SOME WEAKENED GÖDELIAN ONTOLOGICAL SYSTEMS

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Abstract. We describe a KB Gödelian ontological system, and some other weak systems, in a fully formal way using theory of types and natural deduction, and present a completeness proof in its main and specific parts. We technically and philosophically analyze and comment on the systems (mainly with respect to the relativism of values) and include a sketch of some connected aspects of Gödel's relation to Kant.

Kurt Gödel proposed in some of his notes [11] an ontological proof of the necessary existence of a God-like being. He used S5 second-order modal logic with some axioms and definitions added. Following an initiative from a paper by Hájek ([15], p. 134), we would like to examine some weakened Gödelian ontological systems, and that from the philosophical and technical point of view. 'Weakened' means here weaker than Gödel's original S5 based system from 1970. We use second-order logic with third-order constants, adding modalities and some or all Gödel's axioms. A KB based second-order modal logic with Gödel's ontological axioms is especially interesting since it can block the modal collapse of the system in a very simple way, allows contingencies, and preserves value absolutism. Since we want to leave the third-order property of "positiveness" undefined, we do not use Cocchiarella's second-order modal logic [4], which can be used only on the proviso that all third-order properties are defined by second-order properties, for example, in the way Anderson defines "positiveness" (in [3], pp. 297, 303).

We first generally describe the language, semantics and systems of a class of Gödelian ontological logics, and then we consider some special logics of that class. The language and semantics are restricted to at most third-order types. We use world-dependent first-order domains, first-order constants are rigid, and otherwise there are no extensional types. We also use a concept of intensional ultrafilter. A Fitch-style natural deduction system is applied.[1]

[1] The language and semantics are in some aspects similar to those of Fitting [5] and Gallin [7].

§1. Language and semantics.

1.1. Language. In the language we have an appropriate typing mechanism where third-order types are the highest types. In particular, we have a third-order property $\mathcal{P}$ ("positivness") and λ-abstractor. For simplicity, we omit the intensionality sign, since any ambiguity will be solved by the interpretation and derivation rules.[2]

Definition 1.1 (Type). *0 is a type. If $\tau_1, \ldots, \tau_n$ are types, then $< \tau_1, \ldots, \tau_n >$ is a type. The highest argument type τ_i is $< 0, \ldots, 0 >$.*

Definition 1.2 (Order). *Type 0 is first-order. If the highest of the orders of types $\tau_1, \ldots, \tau_n$ is n, then the type $< \tau_1, \ldots, \tau_n >$ is $n + 1^{th}$-order.*

1.1.1. *Vocabulary.* Individual and relation constants: $a, b, \ldots, a_1, \ldots$; $A^n, B^n, \ldots, A^n_1, \ldots$; $\mathcal{A}^\tau, \mathcal{B}^\tau, \ldots, \mathcal{A}^\tau_1, \ldots$; variables: $x, \ldots, x_1, \ldots$; $X^n, \ldots$, $X^n_1, \ldots$; logical constants $=^{<0,0>}$, $\mathcal{P}^{<<0>>}$; operators: $\neg, \rightarrow, \forall, \lambda, \Box$ (other operators can be defined in a usual way.

1.1.2. *Formation rules.* Term and formula have to be defined together because of their definitional interdependency.

Definition 1.3 (Term and formula).

- *Individual and relation constants and variables are terms,*
- *if ϕ is a formula, t a term of type $< \tau_1, \ldots, \tau_n >$, and $t_1, \ldots, t_n$ terms of types $\tau_1, \ldots, \tau_n$ respectively, then $t(t_1, \ldots, t_n)$ is a formula. $\neg\phi$, $\phi \rightarrow \psi$, $\forall\alpha\phi$ and $\Box\phi$ are formulas if ϕ is a formula and α a variable,*
- *terms are also term abstracts of the form $(\lambda\alpha_1 \ldots \alpha_n.\phi)$, where α_i is a variable of some type τ_i and ϕ is a formula. In $(\lambda\alpha_1 \ldots \alpha_n.\phi)$ every subformula of ϕ contains at least one occurrence of at least one α_i, all of $\alpha_1, \ldots, \alpha_n$ occur in ϕ and in ϕ there is no $\forall$ immediately followed by any of α_i.*

The syntactical restriction of lambda abstracts may seem somewhat arbitrary, but it is meant to exclude terms built from closed formulas or from formulas containing closed formulas (see a note about λ-Intro rule in Section 2). Although the restriction also excludes abstracts like, e.g., $\lambda\alpha_1\alpha_2.\forall\alpha_1\phi(\alpha_1\alpha_2)$, this is easy manageable by renaming the variables.

1.2. Semantics. A model is a general (Henkin) model with world-dependent basic domains. Constants can denote, at a world w, objects that do not exist in w.

Definition 1.4 (Model). *A model M is an ordered 6-tuple $< W, R, H, Q, I, A >$ where*

[2] See e.g. [7], p. 71.

- *W is a non-empty set (of "worlds"),*
- $R \subseteq W \times W$,
- *H is a collection $\{D^\tau\}$ of domains where*
  - *D^0 is a non-empty set of individuals,*
  - *$D^\tau \subseteq \wp(D^{\tau_1} \times \ldots \times D^{\tau_n})^W$, where $\tau =< \tau_1, \ldots, \tau_n >$ and $\tau \neq 0$ (D^τ is non-empty),*
- *Q: $W \longrightarrow \wp D^0$ ($Q(w)$ is non-empty),*
- *I is an interpretation function such that for every constant κ^τ, $I(\kappa^\tau) \in D^\tau$; in particular*
  - *$I(=^{<0,0>})$ is a function whose value for each w is the set $\{< d_i^0, d_j^0 >| \ d_i^0 = d_j^0\}$,*
  - *$I(\mathcal{P})$ is a member of $D^{<<0>>}$ whose value for each w is a set of members $d^{<0>}$ of $D^{<0>}$ such that for each w, a value $d^{<0>}(w)$ builds an ultrafilter U over $Q(w)$ and the subset relation in U holds for each w' such that wRw'.*

Additionally*, we can also define that*

- *$I(\mathcal{E}\int\int)$ is a function $d^{<<0>,0>} \in D^{<<0>,0>}$ such that the value $d^{<<0>,0>}$ $(w) = \{< d^{<0>}, d^0 >| \ d^0 \in d^{<0>}(w)$ and for each $e^{<0>}$, if $d^0 \in e^{<0>}(w)$, then for every w', $d^{<0>}(w') \subseteq e^{<0>}(w')$,*
- *$I(N)$ is a function $d^{<0>} \in D^{<0>}$ such that $d^{<0>}(w) = \{d^0 \mid if < d^{<0>}, d^0 >\in I(\mathcal{E}\int\int, w)$, then for each w', $\exists d^0 \in Q(w')$ such that $d^0 \in d^{<0>}(w')\}$,*
- *$I(\mathcal{P})$ is defined as above and, in particular, if $d^{<0>} \in I(\mathcal{P}, w)$, then for each w' such that wRw', $d^{<0>} \in I(\mathcal{P}, w')$, and $I(N) \in I(\mathcal{P})$.*

Definition 1.5 (Variable assignment). *Variable assignment v is a function that assigns an object of type τ to each variable of type τ, i.e., $v(\alpha^\tau) \in D^\tau$.*

Definition 1.6 (Modal ultrafilter). *Modal ultrafilter in w is an ultrafilter U and the subset relation in U holds for each w' such that wRw'.*

Definition 1.7 (Variant of a variable assignment). *A variant $v[d/\alpha]$ is a variable assignment that differs from the variable assignment v at most on α.*

Like term and formula, designation, satisfaction and relative designation of λ-abstract can also be defined only interdependently.

Definition 1.8 (Designation and satisfaction). *Let $[\![t]\!]_v^M$ be a designation of a term t in a model M for a variable assignment v, and '$\models$' a satisfaction sign. Then*

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  - $[\![\kappa]\!]_v^M = I(\kappa)$,
  - $[\![\alpha]\!]_v^M = v(\alpha)$,
- *$M, w \models_v \phi$, iff*

- $M, w \models_v t(t_1 \dots t_n)$ *iff* $< [\![t_1]\!]^M_v, \dots, [\![t_n]\!]^M_v > \in [\![t_1]\!]^M_v(w)$,

  $\vdots$
- $M, w \models_v \forall\alpha^\tau\phi$ *iff for each* $d^0 \in Q(w)$ *and* $d^{\tau\neq 0} \in D^{\tau\neq 0}$, $M, w \models_{v[d^\tau/\alpha^\tau]} \phi$,
- $M, w \models_v \Box\phi$ *iff for each* w' *such that* wRw', $M, w \models_v \phi$.

- $[\![\lambda\alpha_1 \dots \alpha_n.\phi]\!]^M_v = A(v, \lambda\alpha_1 \dots \alpha_n.\phi)$, *where*
  - A *is a function that assigns a member of* D^τ $(\tau =< \tau_1, \dots, \tau_n >)$, *on a variable assignment* v, *to each* λ*-abstract* $\lambda\alpha_1^{\tau_1} \dots \alpha_n^{\tau_n}.\phi$,
  - $A((v, \lambda\alpha_1 \dots \alpha_n.\phi), w) = \{< v'(\alpha_1), \dots, v'(\alpha_n) > \mid M, w \models_{v'} \phi\}$, *where* $v' = v[d_1/\alpha_1, \dots, d_n/\alpha_n]$.

§2. Systems. In the systems we will consider (we call them Gödelian ontological systems) there will be the following rules and axioms:

- Intro and Elim rules for $\neg$, $\to$ (derived rules for $\wedge$, $\vee$ and $\leftrightarrow$),
- Intro and Elim rules for $\forall$; free logic rules for first-order quantification ($E\kappa$ means $\exists\alpha\ \kappa = \alpha$):
  - $\forall$ Elim: $\forall\alpha^0\phi \vdash E\kappa^0 \to \phi(\kappa^0/\alpha^0)$,
  - $\forall$ Intro: $\Gamma \vdash E\kappa^0 \to \phi(\kappa^0/\alpha^0) \Rightarrow \Gamma \vdash \forall\alpha^0\phi$ (κ^0 does not occur in $\Gamma, \forall\alpha^0\phi$).

  Otherwise, $\forall$ Intro and $\forall$ Elim are as usual (all instantiating terms are closed). In proofs we also use derived rules for $\exists$.
- $=$ Intro: $t = t$ (t is a closed term),
- $=$ Elim: $\{t_i = t_j, \phi(t_i)\} \vdash \phi(t_j//t_i)$
  (ϕ is atomic, t_i, t_j are closed terms),
- λ Intro: $\phi(t_1, \dots, t_n) \vdash$
  $(\lambda\alpha_1 \dots \alpha_n.\ \phi(\alpha_1/t_1, \dots, \alpha_n/t_n))(t_1, \dots, t_n)$
  (t_i is a closed term)
  , Restriction: ϕ in $(\lambda\alpha_1 \dots \alpha_n.\phi)$ does not contain closed subformulas.
- λ Elim: $(\lambda\alpha_1 \dots \alpha_n.\phi)(t_1, \dots, t_n) \vdash \phi(t_1/\alpha_1, \dots, t_n/\alpha_n)$
  (t_i is a closed term),
- modal rules for a modal propositional system S,
- Gödel's (or Gödelian) axioms concerning "positivity" and definitions (see sections 3 and 4).

The syntactical restriction of term abstracts in Definition 1.3 serves to restrict the λ-Intro rule not to enable the derivations of term abstracts with ϕ containing closed subformulas. This is one (easy) way to block the unrestricted necessitation of sentences of the system, since Sobel's proof (the only one known) of $\forall X\forall x(Xx \to \Box Xx)$ (in [21], p. 253, and [22], p. 42) is dependent on a vacuous λ introduction, or at least on a λ introduction that leaves a closed subformula within the scope of λ operator (in line 5). The unrestricted necessitation together with T gives the modal collapse of a system. For other ways of blocking the unrestricted necessitation cf., e.g., Anderson [3], Hájek [15] and Fitting [5].

We do not modify Gödel's axioms from 1970 (with Scott's simplification for Axiom 4.3) for the reasons mentioned at the end of Section 3 and at the beginning of Section 4. Some explanations are also added in Section 7.

§3. Magari's and similar systems. Very weak Gödelian ontological systems allow not only for contingencies, but also for a positiveness relativism, which implies the relativism of a God-like being. According to Gödel's interpretation of "positiveness" in a "moral aesthetic sense", positiveness relativism means relativism of moral-aesthetic values.

A very weak Gödelian ontological system is a system of §2 using K and three Gödel's axioms and one definition:

AXIOM 3.1. $\forall X \neg(\mathcal{P}X \leftrightarrow \mathcal{P}\neg X)$,

AXIOM 3.2. $\forall X \forall Y((\mathcal{P}X \wedge \Box\forall x(Xx \rightarrow Yx)) \rightarrow \mathcal{P}Y)$,

DEFINITION 3.1. $Gx \leftrightarrow_{def} \forall X(\mathcal{P}X \rightarrow Xx)$,

AXIOM 3.3. $\mathcal{P}G$.

This is a variant of a system proposed by Magari [19] who uses S5. Let us call the first system MO_K and Magari's original system M_{S5}. Let $X, Y, Z, \ldots = X^1, Y^1, Z^1, \ldots$.

In MO_K, $\Diamond\exists xGx$ is a theorem (provable in a similar way as in Gödel's original proof). Magari claimed that $\Box\exists xGx$ can also be (semantically) proven, but Hájek [15] showed that Magari's claim does not hold for M_{S5} unless a new axiom

AXIOM 3.4. $\forall X \forall Y((\mathcal{P}X \wedge X = Y) \rightarrow \mathcal{P}Y)$

is added. It is so also in MO_K. Let us call the system M_{S5}+Axiom 3.4 the system M^+_{S5}.

We can also fully collapse the modalities of MO_K (and abandon free logic), collapsing also the modality in the Axiom 3.2 in MO_K and thus obtaining

AXIOM 3.2': $\qquad \forall X \forall Y((\mathcal{P}X \wedge \forall x(Xx \rightarrow Yx)) \rightarrow \mathcal{P}Y)$.

In this system, let us call it simply O, we can prove the existence of God in a proof similar to Hájek's for M^+_{S5}:

THEOREM 3.1. $\exists xGx$

PROOF.

1	$\mathcal{P}A$	Assumption
2	$\neg\exists x Ax$	Assumption
3	$\forall x(Ax \to (Ax \vee Bx))$	Tautol., $\forall$ Intro
4	$\mathcal{P}\lambda x(Ax \vee Bx)$	3 Axiom 3.2', λ Intro
5	$(Aa \vee Ba) \to Ba$	red. ad abs. from Aa, 2
6	$\forall x((Ax \vee Bx) \to Bx)$	9 $\forall$ Intro
7	$\mathcal{P}\lambda x Bx$	4, Axiom 3.2', λ Intro
8	$\forall Y \mathcal{P}\lambda x Yx$	7 $\forall$ Intro
9	$\mathcal{P}\lambda x \neg Bx$	8 $\forall$ Elim
10	$\neg\mathcal{P}\lambda x Bx$	9, Axiom 3.1
11	$\exists x Ax$	2–10, $\neg$ Elim
12	$\mathcal{P}A \to \exists x Ax$	2–11 $\to$ Intro
13	$\forall X(\mathcal{P}X \to \exists x Xx)$	12 $\forall$ Intro
14	$\mathcal{P}G \to \exists x Gx$	13 $\forall$ Elim
15	$\exists x Gx$	14, Axiom 3.3 $\to$ Elim

$\dashv$

Adding to O the rules for K, we can prove $\Box\exists x Gx$, and adding to O also the rules for D or T, we can prove $\Diamond\exists x Gx$ (as can be easily seen).

Since in ontological modal logics such as MO_K, MO_{S5}, MO^+_{S5} or O, O_K etc., Gödel's axiom $\forall X(\mathcal{P}X \to \Box\mathcal{P}X)$ is not included, some properties could be positive in some worlds and negative in other worlds, and thus (according to the definition of G) a God-like being, if there is any, can have different properties in different worlds. Also there could be more than one God-like being, although not in one and the same world (cf. Proposition 9.3, provable by Axiom 3.1 and Definition 3.1).

Here is an example for a positiveness relativism.

Example 3.1 (Relativism). *We define a model M where:*

- $W = \{w_1, w_2\}$,
- *R is symmetric,*
- $D^0 = \{a, b, c\}$, $D^{<0>} \subseteq \wp(D^0)^W$, $D^{<<0>>} \subseteq \wp(D^{<0>})^W$, *etc.,*
- $Q(w_1) = \{a, b, c\}$, $Q(w_1) = \{a, b\}$,
- $I(A, w_1) = \{a, b\}$, $I(A, w_2) = \varnothing$, $I(B, w_1) = \{b, c\}$, $I(B, w_2) = \{b\}$, $I(C, w_1) = I(C, w_2) = \{a\}$, *etc.* $I(\mathcal{P}, w_1) = \{I(A), I(B), I(\neg C)\}$, $I(\mathcal{P}, w_2) = \{I(\neg A), I(\neg B), I(C)\}$, *etc.*

In our example, b should be God-like in w_1, and a should be God-like in w_1, since only these objects can possess all positive properties in their respective worlds.

Hájek's analysis, especially in [16], contains the following interesting results. There are subsystems (analogous to the systems considered above) that are relativistic with respect to positiveness, but prove $\Box\exists xHx$ (H for a redefined G). This is a system AO'_0 obtained by an emendation of Anderson's modification (in [3]) of Gödel's system (Anderson allows for indifferent, neither positive nor negative, properties). Hájek's further emendation AOE'_0 (for variable domains) proves $\Box\exists xHx$ and also a theorem corresponding to $\forall X(\mathcal{P}X \to \Box\mathcal{P}X)$, but for a redefined positiveness $P^\sharp$. Interestingly enough, Anderson's subsystem of the first three axioms, as it is presented by Hájek [16, 15], is non-relativistic since it allows for the deduction of $\forall X(\mathcal{P}X \to \Box\mathcal{P}X)$ (and proves $\Box\exists xHx$ too).

We keep here Gödel's axioms 4.1 and 4.2 unmodified because we want to present a God-like being as an "absolute being" in a maximal possible sense (and to see how much contingency we can preserve at the same time). To that end, we take a course somewhat similar to that of Hájek in [15], where he introduces a "cautious comprehension scheme". Thus, instead of allowing indifferent properties, we do not count some "unnatural properties" as properties at all (by a restriction on what to count as λ abstracts). We present a God-like being as a "perfect" being that is "purely" good and "omnipotent" in a (maximal) sense that it possesses only positive, and even no indifferent, properties, and implies only positive consequences. Hence, we retain Axiom 4.1 since it excludes properties that are indifferent with respect to positiveness and since it also makes it possible to prove Proposition 9.2. We retain also Axiom 4.2, according to which God would have no other but positive consequences.

§4. Gödelian ontological KB systems. If we wish to avoid value relativism and conceive God as an absolutely perfect being, we can add the rest of Gödel's axioms and definitions. To ensure the existence of a God-like being we introduce a KB thus obtaining O_{KB}, (on using a B system cf. Adams [2] p. 391 note e, [1] pp. 40-46 and Sobel [22] pp. 38–39):

Axiom 4.1. *= 3.1:* $\forall X\neg(\mathcal{P}X \leftrightarrow \mathcal{P}\neg X)$,

Axiom 4.2. *= 3.2:* $\forall X\forall Y((\mathcal{P}X \wedge \Box\forall x(Xx \to Yx)) \to \mathcal{P}Y)$,

Definition 4.1 (God). *= 3.1:* $Gx \leftrightarrow_{def} \forall X(\mathcal{P}X \to Xx)$,

Axiom 4.3. *= 3.3:* $\mathcal{P}G$,

Axiom 4.4. $\forall X(\mathcal{P}X \to \Box\mathcal{P}X)$,

Definition 4.2 (Essence).
$\mathcal{E}\!\int\!\!\int(X,x) \leftrightarrow_{def} (Xx \wedge \forall Y(Yx \to \Box\forall y(Xy \to Yy)))$,

Definition 4.3 (Necessary existence).
$Nx \leftrightarrow_{def} \forall Y(\mathcal{E}\!\int\!\!\int(Y,x) \to \Box\exists xYx)$,

Axiom 4.5. $\mathcal{P}N$.

Here are some propositions provable corresponding to Gödel's original ontological proof (but taking into account KB rules and free logic rules for first-order quantification). Their proofs are mostly contained in Appendix (Section 9).

PROPOSITION 4.1. $\mathcal{P}(\lambda x.x = x)$ = *Proposition 9.1.*

THEOREM 4.1. $\forall X(\mathcal{P}X \to \Diamond\exists x Xx)$= *Theorem 9.1.*

COROLLARY 4.1. $\Diamond\exists x Gx$ = *Corollary 9.1.*

It is easily seen that that corollary makes O_{KB} a D-system.

PROPOSITION 4.2. $\forall x(Gx \to \forall X(Xx \to \mathcal{P}X))$ = *Proposition 9.2.*

REMARK 4.1. $\forall X(\neg\mathcal{P}X \to \Box\neg\mathcal{P}X)$ = *Remark 9.1.*

THEOREM 4.2. $\forall x(Gx \to \mathcal{E}\!\int\!\!\int(G, x))$ = *Theorem 9.2.*

THEOREM 4.3. $\exists x Gx \to \Box\exists x Gx$ = *Theorem 9.3.*

The main theorems are $\Box\exists y Gy$ and $\exists y Gy$:

THEOREM 4.4. $\exists x Gx$

PROOF.

1	$\Diamond\exists x Gx$	Cor. 9.1
2	$\neg\exists x Gx$	Assumption
3	$\Box$ $\exists x Gx$	(1) Assumption
4	$\Box\exists x Gx$	3 Th.9.3 $\to$E
5	$\neg\Box\exists x Gx$	2 B Reit
7	$\bot$	1,3-5 $\Diamond$ Elim
8	$\exists x Gx$	2-8 $\neg$ Elim

⊣

THEOREM 4.5. $\Box\exists x Gx$

PROOF. Follows from 9.3 and 4.4. ⊣

Relativism is excluded by the theorems about the identity of a God-like being across the worlds and within every world. For the "within" part, cf. Proposition 9.3 of Appendix.

THEOREM 4.6 (Cross-world identity of God). $\forall x(Gx \to \Box Gx)$

PROOF.

1	Ea	Assumption

2	Ga	Assumption
3	$a = a$	$=$ Intro
4	$(\lambda x.x = a)a$	3 λ Intro
5	$Ga \to ((\lambda x.x = a)a \to \mathcal{P}(\lambda x.x = a))$	2, Prop. 9.2, $\forall$ Elim
6	$\mathcal{P}(\lambda x.x = a)$	2,4,5 $\to$ Elim
7	$\Box\mathcal{P}(\lambda x.x = a)$	5 Axiom 4.4
8	$\Box$ $\exists xGx$	Th. 4.4
9	$Gb \wedge Eb$	Assumption
10	$\mathcal{P}(\lambda x.x = a)$	7 K Reit
11	$\mathcal{P}(\lambda x.x = a) \to (\lambda x.x = a)b$	2,9, D.4.1,$\forall$E, $\to$E
12	$(\lambda x.x = a)b$	10,11 $\to$ Elim
13	$b = a$	121 λ Elim
14	Ga	9,13 $=$ Elim
15	Ga	8, 9–14 $\exists$ Elim
16	$\Box Ga$	8–15 $\Box$ Intro
17	$Ga \to \Box Ga$	2–16 $\to$ Intro
18	$\forall x(Gx \to \Box Gx)$	1–17 $\forall$ Intro

$\dashv$

To prove the cross-world identity of a God-like being may, for Gödel himself, appear pointless since he did not use the first-order quantification into a modal context.

Let us also mention that "existence" is a positive property since $\lambda x.\exists yy = x)$ is O_{KB}-deducible from the positivity of $\lambda x.x = x$, which is positive (cf. Proposition 9.1).

The system O_{KB} contains, as its part, a modal collapse of positivities. First, we can prove, in O_{KB}, a quantificational theorem T for positivities:

Proposition 4.3 (Positivity T). $\forall X(\Box\mathcal{P}X \to \mathcal{P}X)$

Proof.

1	$\Box\mathcal{P}A$	Assumption
2	$\neg\mathcal{P}A$	Assumption
3	$\Box\neg\mathcal{P}A$	2 Remark 9.1
4	$\Diamond\exists xGx$	Corr 9.1
5	$\Box$ $\exists xGx$	Assumption
6	$\neg\mathcal{P}A$	3 K Reit

7	$\mathcal{P}A$	1 K Reit
8	$\bot$	4,5–7 $\Diamond$ E
9	PA	2–8 $\neg$ E
10	$\Box\mathcal{P}A \to \mathcal{P}A$	1–9 $\to$ Intro
11	$\forall X(\Box\mathcal{P}X \to \mathcal{P}X)$	10 $\forall$ Intro

$\dashv$

Now the modal collapse for positive properties follows:

PROPOSITION 4.4 (Positivity modal collapse).
$\vdash_{O_{KB}} \forall X(\mathcal{P}X \leftrightarrow \Box\mathcal{P}X)$

PROOF. Axiom 4.4 proves the left to right direction. Proposition 4.3 proves the right to left direction. $\dashv$

O_{KB}, like any Gödelian ontological logic without general modal collapse but with Gödel's axioms and definitions unchanged (only Axiom 4.5 is not needed), contains, besides the logic of contingencies, a logic of invariant positivities. Philosophically, we recognize therein two platonic "worlds": a "world" of unchangeable (eternal) values, and a "world" of contingent appearances. One point of the weakening of ontological systems is precisely to see how much contingency ontology could contain so that it, at the same time, still remains founded on invariant values and on an absolute being.

Without significant changes in the proof, we can also prove the existence and the necessary existence of a God-like *pair*, *triple*, etc., applying, correspondingly, the concept of "positivity" to (n-place) relations. That could be in accord with the theological and philosophical considerations about the plurality of one God (cf., e.g., the Biblical plural *elohim* for God, the doctrine of the triune God, etc.).

§5. Soundness of O_{KB} (and MO_{K}.)

THEOREM 5.1. *O_{KB} is sound with respect to the class C of general symmetrical models with varying domains and with additional clauses for I of Definition 1.4 in Subsection 1.2. I.e., if $\Gamma \vdash_{O_{KB}} \phi$, then $\Gamma \models_C \phi$.*

PROOF. The proof can be given by a mathematical induction with derivation rules as cases. Let us pause only on Gödel's axioms that are sound due to the ultrafilter-like concept of "positivity".

- $\forall X\neg(\mathcal{P}X \leftrightarrow \mathcal{P}\neg X)$. According to the definition of modal ultrafilter, for any object d^τ, $d^\tau \in U$ iff $D^\tau \backslash d \notin U$.
- $\forall X\forall Y((\mathcal{P}X \wedge \Box\forall x(Xx \to Yx)) \to \mathcal{P}Y)$. Let for some terms t_1 and t_2, $M, w \models \mathcal{P}t_1$ and $M, w \models \Box\forall x(t_1(x) \to t_2(x))$. Hence, for any w' such that wRw', $M, w' \models \forall x(t_1(x) \to t_2(x))$. Thus, for w', $[\![t_1]\!]^M_v(w') \subseteq [\![t_2]\!]^M_v(w')$. Since $[\![t_1]\!]^M_v(w) \in [\![\mathcal{P}]\!]^M_v(w)$ and $[\![\mathcal{P}]\!]^M_v(w)$

is an ultrafilter, it follows that also $[\![t_2]\!]_v^M(w) \in [\![\mathcal{P}]\!]_v^M(w)$, and so $M, w \models \mathcal{P}t_2$.

- $\mathcal{P}G$. According to the definition of ultrafilter, ultrafilter is closed under intersection. Since G is closed under conjunction of positive properties, G is positive.

That suffices to prove the soundness of MO_K with respect to the class $C' = C$ without the additional clauses. To prove the soundness of O_{KB} with respect to C, we add two more clauses:

- $\forall X(\mathcal{P}X \to \Box\mathcal{P}X)$. According to the definition of modal ultrafilter, ultrafilter is invariant under the change of worlds in a model.
- $\mathcal{P}N$. According to the definition 1.4 of the model (additional clause for I).

$\dashv$

§6. Completeness of O_{KB} and MO_K. The completeness proof of O_{KB} with respect to the class C of general symmetrical models with varying domains is similar to a completeness proof in Gallin ([7], pp. 74–75, 25–30) and is accommodated for varying domains (cf. also C. Menzel's completeness proof for first-order modal logic in [20], pp. 364–370) and for a KB system. The proof uses a canonical model to prove satisfiability of consistent sets. We give only the characteristic details and modifications. For simplicity, we use the defined quantifier $\exists$.

Important are the concepts of the saturated set, relative consistency and relative j-consistency.

Definition 6.1 (Saturated set). *A set w_i of sentences is saturated iff*
1. w_i is O_{KB}-consistent,
2. for every sentence ϕ, $\phi \in w_i$ or $\neg\phi \in w_i$ (maximality),
3. for every formula $\phi(\alpha^\tau)$ where α^τ is the only free variable, if $\exists\alpha^\tau\phi \in w_i$ then for some constant κ^τ, $\{\phi(\kappa^\tau/\alpha^\tau), E\kappa^\tau\} \subseteq w_i$ if $\tau = 0$, and $\phi(\kappa^\tau/\alpha^\tau) \in w_i$ if $\tau \neq 0$ (ω-completeness),
4. for every sentence ϕ, if $\Diamond\phi \in w_i$ then for some j, $\phi \in w_j$.

We modify Gallin's definition of relative j-consistency introducing the set $\Gamma_j \cup \Box\Diamond\Gamma_j$ for the definition to be appropriate for KB systems.

Definition 6.2 (Relative O_{KB}-consistency). *A sequence $W =< w_0, w_1, \dots >$ is relatively O_{KB}-consistent iff for every j and for every finite set $\Gamma_j \subseteq w_j$, the set $\Gamma_j \cup \Box\Diamond\Gamma_j$ is O_{KB}-consistent.*

Definition 6.3 (Relative j-O_{KB}-consistency). *A sentence ϕ is relatively j-O_{KB}-consistent with a sequence $W =< w_0, w_1, \dots >$ iff W', where $w'_j = w_j \cup \{\phi\}$ and for all $i \neq j$, $w'_i = w_i$, is relatively O_{KB}-consistent.*

Next, we construct, following Gallin, a sequence $W =< w_0, w_1, \dots >$ of saturated sets. For the construction we need, for each τ, infinitely many new constants κ^τ not occurring in w. We start from some O_{KB}-consistent set w. j is a subscript for some set w_j of formulas in some sequence of sets of formulas. k is a superscript for a sequence W^k of sets of formulas. Every member of a sequence W^k can additionally be denoted by a superscript k (w_j^k). $w = w_0^0$ and every other set w_j^0 in the sequence W^0 is empty. Otherwise, W^{k+1} is the sequence obtained form W^k by adding the formula ϕ^k (the formula of the k-th ordered pair $\{\phi^k, j\}_k$, $k > 0$), to a set w_j^k of the sequence W^k. And ϕ^k is added to w_j^k iff ϕ^k is relatively $j - O_{KB}$-consistent with W^k.

In particular, we slightly modify Gallin's construction in the case where $\phi^k = \exists\alpha^\tau\psi$. Then for $\tau = 0$, $w_j^{k+1} = w_j^k \cup \{\phi^k, \psi(\kappa^0/\alpha^0), E\kappa^0\}$, and for $\tau \neq 0$, $w_j^{k+1} = w_j^k \cup \{\phi^k, \psi(\kappa^\tau/\alpha^\tau)\}$, and in either case κ^τ is the first constant of the type τ new to W^k and to ψ. For every W^k there are infinitely many empty w_js. Now, W is the sequence $< w_0, w_1, \ldots >$ such that for each i, $w_i = \bigcup_k w_i^k$.

In the course of the proof, the proposition that every sequence W^k is relatively O_{KB}-consistent is proved by mathematical induction.

W^0 is relatively O_{KB} consistent since every $\Gamma \subseteq w_0^0$ is O_{KB}-consistent and, in KB, $\Gamma \vdash_{O_{KB}} \Box\Diamond\Gamma$.

In the proof of the inductive step, we pause only on the case where $\phi^k = \exists\alpha^\tau\psi$ is added to some w_j of the sequence W^k, producing thus the sequence W^{k+1}, on the assumption that $\exists\alpha^\tau\psi$ is relatively $j - O_{KB}$-consistent with W^k. We assume the inductive hypothesis according to which W^k is relatively O_{KB} consistent. We prove that W^{k+1} is also relatively O_{KB} consistent.

It follows that W^{k+1} is like W^k except that for $\tau = 0$,

$$w_j^{k+1} = w_j^k \cup \{\exists\alpha^0\psi, \psi(\kappa^0/\alpha^0), E\kappa^0\}$$

and for $\tau \neq 0$,

$$w_j^{k+1} = w_j^k \cup \{\exists\alpha^\tau\psi, \psi(\kappa^\tau/\alpha^\tau)\}.$$

In either case, κ^τ is the first new constant.

Suppose now that W^{k+1} is not relatively O_{KB}-consistent. Then, for some n, a finite set $\Gamma_n^{k+1} \subseteq w_n^{k+1}$ is such that $\Gamma_n^{k+1} \cup \Box\Diamond\Gamma_n^{k+1}$ is O_{KB}-inconsistent.

We follow the proof for the *first-order quantification* case. For all n such that $n \neq j$ or if $n = j$ then $\exists\alpha^0\psi, \psi(\kappa^0/\alpha^0), E\kappa^0 \notin \Gamma_j^{k+1}$, it holds that $\Gamma_n^{k+1} = \Gamma_n^k$; hence, for all n such that $n \neq j$ or if $n = j$ then $\exists\alpha^0\psi, \psi(\kappa^0/\alpha^0), E\kappa^0 \notin \Gamma_j^{k+1}$, it holds that $\Gamma_n^{k+1} \cup \Box\Diamond\Gamma_n^{k+1}$ is O_{KB}-consistent.

It follows that some set

$$\Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), \Box\Diamond\psi(\kappa/\alpha), E\kappa, \Box\Diamond E\kappa\},$$

where $\Gamma_j^k = \Gamma_j^{k+1}$, is O_{KB}-inconsistent,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), E\kappa, \Box\Diamond E\kappa\} \vdash_{O_{KB}} \neg\Box\Diamond\psi(\kappa/v)$,

also, $\Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), E\kappa, \Box\Diamond E\kappa\} \vdash_{O_{KB}} \Box\Diamond\psi(\kappa/\alpha)$,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), E\kappa, \Box\Diamond E\kappa\}$ is O_{KB}-inconsistent,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), E\kappa\} \vdash_{O_{KB}} \neg\Box\Diamond E\kappa$,

also $\Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), E\kappa\} \vdash_{O_{KB}} \Box\Diamond E\kappa$,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi, \psi(\kappa/\alpha), E\kappa\}$ is O_{KB}-inconsistent,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi\} \vdash_{O_{KB}} \neg(\psi(\kappa/\alpha) \wedge E\kappa)$,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi\} \vdash_{O_{KB}} E\kappa \rightarrow \neg\psi(\kappa/\alpha)$,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi\} \vdash_{O_{KB}} \forall\alpha\neg\psi$ (since κ is new),

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi\} \vdash_{O_{KB}} \neg\exists\alpha\psi$,

$\Rightarrow \Gamma_j^k \cup \Box\Diamond\Gamma_j^k \cup \{\exists\alpha\psi, \Box\Diamond\exists\alpha\psi\}$ is O_{KB}-inconsistent – contrary to the assumption.

For the *second-order quantification* case, $E\kappa^\tau$ and $\Box\Diamond E\kappa^\tau$ are not added.

In the rest of the proof there are no notable changes except that the canonical model is, according to previously described semantics, a 6-tuple $< W, R, H, Q, I, A >$ (I with additional clauses of Definition 1.4). It is then proved that W is relatively O_{KB}-consistent, that every w_j in W is saturated and that for the canonical model M^C, $M^C, w \models \phi$ iff $\phi \in w$. Suppose, e.g., that ϕ is $\forall\alpha^\tau\psi$. Let $\forall\alpha^\tau\psi \in w$. It follows that for any κ^τ such that $E\kappa^\tau \in w$, $\psi(\kappa^\tau/\alpha^\tau) \in w$ ($\forall$ Elim, maximal O_{KB}-consistency of w). Thus, if $M, w \models E\kappa^\tau$ then $M, w \models \psi(\kappa^\tau/\alpha^\tau)$ (inductive hypothesis), and hence, for any equivalence set $[\kappa^\tau] \in Q(w)$ and for any variable assignment v, $M, w \models_{v[[\kappa^\tau]/\alpha^\tau]} \phi$. Therefore, $M, w \models \forall\alpha^\tau\phi$. Let $\forall\alpha^\tau\psi \notin w$. It follows that $\neg\forall\alpha^\tau\psi \in w$, and then, that $\exists\alpha^\tau\neg\psi \in w$. Hence, for some κ^τ, $\neg\psi(\kappa^\tau/\alpha^\tau), E\kappa^\tau \in w$ (ω-completeness). Thus, $\psi(\kappa^\tau/\alpha^\tau) \notin w$ (O_{KB}-consistency of w). Accordingly, $M, w \vDash E\kappa^\tau$ and $M, w \nvDash \psi(\kappa^\tau/\alpha^\tau)$ (ind. hyp.). Therefore, $M, w \nvDash \forall\alpha^\tau\phi$.

Now the proposition follows that, if a set of sentences is O_{KB}-consistent in O_{KB}, then it is satisfiable in the class C of symmetrical Gödelian ontological models with varying domains. Hereby the completeness theorem for O_{KB} is proved:

THEOREM 6.1. *O_{KB} is complete with respect to the class C of general symmetrical models with varying domains and with additional clauses for I of Definition 1.4 in Subsection 1.2. I.e. if $\Gamma \models_C \phi$, then $\Gamma \vdash_{O_{KB}} \phi$.*

In the completeness proof for MO_K, with respect to the class C' of models (without additional clauses for I), the concept of relative consistency is simple:

Definition 6.4 (Relative MO_K-consistency). *A sequence* $W = < w_0, w_1, \ldots >$ *is relatively* MO_K*-consistent iff for every* j *every finite set* $\Gamma_j \subseteq w_j$ *is* MO_K*-consistent.*

The canonical model for MO_K does not include additional clauses for I of Definition 1.4.

§7. The meaning of "positiveness". The meaning of "positiveness" can be conceived in several ways. First, as already mentioned and as proposed by Gödel ([9], 404), "positiveness" can be conceived in a moral-aesthetic sense and positive properties as moral-aesthetic values. The positivity sub-logic (with S5 and the modal collapse) is a logic of unchangeable (absolute) moral-aesthetic values. That interpretation, as we have seen, does not require all the axioms of the system O_{KB}, e.g., Axiom 4.4 could be abandoned and a moral relativism introduced.

Secondly, the meaning of "positiveness" can also be determined in a logical-ontological ("attributive") way (as is also proposed by Gödel at the end of his notes from 1970, [9], p. 404). "Positive" now means "pure 'attribution' ", which is determined syntactically as a representation by a disjunctive normal form "in terms of elementary properties" having as its member a conjunction not containing negation ([9], p. 404, note 4). There is no possibility of relativism in that approach since that interpretation requires all the axioms of the system O_{KB}, including Axiom 4.4.

If we analyze Gödel's ontological proof on the background of his philosophical reflections (especially in [24], pp. 105–108, 308–318, in [10] and [11], pp. 429–437), it appears that we could describe what Gödel meant by " positiveness" by means of what he calls "the meaning of the world". Gödel conceives "the meaning of the world" as the "separation of wish and fact", or of "force" and fact, and as the overcoming of that separation in the "union" of force and fact (i.e., in a "fulfilled wish"). Now, in the moral aesthetic interpretation, positiveness would be a moral aesthetic wish, and an exemplified positive property would be a fulfilled (moral aesthetic) wish. In the logical-ontological interpretation, positiveness would be the "force" of being ("affirmation of being", [11], p. 433) and exemplified positive property would be a realization of that force. Thus, we can interpret Gödel's axioms as describing the separation of force (wish) and fact, and as describing the force of positivity to overcome the separation. Such an interpretation would be, also methodologically, in accordance with Gödel's phenomenological viewpoint in philosophy, based on the "perceiving" of concepts.[3]

[3] About Gödel's phenomenology in his philosophical views, cf., e.g., H. Wang ([24], pp. 80–81, 164–172, 287–322), D. Føllesdal [6] and R. Tieszen [23].

Accordingly, Axiom 4.1 states the separation of the positive and the negative, i.e., of force (wish) and fact, separating what is morally aesthetically good, from what is not, or separating what is affirmative for being from what is non-affirmative. Axioms 4.2, 4.3 and 4.4 express the force of positivity: positive properties "cause" the positivity of their consequences and intersections, and preserve their positivity in all accessible worlds. The *possibility* of particular positive properties (Theorem 9.1) and of positiveness as a whole (i.e., of a God-like being)(Corollary 9.1) is a "weak" union of force and fact.[4] The force for the final overcoming of the separation lies in the positiveness of "necessary existence" (Axiom 4.5), which "causes" the possibility of a God-like being becoming a necessity. In the final fact that God necessarily exists, we have the final and maximal "union" of "wish" and "fact" (cf. Gödel's "maximum principle for fulfilling of wishes" in [24], p. 312).

It is interesting that in Gödel's ontological system from 1970 the disjunctive property of "being God-like or devil-like", say $\lambda x.(Gx \vee Dx)$, is positive (as remarked by Hájek, [16]), since this property is a consequence of a property of "being God-like" that is itself positive. I think this somewhat astonishing fact of Gödel's system can be justified in the system itself, since, in that system, "positivity" is logically (and also ontologically) stronger than "negativity" in the following sense. "Positivity", not "negativity", is necessarily exemplified (since there can be a world where only a God-like being exists). Further, if we define "devil-like" by $Dx \leftrightarrow_{def} \forall X(\neg\mathcal{P}X \rightarrow Xx)$, then a "devil-like" being cannot even exist if it does not participate in a positive property (of existence and of self-identity), whereas a God-like being (in Gödel's system) in no way possesses or needs to possess any non-positive property. Also, positivity has the "force" of closure over its consequences (axiom 4.2), while "negativity" has no comparable force and can even have positive consequences, since both $\mathcal{P}Y \rightarrow \mathcal{P}(\neg X \vee Y)$ and $\Box\forall X\forall Y(\neg Xx \rightarrow (\neg Xx \vee Yx))$ (where $\mathcal{P}X$ could also hold) are theorems of Gödel's ontological system. Correspondingly, for Gödel, in a formula in the disjunctive normal form, at least one member containing no negation suffices for the formula to be positive ([11], p. 404 n. 4). Therefore, there is in Gödel's system an unbalance between positiveness and negativeness in favor of "positiveness", and thus also between "God-like" and "devil-like" in favor of "God-like". That unbalance makes any disjunction of a positive and a negative property positive. Hájek, in [16], takes a different course and presents a system where, instead of Gödel's axioms 4.1 and 4.2, a single new axiom is introduced that is also in accordance with some of Gödel's remarks about some other variant of the ontological proof.

[4] According to Wang, Gödel says that "possibility is the synthesis of being and nonbeing" and that it is a "weakened form of being" ([24], p. 313).

A kind of a temporal interpretation can also be proposed. Interestingly enough, if modality is interpreted as accessibility in time, the system O_{KB} includes the possibility of time travel (because of B), but does not allow for arbitrary travel through time (e.g., there is no general transitivity in time). Gödel considers the question of time travel as physically undecidable, but philosophical reasons speak, according to Gödel, for the possibility of time travel (cf. [8], pp. 206–207). Let us mention that S4.2, the system for relativistic time proposed by Goldblatt ([14], 45-46), excludes time travel, but, seemingly, does not allow for the ontological proof. In O_{KB}, for the realm of positivities (e.g., of positive values), there is, indeed, no time because of the modal collapse in that part of the system – there are only "eternal" positive values. A full modal collapse, like that intended in Gödel's system (cf. [11], p. 435), completely agrees with Gödel's cosmology where there is no objective "lapse of time" and time is only "subjective" [8, 13].

§8. Gödel's improvement of Kant's moral theology? What Gödel himself might have in mind with his ontological proof is perhaps also to correct what he thinks is an inconsistency in Kant's philosophy:

> His [i.e., Kant's] epistemology proves that God, and so on, have no objective meaning; they are purely subjective, and to interpret them as objective is wrong. Yet he says that we are obliged to assume them because they induce us to do our duty to our fellow human beings. It is, however, also one's duty not to assume things that are purely subjective

(quoted by Wang in [24], pp. 171-172).

According to Gödel, Kant postulates (on moral grounds) God's existence, which he previously, in his "epistemology", did not accept (because of the fallacy Kant saw in the ontological argument). We can leave aside whether this objection is just towards Kant. We only mention that the system with KD45 analyzed by Hájek ([15], pp. 133-134) is, it seems to me, a very near formalization of the type of moral theology proposed, for instance, by Kant, and that it shows us that such a moral theology need not be inconsistent. However, what Gödel wanted to show is, in distinction to Kant, the conformity of epistemological with moral perspectives in the sense that the moral ground for God's existence should also have a theoretical (epistemological and ontological) justification. It is here important to note that Gödel was not satisfied with Kant's dualism of sensual perception and conceptual thinking (cf. [24], pp. 164–165), and that he proposed a kind of conceptual realism (probably best exposed in his *Gibbs Lecture*, [12]).

There is, according to Kant, no inference to conclude that God exists (*Critique of practical reason*, [17], p. 250), there is no theoretical knowledge of God's existence. Kant could have wanted only to show that on the ground of the moral law "I *will* that God ... exists" (*Crit. of pract. r.*, [17], p. 258). Kant requested God's existence (freedom and immortality) as the only way to make moral action meaningful at all - since only God can be the cause of the "highest good", i.e., of the unity of morality and happiness.

According to Kant, God indeed has "objective reality", but only with respect to the moral law and to the "highest good" – as the object of our will. Kant did accept, in his moral philosophy (which is for him true metaphysics), the objectivity of moral concepts, spoke of "intelligible world", and even of a "practical knowledge", but distinguished them strongly from theoretical, i.e., empirical knowledge.

> Morality does not, indeed, require that freedom should be understood, but only that it should not contradict itself ...

(B XXIX in [18]).

To remove the seeming inconsistency, Gödel wanted to secure God's objective reality, proposing (mainly in his unpublished texts) a universal moral-aesthetic ontology and a phenomenological *interpretation and understanding* of the moral-aesthetic phenomenon. That phenomenon primarily consists, as we have already mentioned, in the separation of "fact and wish". The moral- aesthetic "world" is for Gödel not reduced merely to the realm of moral will and decision, as it is for Kant. Nevertheless, Kant, with his transcendental theoretical philosophy, is for Gödel [10] the founder of the phenomenological approach (further developed by Husserl), which Gödel now wants to apply to his moral aesthetic ontology.

§9. Appendix.

In O_{KB} we can prove $\exists x Gx$, $\Box\exists x Gx$:

PROPOSITION 9.1. $\mathcal{P}(\lambda x.x = x)$

PROOF. Cf. Sobel [21], p. 242.

1	$\mathcal{P}(\lambda x.\neg x = x)$	Assumption
2	$\Box$ Ea	Assumption
3	$\neg a = a \to a = a$	Taut
4	$\forall x(\neg x = x \to x = x)$	2–4 $\forall$ Intro
5	$\Box\forall x(\neg x = x \to x = x)$	3-4 $\Box$ Intro
6	$(\mathcal{P}(\lambda x.\neg x = x)\wedge$ $\Box\forall x(\neg x = x \to x = x)) \to \mathcal{P}(\lambda x.x = x)$	Axiom 4.2
7	$\mathcal{P}(\lambda x.x = x)$	5,6 $\to$ Elim

8	$\quad \neg\mathcal{P}(\lambda x.x = x)$	1 Axiom 4.1
9	$\neg\mathcal{P}(\lambda x.\neg x = x)$	1-8 $\neg$ Intro
10	$\mathcal{P}(\lambda x.x = x)$	9 Axiom 4.1

⊣

THEOREM 9.1. $\forall X(\mathcal{P}X \to \Diamond\exists x Xx)$

PROOF. Cf. Scott's proof of his Theorem 1 in [21], p. 257.

1	$\quad \mathcal{P}A$	Assumption
2	$\quad\quad \neg\Diamond\exists x Ax$	Assumption
3	$\quad\quad \Box \quad \neg\exists x Ax$	2 K Reit
4	$\quad\quad\quad\quad Ea$	Assumption
5	$\quad\quad\quad\quad \neg Aa$	3,4 $\neg\exists$ Elim
6	$\quad\quad\quad\quad Aa \to \neg a = a$	ex falso quodlibet
7	$\quad\quad\quad \forall x(Ax \to \neg x = x)$	4–6 $\forall$ Intro
8	$\quad\quad \Box\forall x(Ax \to \neg x = x)$	3-7 $\Box$ Intro
9	$\quad\quad \mathcal{P}A \wedge \Box\forall x(Ax \to \neg x = x)$	1,8 $\wedge$ Intro
10	$\quad\quad (\mathcal{P}A \wedge \Box\forall x(Ax \to \neg x = x)) \to \mathcal{P}(\lambda x.\neg x = x)$	Axiom 4.2
11	$\quad\quad \mathcal{P}(\lambda x.\neg x = x)$	10,9 $\to$ Elim
12	$\quad\quad \neg\mathcal{P}(\lambda x.x = x)$	11 Axiom 4.1
13	$\quad\quad \mathcal{P}(\lambda x.x = x)$	Prop. 9.1
14	$\quad \Diamond\exists x Ax$	2-13 $\neg$ Elim
15	$\mathcal{P}A \to \Diamond\exists x Ax$	1-14 $\to$ Intro
16	$\forall X(\mathcal{P}X \to \Diamond\exists x Xx)$	15 $\forall$ Intro

⊣

COROLLARY 9.1. $\Diamond\exists x Gx$

PROOF. It follows from Axiom 4.3 and Theorem 9.1. ⊣

PROPOSITION 9.2. $\forall x(Gx \to \forall X(Xx \to \mathcal{P}X))$

PROOF. Cf. Sobel's proof (without the existential assumption) of his Theorem 4 in [22], p. 39.

1	$\quad Ea$	Assumption
2	$\quad\quad Ga$	Assumption
3	$\quad\quad\quad Ha$	Assumption

4	$\neg\mathcal{P}H$	Assumption
5	$\mathcal{P}(\neg H) \leftrightarrow \neg\mathcal{P}H$	Axiom 4.1
6	$\mathcal{P}(\neg H)$	4,5 $\leftrightarrow$ Elim
7	$\forall X(\mathcal{P}X \to Xa)$	1,2 Def. 4.1, $\forall$ Elim
8	$\neg\mathcal{P}H \to \neg Ha$	7, $\forall$ Elim
9	$\neg Ha$	6,8 $\to$ Elim
10	Ha	3 Reit
11	$\mathcal{P}H$	4–10 $\neg$ Elim
12	$Ha \to \mathcal{P}H$	3–11 $\to$ Intro
13	$\forall X(Xa \to \mathcal{P}X)$	12 $\forall$ Intro
14	$Ga \to \forall X(Xa \to \mathcal{P}X)$	2–13 $\to$ Intro
15	$\forall x(Gx \to \forall X(Xx \to \mathcal{P}X))$	1–14 $\forall$ Intro

$\dashv$

REMARK 9.1. *From Axiom 4.4 it follows* $\forall X(\neg\mathcal{P}X \to \Box\neg\mathcal{P}X)$ *by Axiom 4.1, and vice versa (also by Axiom 4.1).*

THEOREM 9.2. $\forall x(Gx \to \mathcal{E}\int\int(G,x))$

PROOF. Cf. Scott's proof of his Theorem 2 in [21], p. 258.

1	Ea	Assumption
2	Ga	Assumption
3	Ha	Assumption
4	$Ga \to (Ha \to \mathcal{P}H)$	Prop. 9.2
5	$\mathcal{P}H$	2,3,4 $\to$ Elim
6	$\Box\mathcal{P}H$	6 Axiom 4.4
7	$\Box \mid \mathcal{P}H$	6 K Reit
8	Eb	Assumption
9	Gb	Assumption
10	$\mathcal{P}H \to Hb$	9,10 Def. 4.1, $\forall$ Elim
11	Hb	4,8 $\to$ Elim
12	$Gb \to Hb$	9–11 $\to$ Intro
13	$\forall y(Gy \to Hy)$	8–12 $\forall$ Intro
14	$\Box\forall y(Gy \to Hy)$	7-13 $\Box$ Intro
15	$Ha \to \Box\forall y(Gy \to Hy)$	6-14 $\to$ Intro
16	$\forall Y(Ya \to \Box\forall y(Gy \to Yy)$	15 $\forall$ Intro
17	$Ga \wedge \forall Y(Ya \to \Box\forall y(Gy \to Yy)$	2,16 $\wedge$ Intro

18	$\mathcal{E}\int\int(G,a)$	1,17 Def. 4.2, $\forall$ Elim
19	$Ga \rightarrow \mathcal{E}\int\int(G,a)$	2-18 $\rightarrow$ Intro
20	$\forall x(Gx \rightarrow \mathcal{E}\int\int(G,x))$	1–19 $\forall$ Intro

⊣

Theorem 9.3. $\exists xGx \rightarrow \Box\exists xGx$

Proof. Cf. Scott's proof of his Theorem 3 in [21], p. 258, and Sobel's proof in [21], p., 247 lines 3-13 (and [22], pp. 37–38 lines 3–15).

1	$\exists xGx$	Assumption
2	$Ga \wedge Ea$	Assumption
3	$\mathcal{P}N \rightarrow Na$	2 Def. 4.1, $\forall$ Elim
4	Na	3 Axiom 4.5
5	$\forall Y(\mathcal{E}\int\int(Y,a) \rightarrow \Box\exists yYy)$	2,4 Def. 4.3, $\forall$ Elim
6	$\mathcal{E}\int\int(G,a) \rightarrow \Box\exists yGy$	5 $\forall$ Elim
7	$Ga \rightarrow \mathcal{E}\int\int(G,a)$	Theorem 9.2
8	$\mathcal{E}\int\int(G,a)$	2,7 $\rightarrow$ Elim
9	$\Box\exists yGy$	6,8 $\rightarrow$ Elim
10	$\Box\exists yGy$	1,2-9 $\exists$ Elim
11	$\exists xGx \rightarrow \Box\exists xGx$	1-10 $\rightarrow$ Intro

⊣

Proposition 9.3 (At most one God). $\forall x\forall y((Gx \wedge Gy) \rightarrow x = y)$

Proof.

1	$Ea \wedge Eb$	Assumption
2	$Ga \wedge Gb$	Assumption
3	Ga	2 $\wedge$ Elim
4	Gb	2 $\wedge$ Elim
5	$\forall X(\mathcal{P}X \leftrightarrow Xa)$	1,3, Def. 4.1, Prop. 9.2, $\forall$ Elim
6	$\forall X(\mathcal{P}X \leftrightarrow Xb)$	1,4, Def. 4.1, Prop. 9.2, $\forall$ Elim
7	$\forall X(Xa \leftrightarrow Xb)$	5,6 $\leftrightarrow$ Intro, $\forall$ Intro
8	$(\lambda x.x = b)a \leftrightarrow (\lambda x.x = b)b)$	7 $\forall$ Elim
9	$a = b \leftrightarrow b = b$	7 λ Elim
10	$a = b$	= Intro, 9,10 $\rightarrow$ Elim
11	$(Ga \wedge Gb) \rightarrow a = b$	2-11 $\rightarrow$ Intro
12	$\forall x\forall y((Gx \wedge Gy) \rightarrow x = y)$	1,11 $\forall$ Intro

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