Gödel's Ontological Proof of God's Existence

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Possible witnessing of positive properties

Axioms:

- **(1)** Properties necessarily entailed by *positive* properties are also positive:

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

- **(2)** A property's negation is positive iff the property is not positive:

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Theorem 1: Positive properties possibly have a witness:

$$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$$

Formal proof:

$$
\begin{prooftree}
\AxiomC{$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$}
\UnaryInfC{$\forall\psi.[(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \psi(x)]) \to P(\psi)]$}
\UnaryInfC{$(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \neg\varphi'(x)]) \to P(\neg\varphi')$}
\doubleLine
\UnaryInfC{$(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to P(\neg\varphi')$}
\AxiomC{$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$}
\UnaryInfC{$P(\neg\varphi') \leftrightarrow \neg P(\varphi')$}
\doubleLine
\BinaryInfC{$(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to \neg P(\varphi')$}
\doubleLine
\UnaryInfC{$P(\varphi') \to \Diamond\exists x.\varphi'(x)$}
\UnaryInfC{$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$}
\end{prooftree}
$$

2 Possible existence of a God

Axioms:

- **(3)** Being God is a positive property:

$$P(G)$$

Theorem 2: It is possible that a God exists:

$$\Diamond \exists x. G(x)$$

Formal proof:

$$
\begin{prooftree}
\AxiomC{$P(G)$}
\AxiomC{}
\dashedLine
\RightLabel{Th. 1}
\UnaryInfC{$\forall \varphi. [P(\varphi) \to \Diamond \exists x. \varphi(x)]$}
\UnaryInfC{$P(G) \to \Diamond \exists x. G(x)$}
\BinaryInfC{$\Diamond \exists x. G(x)$}
\end{prooftree}
$$

3 Essentiality of being God

Definitions:

- A property is *essential* for an individual if and only if it holds for that inidividual and necessarily entails every other property that holds for that individual: $\varphi \ ess \ x \leftrightarrow \varphi(x) \wedge \forall \psi. (\psi(x) \to \Box \forall x. (\varphi(x) \to \psi(x)))$

Axioms:

- **(4)** Positive properties are necessarily positive:

$$\forall \varphi. [P(\varphi) \to \Box \ P(\varphi)]$$

Theorem 3: If an individual is a God, then being God is an essential property for that individual:

$$\forall y. [G(y) \to G \ ess \ y]$$

Formal proof:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$G(x)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\psi(x)$}
\UnaryInfC{$G(x) \to \psi(x)$}
\UnaryInfC{$\forall x. [G(x) \to \psi(x)]$}
\RightLabel{Necessitation}
\UnaryInfC{$\Box \forall x. [G(x) \to \psi(x)]$}
\RightLabel{2}
\UnaryInfC{$\psi(x) \to \Box \forall x. [G(x) \to \psi(x)]$}
\UnaryInfC{$\forall \psi. [\psi(x) \to \Box \forall x. [G(x) \to \psi(x)]]$}
\BinaryInfC{$G(x) \wedge \forall \psi. [\psi(x) \to \Box \forall x. [G(x) \to \psi(x)]]$}
\dottedLine
\UnaryInfC{$G \ ess \ x$}
\RightLabel{1}
\UnaryInfC{$G(x) \to G \ ess \ x$}
\UnaryInfC{$\forall y. [G(y) \to G \ ess \ y]$}
\end{prooftree}
$$

Note: the formal proof above uses the necessitation rule of the basic modal logic **K**, instead of using Axiom 4, which corresponds to a restricted form of necessitation. I currently do not know how to formally prove theorem 3 using axiom 4 and not relying on necessitation.

4 Necessity of God's existence

Definitions:

- An individual is a *God* if and only if he possesses all positive properties:

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

- An individual *necessarily exists* if and only if all its essential properties are necessarily witnessed:

$$E(x) \leftrightarrow \forall\varphi.[\varphi \textit{ ess } x \rightarrow \Box\exists x.\varphi(x)]$$

Axioms:

- **(5)** Necessary existence is a positive property:

$$P(E)$$

Theorem A: If there is a God, then there necessarily exists a God:

$$\exists z.G(z) \rightarrow \Box\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{}{\exists z.G(z)}\;1}{G(g)}$$

$$\dfrac{\dfrac{\dfrac{}{G(g)} \qquad \dfrac{\dfrac{}{\forall y.[G(y) \rightarrow G \textit{ ess } y]}\;\text{Th. } 3}{G(g) \rightarrow G \textit{ ess } g}}{G \textit{ ess } g} \qquad \dfrac{\dfrac{\dfrac{P(E) \qquad \dfrac{\dfrac{\dfrac{}{G(g)}}{\forall\varphi.[P(\varphi) \rightarrow \varphi(g)]}}{P(E) \rightarrow E(g)}}{E(g)}}{\forall\varphi.[\varphi \textit{ ess } g \rightarrow \Box\exists x.\varphi(x)]}}{G \textit{ ess } g \rightarrow \Box\exists x.G(x)}}{\dfrac{\Box\exists x.G(x)}{\exists z.G(z) \rightarrow \Box\exists x.G(x)}\;1}$$

Note: Theorem A could be proved more quickly using the necessitation rule. Interestingly, the proof above shows that, by using the given axioms and definitions of god and necessary existence, theorem A can be derived even without the necessitation rule.

5 Necessary existence of a God

Theorem 4: The existence of a God is necessary:

$$\Box\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{\dfrac{}{\forall\varphi.[\Diamond\ldots\Diamond\Box\varphi\leftrightarrow\Box\varphi]}\,\mathbf{S5}}{\Diamond\Box\exists x.G(x)\leftrightarrow\Box\exists x.G(x)} \qquad \dfrac{\dfrac{}{\Diamond\exists x.G(x)}\,\text{Th. 2} \qquad \dfrac{}{\exists z.G(z)\rightarrow\Box\exists x.G(x)}\,\text{Th. A}}{\Diamond\Box\exists x.G(x)}}{\Box\exists x.G(x)}$$

Note: The proof above relies on a theorem of the modal logic **S5**, which is a quite strong modal logic. It would be interesting to try to derive theorem 4 with weaker modal axioms.

6 God's existence

Axioms:

- **(M)** What is necessary is the case:

$$\forall\varphi.[\Box\varphi\rightarrow\varphi]$$

Theorem: There exists a God:

$$\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{}{\Box\exists x.G(x)}\,\text{Th. 4} \qquad \dfrac{\forall\varphi.[\Box\varphi\rightarrow\varphi]}{\Box\exists x.G(x)\rightarrow\exists x.G(x)}}{\exists x.G(x)}$$