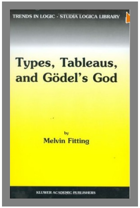
Gödel's Proof of God's Existence

Christoph Benzmüller and **Bruno Woltzenlogel Paleo**

Square of Opposition
Vatican, May 6, 2014



$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi)\to\Diamond\exists x.\varphi(x)]}}{P(G)\to\Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\to_E$$

A gift to **Priest Edvaldo** in Piracicaba, Brazil

First time mechanization and automation of

- (variants of) a modern ontological argument
- (variants of) higher-order modal logic

Work context/history:

- **Proposal:** exploit classical higher-order logic (HOL) as universal meta-logic — cf. previous talks at UNILOG
  - for object-level reasoning (in embedded non-classical logics)
  - for meta-level reasoning (about embedded non-classical logics)
- **Proof of concept:** demonstrate practical relevance of the approach by an interesting and relevant application
- **Experiments:** systematic study of Gödel's argument
- **Relation to Square of Opposition:** should be easy to analyze variants of the Square within our approach

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

What we did:

- A: Pen and paper: detailed natural deduction proof
- B: Formalization: in classical higher-order logic (HOL)
  - Automation: theorem provers LEO-II(**E**) and SATALLAX
  - Consistency: model finder NITPICK (NITROX)
- C: Step-by-step verification: proof assistant COQ
- D: Automation & verification: proof assistant ISABELLE

Did we get any new results? Yes — let's discuss this later!

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntelang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Montag, 09.09.2013 – 12:03 Uhr

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Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

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English Site > Germany > Science > Scientists Use Computer to Mathematically Prove Gödel God Theorem

Holy Logic: Computer Scientists 'Prove' God Exists

By David Knight

picture-alliance/ Imagno/ Wiener Stadt- und Landesbibliothek

Austrian mathematician Kurt Gödel kept his proof of God's existence a secret for decades. Now two scientists say they have proven it mathematically using a computer.

Two scientists have formalized a theorem regarding the existence of God penned by mathematician Kurt Gödel. But the God angle is somewhat of a red herring -- the real step forward is the example it sets of how computers can make scientific progress simpler.

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Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | 1 comments

Do you really need a MacBook to obtain the results? No

Did Apple send us some money? No

Introduction

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Rich history on ontological arguments (pros and cons)

… Anselm v. C. Gaunilo … Th. Aquinas …… Descartes Spinoza Leibniz … Hume Kant … Hegel … Frege … Hartshorne Malcolm Lewis Plantinga Gödel …

Anselm's notion of God:

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"(Necessarily) God exists."

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Different Interests in Ontological Arguments:

- Philosophical: Boundaries of Metaphysics & Epistemology
  - We talk about a metaphysical concept (God),
  - but we want to draw a conclusion for the real world.

- Theistic: Successful argument should convince atheists

- Ours: Can computers (theorem provers) be used . . .
  - . . . to formalize the definitions, axioms and theorems?
  - . . . to verify the arguments step-by-step?
  - . . . to fully automate (sub-)arguments?

Towards: *‘Computer-assisted Theoretical Philosophy”*

(cf. Leibniz dictum — Calculemus!)

Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive ($\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)G(y)$

hence $(\exists x)G(x) \supset N(\exists y)G(y)$

" $M(\exists x)G(x) \supset MN(\exists y)G(y)$ M = possibility

" $\supset N(\exists y)G(y)$

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x)G(x)$ means the system of all pos. prop. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl

$\begin{cases} x = x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. prop. were incon[sistent] It would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth[etic] sense (independently of the accidental structure of the world). Only then the ax. true. It m[ay] also mean pure "attribution" as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive not $x = x$ nor the existence Ax or the exist. of pos. prop.

X i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

Scott's Version of Gödel's Axioms, Definitions and Theorems

Axiom A1 Either a property or its negation is positive, but not both: $\forall \phi [P(\neg \phi) \equiv \neg P(\phi)]$

Axiom A2 A property necessarily implied by a positive property is positive: $\forall \phi \forall \psi [(P(\phi) \wedge \Box \forall x [\phi(x) \supset \psi(x)]) \supset P(\psi)]$

Thm. T1 Positive properties are possibly exemplified: $\forall \phi [P(\phi) \supset \Diamond \exists x \phi(x)]$

Def. D1 A *God-like* being possesses all positive properties: $G(x) \equiv \forall \phi [P(\phi) \supset \phi(x)]$

Axiom A3 The property of being God-like is positive: $P(G)$

Cor. C Possibly, God exists: $\Diamond \exists x G(x)$

Axiom A4 Positive properties are necessarily positive: $\forall \phi [P(\phi) \supset \Box P(\phi)]$

Def. D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties: $\phi \; ess. \; x \equiv \phi(x) \wedge \forall \psi (\psi(x) \supset \Box \forall y (\phi(y) \supset \psi(y)))$

Thm. T2 Being God-like is an essence of any God-like being: $\forall x [G(x) \supset G \; ess. \; x]$

Def. D3 *Necessary existence* of an individ. is the necessary exemplification of all its essences: $NE(x) \equiv \forall \phi [\phi \; ess. \; x \supset \Box \exists y \phi(y)]$

Axiom A5 Necessary existence is a positive property: $P(NE)$

Thm. T3 Necessarily, God exists: $\Box \exists x G(x)$

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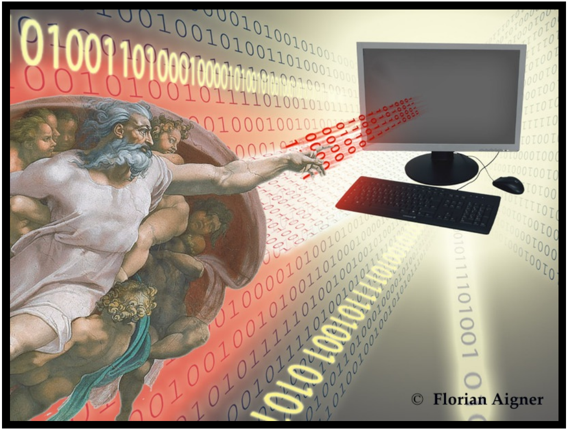
Remainder of this Talk

- Embedding of QML in HOL and Proof Automation (myself)

- Proof Overview (Bruno)
- Experiments and Results (Bruno)
- Conclusion and Outlook (Bruno)



Embedding of QML in HOL and Proof Automation

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

Then use existing HOL theorem provers for reasoning in QML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empirical findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[BenzmüllerRaths, LPAR, 2013]

QML $\varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \supset \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

- Kripke style semantics (possible world semantics)

HOL $s, t \quad ::= \quad C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exist

  interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

  automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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HOL $s, t ::= C \mid x \mid \lambda x s \mid s\, t \mid \neg s \mid s \vee t \mid \forall x\, t$

QML in HOL: QML formulas φ are mapped to HOL predicates $\varphi_{\iota\to o}$

$$
\begin{array}{lcl}
\neg & = & \lambda\varphi_{\iota\to o}\lambda s_\iota \neg\varphi s \\
\wedge & = & \lambda\varphi_{\iota\to o}\lambda\psi_{\iota\to o}\lambda s_\iota(\varphi s \wedge \psi s) \\
\supset & = & \lambda\varphi_{\iota\to o}\lambda\psi_{\iota\to o}\lambda s_\iota(\neg\varphi s \vee \psi s) \\
\Box & = & \lambda\varphi_{\iota\to o}\lambda s_\iota\forall u_\iota\,(\neg rsu \vee \varphi u) \\
\Diamond & = & \lambda\varphi_{\iota\to o}\lambda s_\iota\exists u_\iota\,(rsu \wedge \varphi u) \\
\forall & = & \lambda h_{\mu\to(\iota\to o)}\lambda s_\iota\forall d_\mu\, hds \\
\exists & = & \lambda h_{\mu\to(\iota\to o)}\lambda s_\iota\exists d_\mu\, hds \\
\forall & = & \lambda H_{(\mu\to(\iota\to o))\to(\iota\to o)}\lambda s_\iota\forall d_\mu\, Hds \\
 & & \\
\text{valid} & = & \lambda\varphi_{\iota\to o}\forall w_\iota\varphi w
\end{array}
\quad \text{Ax}
$$

The equations in Ax are given as axioms to the HOL provers!

(Remark: Note that we are here dealing with constant domain quantification)

QML $\quad \varphi, \psi \quad ::= \quad \ldots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \supset \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\, \varphi \mid \exists x\, \varphi \mid \forall P\, \varphi$

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Formalization in HOL

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Example:

QML formula	$\Diamond \exists x G(x)$
QML formula in HOL	valid $(\Diamond \exists x G(x))_{\iota \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\iota (\Diamond \exists x G(x))_{\iota \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge (\exists x G(x))_{\iota \to o} u)$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in QML,
–> we instead prove that valid $\varphi_{\iota \to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Soundness and Completeness: wrt. Henkin semantics

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expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge \exists x Gxu)$

What are we doing?

In order to prove that φ is valid in QML,
–> we instead prove that valid $\varphi_{\iota\to o}$ can be derived from Ax in HOL.

This can be done with interactive or automated HOL theorem provers.

Soundness and Completeness: wrt. Henkin semantics

Example:

QML formula	$\Diamond \exists x G(x)$
QML formula in HOL	valid $(\Diamond \exists x G(x))_{\iota \to o}$
expansion, $\beta\eta$-conversion	$\forall w_\iota (\Diamond \exists x G(x))_{\iota \to o}\, w$
expansion, $\beta\eta$-conversion	$\forall w_\iota \exists u_\iota (rwu \wedge (\exists x G(x))_{\iota \to o} u)$
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What are we doing?

In order to prove that φ is valid in QML,
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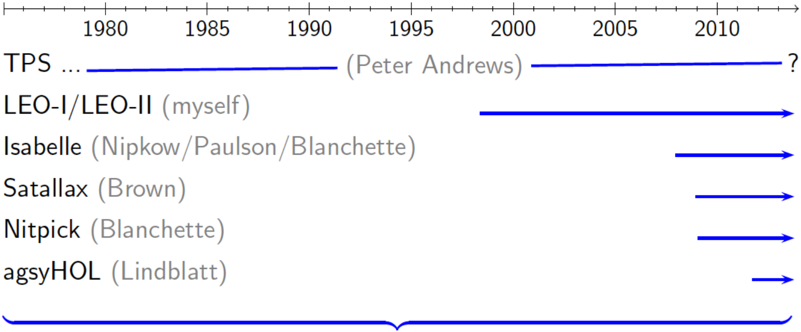
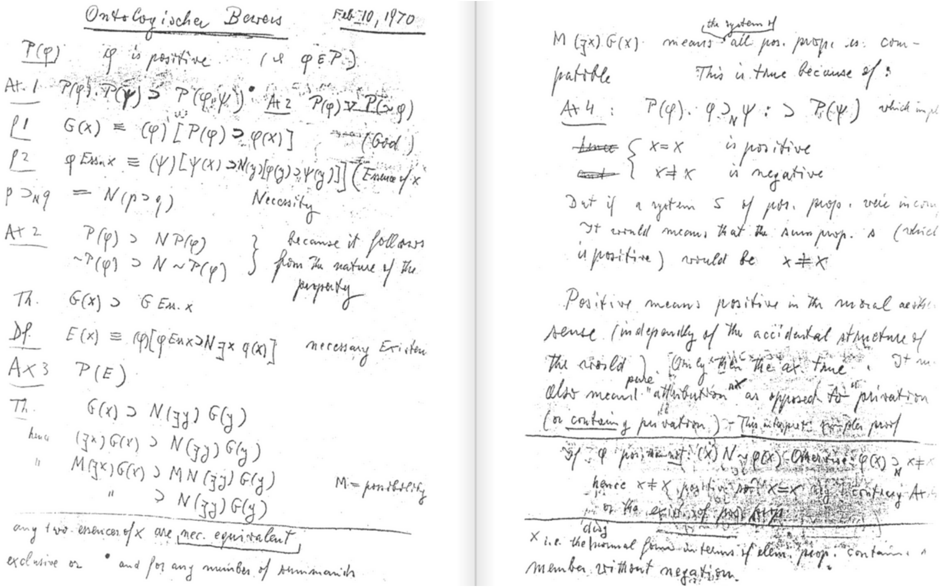
Soundness and Completeness: wrt. Henkin semantics



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
- can be called remotely via SystemOnTPTP at Miami
- they significantly gained in strength over the last years
- they can be bundled into a combined prover HOL-P

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— HOL-P becomes a Universal Reasoner —

Proof Overview
Experiments and Results



Ontologischer Beweis Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$, $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset MN(\exists y) G(y)$

" $\supset N(\exists y) G(y)$ M = possibility

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x) G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl

$x = x$ is positive

$x \neq x$ is negative

But if a system S of pos. props. were incon it would mean, that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth. sense (independently of the accidental structure of the world). Only then the ax. true. It m. also means pure "attribution" as opposed to "privation" (or containing privation). This interpret. simpler proof

If φ pos. not: $(x) N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$ hence $x \neq x$ positive so $x = x$ neg. contrary At ... or the exist. of pos. ...

X i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

T3: $\Box\exists x.G(x)$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\frac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \supset \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

L2: $\Diamond \exists z.G(z) \supset \Box \exists x.G(x)$

$$\frac{\textbf{C1: } \Diamond \exists z.G(z) \qquad \textbf{L2: } \Diamond \exists z.G(z) \supset \Box \exists x.G(x)}{\textbf{T3: } \Box \exists x.G(x)}$$

$$\dfrac{\dfrac{\textbf{S5}}{\forall \xi.[\Diamond\Box\xi \supset \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \supset \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \supset \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

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Proof Overview

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \supset \Box\exists x.G(x)}{\Diamond\exists z.G(z) \supset \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \supset \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \supset \Box\exists x.G(x)}$$

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Proof Overview

$$\textbf{D1: } G(x) \equiv \forall \varphi.[P(\varphi) \supset \varphi(x)]$$

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$$\frac{P(G)}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

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$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \textbf{T1: } \forall\varphi.[P(\varphi) \supset \Diamond\exists x.\varphi(x)]}{\textbf{C1: } \Diamond\exists z.G(z)}$$

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$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \supset \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Natural Deduction Calculus

$$\dfrac{A \vee B \quad \begin{matrix}\overline{A}\\ \vdots \\ C\end{matrix} \quad \begin{matrix}\overline{B}\\ \vdots \\ C\end{matrix}}{C}\ \vee_E \qquad \dfrac{A \quad B}{A \wedge B}\ \wedge_I \qquad \dfrac{\begin{matrix}\overline{A}^{\,n}\\ \vdots \\ B\end{matrix}}{A \supset B}\ \supset_I^n$$

$$\dfrac{A}{A \vee B}\ \vee_{I_1} \qquad \dfrac{A \wedge B}{A}\ \wedge_{E_1} \qquad \dfrac{B}{A \supset B}\ \supset_I$$

$$\dfrac{B}{A \vee B}\ \vee_{I_2} \qquad \dfrac{A \wedge B}{B}\ \wedge_{E_2} \qquad \dfrac{A \quad A \supset B}{B}\ \supset_E$$

$$\dfrac{A[\alpha]}{\forall x.A[x]}\ \forall_I \qquad \dfrac{\forall x.A[x]}{A[t]}\ \forall_E \qquad \dfrac{A[t]}{\exists x.A[x]}\ \exists_I \qquad \dfrac{\exists x.A[x]}{A[\beta]}\ \exists_E$$

$$\neg A \equiv A \supset \bot \qquad \dfrac{\neg\neg A}{A}\ \neg\neg_E$$

$$\dfrac{\alpha : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \dfrac{\Box A}{t : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Box_E$$

$$\dfrac{t : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \dfrac{\Diamond A}{\beta : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Diamond_E$$

$$\Diamond A \equiv \neg \Box \neg A$$

Natural Deduction Proofs

T1 and C1

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \supset \psi(x)]) \supset P(\psi)]}
        }{\forall\psi.[(P(\rho) \wedge \Box\forall x.[\rho(x) \supset \psi(x)]) \supset P(\psi)]}\forall_E
      }{(P(\rho) \wedge \Box\forall x.[\rho(x) \supset \neg\rho(x)]) \supset P(\neg\rho)}\forall_E
    }{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \supset P(\neg\rho)}
    \quad
    \dfrac{\dfrac{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi) \supset \neg P(\varphi)]}}{P(\neg\rho) \supset \neg P(\rho)}\forall_E
  }{(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \supset \neg P(\rho)}
}{
  \dfrac{P(\rho) \supset \Diamond\exists x.\rho(x)}{\textbf{T1: } \forall\varphi.[P(\varphi) \supset \Diamond\exists x.\varphi(x)]}\forall_I
}
$$

$$
\dfrac{
  \dfrac{\textbf{A3}}{P(G)}
  \quad
  \dfrac{\dfrac{\textbf{T1}}{\forall\varphi.[P(\varphi) \supset \Diamond\exists x.\varphi(x)]}}{P(G) \supset \Diamond\exists x.G(x)}\forall_E
}{\Diamond\exists x.G(x)}\supset_E
$$

Natural Deduction Proofs
T2 (Partial)

$$
\dfrac{
\dfrac{\psi(x)^6 \qquad \dfrac{\Pi_2}{\psi(x) \to \Box P(\psi)}}{\Box P(\psi)}\,{\scriptstyle \to_E}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\Box P(\psi)^7}{P(\psi)}\,{\scriptstyle \Box_E} \qquad \dfrac{\Pi_3}{P(\psi) \to \forall x.(G(x) \to \psi(x))}}{\forall x.(G(x) \to \psi(x))}\,{\scriptstyle \to_E}}{\Box\forall x.(G(x) \to \psi(x))}\,{\scriptstyle \Box_I}}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))}\,{\scriptstyle \to_I^7}
}{
\dfrac{\Box\forall x.(G(x) \to \psi(x))}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}\,{\scriptstyle \to_I^6}
}\,{\scriptstyle \to_E}
$$

- Formal encoding(s) of the axioms, definitions, and theorems in Scott's proof script
- Calls to the HOL reasoners mentioned before
- Interactive proofs in proof assistants Isabelle and Coq

The source files of these experiments are available at:

`https://github.com/FormalTheology/GoedelGod/`

Results

- Axioms and definitions are consistent

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Adresses criticisms: modal logic S5 is too strong

$$\forall P.[\Diamond\Box P \supset \Box P]$$

If something is possibly necessary, then it is necessary.

S5 usually considered adequate
(But KB is sufficient! — shown by HOL ATPs)

- Axioms and definitions are consistent
- Logic K is sufficient for proving T1, C and T2.
- For proving the final theorem T3, logic KB is sufficient
- Only for T3 the HOL-ATPs still fail to produce a proof directly from the axioms; thus, T3 remains an interesting benchmark problem; T1, C, and T2 are rather trivial for HOL-ATPs.

- Axioms and definitions are consistent
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Fundamental criticism against Gödel's argument.

Everything that is the case is so necessarily.

Follows from T2, T3 and D2 (as shown by HOL ATPs).

There are no contingent "truths".
Everything is determined.
There is no free will.

Many proposed solutions: Anderson, Fitting, Hájek, …

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- Another implication of Gödel's axioms is monotheism.
- All of the above findings hold for both
  - constant domain semantics and
  - varying domain semantics (for the domain of individuals).

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- Major step towards Computer-assisted Theoretical Philosophy
  - see also Ed Zalta's *Computational Metaphysics* project at Stanford University
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  - remember Leibniz' dictum — *Calculemus!*
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