Gödel's Ontological Proof of God's Existence (Draft)

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Introduction

ToDo: Do also Scott's and Gdel's proofs.

2 Natural Deduction

ToDo: Show and explain here the rules of the calculus we are using.

ToDo: We should use a calculus for the basic modal logic K. Everything else should be stated as axioms.

ToDo: Investigate the relationship between the $\Box_E$ rule and the M axiom (reflexivity). We don't want M to be provable in our calculus.

ToDo: cite a paper that proves soundness and completeness for this calculus.

3 Possibly, God Exists

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall \varphi.[P(\neg \varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box \forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$$

Theorem 1 *Positive properties are possibly exemplified:*

$$\forall \varphi.[P(\varphi) \rightarrow \Diamond \exists x.\varphi(x)]$$

Proof

$$
\begin{prooftree}
\AxiomC{Axiom 2}
\dashedLine
\UnaryInfC{$\forall\varphi.\forall\psi.[(P(\varphi) \land \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$\forall\psi.[(P(\rho) \land \Box\forall x.[\rho(x) \to \psi(x)]) \to P(\psi)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$(P(\rho) \land \Box\forall x.[\rho(x) \to \neg\rho(x)]) \to P(\neg\rho)$}
\doubleLine
\UnaryInfC{$(P(\rho) \land \Box\forall x.[\neg\rho(x)]) \to P(\neg\rho)$}
\AxiomC{Axiom 1}
\dashedLine
\UnaryInfC{$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\neg\rho) \leftrightarrow \neg P(\rho)$}
\doubleLine
\BinaryInfC{$(P(\rho) \land \Box\forall x.[\neg\rho(x)]) \to \neg P(\rho)$}
\doubleLine
\UnaryInfC{$P(\rho) \to \Diamond\exists x.\rho(x)$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$}
\end{prooftree}
$$

Definition 1 *A* God-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \to \varphi(x)]$$

Axiom 3 *The property of being God-like is positive:*

$$P(G)$$

Corollary 1 *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof

$$
\begin{prooftree}
\AxiomC{Axiom 3}
\dashedLine
\UnaryInfC{$P(G)$}
\AxiomC{Theorem 1}
\dashedLine
\UnaryInfC{$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(G) \to \Diamond\exists x.G(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Diamond\exists x.G(x)$}
\end{prooftree}
$$

4 Being God is an essence of any God

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi) \to \Box\ P(\varphi)]$$

Definition 2 *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi\ ess\ x \leftrightarrow \varphi(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

Theorem 2 *Being God-like is an essence of any God-like being:*

$$\forall y.[G(y) \to G\ ess\ y]$$

Proof Let the following derivation with the open assumption $G(x)$ be $\Pi_1[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\neg P(\psi)^1$}
\AxiomC{Axiom 1}
\dashedLine
\UnaryInfC{$\forall\varphi.(\neg P(\varphi) \to P(\neg\varphi))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$\neg P(\psi) \to P(\neg\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$P(\neg\psi)$}
\AxiomC{$G(x)$}
\dottedLine
\RightLabel{D1}
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \varphi(x))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\neg\psi) \to \neg\psi(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\neg\psi(x)$}
\AxiomC{$\psi(x)^2$}
\RightLabel{$\to_E$}
\BinaryInfC{$\bot$}
\RightLabel{RAA^1}
\UnaryInfC{$P(\psi)$}
\RightLabel{$\to_I^2$}
\UnaryInfC{$\psi(x) \to P(\psi)$}
\end{prooftree}
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_2[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\psi(x)^3$}
\AxiomC{$\Pi_1[G(x)]$}
\dashedLine
\UnaryInfC{$\psi(x) \to P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$P(\psi)$}
\AxiomC{Axiom 4}
\dashedLine
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \Box P(\varphi))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\psi) \to \Box P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box P(\psi)$}
\RightLabel{$\to_I^3$}
\UnaryInfC{$\psi(x) \to \Box P(\psi)$}
\end{prooftree}
$$

Let the following derivation without open assumptions be Π_3:

$$
\begin{prooftree}
\AxiomC{$P(\psi)^4$}
\AxiomC{$G(x)^5$}
\dottedLine
\RightLabel{D1}
\UnaryInfC{$\forall\varphi.(P(\varphi) \to \varphi(x))$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\psi) \to \psi(x)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\psi(x)$}
\RightLabel{$\to_I^5$}
\UnaryInfC{$G(x) \to \psi(x)$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^4$}
\UnaryInfC{$P(\psi) \to \forall x.(G(x) \to \psi(x))$}
\end{prooftree}
$$

Let the following derivation with the open assumption $G(x)$ be $\Pi_4[G(x)]$:

$$
\begin{prooftree}
\AxiomC{$\psi(x)^6$}
\AxiomC{Π_2}
\dashedLine
\UnaryInfC{$\psi(x) \to \Box P(\psi)$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box P(\psi)$}
\AxiomC{$\Box P(\psi)^7$}
\RightLabel{$\Box_E$}
\UnaryInfC{$P(\psi)$}
\AxiomC{Π_3}
\dashedLine
\UnaryInfC{$P(\psi) \to \forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_E$}
\BinaryInfC{$\forall x.(G(x) \to \psi(x))$}
\RightLabel{Necessitation}
\UnaryInfC{$\Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^7$}
\UnaryInfC{$\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\to_I^6$}
\UnaryInfC{$\psi(x) \to \Box\forall x.(G(x) \to \psi(x))$}
\end{prooftree}
$$

The use of the necessitation rule above is correct, because the only open assumption $\Box P(\psi)$ is boxed. In the derivation of Theorem 2 below, the assumption $G(x)$ in the subderivation $\Pi_4[G(x)^8]$ is discharged by the rule labeled 8.

$$
\begin{prooftree}
\AxiomC{$G(x)^8$}
\AxiomC{$\Pi_4[G(x)^8]$}
\dashedLine
\UnaryInfC{$\psi(x) \to \Box\forall x.(G(x) \to \psi(x))$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))$}
\RightLabel{$\wedge_I$}
\BinaryInfC{$G(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))$}
\dottedLine
\RightLabel{D2}
\UnaryInfC{$G \ ess \ x$}
\RightLabel{$\to_I^8$}
\UnaryInfC{$G(x) \to G \ ess \ x$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall y.[G(y) \to G \ ess \ y]$}
\end{prooftree}
$$

5 If God's existence is possible, it is necessary

Definition 3 Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi \ ess \ x \to \Box\exists y.\varphi(y)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1 *If there is a God, then necessarily there exists a God:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists z.G(z)$}
\UnaryInfC{$G(g)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\dashedLine
\UnaryInfC{$G(g)$}
\AxiomC{Theorem 2}
\dashedLine
\UnaryInfC{$\forall y.[G(y) \to G \ ess \ y]$}
\UnaryInfC{$G(g) \to G \ ess \ g$}
\BinaryInfC{$G \ ess \ g$}
\AxiomC{Axiom 5}
\dashedLine
\UnaryInfC{$P(E)$}
\AxiomC{}
\dashedLine
\UnaryInfC{$G(g)$}
\dottedLine
\UnaryInfC{$\forall\varphi.[P(\varphi) \to \varphi(g)]$}
\UnaryInfC{$P(E) \to E(g)$}
\BinaryInfC{$E(g)$}
\dottedLine
\UnaryInfC{$\forall\varphi.[\varphi \ ess \ g \to \Box\exists x.\varphi(x)]$}
\UnaryInfC{$G \ ess \ g \to \Box\exists x.G(x)$}
\BinaryInfC{$\Box\exists x.G(x)$}
\RightLabel{1}
\UnaryInfC{$\exists z.G(z) \to \Box\exists x.G(x)$}
\end{prooftree}
$$

6 Necessarily, God exists

ToDo: This section still needs more details. See Coq formalization for more details.

ToDo: this is proven in a way that is slightly different from Gödel's 1970.

Theorem 3 *Necessarily, God exists:*

$$\Box\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\dfrac{\mathbf{S5}}{\forall\varphi.[\Diamond\ldots\Diamond\Box\varphi \leftrightarrow \Box\varphi]}}{\Diamond\Box\exists x.G(x) \leftrightarrow \Box\exists x.G(x)} \qquad \dfrac{\dfrac{\text{Corollary 1}}{\Diamond\exists x.G(x)} \quad \dfrac{\text{Lemma 1}}{\exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\Box\exists x.G(x)}}{\Box\exists x.G(x)}$$

7 God exists

Axiom 6 (M) *What is necessary is the case:*

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Corollary 2 *There exists a God:*

$$\exists x.G(x)$$

Proof

$$\dfrac{\dfrac{\text{Theorem 3}}{\Box\exists x.G(x)} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$