoza, and Leibniz, as examples of the excess, if not outright bankruptcy, of rationalist views in philosophy. Rationalists, on the other hand, have been quick to point out the deceptions and flaws of the senses. They adopt a skepticism about the deliverances of sense experience.

Many rationalists hold that it is only through reason that we can arrive at exact, clear, and distinct knowledge, at knowledge properly so-called. What the senses give us is always inexact, confused, indistinct, or dubious. Leibniz and Descartes, for example, contrasted the clear and distinct conceptions of reason with the confused or deceptive perceptions of the senses. It is by virtue of reason that we can arrive at necessary or a priori truths. It is only through reason that we can arrive at, or at least approach, certainty in our knowledge. Confinement to the realm of sense experience is confinement to the realm of the contingent, the inexact or vague, the a posteriori, and the probabilistic or unreliable.

Empiricists and rationalists have both had visions of the possibility of philosophy as rigorous science, but these visions, of course, differ significantly. Empiricists tend to assimilate philosophy to natural science, or to some favored natural science, while rationalists tend to assimilate philosophy to mathematics or to logic as a priori disciplines. One can trace how this is developed differently in different philosophers.

In Kant's philosophy one finds a more nuanced position on the rationalism/empiricism debate. As is well-known, in his analysis of the a priori conditions for the possibility of experience, Kant separates the contributions of what he calls sensibility, understanding, and reason. The role of *pure* reason in providing knowledge is quite limited. Indeed, Kant's critique of pure reason is a critique of the positions of a host of rationalist metaphysicians who preceded him, including philosophers in the tradition of Leibniz. Kant's own rationalism is quite limited by comparison. It is a "critical" rationalism, conditioned by his empiricism: concepts without the matter and/or forms of sensory intuition are empty. This part of Kant's philosophy, applied to mathematics and logic, was not attractive to Gödel. How could Kant's philosophy do justice to the remarkable developments in mathematics and logic that emerged in the nineteenth and twentieth centuries? The problem for the Kantian view is that we cannot get beyond the two forms of sensory intuition, which Kant associates with arithmetic and Euclidean geometry. Where are the tools in Kant's theory of knowledge that are needed to address the content, meaning, and knowledge we have in the areas of mathematical logic, real analysis, non-Euclidean geometry, modern algebra, topology and, especially, set theory? It is here that Gödel turns to ideas of Husserl, Leibniz, and Plato to fill the breach.

Some of the most famous rationalists in the history of philosophy, especially in connection with mathematics or logic, are Plato, Descartes, Leibniz, Frege, Husserl (commencing with the *Logical Investigations*), Russell (in his realist phase), and Gödel himself. Whatever one might think of the efforts of rationalists in metaphysics, it cannot be denied that rationalist philosophers such as Descartes, Leibniz, Frege, and Gödel have made some of the most profound contributions to mathematics and logic in the history of these subjects. In my view, this is no accident. Not that it is impossible, but one is hard-pressed to find comparable contributions to these sciences by serious empiricists. Consider, for example, Berkeley's remarks on the calculus, Mill's inductivism about mathematics, or Quine's comments about V = L in set theory. These views do not seem to me to initiate important new developments in mathematics.

I would now like to turn to some of the general features that have been associated with reason in the rationalist tradition of philosophy. These features will be in play in subsequent chapters, and I would like to have them on the table in order to provide contrast with alternative views that seem to me to have forgotten about them. What is supposed to be the nature of reason itself? How does it differ from sensation, introspection, imagination, emotion, and the like? We want to consider reason in connection with logic and mathematics in particular.

§ 3. Reason and conception

Let us start with the idea that reason is supposed to be concerned with a kind of *conception*, as distinct from sensation, perception, imagination, emotion or feeling, and so on. We said earlier, for example, that in Euclidean geometry we can *conceive* of objects such as straight lines, circles, and spheres as exact or perfect, even though they cannot be given to us in this manner in sense perception. These objects, and the truths about such objects, in Euclidean geometry are conceptual invariants, not invariants of sense perception. (When I speak of perception I shall mean sense perception.) We will not find a perfectly straight line in sense perception. We will not be able to build a perfectly exact physical sphere. All we can to is to approximate such a construction. In Euclidean geometry we conceive of circles as colorless, but we cannot sense a colorless circle. What can be conceived is quite different from what can be sensed or perceived. This will be true for the typical objects toward which we are directed in our thinking in pure mathematics and logic, but also for objects such as sign tokens (which we perceive with our senses) and sign types (which we do not literally perceive with our senses). Part of what is involved in conception is the capacity to idealize, as we indicated earlier. Capacities for reflection and abstraction are also involved. Formalization is itself a kind of abstraction, as we will see below. Conception is generally much less limited or constrained than perception.

As I shall understand the matter, mere sensation always figures into perception, but perception itself is underdetermined by sensation. A simple way to make this point is to consider objects such as Necker cubes. In viewing the Necker cube below it is easy to

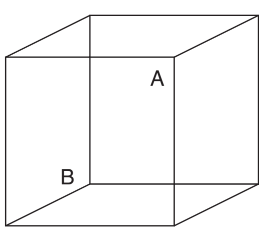

Figure 5.1 Necker cube

see that perception can vary in cases where the sense data remain the same. I can either perceive the cube in such a way that point A is protruding, or I can perceive it in such a way that point B is protruding, but this is not due to any changes in the sense data. Rather, the same sensory material can be informed or interpreted in two different ways. The shift that occurs is something "internal" to me, as it were. One way to describe this experience at the level of phenomenology is to say that we are bringing different concepts to bear in interpreting the sense data. The sense data are informed in different ways by these different concepts. On this description, perception will be a function of sense data and of concepts. Concepts are therefore already involved in perception. This is not to say, however, that we must actively apply concepts or make judgments in order to perceive. On the contrary, concepts are involved in a passive manner in ordinary perception. What makes reason different from perception, however, is that conception in reason need not have any relation at all to sensation. It is possible for conception to be free (see also chapter 6 below) from perception. If we want to think of conception as a feature of pure reason, then we can say that conception is concerned with *pure* concepts. Conception, in this sense, is certainly underdetermined by sense perception. Indeed, we can conceive of many things that we cannot perceive.

Conception (or conceivability) in this sense is certainly related to imagination, but we should not simply identify reason with imagination, or reasoning with imagining. There is a view of imagination according to which imagination always involves the presence of mental images. These mental images would be private and particular to each person. There are other views of imagination or other types of imagination, however, on which subjective mental images do not need to be involved. This kind of imagination could consist of just imagining various (logical) possibilities, whether these are associated with specific mental images or not. In this sense I can imagine, for example, a kind of space that is four dimensional even if I cannot form a mental image of such a space or perceive objects in such a space. Imagination in this sense can certainly accompany acts of reason or of reasoning. Husserl, for example, says in one place that imagination in this sense is the source from which the cognition of eternal truths is fed. Reason can be informed by imagination in this sense. It is through the imagination of a range of possibilities that we might be able to converge on some necessities relative to those possibilities, as in Husserl's view of free variation in imagination. Reason itself, as it is understood in the kind of rationalism I favor, is associated with *truth, evidence, proof, justification*, *universality*, and *objectivity* in a way that imagination is not. On the whole, reason is more constrained or limited than imagination, in the sense that we do not expect everything we imagine to be true, but we do expect the judgments we arrive at on the basis of reason to be true. Imagination is as free as can be. Rationalism, however, is a view in epistemology. It is concerned with sources of *knowledge*.

It is on this basis that we should also distinguish reason from emotion or feeling. To feel that something is true is quite different from knowing it to be true, and from knowing it to be true on the basis of reason in particular. Feeling is also particular, but knowing is often not just particular. Feeling that something is true might at times overlap with knowing that something is true, but it certainly does not guarantee it. Emotions of different types can certainly motivate very powerfully the search for truth but this does not mean that we should run the two things together. In terms of philosophical psychology, we would also want to distinguish the nature of conception from the nature of emotion.

Another feature of reason that is made salient by considerations in phenomenology and philosophical psychology is that, at a certain basic level, sense perception is *passive* or merely *receptive* in a way that conception is not. Conception involves a kind of *activity* or *spontaneity*. Kant, Husserl, and many other philosophers and psychologists have drawn such a distinction. The idea is that in sense perception we do not need to do anything in order to perceive an ordinary object. The perception is, as it were, automatic. There is, in Kantian or Husserlian language, a passive synthesis of data and of partial perceptions of objects in this case. I do not need to take an active role in bringing the partial perceptions of an object together in order to perceive one object. Of course I can take a more active role in perception. This happens, for example, when I take an interest in a perceptual object and set out to explore it, with certain goals in mind. The point, however, is that in the most primitive or "original" modes of perception this does not happen. As we noted in chapters 3 and 6, the monad bottoms out in receptivity. Genetic phenomenology, with its distinction between founding and founded forms of consciousness, seeks to show how there is a genesis of various forms of consciousness from this basis (see e.g. Husserl 1939).

Conception certainly does not just amount to the reception of sense data. In conception there are forms of active synthesis. The mind does something with the data received, such as collecting it, comparing it, reflecting on it, abstracting from it, generalizing on the basis of it, performing imaginative variations on it, reasoning about it logically, idealizing it, and so on. This means that the subject or the ego has, in principle, the capacity to disengage in certain ways from the ongoing flow of sense perception (including emotions or feelings) and to engage in other cognitive activities that may or may not be directly related to the passive flow of sensation. The subject (monad), in short, need not be locked onto sense perception or to particular feelings of an emotional type. Insofar as this is possible it opens up a kind of freedom from sense perception or from particular feelings. There might be more or less powerful sensory stimuli that motivate particular conceptual activities, but such motivation does not necessarily always elicit the activity. The fact that we are not exclusively locked onto sense perception has led some philosophers to hold that some kind of *will* is involved in higher cognitive functions. It is not the case that we must disengage from sense perception, but the fact that it is possible to do so suggests at least the minimal function of a capacity, a will, to do otherwise. *Freedom* from sense perception in this manner means that the subject (monad) can be directed in its activities by, or toward, something other than the immediate sensory stimulus: for example, by or toward conceptions that override the immediate sensory input. The *immediacy* of sensation is contrasted with the mediating capacity of reason, the *mediacy* of conception. Immediate sensory experience is *prereflective*, but conception, by comparison, is *reflective*. Although we might want to recognize that a kind of free will is involved in the activity of conception, as opposed to sensation, we need not take any strong stand at the moment on the large issues about free will and determinism. There are some conceptions of reason that are, of course, attached to a very strong sense of free will but others that are not.

§ 4. Universality and particularity, generality and specificity

The *particularity* or *specificity* of sense perception or introspection has also been contrasted with the *universality* or *generality* which is made possible by conception. In sense perception we are directed toward particular objects or states of affairs in an immediate prereflective manner. There is a long tradition of portraying functions of reason, however, as issuing in what is universal or general. It is reason that makes possible the *abstraction* from the immediacy of sense experience, but also the subsumption of particulars of various types under concepts or universals. There are different theories of abstraction, but what they have in common is that abstraction allows us to omit from our consciousness various particular or specific features of things or states of affairs. In abstracting from a phenomenon we leave certain features out of consideration, or we do not attend to them. Now the elements of a realist's view of abstraction will differ in some respects from the elements of an idealist's or conceptualist's account of abstraction, due to differences in the nature of the objects toward which we are supposed to be directed according to each account. The traditional idealist and the classical conceptualist hold that when we abstract we are directed toward certain mental entities, "general ideas." The platonist, however, will need to provide an account of how abstraction can yield awareness of mind-independent abstract objects. The traditional idealist will not need to do this, but must instead provide an account of abstraction that does justice to the directedness of our thinking in mathematical and logical practice. Similarly, rationalist accounts of abstraction differ from empiricist accounts of abstraction, based on the sources of knowledge that are recognized.

There are clearly some different ways in which we can *generalize* across specific items. Let us consider, in broad outline, three possible types of generalization. First, there are *empirical, inductive generalizations* of different types. These are straightforwardly probabilistic generalizations that we arrive at on the basis of sense experience. Higher-level generalizations in physical theory of the sort included in theoretical hypotheses are not as directly related to sense experience, although they can still be regarded as empirical, contingent and, hence, subject to revision. A second possible type of generalization associated with reason has been noted by some thinkers. It is especially interesting from the perspective of rationalism. Consider generalizations such as "Nothing that is green all over is red," "Anything that is red is colored," and "Anything that is colored has spatial extension." It has been argued that sense experience could not tell against generalizations of this kind. These generalizations, unlike empirical generalizations or testable hypotheses of physical theory, are not subject to revision on the basis of what might come to us in sense experience. The truth of such generalizations is not a mere matter of probability. Husserl, for example, would classify such generalizations as "*material* a priori*" generalizations*. They are "material" generalizations in the sense that they depend on their content or meaning, not merely their form. As a priori, however, they are not empirical, contingent, or probabilistic in nature. Instead, such statements would be part of a priori "regional ontologies." We might assume them in physical theory but we do not subject them to experiment. In Husserl's work, material generalizations are also distinguished from formal generalizations. A third possible type of generalization on this scheme would thus be *formal generalization*. We obtain a purely formal generalization when we abstract from all of the "matter" or content of a statement in order to obtain its pure form. As an example, we might consider the formalization of elementary geometry. Once we have a purely formal judgment we can substitute in it many different "material" expressions. Indeed, many different material generalizations could result from substitutions in formal generalizations. Formalization itself is thus a type of generalization. It is another mode of conception associated with reason.

A rationalist might, therefore, hold that there are both material a priori generalizations and formal a priori generalizations. On this view, sense experience will not tell against the statements of a pure, formal geometry. Similarly, if we have as a principle of a formal system a formula such as $\neg(\exists x)(Px \wedge \neg Px)$, then sense experience will not tell against it. It is immune to revision on the basis of sense experience. The rationalist might say that the formula $\neg(\exists x)(Px \wedge \neg Px)$ is *meant* or intended in such a way that sense experience could not tell against it. We assume such a principle in our work, but we do not treat it as a hypothesis that is to be confirmed or refuted on the basis of experiment. Rather, it is a condition for the possibility of experimentation. It is in this manner that we arrive at what is *intended* in systems of pure geometry or pure logic. This is not to say that a particular formalized geometry does not conform better than others to a range of observations based on sense experience or to an existing theory of the physical universe. Existing theories of the physical universe or observations based on sense experience can, of course, be used to decide which geometry is most fruitful in applications to physical phenomena, but this is not what is meant when it is said that sense experience will not tell against the statements of a pure, formal geometry. We need to distinguish the truths of the geometry as pure geometry from the question whether the geometry has an application or not. The pure mathematician can continue to study what is or is not true in a given geometry, which statements are theorems and which are not, quite independently of whether the geometry does or does not find a use in a physical theory. We should not ban him from thinking that the truths of Euclidean geometry are different from those of, say, hyperbolic geometry.

Before moving on to the next feature of reason in our outline, we might note that rationalists who accept the existence of concepts, essences, universals, or properties, might distinguish different types of such entities from one another, based on the distinction between the material and formal a priori. Thus, one might wish to distinguish material concepts from formal concepts. One might also distinguish inexact from exact concepts. It would be of relevance, furthermore, to distinguish generality from ideality. The point here is that some concepts involve idealization and some do not. The concepts involved in mathematics, logic, and some parts of the physical sciences are typically idealized in certain ways, while the concepts involved in other domains of human endeavor such as everyday experience are not. I will return to some of these matters in the next chapter.

Related to the issue of how the universality/particularity distinction figures into the characterization of reason is the related matter of *systematicity, unification, organization* or *structuring*. Reason is often portrayed as responsible in large part for the presence of these latter features in our experience. The particularities and specificities of our thought or experience are related to one another in various ways, or can be brought into various relations. This is all a function of conceptualization, of bringing particulars under various universals or judgments. Without such structuring, our thought and experience would be punctuate, inchoate, disunified, and arbitrary in certain respects. It would be without reason. Aristotle made this point many years ago, and rationalists and transcendental philosophers have been making it ever since. In mathematics and logic we find these features in their purest form. Axiomatic systems, which abound in mathematics and logic, are the most obvious examples. To axiomatize is just to bring about in a rigorous and careful manner such unification and systematicity. Axiomatization and formalization are thus both to be seen as particular functions of conception made possible by reason. Presentations of mathematics and logic, even when not axiomatized or fully formalized, often follow the form of laying out definitions, theorems, and proofs in sequence. Various fields of mathematics display a remarkable degree of order, systematicity, and organization, or they can be brought into such a form.

§ 5. Reason and objectivity

Yet another feature associated with reason, on many accounts, is *objectivity*. What is objective is not what is particular to this or that thing, or to this or that person. Rather, it is what is *universal* to things or persons. It is something that things or persons have in common, where it will apply to broader or narrower ranges of things or persons. In this sense there might be broader or narrower spheres of universality and objectivity, as in genus/species hierarchies. Earlier we distinguished what is known by inner sense or introspection from what is known by reason on the grounds that what is known by introspection is, in the first instance, specific to each person. What is known about consciousness on the basis of reason, however, should not be subjective, private, and specific in this way. What is subjective is what is particular to each individual subject. We need not deny that there are phenomena answering to this description, just as we need not deny that there are phenomena that do appear in some measure or other to be objective. Objectivity is sometimes characterized in terms of invariance. What is objective is what is invariant across specific differences. It is not relative to persons, places, times, cultures, and so on. The "disinterestedness" of reason is connected with the move away from subjectivity. Rationalists have been among the most ardent critics of different varieties of relativism. One might be a relativist about truth, proof, justification, and evidence. Gödel (Gödel 1946) raises the question, for example, whether there are only relative notions of provability and definability. Absolute notions of provability and definability would stand in contrast to the formalistic view that provability and definability are always relative to a particular formal system. Gödel, in fact, thinks that we have obtained such an "absolute" concept in the attempt to characterize mechanical computability, and that the idea of a Turing machine is an especially clear analysis of this concept.

Rationalists who are platonists will argue that what is objective goes beyond what is merely intersubjective. What is objectively true is not relative to groups of human beings, nor even to the human species as a whole. The claim that truth is relative to the human species as a whole is sometimes called "anthropologism," and I think it is clear that both Husserl and Gödel reject anthropologism about mathematics and logic. Realists or platonists typically hold the strong view that there are mind-independent objects or truths. We find the epistemological correlate of this in the rationalist who holds that we can in principle know about objects or truths that are not merely relative to this or that sets of minds, or even to the minds of human beings as a species.

Objectivity is typically an ingredient in accounts of justification, evidence, and proof. It is not the case that a proposition is true because a particular person says or thinks that it is, or because an important person thinks that it is, or because everyone in a large group says that it is. A result that is alleged to be objective should be one on which different people at different times and in different places will converge. Different subjects (monads) should be able to *repeatedly* arrive at the same result. To insist on objectivity in this sense is to insist on a standard that goes beyond individuals, particular groups, and even humanity as a whole. It is to recognize that there could always be an error in human judgment, so that one should be able in principle to return again and again to verify that a proposition is true. It is to recognize that critical reasoning is indispensable in matters of knowledge. Objectivity in this sense sets a standard for justification and evidence.

As we indicated earlier, rationalists are skeptical about the role that sense perception can play in justification on the grounds of the deceptions of the senses, and the unclear and indistinct nature of sense perception. Sense experience abounds with vagueness and inaccuracy. *Clarity, distinctness*, and *exactness* are only possible in conception. Clarity and distinctness are arguably conditions for the possibility of justification. Without them it is difficult to determine whether reasons offered in attempts at justification are true or false. Rationalists contrast the exactness or precision that is possible through conception with the inexactness and imprecision of sense experience. Mathematics and logic in particular are frequently thought of as operating with exact concepts or objects. We seem to be able to conceive of certain kinds of *perfections* that we could never in principle perceive with our senses. Mathematics and logic are filled with such conceptions. The philosophical psychology of this kind of rationalism recognizes an ability to idealize. One might ask how such idealization is possible or how it works. One way to start on an answer is to argue that it has its origins in everyday activities of perfecting objects and practices. One sees, for example, that a crude measuring stick could be made more accurate, and that it could be made more accurate again. One sees that a surface could be made smoother. What emerges in processes of this type is an awareness of perfecting things (see Husserl's "Origins of Geometry," included in Husserl 1954). In this manner there arises the consciousness of approximation to a standard, to an "ideal." When the standard is *conceived* as something that cannot actually be reached in sensory experience in a finite amount of time by human beings, then we are on the ideal side of the real/ideal distinction that we discussed earlier. The notion of objectivity or objective truth that we considered a paragraph or two ago might itself be viewed as an ideal in this sense.

§ 6. Reasoning and logic

An obvious feature of reason that we have not yet considered is the process of *reasoning* itself. It is in *logic* in a broad sense that reasoning is thematized and studied, including the distinction between valid and invalid reasoning. Human beings clearly have a capacity to reason logically. There are of course different philosophical views of the nature of logic, and there are also many different systems of formal logic. If we look at the concept of reason in relation to these many different systems of formal logic we might wonder which logic we should focus on. For the moment, let me just say that I think there are good reasons, such as those indicated in chapter 2, not to be fixated on classical first-order logic. There are inferences that we can in fact see to be valid, but that could not be known to be valid if the capabilities of human reason were limited to classical first-order logic.

Let us consider a simple example of reasoning. Suppose that someone reasons in the following manner: All As are Bs and all Bs are Cs, therefore, all As are Cs. How do we know that the conclusion follows logically (deductively) from the premises? We could ask Kant's question about this pattern of reasoning: does it conform to sense experience or does sense experience conform to it? Is it a posteriori or a priori? Is it not a condition for the possibility of scientific investigation? The rationalist, in any case, holds that it is not an empirical generalization, and not a theoretical hypothesis of natural science. It is not merely a function of the way we are built psychologically, and it is not a function of linguistic conventions. These are all reductionistic, "leftward" views of the nature of reason. Rather, we know that the conclusion follows from the premises in a deductive argument on the basis of our capacity for reason, on the basis of distinctively *conceptual* activity that is a function of reason. We might merely hypothesize that the inference is valid, but we can also come to *see* that it is valid. This happens all the time. To see that it is valid is not to sense, introspect, merely have a feeling, or merely imagine. Sense perception, introspection, and emotion are not modes of consciousness in which we entertain possibilities and understand necessities relative to possibilities. Reason is precisely such a mode of consciousness. To know that the conclusion follows from the premises is to *understand* (not sense, introspect, or merely feel) that the conclusion is not merely possible given the premises but, based on the concept inclusion or subset relations here, that it is necessary. We grasp the meaning of the relations. Gödel's rationalistic view, as we have seen, is that logic is concerned with abstract "concepts" and the most general and formal relations among such concepts.

Our knowledge that the conclusion of such an argument follows deductively from the premises could be knowledge of something "real" or of something "ideal." In the former case we have the following set of features: the knowledge is probabilisitic in nature, inexact, vague, changeable, posterior to, and dependent on, sense experience or introspection, relative to persons, groups of persons or even the human species, and so on. On the other hand, if it is knowledge of something "ideal," then it does not have these features but instead has the opposite features. In the past century, Frege, Husserl, and Gödel have all argued for a platonic rationalism about logic. We see such arguments in the attacks of Frege and Husserl on psychologism and in their own positive views on the nature of pure logic. Gödel approves of these kinds of arguments of Frege and Husserl and adds some of his own. Gödel and Husserl both make comments about how, in accordance with what Gödel calls the "leftward" *Zeitgeist*, rationalism has been underrated, forgotten, or abused.

§ 7. Rational intuition

Some rationalists have held that there is a kind of *rational intuition*. This is certainly true of earlier rationalists such as Descartes and Leibniz. Descartes and Leibniz got into trouble with this view in metaphysics, where they made large claims about ultimate reality. On the other hand, it can be argued that, on the basis of their magnificent contributions to mathematics and logic, the view served them well in these more circumscribed fields. In his realist phase, Bertrand Russell embraced the idea of a kind of rational intuition, for he claimed that we could know about abstract universals, not only by description, but also by acquaintance (Russell 1912). We have been saying throughout the previous chapters that Husserl holds that there is a kind of rational intuition. In fact, he has probably done more than anyone in recent times to develop an account of it, of categorial or eidetic intuition, in connection with logic, mathematics, and eidetic science generally. Rational *intuition*, on this view, is responsible for objectivity in mathematics and logic, and even in philosophy. The idea is that without rational intuition there are just many possible conceptions without knowledge of which of these conceptions hold of reality and which do not. Our capacity for reason allows us to conceive of many things but, as we saw in the last chapter, Husserl draws a further distinction in the domain of reason between mere conceptions and conceptions that are (partially) filled by categorial or eidetic intuition. This is a distinction between putting forth mere conjectures or hypotheses in seeking knowledge and actually possessing knowledge. Just as knowledge acquired on the basis of the senses requires conception *and* sensory intuition, so knowledge acquired by reason requires conception *and* intuition, only now it is a form of rational intuition of non-sensory invariants that is in play. Intuition is the source of evidence in either case. Husserl says that

> *Category of objectivity and category of evidence are perfect correlates. To every fundamental species of objectivities*—as intentional unities maintainable throughout an intentional synthesis and, ultimately, as unities belonging to a possible "experience"—*a fundamental species of "experience," of evidence, corresponds*, and likewise a fundamental species of intentionally indicated evidential style in the possible enhancement of the perfection of the having of an objectivity itself.
>
> (Husserl 1929, § 60)

We will consider additional elements of this view of intuition in the next chapter.

As we have already noted, there is no place in Kant's philosophy for any kind of intuition that goes beyond the two forms of sensory intuition, time and space, which Kant links to arithmetic and Euclidean geometry respectively. Kant says that sensible intuition is the only possible intuition for us (*CPR* B 307–309). We are not supposed to be capable of "intellectual intuition" which, Kant says at various points in the *Critique of Pure Reason*, would be intuition of the noumenal. But why could there not be a kind of rational intuition that either fulfills or frustrates our conceptions of objects or states of affairs and that (i) encompasses more than the merely sensible, (ii) is not supposed to be intuition of the noumenal, and (iii) is required if there is to be knowledge and not merely empty intention in the sciences of mathematics and logic? A view that goes beyond Kant's is certainly required to make sense of Gödel's remarks on the subject of intuition. If the many wonderful developments in pure mathematics and logic that occurred after Kant's time are to count as epistemic achievements, and if knowledge requires both conception and intuition, as Kant believed, then Kant's own conception of intuition will be much too limited. There is a tremendous body of mathematics that goes beyond Euclidean geometry, arithmetic, and elementary algebra. This body of mathematics exists. It is given to us. If we are to take the given as a transcendental clue then we need to ask how this body of mathematics is possible. What are the conditions for the possibility of its existence? Kant's view is too limited to respond to this question. We presumably do not want to say that none of this mathematics counts as science or as knowledge, or that we cannot distinguish knowledge from illusion in this body of work. We presumably do not want to say that none of it counts as a product of reason. If it is not a product of reason then where do its origins lie? In sensation, pure imagination, purely emotional states? There are no doubt limits to the capacities of reason. In the previous chapter we argued that human reason cannot know about mind-independent$_1$ objects or truths but that it is possible to know about mind-independent$_2$ objects or truths. The development of mathematics and logic since Kant's time shows that Kant set the bar too low.

Platonic rationalists who argue in favor of rational intuition hold that there is intuition of abstract objects such as natural numbers, sets, universals, essences, properties, or other such entities. Gödel says in various places, as we have seen, that there is intuition of concepts. If there is to be any hope of developing such an account of rational intuition then, in my view, we need to steer clear of any kind of mysticism about intuition. Rational intuition should be seen as a component of a broadly scientific worldview. In particular, the sciences of mathematics and logic require a kind of rational intuition, even if we should be wary of claims made on the basis of intuition in metaphysics as a whole, to say nothing of theology. The idea of rational intuition, if it is to have any traction in mathematics and logic, must be inseparable from standards of evidence, proof, and objectivity. It must be a component in the correction or adjustments of belief, of corroboration, repeatability, the development of workable methods in mathematics and logic, and so on.[3]

In the next chapter we will see in more detail how Husserl distinguishes categorial intuition from "straightforward" sensory intuition. Once the distinction is drawn, based on attending to various types of consciousness in a careful way in accordance with the intentional difference principle, it becomes apparent that categorial intuition is quite common. It is perhaps because it is so common that it is frequently overlooked. Straightforward intuition is awareness of objects that does not present them in their categorial form, that is, that does not present them with their parts, properties, relations, and so on. But a lot of our experience and knowledge is therefore, in fact, not straightforward.

§ 8. Rationalistic optimism

Rationalism has at times been connected with a kind of *optimism* about problem solving in the sciences, if not problem solving in general. The optimism in the sciences is manifested in the belief that open scientific problems have solutions. A solution is there, waiting to be found, if only we will work toward finding it. Platonism about mathematics and logic is a natural companion of this kind of rationalistic optimism. Solutions are not invented but are waiting to be discovered, and it is reason that is capable of finding the solutions. This kind of optimism, in fact, drives scientific research. It acts as a regulative ideal. David Hilbert does not promote platonist views in the foundations of mathematics, but a widely quoted passage from some of his work reflects his rationalism and optimism about problem solving in mathematics: "...one of the things that attract us most when we apply ourselves to a mathematical problem is

[3] On reason and intuition, see also Parsons 2000.

precisely that within us we always hear the call: here is the problem, search for the solution; you can find it by pure thought, for in mathematics there is no *ignorabimus*" (Hilbert 1900 and 1926). As we saw in chapter 3, the problem for Hilbert was that he was unable to combine this rationalism, this talk of finding solutions to mathematical problems by pure thought, with his leftward (and anti-platonist) proposal for a foundation of mathematics in a finitistic formalism that appeals to concrete signs that are "intuitively present as immediate experience prior to all thought." Gödel appeals to the incompleteness theorems to show that such a combination is unworkable. How can Hilbert speak about solving problems by pure thought when we are supposed to substitute for pure thought the mechanical manipulation of meaningless finite, concrete sign configurations that are given in sensory perception prior to all thought?

Gödel was also a rationalistic optimist about problem solving in mathematics. In particular, in his later work Gödel agrees with Hilbert that there are no number-theoretic problems that are absolutely undecidable by the human mind. As noted in chapter 3, Gödel thought that if there were absolutely undecidable number-theoretic problems then human reason would be utterly irrational by asking questions it cannot answer, while emphatically asserting that only reason can answer them (Wang 1974, p. 324–25). Human reason would then be very imperfect and in some sense even inconsistent, which contradicts the fact that parts of mathematics that have been systematically and completely developed (e.g. the theory of 1st and 2nd degree Diophantine equations with two unknowns) show an amazing degree of beauty and perfection. In these fields, by entirely unexpected laws and procedures, means are provided, not only for solving all relevant problems, but for solving them in a beautiful and perfectly feasible manner. On Gödel's view, if reason can conceive a problem in mathematics and logic in a clear manner, then it can in principle solve the problem. One can in fact see how this has happened again and again throughout the entire history of mathematics, as Hilbert himself notes (Hilbert 1900).

Gödel's response to the continuum problem in classical set theory points very clearly to another difference between rationalists and empiricists in the foundations of mathematics. Rationalists such as Gödel argue that, on the basis of our capacity for rational intuition, we need to find new axioms of set theory that will be evident and that will enable us to solve the continuum problem. Those who are inclined toward empiricism, on the other hand, often ask why we need new axioms in mathematics. What is the point? We already have all the mathematics we need to do natural science, there is no capacity for empirical knowledge that could make such new axioms of higher set theory evident to us, and so on. Another option for the empiricist is to argue that we should accept or reject axiom candidates in higher set theory on the grounds of Occam's razor, or on grounds of particular probabilistic or inductive considerations. I will consider some of these options in more detail in chapter 8.

One very broad background consideration about reason in Gödel's work is that he evidently thinks that reason is present everywhere in the universe. Appearances to the contrary are just illusions: "Every chaos is merely a wrong appearance" (Gödel 2003b, Vol. V, p. 81). At the other extreme would be a philosophy according to which reason exists nowhere in the universe. The universe is in fact absurd. I think there is evidence to suggest that this later view is false, even if we do not embrace Gödel's hyper-rationalism. The evidence is to be found, as Gödel suggests, in the results of mathematics, logic, and the other sciences. If you think that reason exists in the universe at all, then you will presumably think that a minimal condition imposed by reason is consistency. Inconsistency signifies a lack of reason. It signifies irrationality or non-rationality. Even if we may find ourselves in contradictions at times, we correct for the contradictions in our scientific activities once we are aware of them. The snake/coiled rope example in the last chapter shows how this correction happens automatically in ordinary perception. There can be localized inconsistencies of this type in both perceptual experience and mathematical experience. We can also imagine what it would be like for there to be a global breakdown, as in Husserl's thought experiment in the last chapter, but our experience is in fact not like this. A world in which nothing ever stabilizes, in which there are no invariants (objects) in our perceptual or mathematical experience, would be a "world" in which reason is absent.

Consistency of formal systems is a condition in both of Gödel's incompleteness theorems: if a formal system of mathematics is consistent then it is incomplete, and if a formal system of mathematics is consistent then it cannot prove the statement that asserts its own consistency. In his own view of the philosophical implications of the incompleteness theorems Gödel does not seem to spend a lot of time worrying about what his theorems might mean if this consistency condition fails. What lies behind this, if I am correct, is his global rationalism. I do not deny that one can press rationalism too far, claiming that reason is present when in fact it is absent. The opposite error also cannot be ruled out: claiming that reason is absent when in fact it is present. In chapter 8, I shall offer some further reflections on how we might avoid these errors, insofar as it is possible.

§ 9. Interlude on holism and reason

As we have remarked at various places, many rationalists have held that it is only reason that could allow us to arrive at necessary truths, at a priori knowledge, at clear and distinct knowledge, and at certainty. If we understand analytic judgments to be judgments that are true on the basis of the meaning of the terms that they contain, then it is reason that also makes it possible to see or understand that these judgments are true. There have, of course, been recent challenges on the grounds of certain types of holism to the old divisions between the analytic and the synthetic, the necessary and the contingent, the a priori and the a posteriori, the indubitable and the probabilistic. On these holistic views there are no such sharp divisions. Rather, there are only continuous gradations, and nothing in the overall web of belief is supposed to be fixed once and for all in accordance with these categories. The very distinction between the rational and the empirical would be subject to such a holistic view. It is in the work of Nelson Goodman, Willard van Orman Quine, Morton White, Richard Rorty, and others that versions of this kind of holism have been most fully developed. I will come back to Quine's views in particular in chapter 8, but the comment I want to make about this now is that I think this kind of pragmatic holism, insofar as it is supposed to extend across sense perception *and* pure mathematics and pure logic, presents us with another "leftward" view of reason. It is persuasive as a view of how our thinking with concepts pertaining to sense perception can be revised, corrected, and adjusted on the basis of sense perception, and of how the deliverances of sense perception can be revised, corrected, and adjusted on the basis of our thinking with concepts pertaining to sense perception, but it is not at all persuasive if we recognize concepts and modes of conception in our knowledge that do not direct us toward sensory objects at all, but rather direct our thinking toward objects of an altogether different kind. As we saw above, Husserl claims that a fundamental species of experience corresponds to every fundamental species of objectivities. "Category of objectivity and category of evidence are perfect correlates." It is not the case that all objects in our various practices are *meant* as sensory objects or even more generally as objects of natural science. This idea of *tracking how objects are meant* in our practices is really the central point of the intentional difference principle, to be discussed below. Some of the thinking in which objects are not meant as objects of natural science is nonetheless perfectly sound scientific thinking. If this is correct then what pragmatic holism does is to reduce reason to empirical, applied, or pragmatic reason. It forgets about, or is blind to, other features and capacities of reason. I return to this topic in chapter 8.

§ 10. Human reason exhibits intentionality

The notion of intentionality takes center stage in Husserl's phenomenology, whereas it is simply absent, or at best marginal, in the work of Plato, Leibniz, the traditional rationalists, and Kant. If we examine how human reason and conception functions in mathematics and logic then we find, as a central feature, that it exhibits intentionality. Intentionality is a broader notion than the common notion of intention. Franz Brentano, who was one of Husserl's teachers, is famous for arguing that if there is a distinctive mark of the mental then it is that mental phenomena exhibit intentionality while physical phenomena do not. This means that mental phenomena—various states of consciousness—display "aboutness" or "directedness."[4] They are directed toward objects. As we saw in chapter 1, Gödel takes note of this division. A belief, for example, is always a belief about some object or state of affairs. Purely physical states, however, are not "about" anything. They do not have this characteristic of meaning or referring to something or other. Brentano's thesis need not be construed as committing us to a

[4] Husserl developed and modified his theory of intentionality in various ways over a long period of time. In recent times, John Searle (Searle 1983) has put forward a theory of intentionality that is similar in a number of ways to Husserl's account, but Searle does not consider the intentionality of mathematical consciousness.

metaphysics of substance dualism. In fact, under the terms of Husserl's phenomenological *epochē* it could not. What I want to note at the moment, however, is that it seems very difficult to deny that consciousness, at least in the forms most relevant to scientific thinking, exhibits intentionality. If we can speak about beliefs at all, for example, then what would it be like for a belief to not be about something or other? Imagine that you have a belief but that it is not about anything. This seems absurd.

Suppose we fasten firmly onto this idea of directedness. Once we take it seriously then we see that there are indeed many forms of conscious directedness. A number of these forms will be relevant to science, and in this book I am most interested in the types of conscious activities involved in scientific thinking. In particular, sensory perception is only one form of conscious directedness, and it is not the only form involved in scientific thinking. It is evidently a condition for the possibility of some forms of scientific thinking about nature but not necessarily for all forms of scientific thinking. There is also conscious directedness involving abstraction, idealization, reflection, variation, axiomatization, formalization, and other "higher-order" cognitive activities. Our awareness is directed in different ways in imagination, memory, mathematical thinking, and so on. Following the style of phenomenology, let us say that in mathematical thinking I am directed toward mathematical objects and states of affairs, in sensory perception I am directed toward sensory objects and states of affairs, in imagination I am directed toward imaginary objects and states of affairs, and so on. At the outset it will not be a good idea to automatically run the different types of object-directedness into one another. We ought not to suppose from the outset, for example, that mathematical thinking just is a species of imagination. Rather, the various connections between the two would need to be investigated and clarified.

In making these remarks I want to hold that object-directedness does not require that there be an object. Nothing about saying that imagination is object-directed requires us to say that imaginary objects exist. The same is true for states of belief. Having a belief about something does not entail that there is a corresponding object of belief. A number of years ago I believed that I saw a house on a street in the Hollywood hills but it turned out, to my surprise, to be merely two walls of a house on a movie set. It is not like there had to be a house in this instance in order for my belief to be about what it was about. Here one can follow Husserl in holding that it is the "content," or what Husserl calls the "noema," of my belief that made the object-directedness possible even if there was no house. What makes the directedness of the conscious act possible is not the existence (or "intentional inexistence" or "non-existent being") of an object, but rather the "content" or "meaning" associated with the act. Let us call the contents or meanings expressed by predicates *concepts*. As we saw in chapter 3, Gödel says in his 1961 paper (Gödel *1961/?, p. 383) that Husserl's phenomenology would have us direct our attention onto our own acts in our use of concepts, onto our powers of carrying out our acts, and so on. It is this kind of approach that Gödel thinks is promising. I want to now provide some further elaboration on this kind of view.

Gödel makes a number of remarks on the nature of concepts at various places in his writings from the 1940s through the 1970s. Concepts are mentioned, as we have seen, in his general views on logic. Logic, Gödel says, is about the most general abstract (and "formal") concepts and the relationships of these concepts to one another (see, e.g. Gödel *1953/59, p. 354). Sense perception, on the other hand, is about particular objects and their properties and relations. In a passage written prior to the 1961 paper, as we noted in chapter 3, Gödel likens the conceptual content of sentences to Frege's notion of sense (*Sinn*) (Gödel *1953/59, p. 350). Concepts are abstract, objective intensional entities (see, e.g. Gödel 1944, *1953/59, *1961/?). They are not merely subjective and, unlike conscious states, they do not themselves have temporal phases. Concepts are not sets, although one might conjecture that every set is the extension of some concept. Concepts are entities *sui generis*. One could make a list of things with which they are not to be identified or to which they are not to be reduced. Gödel rejects those forms of reductionism that ignore or seek to eliminate the intuition of such concepts.

By the time of the 1961 paper Gödel had studied Husserl's philosophy for several years, and he says that phenomenology would have us direct our attention onto our own acts in our use of concepts. A standard way to individuate the structure of mental acts that exhibit intentionality is through statements such as "I believe that Pa," "I know that Pa," "I remember Pa," and the like. In each of these examples we have a conscious subject or monad ("I"), a type of consciousness (e.g. belief), and the content or meaning of the act, expressed by "Pa." As indicated a moment ago, let us call the content or meaning (*Sinn*) expressed by "Px" a concept. The type of consciousness (e.g. believing, imagining, remembering, perceiving) is called the "thetic character" by Husserl in his book *Ideas I,* while what is expressed by Px is called the "noematic *Sinn*." In *Ideas III* (Husserl 1952b, p. 89) Husserl says "The noema is nothing but a generalization of the idea of meaning (*Sinn*) to the field of all acts." Similar remarks are made in *Ideas I* (see, e.g. Husserl 1913, § 90–91) and other writings. We can think of the "x" in Px as a variable that can be replaced by singular terms. Husserl says that in the noematic *Sinn* there is a "determinable x" by virtue of which we can be directed toward the same object under different concepts. The same object can, and typically does, take on different "predicates" in our experience, as in x is P, Q, R, S, and so on. In certain circumstances, names come to be used in place of the determinable x. There can also be direct-object constructions, as when I say "I perceive the tree," "I perceive the green tree," or "I imagine the purple tree." If the form here is "I perceive x", then we still have a monad and a thetic character and a kind of noematic *Sinn*, but in this case the noematic *Sinn* is expressed by a singular term. The same noematic *Sinn* (content) might be combined with different thetic characters, or the same thetic character might be combined with different noematic *Sinne* (contents), albeit not arbitrarily. Husserl sometimes refers to the thetic character and the noematic *Sinn* together as the "full noema."

Gödel's association of the conceptual content of sentences with Frege's notion of sense (*Sinn*) in the 1950s is arguably compatible in certain relevant respects, to be noted below, with his later turn to Husserl's philosophy.[5] For purposes of terminological uniformity with Gödel's texts, I will sometimes use the expressions "concept," "content," or "meaning" in place of "noema," "noematic *Sinn*," or Husserl's other special language for components of the noema. I will also sometimes use the term "intention" in place of these expressions.

Now to direct our attention onto our own acts in our use of concepts, according to Husserl, will include reflection on the noematic *Sinne* of our acts. This kind of direction of our attention is accomplished through phenomenological reflection. The *epochē* would have us shift our attention from objects to the consciousness of objects. Experience of objects is possible thanks to the intentionality of consciousness, but the structure of intentionality is such that we are always directed toward objects on the basis of our concepts. In shifting our attention from objects to our experience of objects we can thus shift our attention to the concepts by virtue of which we are directed toward objects in our experience.

For the moment, we can say that there are different types of consciousness and that with these different types of conscious states there will be associated different contents or concepts that are responsible for our being directed in just the way that we are. While there is always some concept or other associated with an intentional state there may or may not be an object or state of affairs corresponding to the concept. It is important to note that in speaking only of object-directedness we have a great deal of freedom that we would not have if we thought we had to first decide all of the details about evidence for the existence of objects. Instead, we will be speaking about what types of object-directedness there are, and we will come back to some questions about knowledge and evidence later, and especially in the next chapter. In order for the belief to be justified we would need to have evidence for it, that is, we would need to have evidence for the object or state of affairs toward which we are directed. Our concepts would need to be filled to some extent. This evidence can come in different degrees and types.

In claiming that reason in mathematics and logic exhibits intentionality I want to say that in these fields our consciousness is directed toward mathematical and logical objects by virtue of mathematical and logical concepts. Although all acts that exhibit intentionality employ concepts, scientific reason in particular is concerned with the

[5] There is a voluminous literature on the relationship of Husserl's notion of noematic *Sinn*, which is used in a general theory of consciousness that also includes language, to Frege's notion of *Sinn*, which is used primarily by Frege in connection with his study of logic and language. For some of the central literature see Føllesdal 1969, Mohanty 1982, the collection edited by Dreyfus 1982, Smith and McIntyre 1982, McIntyre 1987. There are, of course, a number of important differences in the views of Frege and Husserl on meaning and reference but my account will depend on only a few particular points of agreement. See also Drummond 1990, Hill and Rosado-Haddock 2000.

type of conception discussed earlier that is distinct from ("outer") sensation, "inner sense," pure imagination, emotion, and some other cognitive capacities.

I would also like to highlight the connection between intentionality and meaning theory that is relevant to this view. I said above that we can use the notion of meaning in place of "content" or "concept." We are directed toward objects by way of the meanings associated with our acts. We are meaning-bestowing or meaning-constituting beings. To be directed at all, to be conscious, is to mean. We saw in chapters 1 and 4 how Husserl speaks of "intentional constitution." The monad produces the meaning of being of its objects in mathematical and logical practice in a rationally motivated manner through intentional constitution. It is thanks to the fact that human consciousness exhibits intentionality that, after the *epoché*, we can refer to ourselves as beings who constitute the meaning of being through our cognitive activities.

Gödel's idea of clarifying the meaning of the primitive terms of mathematics on the basis of the phenomenological method thus amounts to a call to reflect on the basic abstract concepts, such as the concept of "set," by virtue of which we are directed toward objects in our thinking in mathematical practice. The idea is to unfold more of what is in the horizon of the concept, to find limitations of the concept, to understand the relation of the concept to other concepts, and so on.

§ 11. Minds and brains again

The kind of language I have been using throughout this chapter and elsewhere in the book is mind language. Mind language in this style is very different from brain language. Brain states, viewed as purely physical phenomena, do not exhibit aboutness. It is not part of the very nature or essence of a particular neurochemical activity that it somehow means or refers to something outside of itself. In describing this neurochemical activity in the language of natural science, we should not find that in addition to neurons and neurochemical interactions there are "contents," concepts, noemata, meanings, and so on. This is simply the wrong level of description. It is widely agreed that the sentences of neuroscience should, insofar as possible, be shorn of the intensional idiom. One aims for a thoroughgoing extensionalism. The logic of the language of mental phenomena is clearly different from the logic of the language of purely physical phenomena. We can see how this difference is marked, as we have noted, in the failure of certain principles of extensional logic (e.g. substitutivity *salva veritate* and existential generalization) when we try to apply these principles to statements expressing the intentionality of consciousness.

Gödel once suggested that the argument that mental procedures are mechanical procedures is valid if one assumes that (i) there is no mind separate from matter and (ii) the brain functions basically like a digital computer (see Wang 1974, p. 326). Gödel evidently thought that (ii) was very likely but that (i) was a prejudice of our time that might actually be disproved. It is interesting to compare this with the position of Roger Penrose in his book *Shadows of the Mind* (Penrose 1994; see also Feferman 1996).

Unlike Gödel, Penrose denies (ii). He holds that the missing science of consciousness is to be a form of neuroscience that recognizes non-computational brain processes. Penrose thinks that Gödel's incompleteness theorems imply some kind of platonism. He also uses the incompleteness theorems to argue that there are non-computational procedures for *knowing* mathematical truths, but this is immediately equated with the view that there must be non-computational *brain* processes (Penrose 1994). It is fair to ask Penrose the question that immediately gets us into a corner: how could the brain be stimulated by abstract objects or, rephrasing it somewhat, what could the relationship between the brain (which for Penrose is not a computer) and abstract objects be? Given what a brain is and given the kinds of physical interactions in which it can be involved, and given that an abstract object is typically understood to be a non-physical object of some kind, it would be a category mistake to suppose that brains could have access to abstract objects. One might expect him to address issues about non-computational *mental processes*, where these are not immediately equated with brain processes, but there is no phenomenology of consciousness in Penrose's work. In particular, there is no phenomenology of the consciousness of abstract objects. There is neuroscience with its "natural attitude" on one side, and platonism on the other, and not the slightest hint how the two could be related. One might therefore think that the missing science of consciousness cannot consist only of neuroscience (see Tieszen 1996 and 2005a).

Gödel's remarks, on the other hand, suggest that brain processes are computational, but mental processes are not. Although Gödel directs our attention to Husserl's phenomenology, there is very little discussion in his own work of how this would be possible. His remarks at least show that he does not feel compelled to immediately say that brain processes are non-computational if the incompleteness theorems suggest that mental processes are non-computational. Of course there are also problems with Gödel's view. One of the worries, especially given the paucity of his remarks on the subject and some of his religious views, is that in questioning (i) he may be holding a kind of substance dualism. I have already indicated that I think we should steer clear of substance dualism. I do not see how any such metaphysical view could be affirmed if we adhere to the phenomenological *epoché*. In relation to (i), I will not venture beyond the following separation: consciousness exhibits intentionality and brain states do not exhibit intentionality.

As I indicated earlier, Gödel seems to think that the non-computational character of mental processes is connected directly with the ability of the mind to grasp abstract objects. As I construe this, Gödel's anti-mechanism about reason, his interest in the phenomenology of consciousness, and his platonism about concepts are all part of one package. Penrose, whose argument is more like that of J.R. Lucas (Lucas 1961, 1970), does not make these connections. As will become even clearer in chapter 7, I think that Gödel is on the right track here.

§ 12. The intentional difference principle

If there are different forms of conscious directedness, then what kind of territory can be mapped out regarding these different forms? There is a fundamental principle that we need to take seriously if we are to answer this question. I will call it the *intentional difference principle* (IDP). In formulating the principle I will use the term "noema" to stand for what was called above the full noema of an act.

IDP: *Each noema yields, prima facie, a different way of being directed toward an object or state of affairs. We are to take the directedness of consciousness just as it presents itself and not as something else.* There can, of course, be many different noemata that direct us toward the same object, and there can also be many noemata that do not direct us toward the same object. Noemata can differ from one another on account of either their thetic character or their noematic *Sinne*. If, for example, I express the noematic *Sinn* of one act as "x is a triangle" and the noematic *Sinn* of another act as "x is a natural number" then clearly these two *Sinne* direct our thinking toward different objects. In some cases it will be obvious to everyone, from the context of the experience, that two *Sinne* do not direct us toward different objects or states of affairs, while in other cases it will be obvious that they do. In the process of acquiring knowledge in the sciences, the claim that two noematic *Sinne* Φ and Ψ do not direct us toward different objects must often be shown. In fields such as mathematics and logic it should, in fact, be proved if possible.

It also follows from our remarks above that a shift in thetic character makes for a shift in the type of conscious directedness. Remembering is different from perceiving, and both are different from imagining. Perceiving is different from judging. (I use the term "perception" here, as usual, to refer only to sensory perception.) In the discussion below we will see that consciousness of abstract or ideal objects involves certain shifts in thetic character. Given what we just said, the thetic character involved in the consciousness of abstract objects cannot be perception. Indeed, it would be a category mistake to substitute an expression with mathematical content, such as $5 + 7 = 12$, for "S" in the scheme "I perceive that S," where perception is understood as sensory perception. The content and the thetic character are from incompatible categories. In mathematical practice numbers are not meant as sensory objects that are related to one another in a physical manner. This is evident from the way in which we speak about these objects in number theory.

It is important to note that the fact that *a cognitive act is directed in a particular way means that it is not directed in other ways*. If my thinking is directed by "x is a triangle," then there are a host of ways in which it is not, and cannot, be directed. Noematic *Sinne* or concepts always present us with a perspective on an object or situation. Human reason, like other modes of awareness, is perspectival, and we are finite beings (monads) and cannot take all possible perspectives on an object, state of affairs or domain. We do not experience everything all at once. Our knowledge is thus typically incomplete in certain ways, although we might see how we could continue to perfect it.

Some contents will be compatible with a given content, but others will not. Noematic *Sinne* can be consistent with one another, imply one another, and so on, as indicated in the example in the next paragraph. There will also be categories or regions of noematic *Sinne* appropriate to particular kinds of object-directedness, and other categories of noematic *Sinne* appropriate to other kinds of object-directedness. Not every kind of object, for example, is meant as a causal object. The category of causality is not operative in all forms of directedness.

Noematic *Sinne,* as we said, are to be understood as intensional entities, and as such their identity is not to be determined extensionally. Intensionality is linked to intentionality in the following way: expressions for mental acts that exhibit intentionality (e.g. knowledge, belief, hope) create intensional contexts, that is, contexts in which principles of standard extensional logic, such as substitutivity *salva veritate* and existential generalization, fail. As we have indicated, one can think of contents of mental acts as meanings or intensions.

Various relationships that have been noted between intensions (*Sinne*) and extensions are directly applicable in this theory of consciousness. Consider, for example, the intensions or concepts "x is a rectangle," "x is a quadrilateral," "x is a parallelogram," "x is a polygon," and "x is a square." These concepts can be ranked or ordered in terms of increasing and decreasing intension and increasing and decreasing extension. Let us say that a set of concepts can be put into an order of increasing intension if each concept in the series except the first includes more properties or determinations than the one preceding it. In this manner we move back and forth between levels of generality and specificity. The concepts rectangle, quadrilateral, parallelogram, polygon, and square can thus be put in order of increasing intension as polygon, quadrilateral, parallelogram, rectangle, and square. The order of decreasing intension is simply the reverse. As more properties are added, the extension of the concept decreases. The set of things to which the concept applies has a smaller extension. The extension increases when a set has more members. Thus, concepts can often be put in order of increasing or decreasing extension. The order of increasing intension is usually the same as that of decreasing extension, and the order of decreasing intension is usually the same as the order of increasing extension. It is possible for concepts to be ordered in increasing intension and yet not to exhibit an order of decreasing extension when all of the concepts have empty extensions. It can also happen that the intension increases with each successive concept in the ordering and yet the extension does not decrease because none of the concepts has the empty extension but they all have the same extension.

A person (monad) who *knows* how to put these concepts in the order indicated knows that if something is a rectangle then it is a polygon, and so on. It is also known that if something is a polygon it is not necessarily a rectangle, although it is *possible* that it is a rectangle. Its being a rectangle is in the horizon of the concept of a polygon. In short, the narrower intension (*Sinn*) implies the wider intension, but the converse does not hold. All of this knowledge is a function of reason, not of sensation, introspection, emotion, or mere imagination. Once one has knowledge of the ordering, but not otherwise, we could say that if something is *meant* as a square, for example, then it is meant as a polygon, although to mean something as a polygon is not necessarily to mean it as a square. The meaning constitution is not at all arbitrary but is quite structured. Imagine what our experience in geometry would be like if this were not true. Our thinking can be directed by any one of these concepts or intensions, in which case this thinking will be systematically related to our thinking that is directed by the other concepts. All of this ordering, structuring, or organization is a function of human reason.

Of course a person might not know how to order the concepts, which is to say that the higher-level intention to order the concepts is not fulfilled. This would happen if one does not understand the meaning of one or more of the concepts to be ordered. In short, a person might have the intention to order the concepts but not know how to do it. To *know* the ordering, on the account we have been suggesting, just is to have a categorial *intuition* (not a hunch) that one otherwise lacks. There is nothing mystical about intuition here. On the contrary, intuition is a condition for the possibility of objectivity.

Now it is clear that sensory perception is one type of conscious directedness. We are not, however, directed exclusively to sensory objects. To take the IDP seriously is to hold that in sensory experience we are directed toward sensory objects, in mathematical experience we are directed toward mathematical objects, in imagination we are directed toward imaginary objects, and so on. This should be our starting position. It might turn out that some of these types of object-directedness can be reduced to others, but in each case that will remain something to be shown. The starting position is non-reductionistic. In thinking of how to solve a problem in a number theory textbook, for example, my thinking does not seem to be directed toward sensory objects or states of affairs, although there may be some sensory experience in the background. If in our experience we were directed exclusively toward sensory objects or the objects of "inner sense," then it is not clear how sciences such as mathematics and logic would be possible at all.

§ 13. Elaborating on links Between Plato, Leibniz, Kant, and Husserl

Let us now explicitly link a number of the ideas in this and previous chapters with the views of Plato, Leibniz, Kant, and Husserl that were of interest to Gödel. The picture that emerges of how we are conscious of mathematical reality is roughly as follows: the reality of which human subjects are aware in pure mathematics consists of abstract mind-independent objects, in the sense of mind-independence$_2$. It is platonic reality, but in the sense of constituted platonism, which is developed on the basis of Husserl's extension of ideas in Kant's transcendental philosophy. Gödel is interested in Plato, but he wants to use Husserl's transcendental phenomenological method to appropriate

Plato's objectivism. At the same time, human subjects who know about mathematical objects are viewed as transcendental subjects once the *epochē* has been engaged, and transcendental subjects in their full concreteness are said to be monads. Both Husserl and Gödel are, of course, thinking of Leibniz here.

In his "Preface to the General Science" (1677), "Preface to a Universal Characteristic" (1678–79) and other writings on the universal characteristic, Leibniz employs the analogy of a telescope (Leibniz 1951, 1966, 1989). With the naked eye we do not always see very clearly. Building a tool such as a telescope allows us to see much more clearly. In the case of mathematics and logic we also need to build a tool, or ideally even a method, that will allow us to see clearly what is mathematically or logically true and what is not, what is valid and what is not, what is consistent and what is not, and so on. For Leibniz, whose vision is nothing less than grand, this tool will be the universal characteristic and *calculus ratiocinator*. The universal characteristic is supposed to do for us in mathematics and logic what the telescope does for the naked eye in perception. The point of the universal characteristic, moreover, is evidently to allow us to *mechanically calculate* the answers to our questions about what is valid and what is not, what is consistent and what is not, what is logically equivalent and what is not, and so on. Leibniz says that once the universal characteristic has been developed we should soon settle disputes about these matters by taking pen to paper and calculating. If someone should doubt my results, Leibniz writes in "Preface to the General Science" (1677) and elsewhere, then I should say to him "Let us calculate, Sir!" The universal characteristic, once created, will be the greatest instrument of reason. It will exalt reason no less than the telescope serves to perfect our vision.

As we have seen, Gödel thinks the incompleteness theorems and related results show that while Leibniz's program of the *calculemus* may enjoy some success, it cannot be carried out in full. The general view of mathematical knowledge is thus different from the idea that motivates Leibniz's method of the universal characteristic and the *calculus ratiocinator*. In mathematics we will not always be able to decide mathematical problems on the basis of mechanical calculation. Mathematical reality consists of abstract mind-independent objects that cannot be accessed (only) through mechanical calculation. Mechanical calculation alone cannot give us insight into abstract mind-independent objects. Rather, we must reflect on and employ (abstract) meanings or concepts. Leibniz's notion of reason, insofar as it is represents reasoning as sheer calculation or computation, is inadequate. Following Plato's rudimentary objectivism, but updated by Husserl's transcendental phenomenology, it is a platonic rationalism that we want.

We can nonetheless extend the telescope analogy somewhat, as Frege does in his famous paper on sense and reference (Frege 1892). We saw in an earlier chapter how in one of his unpublished papers from the 1950s, before he had read Husserl, Gödel associates his notion of concept with Frege's notion of sense. The notion of sense (*Sinn*) also plays a role in Husserl's theory of intentionality, as indicated in this chapter. Frege says that signs express a sense (*Sinn*) and designate a reference, although they will not always have a reference. In grasping a sense we are not assured of a reference. It is in the sense of a sign that the mode of presentation is contained. The sense serves to illuminate a single aspect of the reference, supposing the sign to have reference. Frege says that comprehensive knowledge of the reference would require us to be able to say immediately whether any given sense belongs to it, and that to such knowledge we never actually attain. Both the reference and the sense of a sign are to be distinguished from the associated (subjective) idea. One man's idea is not that of another. Thus, there can be a variety of differences in the subjective ideas associated with the same sense. The sense, however, may be the common property of many subjects and is therefore not a part or a mode of the individual mind. Frege says that one can hardly deny that humankind has a common store of thoughts which is transmitted from one generation to another. Thus, the r*eference* is the object itself which we designate by means of the sign, while the *idea* is wholly subjective. In between lies the *sense*, which is not subjective like the idea, but is not yet the object itself.

Now suppose that someone observes the Moon through a telescope. Think of the Moon as the reference. It is the object of the observation, mediated by the real image projected by the object on the glass in the interior of the telescope, and by the retinal image of the observer. The optical image on the glass is like the sense, and the retinal image is like the subjective idea or experience. The optical image in the telescope is indeed one-sided and dependent upon the standpoint of observation, but it is still objective because it can be used by different observers. Each of these observers, however, would have his or her own retinal image on account of the diverse shapes of the observer's eyes and other subject-specific features.

Elaborating on the telescope analogy in connection with Husserl and Gödel, we can construct the following picture of how the monad is conscious of mathematical reality. For Husserl, the picture applies whether the objects are "real" or "ideal." In the case of mathematics the objects are ideal. As we said, the domain of which we are aware in mathematics consists of abstract or ideal mind-independent objects, in the sense of mind-independence$_2$. Human consciousness exhibits intentionality. The monad is directed toward mathematical objects (referents) on the basis of the meanings (or concepts or *noemata*) of founded acts of reason. The referent is given from one perspective by the meaning. The meaning is, as it were, a mode of presentation of the referent. The meaning is like the optical image in the telescope, whereas the "real" (in Husserl's sense) subjective idea is like the retinal image. It occurs in time and is specific to each different subject. Husserl refers to "real" mental phenomena as *noeses*. Thus, the referent, if there is one, is objective. It is an invariant across different meanings (*noemata*) and different monads who have different *noeses* associated with those meanings. The meaning (*noema*) is also an invariant through the different ideas (*noeses*) that may be associated with it in different monads. The meaning is objective and invariant across different monads, but the subjective ideas of one monad are different from the subjective ideas of other monads. One important point of difference between Frege and Husserl is that Husserl thinks the reference associated with a sentence (a proposition, if you like) is a state of affairs, not a truth value. Frege is focused on logic and evidently thinks that "the True" and "the False" suffice for this purpose. Husserl, however, is attempting to account for our experience. In our experience we are not directed toward "the True" or "the False" in the thinking we express in sentences, but rather we are directed toward states of affairs (*Sachverhalten*). Husserl's semantics is in fact even more nuanced, for he appears to hold that states of affairs are themselves based on situations (*Sachlage*).[6]

We are thus conscious of abstract mind-independent$_2$ objects on the basis of the meanings associated with our founded acts of mathematical cognition. The monad is responsible for the constitution of the meaning of being of its objects, and in sciences such as mathematics and logic this is a non-arbitrary, rational constitution. Some further details, however, are needed. It does not follow from the fact that we can be directed toward objects that the objects actually exist. In order to claim that the objects exist we need to have evidence (*Evidenz*) for them, and evidence is provided by intuition. Our mathematical intentions must be (partially) fulfilled, fulfillable, or frustrated. Short of that we do not have knowledge of the mathematical objects toward which we are directed. This we will discuss in further detail in the next chapter.

There are many abstract mind-independent objects in mathematics that we will not see at all at a given stage of our experience or that we will not see clearly in our basic, untutored experience. The unaided eye (i.e. unaided reason) is limited in what it can see. The development of mathematics over its long history depends on the fact that we have built up our conceptual tools to see more of mathematical reality or to see mathematical reality more clearly. This is just what we need to continue to do. In accordance with the kind of rationalism we find in Leibniz and Husserl, it is possible to clarify concepts or meanings in mathematics and logic. It is in these sciences that we can in principle attain clarity and exactness. This will not be a matter of just developing the universal characteristic and *calculus ratiocinator*. Mechanical calculation has its place, but the claim is that it is not the whole story when it comes to how we know things in mathematics. We know things in mathematics on the basis of acts of reason, but reason is not exhausted in mechanical calculation. As Gödel says, we have to focus "more sharply on the concepts concerned by directing our attention in a certain way, namely onto our own acts in the use of these concepts, onto our powers in carrying out our acts, etc." In his 1961 text Gödel says we should look to Husserl's transcendental phenomenology for help with this task. Transcendental phenomenology takes such concepts as data instead of eliminating them or covering them over with leftward philosophical schemes. The incompleteness theorems do not imply that we should give up on the goal of deciding open mathematical problems that can be formulated clearly. They do not imply that there are absolutely undecidable propositions. We just need to continue cultivating the capacity for reason that we already possess, without undermining this capacity with (leftward) philosophical doctrines that imply we are capable

[6] See, e.g. chapters 13 and 14 by Rosado-Haddock in Hill and Rosado-Haddock 2000.

of so much less. What we need to do to solve even such stubborn problems as CH, to use the telescope analogy, is to build better conceptual tools, a better conceptual telescope whose lens is not distorted by leftward philosophical schemes, in order to intuit more of abstract mind-independent$_2$ mathematical reality. It is in this manner that we will exalt reason.

In the next chapter I turn to some of the distinctive features of acts of reason as *founded* acts of consciousness. This will bring us one step closer to understanding how it is possible for human minds to know about abstract mind-independent objects of mathematics and logic.

6

Constituted Platonism, Reason, and Mathematical Knowledge

In this chapter I address a central issue for Gödelian platonic rationalism: how can human minds know about mind-independent abstract or ideal objects and truths of mathematics and logic? The groundwork for this has been prepared with our discussion of the *epochē*, the intentionality of human consciousness, constituted platonism, and the intentional difference principle. It is argued below that the human mind (monad) can know about mind-independent abstract objects and truths, in the sense of mind-independence$_2$, on the basis of *reason*. Acts of reason in mathematics and logic, viewed genetically, are *founded* on lower-level cognitive activities, in particular on sense perception. The chapter opens with a discussion of founding and founded types of conscious directedness and meaning constitution. Once this part of the chapter is complete I will discuss the question of how knowledge of transcendent abstract objects and truths of mathematics and logic is possible. Taking mathematical practice seriously is an important part of my argument. The idea is to start with actual classical mathematical practice, including practice in higher set theory, and then ask how it is possible. The distinction between empty and fulfilled intentions will be employed at this level, in conjunction with the points we have already made about constituted platonism, in order to explain how knowledge based on intuition is at work in the kind of cognition involved in mathematical and logical practice. The view of intuition here is far from being mystical. On the contrary, it is at the center of a theory of knowledge and objectivity in the sciences of mathematics and logic. Once we describe in more detail the difference between categorial (or eidetic) intuition and straightforward sensory intuition it will become clear just how commonplace categorial intuition is. I then take up the manner in which rational or "eidetic" intuition in mathematics and logic is analogous to sensory intuition, and dispel some misunderstandings about the analogy. In chapter 8, I will address several points about the limits of reason and science in order to ward off worries about falling back into naive and unfounded views of the sort that afflicted earlier forms of rationalism.

§ 1. Reason and founded acts of consciousness

Let us start by focusing on some points in the phenomenology of sensory perception. Sensory perception is of concrete objects, but it appears that a kind of *abstraction* is involved at even the lowest levels of awareness of sensory objects. There is a constantly changing flow of sensory input but in ordinary sensory perception we are not directed toward this sensory material. We are directed instead toward particular objects that are experienced as identical or invariant through this constantly changing flow of sensory input. There is *one* object that is "*formed*" or "*synthesized*" out of the multiplicity of data reaching our senses. The notion of abstraction here can still be kept fairly simple: "abstraction" simply means "not attending to" something. Here we can already speak of a kind of abstraction that takes place *passively* or automatically in our experience. Thus, to not attend to the complex flow of sensory material itself in sensory experience is to abstract from it, although the abstraction in this case does not involve a separate act. It is not an *act* of abstraction that we perform. There are, however, acts of abstraction, as we will see in a moment, and these are associated with the kind of activity or "spontaneity" involved in reason. In sensory perception we are not directed toward the constantly changing flow of sensory input, even though it is part of the concrete whole of the experience. To perceive a concrete sensory object, we might say, is already to abstract from a difference (or some differences). This is evidently what Gödel had in mind when in the passage cited at the beginning of chapter 4 he wrote about how, even in the case of physical experience, we *form* our ideas of objects on the basis of something that is immediately given. Sensory input is immediately given but is "formed" in certain ways. Gödel then says that in the case of mathematical intuition we also form our ideas of objects on the basis of something which is given except that here it is not, or not primarily, the sensations. Mathematical intuition need not be conceived as a faculty giving an *immediate* knowledge of the objects concerned. Gödel says that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, that is, the idea of the object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. We do not create everything in our experience. What is given, however, is only a moment of the whole of the experience. The experience is underdetermined by what happens to be given. He goes on to say that evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas. One of his remarks about forming our ideas of objects (Gödel 1964, p. 268) is that this is the function of a kind of synthesis, of generating unities out of manifolds (e.g. *one* object out of its various aspects). According to Husserl, this kind of synthesis is present in both sense perception and the type of conception involved in mathematics and logic, although it is a passive synthesis in the case of the most basic kind of sense perception, and an active synthesis in the case of acts of reason. All of the ideas of Gödel mentioned here will be found in our account below.

Sense perception might be a founding stratum for certain types of scientific thinking but it is not the same thing as scientific thinking. Rather, we should say that scientific thinking in the case of natural science is "founded" on sense perception. Scientific thinking of this type employs theories and conceptions that are built up on the basis of sense perception. It utilizes unique capacities of reason. The intentionality of reason, unlike the intentionality of the lowest levels of sense perception, involves activity and not mere passivity, spontaneity and not mere receptivity, volition or will, various forms of abstraction, reflection, generalization, variation, and idealization. Relative to sensory intuition, the correlative type of intuition would be mediate, not immediate. The intentionality of reason involves the organization of concepts, rigor, order, clarity, and a drive for discovery. At the highest levels in mathematics and logic it involves proof, justification, ideation, and the search for necessities and not mere possibilities, exactness, axiomatization, and formalization. The nature of the constitution of the meaning of being in the case of acts of reason in mathematics and logic involves these various kinds of founded acts, with their correlative contents and invariant objects.

In the *Logical Investigations* (*LI*), *Experience and Judgment* (*EJ*), and other works, Husserl gives some examples of how abstraction and generalization extend beyond sensory experience, and he begins to distinguish the different types of abstraction and generalization that are involved in different types of conscious directedness. In the Sixth *Logical Investigation*, which Gödel recommended, Husserl characterizes the difference between ordinary "straightforward" sensory perception and "categorial intuition" (see Husserl 1973, Investigation VI, §§ 40–52). Pure categorial intuition includes intuition of objects such as natural numbers, sets, and states of affairs. A related distinction is drawn between sensory and categorical abstraction. There can also be forms of abstraction in which sensory and categorial elements are mixed. There are extended discussions of these ideas in the *Logical Investigations* and other writings, and *Experience and Judgment* is a book-length treatment of the genesis of the constitution of categorial or "predicative" objects from "prepredicative" experience. I will only be able to scratch the surface in this chapter, but it is now time to follow Gödel's suggestion and to investigate categorial or eidetic intuition in more detail.

In the *LI* Husserl makes the following kinds of observations to distinguish straightforward sensory perception from the consciousness of abstract or ideal objects. In sensory perception the external object appears at once, as soon as our glance falls upon it. The unity of perception in this case does not arise through our own active bringing together of the parts that we perceive, but is rather an immediate fusion of part-intentions without the addition of any kind of thetic character other than perception. In ongoing straightforward perception, the sense perception is merely extended. The unification of percepts is not the performance of some new act through which there is consciousness of a new object. The same object is meant in the extended act that was meant in the part-percepts taken singly. There is as it were a passive unity or synthesis of identification through these acts, but this is not the same thing as a separate *act* of identification. In ordinary perception we are directed toward the "real" perceptual object, not toward an ideal or abstract identity. The identification that occurs in ordinary ongoing *perception* is not like the case, for example, where we *judge* that a = a or a = b. Different thetic characters are involved here and, hence, a different kind of directedness. Husserl says that the latter situation involves a new *relational* activity that is not involved in straightforward perception. The perceptual series can be used to *found* such a new relational act when we articulate our individual percepts and relate their objects to one another. In this latter case the unity of continuity holding among the individual percepts provides a basis for a consciousness of identity itself, but the latter type of consciousness is different from straightforward perception. It involves a judgment, not mere perception.

As we saw above, Husserl distinguishes "real" from "ideal" objects. In clarifying what a straightforward percept is we are also clarifying what a sensible or "real" (as opposed to "ideal") object is. A real object is just the possible object of a straightforward perception. Sensible objects are, in general, the possible objects of sensible intuition and sensible imagination. We can then also define real part, real piece, real moment, and real form. Each part of a real object is a real part. In straightforward perception the whole object is explicitly given while each of its parts is implicitly given. Every concrete sensory object and every piece of such an object can be perceived in explicit fashion. As mentioned in chapter 4, Husserl distinguishes "pieces" of objects (independent parts) from "moments" (non-independent parts) of objects (Husserl 1973, Investigation III). A moment is just a part of an object, the existence or perception of which depends upon the existence or perception of the whole of which it is a part. Moments of objects, unlike pieces, are incapable of separate being. Husserl thus holds that moments are "abstract" in the sense that we may be able to consider them by themselves in thinking and in language, even though they cannot be taken to exist by themselves. The apprehension of a moment, and of a part generally, *as* part of a whole already points to a founded (and, hence, no longer straightforward sensory) act, since part-whole awareness already involves a relational kind of act. Husserl says that in these cases the sphere of "sensibility" has been left behind and the sphere of "understanding" or "reason" entered. "Understanding," as opposed to sensibility, can be defined as the capacity for categorial acts, that is, acts directed toward categorial objects. It is in understanding (in this sense) that we rise up to scientific thinking.

When a sensory object is apprehended in a straightforward manner, it simply stands before us. The parts that constitute it are in it but are not made our explicit objects. We can also, however, grasp the same object in an explicating fashion in articulating acts in which we put certain parts into relief. Relational acts can then bring the parts into relation to one another or to the whole. This is a new kind of active "synthesis" that we bring to the situation, since relational acts are not themselves straightforward acts of perception. Only through such new modes of interpretation will the connected and related members assume the character of "parts" or of "wholes." Now the articulating acts and the act we call "straightforward" are experienced together in such a way that new objects, the relationships of the parts, are constituted. What we have here, therefore, are some of the most basic categorial objects given to us in categorical acts. All such relationships of wholes to parts or parts to parts are of a categorial "ideal" nature. At this stage, further cognitive activities can take place. Suppose, for example, that we now abstract from the contents of particular part/whole relationships and *formalize* the basic relation "x is a part of y" as "$x \leq y$." To *judge* that $x \leq y$ is no longer part of straightforward sensory perception. Rather, it requires a detachment from the here and now of the ongoing sensory perception itself and a reflection on it. In accordance with the intentional difference principle (IDP) discussed in chapter 5, § 6, we are directed in a different way. In a similar manner, external relations such as "a is to the right of b" or "a is larger than b," are given as states of affairs in founded acts, and we might then go on to represent these relations in a formal manner. In these sections of the Sixth Investigation of the *LI*, Husserl gives two additional examples of the consciousness of "higher-order" objects: collectiva and disjunctiva (see § 51).

Husserl thinks that consciousness of this kind requires a voluntary activity of the transcendental ego (monad), a modification of the will that is a function of the kind of interest or goal that we have. There is, as it were, an ability and a will to detach from the ongoing sensory directedness. This is necessary if there is to be *knowledge* of objects or states of affairs, for knowledge requires the capacity to return again and again to the same objects or states of affairs over time. We must be able to repeat the acts and processes that give us the object or state of affairs. This is not involved in our experience if we are just caught up in the flow of sensory experience moment after moment. A will to *know* is distinct from merely sensing or even from merely believing.

What we have in these cases is a shift in directedness (*noema*) from concrete sensory perception to the awareness of something else, an abstract object or state of affairs. To be conscious of the fact that "the pen is to the left of the computer" is no longer to be engaged or immersed in straightforward perception itself. An abstraction has taken place in which some features have been lifted out of the concrete sensory experience. In such an abstraction we stand in a different relation to our sensory environment. We are, so to speak, more distant from it. In being at an even further remove from the immediate sensory environment, the gap between the human subject and the object or state of affairs becomes even more apparent. As Heidegger (Heidegger 1927) emphasized, this gap between subject and object does not seem to be present at all when we are immersed in the most basic kinds of everyday skilled activities and engaged practices. It is precisely because there is such a strong emphasis on skilled activities and engaged practices in Heidegger's work, however, that one finds little help there with problems about conscious directedness toward abstract mathematical and logical objects, and related problems concerning scientific reason.

In the examples at hand there is already a form of directedness toward an abstract state of affairs, albeit one that is not very far removed from concrete sensory experience. We are simply adhering to the IDP here. We are to take the object-directedness just as it presents itself and not as something else. We are to attend to what is meant in our acts and how it is meant. As a consequence we must now allow that there is directedness toward an object or state of affairs that is "abstract." It is useful, however, to distinguish *pure* abstract objects or states of affairs from those that are mixed with sensory components. Husserl discusses such a distinction at some length.

In the examples of categorial awareness mentioned thus far the synthetic acts are so founded on straightforward percepts that our awareness is subsidiarily directed to the objects of the founding percepts insofar as it brings them into a relational unity. There are, however, other kinds of categorial acts in which the objects of the founding acts are not intended by the *noema* of the founded act. In this case there is also a kind of "abstraction," but it is not an abstraction that amounts, for example, to setting some real moment of a sensory object into relief. Instead, Husserl claims, there can be consciousness of a genuine universal as an "ideal" object. This level of Husserl's analysis would have been of special interest to Gödel since Gödel says in the Gibbs Lecture, as we saw above, that he thinks Aristotelian realism in the case of mathematics and logic is not tenable (Gödel *1951, p. 321–22). On the Aristotelian view, concepts or other abstract objects would have to be parts or aspects of material things. In Husserl's language they would be non-independent parts (moments) of "real" objects. Gödel says that he is interested, however, in showing that platonism about concepts is the only tenable option in the foundations of mathematics and logic.

How should we understand the claim that there can be consciousness of a genuine universal as an "ideal" object? It seems, for example, that we can be directed toward the redness of a particular rose or toward the redness of a particular wagon. This is directedness toward a non-independent part of a "real" object in each case. This would be different, however, from being directed in our awareness toward the ideal universal or essence "redness" itself. To *mean* something universal is different from meaning some sensory particular or some "real" part of a sensory particular. This is just the IDP at work again. We might become aware of the identity of a universal on the basis of different individual intuitions, as in the case just mentioned. Thus, we would again have an abstraction from a difference, but now at an even higher level. The essence or universal is *meant* as the ideal unity through this multiplicity. Even if this kind of abstraction, which Husserl sometimes calls "ideational" or material "eidetic" abstraction, rests on what is individual, it does not for that reason *mean* what is individual. The awareness of the essence as an essence is not awareness of a sensory individual or a sensory part. Moreover, it is not awareness of a mental entity. It is not a deliverance of inner sense or introspection but of reason. An essence is meant as something objective, not subjective. In accordance with constituted platonism, it is possible to mean and even intuit such objects (invariants) as mind-independent, but only in the sense of mind-independence$_2$.

In mapping out the territory of conscious directedness we are thus led to distinguish between different types of abstraction. These are, in effect, different types of meaning constitution. There can be abstraction of essences pertaining to sensory objects (e.g. redness). Husserl holds that such essences are inexact or vague. In *Ideas I* (Husserl 1913, § 74) he says these are "morphological" essences. Unlike such morphological essences, however, mathematical and logical essences are held to be exact or precise. They involve certain kinds of idealizations. There can be mixed acts of understanding or reason in which sensory elements are combined with categorial forms. For example, we can consider the relational act of taking an object x to be an element of a collection y (in the sense of a set), which we represent formally as $x \in y$. There could then be directedness toward a set (collection) of chairs. Chairs are concrete sensory objects but sets themselves, as mathematical objects, are not. There could also be directedness toward a set of natural numbers in our thinking. In the latter case, no sensory elements are involved at all. We have, as it were, pure categorial abstraction. Pure logic, pure arithmetic, pure geometry, pure set theory, and so on, contain no sensory concepts in their theoretical fabric. In being directed by "x is a triangle" or "x is a natural number," for example, we are not directed toward sensory objects. We do not *mean* sensory objects. We can see that sensory objects are not meant in this case by just reading books on geometry or number theory or holding discussions with geometers or number theorists and taking note of which kinds of properties they in fact predicate of their objects and which kinds of properties they do not predicate of their objects.

In his analyses of abstraction Husserl also employs a form/matter distinction, where "matter" does not refer to sensory or physical matter but, roughly put, just to any judgment or proposition with content. For example, the following pair of propositions contain references to exact but "material" Euclidean essences: "all triangles are three-sided" and "all rectangles are four-sided." If we say that these two propositions have the *same logical form*, say $(\forall x)(\Phi x \rightarrow \Psi x)$, then once again we are abstracting from a difference. In this case, however, we are dealing with yet another kind of abstraction, a formal abstraction (see, e.g. Husserl 1913, 1929). We are moving from "material" propositions to the logical form of those propositions. Formal abstraction is thus different from the kind of abstraction involved in obtaining (exact or inexact) "material" essences. The relation of form to "matter" is not the same, for example, as the relation of genus to species or of species to instance. If an object is red then it is colored, and if something is colored then it is in some sense extended. There is an abstraction here as we move from the more to the less specific, but it is different from a formal abstraction. Now if we consider each of the three cases—"all triangles are three-sided," "all rectangles are four-sided," and $(\forall x)(\Phi x \rightarrow \Psi x)$—we see that the mind is directed differently in each case. In the case where I think that $(\forall x)(\Phi x \rightarrow \Psi x)$ the directedness, as purely formal, is quite indeterminate, although it is not completely lacking in determinateness. It is, for example, different from the directedness involved in thinking that $(\exists x)(\Phi x \wedge \Psi x)$.

Similarly, when we say that the propositions "all primes >2 are odd" and "all composites >2 are even" have the same logical form we are abstracting from a difference. The natural numbers themselves would be abstract objects (invariants) and so would the concept of natural number and the concepts or essences "being prime," "being odd," etc. Where we say that "the pen, the pencil, and the eraser are *three*" and that "the cup, the cigarette, and the match are *three*" we are abstracting from some differences. According to Husserl, each natural number is itself an ideal or eidetic particular.

Husserl begins to develop in this manner, and on the basis of the IDP, an account of directedness toward the abstract objects of pure mathematics and pure logic. To not take the IDP seriously is to lose track of basic facts about human consciousness. We are to take the directedness of consciousness just as it presents itself and not as something else. If I express the content of an act of consciousness as "x is a primitive recursive functional" and the content of another act as "x is a transfinite set" then these two contents clearly yield different kinds of directedness, although there might be some relations between the contents. The fact that a cognitive act is directed in a particular way means that it is not directed in other ways. If my thinking is directed by "x is a primitive recursive functional" then there are a host of ways in which it cannot be directed. Some contents will be compatible with a given content and others will not. As we have been saying, contents are intensional entities and, as such, their identity is not to be determined extensionally.

Contents always present us with a perspective on an object or situation. Consciousness is perspectival and we are finite beings (monads) and cannot take up all possible perspectives on an object, a state of affairs, or domain. We do not experience everything all at once. Our knowledge is thus typically incomplete in certain ways, although we might see how we could continue to perfect it. The incompleteness theorems can be viewed as establishing this last statement, under certain specific conditions, on a scientific basis.

Some of the most important concepts that are relevant to Gödel's work are as follows: "x is a sign token," "x is a sign type," "x is a natural number," "x is an arithmetic function," "x is a primitive recursive functional of finite type," "x is an ordinal $< \epsilon_0$," "x is a predicatively defined set," "x is an impredicatively specified set," "x is a constructible set," and many other concepts of higher set theory. With each of the contents (meanings) mentioned we have a different form of directedness. In the case of "x is a sign token" the directedness is toward a sensible, physical object. In the case of "x is a sign type" this is no longer true. There is already a shift in content and thetic character. There is still a close relation to sensible, physical objects in this case, however, so that sign types might be referred to as "quasi-concrete" (see Parsons 1980 and 2008, especially chapters 5 and 7). In the other cases we have forms of directedness that have been built up in non-arbitrary ways on the basis of further acts of abstraction, reflection, generalization, idealization, and imaginative variation.

In the case of the employment in classical mathematics of the concept "x is a natural number" and the other contents just listed, the directedness is toward entities that cannot be construed as physical or mental. As was suggested above, we might appeal to the incompleteness theorems themselves to argue for this. In the analysis of mathematical cognition we also appeal to the IDP. To be directed by "x is a natural number" is not to be directed toward a sensible, physical object, although there is a sense in which we can represent the objects toward which we are directed in this case by sensible, physical objects such as sign tokens. The properties of sensible, physical objects, however, are different from the properties of natural numbers. Sensible, physical objects have, for example, spatial and temporal properties. In pure number theory, however, we are not directed toward objects in physical space. As we suggested above, there are categories of contents, along with contents and thetic characters, that are either compatible or incompatible with one another. In being directed by "x is a natural number" we are also not directed toward objects with temporal properties. In the science of classical number theory, as it is actually practiced, natural numbers are not meant as entities that have temporal duration. They are therefore unlike mental processes (*noeses*), since mental processes (*noeses*) do have temporal duration. Of course there will be mental processes and subjective ideas (*noeses*) associated with all of these contents if there is to be directedness at all but, by the IDP, we should not substitute the mental process or the subjective idea for the content (*noema*) of the process and, hence, for the natural number that is intended by the mental process. The mental process or idea is subjective. It belongs to a particular subject at a particular time and place. The natural number is objective. It is, in other words, not immanent to consciousness but is meant as transcendent. To say it is meant as transcendent is not only to say that it is not itself a mental entity or process but also that we do not know everything we could possibly know about it. Our knowledge of it is not complete. It is also to say that we cannot shape it through acts of will in any way we like. It resists us in certain ways. All of these features point to the abstractness and mind-independence of the intended objects in these cases.

Formalization, as one type of abstraction, amounts to lifting the form off of material propositions. We could then focus on the forms of propositions in a theory, obtaining an entire theory-form. The theory-form arrived at in this manner is intended or meant as "ideal," not as "real." Thus, we are not speaking of a formal system here as a concrete object that is determined by various sensory qualities associated with sign tokens (e.g. specific marks on paper with specific colors, etc.) or even sign types. Once we have a theory-form it can be rematerialized in many possible ways. Think of all the ways in which the simple form $(\forall x)(\Phi x \rightarrow \Psi x)$ can be filled in with different content. There can obviously be many different categories or regions of such material fillings. One very broad distinction among these, as we indicated, is that some will involve exact essences and some will involve inexact essences. Formal abstraction is to be distinguished from material eidetic abstraction, and the latter may yield exact or inexact essences, depending on whether idealization is involved or not. Sensory essences are vague or inexact ("morphological"), while the concepts of classical logic and mathematics are held to be exact. A kind of perfection or ideality is involved in mathematical concepts that is not involved in concepts of "real" sensory objects. Correlatively, the laws of mathematics and logic are exact, while not all of the laws of the empirical sciences have the same kind of exactness and precision. Instead, they involve probabilistic elements and are more or less vague generalizations from, or typifications of, sense experience. Indeed, as a good rationalist, Husserl thinks that the exact sciences establish norms toward which the natural sciences strive in trying to become more exact, more precise, more clear, distinct and certain, or more perfect. The sciences of the "real" approximate the sciences of the "ideal" more or less closely.

Once we have obtained some abstractions, then these abstractions can figure into further abstractions, reflections, and variations. Without going into details here, we can say that in the development of the sciences of mathematics and logic, abstractions and idealizations are built on top of abstractions and idealizations in a structured and organized way. They are not just thrown together haphazardly. In pure mathematics these abstractions no longer involve sense data but, as Gödel indicates in the passage quoted above, they are themselves the "hyletic data" or the basis for further higher-level acts of cognition. They are conditions for the possibility of higher-level acts of scientific cognition. Although they do not fully determine these higher-level acts and their contents, they do constrain them in certain ways. In some of his writings Gödel says that in mathematics it is arithmetic that is the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception in the case of sensory knowledge (e.g. Gödel 1944, p. 121). It provides a kind of founding stratum and constraint on our mathematical thinking. Presumably we should not be prepared to regard concepts or principles that contradict arithmetic as mathematical knowledge, while concepts or principles at higher levels that enhance, extend, or build upon our knowledge of arithmetic should, under the right conditions, be embraced.

I would argue, for reasons of this sort, that the phenomenology of the levels of human consciousness indicates that the human mind can, in a systematic and non-arbitrary way, abstract universals from concrete particulars, can subsume concrete particulars under universals, can relate universals to universals, or concrete particulars to concrete particulars. According to the IDP, to *mean* something universal is different from meaning some sensory particular. We might become aware of the identity of a universal on the basis of different individual intuitions. Thus, we would have an abstraction from some differences. The universal would be given as the ideal unity (invariant) through this multiplicity. Even if an abstraction rests on concrete individuals, it does not for that reason *mean* what is individual. Directedness toward a universal as a universal is not directedness toward a sensory individual.

The abstractions and idealizations involved in concepts of the infinite or transfinite are especially significant. Husserl argues that the concept of the (denumerable) infinite already has its origins in sensory perception, in the sense that we can come to see through reflection on our perceptual experience that we could always perceive more of a given perceptual object (Husserl 1939, § 51). Perception is perspectival and we never have a complete perception of an ordinary physical object. We actually have only a finite number of perceptions of an object, but it is part of the idea of perception that *we could always have yet another perception* of the object. Having a complete perception of a physical object, seeing it from every possible perspective, and coming to know everything about it that could be known, is already an idealization. Husserl says that the structure "and so on" that is so prevalent in mathematical and logical concepts thus already has a founding stratum in sense perception, even if it is not itself thematized in perception.

The conception of the transfinite involved in modern Cantorian set theory, however, involves further abstractions and idealizations. I will say a few more things about this below, but at the moment, setting aside questions about evidence, I will only note that I think we can at least be *directed* in our founded acts of reason toward transfinite sets. Cantorian set theory, as a part of mathematical practice, in fact exists. It is not meaningless. If it were, then no one could work in it and have their thinking directed by the basic concepts in it in order to obtain the many results that have in fact been obtained. One finds definitions, proofs, and theorems in higher set theory. These are products of reason and not of sense perception, "inner sense," mere imagination, or pure emotion. I think we do and can *mean* the infinite in set theory, even as actual and completed. We can look to set-theorists and the texts on set theory to determine how the objects are meant in this context. We will then find that they are meant in certain ways but not others. At a general level, the objects of set theory are not meant, for example, as spatial objects (in the sense of physical space), temporal objects, causal objects, and so on. No spatial, temporal, or causal language is used in describing the properties and relations of the objects or truths of set theory in our practice in this domain of thinking. The question we can then address, according to Husserl, is this: what are the conditions for the possibility of the constitution of the meaning of being of the objects of set theory, as this is found in set-theoretic practice? How is this meaning constituted in a rational, non-arbitrary way, in accordance with the general dictates of science?

The more difficult questions concern not the prima facie directedness and meaningfulness of set theory, but the kind of evidence we can have in set theory, the kind of intuition that can be operative in this domain of cognition. We have already seen how Gödel agrees that we cannot have a complete intuition of the objects of transfinite set theory as individuals. He suggests, rather, that we can have a more or less clear intuition of the basic concepts of set theory, of the concepts that direct us toward the objects. In his paper on Cantor, he argues that this intuition of concepts is of course only partial and in need of further clarification. At the same time, he wants to hold that we do have some *knowledge* in set theory. Not everything is conjecture, mere hypothesis, or unrealized expectation. Even if we do not have a perfectly clear and distinct grasp of the basic concepts of set theory we are still able in the better understood parts of the theory to distinguish knowledge from illusion. We can, for example, distinguish correct from incorrect proofs, solve particular problems, and so on. The picture that emerges from the work of Gödel and Husserl is that we have an intuition of the concepts that make set theory possible, that the concepts themselves are exact but the intuition of them is only partial and is not fully determinate. The intuition can in principle be made determinate, clear, and exact enough to obtain solutions to particular problems in which the concepts figure. We can clarify our grasp of the concepts (see Tieszen 2002). The intuitions can be expressed in formal systems, but operating only in the purely formal system will not yield solutions to open problems because one then abstracts away from the meaning or content of the problems. If we hold to platonic rationalism about mathematics and logic, then we can certainly recognize that there are undecided problems in these domains, but the problems need not be *absolutely* undecidable because we might be able to further develop our intuition in order to solve them. According to the rationalism of such a platonic rationalism it is in principle possible for reason to solve a mathematical problem if it can conceive the problem clearly, that is, if the problem has a clear meaning for us. I will have more to say below about the kind of evidence we can have in higher set theory.

§ 2. Formalization with and without reason

Suppose we try to view all of mathematics in terms of concrete, "real" formal systems, as Hilbert and Carnap tried at one time to do. If these programs had been successful then we could presumably eliminate all references to pure abstract objects in mathematics and logic, and eliminate appeals to categorial or rational intuition. The finite sign configurations of the "real" formal system, and the rules (or conventions) for manipulating the sign configurations, are all given, at least as tokens, in what we have been calling straightforward sensory perception. The conscious directedness in this case would be toward something given (at least in principle) in straightforward sensory perception. A "proof," on this interpretation, is just a finite sequence of finite sign configurations called "sentences," formed on the basis of the finite sets of rules or conventions of a particular formal system. In effect, the idea is to reduce the conscious directedness toward abstract objects that is part of mathematics as it is given and practiced to directedness toward concrete sign configurations and purely combinatorial operations on such objects, or to substitute the latter for the former. Given what we have said above, we might think of this as formalization without reason.

In Hilbert's case such formalization is not possible, because if C is Hilbert's finitist, concrete mathematics, as in chapter 2, then it follows from the first incompleteness theorem that if C is consistent, then the Gödel sentence for C cannot be decided by C even though it is true. It follows from the second incompleteness theorem that if C is consistent, then C cannot prove CON(C). Given the way we characterized C in chapter 2, it follows that deciding the Gödel sentence for C or proving CON(C) must require objects or states of affairs that *cannot* be completely represented in space-time as finitary, concrete, real and straightforwardly intuitable. In other words, deciding the Gödel sentence for C, or proving that CON(C) requires an appeal to the meanings or content of sign configurations, to objects or states of affairs that are in some sense infinitary, ideal, or abstract, and not straightforwardly intuitable.

Either there are such "abstract" entities or not. Suppose that there are no such entities or that there is no consciousness of such entities. Then it would follow, since we must remain within the perspective of C, that we must stop or "become static" with respect to deciding the Gödel sentence for C or obtaining a proof of CON(C).

That is, we could not decide some clearly posed mathematical problem. I mean we could not decide it unless the decision were to be made arbitrarily or on grounds that do not employ mathematical reason. However, for some T (e.g. PA) we in fact do have proofs of CON(T) and decisions of related problems. Contradiction. Therefore, there are such entities and we must have some consciousness of them.

The level of abstraction required to prove Con(C) is not very substantial compared to some parts of mathematics. It would certainly be more substantial, for example, in the case of consistency proofs for real analysis or in proofs of the consistency of ZF + the continuum hypothesis (CH) or of ZF + $\neg$ CH. In these later cases we ascend to even greater heights of abstraction, idealization, and reflection, in order to extend the science of mathematics with new concepts, methods, and results.

We might also think of the role of formalization in Carnap's view of mathematics as syntax of language as a case of formalization without reason. In Carnap's case, it is also not possible to reduce the conscious directedness toward abstract objects that is part of mathematics as it is given and practiced to directedness toward concrete sign configurations and purely conventional combinatorial operations on such objects, or to substitute the latter for the former. The second incompleteness theorem shows that it is not possible, as Carnap requires, to distinguish the allegedly pure linguistic conventions that govern truths of mathematics and logic from empirical truths unless we have a consistency proof for the linguistic conventions. For if the linguistic conventions are inconsistent then they will imply all sentences, including all empirical sentences. We could no longer have Carnap's sharp division between empirical sentences and sentences of logic and mathematics. The consistency proof would itself have to be either mathematical or empirical in nature. If it is to be a mathematical consistency proof then, as in the Hilbert case, it must involve objects or concepts that are no longer concrete and finitary in nature, and some modes of knowledge other than immediate sensory perception must be in play. Some kind of content or appeal to meaning must be involved. If, on the other hand, the consistency proof is empirical in nature, then we have to rely on empirical assertions to support the claim that the syntactical conventions are consistent. These empirical assertions have content. They are about something. In this case the claim that mathematical assertions are a priori, void of content, and true on the basis of linguistic conventions is undermined.

We could say that the incompleteness theorems, read against the philosophical background from which they emerged, show that the conscious directedness toward abstract objects that is part of mathematics as it is given and practiced cannot be reduced to directedness toward concrete sign configurations and purely combinatorial operations on such objects. Mathematical intuitions (or, if you like, categorial intuitions) of the type described in this chapter cannot be eliminated. Put in terms of the IDP, we cannot simply substitute *noemata* pertaining to concrete syntax and the properties of concrete syntax for *noemata* of the sort that are found in standard meaningful mathematical (as distinct from purely formal) practice. There is a sense in which we could do this, however, if our formalized theories were shown on finitist grounds to be consistent and complete, for then we could in a finitistically acceptable way identify formal provability with mathematical truth. Gödel therefore concludes that we cannot eliminate either the mind or abstract objects if we want a satisfactory foundation of mathematics. Many people have the sense that meaningful mathematics extends beyond C. From my point of view this simply means that many mathematical statements express *noemata* that are not captured in purely syntactical terms. There are, as it were, intended abstract *meanings* and objects of mathematical theories that are taken as data in their own right. Note, by the way, that it does not follow that strict formalization is not important or not useful. It is very important but it just does not give us the whole picture. It is part of a larger whole.

The incompleteness theorems show that mathematical (e.g. arithmetical) truth (and even proof) cannot be understood completely in terms of a purely formal "real" notion of proof. Directedness toward concrete formal proofs is, by the IDP, different from directedness toward mathematical truths. Purely formal proofs in this sense are concrete and are given in straightforward perception. Moreover, they are always relative to a particular formal system. It is possible that there is some kind of "absolute" concept of provability or of truth, but it has to be distinct from the purely formal notion of proof since the latter is always relative to a particular, specified formal system. In any case, there is certainly an informal, pre-formal, or "contentual" notion of proof at work in our mathematical thinking. After all, strict formal systems were first created only about one hundred years ago, but the entire history of mathematics is replete with proofs that are typically quite rigorous. A kind of informal rigor has been present in mathematics for a long time.

In various places in his writings, going back to the 1930s, Gödel distinguishes the purely formal and relative concept of proof from the concept of proof as "that which provides evidence." In one paper from the 1930s (see, Gödel 193?, p. 164), for example, Gödel already says that the incompleteness theorems show that in the transition from evidence to pure formalism something is lost, and that the incompleteness theorems therefore do not undermine Hilbert's rationalistic optimism, that is, his conviction in the solvability of every precisely formulated mathematical question. In the 1961 text that we discussed in chapter 3 we saw that Gödel regards solvability in this sense as one of the two rationalist elements in Hilbert's thinking. The second element, which also eludes us if we translate it into a purely formal context, is that a proof is supposed to provide a secure grounding of a proposition. The problem here, as we saw, is that purely formal proofs in mathematics cannot provide such a secure grounding unless we know that the formal systems in which they are carried out are consistent, but we can only prove that they are consistent by using means that are more powerful that those available in the given formal system, and we have then stepped outside of the finitisitic formalist's conception of knowledge. In various writings Gödel thus says that the concept of proof as that which provides evidence is an "abstract concept" of proof. Elsewhere, he gives as an example of an "abstract concept" the concept of proof, understood in the non-formalistic sense of "known to be true." A concept is said to be "abstract" if it does not refer to sensory objects (see, e.g. Gödel *1951, p. 318, footnote 27). A similar view is expressed in version III of the Carnap paper (Gödel *1953/59, p 341, footnote 20), where Gödel says that the concept of proof, in its original contentual meaning, is an abstract concept. This is the concept according to which a proof is not "a sequence of expressions satisfying certain formal conditions, but a sequence of thoughts convincing a sound mind." Here Gödel says that the abstract and the transfinite concepts together form the class of "non-finitary" concepts. As late as 1972 (Gödel 1972, p. 273, footnote e), Gödel refers to the concept "p implies q" as an "abstract concept" when it is understood in the sense of "From a convincing proof of p a convincing proof of q can be obtained."

If we combine these comments of Gödel on proof and evidence with the Husserlian view that it is rational intuition that provides evidence in mathematics, then it is natural to understand proofs in the "abstract" sense as the fulfillments of mathematical intentions. In other words, proofs are just expressions of mathematical intuitions. They are a source of objectivity in mathematics. Mathematical intentions without proofs are empty, but "proofs" without mathematical intentions, such as purely formal proofs, are blind. This will be discussed in more detail below (see also Tieszen, forthcoming).

These comments about mathematics and formalization *with reason* should not be especially surprising. It seems that for a wide range of cases we are more adept at identifying and responding to abstract data than we are to identifying and responding to the immediate, individual concrete phenomena in which they are expressed or exemplified. In written and spoken communication it is typically not the written or sounded physical word tokens toward which we are directed. We are not directed, for example, toward the sizes of sign tokens, colors of the tokens, etc. We are also not directed toward the word *types* (which are already minimally abstract). Of course we could be so directed, but then it would be for a different set of interests or purposes. What typically happens in ordinary communication is that these things recede into the background and are often not noticed at all. We might find ourselves directed toward one of these features of the sign tokens if, for example, it somehow blocks our grasp of a person's meaning. What we are directed toward and attempt to grasp, unless there is some kind of breakdown, are the meanings of the expressions. If it is the sign tokens that are concrete, then it appears that the meanings toward which we are directed could not count as concrete. They are not given to straightforward sensory perception. Now consider all of the things we do not notice about the physical, concrete sign tokens or sounds in ordinary communication. There is an abstraction at work here. To grasp the meaning is again to abstract from many underlying differences in the immediate, individual concrete phenomena through which the meaning is expressed. We are again to think of this in terms of the IDP and shifts in directedness. To try to interpret away such abstract data in our experience is to lose track of some basic facts about our conscious life.

§ 3. Knowledge of abstract mind-independent objects in mathematics

Let us now address in more detail the question of how knowledge of mind-independent abstract or ideal objects is possible. It will be useful to start with a brief recapitulation of the ideas on constituted platonism from chapter 4. We discussed the phenomenological *epochē* and the idea that human consciousness constitutes the meaning of being of the world of experience of (i) physical phenomena, (ii) mental phenomena, and (iii) abstract or ideal phenomena. Focusing on (iii) in particular, we said that abstract or ideal objects are constituted as such by consciousness, by the monad. Whatever things are, mathematical or logical objects included, they are as experienceable things. It is experience alone that prescribes their sense. The genuine concept of the transcendence of mathematical or logical objects can only be derived from the contents of mathematical or logical experience itself, and for expressions of this experience we must turn to the texts and discourses found in mathematical practice. Nothing exists for me otherwise than by the actual and potential performance of my own consciousness. We said that whatever is given as an existing object in mathematics or logic is something that has received its whole sense of being from my intentionality. There is no conceivable place where the life of consciousness could be broken through so that we might come upon a transcendent mathematical or logical object that had any other sense than that of an intentional unity (invariant) making its appearance in the subjectivity of consciousness. If what is experienced has the sense of "transcendent being," then it is experience itself that constitutes this sense. If what is experienced has the sense of being "ideal," "non-mental", "acausal," "unchanging," "non-spatial," "partially given," and "non-physical", then it must be mathematical consciousness itself that, in a non-arbitrary manner, constitutes this sense. If mathematical or logical objects are considered to be objects that existed before we became aware of them, and that would exist even if there were no human subjects, then it must be the case that this sense of mathematical or logical objects is constituted in mathematical practice in a motivated and non-arbitrary manner. If transfinite sets are considered to be abstract, mind-independent objects that cannot be constructed individually, and cannot be given to us as actual completed totalities, then it must be the case that they are constituted with this meaning by the monad. If we consider the general features of mathematical or logical platonism that we outlined in § 3 of chapter 4, then we can say that mathematical or logical objects in classical mathematics possess these features, except that we must add the crucial qualification that they are constituted non-arbitrarily in this manner in the consciousness of the transcendental subject (monad).

It was in this manner that we arrived at the wholly unique kind of "platonism" about mathematics or logic that I called "constituted platonism." In a remarkable new twist in the age-old debate about platonism, we look to the transcendental ego (monad) as the source or origin of platonism about logic and mathematics, where logic and mathematics are built up non-arbitrarily through founded acts of abstraction, idealization, reflection, variation, and so on. Just as the "realism" about physical objects is not a naive realism, so this unique kind of platonism about mathematical objects is not a naive platonism.

It was on the basis of the *epochē* that we distinguished a weak from a strong sense of "appearance-independence" or "mind-independence." There could not be mind-independent objects in the strong or absolute sense of lying outside of all possible experience (or appearance). We simply cannot say anything about the possibility of such radically independent things-in-themselves. There can be no knowledge of objects that are mind-independent in the stronger sense. The "unbridgeable gap" between the human subject and the abstract objects of mathematics and logic that critics of platonism never fail to mention certainly applies to this notion of mind-independence. On the other hand, there are mathematical objects that are mind-independent in a weaker sense, according to which objects are meant as non-mental invariants in a manifold of experience. There *can* be knowledge of objects (invariants) in this sense. We could be mistaken about objects in our experience, so that we could at some later stage come to see that we had been under an illusion, that we had mere appearances at an earlier stage. The distinction between weak and strong senses of "mind-independence" or "mind-dependence" will then affect the formulations of mathematical platonism and mathematical idealism. In transcendental phenomenology we must set aside the strong (or naive) sense of mind-independence.

We represented these distinctions in a diagram that indexed the notions of mind-dependence and mind-independence. Mathematical or logical objects are

mind-dependent$_1$		mind-independent$_1$
$\Box$		$\Diamond$
/ \\		
mind-dependent$_2$	mind-independent$_2$	
#	*	

Figure 6.1 Positions on realism and idealism

We said that it is not possible to hold that abstract objects are mind-dependent$_1$ and mind-independent$_1$. Most of the debate in the philosophy of mathematics literature, including recent debate, seems to take place at this level. For example, Brouwer might be placed at $\Box$, and some philosophers seem to think that Gödel belongs at $\Diamond$. It is also inconsistent to say that abstract objects are mind-dependent$_2$ and mind-independent$_2$. Formulated in this way, mathematical platonism and mathematical idealism are still incompatible. It is not inconsistent, however, to hold that abstract or ideal mathematical objects are mind-dependent$_1$ and mind-independent$_2$. Indeed, mind-independence$_2$ falls under mind-dependence$_1$. What this means is that mathematical or logical platonism, in this sense, is compatible with transcendental phenomenological idealism. Platonism in this sense, which I called "*constituted platonism*" or "*constituted realism*," is concerned with non-arbitrarily or rationally motivated constituted mind-independence in our experience in mathematical practice. What we are now to investigate from within the *epochē* is the *constitution of the sense* (*Sinn*) of mind-independence on the basis of founded acts of cognition. We need to investigate the rationally motivated *constitution of the meaning of the existence of ideal mind-independent mathematical objects*.

If Gödel is at the position marked by "◇" in our diagram, then I think he is indeed in serious trouble. It is an indefensible position. It is an altogether different matter, however, if he really does want to take the phenomenological method of the *epochē* seriously and is in the position marked by "$*$". The position marked by "$*$" includes the new combination of ideas that Gödel liked in Plato, Leibniz, Kant, and Husserl, as discussed in chapters 1, 3, 4, and 5. In accordance with *transcendental* phenomenology (Husserl relative to Kant), the *monad* (Husserl relative to Leibniz) constitutes the meaning of being of the objects of mathematics and logic as *abstract* and *mind-independent* (Husserl relative to Plato).

As we look back from this viewpoint, we can say that the standard formulations of the positions of mathematical platonism and mathematical idealism are too simple. They are ambiguous. If we make the distinctions just indicated then the assertion that abstract objects are mind-independent$_1$ is naive (or pre-critical) mathematical or logical platonism and is untenable. The assertion that mathematical or logical objects are mind-dependent$_1$, with no further qualification, is naive (or pre-critical) idealism and is untenable. The third position that we outlined combines a transcendental phenomenological idealism and a mathematical or logical platonism in which neither the platonism nor idealism is any longer naive. We have left naive metaphysics behind. It also follows that transcendental phenomenological idealism is not compatible with naive mathematical or logical platonism.

While it is not possible for human subjects to know about objects that are mind-independent$_1$, it *is* possible for there to be knowledge of objects that are mind-independent$_2$. We look to the constituting activities of consciousness that make this possible. Furthermore, there can be directedness toward objects that are mind-independent$_2$ and *abstract*, because our mental life is not locked onto only what is given in the most basic founding acts of sense perception or in introspection. We look to structures and capacities of consciousness that make this possible. There are founded acts of *reason* involving acts of abstraction, reflection, idealization, formalization, variation, and so on, in which, in accordance with the IDP, we are directed toward and mean such abstract or ideal objects. It is certainly not possible to know about objects that are mind-independent$_2$ and abstract on the basis of sense perception or "outer sense." What we can be directed toward and know about on the basis of sense perception is various kinds of physical objects in space and time, along with properties of, and relations concerning, such objects. It is also not possible on the basis of introspection or "inner sense" to know about objects that are mind-independent$_2$ and abstract. On the basis of introspection or inner sense we can be directed toward, and know about, some of our own private particular mental phenomena, and even here one has to exercise caution in claims about what is known. In any case, knowledge of something private, subjective, and particular is distinct from the knowledge delivered by reason, which is knowledge of something objective and universal. It is also not possible to know about objects that are mind-independent$_2$ and abstract on the basis of pure imagination. Scientific reason in mathematics and logic is much more constrained than pure imagination. Reason is concerned, for example, not only with the imagination of pure possibilities, but with finding necessities relative to possibilities. There is no scientific knowledge that is a function of mere imagination. Furthermore, it is not possible to know about such objects on the basis of a mere feeling, a hunch, or some kind of mystical intuition.

It is reason that makes possible the directedness toward, and knowledge of, such objects. Mathematics and logic are our best examples of domains of knowledge. It is in these sciences that we typically have the most exact, clear, distinct, unchanging, and certain knowledge. Mere conception, conjecture, or hypothesis does not count as knowledge. Knowledge requires more that mere intention. What it means to say that there is *knowledge* of the objects or truths of these sciences is that we are not only directed by intentions toward the invariants that we conceive in mathematics, but that some of these intentions can be fulfilled, partially fulfilled, or frustrated. Knowledge involves intentionality, and some of our mathematical intentions can be (partially) fulfilled in intuition or frustrated. When we lack knowledge they are neither fulfilled nor frustrated.

We characterize rational (categorial) intuition in mathematics as the fulfillment, partial fulfillment, fulfillability, or frustration of mathematical intentions P, where these intentions are the products of founded acts of scientific reason. Intuition is defined in terms of fulfillment of intentions. It is the source of evidence and objectivity. There is nothing mystical about it. We regard frustration as a particular kind of fulfillment, namely the fulfillment of the negation of the original intention P, so that we have $\neg$ P. The original intention is seen to be illusory. Rational intuition is governed by consistency, $\neg (P \wedge \neg P)$, and can sometimes be extended or cultivated to a point at which we can *decide* for a given intention P whether P or $\neg$ P. The decision need not be purely mechanical (see also chapter 7), but it is also not arbitrary. It involves reason and directedness toward abstract or ideal objects or meanings by way of contents of our acts, and it is made by finite beings (monads). There is rational intuition in the context of mathematics when some invariant object or generalization is not only conjectured or postulated, not only merely intended, but is made *present* to consciousness, just as an ordinary perceptual object can be merely intended or can be made present to consciousness. That which is meant but is absent in the mere intention or conjecture is made present. We "see" it. It is revealed, as in the example of putting the geometric concepts in § 6 of the last chapter in order. In some cases the revelation is only partial, indistinct, not completely clear. In cases of the frustration of an intention, an altogether new and even unexpected or surprising invariant may thrust itself upon us.

On this account, rational intuition in mathematics is understood in terms of the fulfillment of mathematical intentions, or the filling of mathematical concepts. It is important not to associate other meanings with the term "intuition," especially since this term has been used and abused in so many ways over the years. The argument is that if rational intuition were not present in mathematics, including higher set theory, then mathematics would appear to us as no more than a set of empty hypotheses, conjectures, expectations, or problems. As a matter of fact, however, mathematical practice does not appear this way. Expectations in mathematics, including higher set theory, are realized or frustrated, problems are solved rationally and scientifically in one way or another, conjectures and hypotheses are confirmed or refuted. Proofs are found. Proofs in mathematics are fulfillments of mathematical intentions. They are expressions of mathematical intuitions. In short, the intention/fulfillment structure is part of mathematical cognition, just as it is part of perceptual cognition. There are not only empty intentions in mathematics but, where there is knowledge, there are also intuitions. There is a direct analogy with sense perception: sense perception would appear to us as nothing more than a set of perceptual conjectures, unconfirmed hypotheses, or unrealized expectations if there were no sensory intuition. But it does not appear this way to us. Our practice shows that it does not.

Based on our earlier discussion of the difference between categorial intuition and "straightforward" sensory intuition, which depends on the intentional difference principle, it becomes apparent that categorial intuition in many domains is quite common. It could be argued that we are blind to categorial intuition or to the distinction between categorial intuition and "straightforward" sensory intuition to the extent that our study of different types of human consciousness is blunted by philosophies of mind that are insensitive to first-person conscious experience. As we have indicated, straightforward intuition is awareness of objects that does not present objects in their categorial form. It does not present objects with their parts, properties, relations, and so on, but in much of our (more reflective) experience and knowledge, objects in fact are presented with their various parts, properties, relations, and so on.

As we indicated in chapter 4, within or after the reduction we still have an appearance/reality distinction, and a knowledge/illusion distinction. We still have the distinction between existence and truth on the one hand and mere intention on the other. It is intuition that makes the difference. It is responsible for objectivity. The monad exhibits intentionality, and can have mere intentions (thoughts) in mathematics but also knowledge, where knowledge requires at least the partial fulfillment in intuition of mathematical intentions. We can distinguish knowledge from illusion even in higher set theory. The invariants in mathematics are meant as abstract or ideal, not as sensory or as mental in nature. This is not naive idealism. The invariants are not constituted or not meant as "real," as temporal, changing, as mental, etc. The objects and truths are not constituted as created by the mind. They are not meant that way. These invariants are *not sensed* in "outer sense" or introspected in "inner sense" but are *conceived*, reasoned about, are held to "exist," not just to be mere appearances or illusions, and are given as invariants *in our experience* in mathematical and logical practice in the sense of mind-independence$_2$. These invariants are not forever cut off from consciousness, from the mind, from knowledge, as in naive platonism (mind-independence$_1$). The consciousness of them is constituted by reason, through the intentionality of reason (intentionality and conception), the processes and structures of reason, which means the constitution is systematic, non-arbitrary, constrained, motivated, concerned with proof, justification, intrasubjective and intersubjective repeatability, convergence on the same results, clarity, exactness, rigor (not necessarily only formal), order, and discovery. In constituted platonism, but not naive platonism, the constituting activities of reason and the constituted objects are brought into correlation. They require one another. In naive platonism there is no such correlation.

Let us now recall Husserl's (Husserl 1929, § 100) formulation of the plight of the platonist that we quoted in chapter 1:

> no one ventured, or had the courage to venture, to take the *ideality of the formations with which logic is concerned* as the characteristic of a separate, self-contained, *"world" of ideal Objects* and, in so doing, to come face to face with the painful question of how subjectivity can in itself bring forth, purely from sources appertaining to its own spontaneity, formations that can be rightly accounted as ideal *Objects* in an ideal "world."

> For only then was one faced with the unintelligibility of *how ideal objectivities* that originate purely in our own subjectivities of judgment and cognition, that are there originaliter in our field of consciousness purely as formations produced by our own spontaneity, *acquire the being-sense of "Objects,"* existing in themselves over against the adventitiousness of the acts and the subjects. How does this sense "come about," how does it originate in us ourselves? And where else could we get it, if not from our own sense-constituting performance?

We want to know how subjectivity can in itself bring forth from sources pertaining to its own spontaneity, its own capacity for reason, formations that can be rightly accounted as ideal objects in an ideal "world." How does this meaning or sense come about? How does it originate in us? Where could we get this sense if not from our own meaning-constituting performance? We now have a glimpse of how these questions can be answered. How can the mind know about abstract, transcendent, mind-independent objects? The short answer is that this knowledge has its origins in our founded, rational, constrained meaning-constituting activities. Human reason, through its intentionality, constitutes the consciousness of such objects as having these characteristics. The mind constitutes the meaning of being of such objects. It is not like naive platonism. The mind intends or means objects this way in this domain of cognition. What we are describing here is a position about mathematical or logical *experience* in mathematical practice. *Constituted platonism is concerned with the kind of directedness (intentionality) involved in thinking and problem-solving in the practice of mathematics and logic.* We are focusing on a domain of intentional directedness in a certain practice, with its characteristics. Moreover, at the fundamental level what we are analyzing here is the *genesis* of mathematical and logical platonism. The way to bridge the gap between human subjectivity (monads) and mathematical and logical objectivity is to fill in the account of the kinds of founded intentional acts and processes—collection, comparison, abstraction, idealization, reflection, variation, and the like—that make the constitution of the meaning of being of mathematical and logical objects possible. These are the kinds of intentional acts and processes without which mathematics and logic, as actually given and practiced, would not be possible.

In order to understand the import of constituted platonism, consider the following two statements:

(i) There exist mathematical objects that are mind-independent and abstract.
(ii) Monads constitute the meaning of being of mathematical objects as mind-independent and abstract.

According to constituted platonism, we are not entitled, without qualification, to assert (i), but we are entitled to assert (ii). In like manner, we cannot assert without qualification that there are mathematical objects that exist even if there are no minds, or even if there never had been or never were any minds in the universe. We cannot say that there are actual transfinite sets that cannot be given to us as completely as individuals. These kinds of assertions need to be qualified in order to avoid naive platonism. What we *can* say is that in the science of mathematics *we constitute the meaning of being* of mathematical objects in a rationally motivated and non-arbitrary way as mind-independent and abstract. Similarly, *we constitute the meaning of being* of mathematical objects in a rationally motivated and non-arbitrary way as objects that exist even if there are no minds, or even if there never had been or never were any minds in the universe. We *constitute the meaning of being* of some objects in mathematics as actual transfinite sets that cannot be given to us completely as individuals. We have now shifted to the matter of meaning constitution, and each of these types of meaning constitution *is* possible. We ask the following kinds of questions: If mathematical objects are meant as objects that exist even if there are no minds in the universe, then where could this meaning come from? It could only come from us. It must be constituted by us. Moreover, meaning constitution is not a free-for-all in sciences such as mathematics and logic. The meaning constitution does not, and cannot, go just any way that we will it to go. It depends on strictures of reason itself, strictures involving rigor, evidence and justification, and not fantasy, emotion, wishful thinking, and so on.

The fundamental tenet of transcendental idealism, as we saw in chapter 4, is that there can be no object that is not the object of some possible consciousness. As I am reading this tenet, it rules out mind-independent$_1$ objects. Are we now saying, contrary to the principle of transcendental idealism, that there can be objects that are not the objects of some possible consciousness? No. We do not hold that there are mind-independent$_1$ objects. This would be a flat contradiction. We are saying that it is possible for the monad to constitute the meaning of being of objects of (classical) mathematics and logic as ideal and mind-independent, but now we are only talking about mind-independence$_2$. With this distinction the threat of contradiction vanishes.

The task is then to explain the conditions for the possibility of such meaning constitution in various parts of mathematics.

Knowledge of abstract objects has its origins in human reason, but constituted platonism is of course different from a host of other positions in the philosophy of mathematics and logic. Consider, for example, the position known as fictionalism. I think that fictionalism about mathematics is, in Gödel's language, a reflection of the leftward *Zeitgeist.* It is another variant of a leftward position on which the capacities of the mind are downsized or misconstrued. The philosophical psychology of fictionalism about mathematics blurs distinctively different cognitive structures and capacities that are revealed if we abide by the IDP and engage in a careful phenomenology of the various modes of human consciousness. The thinking or directedness involved in mathematics is not like the directedness involved in fiction. Fiction, in accordance with the IDP, has its own kind of meaning constitution or directedness. The thetic characters and contents involved in fictional thinking are different from those involved in mathematical thinking. If we consider examples of what would be regarded widely as fiction, then we see that the kind of meaning constitution or directedness involved in mathematics on the one hand and fiction on the other involves different skill sets and goals. The type of directedness involved in fiction does not involve proof and the kind of skepticism needed to distinguish correct from incorrect proofs. It is not concerned with issues of justification or with applications that are used to predict events and control the natural world. It does not have a cumulative history in which later results are built upon and require earlier results, it does not involve the collaborative work of communities of investigators on problems in the field, and it does not exhibit the dynamics of, and methodologies for, solving open problems that are found in mathematical thinking. It does not require symbolic or special formal notation in order to advance beyond all but the most primitive thinking in the domain. In general, the nature of the thinking involved in fiction is less constrained than mathematical thinking. I would argue that different cognitive capacities—reason, imagination, memory, volition, association, emotion—are involved in the two domains in different ways.

Constituted platonism is also quite different from any kind of conventionalism. Conventionalism about mathematics is just another leftward position. A central conclusion of chapter 3, based on Gödel's arguments against Carnap, is that mathematical truths are not just a function of syntactical conventions. We cannot substitute contents or *noemata* that direct us toward finite sign configurations and mechanical manipulation of such objects for contents or *noemata* that direct us toward abstract mathematical objects. Although Gödel's argument was directed toward Carnap's earlier views, I think it would not be difficult to develop arguments against other types of conventionalism about mathematics on the basis of our form of platonic rationalism. Once again, different thetic characters and contents are involved in thinking that is based on mere convention.

Constituted platonism is also not psychologism or any of the other leftward "isms" about mathematics and logic. It is a position about the a priori origin of mathematical and logical consciousness.

§ 4. An example of illusion and rational intuition in mathematics

Many examples of the operation of rational intuition in mathematics, of fulfillment and frustration in mathematics, could and should be considered. I think it is important to present such examples in order to fully illustrate the view and its nuances. Let us consider here, as one special example, Gödel's remarks on illusion and the paradoxes of set theory. This example should be compared with the snake/coiled rope example in chapter 4 in order to see some of the analogies between sense perception and rational intuition. It is an example involving the frustration of a set-theoretic intention that leads to a new, corrected intuition.

First, a brief word about the analogy with sense perception is in order. I think we need to distinguish the types of perceptual illusion that are correctable from those that are not correctable. When Gödel speaks of illusions in sense experience and mathematical experience, he does not seem to make such a distinction. In his paper on Cantor's continuum hypothesis (Gödel 1964, p. 268) he says that the set-theoretic paradoxes are hardly any more troublesome for mathematics than are the deceptions of the senses for physics. Now in the Gibbs Lecture (Gödel 1951, p. 321) he uses, as an example of a perceptual illusion, the case of a rod immersed in water. The argument he makes in this context is that the paradoxes of set theory do not refute platonism, just as the fact that our visual perceptions sometimes contradict our tactile perceptions (as in the case of the immersed rod) does not refute the claim that an outer physical world exists. In neither case should the presence of a contradiction make us doubt that there is a mind-independent reality.

Now in the set-theoretic case it appears that we can make a correction in the manner in which we are directed toward sets in our thinking. In the presence of the contradiction (paradox), we can shift to a new perspective on sets that does not appear to be contradictory. We abandon the naive comprehension principle in favor of, say, a replacement axiom. In the case of a rod immersed in water there is nothing we can do to eliminate the contradiction between the visual perception and the tactile perception. I can know that the rod is not really bent, but I cannot come to literally see that it is not bent. The knowledge that the rod is not really bent depends on tactile perception, certain conceptual contributions, and perhaps other data, and not on what is given in visual perception. These other contributions override, but do not correct, the visual perception in this case. There are many similar kinds of perceptual illusions that are not correctable, such as the Müller-Lyer illusion. Gestalt psychologists have highlighted many of these in their work.

Now compare these kinds of cases in which there is a non-correctable element in perception with the snake/coiled rope example that we considered in chapter 4. In this latter kind of case the perceptual illusion is correctable. Indeed, it is corrected by further perceptual intuitions, by what is given in sense perception. The intention that directs me toward the snake is frustrated in light of further perceptual intuitions. A material contradiction arises in my experience: it is not possible for there to be an x in my experience such that x is a snake and x is a coiled rope. The illusion is detected thanks to subsequent intuitions and a correction is made. Thus, I think we can draw an analogy between the snake/coiled rope example and the example in set theory of shifting from the naive comprehension principle to a modified principle that does not appear contradictory. In the set-theoretic case the illusion is detected on the basis of rational intuition, not sensory intuition. An initial intuition is corrected and adjusted by subsequent experience.

Let us consider the Russell paradox in particular. Suppose that at a certain stage of your experience in mathematics you take yourself to be directed toward a domain of sets that is determined by the generalization expressed in the "axiom" $(\exists x)(\forall y)(y \in x \leftrightarrow Py)$, along with the other standard axioms of set theory. At a later stage, however, you become aware of the fact that this generalization is illusory because it is possible to derive a contradiction from it. The need for a correction here depends only on the logical derivation of the contradiction: 1) Assume that $(\exists x)(\forall y)(y \in x \leftrightarrow Py)$. 2) $(\exists x)(\forall y)(y \in x \leftrightarrow y \notin y)$, from 1 by substitution. 3) $(\forall y)(y \in r \leftrightarrow y \notin y)$, from 2 by substitution. 4) $r \in r \leftrightarrow r \notin r$, from 3 by substitution. We might gloss this by saying that it is taken to be true at the earlier stage of the experience that for any concept Py there is a set containing all and only the objects falling under Py. But then what of the concept $y \notin y$? There can be no invariant object—no set r—answering to the intention "the set of all nonself-membered sets." The explosion of the *noema* that results from the contradiction has nothing to do with sense experience or the methodologies of the natural sciences.

What usually happens is that our experience in mathematics settles down so that we do not have a continuous series of illusory experiences of this sort. Otherwise, no world of mathematics would be constituted. Instead, there is typically a more or less harmonious (consistent) course of experience involving the transcendent objects of mathematics, or else we would be staring in the face of meaninglessness and irrationality itself. We can see here how it is possible to make an appearance/reality distinction after the *epochē*. Looking back on the experience, we can say that there was merely an appearance of a domain determined by the naive comprehension principle at the earlier stage in the experience, and that what we have "in reality" is a domain determined instead by a corrected and different generalization of the sort expressed in a separation or replacement axiom. We can distinguish knowledge from illusion, where knowledge depends on the evidence provided by intuition. The generalized intention by virtue of which we were directed toward the domain determined by the naive comprehension axiom is shown to be unfounded and frustrated.

Now consider the domain of objects toward which we are directed in our thinking by the corrected or modified generalization involved in the replacement axiom, along with the other axioms of ZFC set theory. Are we entitled to hold that our intentions in this case are fulfilled or at least partially fulfilled in intuition? Is there at least some filling of our concepts in this domain? Do we have at least some evidence in this domain? A full-scale treatment of these questions could fill another book. Here I will provide only a few general observations.

I think that in his later work Gödel would certainly answer these questions in the affirmative. The axioms of ZFC are not just meaningless finite sign configurations that we manipulate according to a finite set of formal rules. As Kreisel (Kreisel 1967, p. 144) has put it, Zermelo's analysis in his 1930 paper (Zermelo 1930) furnishes an instance of a rigorous discovery of axioms for the notion of set: "the intuitive notion of cumulative type structure provides a coherent *source* of axioms; our understanding is sufficient to avoid an endless string of ambiguities to be resolved by further basic distinctions, like the distinction ... between abstract properties and sets *of* something." Kreisel says, against pragmatism, that one does not have to put up with an ad hoc collection of different axioms for different purposes.

The axioms of ZFC, understood against the background of an iterative concept of set, direct our thinking in a certain manner that is non-arbitrary, rationally motivated, and replete with methods and proofs. We do not need to adopt a complete leftward skepticism about higher set theory just because some paradoxes turned up in early conceptions of the subject. The paradoxes, Gödel says, were used as a pretext for a leftward upheaval. One can read Gödel's emphasis on a "genetic" or "iterative" conception of set as bearing out the claim that, relatively speaking, we do have some kind of a rational procedure for constituting the objects of higher set theory, for sets are viewed as objects for which we (could) have evidence at various stages of our (possible) experience, even if we cannot be conscious of these objects as complete individuals. It is because we are directed toward objects such as sets by our meanings or intensions that it is possible to reason with intensions of mathematical expressions, even if the extensions of these expressions cannot be fully given in intuition.

Gödel is, in effect, arguing for such partial fulfillment of our set-theoretic intentions in the context of ZF in his paper on Cantor's continuum problem (Gödel 1947/64). I think it is important and useful to read Wang's comments (Wang 1974, chapter VI, and 1977) on the intuitive foundations of set theory in terms of fulfillment or partial fulfillment of our set-theoretic intentions in ZF. Much of the other literature that explicates and discusses the iterative conception of set can also be read this way.[1] There are of course many nuances and I think the literature shows that we do not yet have anything like a definitive account of the matter. As indicated in chapter 4, it is in cases

[1] See, e.g. especially Shoenfield 1967, Boolos 1971, Scott 1974, Parsons 1977, Hallett 1984, Tait 1990, 1998, 2005a, Kanamori 1997, Potter 2004, Koellner 2006 and 2009. Maddy's work (Maddy 1988a and 1988b) on what goes into making the axioms believable is also very useful. See also Martin 1998 and 2005.

of this type that the real work of constitutional analysis must begin. As Kreisel says, probably the first step is "*to recognize the objectivity of the basic notions* (subset, power set)... and then, if possible, to give a phenomenological analysis of these notions" (Kreisel 1969, p. 97).[2] By way of contrast, imagine what our experience in ZFC would be like if no such conception of set were associated with its axioms, but if we instead literally regarded the axioms as meaningless strings of signs to be manipulated according to purely formal definitions and rules of inference.

A number of general points can be made in favor of the idea that there is partial fulfillment of our concepts in this context. First, we can engage in some meaning clarification, as Gödel does, by drawing a sharp distinction between a "logical" conception of sets, according to which a set is obtained by "dividing the totality of all existing things into two categories" given a particular predicate, and a "mathematical" conception of sets according to which a set is obtained in stages by iterated applications of the operation "set of" to some given objects (Gödel 1947/64). On the iterative or mathematical view of sets, we build up sets by applying the "set of" operation to some given objects. I start with some given objects and form the set of those objects. I might then form another set of objects from the given objects. Now I can form the set consisting of these two previously formed sets. These and other processes of set formation governed by the axioms, which we can think of as (partial) fulfillment procedures, can continue indefinitely. Paul Bernays even described the mathematical conception of sets as "quasi-combinatorial" (Bernays 1935). In the case of the "logical" conception of set we have nothing that even approaches a fulfillment procedure in this sense. Gödel points out that our experience with the mathematical conception has not led to antinomies, while the same is not true in the case of the logical conception. The idea of having a procedure for obtaining objects comes out in a more determinate way in the case of the "constructible" sets in Gödel's consistency proof of axiomatic set theory with Choice and the Continuum Hypothesis, for the constructible sets are those which can be obtained by iterated application of the operations given by the axioms presented in Gödel 1940. Gödel thus speaks of the constructible sets as formed at stages in a "generating process," and of the generating process as continuing into the transfinite. Gödel made the interesting remark to Kreisel that he used "L" in the axiom "V=L" (i.e. every set is constructible) to stand for "lawlike." The procedure in the case of hereditarily finite sets, for example, is even more determinate.

In full set theory Gödel is speaking of transfinite sets, and this certainly involves various idealizations and abstractions that go beyond more elementary parts of mathematics. Although we do not need to claim that all of the members of transfinite sets can be given to us, this still involves extensions of the concepts of evidence, fulfillment, and of a procedure far beyond what is presently recognized as constructive in mathe-

[2] See Tieszen 1992 and also Hauser 2002, 2005, and 2006.

matics, as we indicated in chapter 2. What, however, was the motivation for going beyond constructivism in chapter 2? One of the main reasons was that we have rigorous, non-arbitrary, meaningful mathematical and metamathematical results that can be obtained only by leaving behind the limitations of finitism and constructivism. Gödel is arguing that we have some type or degree of evidence for these objects that we would not possess at all with the logical conception of set. For while the notion of a process or procedure is greatly extended in the case of the sets Gödel has in mind, there is still a significant difference from the logical conception of set, in which case there is no procedure of any kind for obtaining the objects at various stages from given objects. Surely there is a difference in evidence between having a concept, such as the naïve concept of set, which is provably contradictory, and having a concept of set which at this stage of our experience is not known to be contradictory but which, on the contrary, has some stability and history in our experience, and has supported many results and the developments of many methods.

Gödel's papers on constructive mathematics show that he distinguishes different degrees of evidence for mathematical objects. In chapter 2 we noted some of his remarks on degrees and types of evidence in mathematics. Thus we might say that at the present stage of our experience in set theory we do have evidence for objects such as transfinite sets, as these objects are understood on an iterative conception, but that, on the basis of criteria such as adequacy, clarity, distinctness, and apodicticity, it is different from the kind of evidence we have for finite objects such as natural numbers. I think this does not mean that we need to give up on the ideal of security or certainty in mathematics, even at the level of higher set theory. We do not need to abandon the rationalist idea that a proof should provide a secure grounding for a proposition, or that it should be a sequence of thoughts that convinces a sound mind. It is just that mathematical intuition does not yet deliver complete clarity and distinctness everywhere in set theory. Gödel of course suggests that, even if objects such as transfinite sets cannot be completely given as individual objects in intuition, there can still be reflection on and scientific analysis of the concepts involved.

In the case of the paradox, subsequent experience shows that there was no domain corresponding to the generalization involved in the naive comprehension principle. It is not the case that I was really aware of sets as determined by the naive comprehension axiom at the earlier stage. It only appeared that I was. What seems to be mind-independent, given the evidence thus far, is the domain determined in part by the new, modified generalization. This could in principle be overturned, however, in future experience. That is, there might be mathematical evidence (experience) in the future that would show us that it is also not a correct axiom. Its being a correct axiom is not absolute, even if the sets toward which we are directed by it are given as existent and mind-independent in accordance with all of our evidence thus far. Our evidence that the axiom is correct is in this sense presumptive. What is given in mathematical intuition in this kind of case could be overturned by subsequent mathematical (but not sensory) intuition.

While allowing that mathematical axioms would typically be overturned, if they are overturned at all, by virtue of mathematical intuition, one might ask whether there are special cases in which a tentatively held or presumed axiom could be overturned in sensory perception. For example, in the topology of knots, a tentative axiom might lead to a conjecture that one knot could not be turned into another, but we might then in practice find a way of doing it. My analysis of the "practice" in which we might overturn a tentative *mathematical axiom* by finding a way to transform one knot into another is that, strictly speaking, this could not be done on the basis of straightforward sense perception. What Husserl calls "categorial" acts would have to be involved. Sense perception is, of course, involved in handling the physical knot and manipulating it, but I would argue, on the basis of the analysis above, that sense perception is only a moment of the categorial awareness that must be present. The acts that make it possible to overturn an *axiom* would already have to be categorial. Hence, mathematical intuition is needed for the determination. This happens to be a case in which categorial acts are still mixed with a sensory moment. It seems that cases of this kind are typically found only at the very earliest stages in the origin of mathematical theories.

What our example of the paradox shows, and this would also have to be true of other examples, is that from within the *epochē* everything is indeed understood as appearance or phenomenon, and that appearances are corrected or verified only by further appearances. Within the sphere of appearance, however, we can still distinguish the "real," the transcendent, or the mind-independent from the "merely apparent," the immanent, or the mind-dependent, on the basis of what stabilizes or becomes invariant in our experience. We can still distinguish knowledge from illusion. This is a key idea of transcendental phenomenology, and it holds for both sensory experience and mathematical experience. There are illusions and corrections and refinements in mathematical experience, just as there are in sensory experience. We cannot somehow get outside of appearances, however, to an appearance-independent mathematical thing-in-itself. We can just drop the idea of a noumenal platonic mathematical world. The "existent" will be that for which we have evidence across places, times, and human monads. This will not hold for the "merely apparent." Rational justification depends on evidence. Imagine a form of mathematical experience in which nothing ever stabilizes or becomes invariant. This would be a form of mathematical experience that is without reason, that is, it is not mathematical at all. It would be experience in which there is no order and no rational connection among the contents of consciousness. No world of mathematics would be constituted. We are nonetheless not entitled to say that what is stable or invariant in the field of consciousness is the final, absolute reality. At best, the notion of "absolute reality" might be preserved as an infinite ideal. Thus, transcendental phenomenology recognizes an appearance/reality distinction after the reduction that allows for a kind of realism, only it is not naive realism. It is also not a naive idealism for the same reason: it makes an appearance/reality distinction after the *epochē*.

Of course there are many open problems, conjectures, hypotheses, and expectations in mathematics. Consider the difference, mentioned earlier, between the statement of the continuum hypothesis (CH) itself and the statement that CH is consistent with the axioms of ZFC. We have a proof of the latter statement. It is not on the same level as CH. Statements of open problems, conjectures, hypotheses, and expectations in the science of mathematics are not, however, meaningless. Our thinking in such cases is typically not meaningless or without directedness. Rather, our mathematical meaning-intentions are simply neither fulfilled nor frustrated. They are "empty." We do not have the relevant knowledge or evidence. We cannot yet distinguish knowledge from illusion. Our thinking is directed by way of the *noemata* or contents of our acts in these cases, but we do not know which invariants exist or, in the case of propositions, which are true. This parallels the situation in which we do not have knowledge in sense perception but only expectation or conjecture.

Gödel was especially interested in this situation in the case of set theory. One can see his rationalistic platonism as strongly motivated by open problems in mathematics, and in set theory in particular. In his paper on Cantor's continuum hypothesis, he says that reflections on the (iterative) concept of set "of a more profound nature than mathematics is used to giving" (Gödel 1964, p. 257) are required in order to solve open problems, such as CH, in set theory. We need to further clarify our intuition of basic concepts of set theory (see Wang 1974, pp. 187–219). Consider, for example, Cantor's characterization of the primitive concept of set. In 1883 Cantor "defined" a set as "a Many that allows itself to be thought of as a One."[3] In order to have a set, it has to be possible to think of the many (multiplicity) as a one without contradiction. Although we have a group of axioms (or groups of axioms) regarding such a conception of set, and many interesting and remarkable results, the conception is still quite open in some respects. There appears to be a yawning gap and a vast range of indeterminacy between what we presently know, based on the results that have been proven, and the limit condition that *any* multiplicity that can be thought of as a one without contradiction is a set. We do not fully understand this latter generalization. Gödel of course wanted to eliminate some of this indeterminacy by finding a new evident axiom or axioms that would enable us to solve the continuum problem and other open problems of set theory. One would not want a "decision" of CH to depend on an accidental property of sets, a property one could imagine sets not possessing. Hence, one might ask, for example, whether it is essential to the (iterative) concept of set that sets be well-founded, constructible, that there be measurable cardinals, supercompact cardinals, that the axiom of determinacy hold, or that the axiom of quasi-projective determinacy hold. One can formulate axioms that decide CH but, at least at present, none of these is

[3] See Cantor 1883. A nice history of set theory can be found in Ferreirós 1999. For some historical and conceptual connections between Cantor and Husserl in particular, see Hill 1997.

(intersubjectively) seen as essential.[4] They do not force themselves upon us. It need not be the case that an axiom candidate forces itself upon us in one fell swoop. An axiom might become evident only as we build up new groups of concepts. We have some understanding of the meaning-intentions expressed by proposed new axioms P, but none of these meaning-intentions can presently be regarded as fulfilled. Gödel's view is that we do not, and cannot, just constitute new axioms, solutions to open problems, and the development of new areas of mathematics in any way we like. The constitution is always constrained or determined in some respects, just as sense perception is always constrained and determined in some respects, even if it is underdetermined by the existing "matter." This will presumably be true whether a new axiom is justified intrinsically or extrinsically.

The philosophical psychology of open problems in mathematics has its own distinctive features. New conceptions are needed in order to solve open problems. We have been saying that what we intuit is always a (partial) function of our concepts. It is by building up new groups of concepts that we are able to see new things, things we would not be able to see without these concepts, as in our earlier telescope analogy. As in other parts of science, we expect new things to be revealed to us on the basis of new founded acts of cognition (see also chapter 7). It seems that various unconscious factors are involved in the creative process but, insofar as one can consciously and actively seek to develop new conceptions, the role of imagination and free variation is very important. Solving open problems is not just a matter of reason, but rather of the imagination of possibilities in the service of reason. The solutions to open problems and other new developments in mathematics should be a function of the capacity of the monad for putting imagination in the service of reason, not of the capacity of the monad to sense physical objects, to merely introspect, to produce mere conventions, fiction, pure formalisms, and so on. In the case of set theory, the idea would be to attempt to intuit enough of the essence of the concept of set through the determination of new properties of sets to allow a decision of CH.

To close this section, I would like to note that some readers of Gödel's comments on Husserl seem to be under the impression that Husserl's phenomenology is supposed to provide the magical solution to problems such as CH. In my view, this is setting the bar much too high. No philosophy can offer a method for solving deep open problems in mathematics. What I think can happen is that certain philosophical attitudes or positions are much more conducive to, and helpful for, thinking about such problems, while other positions are probably even harmful in this regard. Gödel's particular remarks about the iterative concept of set are only a part, and arguably not even the most important part, of a much broader picture of the philosophy of mathematics and

[4] The technical and philosophical work concerning new axioms is fascinating but I cannot consider it in any detail here. See, e.g. Steel 2000, Maddy 2000, Feferman 2000, H. Friedman 2000, and Woodin 1994, 2001a, 2001b, 2005.

logic that emerges from his engagement with the ideas of Husserl, Leibniz, Plato, and Kant that I have been developing in this book.

§ 5. Sensory intuition and rational intuition: the analogy

On what grounds can we hold that there is an analogy between sense perception and rational intuition, given the position we have outlined? In both cases finite, limited conscious subjects (monads) are directed toward objects by virtue of the contents or meanings of their acts. In both cases there is intentionality, aboutness. There is an act/*noema*/*noesis*/hylé (or "matter")/object structure. Both mathematics and empirical science have content. It is not the case that one has content but not the other. In both cases the objects toward which subjects are directed are invariants through a manifold of experience, but in the one case the invariants are given in sense experience and in the other case they are given in mathematical experience. In both cases these invariants are meant as transcendent and mind-independent$_2$. They are not meant as mental entities. The invariants are known perspectivally and, typically, only partially or incompletely. We could come to know more about the objects, their properties, their relations to other objects, by extending our experience. In both cases there can be empty intentions that direct us toward objects. These intentions determine an associated horizon of possible experience compatible with the intention. The intentions can be fulfilled or frustrated in future courses of experience. In both cases there can be knowledge but also illusion. There is a possibility in ongoing experience of correction, refinement, and adjustment. What it means to say that objects exist in either case is that invariants have emerged and acquired stability and a history in our experience in the domain of experience in question. A world has been constituted. Both sensory intuition and rational intuition are regulated by the possibility of improving our understanding and deciding currently undecided problems relevant to their domains, insofar as there is an interest in knowledge acquisition. In neither case are the objects toward which we are directed our own free creations. Both types of intuition involve constraints. There are phenomena that resist our will, so that we are "forced" in certain directions. Surprise is possible in both cases. The horizons of our acts in either sense perception or rational intuition, for example, include certain possibilities but exclude others. If further experience presents something that is inconsistent with the range of possible experience in the horizon of an act, then a shift to a new intention or *noema* is required.

One of the differences between sensory and rational intuition is that rational intuition requires *founded* acts and contents. As discussed in §1of this chapter, there are founding and founded modes of intentionality in our experience. The invariants toward which we are directed in the two cases are, of course, different. We adhere to the IDP. The invariants are constituted as having different properties in our experience.

In the case of physical objects they are meant as causal, changing, temporal, spatial, inexact, and "real." They come into being at some time and eventually pass away. In the case of objects of classical mathematics they are meant as acausal, unchanging, non-spatial, omnitemporal, exact, and ideal. They are constituted as objects that existed before we became aware of them, and that would exist even if there were no human subjects. They are constituted as objects for which we need not have expressions at a given time in this or that language, and that need not be objects of human consciousness. If there is to be *knowledge* of such objects then of course there must be consciousness of the objects, and there must be a means for expression of this consciousness.

On this kind of view, Gödel can say that "the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much reason to believe in their existence." They are just as necessary in obtaining a satisfactory system of mathematics as physical bodies are in obtaining a satisfactory theory of sense perceptions (Gödel 1944, p. 128). In the case of the reality toward which we are directed in our thinking in ZF set theory, Gödel says, as we saw, that "Cantor's conjecture must be either true or false, and its undecidability from the axioms known today can only mean that these axioms do not contain a complete description of this reality; and such a belief is by no means chimerical, since it is possible to point out ways in which a decision of a question, even if it is undecidable from the axioms in their present form, might nevertheless be obtained" (Gödel 1947, p. 181). The way to decide Cantor's conjecture is to obtain a more complete description of the set-theoretic reality involved through clarification of our intuition of the basic concepts by virtue of which we are directed toward that reality.

Gödel (Gödel *1953/59-III, p. 352) says that the "inexhaustibility" of mathematics, which we discussed in chapters 2 and 3, makes the similarity between reason and the senses even closer because it shows that there exists a practically unlimited number of independent "perceptions" also of this "sense" (reason). He continues,

> The inexhaustibility of mathematics appears, not only through foundational results [such as the incompleteness theorems], but also in the actual development of mathematics, e.g. in the unlimited series of axioms of infinity in set theory, which are analytic (and evident) in the sense that they only explicate the content of the general concept of set. That such a series may involve a great (perhaps even infinite) number of actually realizable independent rational perceptions is seen in the fact that the axioms concerned are not evident from the beginning, but only become so in the course of the development of mathematics.

In order to understand the first transfinite axiom of infinity, for example, one must first have developed set theory to a considerable extent. It is worth noting that Gödel is speaking here of the intrinsic necessity of axioms, as distinct from extrinsic justification. This distinction is discussed in more detail in chapter 8.

This analogy, Gödel says, comes pretty close to the true state of affairs, except that this additional "sense"—reason—is not counted as a sense by its detractors, because its objects are quite different from those of the other senses. Gödel says that, with sense perception, we know particular objects and their properties and relations, while with mathematical reason we perceive the most general (namely "formal") concepts and their relations which are separated from space-time reality, insofar as space-time reality is completely determined by the totality of particularities without any reference to the formal concepts. Mathematical concepts can be introduced by rules for handling symbols on the basis of empirical facts concerning physical symbols, but in reality the situation is just the opposite: the rules for the use of symbols are so chosen that they express properties of previously conceived mathematical concepts or objects.

Despite their remoteness from sense experience, Gödel says, "we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true." If Gödel is claiming here that an iterative conception of set, presumably a maximal iterative conception, forces all of the ZFC axioms on us, then he is overstating the case. Even so, we cannot take sets to be just anything. The objects toward which we are directed in set theory are not our own free creations. There is resistance to our ability to will anything we like. Our intuition of sets is constrained or "forced" in certain ways, just as our intuition of physical objects is constrained in certain ways. We cannot change our intuition at will, or make an object be anything we want it to be. In the case of sets, numbers, mechanical procedures, geometrical objects, and so on, this is shown by the way our concepts of these objects have been sedimented over time, and by the process of determining necessities from among the possibilities regarding these objects. Our experience with these objects has stabilized enough to permit, for example, of some degree of formalization, axiomatization, solution of open problems, and other research developments. Gödel thus says that

I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics. That new mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis are perfectly possible was pointed out earlier. (Gödel 1964, p. 268)

Finally, as we saw above, Gödel says that

It should be noted that mathematical intuition need not be conceived as a faculty giving an *immediate* knowledge of the objects concerned. Rather, it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, i.e., the idea of the object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only reproduce and combine those that are given. (Gödel 1964, p. 268)

Mathematical intuition is not "immediate" in the way that sensory intuition is. We have discussed some of the ideas behind this claim earlier in this chapter and also in chapter 5. Sensory intuition involves passive synthesis of part-percepts while, genetically speaking, rational intuition is a founded form of intuition that requires the active or "spontaneous" employment of reason in an effort to achieve *knowledge* as an abiding acquisition. We do not freely create all of the elements involved in the intuition of objects. In the case of either sensory or rational intuition, our awareness of objects is formed on the basis of given data or given "material," but the given data in the case of mathematical intuition is not, or not primarily, the sensations. Formal and structural features are already involved in sense perception. What Gödel says about how our awareness of objects is formed on the basis of given data in mathematics is that "Evidently the 'given' underlying mathematics is closely related to the abstract elements contained in our empirical ideas." Once we obtain some abstractions from sensory experience, in the various ways indicated in § 1, then these very abstractions provide the material in our thinking for further abstractions, generalizations, idealizations, variations, and so on. Our higher-level acts of directedness are built up in this manner. "It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality" (Gödel 1964, p. 268). The presence of sensations in us, on the basis of which we form our awareness of physical objects, is due to a certain kind of relationship between ourselves and reality. The presence of the abstract data on the basis of which we form our awareness of mathematical objects is due to another kind of relationship between ourselves and reality. This other kind of relationship between ourselves and reality is a founded relationship. On the reading I am suggesting, the relationship is not to be understood in terms of the kind of directedness (intentionality) involved in sense perception or "inner sense," but rather in terms of the kind of intentional directedness due to founded acts of reason. It is not purely sensory and passive. In the case where there is evidence for mathematical objects, it is a relationship of the constituting subject (monad) to the constituted objects. It is the relationship that is revealed through the examination of the IDP at higher levels of human knowledge. In the next sentence of the passage I have been quoting, Gödel says that the question of the objective existence of the objects of mathematical intuition is an exact replica of the question of the objective existence of the outer world. What would be relevant in both cases is whether we have evidence for the objects (invariants) of our cognitive acts as this would be provided in sequences of acts carried out through time, whether we have procedures for the fulfillment of our intentions.

Looking back from this perspective, it is possible to see how we have entangled ourselves in certain insurmountable problems about mathematical knowledge. One of the central problems is supposed to be this: how could we be causally related to abstract or ideal objects? The question assumes that only a causal theory of knowledge or of reference is in play. The answer is that we, of course, cannot be causally related to abstract or ideal objects. We do not need to be causally related to every object toward which we are directed in our thinking in order to have knowledge. *Sense perception* may involve causal connection, but *conception* does not need to involve causal connection in order to be veridical or in order to be justified. The methodology of the *epochē* shows us that there is a deeper invariant structure of human consciousness that runs through all modes of conscious directedness involved in knowledge, and that the category of causality is appropriate for, and specific to, some of these modes but not others. What comes to the fore in all cases of knowledge is whether we do, or do not, have evidence for objects or invariants as this is provided in sequences of acts carried out through time. In the case of sense perception, we can point out that the only way we could even identify the perceptual object to which we (believe we) are causally related is through ongoing perceptions that either correct, or fail to correct, that identification. Thus, at a later stage in my experience I might see that the object I perceived earlier as a snake is really a coiled rope. What is regarded as the correct attribution—"I am causally related to a coiled rope"—is a function of what stabilizes as an invariant in our sensory experience. The category or concept of causality is involved in the constitution of sense perception but it is not involved in every kind of constitution. It is the wrong category in the case of mathematical experience. In mathematical experience objects are not meant as causal. To apply this category in the case of mathematics violates the thetic character and the content of acts of mathematical cognition.

It is possible to explain why the category of causality has been misapplied in this way. One reason is simply that the focus of a lot of recent epistemology has been exclusively on sense perception. This is probably related to the fact that we have developed an interpretation of the world in natural science that has been very successful in its own domain, the domain of the sensed natural world, of causal relations between natural objects, although even here the role of causality has in recent times been faced with some anomalies. On the whole, though, we have had tremendous success in natural science. Many things have been revealed to us through the interpretation of the world involved in natural science. This interpretation indeed tends to crowd out all others and we then insist on applying it everywhere, which of course covers over, or forgets about, human subjectivity and the directedness of human minds toward abstract objects of logic and mathematics. To the extent that the interpretation becomes exclusive, the problem of "knowing" about abstract objects in mathematics becomes intractable (see chapter 7). We find ourselves asking how brains could be causally related to abstract objects.

From my point of view, it is not at all surprising why this question cannot be answered. There is a reason for wanting to suspend or bracket natural sciences of the mind such as neuroscience, with their adoption of the natural attitude. The reason is, of course, not to avoid neuroscience in particular or natural sciences of cognition in general. We need such important sciences. The reason is rather to avoid a reductionistic, eliminativist, and one-sided *philosophy of mind* that forgets about human consciousness and intentionality. Such natural sciences are directed toward their own objects but they abstract away from, or forget about, analyses of *experience of objects*. They presuppose that we are related to objects, and are therefore not themselves in a position to account for this relation.

I would argue that Benacerraf's (Benacerraf 1973) frequently cited problem of squaring a causal account of knowledge with a platonic view of mathematical objects is simply an artifact of the leftward *Zeitgeist*.[5] It is a problem for reductionistic and eliminativist forms of empiricism and naturalism that fail to take mathematics seriously as its own form of knowledge with its own forms of evidence, and that fail to recognize the intentionality of human consciousness and the many distinguishable capacities of the human mind, including the capacity for the founded acts of scientific reason that are involved in mathematics and logic.

Another alleged problem for platonism is supposed to arise if we ask how a platonic entity—an abstract object—can interact with, or participate in, the physical world. My response to this is that the language of interaction or participation is inappropriate, and should not be used in describing constituted platonism. Platonic entities cannot interact with, or participate in, the physical world. There is little reason to doubt that in some kinds of acts of formal abstraction we can see physical phenomena as more or less having certain forms or structures, but this does not mean that there has to be an "interaction" between two different kinds of entities. There is no need for such interaction or participation. The argument is that we should instead observe the IDP and consider how the cognitive processes of formal abstraction and free variation work in our experience. The problem is that "interaction" is already a term or concept that belongs on the physical side of the division. Certain concepts or categories of concepts are perfectly appropriate in the case of the directedness toward physical objects that are not appropriate in the case of other types of directedness. Concepts, meanings, forms, essences, and the like do not interact with one another or with particulars of any type. Physical things evidently do interact with one another. So we are taking a category that applies to physical things and applying it where it has no application, that is, either to platonic invariants or across the division between platonic invariants and physical invariants. The same point can be made about other supposed relations, such as instantiation or exemplification, insofar as they are understood as involving such cross-categorial applications. There is a tendency among certain critics of platonism to take a category that applies to physical invariants, apply it to platonic invariants, express puzzlement at the result, and thereby dismiss platonism. At bottom, it is really just question-begging.

To conclude this section I want to note briefly two additional analogies that might be drawn. First, we have seen that in some of his writings Gödel extends the analogy

[5] Many recent manifestations of the leftward *Zeitgeist* in mathematics—attempts at nominalist interpretations of mathematics in particular—are covered in detail in Burgess and Rosen 1997.

between sensory and rational intuition by suggesting that arithmetic in mathematics is the domain of the kind of elementary indisputable evidence that may be most fittingly compared with sense perception in the case of sensory knowledge (Gödel 1944, p. 121). It provides a kind of founding stratum and constraint on our mathematical thinking. Presumably we should not be prepared to regard concepts or principles that contradict arithmetic as mathematical knowledge, while concepts or principles at higher levels that enhance or extend our knowledge of arithmetic should, under the right conditions, be embraced. As we saw in chapter 2, Gödel does distinguish degrees of evidence in mathematics in some of his writings.

Second, it can be noted that if we are realists about the physical world and realists about the conceptual world then another interesting analogy is possible: just as we can try to understand what is *probably* the case about an existing physical world, so we can try to understand what is *probably* the case about an existing conceptual world. In the case of the conceptual world of mathematics and logic, probabilistic evidence is generally not regarded as a substitute for rigorous proof. I do not think that Gödel means to suggest that it is or can be. Rather, I think the idea is that we should not dismiss such evidence out of hand as we are trying to make progress in acquiring mathematicaI knowledge (see also chapter 8). It can play a role in determining the truth about abstract, mind-independent objects.

7

Minds and Machines

There is a substantial literature on the implications of Gödel's incompleteness theorems for the question whether human minds are machines. Turing, Post, and a number of other figures apart from Gödel weighed in on this question right from the beginning. Anticipating by a decade the work of Gödel, Church, and Turing on mechanical decidability, Emil Post had reached the conclusion that "mathematical thinking is, and must be, essentially creative" (Post 1941). This fact, he says, "makes of the mathematician much more than a kind of clever being who can do quickly what a *machine* could do ultimately. We see that a *machine* would never give a complete logic; for once the machine is made *we* could prove a theorem it does not prove." As a consequence of his own work Post became increasingly interested in the "psychological analysis of the mental processes involved in combinatory mathematical processes."

In a widely known paper "Computing Machinery and Intelligence" (Turing 1950), and in other papers around the same time (Turing 1948, 1951a, 1951b, 1952), Turing considered the question whether the incompleteness theorems could be used to show that machines have disabilities to which the human intellect is not subject. Turing considers and dismisses this as an objection to his own view that we should be able to build machines that pass the imitation test (now called the Turing test) for human intelligence. In an earlier well-known technical paper, however, Turing considered the possibility of using ordinal logics to overcome incompleteness and he made some interesting remarks on the role that intuition and ingenuity might play in mathematical reasoning (Turing 1938).

J. R. Lucas (Lucas 1961, 1970) and Roger Penrose (Penrose 1989, 1994) are perhaps the figures who are best known in more recent times for arguing that the incompleteness theorems imply that human minds are not computing machines. Their work has of course prompted many responses. I think it is safe to say, however, that as of this writing very little of the philosophical literature on minds and machines has addressed Gödel's own recently published papers and his own philosophical interests and preferences. Only now is this material starting to be taken into consideration.

What I want to do in this chapter is to approach the issue of minds and machines in terms of what has been said in earlier chapters about the human mind and human reason (the "monad"). These ideas about the human mind and human reason have been developed on the basis of Gödel's philosophical writing on the incompleteness theorems and his other foundational results. As indicated in chapter 1, Gödel wants to develop, in connection with his results, a defensible form of platonic rationalism that combines certain ideas in Plato, Leibniz, and Kant, using the phenomenological method of *epochē* and some other central ideas in Husserl's philosophy. Such a combination of views, following Gödel, was first sketched in chapter 1, and the sketch has been elaborated substantially in subsequent chapters. It is now time to apply and expand the position in connection with the claim that the mind is a machine or a computational information processor. I will first set out some of Gödel's central remarks on minds and machines. A brief characterization of what machines in fact do, according to the deep and fruitful analysis of mechanical computability provided by Turing, will follow. An important part of my argument will then be presented, consisting of an analysis of the origins of the idea that minds are machines. This kind of genetic analysis shows us how the identification of human minds with computational machines came about in the first place. Some of the central consequences will be spelled out by contrasting the features of Turing machines with the features of human minds and human reason that have been presented in earlier chapters of the book. At the end of the chapter I want to distance the non-mechanist view of reason that I present from some dubious metaphysical forms of anti-mechanism, and to offer a few words in support of the rationalistic optimism of Hilbert and Gödel according to which there are no absolutely undecidable mathematical problems.

§1. Some remarks of Gödel on minds and machines

It is interesting that in the introductory section of his doctoral thesis of 1929 Gödel already mentions the possibility of the incompleteness of axiomatic formal systems of mathematics. He says there that

> we cannot at all exclude out of hand ... a proof of the unsolvability of a problem if we observe that what is at issue here is only unsolvability by certain *precisely stated formal* means of inference.
> (Gödel 1929, p. 63)

This remark was omitted from the published version of Gödel's completeness result in 1930 but it shows that he was already cautious about identifying (un)solvability with purely formal or mechanical (un)solvability. Other remarks from the 1930s, after the announcement of the incompleteness theorems in 1931, also show that Gödel was inclined from an early date to separate the question of mechanical decidability from decidability by human reason. He says, for example, that

> The generalized undecidability results do not establish any bounds for the powers of human reason, but rather for the potentialities of pure formalism in mathematics ... Turing's analysis of mechanically computable functions is independent of the question whether there exist finite *non-mechanical* procedures ... such as involve the use of abstract terms on the basis of their meaning.
> (Gödel 1934, p. 370)

Note that while the procedures Gödel speaks of here are finite they are evidently not mechanical, because they involve the use of "abstract terms on the basis of their meaning." We have seen this idea on a number of occasions in earlier chapters.

In another interesting comment from sometime in the 1930s, Gödel (Gödel 193?) says that the limitations established by his incompleteness theorems could be interpreted as (i) establishing that certain problems are absolutely undecidable or, on the other hand, (ii) they might only show that something was lost in the transition from understanding proof as something that provides *evidence* to understanding proof in the sense of pure formalism (Gödel 193?, p.164). In his latter writings it is clear that Gödel thinks we should embrace alternative (ii) because undecidable number-theoretic sentences of the type produced by the incompleteness theorems for a given formal system are always decidable by evident inferences not expressible in the given formal system. The new inferences will be just as evident as those of the given formal system. These comments are, of course, related directly to Gödel's distinction, discussed in earlier chapters, between purely formal proofs and the contentual, "abstract" concept of proof according to which a proof is "that which provides evidence." In this paper from the 1930s, Gödel already says that the result would be that "it is not possible to formalize mathematical evidence even in the domain of number theory, but the conviction of which Hilbert speaks [that all mathematical problems are decidable] remains entirely untouched." Gödel says in Gödel 193? that it is not possible to mechanize mathematical reasoning. It will never be possible to replace the human mathematician by a machine, even if we confine ourselves to number-theoretic problems.

In chapter 1 we already noted Gödel's remarks on "Leibniz's program" in his 1939 lectures on logic at Notre Dame. He says that Leibniz's program of the *calculemus* cannot be carried through in full. Referring to what is now called Church's theorem, he says that we know, for the full predicate calculus, that "the human mind will never be able to be replaced by a machine already for this comparatively simple question to decide whether a formula is a tautology or not."

In the 1951 Gibbs Lecture, to be discussed in more detail below, the central implication of the incompleteness theorems is formulated as a disjunction:

> Either mathematics is incompleteable in this sense, that its evident axioms can never be comprised in a finite rule, that is to say, the human mind (even within the realm of pure mathematics) infinitely surpasses the powers of any finite machine, or else there exist absolutely unsolvable Diophantine problems. (Gödel *1951, p. 310)

This disjunction is described as a "mathematically established fact" and Gödel says that it is of "great philosophical interest." In some notes in the *Nachlass* he says he wants to show that the human mind does in fact infinitely surpass the powers of any finite machine. Skipping ahead to the 1960s, we find a host of similar comments. As we saw above, in his 1961 manuscript (Gödel *1961/?, p. 385) on Husserl, Gödel says that in the systematic establishment of axioms of mathematics, new axioms again and again become evident, even though they do not follow by formal logic from those previously established. The incompleteness theorems do not, however, exclude the possibility that every clearly posed yes or no question is solvable through such a development of new axioms, for what a machine cannot imitate is the fact that new axioms become evident to us on the basis of the (abstract) meaning of the primitive concepts. This view of mathematical knowledge is obviously quite different from the view that would result from Turing's analysis.

Asked about the philosophical implications of his incompleteness theorems in the early 1960s, Gödel wrote the following response in 1962 in a draft letter that presents a nice summation of his view:

> My theorems only show that the *mechanization* of mathematics, i.e., the elimination of the *mind* and of *abstract* entities, is impossible, if one wants to have a satisfactory foundation and system of mathematics.
>
> I have not proved that there are mathematical questions undecidable for the human mind, but only that there is no *machine* (or *blind formalism*) that can decide all number theoretic questions (even of a certain very special kind).
>
> Likewise it does not follow from my theorems that there are no *convincing* consistency proofs for the usual mathematical formalisms, notwithstanding that such proofs must use modes of reasoning not contained in those formalisms. What is practically certain is that there are, for the classical formalisms, no conclusive *combinatorial* consistency proofs (such as Hilbert expected to give), i.e., no consistency proofs that use only concepts referring to finite combinations of symbols and not referring to any infinite totality of such combinations.
>
> ...It is not the structure itself of the deductive systems which is being threatened with a breakdown, but only a certain *interpretation* of it, namely its interpretation as a blind formalism.
>
> ... our knowledge of the abstract mathematical entities themselves (as opposed to the *formalisms* corresponding to them) is in a deplorable state. This is not surprising in view of the fact that the prevailing bias even denies their existence. (Gödel 2003b, Vol. V, p. 176–177)

Much of what I have written in previous chapters can be read as a commentary on these remarks. Gödel thinks that human reason (the "monad") may be able to use abstract terms on the basis of their meaning, whereas this is not a possibility in the case of a purely formalistic or mechanistic view of mathematics.

A similar theme is sounded in the following very late passage, although Gödel again adds that human reason is capable of constantly developing its understanding of abstract terms:

> Turing in his 1937...gives an argument which is supposed to show that mental procedures cannot go beyond mechanical procedures. However, this argument is inconclusive. What Turing disregards completely is the fact that *mind, in its use, is not static, but constantly developing*, i.e., that we understand abstract terms more and more precisely as we go on using them, and that more and more abstract terms enter the sphere of our understanding.
>
> (Gödel 1972a, p. 306)

In accordance with some of our phenomenological observations in earlier chapters, human monads, but not machines, are or could be "constantly developing" their understanding of "abstract terms." Gödel's comment about the constant development of the mind might be understood as follows. The mind can constantly clarify and develop its grasp of abstract concepts without diagonalizing outside of the abstract concept(s) it is intuiting. *We* do not diagonalize out of the abstract concept(s) we are using the machines to (partially) capture. There can be a constant development of machines, on the other hand, but only by diagonalizing out of each particular machine under consideration. Machines are concrete entities that do not have access to abstract concepts. What amounts to an essential change in the machine, a change that requires a new machine, is for us only an incidental change in, and clarification of, the awareness of the concept(s) we are using the machines to capture. If we start with PA, for example, then we can think of the undecidable Gödel sentence G_{PA} as an axiom that could be added to PA. We see that G_{PA} is true on the basis of our grasp of the concept of the natural numbers. More and more "axioms" of this type could be added to the successive formal systems obtained in this way, and one could argue that these "axioms" are evident to us on the basis of our understanding of the concept of the natural numbers. We are directed through this sequence on the basis of our understanding of this concept. There is no reason to stop with respect to deciding these Gödel sentences. We could not proceed strictly on the basis of the given formal system or machine, however, because the Gödel sentence obtained for that system is formally independent of the system. Instead, we refer through PA to what is not enclosed in PA. PA does not adequately capture our arithmetic intentions in the first place.

I would now like to return to the disjunction, quoted above, that Gödel states in the 1951 Gibbs Lecture (see also Tieszen 2006). This disjunction is stated in various places in Gödel's writings and in notes in the *Nachlass*.[1] The disjunction is presumably a mathematically established fact because Gödel is thinking of it as a reformulation of his incompleteness theorems. Suppose that the human mind is a finite machine (in some sense) and there are for it no absolutely undecidable Diophantine problems. Well-defined or effectively given formal systems can be viewed as Turing machines, and Turing machines can be viewed as well-defined formal systems (see also Gödel's remark on this in the 1964 Postscript to Gödel 1934, pp. 369–370). Given this relationship between formal systems and Turing machines we can substitute the notion of Turing machine (TM) for "finite machine" in this supposition. Then what the incompleteness theorems show is that the supposition could not be true. It could not be true that

[1] See, for example, Gödel 2003b, Volume V, p. 80 and p. 160. See also Wang 1974, pp. 324–5. Gödel allowed the following statement of his view to appear in the 1963 publication on *Mathematics* in the *Life* Science Library: "'Either mathematics is too big for the human mind', he says, 'or the human mind is more than a machine'. He hopes to prove the latter" (see van Atten and Kennedy 2003, p. 460 for full reference).

(1) the human mind is a finite machine (a TM) and there are for it no absolutely undecidable Diophantine problems.

The incompleteness theorems show that if we think of the human mind as a TM then there is for each relevant TM some "absolutely" undecidable Diophantine problem. The denial of the conjunction (1) is, in so many words, Gödel's disjunction. In formulating the negation of (1) Gödel says that the human mind "infinitely surpasses the powers of any finite machine." One reason for using such language is evidently that there are denumerably many different Turing machines, and for each TM there is some absolutely unsolvable Diophantine problem of the type Gödel mentions. So Gödel's disjunction, understood in this manner, is presumably a mathematically established fact. It is not possible to reject both disjuncts. If (1) cannot be true then certain views about mathematics must be rejected. As Gödel notes in his 1961 manuscript (Gödel *1961/?) and elsewhere, we cannot be mechanists about the mind or, in the appropriate sense, strict formalists, and yet be unmitigated optimists about mathematical problem-solving.

Three possibilities are left open by the disjunction:

(2) the human mind does not infinitely surpass the powers of any finite machine (TM) and there are absolutely unsolvable Diophantine problems. That is, the human mind is a finite machine and there are for it absolutely unsolvable Diophantine problems.
(3) the human mind infinitely surpasses the powers of any finite machine and there are absolutely unsolvable Diophantine problems.
(4) the human mind infinitely surpasses the powers of any finite machine and there are no absolutely unsolvable Diophantine problems.

Each of these positions has its adherents. Universal computationalists, who think that everything in the universe can be understood in terms of mechanical computation, have been prepared to accept (2). This seems to be the position, for example, of Stephen Wolfram (Wolfram 2002). On this option it appears that if the human mind is a finite machine (a TM), then we must say that there are for it absolutely undecidable problems. But why would we want to accept the consequent of this conditional in the absence of conclusive evidence for the antecedent? I take it that nothing definitive can be said at this time in favor of the truth of the antecedent. The eminent logician Solomon Feferman, on the other hand, has argued for (3) (Feferman 2006). Gödel has some interesting things to say about each of these possibilities, but it is clear from some of his writings and comments he made to Hao Wang that, at least later his career, he favors option (4).

In my discussions with Hao Wang in the mid 1980s, Wang told me that Gödel hoped to use ideas in Husserl's phenomenology to show, among other things, that the human mind is not a machine. A passage of a draft letter found in the Gödel *Nachlass* by Mark van Atten and Juliette Kennedy confirms Wang's comments (see van Atten

2006, 256–258, and van Atten and Kennedy 2003, p. 460). The letter was probably drafted around 1963:

> Before my results had been obtained it was conjectured that any precisely formulated mathematical yes or no question can be decided by the mechanical rules of logical inference on the basis of a few mathematical axioms. In 1931 I proved that this is not so, i.e., no matter what and how many axioms are chosen there always exist number-theoretic yes or no questions which cannot be decided from these axioms. Combining the proof of this result with Turing's theory of computing machines, one arrives at the following conclusion: Either there exist infinitely many number theoretic questions which the human mind is unable to answer or the human mind contains an element totally different from a finite combinatorial mechanism (such as a nerve net acting like an electronic computer). I hope to be able to prove on mathematical, philosophical, and psychological grounds that the second alternative holds.
>
> … It follows from my results that a consistency proof for any part of mathematics containing number theory is impossible if in this proof one confines himself to *concrete* (i.e., empirically meaningful) combinatorial properties and relations of formulae considering formulae to be finite strings of symbols without meaning.
>
> Rather, the use of *abstract* mathematical concepts (which have no immediate empirical meaning) and certain immediate insights about them are needed in order to succeed, such as the concept of a "rightly convincing proof" or of a "*procedure*" which applied to such a proof (or to a "procedure of the I order") yields another such proof (or procedure of the I order) or the "existence" of an integer with a given property no matter whether there is a way of finding it.

In the *Nachlass*, Gödel gives ten reformulations of the sentence that says "I hope to be able to prove on mathematical, philosophical, and psychological grounds that the second alternative holds." Several of these are especially relevant to understanding Gödel's interest in Husserl's phenomenology. One of the formulations reads:

> I conjecture that the second alternative is true and that the transformation of certain aspects of traditional philosophy into an exact science will lead to its proof. I am now working toward such a transformation.

This formulation is especially interesting for its links to Husserl's idea of philosophy as a rigorous, a priori science, which is of course also related to aspects of Leibniz's rationalism that were appealing to Gödel. Another formulation of Gödel reads:

> I conjecture that the second alternative is true and perhaps can be verified by a phenomenological investigation of the processes of reasoning.

In a third formulation, he says that,

> I hope to show by an investigation of the basic ideas underlying all our thinking that the second alternative holds and that systematic but non-mechanical methods for the decision of mathematical questions exist which make it probable that any mathematical yes or no question *can* be answered by the human mind.

Gödel thus hopes to show that the human mind contains an element totally different from a finite combinatorial mechanism, and he suggests the possibility of verifying this in a rigorous way on the basis of a phenomenological investigation of the processes of reasoning. He wants to use phenomenological considerations to investigate the matter of the decidability of mathematical problems posed by human reason. Human reason, on this view, is not to be understood in a completely mechanical manner, contrary to Leibniz and others.

The idea is that the human monad must use systematic and finite but non-mechanical methods for the decision of open problems in mathematics (sometimes he says number theory), based on a grasp of the abstract meanings of the terms involved. If human reason can know about mind-independent ideal concepts or objects on the grounds of categorial intuition or *Wesensanschauung*, then finite human minds, even though they might use such forms of intuition systematically, could not be (Turing) machines. It would not be possible to completely formalize mathematical evidence. It is on this basis, which appeals to ideas in Husserl but is distinct from the positions of Leibniz and Hilbert, that Gödel retains his rationalistic optimism about mathematical problem-solving.

§ 2. Turing machines and mechanical computability

We know what Turing machines are. Turing gives us a wonderfully clear characterization. As we noted in chapter 2, Gödel even describes Turing's analysis of the concept of mechanical computability as a good example of the clarification of the meaning of a concept. The concept was brought into the correct perspective, he says, with the work of Turing. With the proofs of the equivalence of the various characterizations of the concept of mechanical computability we have been able to bring out the essence of the concept. An essence is just an invariant or identity through variation.

Without going into details here we can note that Turing machines can be represented as finite lists of instructions, where each instruction indicates an input state, an input symbol, an output symbol, an output action, and an output state.[2] It suffices to regard the input and output symbols as 1s and 0s. The input and output symbols all appear on a tape that is divided into discrete cells that are either blank or that contain a 1 or a 0. The tape can contain as much space as is needed for a computation. The bit strings on the tape are then processed in accordance with the TM instructions. Once we have specified what TMs are, it is reasonable to accept the Turing-Church Thesis that a function is mechanically computable if and only if it is Turing computable. *Turing computability must involve only completely and explicitly given finite, concrete sign configurations in space-time and purely mechanical, finitary operations on these objects without reference to the meaning, if any, of the symbols*. The power of Turing machines derives from

[2] Details can of course be found many places in the literature. See, e.g. Taylor 1998. See also Sieg 2002 for an analysis in connection with some ideas of Gödel.

the fact that they deal only with such sign configurations. The meaning or interpretation of the sign configurations, should there be one, plays no role at all in what a TM does.

Computational functionalists have argued that what we need to do in order to understand human reason, intentionality, consciousness and all of the features we have indicated, is to build a TM that appears to third-person observers to exhibit these features. We only need to build a TM that passes the imitation test. If we can construct a TM that passes the Turing Test for these features of human reason and intentionality then allegedly there is nothing more to actually having these features of human reason and intentionality. Of course some philosophers of mind do not want to be so closely wedded to the Turing Machine model. It is enough that we have some purely physical entity or other (such as, for example, a brain) to which the features in question can be reduced. The set of allegedly suspicious features of consciousness and intentionality then reduces to the purely physical features that are allegedly not suspicious, as noted in § 1 of chapter 5. In short, the prospect is held out that the apparent differences between human minds (monads) and machines do not clinch the matter. The right kind of TM, obtained by the right kind of engineering, will exhibit intentionality, understanding, and so on.

There are many possible objections to this latter claim. In the case of the Turing machine model, for example, John Searle is famous for arguing on the basis of his Chinese room thought experiment, that the Turing test is not a test for intelligence anyway, and that programmed machines do not have (intrinsic) intentionality while human consciousness does exhibit intentionality (Searle 1980; see also Searle 1983). According to Searle, this difference is real, not merely apparent. The arguments on both sides of the Chinese room debate then proliferate. I want to take up a different approach. I want to ask how one could get into a position in the first place to think that human minds really just are Turing Machines. The argument I develop will apply not only to the TM model of minds but also to many other forms of physicalistic reductionism.

In light of the problems that have surfaced for the mechanistic view of human minds as computing machines, it is sometimes suggested that human minds are computing machines but not in the sense of Turing machines. They are, for example, machines that need to be embodied and to have some connection to the environment, to learn, etc. My response to this is to point out that we have a clear conception of what a computing machine is according to Turing's characterization, and that it is not clear what a computing machine would be that is not a Turing machine. If it is embodiment, a connection to the environment, or learning that is at issue then these phenomena must be regimented in terms of Turing machine computation or else it is not clear that we are still talking about a computing machine. There seems to be a tendency, especially among philosophers, to slip vague or even non-computational ideas into the notion of what a "computing machine" is supposed to be in order to deal with the objections that have been raised to the view that human minds are computing machines. It always looks like hand waving. There is also discussion in the literature of "hypermachines," quantum computers, super-recursive algorithms, Darwin machines, and the like, but I think the same point can be made about the clarity of these conceptions.

§3. The empiricist Zeitgeist and the rise of computational mechanism

As we have seen, Gödel refers to the bias or prejudice of the modern "leftward" *Zeitgeist* at several places in his writings.[3] He suggests that, philosophically speaking, the incompleteness theorems disrupt the *Zeitgeist*.

The mechanist view of the human mind or of human reason is certainly part of the spirit of our times. We see many forms of this view all around us. There are lots of technical issues about the relation of minds to machines that one can go into. It is possible to spend years studying these and yet to feel that the technical discussions are often inconclusive. Part of the reason for this is that basic assumptions on which the debate is predicated are not questioned. What I want to do in this section is to take Gödel's references to the bias or prejudice of the leftward *Zeitgeist* seriously. Modern mechanism is part of the empiricist, scientific perspective that has led to incredible success in many domains of human endeavor. This perspective has revealed many things to us that we would otherwise not have seen. One can argue that it has also concealed important things. In order to have a firmer grasp of what it conceals we need to consider how it originated. Thus, in this section I will briefly discuss the philosophical origins of the modern mechanist view in order to make clearer the structure of revelation and concealment that it embodies. One might think that Gödel would have to look far into the future of human development to see a way out of the hypothesis that all human cognitive functions are mechanizable (see also Webb 1990). Perhaps we can learn just as much about the view that human minds are machines by looking to the past.

As Gödel suggests, the empiricist Z*eitgeist* is a perspective on the world that has been in a process of development since the Renaissance. One could trace its development in detail. For example, a very important stage in its development occurred during the Scientific Revolution (see Husserl 1936). The mechanist perspective is, in effect, an interpretive scheme that has been built up as part of the general development of the scientific worldview. It is a scheme we use to interpret the world and objects in the world. Let us first consider some general features involved in the leftward *Zeitgeist*, and then discuss computational mechanism against this background.

[3] See, e,g. Gödel *1961/? and the letter to Rappaport quoted above. Hao Wang also discusses this in Wang 1974, 1987, and 1996.

Our thinking and awareness in the natural sciences is, of course, what we have called founded thinking and awareness. What we need to do, therefore, is to consider some of the general features involved in our founded understanding of the world on the basis of the natural sciences. These features are, in various combinations, conditions for the possibility of natural science, and they concern the following aspects of our experience, aspects that can overlap and condition one another:

1. The central epistemic idea of empiricism or naturalism is that all knowledge is derived from sensory (external or outer) experience. Evidence in natural science is based on sensory experience. Natural sciences will typically seek to determine causal relations and proffer causal explanations in domains of inquiry that are based on sensory experience, although there are perhaps exceptions in domains such as quantum physics.
2. The distinction between quantitative and qualitative aspects of our experience of the world, and the use of calculational or mechanical techniques with the quantitative aspects.
3. The distinction between formal and "material" aspects of our thinking and understanding (where calculation can also be used with the formal aspects) along with a related distinction between form and meaning.
4. The role of idealization and abstraction.
5. The related distinction between the universal and the particular, or between the general and the specific, with the idea that natural *science* is to seek generalities, uniformities, or universal laws concerning natural phenomena in different domains.
6. The fact that there are prereflective and immediate forms of experience and also more reflective, mediate forms of experience.
7. The fact that science requires "objectivity," so that some way of separating the objective from the subjective is called for by modern science.

I will not say much here about point 1. In one form or another it just establishes a baseline for *natural* sciences. Sense experience is perfectly appropriate for empirical sciences. Concerning point 2, one of the central features involved in many of the natural sciences is calculative thinking. Not all types of thinking appear to be calculative but calculative thinking is a condition for the possibility of many of our sciences. One simply cannot engage in vast domains of natural science without calculative methods and concepts. It can, of course, take a great deal of training and specialization to master and develop these methods and concepts, and the methods and concepts will themselves take on more or less value as a function of how much work they do, the range of their application, how efficient they are, and so on. Calculative thinking requires that we be able to distinguish quality from quantity in phenomena. One must be able to quantify phenomena to make them amenable to calculational techniques. This emphasis on the mathematization of experience is clearly present at the beginning of modern natural science in the distinction between so-called primary and secondary qualities. Such a distinction can be found in the work of Galileo, Descartes, Locke, and others. It has been argued that the distinction is present even in ancient Greek philosophy. In Galileo's work, for example, number, shape, magnitude, position, and motion are taken to be primary qualities and colors, tastes, smells, and warmth/cold to be secondary qualities. The former properties are seen as objective features of experience, while the latter are viewed as subjective. Indeed, the primary qualities are just those that are mathematizable and, in Galileo's view, are absolute and immutable, while the secondary qualities are sensory, relative, and fluctuating. Knowledge is supposed to be concerned with primary qualities, but opinion and illusion are generally associated with secondary qualities. One might hold that the primary qualities inhere in the objects themselves, while secondary qualities do not. The primary qualities are tightly linked with third-person, empirical observation. They are the "objective" features of the world of causes and effects.

The features of quantification and calculation are attended by the feature involved in scientific understanding mentioned in point 3: the shift from "contentual" or "material" aspects of thinking and understanding to formal aspects. The quantifiable aspects of our experience are represented in mathematical and logical formulas. Mathematics, mathematical physics, chemistry, engineering, and many of the other pure and applied sciences require that we discern the form or structure of phenomena. In attempting to determine the form or structure of a phenomenon, a kind of formal abstraction takes place. What we abstract from, what is not needed, is what I have called the "content" or "matter" associated with the phenomenon. One of the interesting outgrowths of mathematization is that once we have worked out the appropriate mathematics for the scientific treatment of a phenomenon, we can often mechanize the mathematics.

What we have said thus far is that with the modern understanding of the world in natural science there is often a focus on quantitative aspects of our experience, where computational techniques are used with the quantitative features that have been abstracted. The understanding of the world in natural science, in a similar vein, involves a shift to formal or structural features of experience in which we abstract from content or certain aspects of meaning. These shifts, as indicated in point 4, are attended by a kind of idealization. Everyday experience is inexact and imprecise in a variety of ways. With the shift to quantification and formalization we obtain a kind of precision and exactness that is otherwise not available to us. This move toward the more exact and precise involves us in various idealizations. We leave behind some of the complexity and richness, but also the imperfection, of the plenum of everyday experience. The scientific understanding of the world is thus typically an understanding in which various idealizations of the world are at work.

According to point 5, natural science requires that we be able to distinguish universal from particular features in our experience. Natural science is all about finding regularities, generalizations, or lawlike features of the world on the basis of our particular sensory experiences. Points 2, 3, and 4 are all involved in making this possible.

As I have been indicating, the understanding of the world provided by natural science involves various kinds of abstraction. It requires us to abstract from a larger whole, that is, the whole of our experience. In our discussion of wholes and parts in chapter 6 we distinguished "pieces" (independent parts) from "moments" (non-independent parts). What makes a part of a whole a piece is just that it can exist independently of the whole of which it is a part, while this is not possible in the case of moments. Moments are abstractions that are "founded" on larger given wholes. Now quantification, formalization, generalization, variation, and the like are moments of our experience. They are founded on our experience as a whole, where this experience also includes qualitative, contentual, non-calculational, "meaningful," referential, and particular or specific aspects. The modern understanding of the world in natural science is thus a founded understanding of the world. This means that there is a deeper, founding whole on which it depends and of which it is a part. Husserl calls the deeper founding stratum of everyday practices and perception the "lifeworld" (*Lebenswelt*). A conception such as this can also be found in the work of other philosophers. Wilfrid Sellars, for example, distinguishes what he calls the "manifest image" of the world from the "scientific image." This leads us to point 6. The founded understanding of the world that we find in natural science and modern technology requires the various kinds of reflective activities we have been discussing. The modern scientific understanding of the world is, I would like to argue, a more reflective form of understanding that involves us in various abstractions and idealizations. There are, however, also prereflective and more immediate forms of understanding or awareness. These are forms of understanding or awareness that do not involve all of the abstractions and scientific theorizing that are in the background of the understanding of the world in natural science.

To abstract features of our experience is not itself to be engaged in experience in the same way that we would be were we not abstracting. Abstracting features of experience already requires, as we said, taking a more *reflective* stance on our experience. Indeed, we might draw a general (albeit relative) distinction between reflective and prereflective modes of experience. Prereflective modes of experience would be more *immediate* forms of experience. They would not involve the kind of mediation that attends higher levels of generalization, abstraction, variation, and theory construction. We already used the language of "founding" in chapter 6 when we said that abstractions are founded on some whole. So the features we abstract from our experience are founded on some larger whole of experience. There is, as it were, a founding level of experience, and then also founded forms of experience. The interpretive scheme of the natural sciences that we have been discussing must count as providing us with a founded form of experience. It is a scheme that is built up over time out of abstractions that involve more reflective, mediate, and theoretical stances on our experience. There are, as it were, layers of thinking, directedness, and experience. There can be, and has been, lifeworld experience without natural science.

What this higher-level interpretive scheme yields, however, is just the kind of distinction noted in point 7. Points 2–6, which are concerned with quantification, calculation, formalization, idealization, exactness, precision, and generalization, all involve a more reflective, mediated perspective on the world. Along with point 1, they are all features that allow us to separate what is objective from what is subjective. The scientific understanding of the world involves us in a higher degree of objectification of the world. It is thanks to these features that other commonly recognized aspects of objectivity are possible, such as intersubjective agreement on methods and results and repeatability of calculations, experiments, procedures, and the like. As mentioned earlier, it was the intention of Galileo and other founders of modern natural science to distinguish what was absolute and immutable from what was relative, fluctuating, and due solely to subjective sensory experience. Knowledge is supposed to be concerned with the former characteristics, while the rest is a matter of opinion and illusion. It is a corollary of our earlier analysis that this search for "objective" characteristics itself involves a kind of abstraction from our experience. The point is precisely to excise the subjective aspects of experience. What we obtain with the components of the interpretive scheme is a kind of objectivity that would otherwise be lacking in our epistemic enterprises. We can leave behind the inner sensings, feelings, thoughts, and subjective perspectives, and focus on the outer observable phenomena that would, in principle, be available to all. What the interpretive scheme yields is just the third-person stance on the world. In short, the intention behind it is precisely to abstract from human subjectivity, to minimize subjectivity and maximize objectivity.

We have now specified some of the central elements of the scientific understanding of the world on the basis of the natural sciences. The features we have discussed, taken as a whole, give us a particular perspective on the world. They provide a way of interpreting the world. This interpretation reveals the world to us in a certain way. It is by these means that we approximate an exactness, clarity, and distinctness in our knowledge that is not part of our everyday, informal understanding of the world. Indeed, an interpretive scheme comprised of these components has a normative character. In light of the successes of mathematical natural science and modern technology, we come to believe that we *should* quantify, formalize, and idealize.

This kind of interpretive scheme is routinely applied to nature and everything in nature. We can see how it is at work in the various natural sciences. It conditions what is revealed to us, and the revelations of natural science have indeed been very powerful and successful. Great advances in what Gödel would call the "leftward" direction have been made on many fronts.

Pure formalism and modern computational mechanism have developed against this background. They arise from refinement of features 2, 3, and 4. We discover with mathematization that we can make various aspects of our thinking amenable to purely mechanical, computational techniques. With pure formalism and mechanism there are further abstractions: one abstracts from the meaning, reference, and origins of the symbolic languages that are essential to the mathematization of nature. The idea is precisely to abstract away from the meaning of signs, and to focus on the mechanical manipulation of sign configurations solely on the basis of precisely specified finite sets of rules, in just the manner that is encapsulated in the idea of a Turing machine.

It is possible for human beings to be directed toward concrete, finite sign configurations (as tokens), and we can clearly manipulate these objects in a purely mechanical way on the basis of finite sets of rules. We can do this without needing to understand the meaning (if any), reference (if any), or origin of the sign configurations. This is one form of conscious directedness for human beings. There is a sense in which we can view such a purely syntactical environment as a world of its own. It might be a world in which it seems supremely important to solve the calculational problems at hand, and yet one can do this, without even knowing what the calculations are about. Not only can we do this but we have obviously developed very powerful technical extensions of this capacity in our existing computers which, incidentally, no one now regards as intelligent. We can offload this capability and have our machines execute the operations involved. What we have been able to do is to take a capacity we have and amplify and extend it in our computer technologies. In recent times we see that vast input-computation-output structures like this can be completely computerized.

There is a kind of calculational meaning at work when we are engaged in such purely formal or mechanical thinking, but it is quite different from knowing the origins or references of the symbols we are manipulating. In his theory of meaning, Husserl calls this calculational meaning the "games meaning" of signs. This is the kind of meaning that chess pieces take on by virtue of the rules of the game of chess. A number of philosophers have introduced other expressions for this kind of meaning, especially in the late nineteenth and early twentieth century when formalism was first emerging. Husserl distinguishes the "games meaning" involved in manipulation of symbols according to rules in formal systems, from meaning as intention and meaning as the fulfillment of intention. His central notion of meaning, which is part and parcel of his theory of the intentionality of human consciousness, is that of meaning-intention. The meaning as intention not only points to a referent but in some cases the referent itself may be intuitively present to us.[4] We, or our computers, can however also perform calculations quite independently of knowing what they are about. They might be about the space shuttle, or nuclear weapons, or economic models concerning homelessness, or problems concerning environmental pollution.

A vast technization and mechanization can thus emerge in the sciences. Syntax is now sharply distinguished from semantics, and the two subjects can be pursued independently or alongside one another. The modern theory of computation arises. Gödel shows how it is possible to "arithmetize" the syntax of precisely specified formal

[4] One might again note the parallel with Frege, in whose theory of meaning the notion of "sense" plays a central role, where we are directed toward a referent by way of a sense, and where Frege criticizes the formalism of Thomae, Heine, and Hilbert in mathematics and logic. Husserl allows for the "games meaning" of the formalist in his theory of meaning, but also wants to keep it in its proper place.

languages. Among the many outstanding results of this entire development, as noted above, is the emergence of a sharp concept of mechanical procedure, as characterized by Turing, along with the fact that a host of alternative characterizations of the concept of mechanical procedure can be proved to be equivalent. Many other interesting, important, and beautiful results emerge in the theory of computation.

§ 4. Limitations on mechanistic and naturalistic conceptions of consciousness

The distinctions that lie behind the empiricist, scientific worldview and modern mechanism allow us to separate the subjective from the objective. They are, in fact, used for just this purpose. With quantification, calculation, formalization, idealization, and exactness we obtain intersubjective agreement on methods and results, including repeatability of calculations, experiments, and procedures. We obtain a kind of objectivity, and objectivity is what we seek everywhere in the modern sciences.

Now what happens when this kind of interpretive scheme is turned back around on human beings in particular? It is not surprising that what is "revealed" to us is that the nature of human being is quantifiable, formalizable, and computational. The human mind itself (not just the body) is a machine. It can, or should, be understood in terms of the kind of organization of causal relations we find in the case of quantified, formalized, computational states (see § 1 of chapter 5). We can then ask about the specific computational architecture involved, the computational structure of the different forms of cognition, and so on. The hope is that we might in this way finally develop a science of the mind.

When we turn this interpretive scheme back around on ourselves we thus find that, true to our intentions to eliminate human subjectivity, we have eliminated human subjectivity (minds), along with all of its (their) complexity and detail. In its stead we have a purely objectified subject, merely the outer shell as it were. Consciousness, the very essence of subjectivity, disappears. The *Zeitgeist* thus tends, as Gödel says, toward the elimination of the human mind (monad) and its most important features. At earlier stages in the development of the modern *Zeitgeist*, the human body was interpreted as a machine, with the effect that the "lived body" and bodily intentionality were ignored. The distinction between the human body as a purely material thing (*Körper*) and the lived body (*Leib)* as a source of intentionality and meaning conferral was covered over (see, e.g. Husserl 1952a). As the scheme was extended and augmented, the human mind also came to be interpreted as a machine.

Thus, we develop in the sciences an interpretive scheme the goal of which is to absolutely minimize subjectivity and to maximize objectivity, and when we apply this interpretive scheme to the human mind we see that we achieve just this effect. The problem is that we are forgetting what this interpretive scheme abstracts from or leaves behind in the first place. It is not a foundation but is rather already a *founded*, reflective scheme that depends on making the abstractions we have noted (e.g. quantitative from qualitative features, primary from secondary qualities, form from content) and then forgetting about the whole from which they were abstracted. Hence, it can become a limited or one-sided view that conceals much that is important about human cognition. The key point is this: the claim that the human mind (monad) is a finite machine *depends* on the fact that human beings whose cognitive acts exhibit intentionality have developed a particular interpretive scheme in the first place, a scheme which they have then applied to themselves. We have, in effect, taken an important and fruitful interpretive scheme and applied it beyond its legitimate boundaries. In so doing, we substitute a part of what we can do—namely, act as computers—for the whole. At the founding level of all of this, however, we have human subjects with intentionality who build up ways of understanding the world through their manifold capacities for interpretation. The claim that the human mind is a finite machine (a TM) or a finite combinatorial mechanism rests on a development that presupposes human intentionality, directedness, the capacity for acts of abstraction, and so on. Our awareness of our own consciousness, however, does not depend on building up layers of scientific theory, abstraction, idealization, and so on. At the prereflective, pre-scientific level, humans are already conscious interpreters of the world who are directed toward various goals.[5]

The problem with the computational view of the human mind is that it depends on a rather narrow and naive or one-sided conception of knowledge and understanding. Human subjects ("monads") in their full concreteness (as opposed to abstract conceptions of humans) cannot be fully understood through the abstractions of science and technology. Rather, consciousness precedes scientific theory and makes natural science and technology possible in the first place. We have an immediate knowledge of our own consciousness that does not require the mediated ways of knowing that are part of science and modern technology.

The argument is that if we are to see things whole, then we must keep both objectivity and subjectivity in the picture. The interpretive scheme involved in natural science provides us with a founded understanding of the world, and there is a deeper, founding whole on which it depends. This deeper founding stratum of everyday practices and perception, as noted above, is called the "lifeworld" in Husserl's philosophy. There are prereflective and more immediate forms of understanding and knowing. These are forms of understanding and knowing that do not involve all of the abstractions of the interpretive scheme we have been discussing. On the view I am developing the interpretive scheme is not foundational but is itself founded on our lifeworld experience. In this manner, we can turn the tables on scientism, on the view

[5] As mentioned above, it is sometimes claimed that even if the human mind is not a Turing machine it is nonetheless some other kind of machine. I think the argument presented in this section can be extended to cover any other conceptions of machines that involve the abstractions that have been noted above.

that the sciences and technology provide the fundamental way of knowing and understanding the world, and that value natural science and technology above all else.

In the mechanist *Zeitgeist* we have an alignment of objectivism, empiricism, and computational mechanism. Thanks to the overstated objectivism, there are no human minds or human subjects. Thanks to the overstated empiricism, there are no abstract objects, and there can be no knowledge of abstract objects. Thanks to the overstated computational mechanism, there is no meaning or reference. The predominant perspective of our times seeks, as Gödel suggests, to eliminate the mind and abstract entities. The mechanist *Zeitgeist* is an interpretive scheme that both reveals and conceals. This is the answer to the question of how we could get into a position to think that the differences between Turing machines and human minds are not real, and that human minds (monads) just are TMs. We focus on what the mechanist *Zeitgeist* reveals and forget about what it conceals. This is part of how I would explain the one-sidedness or prejudice (in a hermeneutical sense) of the *Zeitgeist* that Gödel mentions in some of his remarks. Consciousness has the potential to trap itself. We cannot, however, constitute the human mind as a machine without landing in a paradox: the mind constitutes the meaning of being, but it is a condition on being a TM that TMs abstract away from meaning.

Computational functionalism abstracts some features of our experience and treats them as real and as the source of truth when in fact other features are also real and can also be a source of truth. The features that are left behind by the abstractions are no less real. Now if we view quantity as real or important and quality as unreal or unimportant then we are taking something as a piece (i.e. quantity) that is really only a moment. We have confused a piece with a moment. Similarly, if we view formalization or computation as giving us what is real, while content is taken to be unreal or unimportant, then we are taking what is only a moment of our experience as a piece. This is precisely to recede from a more holistic view of the world. We can understand holism in terms of such a theory of wholes and parts. Does this mean that the human mind is not a machine? Yes. The claim that the human mind is a machine is based on a partial understanding of the human mind. It is a view that takes a part for the whole. It is reductionistic in this sense. It is, in intention, an eliminative reductionism insofar as it wishes to eliminate from epistemology, ontology, or methodology, the features from which it abstracts.

I suggest the following alternative picture. The human mind can, in fact, function in some ways as a machine or computer. We do this when we are following algorithms or doing mechanical calculations. We can then imagine significant extensions of this ability. We are quite capable of understanding and reacting to what I have called the "games meaning" of symbols. We then devise computing machines that can do all of this for us. We thus technically enhance our computational abilities in various respects. What humans can do in addition to this is to understand the contentual meaning of signs, refer to objects in the world, become directly acquainted with some of these objects, understand the origins of particular formulas, and so on. The human mind is not only, or not fundamentally, a machine.

§ 5. Turing machines and human reason

If the analysis in the previous chapters is on the right track, then the features of human consciousness and human reason that we have laid out are among those that are concealed or forgotten. It was pointed out earlier how it is possible to describe very clearly what Turing Machines do. If the human "monad" is engaged in the kinds of activities we have attributed to it in earlier chapters, then it is not a Turing Machine. It is not *any* Turing Machine. There is a glaring mismatch between mechanical computability and human reason. To understand what a monad is in our sense is to refrain from applying the interpretive scheme of the natural sciences to all aspects of human being. Phenomenology requires that we bracket or suspend such an all-inclusive interpretation.

Let us therefore briefly enumerate the differences between human monads and Turing machines that have emerged in the discussion up to this point. On the basis of the argument we have been developing, the claim is that these are genuine differences. To think that they are not is to embrace the abstractions involved in the interpretive scheme we have discussed, and then to forget about that from which we have abstracted. The enumeration depends on standard descriptions in the theory of computablility about what TMs do and what they do not do. Hence, taking stock of what we said about human reason, on the one hand, and what Turing machines do on the other, we can say that Turing machines do not have intentionality. Any TM operates only on meaningless, finite sign configurations. TMs derive their power and their appeal to reductionistic physicalists from just this fact. TMs are not conscious of anything. This is just the so-called "hard problem" with computational functionalism that is so widely discussed in the recent philosophy of mind literature, as we noted in §1 of chapter 5. The concepts of consciousness and intentionality do not enter into the description of TMs at any stage. In a theory of computability class you would misunderstand what TMs are if you thought otherwise. The symbols involved in TM computations are not *about* anything, except insofar as we interpret them as being about something or other. For a TM, they are not even symbols. Because TMs do not have consciousness that exhibits intentionality, they do not have a perspectival form of consciousness, or a field of consciousness that is organized into background and foreground, or that has other Gestalt characteristics. TMs are not directed by, or toward, meanings. They do not bestow meaning. They do not constitute the meaning of being of anything. At a suitable level of description, TMs can manipulate concrete finite sign tokens, but they do not engage in intentional acts of abstraction, in the conception of perfections, in idealization, reflection, formalization, axiomatization, and imaginative variation. They cannot imagine possibilities and find necessities relative to these possibilities. TMs do not arrange *meanings* or contents into genus/species structures. They cannot relate abstract universals to particulars, or abstract universals to abstract universals in a systematic way, but they can only relate certain kinds of concrete particulars to other concrete particulars in a rule-governed, mechanical way. Thus, a kind of informal rigor seems to be possible in human reason that is not possible in TM computation.

Since TMs are not described as being conscious or as having intentionality, they do not have empty, fulfilled, or partially fulfilled intentions. TMs do not have expectations that can be either fulfilled or frustrated. Unlike "monads," they are not goal-directed. In the theory of computability, none of these things figures into the description of Turing machines. They do not possess or lack evidence. They are not convinced by anything. When computing with what *we* call mathematical or logical information, they are not directed toward abstract and ideal mind-independent$_2$ objects and truths. They do not constitute the meaning of being of the objects of mathematics and logic as abstract and mind (or machine)-independent. There cannot be anything like constituted platonism for TMs. They do not conceive and know about mathematical and logical invariants that are abstract and mind-independent$_2$. They do not intuit such objects, that is, have fulfilled intentions with respect to such objects. TMs do not find, or provide, proofs in what Gödel calls the "abstract" or contentual sense of "proof."

The finite "monad" might be able to make systematic decisions on the basis of a grasp (intuition) of abstract meanings, but no TM could do such a thing. Human cognitive acts and processes in mathematics are finite at any given stage and display a high degree of systematicity when brought into a rigorous form, but what they are *about* is abstract objects and concepts. It is because we are directed toward objects by meanings or intensions that the possibility arises of reasoning with intensions of mathematical expressions, even when the extensions of these expressions cannot be fully given in intuition. The possibility arises of reasoning with meanings of expressions that are about infinitary objects. The objects or concepts toward which we are directed generally transcend the mind or our intuition, in the sense that there is more to them than we can grasp at any given stage of our experience. We nonetheless make nonarbitrary decisions against a vast mathematical background that is not explicitly or completely known. We are continuously clarifying and extending our partial and incomplete intuition into these objects and concepts. Reason, in other words, can operate with a *partial* intuition of objects. Human reason constantly operates with partial information, with the divisions between the implicit and explicit, the exact and the inexact, and with what is present and what is absent. In the theory of computability, TMs are never described as functioning with partial, implicit, or inexact data and instructions.

The claim, to reiterate, is that to think the differences just enumerated are not real is to fall back under the spell of the leftward *Zeitgeist*, whose one-sidedness has been exposed by the analysis of its origins. It is to forget what the mechanist *Zeitgeist* conceals.

§ 6. Objectivity about the mind without eliminating subjectivity

The point about computational functionalism is actually more general. The very features that make the interpretive scheme of the natural sciences so powerful and successful in many domains of inquiry are what make it powerless when we turn it on the phenomenon of consciousness itself. Computational functionalism suffers from the same kind of flaw as a number of earlier leftward attempts to develop a science of the mind. The reason that the problem of consciousness is invariant through the identity theory of mind, neuroscientific reductionism, behaviorism, and computation functionalism is that, with some variations, they all do basically the same thing. Each takes a part for the whole. The point of the interpretive scheme is precisely to abstract from human subjectivity, but then one cannot use the interpretive scheme to account for (all aspects of) human subjectivity. This is the contradiction revealed by our genetic analysis. Behaviorists, for example, wanted to analyze human being in terms of third-person observable stimulus-response models, operant conditioning, and various dispositions to behave. This is simply a variant of the interpretive scheme I have described. But just as the human mind cannot be understood only in terms of observable behavior or non-conscious brain states, so it cannot be understood only in terms of computational functionalism. If we keep applying variants of the interpretive scheme of the natural sciences to ourselves and supposing that they give us the whole picture, then we can expect the problem of consciousness to persist. It will not help to appeal to evolutionary biology. Evolutionary biology is itself a founded interpretive scheme. It cannot be used to eliminate or reduce consciousness or intentionality. On the contrary, it is itself a theory that presupposes consciousness and intentionality. The point here is not at all to reject evolutionary biology but only to argue, without rebounding into dubious metaphysical or religious beliefs, that it cannot by itself tell the whole story of human consciousness.

The kind of genetic analysis I am presenting would predict that viewpoints would emerge that question the reality of human consciousness, qualia, intentionality, and the like. In fact, this is just what has happened. Daniel Dennett, for example, wishes to "quine qualia" (Dennett 1988). Qualia are first-person, subjective sensory experiences of colors, tastes, smells, and the like, that is, just the secondary qualities that were to be set aside in the development of the scientific worldview in the first place. Dennett's idea, which is thus not a great surprise, is to show that qualia do not exist, by following the Quinean strategy of eliminating references to any purported phenomena that are not indispensable to natural science. Paul and Patricia Churchland (Churchland 1983, 1986), Georges Rey (Rey 1983), and others have at times used language that suggests we should quine consciousness itself. In a similar vein, there have been challenges to the claim that intentionality is a genuine feature of reality. From my point of view, however, we have to ask how anyone could deny the reality of these things unless they were already in the grip of a founded interpretive scheme, which Gödel presumably thinks is part of the *Zeitgeist*, that tells us there can be no such things. One of the dangers of the interpretive scheme of the mathematical natural sciences is that it can blind us to the existence of phenomena that would otherwise be obvious. It can, under the right circumstances, cover over or conceal aspects of reality. Indeed, one might say that it can create prejudices of various types. One unconsciously accepts various background assumptions and principles that are passed down through the generations, without subjecting the assumptions and principles to critical scrutiny.

Once again, I am not arguing that we should not try to understand human beings through their observable behavior, through neuroscience, computational functionality, evolutionary biology, and so on. We should of course study human behavior, engage in neuroscience, study computational aspects of cognition, and accept evolutionary biology. The point is rather to keep these studies in perspective.

The views of knowledge and existence associated with the interpretive scheme of the natural sciences, in other words, need to be carefully scrutinized. It might be held that only the rigorous, the exact, or the precise statement can be true. Knowledge is confined to the quantifiable, the exact, and so on. The rest is opinion or illusion, and is unreliable. The qualitative, contentual, and non-computational are shunted off to the margins, and the quantitative and computational occupy the center of our attention. They are, as it were, in the foreground of modern awareness. There is inattention to the other features of our experience. What has our attention is what is important. Our intentions, goals, and purposes are fixed accordingly. What is valued and what is not valued is thus also fixed in this way. The features of our experience that have been forgotten or marginalized are valued less than the features at the fore.

The effect of the leftward *Zeitgeist*, if the argument above is correct, is to abstract away from and then forget about human subjectivity, in an effort to obtain an objective perspective on the world. There is a tendency in this process to create a false dilemma: we can countenance either objectivity or subjectivity but not both. What is attributed to subjectivity is illusory, unreliable, fluctuating, relative, inexact, and so on. But why can there not be some objectivity about human subjectivity, or about human consciousness, without making consciousness disappear? This would be an objectivity about subjectivity without the physicalistic reductionism. It would be an objectivity about subjectivity that depends on findings of reason, not on the particularities of introspection. The difference between reason, when it is directed toward human consciousness, and introspection, is that introspection gives us what is particular and private to individual subjects. Introspection yields the kind of data that shows how individual subjects are different from one another. Reason delivers what is universal. The claim that many types of human consciousness exhibit intentionality, for example, is not a deliverance of introspection. It is not something that is true for one person but false for another. It is not some fleeting, relative feature of human consciousness. On the contrary, the claim that, for example, there can be a belief even though the belief is not about anything, yields a (material) contradiction on the concept of belief. It is reason that makes possible the awareness of contradictions (impossibilities), not introspection. It appears that there are many other such universal features of human consciousness. There could not be consciousness of objects in our experience, for example, if various other kinds of structures were not in place, such as a retention-protention structure, certain types of memory, and so on (see, e.g. Husserl 1928 and 1962). It is such universal and hence objective features of human consciousness that eidetic phenomenology seeks to display.

A skepticism according to which everything about human subjectivity is illusory, relative, or fluctuating is skepticism gone awry. There can be some objectivity in the phenomenology of the human mind, for certainly there are common structures of cognition that make sciences such as mathematics possible. We can look for the invariants in human cognition and at the same time do justice to the mind and its forms of directedness, its capabilities for abstraction, and so on. In short, we need to keep both subjectivity and objectivity in the picture.

I would like to conclude this section with two brief remarks about metaphysics, minds, and machines. First, how should we think about realism or platonism in relation to what Turing machines can do? My remarks here parallel the earlier remarks about strict formalism. Start with realism about the physical world. Consider the existence of a house. Can I write a house into existence with a bunch of programming code? The realist will hold that you cannot write a house into existence in this manner. A real house is machine-independent, and hence mind-independent if minds are machines. At best, you can create something on your monitor or some other peripheral device that seems like a house. You can create a better or worse simulation. A real house is something that exists outside of, or independently of, the virtual world generated by the computer. It is something that transcends this virtual world. The claim of the realist is that with computers you can never get outside of the simulation to the reality. Similarly, in the case of mathematics, the claim of the platonist is that you cannot write a mathematical object or truth into existence on the basis of a bunch of programming code. A natural number, a set, or a mathematical truth must exist outside of or independently of the world generated by the computer. It is something that transcends this virtual world. It transcends syntax.

Given our distinction between naive platonism and constituted platonism, we need to put a somewhat finer point on these remarks. We need to keep in mind that the notions of independence, transcendence, or of "being outside of" carry different meanings. According to constituted platonism we can never get outside of appearances to things-in-themselves. We must set aside the metaphysics of naive platonism. We cannot know about, or be aware of, objects that are mind-independent$_1$. This might seem similar to the claim that, like computers, we cannot get outside of our own virtual world to something that is independent of that world. The whole point of constituted platonism in mathematics, however, is that we *can* be aware of, and know about, objects that are mind-independent, not in the absurd sense of mind-independence$_1$ but rather in the sense of mind-independence$_2$. What is mind-independent$_2$ is not meant as our own creation. It is meant as something that is not created by us and that is not created by a computer (or a formal system). It *is* possible to get outside of appearances to reality, as we saw in the case of the snake/coiled rope example, but we just cannot understand reality in terms of completely inaccessible things-in-themselves. In relation to computers, constituted platonism is thus the claim that you cannot write an independent$_2$ *mathematical* object (invariant) into existence on the basis of a bunch of programming code. A natural number, for example, is not meant that way. Mathematical objects transcend syntax. Computers, unlike monads, do not constitute the meaning of being of anything.

The second remark I want to make on metaphysics, minds, and machines concerns Gödel's suggestion, mentioned above, that the argument that mental procedures are mechanical procedures is valid if one assumes that (i) there is no mind separate from matter and (ii) the brain functions basically like a digital computer (see Wang 1974, p. 326). Gödel commented that (ii) was very likely but that (i) was a prejudice of our time that might actually be disproved. My worry about Gödel's suspicions concerning (i), especially given the paucity of his remarks on the subject and some of his comments about religion, is that he perhaps accepts a kind of substance dualism about mental and physical phenomena. I want to avoid substance dualism. I have now argued that one need not be committed to it in order to make the point about differences between minds and machines. In relation to (i), I said I would not venture beyond the fact that human consciousness exhibits intentionality, and brain states do not exhibit intentionality. If we hold only to this distinction and we adopt Husserl's method of the *epochē*, then we can say that human minds are not TMs without taking a death-defying leap into a new metaphysics. Perhaps Gödel really believed, or at least wanted to believe, that the mind is a disembodied or non-physical spirit, some kind of real thing-in-itself behind all possible appearances. On the view I have developed, we do not, and cannot, claim or presuppose any such thing. We need to bracket such a claim. Although one can perhaps entertain such a conception of the mind or monad, there is no evidence for such a view. We have no appearance of such a mind or monad in our experience. There is no intuition, in the sense of a fulfilled intention, of such a mind. What we retain is not a questionable metaphysics of the mind but rather a rich phenomenology of human consciousness, along with a variety of empirical approaches to understanding the human mind, in neuroscience, observable behavior, cognitive psychology, computational models, and evolutionary biology.

I agree that it is difficult to see how all of these perspectives on the mind cohere, but I think we should just admit that we are dealing with a very complex phenomenon and continue to pursue each approach, rather than attempting to explain away the difficulties in order to make ourselves philosophically comfortable. As Thomas Nagel has put it, it is natural to feel victimized by philosophy, but the defensive reaction to philosophy found in deflationary theories such as positivism and pragmatism goes too far (Nagel 1986, pp. 11–12). There is a persistent temptation to turn philosophy into something less difficult and more shallow than it is.

§7. Rationalistic optimism and decidability as an ideal

How does the non-mechanist view of the human mind we have been discussing point us in the direction of option (4) instead of option (3) among the possibilities allowed by Gödel's disjunction? I would like to make a few brief comments on this issue before concluding this chapter.

In our attempt to find a workable combination of leftward and rightward views, the contrast with blind formalism or blind mechanism is striking. Why will it not be possible to replace human mathematicians with machines? In blind mechanism there is no directedness by way of meaning, including no directedness by way of acts of abstraction, idealization, reflection, imaginative variation, and so on. There is no directedness by, or toward, abstract objects or concepts. This makes a difference to questions about absolute undecidability. According to the argument above, the capacities of human reason are not identifiable with the capacities of mechanical computation. Human reason is capable of making systematic but non-mechanical decisions in mathematics in a finite amount of time and space on the basis of grasping abstract mind-independent$_2$ concepts or objects. Open problems in mathematics can, in accordance with what we have said, be viewed as expressions of meaning-intentions that can be either fulfilled or frustrated. We can see a sentence such as $2^{\aleph_0} = \aleph_1$ as an expression of an intention that we expect to be either fulfilled or frustrated, instead of viewing it as no more than an uninterpreted string of signs, such that either $2^{\aleph_0} = \aleph_1$ or $\neg\, 2^{\aleph_0} = \aleph_1$ can be derived from other uninterpreted sign configurations in a mechanical way on the basis of uninterpreted sets of "rules." We cannot rule out the prospect of finding necessities relative to possibilities of imaginative variation that might decide the problem.

It is the nature of human reason in its scientific practice to proceed from what is given and constantly hypothesize or to intend states of affairs that are not yet realized but that may be realized. An open horizon is built into our meaning-intentions. This drive toward the resolution of open problems, the will to know as we described it in chapter 5, may be viewed as part of our cognitive make-up. In the context of our remarks above, it amounts to the effort to come to know more about a transcendent or mind-independent mathematical reality on the basis of continuous analysis and clarification of the abstract concepts (or meanings) by virtue of which we are directed toward that reality. Optimism about solving open problems is, on this view, accompanied by a kind of mathematical and logical platonism.

I would like to suggest that Hilbert's optimism about mathematical problem-solving, an optimism that Gödel does not want to abandon, can be retained if we shift to this kind of platonic rationalism. Rationalistic optimism about mathematical problem-solving can be viewed as a regulative ideal in a quasi-Kantian or Husserlian sense. The ideal of decidablility regulates our scientific thinking. It is the ideal of coming to decide clearly posed "yes" or "no" questions about the universe of mind-independent abstract mathematical and logical objects. To say that it is a regulative *ideal* does not, of course, imply that we will actually complete the task of solving all open mathematical problems at some point in human history. Even if there are unsolved problems at any given stage of mathematical research it need not follow that there are absolutely unsolvable mathematical problems. It is precisely because the real completion of our knowledge is not, or may not be, possible that decidability can be considered an "ideal." Nonetheless, it stands as a postulate of reason. It is a projection of completeness, "lying at infinity." It is the idea of perfection of fulfillment as regards the aims of mathematical reason. Such an ideal is what lies behind not halting with respect to deciding clearly formulated mathematical problems. We might say that it is only because we are aware of such an *ideal* in the first place that we realize that we have fallen short, that our knowledge is incomplete. It is this fact about human cognition, this awareness of an ideal, that is a condition for the possibility of the awareness of open problems.

Measuring our knowledge against an ideal of decidability (by reason) is, in effect a condition for the possibility of mathematics as a science, in the sense that the scientific endeavor would collapse (or not even commence) were such an ideal not regulating our behavior. Why would we pursue a problem if we thought that there could be no answer to it? We presumably do not want to give up on Hilbertian optimism as such a regulative ideal. If we were not driven by this ideal, would we perhaps have given up on solving such problems as whether there is a consistency proof for arithmetic or real analysis, whether CH is consistent with ZFC, or whether $\neg$ CH is consistent with ZFC? These are problems I mentioned earlier that appear to require developments in our rational capacities, or what Gödel calls an ascent to "higher" states of consciousness.

If we eschew blind mechanism, if the human mind is not a finite machine, then it is not clear that there are pressing reasons to hold that there are absolutely undecidable Diophantine problems. It has been suggested by some that perhaps set theory will split around CH, so that there will be one set theory in which $2^{\aleph_0} = \aleph_1$ and another in which $\neg\, 2^{\aleph_0} = \aleph_1$, on the analogy with the split in geometry around the parallel postulate. Although I will not take the matter up here, it can be noted that Gödel (Gödel 1964), Kreisel (Kreisel 1971), and Wang (Wang 1974, pp. 205–206) have argued that the situation is not analogous to the split in geometry around the parallel postulate. In any case, it is possible to say enough about the *meaning* of CH to cast doubt on this view. In the case of Diophantine problems of the type Gödel has in mind, it is highly unlikely that there could be an analogous kind of split. One might think the case for rejecting absolute undecidability is more compelling in connection with Diophantine problems, but this kind of optimism might also be extended to any mathematical problem that can be clearly formulated, for we must allow *any* method for producing conclusive evidence that human reason can conceive, and it is difficult to determine in advance what this might include.

Gödel argues, as we saw earlier, that the notion of reason collapses in a kind of contradiction if it is possible for reason to pose clear "yes" or "no" problems in mathematics and logic that are absolutely undecidable by reason. We see the fruits of reason all around us in the amazing progress that has been made in mathematics, logic, and, properly understood, the natural sciences.

8

Reason, Science, and Evidence

In this final chapter I would like to take up two important topics concerning science, evidence, and the type of platonic rationalism that was presented in earlier chapters. First, I want to discuss in more detail the contrast between the holistic empiricism about mathematics and logic that is due to W.V.O. Quine on the one hand and, on the other hand, the kind of rationalism that I think is entailed by Gödel's interest in developing ideas in Plato and Leibniz on the basis of the phenomenological method of Husserl. Some of the differences have been mentioned in chapter 3, 5, and elsewhere. In chapter 5 in particular it was suggested that the kind of pragmatic holism about mathematics and logic due to Quine, Goodman, and others is another problematic "leftward" view of reason. I want to expand on this claim in the present chapter. Quine's holistic views on logic and mathematics are worth considering in connection with Gödel for several reasons. They developed in response to Carnap and they have been very influential in philosophy since the 1950s. They still exert a lot of influence on a number of major scholars in the philosophy of mathematics and the philosophy of logic, including some scholars who have written at length on Gödel's work. Quine's views have also been associated with Gödel's ideas about extrinsic justification of mathematical axioms, especially axioms of set theory, in some of the literature on the intrinsic/extrinsic evidence distinction. Since I think there are problems with assimilating Gödel to Quine on the matter of holism and extrinsic evidence, it will be worthwhile to address some of these issues before bringing the book to a close.

Second, Gödel suggests in his 1961 text that he wants to find a workable, properly balanced combination of leftward and rightward directions in philosophy and the foundations of mathematics and logic. We have seen many of his claims about the excesses of the leftward direction, but we also want to avoid the excesses of earlier or traditional forms of rationalism. I would like to close this book with a number of remarks about how the rationalistic platonism I have developed can address this issue. This matter is made all the more important, I think, by the fact that Gödel either held, or is alleged to have held, some peculiar and rather dubious views about angels, demons, ghosts, metaphysical mind/body dualism, immortality of the soul, various religious doctrines, and the like. Many of these views, it seems to me, are of an altogether different type from rationalistic platonism about logic, mathematics, and science. As I have been saying throughout this book, mathematics and logic are our most exact and rigorous sciences, but they are not natural sciences. Any defensible form of rationalistic platonism should, in my view, be linked to the requirements of science and evidence. The requirements of science and evidence, however, are not coextensive with conditions on natural science. The natural sciences only approximate the rigor of the eidetic sciences of mathematics and logic. I want to put aside Gödel's comments on angelology, theology, demonology, and so on. I think they have little bearing on finding a defensible form of platonic rationalism about mathematics and logic. It is perhaps also worth mentioning that some of Gödel's critics are quick to emphasize his bouts of mental illness, as though these episodes are somehow a *reductio* of his views on platonic rationalism about mathematics and logic. I think that this kind of suggestion has to be strenuously resisted. The latter part of the present chapter is intended to add an exclamation point to this resistance.

In chapters 4, 5, and 6, I argued for a new kind of platonism, based on some of Gödel's remarks about the importance of using Husserl's phenomenological method. Constituted platonism is not naive platonism. It provides a more balanced account of the platonism that appears to be part of mathematics as it is actually practiced. We ask about the rationally motivated, non-arbitrary constitution of the meaning of the texts and discourses that we find in much of actual mathematical practice. We focus on the practice of mathematics, on how mathematicians mean their objects, properties, and truths, not on the distortions that arise from applying some pre-conceived philosophical scheme, such as logical positivism, to mathematics. The balancing of platonism in constituted platonism is, at the same time, a balancing of our conception of reason. Human reason cannot know about abstract mind-independent$_1$ objects and truths. Such objects and truths would be altogether beyond our ken were they to exist. Rationalism in this broad sense, qualified by the phenomenological method, is not naive. In the second part of the chapter, I will add to this point a number of further conditions on what should count as limits of reason, science, and evidence, if we are to develop a defensible combination of what Gödel calls the leftward and rightward directions in philosophy.

§ 1. Gödel and Quine: some comparisons

We can start by recalling some basic points in Gödel's critique of Carnap. In chapter 3 we saw that Gödel wanted to draw three main conclusions about Carnap's view of mathematics as syntax of language: first, we cannot replace mathematical intuition with conventions about the use of symbols. The conventionalism and nominalism involved in Carnap's early view of mathematics is undermined by the incompleteness theorems. In his later writings Gödel recommends, as an alternative, that we read Husserl's work on categorial or "eidetic" intuition. Second, it could not be the case that mathematics does not describe any existing objects or facts. Mathematical propositions could not be empty tautologies that are void of content. The truths of mathematics could not be based solely on linguistic conventions. Rather, true mathematical propositions and true empirical propositions are both about objects or facts, and there are analogies in our knowledge of such objects, although the objects in the one case are concrete and are given in sense perception, and in the other case are abstract and are given in mathematical intuition. Third, Gödel thinks that the logical positivist's way of trying to establish the compatibility of the a priori and the a posteriori, the mathematical and the empirical, the analytic and the synthetic, will not work.

This last point, it was noted, is also at the center of Quine's famous criticism of the logical positivists, but I think that Gödel's reasons for holding it are different from Quine's. Gödel says that there is one ingredient of Carnap's incorrect theory of mathematical truth that is correct and discloses the true nature of mathematics, namely, it is correct that a (pure) mathematical proposition says nothing about physical or psychical reality existing in space and time, because it is already *true owing to the meaning of the terms occurring in it, irrespective of the world of real things*. What is wrong with Carnap's view, according to Gödel, is that the meaning of the terms (that is, the abstract concepts they denote) is asserted to be something man-made, and consisting merely of linguistic conventions. In contrast with Carnap, but also with Quine, Gödel says

> The truth, I believe, is that these concepts form an objective reality of their own, which we cannot create or change, but only perceive and describe. Therefore, a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says something about relations of concepts. (Gödel *1953/59, p. 320)

Gödel contrasts his own view of analyticity with Carnap's: "analytic" does not mean "true owing to our definitions," but rather it means "true owing to the nature of the concepts" occurring in mathematical statements (Gödel *1951, p. 321). Analyticity is a matter of the relation of abstract concepts to abstract concepts. The necessity associated with mathematical truths is presumably a function of this fact. It is not man-made. We are forced or constrained in certain ways by the concepts. It is also not a matter of the relation of abstract concepts to sense perceptions or to the empirical content of the natural sciences. In the Gibbs Lecture Gödel says that his concept of analyticity is so far from meaning "void of content" that it is possible that an analytic proposition might be undecidable, or decidable only with a certain probability. This is because our knowledge of the world of abstract concepts may be as limited and incomplete as that of the world of physical objects.

A number of differences between Gödel and Quine emerge from these comments that Gödel makes about Carnap. First, Gödel says that mathematical propositions say nothing about physical or psychical reality existing in space and time, because they are already true owing to the meaning of the terms that occur in these propositions, irrespective of the world of real things. They are true on the basis of the relations of the abstract concepts that they denote. Quine would deny such a claim. I think it is compatible with Gödel's view to hold that mathematical propositions *do* say something about the physical world in their *applications* to the physical world. It is just that their truth or falsity as mathematical propositions is not determined by whether they have such applications or not. We should not prohibit the mathematician from distinguishing true from false propositions in some geometry, for example, because the geometry has no applications to the physical world. Gödel's claim that these abstract concepts form an objective reality of their own that we cannot create or change would be challenged by Quine, as would the claim that we can perceive (or intuit) and describe this objective reality. Quine's platonism about mathematics, unlike Gödel's, is purely a function of accepting the ontological commitments of the mathematics needed in order to engage in natural science. If the mathematics that is indispensable to natural science involves us in quantification over abstract objects, then we should accept the existence of such objects. On the other hand, if such quantification is dispensable, then we need not accept the existence of objects of this type. This is far from the view that in mathematics there is an unchangeable objective reality that we can only perceive and describe. Moreover, there is no notion of the "perception" or intuition of abstract objects or concepts in Quine's work. Quine has no notion of rational, categorial, or eidetic intuition, and he does not recognize a correlative form of evidence that is unique to mathematics.

Generally speaking, the elements of Gödel's rationalism about mathematics and science contrast rather sharply with Quine's empiricism about mathematics and science. Gödel is interested in the idea that philosophy itself can become a rigorous a priori science, whereas Quine is interested in naturalizing philosophy. For Quine, as we will see below, it is not at all clear how we are to understand the meaning or content of unapplied parts of mathematics, including higher set theory. It could evidently not be imbibed from empirical content. Gödel would evidently agree that it could not be imbibed from empirical content, but in his later work he gives every indication that he thinks higher set theory is meaningful and does have its own content. Gödel's remarks about combining ideas of Plato, Leibniz, and Husserl suggest that he wants to use Husserl's phenomenology to bring about an improvement and realization of the Leibnizian idea of a universal ontology as the systematic unity of all conceivable a priori sciences on a new, non-dogmatic foundation through the use of the transcendental phenomenological method. This seems to be worlds apart from Quine's vision of philosophy, mathematics, and logic.

To some readers, Gödel seems closest to Quine in some of the comments he makes in his paper on Russell (Gödel 1944) and in some of his related remarks on "extrinsic" justification of axioms in his paper on how the continuum problem might be solved (Gödel 1964). There are also some remarks in the same vein in Gödel's work on Carnap (see, e.g. Gödel *1953/59, p. 347, including footnote 33). I think it is not at all clear, however, that the comments in question could be regarded as Quinean. In the Russell paper he notes how Russell compares axioms of mathematics and logic with laws of nature, and mathematical evidence with sense perception. Axioms need not be evident in themselves. Their justification could lie in the fact that they make it possible for "sense perceptions" to be deduced. Gödel is not literally linking mathematics to sense perception in these comments. That is why he puts "sense perceptions" in scare quotes. He says that, in mathematics, the domain of elementary indisputable evidence that may be most fittingly compared with sense perception is arithmetic (Gödel 1944, p. 121). There is evidence in sense perception but there is also evidence in mathematics. What he is saying is that, instead of being intrinsically plausible the justification of axioms could lie in the fact that they make it possible for arithmetic, that is, mathematical, results to be deduced.

In the draft of the Carnap paper just cited, Gödel says that in whatever way mathematics is built up, one always needs certain undefined terms and certain axioms (deductively unprovable) assertions about them. For these axioms there exists no other rational (and not merely practical) foundation except that either (i) they (or propositions implying them) can be directly perceived to be true owing to the meaning of the terms or by an intuition of the objects falling under them or (ii) they are assumed, like physical hypotheses, on the grounds of inductive arguments, for example, their success in applications (Gödel *1953/59, p. 347). In a manner similar to his comments in the Russell paper, Gödel says that in case (ii), insofar as it may be judged to actually occur, mathematics still has content, and that the mathematical character of the axioms (in spite of their inductive foundation) appears in the circumstance that they have consequences in that part of mathematics covered by case (i), i.e. in which an immediately understandable clear meaning of the primitive terms is perceived. An example would be axioms of infinity that have number-theoretic consequences. Gödel's example here of what he means by "success" of a non-evident mathematical axiom refers only to pure mathematics, and it amounts to the fact that many of the consequences of the non-evident axiom can be verified on the basis of the evident axioms, the proofs however being more difficult, and further that it solves important problems not solvable without the axiom. (By the second incompleteness theorem, the consistency of such a non-evident axiom would be indemonstrable on the basis of the evident axioms, provided it solves any problem of Goldbach type.) Here again Gödel is not linking "success" or "consequences" to sense perception or natural science. "Success," "consequences," and "applications" are, rather, intra-mathematical in nature. Gödel also seems hesitant, for good reasons I think, about whether it ever actually happens in mathematics that axioms are assumed like physical hypotheses on the grounds of inductive arguments.

Turning to the Cantor paper (Gödel 1964), Gödel considers how we might find a new axiom or axioms that will decide the continuum hypothesis. He devotes most of his attention in the paper to the question of how we might find an axiom that would be accepted on the grounds of its intrinsic necessity, but then he says that even if we disregard "the intrinsic necessity of some new axiom, or even in case it has no intrinsic necessity at all, a probable decision about its truth is also possible in another way, namely, inductively by studying its 'success'" (Gödel 1964, p. 261; see also p. 267). What is meant by success in this context, we are told, is fruitfulness in consequences, in particular in "verifiable consequences." It is important to be careful about the notion of verifiable consequences here. What he tells us, as in the Carnap paper, is that fruitfulness in verifiable consequences means that there are consequences demonstrable without the new axiom whose *proofs* with the help of the new axiom are considerably simpler and easier to discover, and that make it possible to contract into one *proof* many different proofs. As an example, Gödel now mentions the classical axioms for real numbers that are rejected by intuitionists. He says that these axioms have been verified to some extent because analytical number theory frequently allows us to prove number-theoretic theorems that, in a more cumbersome way, can subsequently be verified by elementary methods. Here again he is speaking about consequences of (new) axioms for mathematics itself. Gödel goes on to say that there might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems that, whether they are intrinsically necessary or not, they would have to be accepted at least in the same sense as any well-established physical theory. This seems to be the same kind of analogy he is making in the Russell paper. The "extrinsic" justification of the axioms is tied to *mathematical* evidence, not to perceptual evidence, or evidence in the natural sciences.

Later in the Cantor paper (Gödel 1964, p. 269) Gödel returns to the issue of intrinsic/extrinsic justification and says that besides mathematical intuition there exists another criterion of the truth of mathematical axioms, although it is only probable: namely, their fruitfulness in mathematics and also, possibly, in physics. Because he now mentions the potential for fruitfulness in physics, he appears to be allowing for the consequences of new axioms for natural science. It does indeed seem possible that new axioms in mathematics could have fruitful applications in physics. Gödel himself, however, gives no examples of what this could mean. The examples of fruitfulness that he does provide are concerned with mathematics, as just indicated. Gödel says that the criterion of fruitfulness might become decisive in the future, but that it cannot yet be applied to the specifically set-theoretic axioms, such as those referring to large cardinal numbers, because very little is known about their consequences in other fields. When he mentions an example involving set theory it is that the simplest case of the application of this fruitfulness criterion arises when some set-theoretical axiom has number-theoretic consequences verifiable by computation up to any given integer. At the time that he wrote these remarks Gödel added that it was not possible to make the truth of any set-theoretic axiom reasonably probable in this manner on the basis of what was then known.

It seems that Gödel thinks that intrinsic and extrinsic justification of axioms could both have a place in mathematics. There is intrinsic justification when it is possible for there to be intuition of the relations of abstract concepts to one another or intuition of objects falling under concepts. Intrinsic justification is tied up with mathematical intuition of abstract objects and with analyticity. In this case, truth is based on the relations of mind-independent abstract concepts (or meanings) to abstract concepts (or meanings), or on relations of concepts to objects falling under concepts, independently of sense experience. In this case we are dealing with necessary truths. Gödel evidently thinks that extrinsic justification is still possible even when axioms are not supported directly by intuition and are not seen to be true on the basis of concept to concept relations. Here the evidence is probabilistic, being based on fruitful consequences. In the examples he actually gives, the idea appears to be that some parts of mathematics will be regarded as intrinsically supported on the basis of mathematical intuition and that other parts of mathematics not so supported might nonetheless be justified on the basis of the extrinsic criterion of fruitfulness. The part of mathematics regarded as intrinsically supported will be more limited relative to the part that is extrinsically justified. Suppose, for example, that someone thinks that axioms of (some part of) intuitionism or some other form of constructivism or even platonism about the natural numbers are intrinsically justified. The idea is that relative to such views of mathematical evidence we might still have some justification for axioms not supported on these grounds, only now it will be extrinsic justification. Extrinsic justification might thus lead from acceptance of a more limited view of mathematics to acceptance of a more extended view, albeit on probabilistic grounds. Similarly, if we are skeptical of full set theory we might still regard some of its axioms, or a new axiom, to be "verified" on such extrinsic grounds.

Quine, by way of contrast, does not have a place for intrinsic justification of axioms at all. The idea that extrinsic justification is probabilistic and evidently does not involve analyticity or mathematical intuition seems, on the surface, to be more Quinean in character, except that, as we have noted, Gödel's actual examples are intra-mathematical. For Gödel, the success or fruitfulness of an axiom means that there are consequences demonstrable without the new axiom whose proofs with the help of the new axiom are considerably simpler and easier to discover, and that make it possible to contract into one proof many different proofs. "Success" refers to the fact that many of the consequences of the non-evident axiom can be verified on the basis of the evident axioms, the proofs however being more difficult, and further that a non-evident axiom solves important problems not solvable without the axiom. Gödel does not suggest in his writings, as Quine does, that we choose axioms on the basis of what is indispensable or dispensable to natural science or with an eye to minimizing ontological commitments (see below).

It seems to me that Gödel could not favor extrinsic justification alone, for many of his ideas that we have discussed in this book would then have to be abandoned. On the other hand it appears that, although he may favor it, he is not prepared to settle for intrinsic justification alone. What is the motivation for recognizing extrinsic justification of (new) axioms? I conjecture that it is to show that even without intrinsic justification we can still make progress in mathematics. Considering CH in particular, the idea is that, even without intrinsic evidence, we are not in a hopeless situation. Deciding CH is not just arbitrary, as though it were a matter of flipping a coin. We cannot go any way we like with it. Probabilistic evidence or weight can be considered, even if necessity is not (yet) on the horizon. There is still a reason to go one way rather than another. We can try to learn about what is probably true of the abstract world of sets. It is not as though there is no content or no meaning in this case. It is not as though CH is so vague that we cannot get some traction in thinking about it.

With respect to the prospect of finding new axioms in set theory that settle CH, it might be suggested that Gödel actually put forward two programs, the intrinsic program and the extrinsic program. One might then argue for one or the other of these, or claim that both should continue to be pursued (see, e.g. Koellner, forthcoming). Would we regard CH as settled, however, if the decision depends on an axiom that is only extrinsically justified? I think the point here is that under such conditions we would regard CH as probably true or probably false, but not that the problem is solved once and for all. In any case, I do not think that we are in a position to rule out the possibility of intrinsic justification of a new axiom in set theory that settles CH, even if some of the existing suggestions about how to proceed (e.g. by ways of axioms of infinity) have not been successful. Perhaps we simply do not yet have the right concepts. With the acquisition and application of new concepts, a new axiom might gradually become evident to us. I have to confess, however, that it is not completely clear to me how the two criteria of the truth of mathematical axioms are supposed to ultimately be related to one another in Gödel's thinking.

§ 2. Quinean holistic empiricism

In order to develop a deeper appreciation of the contrast between the views of Gödel and Quine, let us examine Quine's views in more detail, especially his considered position as it is encapsulated in his latter writings (see especially Quine 2008a and 2008b, but also the earlier work 1951, 1960, 1970, 1974, 1984, 1986, 1992 and 1995). Quine describes how his holistic empiricism emerged in response to Carnap's work. He says that Carnap's adherence to analyticity was due largely to his philosophy of mathematics. There were two problems for Carnap's view of mathematics. One of the problems concerned the explanation of the necessity of mathematical truth. The other problem concerned the lack of empirical content in mathematics. Since mathematics is supposed to lack empirical content, how could an empiricist accept mathematics as meaningful? Analyticity, Quine says, was Carnap's answer to both problems.

In his later work Quine responds to both problems with his "moderate" holism. His own view of mathematics, as we noted, is pegged on the natural sciences. Quine argues that when a prediction in the natural sciences fails, it is a body of theoretical sentences that together imply the failed prediction. He says that his earlier holism was needlessly strong. We need not hold that the unit of empirical significance is the whole of science. In later writings he invokes not the whole of science but, rather, clusters of it that are just inclusive enough to imply an observable effect of an observable experimental condition. The body of theoretical sentences that imply a false prediction will include not only sentences from biology, physics, or whatever, but also from mathematics. It will include, for example, sentences from arithmetic.

Quine says his moderate holism clears up the problem of the necessity of mathematical truth without the help of analyticity. The solution to the problem is to say that we are free to choose which of the implying body of sentences to revoke, but that we are also free to spare arithmetic. We will in fact spare it because to upset it would reverberate through all branches of science. Scientists observe a maxim of minimum mutilation. This explains the quality of necessity that is felt to attach to arithmetic truth. The necessity of these truths lies simply in our freedom of choice. We are free to safeguard those tenets and revise others instead when revision is called for (Quine 2008a, p. 335). Quine sometimes speaks as though this point is supposed to hold for any other truths of mathematics but, as we will see below, there will be a problem for such a view in the case of unapplied parts of mathematics. Quine says we will not in fact choose to revoke a purely mathematical sentence because to do so would reverberate excessively through the rest of science. The necessity of mathematics lies in our determination to make revisions elsewhere instead. There is no deeper sense of necessity anywhere.[1]

Concerning the problem of lack of content, the solution is as follows: insofar as mathematics is applied in natural sciences it shares empirical content. Arithmetic and differential calculus partake of the empirical content imbibed from the implication of observation categoricals. Mathematics does not imbibe empirical content in the inductive way that Mill supposed, but it imbibes it in the hypothetico-deductive manner of theoretical science. As Quine notes, however, this raises another problem. What should we say about inapplicable parts of mathematics, such as higher set theory? These parts of mathematics share no empirical meaning because they are not applied. Here Quine, unlike Gödel, says he sympathizes with the empiricist in questioning the meaningfulness of these parts of mathematics. We keep their sentences as meaningful, but only because they are built of the same vocabulary and grammatical constructions that are needed in applicable mathematics. It would be an intolerably pedantic tour de force to gerrymander our grammar in such a way as to account the inapplicable flights ungrammatical while preserving the applicable parts. This would be a thankless task.

But now how are we to understand truth and falsity for the inapplicable sentences? Here we can get some guidance from a maxim that already serves the natural scientist, namely, Occam's razor: where a choice is otherwise undetermined, opt for economy. Quine says he wants to minimize the cleavage between mathematics and natural science, and that this tendency is abetted by his point above about shared empirical content and also by his questioning of the analytic/synthetic distinction.

The inapplicable parts of mathematics are not wholly inscrutable. The main axioms of set theory, Quine says, are generalities operative already in the applicable part of the domain. Further sentences such as the continuum hypothesis and axiom of choice,

[1] As we saw in chapters 2 and 3, Gödel is evidently prepared to recognize different degrees of evidence in different parts of mathematics, e.g. in arithmetic as distinct from set theory. Even so, I do not think he could hold that necessity in mathematics is a function of observing the principle of minimum mutilation. The later principle, as just suggested, has no application at all in unapplied parts of mathematics. On Quine's view it appears that the quality of necessity could not (be felt to) attach to any statements of unapplied mathematics. But are such statements supposed to therefore all be on a par, as though there is no difference between the theorems and non-theorems in these parts of mathematics? See below.

which are independent of the main axioms, can still be submitted to considerations of simplicity, economy, and naturalness that contribute to the molding of scientific theories generally. Such considerations, Quine says, support Gödel's axiom of constructibility, V = L. This axiom "inactivates the more gratuitous flights of higher set theory," and incidentally it implies the axiom of choice and the CH. Quine says it is all a matter of tightening and streamlining our global systems of the world. This view is, of course, very different from Gödel's position in his Cantor paper. Gödel does mention the possibility of choosing as axioms those propositions that have the most fruitful consequences in physics, but V = L does not have fruitful consequences in physics. In fact, it has no consequences in physics. Quine chooses it because it inactivates the more gratuitous flights of higher set theory, not because it has fruitful consequences in physics.

In another remark that is quite at odds with Gödel, Quine says of the vast proliferations of inapplicable mathematics that "these domains are integral to our overall theory of reality only on sufferance." It is left to us to try to assess these sentences as true or false, if we care to. Many are settled by the same laws that settle applicable mathematics. For the rest, we are to settle them as far as is practicable by considerations of economy, on a par with the decisions we make in natural science when trying to frame empirical hypotheses worthy of experimental testing (Quine 2008a, p. 468).

In his later writings Quine allows that there is a common-sense notion of analyticity (Quine 2008a, p. 395). On the view in the *Roots of Reference* (Quine 1974), for example, *logical* truths are analytic. It needs to be kept in mind that what Quine means by "logic" is classical first-order logic with identity. If logical truths are analytic, that is, true by the meanings of words, then what should we say of proposed revisions, such as the revisions that would eliminate the Principle of the Excluded Middle (PEM) as a law of logic? If we make the kinds of revisions that lead to the rejection of the PEM, is this a change of theory or just a change of subject, a change in the meaning of our words? Quine's answer is that for logic (in his sense), a change of theory is a change of meaning. Repudiation of PEM would be a change of meaning, and no less a change of theory for that (Quine 2008a, p. 396).

Mathematical truths are supposed to share in the empirical meaning of sciences where they are applied. They do so by participating with other sentences in jointly implying observation categoricals. This cannot be said of logical truths. Any sentence already implies any logical truth and thus gains no further implying power by being conjoined with it (Quine 2008a, p. 432). So the grounds for revision of logic are evidently different from the grounds for revision of mathematics.

§ 3. Against holistic empiricism about mathematics and logic

Let us now turn to a critical examination of these claims (see Tieszen 2000, and also Parsons 1983b, 1995b). The first point I want to take up concerns Quine's claim that the quality of necessity is felt to attach to mathematical truths because choosing to revise mathematical sentences would reverberate excessively through the rest of science. We observe a maxim of minimum mutilation. The necessity of mathematics lies in our determination to make revisions elsewhere instead. Does it follow from Quine's view that the quality of necessity is not felt to attach to truths of inapplicable mathematics? Presumably this quality is not present in inapplicable parts of mathematics because, in this case, we do not have truths that will be included in bodies of theoretical statements that imply false predictions in natural sciences. Now we need to keep in mind how much of mathematics is unapplied. Quine allows at one point that there are, in fact, extensive proliferations of inapplicable mathematics. This later point seems to me to be correct. Consider the vast segments of geometry, number theory, topology, algebra, category theory, set theory, and so on for which there are no known applications in natural science. Is it correct that the truths in these areas are not felt or held in mathematical practice to be necessary, as though all sentences formulated in the languages of these parts of mathematics are on a par? To answer "yes" to this question seems to me to show a blatant disregard for actual mathematical practice. Moreover, assuming it is applicable at all, Occam's razor cannot generate such feelings or experiences of necessity. It could not have the force in inapplicable mathematics that the maxim of minimal mutilation has in applied mathematics. In any case, I cannot think of any examples where Occam's razor has been used in actual mathematical practice in pure mathematics in order to trim or reject parts of the subject because they do no work in the natural sciences.

It seems to me that the quality of necessity, however, *is* in fact felt or experienced in connection with many inapplicable parts of mathematics. After all, we have various definitions, theorems, and methods here too. We have a distinction between possibility and necessity, conjecture and proof, and so on. How to explain this sense of necessity? For Gödel the explanation lies in the relations, which can be grasped by reason, of abstract mind-independent concepts or meanings to other abstract mind-independent concepts or meanings, or it lies in the relations of such concepts to objects that fall under the concepts. The necessity is not man-made. It is not a function of man-made linguistic conventions but it is also not a function of our *choices* (which are man-made) to revoke some sentences but not others. Revocation on the basis of disturbances in the natural sciences plays no role in inapplicable mathematics. We saw above how Gödel argues that we do not have free choice everywhere in mathematics. We are forced or constrained in some ways. This is part of what it means to be a rationalistic realist or platonist about mathematics. It is part of what it means to speak of the objectivity of mathematics, including inapplicable mathematics. The constraints in mathematics are a function of how abstract mind-independent concepts or meanings are related to one another. For Gödel the necessity of mathematical truths, including those of inapplicable mathematics, lies in our *lack of freedom* of choice. On what grounds are we forced if the natural sciences are undisturbed by our exact, clear, interesting, intersubjectively stable, justified and sometimes beautiful mathematics? Inapplicable mathematics is just as much a product of human reason as applicable mathematics. It is not the case that applicable mathematics is a function of reason and inapplicable mathematics is not. Our thinking in well-established parts of inapplicable mathematics is meaningful, directed, based on proof and exact methods.

Quine allows that inapplicable mathematics, unlike applicable mathematics, shares no empirical content or meaning. He sympathizes with the empiricist, as we would expect from a "leftward" oriented position, in questioning the meaningfulness of all inapplicable parts of mathematics. This too seems to me to be totally out of sync with mathematical practice. Quine says that we keep the sentences of these parts of mathematics as meaningful but only because they are built out of the same vocabulary and grammatical constructions that are needed in applicable mathematics. But what kind of meaning or content could this be? It could not be empirical content. Since Gödel evidently recognizes content or meaning that is not empirical in nature, content that arises from abstraction, idealization, reflection, variation, and so on, he will not have the same problem trying to explain this matter. Further, if we are directed in our thinking in mathematics by the meanings or contents of our acts, then Gödel will also have no problem explaining how practitioners in inapplicable mathematics could find their work to be meaningful and their thinking to be directed in certain ways and not others, how they could make conjectures but also find proofs, and so on. If our primary engagement with mathematics, however, is to use parts of mathematics such as differential or integral calculus in order to solve problems concerning the natural world, then Quine's remarks on mathematics are somewhat more plausible. This is the perspective of the engineer or natural scientist, in which practical or applied reason is at work. In this case our thinking *is* directed toward the world of nature. But this is far from encompassing all of mathematics, and even here the concepts and methods of calculus remain invariant across many different physical problems.

In the case of inapplicable mathematics, Quine urges Occam's razor when the choice is otherwise undetermined. It is not clear to me that there really ever are such cases. It is difficult to imagine a situation in actual mathematical practice in inapplicable mathematics in which our choices are so undetermined that the only option left is the use of Occam's razor. Notice also that Occam's razor is also used in a quite specific manner: to trim the parts of mathematics that do no work in natural science. The scenario would perhaps be somewhat more plausible if mathematics amounted to nothing more than playing games with meaningless symbols. This is not, however, what actual mathematical practice is like in these areas. The thinking is these areas is meaningful, directed, and typically has a history that is replete with various methods, proofs, theorems, required skill sets, and the like. There are plenty of determining features in play.

This is one of the reasons why Quine's remarks on deciding in favor of V = L in set theory are so implausible. Imagine the following news flash: the continuum hypothesis has been decided! When you read the story under the headline you find that CH is true because it is implied by V = L and that V = L is to be accepted on grounds of simplicity, economy, and naturalness of the type that contributes to the molding of scientific theories generally. You find that domains of inapplicable mathematics are integral to our overall theory of reality only on sufferance. You might have expected to read a *proof* of CH upon seeing the headline. Instead of finding a proof, however, you find the philosophical but not mathematical points just mentioned. You find, in other words, another case in which some pre-conceived leftward philosophical position has pushed its way in front of actual mathematical work. It is a position that might in fact subvert actual mathematical work. You find that V = L inactivates the more "gratuitous" flights of higher set theory. These parts of set theory are gratuitous, however, only because they do no work that we know of in natural science. Adopting V = L supposedly streamlines and tightens our global systems of the world, but what Quine means by "the world" here is just the world of natural science. It seems to me that, from Gödel's point of view, such a leftward position would just have us turn our back on the exact, abstract mind-independent$_2$ concepts or objects with respect to which mathematical problems are to be solved one way or the other on the basis of our capacity for motivated, founded forms of rational directedness. It is a position that makes us blind to such a possibility. Upon reading the story under the headline, however, it should be apparent that open mathematical problems are, in fact, not decided in mathematical practice in the manner suggested by Quine's view. Let us hope, for the sake of mathematics, that considerations of this type do not come to be seen as standards for deciding open problems in mathematics.

Gödel, I think, does not want to give up on finding rigorous proofs in order to solve mathematical problems. This is one of the "rightward" aspects of mathematics that he wants to preserve, according to the 1961 text. The notion of provability need not reduce to the purely formal notion. Rather, Gödel is concerned with the abstract concept according to which a proof is supposed to provide evidence for a proposition. We cited some texts on this in chapter 6. As I indicated a few pages ago, it is not clear that Gödel's remarks on the probablistic or inductive criterion of truth of mathematical axioms is similar to Quine's view. His remarks need not entail an abandonment of rationalistic platonism. Just as we can try to determine what is probably true of the world of sensory objects and natural science, so we can try to determine what is probably true of the world of abstract mind-independent concepts and objects. This would be probabilistic reasoning with respect to the world of abstract mathematical concepts, not with respect to the world of natural science, as in Quine. I do not think that Gödel is suggesting that we settle for probabilistic evidence in place of the evidence provided by mathematical proof. In appealing to probabilistic evidence, we are still seeking objectivity about, and reasons concerning, the world of concepts, just as we use probabilistic reasoning to seek objectivity about the world of sensory objects in the natural sciences. This might even help us to the kind of evidence provided by mathematical proof. Moreover, probability theory itself is arguably a product of reason. It is not reducible to empirical psychology, biology, and so on.

In discussing the inapplicability of higher set theory, Quine tells us that the main axioms of set theory are generalities that are already operative in the applicable part of the domain. If we allow that this is true in the case of set theory, we still need to ask whether the point holds for generalities in the many other parts of inapplicable mathematics, for example, for various areas of geometry, algebra, topology, category theory, function theory, and so on. How, for example, are the generalities of some n-dimensional geometry, for sufficiently large n, operative already in the applicable part of the domain, if there is no applicable part of the domain? One can nonetheless obtain various results about these geometries. Our thinking about them appears to be meaningful and directed in certain ways but not others, even if they lack empirical content. We can distinguish possibility from necessity, and mere conjecture from proof, in these areas of mathematics.

The reasons given by Quine for accepting V = L are suspect on other grounds as well. Quine relies on the fact that CH and the axiom of choice (AC) are independent of the other standard axioms of set theory, axioms that are supposed to be operative in the applicable parts of the domain. We can of course read a *proof* of this fact, part of which is due to Gödel (Gödel 1940) and the other part of which is due to Cohen (Cohen 1966). But how could we have possibly arrived at these independence results on the basis of Quine's view? Showing that CH is consistent with the other axioms of set theory, as Gödel did, requires that we take the classical ordinals of ZF as given. Quine would have presumably already inactivated the parts of set theory that imply the existence of such ordinals. We would not have them at our disposal for Gödel's proof of the consistency of CH with the other axioms of ZF. Hence, one can ask how it is supposed to be known and thereby assumed, as it is by Quine, that CH and AC are independent of the other axioms of set theory. It is not clear what a result such as Gödel's could mean or how it could be obtained if we take Quine's view seriously. Furthermore, we can ask about the impact that a view such as Quine's would then have had on work that came later, such as Cohen's results. Unlike Gödel, Quine would also feel no need to call for new axioms in areas such as higher set theory. What would be the point of calling for new axioms in inapplicable parts of mathematics? None are required for existing natural science. Gödel, on the other hand, thinks we need new axioms in order to solve open mathematical problems on the basis of reason. As a kind of empiricism, Quine's view has run out of steam long before reaching this point. As a rationalistic optimist, Gödel is just getting started.

As I said earlier, I think this kind of holistic empiricism, insofar as it is supposed to extend across sense perception *and* mathematics and logic, presents us with another "leftward" view of reason. It is another variation on the empiricist *Zeitgeist*. It is plausible for a particular category of concepts, namely, concepts that pertain to sense perception, provided this is understood in the following manner: our consciousness involving these concepts can be revised, corrected, and adjusted on the basis of sense perception. Moreover, in the manner of reflective equilibrium, we can correct and adjust for the deliverances of sense perception on the basis of concepts pertaining to sense perception. As I read Husserl, he could agree with all of this. The view is not at all persuasive, however, if concepts of either applicable or inapplicable mathematics and logic are supposed to be revisable on the basis of sense perception or the needs of natural science.

On Quine's view we are to accept the mathematics that is indispensable to the natural sciences. The rest of mathematics is dispensable. Since mathematics is dependent on the needs of the natural sciences we are evidently supposed to believe that its existence claims and its truths change as the needs of natural science change. What we regard as true in mathematics varies as the needs of natural science vary, as though mathematical results do not remain invariant across variations in natural science. There is a remarkable relativity of the notion of mathematical truth here. Compare, for example, the mathematics needed for Aristotelian physics with the mathematics required for the theory of relativity or quantum mechanics. Very little if any of the mathematics required for the theory of relativity is required for Aristotelian physics. The alternative view that I suggest respects the autonomy of mathematics and logic as sciences. It does not make mathematical existence or mathematical truth relative to the needs of natural science. It makes these properties relative only to mathematics itself, with *its* concepts, methods, results.[2] What is meant or intended or constituted as abstract and mind-independent is not dependent on the natural sciences. The rational motivation for this meaning-bestowal does not derive from this source.

The notion of the intentionality of consciousness, including mathematical consciousness, is not front and center in Quine's view of the world. Indeed, it tends to disappear or to be suppressed in his writings. Consequently, there is no view of an independent kind of directedness in mathematical thinking, of (the genesis of) the kind of content that could make this possible, of the conjecture/proof distinction in applicable and inapplicable mathematics, of the related notions of evidence and (rational) intuition, and of the possibility of the constitution of the meaning of being of objects of mathematics and logic as ideal, exact, abstract, mind-independent, and so on. If, in accordance with the IDP, we recognize concepts and modes of conception in our knowledge that do not direct us toward sensory objects at all, but rather direct our thinking toward objects of an altogether different kind, then *Quine's view appears to reduce reason to empirical, applied, or pragmatic reason*. On the alternative I have described in earlier chapters we recognize the intentionality of reason and the capacity of reason for acts of abstraction of different types, acts of idealization, of reflection, variation, and so on. The genetic analysis sketched in chapter 6 shows us that it is acts of this type that make possible the directedness of consciousness toward objects of an altogether different kind from sensory objects. It is not the case that all objects are *meant* in our thinking as sensory objects, or even more generally as objects of natural science. This does not exhaust the nature of the directedness of our scientific thinking. Taking the

[2] In this I am in agreement with a central theme in Maddy 2009.

IDP seriously, we can say that some objects are meant as exact, abstract, mind-independent, and so on. Since concepts of pure mathematics are not *about* sensory objects or objects of natural science, they are not subject to revision on the basis of sense perception or natural science. There are different categories of concepts or *noemata*, concepts or *noemata* that are about different things. Category of evidence and category of objectivities are perfect correlates. Mathematics and logic are subjects in their own right, and the concepts employed in these sciences are at a further remove from sense experience and natural science. The thetic character and the content of mathematical acts differs from perceptual acts. Some of the thinking in which objects are not meant as objects of natural science is nonetheless perfectly sound scientific thinking. If this is correct, then what holistic empiricism does is to forget about, cover over, or obscure other features and capacities of reason. Reason makes possible objectivity not only in natural science, but also in applicable and inapplicable mathematics and logic. Quine's practical reason does not account for objectivity, truth, or the conjecture/proof structure in either applicable or inapplicable mathematics. The notion of reason is thus not very robust if we go Quine's way. It seems quite likely that Gödel would follow Husserl's diagnosis of a "superficialization" of reason in this case. In his Vienna Lecture of 1935 Husserl (see English translation of Husserl 1954, Appendix A) says that "The reason for the failure of rational culture...lies not in the essence of rationalism itself but solely in its being rendered superficial, in its entanglement in 'naturalism' and 'objectivism'." Husserl's point about naturalism here should be clear from our discussion above, and the point about "objectivism" is that we will face insurmountable problems if objectivity is not brought into relation with human subjectivity, that is, with monads as conscious subjects of experience. Constituted platonism does bring objects into relation with subjects.

I do not deny that our understanding of concepts of mathematics and logic can be corrected, adjusted, or revised in various ways over time. Parts of mathematics that are well-understood are by now quite stable. It seems quite unlikely, for example, that there will be any revisions in our understanding of arithmetic. In any case, these revisions would not be induced on the basis of sense perception or the needs of natural science. A priori knowledge is usually understood as knowledge that is not revisable on the basis of future courses of sense experience. Mathematical knowledge might be a priori and yet still revisable if the revisions are independent of sense experience. Revision in this case would instead depend on capacities unique to human reason. It is best to consider actual examples of corrections or revisions of our understanding in mathematical practice. The history of the subject provides many examples. How do we come to see or know that there was a mistake in a definition, a proof, an axiom, a theorem, in the statement of a problem, or in the solution of a problem? On my view, this is determined by concepts, capacities, and methods unique to mathematics itself. Mathematics and logic are autonomous sciences, which is not to say that they cannot have deep and rich applications.

In this section I have been rather critical of Quine's views on mathematics. I want to note, however, that I think some of his ideas are developed in a much more attractive form in the work of Dagfinn Føllesdal (see Føllesdal 1988 and Smith 1994). Føllesdal is known for bringing the ideas of Quine and phenomenology into contact in an interesting way.

§ 4. Evidence and limits of reason

Quine is certainly correct in wanting to draw boundaries on what can count as science and evidence. The problem is that he just draws them too narrowly. Gödel and Husserl both point out various problems with, and limitations on, various empiricist views of mathematics and logic. This does not of course imply that we should jettison empirical science. On the contrary, empirical science is perfectly legitimate in its own domain. Reason alone cannot deliver the information that is delivered by the senses, unless sense experience can be completely eliminated in favor of pure reason in natural science, which seems unlikely in spite of some of Leibniz's suggestions. Reason does not give the particulars of sense experience, and it is not clear how it could on its own shape the generalizations that depend on these particulars. It is rather a matter of seeing empirical science in its proper perspective, of realizing that there are certain things that it cannot by its own nature tell us, just as there are certain things that reason alone cannot tell us. It seems to me that, on the one hand, we need to avoid a scientism according to which we do not know or understand anything unless it is based on the natural sciences while, on the other hand, we need to also avoid the dubious elements that have been part of earlier forms of rationalistic metaphysics. We need to go between these extremes. I think we also need to put up barriers against the questionable and perhaps even superstitious beliefs that are expressed in some of Gödel's comments on immortality of the soul, religion, and the like. It is a question of separating the wheat from the chaff in Gödel's pronouncements. Difficult as it may be to circumscribe the limits of reason, it is arguable that in his philosophical writings Gödel could have done more to specify the difference between what can be rationally supported and what cannot be rationally supported, between what counts as evidence and what does not. He alludes to the excesses of earlier forms of rationalism in the 1961 text in which he directs our attention to Husserl's philosophy, but he has little to say about how we might achieve a better balance with this approach. I want to now take this up in somewhat more detail. I will focus not so much on the notion of reason in general, but rather on the somewhat more manageable task of considering the function of reason in mathematics and logic.

The question of the limits of reason in mathematics and logic is, in some ways, much more difficult than the question of the limitations of empiricist views of mathematics and logic, for it is connected in a deep manner with the open-endedness of what we can hope to achieve through the cultivation of our rational capacities. It is connected with the capacity for imagination and imaginative variation. Science in a broad sense is a function of reason, but could our early human ancestors have foreseen the development of products of reason such as differential and integral calculus, quantum mechanics, the theory of relativity, modern geometry, group theory, or set theory? We said earlier that the balancing of platonism in constituted mathematical platonism is, at the same time, a balancing of our conception of mathematical reason. The *epochē* can be seen as imposing a kind of skepticism about metaphysical schemes of the past, but it is not the extreme empiricist skepticism of the logical positivists. Reason constitutes the meaning of being of its objects in mathematics and logic in a motivated, systematic, and non-arbitrary manner. It cannot know about, or have evidence for, objects that are non-concrete and mind-independent$_1$. Such objects and truths, were they to exist, would be beyond our ken. We cannot intuit such objects. Reason cannot know about mathematical objects or truths independently of the way they are given to us with evidence in conception. Rationalism in this broad sense, qualified by the phenomenological method, is not naive. The same point should apply across the board in metaphysics. If angels, demons, ghosts, souls, gods, God, and so on are objects that are non-concrete and mind-independent$_1$ then reason cannot know about, or have evidence for, such objects. Reason cannot know about, intuit, or have evidence for what Kant would have called noumenal objects. We are not in a position to distinguish knowledge from illusion in this case. Decidability with respect to such objects, on the basis of reason, is not in the cards. This follows from the position we described in earlier chapters of this book, and it constitutes an important limitation on reason, a limitation that places many objects of traditional rationalistic metaphysics out of reason's reach.

A central limitation on reason is that reason is confined, and always will be confined, to appearances. It is not that sense perception is confined to phenomena and reason is not, as though reason could somehow break through to the noumenal world. There is no intellectual intuition of the noumenal. This means that large metaphysical claims of the type associated with traditional rationalism, claims about non-material souls and so on, are not possible. Reason is no different from other modes of human consciousness in being limited to appearances and to what can be given with evidence as invariant in the manifold of appearances.

On the other hand, we argued that reason can know about, and have evidence for, at least some abstract mind-independent$_2$ objects and truths in mathematics and logic. We can understand objectivity without appealing to things-in-themselves. Decidability on the basis of reason, in this case, is not ruled out. Here we are in a position to distinguish knowledge from illusion. The constitution of the meaning of being of these objects and truths is built up in founded acts, in accordance with the strictures of science and evidence. This has not been the case with our conceptions of angels, demons, ghosts, souls, and God, and it is not apparent that it could be the case. It is clear, I think, that we do not have evidence for everything toward which we may be directed in our thinking. I certainly do not deny that we can *think* about angels, demons, ghosts, souls, gods, or God. Science, however, depends on evidence. There are theories of evidence that will help us to rule out many dubious non-concrete objects without eliminating all non-concrete objects. I prefer a view of science according to which science reflects the activities of critical reason in a wide sense. Science, in this sense, should put up barriers against superstition. In some cases we could rule out evidence for objects on a priori grounds because, for example, the *noematic Sinne* of our acts are formally or materially contradictory. In other cases we cannot rule out the existence of objects on purely a priori grounds. We can hold that there is just nothing at this time that counts as evidence for the objects, and continue to critically examine the matter of whether there could possibly be anything that could count as evidence for them. When we have evidence, in the primary case, our empty intentions or *noemata* must be filled in intuition to various degrees (Husserl 1973, Investigation VI, §§ 1–29). There must be at least partial fulfillment of our intentions. Mere consistency of our conceptions is not enough. I doubt that there is any concept of an object of metaphysics such that the mere existence of the concept, or the mere consistency of the concept, guarantees the existence of the object.[3] The presence of a consistent intention does not by itself imply the presence of the object of the intention (see also Tieszen 2005c, chapter 1). Consistency alone does not make for science and evidence. We might have logically consistent conceptions or theories about God, angels, demons, souls, and so on, but this does not mean we have a science of these things or that we have evidence for these things.

Let us briefly enumerate some standard conditions on evidence that are involved in building up the monad's constitution of the meaning of being of at least some abstract mind-independent$_2$ invariants in mathematics and logic, including unapplied parts of these subjects. These conditions are typically not involved in the genesis of our conceptions of angels, demons, ghosts, souls, and God.

First, evidence cannot be merely anecdotal. There is also no such thing as evidence that is evidence for only one person. When we have evidence we can *repeat at will* the experience of the object or truth in question. The repeatability is both intrasubjective and intersubjective. The same subject (monad) can repeat the experience at will at different times, and different subjects (monads) can repeat the same experience at different times and places under suitably controlled conditions. This is what it means to speak of objects as invariants and it is required for the constitution of objectivity. In science there is convergence on invariants across maximal variation in subjects (monads) capable of reason, of conceptualization. This is an ideal that we at least approximate in our best scientific practices and that we seek to approximate in the parts of science that are currently unsettled and in development.

If we compare these conditions with what we find in the domain of religious belief, for example, then the contrast is rather stark. Is something like rational theology in a Leibnizian sense possible? Gödel might have believed that it was. His rationalistic optimism perhaps encompassed such a view. Leibniz held the optimistic belief, for

[3] I reserve discussion of Gödel's ontological argument for the existence of God (Gödel *1970) for another occasion.

example, that we could reconcile Roman Catholicism with Protestantism. The problem is that there are many subdivisions within Protestantism and there are also, if only unofficially, subdivisions within Roman Catholicism, not to mention the difference with the Eastern churches. The problem for rational theology and reconciliation is greatly compounded by the existence of Islam, Judaism, Jainism, Hinduism, Sikhism, Buddhism, Shintoism, Taoism, and other religious worldviews, along with the subdivisions within these religions or religious worldviews. Where is the constitution of objectivity here? Can we really expect to bring order, clarity, systematization, and so on into this set of beliefs? Followers of these different conglomerations of belief allege invariants, but the problem is that monads seem to be far from converging on what the invariants really are, whereas there is remarkable convergence in many areas of mathematical practice.

As I portrayed the matter in chapters 5 and 6, rational intuition is concerned with objectivity and justification, but justification is not the function of imagining, desiring, hoping, wishing, and so on. In mathematics and logic, the evidence involved in justification frequently comes in the form of proofs, and some of these are proofs of existence statements. There is a distinction between conjecture and proof, problem and solution, expectation and realization, absence and presence. The proofs I have in mind here are the kinds of proofs one finds in mathematical practice, where this practice is not always manifested in axiom systems. Proofs, as we have said, need not be understood as purely formal proofs. We can say that Gödel's theorems show us that the purely formal concept of proof ought not to be identified with the concept of proof according to which a proof is what provides evidence. Indeed, proofs have been present in mathematics from very early times and have been considered to provide evidence since these times, even though they were not cast as purely formal proofs.

Now consider, on the one hand, the possibility of repeating the experience provided by a proof of an existential statement in mathematics and, on the other hand, the possibility of repeating the experience in which an existential claim is made about a ghost, a demon, a soul, or even God. The latter kinds of experience are certainly not everyday occurrences. It is not clear that the condition of repeatability applies at all. There does not seem to be intersubjective convergence on these phenomena. The harmony of the monads that is a condition for the constitution of objectivity does not seem to be present. In angelology, demonology, theology, and so on, there does not appear to be a conjecture/proof structure. In sense perception and, analogously, in mathematics, not everything appears to us as a problem, conjecture, hypothesis, or unrealized expectation, but outside of science, in questions about ghosts, demons, souls, God, and so on, virtually everything does appear this way. In mathematics and logic we can find necessities relative to sets of possibilities. We are not dealing in mere possibilities.

These points are related to the fact that, unlike superstition, science tries to use exact language, language that can be evaluated with respect to truth and falsity. In mathematics and logic in particular, one strives for exactness, clarity, and distinctness in thought and language. Justification, which is the business of reason, requires clarity, for otherwise we cannot make judgments about whether premises and conclusions in arguments are true or false. Rigor, order, and systematicity are ideals of science. There are types of superstitious belief that willfully forego these desiderata. Hypotheses and conjectures in science cannot be so broad that they are incapable of being falsified. There should be no ad hoc modifications of hypotheses, conjectures, or solutions. Simplicity is valued over complexity, all things being equal.

There should be no place in sciences such as mathematics and logic for beliefs that exist primarily to satisfy the emotions or wishes of the subject. The magical, the occult, and the mysterious, although they may fascinate or charm, are far from providing evidence. There is no place in science for any kind of mystical intuition. Mental laziness has no evidential value. When we cannot explain, prove or disprove something in the sciences, it will not do to simply give up and glibly attribute the mystery to God, to what is in the Tarot cards, or to the astrological features of the stars.

Integrity and honesty in gathering and presenting evidence is an ideal of science. Faking evidence is intolerable in scientific investigations, but is not always treated with disdain among purveyors of superstitious belief. Mathematicians and logicians are problem solvers who will often display great persistence in finding out what went wrong and why if their expectations or conjectures are frustrated. Mathematicians will be vigilant about noting inconsistencies with other established parts of mathematics. There is a persistence and a will to root out bias, prejudice, one-sidedness, and incorrect or partial perspectives that is all too frequently lacking in the case of those who promote questionable metaphysical schemes. Willingness to be proved wrong and then adjust accordingly is an ideal of the scientific attitude. The skill sets and goals of mathematicians and logicians are developed around this attitude and are typically quite different from the skill sets and goals of those who are directed in their thinking toward angels, ghosts, souls, and God.

The constitution of the meaning of being of at least some abstract mind-independent$_2$ objects and truths in mathematics and logic, including unapplied or inapplicable parts of these subjects, builds on previous results, in which earlier work is rejected, replaced, reformulated, or adjusted around particular invariants. The history of mathematics displays a cumulative, fruitful, and evolving character that is lacking in many domains of dubious metaphysics. This also contrasts rather sharply with superstitious belief.

One could point to many other distinctive features of mathematics as science. In addition to proofs, we might be able to trace certain mathematical concepts back to their origins in the lifeworld. We should consider not only the movement of thought from the particular to the universal or from the "material" to the formal, but also the reverse direction in which we see many important and fruitful applications of mathematical concepts and forms in mathematics itself and in the natural sciences. In our discussion of the genesis of mathematical directedness we noted that there are mixed categorial forms. The remarkable applications both within and outside of mathematics distinguish the science of mathematics from other conceptual frameworks that might be thought to involve non-concrete objects. Although the existence of applications distinguishes mathematics and logic from superstitious belief, this need not be taken as the only criterion of truth or objectivity.

As I have said at various points in this book, I would like to develop an account of our consciousness of abstract mind-independent objects that is situated in a scientific setting. The setting, however, should not be so narrow as to include only the natural sciences. Many of the conditions just described are conditions on science in general, not just on natural sciences. I particular, I think our view of consciousness needs more breadth than the natural sciences can provide. We need to include a rich phenomenology of human consciousness, along with the idea that mathematics and logic are autonomous sciences (an idea in which the IDP will again have a role to play). One would then provide careful descriptions and analyses of the types of conscious directedness involved in logic, mathematics, and the natural sciences, where these descriptions and analyses should be subjected to rational, critical analysis and development. Science in this sense depends on a broad conception of reason and critical analysis that includes many scientific specializations, many perspectives on Being, but that cannot be corralled into, or reduced to, any one of them in particular. Thus, my view of the scientific endeavor is broader than that of naturalistic reductionism but it is not so broad as to include some of Gödel's's alleged and very speculative views on religion, angels, immortality of the soul, and the like. There are also reports on some of Gödel's ideas that make him out to be rather superstitious and I prefer to avoid such associations.

The new combination of leftward and rightward ideas I have been sketching in this book does not imply any kind of irrationalism. On the contrary, it is possible to embrace it while adopting a healthy skepticism toward superstition, mysticism, prophecy, and the like. We adopt a rationalist skepticism to accompany our broader view of reason, science, logic, mathematics, and evidence, instead of an exaggerated empiricist skepticism that eliminates the mind and its directedness toward abstract objects in the sciences of mathematics and logic. We include critical reason but not pure, uncritical reason. We see the worries that led to finding a foundation for mathematics exclusively in concrete mathematics as exaggerated. Of course we do not want idealizations, abstractions, and forms of reflection that lead to inconsistency. Instead, we want to open up a space in which we allow the broadest set of mathematical possibilities and necessities compatible with scientific thinking.

§ 5. How to exalt reason

Is reason present everywhere in the universe or nowhere? We seem to have counterevidence for the claim that it is present nowhere in the universe. Gödel points especially to the existence of mathematics and logic to support the claim that reason is present in the universe. In fact, he makes comments in some places that suggest he thinks reason and meaning are present everywhere in the universe, and that appearances to the contrary are just illusions. As he says in a draft of an unsent letter to a student (Gödel 2003b, Vol. V, p. 81), "Every chaos is merely a wrong appearance," and "The appearance of a chaos is also deceptive. The world is very orderly despite its often chaotic appearance." These kinds of remarks seem to be connected with some of his theological beliefs. They would presumably follow from Leibniz's metaphysical view of the existence of a central monad. I am not willing to commit to this kind of theological hyper-rationalism. It is not clear to me that we need to accept the dichotomy that reason is either present everywhere or nowhere. There seems to be counterevidence to both claims. We do not want to claim that it is present when in fact it is not, nor do we want to claim that it is not present when in fact it is. It is again a matter of finding the proper balance. I am prepared to settle for the view that reason and meaning are potentially present when human consciousness is present. We look to the human subject (monad) as a source of reason and meaning, not to some other source. Constituted platonism results from meaning constitution, and the only meaning constitution to which we evidently have access is that which has its origins in human consciousness.

Leibniz's idea that we should seek to exalt reason through the development of a methodology that would help us to see more clearly does not seem to me to simply be a mistake. His own version of it, insofar as it depended solely on mechanical calculation, is just limited. It is also hampered to some extent by dubious metaphysical claims. Dubious rationalist metaphysical schemes will not help us to exalt reason, but will only degrade and devalue reason. The way to exalt reason in mathematics and logic is through formalization (including proof-theoretic studies), axiomatization, reflection, idealization, abstraction, imaginative variation, the determination of necessities relative to sets of possibilities, and the related cultivation of intuition of abstract concepts. It is to keep practice in mathematics and logic free of "leftward" reductionistic philosophical schemes. In the background of mathematics and logic this will include the non-reductionistic, phenomenological investigation of human consciousness on the basis of our capacities for reason, for finding structure and invariant features. It will also include the rational phenomenological investigation of the lifeworld (*Lebenswelt*) on which the sciences of mathematics and logic are founded.

We might hope to exalt reason, as just suggested, if we can keep practice in mathematics and logic free of "leftward" reductionistic philosophical schemes. Since the time of Husserl and Gödel such leftward positions have only proliferated. In addition to fictionalism and new versions of naturalism, empiricism, and formalism, we also have modal nominalism, modal fictionalism, and other positions. The *Zeitgeist*, Gödel would no doubt argue, has not abated.[4] Many of these newer philosophical positions have not been discussed explicitly in this book. They are, from my point of view, many variations on a theme. They certainly deserve to be analyzed and

[4] There are, of course, some exceptions. See, e.g. Katz 1998, 2002, and much of the work of George Bealer. Katz and Bealer argue for forms of rationalistic realism, but they do so independently of Gödel's work. As I indicated in the Preface, I think Gödel's work adds a new dimension to the debate.

evaluated. This is material for further work. Meanwhile, I think I have provided a basis for seeing how some fundamental objections to these views can be developed. I would like to set out the kind of platonic rationalism about mathematics and logic that I have presented in this book as an alternative, or at least as a path to an alternative. I cannot with assurance claim that it is Gödel's own view. It is an attempt to develop, after Gödel, a defensible version of platonic rationalism that is based on Gödel's ideas about using Husserl's transcendental phenomenology to preserve in a new and updated combination some central elements of Plato's rudimentary objectivism, Leibniz's rationalism, and Kant's transcendental philosophy.

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