Computer-Assisted Analysis of Emendations of Gödel's Ontological Proof

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The axioms in Gödel's ontological argument [11, 16] (cf. Appendix A) entail what is called the *modal collapse* [17, 18]: the formula $\varphi \rightarrow \Box\varphi$, abbreviated as MC, holds for any formula φ and not just for $\exists x. God(x)$ as intended. This fact, which has recently been confirmed with higher-order automated theorem provers [3, 6], has led to strong criticism of the argument and stimulated attempts to remedy the problem. Hájek [14, 13] proposed the use of cautious instead of full comprehension principles, and Fitting [10] suggested that greater care is necessary in the semantics of higher-order quantifiers in the presence of modalities. Others, such as Anderson [2, 1] and Bjordal [7], proposed slight emendations of Gödel's axioms and definitions. They require neither comprehension restrictions nor more complex semantics. Therefore, they are technically simpler to analyze with computer support. We have formalized them using the proof assistant Isabelle/HOL [15] together with the automated higher-order reasoners Leo-II [5], Satallax [9], Metis [12], and Nitpick [8]. Our formalizations[1] employ the embedding of higher-order modal logic (HOML) in classical higher-order logic (HOL) as introduced in previous work [3, 6, 4]. We explored the effect of different domain conditions on the provability of lemmas, theorems and even axioms. This was motivated by a controversy between Hájek and Anderson regarding the redundancy of some axioms in Anderson's emendation. In *constant domain semantics*, the individual domains are the same in all possible worlds. In *varying domain semantics*, the domains may vary from world to world. This variation is technically encoded with the help of an existence relation expressing which individuals actually exist in each world. Quantifiers are then uniformly formalized as *actualist quantifiers* (i.e. guarded by the existence relation). Our main results are summarized in the following sections.

1. Anderson's Emendation (cf. Appendix B)

Constant domain semantics. Gödel's theorems T1, C and T3' can be quickly automated (in logics K and KB, respectively), Gödel's axioms A4 and A5 are proven redundant (the former one in logic K4B and the latter one already in K). Moreover, a trivial countermodel (consisting of two worlds and two individuals) to MC generated by Nitpick (for all mentioned logics); this model also shows the consistency of the axioms.

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Varying domain semantics. : In this setting we obtain the same results as above.

Mixed variant. (varying domain quantifiers are used only in the definitions of essence and NE; cf. Fitting's comments to Anderson in [10]): Also in this setting we obtain the same results as above. However, if a varying domain quantifier is used only in the definition of NE, then the situation changes slightly. Now axiom A5 is no longer provable and a countermodel is reported by Nitpick. The remaining results are as before.

2. Bjordal's Emendation (cf. Appendix C)

Constant domain semantics. Gödel's axiom A2, A3 can be quickly automatically derived in logic K from Bjordal's definition B:D. A4 can be proved in logic T (reflexivity). Proving Gödel's D1 from B:D is possible in logic K4. Conversely, the proof that B:D follows from D1, A2, A3 and A4 is possible already in logic K. Hence, Bjordal's lemma B:L1 holds in logic S4. The provers also show that theorem T3 follows from B:D, B:A1 and B:A2 already in logic KB. Modal collapse does not follow in Bjordal's setting as Nitpick demonstrates with a countermodel (consisting of two worlds and one individual).

Varying domain semantics. : The results are the same as above.

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3. Conclusions

Anderson's emendation (cf. Appendix B; , which we have analysed for different domain conditions. These variations were motivated by various comments on Anderson's work in the literature.

In this approach full comprehension is naturally "built-in" since the underlying HOL supports λ-abstraction.

Summary (what else can we say here, feel free to add): Using our approach, the formalization and (partly) automated analysis of different variants of Anderson's and Bjordal's emendations of Gödel's ontological argument has been surprisingly straightforward. The provers confirmed the claimed results and in a few cases they have even contributed some novel insights. The weakening of the comprehension principles would clearly constitute another

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[1]The formalizations are available in the subdirectories `Anderson` and `Bjordal` at `https://github.com/FormalTheology/GoedelGod/blob/master/Formalizations/Isabelle/`.

interesting parameter for further experiments. However, this seems hard to achieve in our approach, since full comprehension is naturally built-in.

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@Bruno: I think we need more compact presentation of the proofs.

Appendix A. Scott's version of Gödel's ontological argment

A1 Either a property or its negation is positive, but not both:

$$\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

D1 A *God-like* being possesses all positive properties:

$$G(x) \equiv \forall\varphi[P(\varphi) \rightarrow \varphi(x)]$$

A3 The property of being God-like is positive:

$$P(G)$$

C Possibly, a God-like being exists:

$$\Diamond\exists xG(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\ P(\varphi)]$$

D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:

$$\varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

T2 Being God-like is an essence of any God-like being:

$$\forall x[G(x) \rightarrow G\ ess\ x]$$

D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE(x) \equiv \forall\varphi[\varphi\ ess\ x \rightarrow \Box\exists y\varphi(y)]$$

A5 Necessary existence is a positive property:

$$P(NE)$$

L1 If a god-like being exists, then necessarily a god-like being exists:

$$\exists xG(x) \rightarrow \Box\exists yG(y)$$

L2 If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists xG(x) \rightarrow \Box\exists yG(y)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists xG(x)$$

Appendix B. Anderson's Emendation

A:A1 If a property is positive, its negation is not positive:

$$\forall\varphi[P(\varphi) \rightarrow \neg P(\neg\varphi)]$$

A2 A property necessarily implied by a positive property is positive:

$$\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

T1 Positive properties are possibly exemplified:

$$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$$

A:D1 A *God-like* being necessarily possesses those and only those properties that are positive:

$$G_A(x) \equiv \forall\varphi[P(\varphi) \leftrightarrow \Box\varphi(x)]$$

A3' The property of being God-like is positive:

$$P(G_A)$$

C Possibly, a God-like being exists:

$$\Diamond\exists x G(x)$$

A4 Positive properties are necessarily positive:

$$\forall\varphi[P(\varphi) \rightarrow \Box\ P(\varphi)]$$

A:D2 An *essence* of an individual is a property that necessarily implies those and only those properties that the individual has necessarily:

$$\varphi\ \mathit{ess}_A\ x \equiv \forall\psi[\Box\psi(x) \leftrightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y))]$$

T2' Being God-like is an essence of any God-like being:

$$\forall x[G_A(x) \rightarrow G_A\ \mathit{ess}_A\ x]$$

D3' *Necessary existence* of an individual is the necessary exemplification of all its essences:

$$NE_A(x) \equiv \forall\varphi[\varphi\ \mathit{ess}_A\ x \rightarrow \Box\exists y\varphi(y)]$$

A5' Necessary existence is a positive property:

$$P(NE_A)$$

L1' If a god-like being exists, then necessarily a god-like being exists:

$$\exists x G_A(x) \rightarrow \Box\exists y G_A(y)$$

L2' If possibly a god-like being exists, then necessarily a god-like being exists:

$$\Diamond\exists x G_A(x) \rightarrow \Box\exists y G_A(y)$$

T3' Necessarily, a God-like being exists:

$$\Box\exists x G_A(x)$$

Appendix C. Bjordal's Alternative

In Bjordal's emendation G (God-like) is taken as primitive and P (Positive) is defined (cf. definition D).

B:D A formulas ϕ is positive iff it is necessarily the case that anything which is God-like has the property ϕ.

$$P(\phi) \equiv \Box\forall x(G(x) \to \phi(x))$$

B:L1 D is logically equivalent in S4 with the union of Gödel's definition D1 and axioms A2, A3 and A4.

$$D \leftrightarrow D1 \wedge A2 \wedge A3 \wedge A4$$

The proof splits into the two implication directions B:L1$^{\rightarrow}$ and B:L1$^{\leftarrow}$. B:L1$^{\rightarrow}$ can be further split into four single steps.

B:D2 ϕ is a maximal composite of object x's positive properties iff x has ϕ and ϕ is positive and all positive properties ψ which x has are such that is necessarily the case that all objects which have ϕ also have ψ.

$$MCP(\phi, x) \equiv (\phi(x) \wedge P(\phi)) \wedge \forall\psi((\psi(x) \wedge P(\psi)) \to \Box\forall y(\phi(y) \to \psi(y)))$$

B:D3 x has the N-property iff x is such that if ϕ is a maximal composite of x's positive properties then it is necessary that some object y has the property ϕ.

$$N(x) \equiv \forall\phi(MCP(\phi, x) \to \Box\forall y\phi(y))$$

B:A1 If a property is positive, its negation is not positive:

$$\forall\varphi[P(\varphi) \to \neg P(\neg\varphi)]$$

B:A2 The N-property is positive.

$$P(N)$$

T3 Necessarily, a God-like being exists:

$$\Box\exists x G(x)$$

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