Gödel's God in Isabelle/HOL

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A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$
D1	A *God-like* being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists x G(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box\, P(\phi)]$
D2	An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G\ ess.\ x]$
D3	*Necessary existence* of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists x G(x)$

Figure 1: Scott's version of Gödel's ontological argument [11].

1 Introduction

Dana Scott's version [11] of Goedel's ontological argument [8] is formalized in quantified modal logic KB (QML KB) within the proof assistant Isabelle/HOL. QML KB is modeled as a fragment of classical higher-order logic (HOL); thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benzmüller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (properties).

The gaps in Scott's proof have been automated with Sledgehammer [5], performing remote calls to the higher-order automated theorem prover LEO-II [3]. Sledgehammer then suggests the Metis [9] calls. The Metis proofs are verified by Isabelle/HOL. For consistency checking, the model finder Nitpick [6] has been employed. The successfull calls to Sledgehammer (normally, they automatically eliminated by Isabelle/HOL) are deliberately kept in the file for demonstration purposes.

Isabelle is described in the textbook by Nipkow, Paulson, and Wenzel [10] and in tutorials available at: http://isabelle.in.tum.de.

1.1 Related Work

The formalization presented here is related to the THF [13] and Coq [4] formalizations at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/.

An older ontological argument by Anselm was formalized in PVS by John Rushby [14].

2 An Embedding of QML KB in HOL

The types i for possible worlds and μ for individuals are introduced.

typedecl i — the type for possible worlds
typedecl μ — the type for indiviuals

Possible worlds are connected by an accessibility relation r.

consts $r :: i \Rightarrow i \Rightarrow bool$ (**infixr** r *70*) — accessibility relation r

QML formulas are translated as HOL terms of type $i \Rightarrow bool$. This type is abbreviated as σ.

type-synonym $\sigma = (i \Rightarrow bool)$

The classical connectives $\neg, \wedge, \rightarrow$, and $\forall$ (over individuals and over sets of individuals) and $\exists$ (over individuals) are lifted to type σ. The lifted connectives are $m\neg$, $m\wedge$, $m\rightarrow$, $\forall$, and $\exists$ (the latter two are modeled as constant symbols). Other connectives can be introduced analogously. We exemplarily do this for $\vee$, $\leftrightarrow$, and $=$. Moreover, the modal operators $\Box$ and $\Diamond$ are introduced. Definitions could be used instead of abbreviations.

abbreviation *mnot* $:: \sigma \Rightarrow \sigma$ $(m\neg)$ **where** $m\neg\ \varphi \equiv (\lambda w.\ \neg\ \varphi\ w)$
abbreviation *mand* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\wedge$ *65*) **where** $\varphi\ m\wedge\ \psi \equiv (\lambda w.\ \varphi\ w \wedge \psi\ w)$
abbreviation *mor* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\vee$ *70*) **where** $\varphi\ m\vee\ \psi \equiv (\lambda w.\ \varphi\ w \vee \psi\ w)$
abbreviation *mimplies* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\rightarrow$ *74*) **where** $\varphi\ m\rightarrow\ \psi \equiv (\lambda w.\ \varphi\ w \longrightarrow \psi\ w)$
abbreviation *mequiv*$:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\equiv$ *76*) **where** $\varphi\ m\equiv\ \psi \equiv (\lambda w.\ (\varphi\ w \longleftrightarrow \psi\ w))$
abbreviation *meq* $:: 'a \Rightarrow 'a \Rightarrow \sigma$ (**infixr** $m=$ *50*) **where** $x\ m=\ y \equiv (\lambda w.\ x = y)$
abbreviation *mforall* $:: ('a \Rightarrow \sigma) \Rightarrow \sigma$ $(\forall)$ **where** $\forall\ \Phi \equiv (\lambda w.\ \forall x.\ \Phi\ x\ w)$
abbreviation *mexists* $:: ('a \Rightarrow \sigma) \Rightarrow \sigma$ $(\exists)$ **where** $\exists\ \Phi \equiv (\lambda w.\ \exists x.\ \Phi\ x\ w)$
abbreviation *mbox* $:: \sigma \Rightarrow \sigma$ $(\Box)$ **where** $\Box\ \varphi \equiv (\lambda w.\ \forall v.\ \ w\ r\ v \longrightarrow \varphi\ v)$
abbreviation *mdia* $:: \sigma \Rightarrow \sigma$ $(\Diamond)$ **where** $\Diamond\ \varphi \equiv (\lambda w.\ \exists v.\ w\ r\ v \wedge \varphi\ v)$

For grounding lifted formulas, the meta-predicate *valid* is introduced.

abbreviation *valid* $:: \sigma \Rightarrow bool$ ([-]) **where** $[p] \equiv \forall w.\ p\ w$

3 Gödel's Ontological Argument

Constant symbol P (Gödel's 'Positive') is declared.

consts $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms $A1(a/b)$: $\forall\varphi[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$ (Either a property or its negation is positive, but not both.) and $A2$: $\forall\varphi\forall\psi[(P(\varphi) \wedge \Box\forall x[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ (A property necessarily implied by a positive property is positive).

axiomatization where
$A1a$: $[\forall (\lambda\varphi.\ P\ (\lambda x.\ m\neg\ (\varphi\ x))\ m\!\rightarrow\ m\neg\ (P\ \varphi))]$ **and**
$A1b$: $[\forall (\lambda\varphi.\ m\neg\ (P\ \varphi)\ m\!\rightarrow\ P\ (\lambda x.\ m\neg\ (\varphi\ x)))]$ **and**
$A2$: $[\forall (\lambda\varphi.\ \forall (\lambda\psi.\ (P\ \varphi\ m\!\wedge\ \Box\ (\forall (\lambda x.\ \varphi\ x\ m\!\rightarrow\ \psi\ x)))\ m\!\rightarrow\ P\ \psi))]$

We prove theorem T1: $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$ (Positive properties are possibly exemplified). T1 is proved directly by Sledgehammer with command *sledgehammer* [*provers* = *remote-leo2*]. Sledgehammer suggests to call Metis with axioms A1a and A2. Metis sucesfully generates a proof object that is verified in Isabelle/HOL's kernel.

theorem $T1$: $[\forall (\lambda\varphi.\ P\ \varphi\ m\!\rightarrow\ \Diamond\ (\exists\ \varphi))]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis A1a A2*)

Next, the symbol G for 'God-like' is introduced and defined as $G(x) \leftrightarrow \forall\varphi[P(\phi) \rightarrow \varphi(x)]$ (A God-like being possesses all positive properties).

definition $G :: \mu \Rightarrow \sigma$ **where** $G = (\lambda x.\ \forall (\lambda\varphi.\ P\ \varphi\ m\!\rightarrow\ \varphi\ x))$

Axiom $A3$ is added: $P(G)$ (The property of being God-like is positive). Sledgehammer and Metis then prove corollary C: $\Diamond\exists x G(x)$ (Possibly, God exists).

axiomatization where $A3$: $[P\ G]$

corollary C: $[\Diamond\ (\exists\ G)]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis A3 T1*)

Axiom $A4$ is added: $\forall\phi[P(\phi) \rightarrow \Box\ P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where $A4$: $[\forall (\lambda\varphi.\ P\ \varphi\ m\!\rightarrow\ \Box\ (P\ \varphi))]$

Symbol *ess* for 'Essence' is introduced and defined as

$$\varphi\ ess.\ x \leftrightarrow \varphi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\varphi(y) \rightarrow \psi(y)))$$

(An *essence* of an individual is a property possessed by it and necessarily implying any of its properties).

definition $ess :: (\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$ (**infixr** *ess 85*) **where**
$\varphi\ ess\ x = \varphi\ x\ m\!\wedge\ \forall (\lambda\psi.\ \psi\ x\ m\!\rightarrow\ \Box\ (\forall (\lambda y.\ \varphi\ y\ m\!\rightarrow\ \psi\ y)))$

Next, Sledgehammer and Metis prove theorem $T2$: $\forall x[G(x) \rightarrow G\ ess.\ x]$ (Being God-like is an essence of any God-like being).

theorem $T2$: $[\forall (\lambda x.\ G\ x\ m\!\rightarrow\ G\ ess\ x)]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis A1b A4 G-def ess-def*)

Symbol *NE*, for 'Necessary Existence', is introduced and defined as

$$NE(x) \leftrightarrow \forall\varphi[\varphi\ ess.\ x \rightarrow \Box\exists y\varphi(y)]$$

(Necessary existence of an individual is the necessary exemplification of all its essences).

definition $NE :: \mu \Rightarrow \sigma$ **where** $NE = (\lambda x.\ \forall (\lambda\varphi.\ \varphi\ ess\ x\ m\!\rightarrow\ \Box\ (\exists\ \varphi)))$

Moreover, axiom *A5* is added: $P(NE)$ (Necessary existence is a positive property).

axiomatization where *A5*: $[P\ NE]$

The B axiom (symmetry) for relation r is stated. B is needed only for proving theorem T3 and for corollary C2.

axiomatization where *sym*: $x\ r\ y \longrightarrow y\ r\ x$

Finally, Sledgehammer and Metis prove the main theorem *T3*: $\Box\exists x G(x)$
(Necessarily, God exists).

theorem *T3*: $[\Box\ (\exists\ G)]$
sledgehammer [*provers = remote-leo2*]
by (*metis A5 C T2 sym G-def NE-def*)

Surprisingly, the following corollary can be derived even without the T axiom (reflexivity).

corollary *C2*: $[\exists\ G]$
sledgehammer [*provers = remote-leo2*](*T1 T3 G-def sym*)
by (*metis T1 T3 G-def sym*)

The consistency of the entire theory is checked with Nitpick.

lemma *True* **nitpick** [*satisfy*, *user-axioms*, *expect = genuine*] **oops**

4 Additional Results on Gödel's God.

Gödel's God is flawless: (s)he does not have a non-positive property.

theorem *Flawlessness*: $[\forall\,(\lambda\varphi.\ \forall\,(\lambda x.\ (G\ x\ m{\rightarrow}\ (m\neg\ (P\ \varphi)\ m{\rightarrow}\ m\neg\ (\varphi\ x))))]$
sledgehammer [*provers = remote-leo2*]
by (*metis A1b G-def*)

There is only one God: any two God-like beings are equal.

theorem *Monotheism*: $[\forall\,(\lambda x.\forall\,(\lambda y.\ (G\ x\ m{\rightarrow}\ (G\ y\ m{\rightarrow}\ (x\ m{=}\ y)))))]$
sledgehammer [*provers = remote-satallax remote-leo2*]
oops

5 Modal Collapse

Gödel's axioms have been criticized for entailing the so-called modal collapse. The prover Satallax [7] confirms this. However, sledgehammer does not seem to be able to determine which axioms, definitions and previous theorems are used by Satallax; hence it suggests to call Metis using everything, but this (unsurprisingly) fails. Attempting to use "Sledegehammer min" to minimize Sledgehammer's suggestion does not work. Nevertheless, calling Metis with T2, T3 and ess$_d ef does work.$

lemma *MC*: $[\forall\,(\lambda\varphi.(\varphi\ m{\rightarrow}\ (\Box\ \varphi)))]$
using *T2 T3 ess-def sym* **sledgehammer** [*provers = remote-satallax*]
oops

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