Gödel's God in Isabelle/HOL

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November 1, 2013

A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$
D1	A *God-like* being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists x G(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box\ P(\phi)]$
D2	An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G\ ess.\ x]$
D3	*Necessary existence* of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists x G(x)$

1 Introduction

A formalization and (partial) automation of Dana Scott's version [11] of Goedel's ontological argument [8] in quantified modal logic KB (QML KB) is presented. QML KB is in turn modeled as a fragment of classical higher-order logic (HOL). Thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benzmüller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (poperties).

The formalization presented here has been carried out and verified in the Isabelle/HOL proof assistant; for more information on this system see the textbook by Nipkow, Paulson, and Wenzel [10]. More recent tutorials on Isabelle can be found at: http://isabelle.in.tum.de.

Some further notes:

1. This LaTeX text document has been produced automatically from the Isabelle source code document at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Isabelle/GoedelGodSession with the Isabelle build tool.

2. The formalization presented here is related to the THF [13] and Coq [4] formalizations at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/.

3. All reasoning gaps in Scott's proof script have been automated with Sledgehammer [5], performing remote calls to the higher-order automated theorem prover LEO-II [3]. These calls suggest the Metis [9] calls as given below. The Metis proofs are verified in Isabelle/HOL.

4. For consistency checking, the model finder Nitpick [6] has been employed.

2 An Embedding of QML KB in HOL

The types i for possible worlds (or states) and μ for individuals are introduced.

typedecl i — the type for possible worlds
typedecl μ — the type for indiviuals

Possible worlds are connected by an accessibility relation r.

consts $r :: i \Rightarrow i \Rightarrow \mathit{bool}$ (**infixr** r *70*) — accessibility relation r

The B axiom (symmetry) for relation r is stated. B is needed only for proving theorem T3.

axiomatization where *sym*: $x\ r\ y \longrightarrow y\ r\ x$

QML formulas are identified with certain HOL terms of type $i \Rightarrow \mathit{bool}$. This type will be abbreviated in the remainder as σ.

type-synonym $\sigma = (i \Rightarrow \mathit{bool})$

The classical connectives $\neg, \wedge, \rightarrow$, and $\forall$ (over individuals and over sets of individuals) and $\exists$ (over individuals) are lifted to type σ. The lifted connectives are $m\neg$, $m\wedge$, $m\Rightarrow$, $\forall$, Π, and $\exists$. Further connectives could be introduced analogously. Definitions could be used instead of abbreviations.

abbreviation *mnot* $:: \sigma \Rightarrow \sigma$ $(m\neg)$ **where** $m\neg\ \varphi \equiv (\lambda w.\ \neg\ \varphi\ w)$
abbreviation *mand* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\wedge$ *79*) **where** $\varphi\ m\wedge\ \psi \equiv (\lambda w.\ \varphi\ w \wedge \psi\ w)$
abbreviation *mimplies* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\Rightarrow$ *74*) **where** $\varphi\ m\Rightarrow\ \psi \equiv (\lambda w.\ \varphi\ w \longrightarrow \psi\ w)$
abbreviation *mforall-ind* $:: (\mu \Rightarrow \sigma) \Rightarrow \sigma$ $(\forall)$ **where** $\forall\ \Phi \equiv (\lambda w.\ \forall x.\ \Phi\ x\ w)$
abbreviation *mforall-indset* $:: ((\mu \Rightarrow \sigma) \Rightarrow \sigma) \Rightarrow \sigma$ (Π) **where** $\Pi\ P \equiv (\lambda w.\ \forall x.\ P\ x\ w)$
abbreviation *mexists-ind* $:: (\mu \Rightarrow \sigma) \Rightarrow \sigma$ $(\exists)$ **where** $\exists\ \Phi \equiv (\lambda w.\ \exists x.\ \Phi\ x\ w)$
abbreviation *mbox* $:: \sigma \Rightarrow \sigma$ $(\Box)$ **where** $\Box\ \varphi \equiv (\lambda w.\ \forall v.\ \neg\ w\ r\ v \vee \varphi\ v)$
abbreviation *mdia* $:: \sigma \Rightarrow \sigma$ $(\Diamond)$ **where** $\Diamond\ \varphi \equiv (\lambda w.\ \exists v.\ w\ r\ v \wedge \varphi\ v)$

For grounding lifted formulas, the meta-predicate *valid* is introduced.

abbreviation *valid* $:: \sigma \Rightarrow \mathit{bool}$ ([-]) **where** $[p] \equiv \forall w.\ p\ w$

3 Gödel's Ontological Argument

Constant symbol P (Gödel's 'Positive') is declared.

consts $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms $A1(a/b)$: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$ (Either a property or its negation is positive, but not both.) and $A2$: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ (A property necessarily implied by a positive property is positive).

axiomatization where
  A1a: $[\Pi\ (\lambda\Phi.\ P\ (\lambda x.\ m\neg\ (\Phi\ x))\ m\Rightarrow\ m\neg\ (P\ \Phi))]$ **and**
  A1b: $[\Pi\ (\lambda\Phi.\ m\neg\ (P\ \Phi)\ m\Rightarrow\ P\ (\lambda x.\ m\neg\ (\Phi\ x)))]$ **and**
  A2: $[\Pi\ (\lambda\Phi.\ \Pi\ (\lambda\psi.\ (P\ \Phi\ m\wedge\ \Box\ (\forall\ (\lambda x.\ \Phi\ x\ m\Rightarrow\ \psi\ x)))\ m\Rightarrow\ P\ \psi))]$

We prove theorem T1: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$ (Positive properties are possibly exemplified). T1 is proved directly by Sledghammer with command *sledgehammer* [*provers* = *remote-leo2*]. This successful attempt then suggests to instead try the Metis call in the line below. This Metis call generates a proof object that is verified in Isabelle/HOL's kernel.

theorem *T1*: $[\Pi\ (\lambda\Phi.\ P\ \Phi\ m\Rightarrow\ \Diamond\ (\exists\ \Phi))]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis A1a A2*)

Next, the symbol G for 'God-like' is introduced and defined as $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$ (A God-like being possesses all positive properties).

definition $G :: \mu \Rightarrow \sigma$ **where** $G = (\lambda x.\ \Pi\ (\lambda\Phi.\ P\ \Phi\ m\Rightarrow\ \Phi\ x))$

Axiom *A3* is added: $P(G)$ (The property of being God-like is positive). Sledgehammer and Metis then prove corollary C: $\Diamond\exists xG(x)$ (Possibly, God exists).

axiomatization where *A3*: $[P\ G]$

corollary *C*: $[\Diamond\ (\exists\ G)]$
sledgehammer [*provers* = *remote-leo2*] **by** (*metis A3 T1*)

Axiom *A4* is added: $\forall\phi[P(\phi) \rightarrow \Box\ P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where *A4*: $[\Pi\ (\lambda\Phi.\ P\ \Phi\ m\Rightarrow\ \Box\ (P\ \Phi))]$

Symbol *ess* for 'Essence' is introduced and defined as $\phi\ ess.\ x \leftrightarrow \phi(x)\wedge\forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$ (An *essence* of an individual is a property possessed by it and necessarily implying any of its properties).

definition *ess* $:: (\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$ (**infixr** *ess 85*) **where**
  $\Phi\ ess\ x = \Phi\ x\ m\wedge\ \Pi\ (\lambda\psi.\ \psi\ x\ m\Rightarrow\ \Box\ (\forall\ (\lambda y.\ \Phi\ y\ m\Rightarrow\ \psi\ y)))$

Next, Sledgehammer and Metis prove theorem *T2*: $\forall x[G(x) \rightarrow G\ ess.\ x]$ (Being God-like is an essence of any God-like being).

theorem *T2*: $[\forall\ (\lambda x.\ G\ x\ m\Rightarrow\ G\ ess\ x)]$
sledgehammer [*provers* = *remote-leo2*] **by** (*metis A1b A4 G-def ess-def*)

Symbol *NE*, for 'Necessary Existence', is introduced and defined as $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$ (Necessary existence of an individual is the necessary exemplification of all its essences).

definition *NE* $:: \mu \Rightarrow \sigma$ **where** $NE = (\lambda x.\ \Pi\ (\lambda\Phi.\ \Phi\ ess\ x\ m\Rightarrow\ \Box\ (\exists\ \Phi)))$

Moreover, axiom *A5* is added: $P(NE)$ (Necessary existence is a positive property).

axiomatization where *A5*: $[P\ NE]$

Finally, Sledgehammer and Metis prove the main theorem *T3*: $\Box\exists xG(x)$ (Necessarily, God exists).

theorem *T3*: $[\Box\ (\exists\ G)]$

sledgehammer [*provers* = *remote-leo2*] **by** (*metis A5 C T2 sym G-def NE-def*)

corollary *C2*: [$\exists$ *G*]
sledgehammer [*provers* = *remote-leo2*] **by** (*metis T1 T3 G-def sym*)

The consistency of the entire theory is checked with Nitpick.

lemma *True* **nitpick** [*satisfy*, *user-axioms*, *expect* = *genuine*] **oops**

It has been critisized that Gödel's ontological argument implies what is called the modal collapse. The prover Satallax [7] can indeed show this, but verification with Metis still fails.

lemma *MC*: [p $m\Rightarrow$ ($\Box$ p)]
using *T2 T3 ess-def sym* **sledgehammer** [*provers* = *remote-satallax*] **oops**

Acknowledgments: Nik Sultana, Jasmin Blanchette and Larry Paulson provided very important help wrt consistency checking in Isabelle. Jasmin Blanchette instructed us on how to produce latex documents from Isabelle sources, and he showed us useful tricks in Isabelle.

4 Further results on Gödel's God.

abbreviation *mequals* :: $\mu \Rightarrow \mu \Rightarrow \sigma$ (**infixr** *m= 90*) **where** x $m=$ $y \equiv \Pi$ (λ φ.(φ x $m\Rightarrow$ φ y))

Gödel's God is flawless, that is, he has no negative properties.

lemma *Flawless*: [Π ($\lambda\varphi$. $\forall$ (λx. (G x $m\Rightarrow$ ($m\neg$ (P φ) $m\Rightarrow$ $m\neg$ (φ x)))))]
sledgehammer [*provers* = *remote-leo2*] **by** (*metis A1b G-def*)

Moreover, it can be shown that any two God-like beings are (Leibniz-)equal, that is there is there only one God-like being.

lemma *Monotheism*: [$\forall$ (λx. $\forall$ (λy. ($G(x)$ $m\Rightarrow$ ($G(y)$ $m\Rightarrow$ (x $m=$ y)))))]
sledgehammer [*provers* = *remote-leo2*] **by** (*metis C sym T2 ess-def*)

Add-on: We briefly show that Leibniz equality denotes equality.

lemma *test2*: $x = y \Longrightarrow$ [x $m=$ y]
sledgehammer [*provers* = *remote-leo2*]
by *metis*

lemma *test1*: [x $m=$ y] $\Longrightarrow x = y$
sledgehammer [*provers* = *remote-satallax*] **oops**

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