Formalization, Mechanization and Automation of Gödel's Proof of God's Existence

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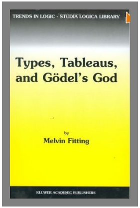

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\dfrac{\text{Theorem 1}}{\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{P(G) \rightarrow \Diamond\exists x.G(x)}\,\forall_E}{\Diamond\exists x.G(x)}\,\rightarrow_E$$

A gift to **Priest Edvaldo** and his church in Piracicaba, Brazil

Part A:

Informal Proof and Natural Deduction Proof

Gödel's Manuscript (1970)

Ontologischer Beweis   Feb 10, 1970

$P(\varphi)$ φ is positive (or $\varphi \in P$)

At. 1 $P(\varphi).P(\psi) \supset P(\varphi.\psi)$ At 2 $P(\varphi) \vee P(\sim\varphi)$

P1 $G(x) \equiv (\varphi)[P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess } x \equiv (\psi)[\psi(x) \supset N(y)[\varphi(y) \supset \psi(y)]]$ (Essence of x)

$p \supset_N q = N(p \supset q)$ Necessity

At 2 $P(\varphi) \supset NP(\varphi)$
$\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess. } x$

Df. $E(x) \equiv (\varphi)[\varphi \text{ Ess } x \supset N \exists x\, \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y)\, G(y)$

hence $(\exists x)\, G(x) \supset N(\exists y)\, G(y)$

" $M(\exists x)\, G(x) \supset MN(\exists y)\, G(y)$ M = possibility

" $\supset N(\exists y)\, G(y)$

any two essences of x are nec. equivalent

exclusive or • and for any number of summands

$M(\exists x)\, G(x)$ means the system of all pos. props. is compatible. This is true because of:

At 4: $P(\varphi).\varphi \supset_N \psi : \supset P(\psi)$ which impl

$\left\{ \begin{array}{ll} x = x & \text{is positive} \\ x \neq x & \text{is negative} \end{array} \right.$

But if a system S of pos. props. were incom[patible] It would mean that the sum prop. s (which is positive) would be $x \neq x$

Positive means positive in the moral aesth[etic] sense (independently of the accidental structure of the world). Only then the ax. true. It m[ay] also mean pure "attribution"* as opposed to "privation" (or containing privation). This interpret[ation] simpler proof

If φ pos. not: $(x)\, N \sim \varphi(x)$. Otherwise: $\varphi(x) \supset_N x \neq x$

hence $x \neq x$ positive not $x = x$ [illegible] At [illegible]

or the exist. of pos. [illegible]

* i.e. the disj. normal form in terms of elem. prop. contains a member without negation.

A1	Either a property or its negation is positive, but not both:	$\forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall \phi \forall \psi [(P(\phi) \wedge \Box \forall x [\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall \varphi [P(\varphi) \rightarrow \Diamond \exists x \varphi(x)]$
D1	A *God-like* being possesses all positive properties:	$G(x) \leftrightarrow \forall \phi [P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond \exists x G(x)$
A4	Positive properties are necessarily positive:	$\forall \phi [P(\phi) \rightarrow \Box\, P(\phi)]$
D2	An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi \ ess.\ x \leftrightarrow \phi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x [G(x) \rightarrow G \ ess.\ x]$
D3	*Necessary existence* of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall \phi [\phi \ ess.\ x \rightarrow \Box \exists y \phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box \exists x G(x)$

T3: $\Box \exists x. G(x)$

Proof Overview

$$\frac{\textbf{C1: } \Diamond \exists z. G(z)}{\textbf{T3: } \Box \exists x. G(x)}$$

Proof Overview

$$\frac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

$$\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)$$

$$\frac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Proof Overview

$$\dfrac{\dfrac{\mathbf{S5}}{\forall \xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

$$\dfrac{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x) \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall \xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\dfrac{\dfrac{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

D1: $G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$

D3: $E(x) \equiv \Box\exists y.G(y)$ (cheating!)

$$\dfrac{\dfrac{\dfrac{P(E)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$\color{red}{\textbf{D3: } E(x) \equiv \Box\exists y.G(y)}$ (cheating!)

$$\dfrac{\dfrac{\dfrac{P(E)}{\textbf{L1: } \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

Proof Overview

$$\textbf{D1: } G(x) \equiv \forall\varphi.[P(\varphi) \to \varphi(x)]$$

$$\textbf{D3: } E(x) \equiv \Box\exists y.G(y) \qquad\qquad \textbf{D3: } E(x) \equiv \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\dfrac{\textbf{T2: } \forall y.[G(y) \to G\ ess\ y] \qquad P(E)}{\textbf{L1: } \exists z.G(z) \to \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \to \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \to \Box\xi]}}{\textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1: } \Diamond\exists z.G(z) \qquad \textbf{L2: } \Diamond\exists z.G(z) \to \Box\exists x.G(x)}{\textbf{T3: } \Box\exists x.G(x)}$$

$$\textbf{D1:}\ G(x) \equiv \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

$$\textbf{D2:}\ \varphi\ ess\ x \equiv \varphi(x) \wedge \forall\psi.(\psi(x) \rightarrow \Box\forall x.(\varphi(x) \rightarrow \psi(x)))$$

$$\textbf{D3:}\ E(x) \equiv \forall\varphi.[\varphi\ ess\ x \rightarrow \Box\exists y.\varphi(y)]$$

$$\dfrac{\dfrac{\textbf{A3}}{P(G)} \qquad \dfrac{\dfrac{\textbf{A2}}{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]} \qquad \dfrac{\textbf{A1a}}{\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]}}{\textbf{T1:}\ \forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]}}{\textbf{C1:}\ \Diamond\exists z.G(z)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textbf{A1b}}{\forall\varphi.[\neg P(\varphi) \rightarrow P(\neg\varphi)]} \qquad \dfrac{\textbf{A4}}{\forall\varphi.[P(\varphi) \rightarrow \Box\ P(\varphi)]}}{\textbf{T2:}\ \forall y.[G(y) \rightarrow G\ ess\ y]} \qquad \dfrac{\textbf{A5}}{P(E)}}{\textbf{L1:}\ \exists z.G(z) \rightarrow \Box\exists x.G(x)}}{\Diamond\exists z.G(z) \rightarrow \Diamond\Box\exists x.G(x)} \qquad \dfrac{\textbf{S5}}{\forall\xi.[\Diamond\Box\xi \rightarrow \Box\xi]}}{\textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}$$

$$\dfrac{\textbf{C1:}\ \Diamond\exists z.G(z) \qquad \textbf{L2:}\ \Diamond\exists z.G(z) \rightarrow \Box\exists x.G(x)}{\textbf{T3:}\ \Box\exists x.G(x)}$$

Natural Deduction Calculus

$$
\dfrac{A \vee B \quad \begin{array}{c}\overline{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\overline{B}\\ \vdots \\ C\end{array}}{C}\ \vee_E
\qquad
\dfrac{A \quad B}{A \wedge B}\ \wedge_I
\qquad
\dfrac{\begin{array}{c}\overline{A}^{\,n}\\ \vdots \\ B\end{array}}{A \rightarrow B}\ \rightarrow_I^n
$$

$$
\dfrac{A}{A \vee B}\ \vee_{I_1}
\qquad
\dfrac{A \wedge B}{A}\ \wedge_{E_1}
\qquad
\dfrac{B}{A \rightarrow B}\ \rightarrow_I
$$

$$
\dfrac{B}{A \vee B}\ \vee_{I_2}
\qquad
\dfrac{A \wedge B}{B}\ \wedge_{E_2}
\qquad
\dfrac{A \quad A \rightarrow B}{B}\ \rightarrow_E
$$

$$
\dfrac{A[\alpha]}{\forall x.A[x]}\ \forall_I
\qquad
\dfrac{\forall x.A[x]}{A[t]}\ \forall_E
\qquad
\dfrac{A[t]}{\exists x.A[x]}\ \exists_I
\qquad
\dfrac{\exists x.A[x]}{A[\beta]}\ \exists_E
$$

$$
\neg A \equiv A \rightarrow \bot
\qquad
\dfrac{\neg\neg A}{A}\ \neg\neg_E
$$

$$\dfrac{\alpha : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Box A}\ \Box_I \qquad\qquad \dfrac{\Box A}{t : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Box_E$$

$$\dfrac{t : \boxed{\begin{array}{c}\vdots\\ A\end{array}}}{\Diamond A}\ \Diamond_I \qquad\qquad \dfrac{\Diamond A}{\beta : \boxed{\begin{array}{c}A\\ \vdots\end{array}}}\ \Diamond_E$$

$$\Diamond A \equiv \neg\Box\neg A$$

$$
\begin{prooftree}
\AxiomC{\textbf{A2}}
\dashedLine
\UnaryInfC{$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$\forall\psi.[(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \neg\rho(x)]) \rightarrow P(\neg\rho)$}
\doubleLine
\UnaryInfC{$(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \rightarrow P(\neg\rho)$}
\AxiomC{\textbf{A1a}}
\dashedLine
\UnaryInfC{$\forall\varphi.[P(\neg\varphi) \rightarrow \neg P(\varphi)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(\neg\rho) \rightarrow \neg P(\rho)$}
\doubleLine
\BinaryInfC{$(P(\rho) \wedge \Box\forall x.[\neg\rho(x)]) \rightarrow \neg P(\rho)$}
\doubleLine
\UnaryInfC{$P(\rho) \rightarrow \Diamond\exists x.\rho(x)$}
\RightLabel{$\forall_I$}
\UnaryInfC{\textbf{T1:} $\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{\textbf{A3}}
\dashedLine
\UnaryInfC{$P(G)$}
\AxiomC{\textbf{T1}}
\dashedLine
\UnaryInfC{$\forall\varphi.[P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)]$}
\RightLabel{$\forall_E$}
\UnaryInfC{$P(G) \rightarrow \Diamond\exists x.G(x)$}
\RightLabel{$\rightarrow_E$}
\BinaryInfC{$\Diamond\exists x.G(x)$}
\end{prooftree}
$$

$$
\dfrac{
\dfrac{\psi(x)^6 \qquad \dfrac{\Pi_2}{\psi(x) \to \Box P(\psi)}}{\Box P(\psi)}\,{\to_E}
\qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{\Box P(\psi)^7}{P(\psi)}\,\Box_E
\qquad
\dfrac{\Pi_3}{P(\psi) \to \forall x.(G(x) \to \psi(x))}
}{\forall x.(G(x) \to \psi(x))}\,{\to_E}
}{\Box\forall x.(G(x) \to \psi(x))}\,\Box_I
}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))}\,{\to_I^7}
}{
\dfrac{\Box\forall x.(G(x) \to \psi(x))}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}\,{\to_I^6}
}\,{\to_E}
$$