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Semantic Analysis of some Variants of Anderson-like Ontological Proofs

Abstract. The aim of this paper is to prove strong completeness theorems for several Anderson-like variants of Gödel's theory wrt. classes of modal structures, in which: (i). 1^{st} order terms order receive only *rigid* extensions in the *constant objectual* 1^{st} order domain; (ii). 2^{nd} order terms receive *non-rigid* extensions in *preselected world-relative objectual* domains of 2^{nd} order and *rigid* intensions in the *constant conceptual* 2^{nd} order domain.

Keywords: Ontological proof, 2^{nd} order modal logic, positive properties, completeness.

1. Introduction

The basis of all theories to be discussed in this paper is so called *Gödel ontological proof* (see [6] and [13]) which will be presented below first informally and next in a more formal manner.

The informal presentation of Gödel's theory.

Gödel uses a kind of modal language with a 2^{nd} order notion of a *positive property* as a primitive, which he introduces with no elaborate clarification. However, his terse and sometimes cryptic explanations yield that he offers two readings of this notion: (1). positive in a *moral — aesthetic* sense. The positiveness in this sense is independent of the *accidental structure of the world*; and (2). positive in a sense of *pure attribution.* The positiveness in this sense is said to be *opposed to privation.*

The additional three concepts are introduced by the definitions:

Definition 1. A *God* is any being that has every positive property;

Definition 2. A property A is an *essence* of an object x if and only if A entails every property of x;

Definition 3. An object x has the property of *necessarily existing* if and only if its essence is necessarily exemplified.

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These above concepts are characterized by the following axioms:

Axiom 1. Conjunction of positive properties is also positive;

Axiom 2. A property or its complement is positive;

Axiom 3. If a property is positive, then its complement is not positive;

Axiom 4. If a property is positive, then it is necessarily positive;

Axiom 5. The property of necessary existence is a positive property;

Axiom 6. Any property entailed by a positive property is positive.

The above set of definitions and axioms has been proposed by Gödel with a view to proving, by means of an appropriate modal logic of the 2^{nd} order, that:

Theorem. A God necessarily exists.

The formal presentation of Gödel's theory.

A language capable of expressing Gödel's axioms should be equipped with a 2^{nd} order unary predicate $\mathbf{P}$, where $\mathbf{P}(\alpha)$ is to be read: *the property α is positive*, a necessity symbol L, two sorts of variables: $x, y, z, \ldots$ (1^{st} order), $\alpha, \beta, \gamma, \ldots$ (2^{nd} order), Boolean operators: $\cap, -$ (intersection and complementation), customary logical symbols such as: $\wedge, \vee, \rightarrow, \leftrightarrow, \neg$ (conjunction, disjunction, implication, biconditional, negation) and quantifiers $\forall, \exists$ for both sorts of variables.

The *Gödel's theory* **G0** is based on the following set of definitions and axioms:

$$\mathsf{G}(x) \overset{\text{df}}{=} \forall\alpha[\mathbf{P}(\alpha) \rightarrow \alpha(x)] \tag{1.1}$$

$\mathsf{G}(x)$ is read: *x is God-like* or simply *x is a God*

$$\alpha \ \mathsf{Ess}\ x \overset{\text{df}}{=} \forall\beta\big[\beta(x) \rightarrow \mathsf{L}\forall y[\alpha(y) \rightarrow \beta(y)]\big] \tag{1.2}$$

$\alpha \ \mathsf{Ess}\ x$ is read: *a property α is an essence of entity x*

$$\mathsf{NE}(x) \overset{\text{df}}{=} \forall\alpha[\alpha \ \mathsf{Ess}\ x \rightarrow \mathsf{L}\exists y\alpha(y)] \tag{1.3}$$

$\mathsf{NE}(x)$ is read: *x necessarily exists*

$$\mathbf{P}(\alpha) \wedge \mathbf{P}(\beta) \rightarrow \mathbf{P}(\alpha \cap \beta) \tag{1.4}$$

$$\mathbf{P}(\alpha) \vee \mathbf{P}(-\alpha) \tag{1.5}$$

$$\mathbf{P}(\alpha) \rightarrow \neg\mathbf{P}(-\alpha) \tag{1.6}$$

$$\mathbf{P}(\alpha) \to \mathsf{L}\mathbf{P}(\alpha) \tag{1.7}$$

$$\mathbf{P}(\mathsf{NE}) \tag{1.8}$$

$$\mathbf{P}(\alpha) \wedge \mathsf{L}\forall x[\alpha(x) \to \beta(x)] \to \mathbf{P}(\beta) \tag{1.9}$$

Our observation.

It is clear that the above axioms of K. Gödel leave a degree of freedom in interpreting the necessity symbol L. If, however, our theory meets the following rather natural condition:

- no new formula *not* containing the symbol L can be proved if the axiom $\mathsf{L}\boldsymbol{\phi} \leftrightarrow \boldsymbol{\phi}$ is added to the theory

then the axioms: (1.4) – (1.9) are too weak to prove the much needed sentence: $\exists x \mathsf{G}(x)$. To see this, assume that variables of 1^{st} order range over the set of natural numbers ω, variables of 2^{nd} order range over 2^{ω} and $\mathbf{P}$ is interpreted as a non-principal ultrafilter of the Boolean algebra 2^{ω} containing all co-finite sets of natural numbers. Then, assuming that $\mathsf{L}\boldsymbol{\phi} \leftrightarrow \boldsymbol{\phi}$, for every formula $\boldsymbol{\phi}$, we get that the axioms (1.4) – (1.9) are satisfied and the sentence $\exists x \mathsf{G}(x)$ is false because the intersection of all co-finite sets is empty.

Anderson's modification of Gödel's theory.

In this paper we shall be mainly dealing with the *Anderson theory* **A0**, which — in relation to the language of Gödel's theory — requires a slightly modification: the Boolean operator $\cap$ is dropped.

The theory **A0** is based on the following set of definitions and axioms:

$$\mathsf{G}(x) \overset{\text{df}}{=} \forall \alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)] \tag{1.10}$$

$$A \preccurlyeq B \overset{\text{df}}{=} \mathsf{L}\forall x[A(x) \to B(x)] \tag{1.11}$$

$A \preccurlyeq B$ is read: A *entails* B, where A, B are terms of 2^{nd} sort

$$A \;\mathsf{Ess}\; x \overset{\text{df}}{=} \forall \beta\big[\mathsf{L}\beta(x) \leftrightarrow (A \preccurlyeq \beta)\big] \tag{1.12}$$

$$\mathsf{NE}(x) \overset{\text{df}}{=} \forall \alpha[\alpha \;\mathsf{Ess}\; x \to \mathsf{L}\exists y \alpha(y)] \tag{1.13}$$

$$\mathbf{P}(\mathsf{G}) \tag{1.14}$$

and the axioms (1.6)–(1.9) in addition.

Our contribution.

In the first part of this paper we shall introduce a number of formalized axiomatic theories similar to the Anderson's theory **A0** and corresponding classes of model structures which all have constant *objectual domains* of 1^{st} order and constant *conceptual domains* of 2^{nd} order. Terms of 1^{st} order will receive *rigid* interpretation, i.e. they will have the same extensions in all possible worlds in the domain of 1^{st} order (i.e. in the domain of so called *existing objects*) and terms of 2^{nd} order will be given a special treatment motivated by Carnap's method of extension and intension (see [3, 4]); so that terms of 2^{nd} order will receive *extensions* and *intensions*. Extension of a term of 2^{nd} order will be allowed to vary from one possible world to another but it is always required that the extension of a term of 2^{nd} order at a world w belongs to the preselected objectual domain of 2^{nd} order related to that world (i.e. members of these preselected domains are called *existing properties*). Intensions of terms of 2^{nd} order will be *rigid* and they will be identified with members of the domain of 2^{nd} order (i.e. of the domain of so called *conceptual properties*). We shall consider only monadic theories (i.e. allow only monadic variables of 2^{nd} order) but it is easy to realize that the general case, with 2^{nd} order variables of arbitrary arity, can be dealt with in just the same manner, simply by introducing more complicated notations. In the second part, we shall prove completeness theorems for our theories wrt. appropriate classes of model structures. The reader interested in proofs of formulas claimed to be derivable from particular sets of axioms will find them in the appendix.

Other modifications of Gödel's theory and related works.

We wish to mention here two interesting modifications of Gödel's theory, one due to D. Scott (see [13]) and second to M. Fitting [5], each capable of proving the formula $\mathsf{L}\exists x\mathsf{G}(x)$.

The language of *Scott's theory* differs from the language of Gödel's theory only in that the Boolean operator $\cap$ is also dropped. Another difference concerns the definition of an essence: in Scott's theory the essence of an entity x is a property that entails every property that x possesses, and additionally, x itself possesses this property. Moreover, axiom (1.4) is deleted, axioms (1.5) and (1.6) are unified in the form: $\mathbf{P}(\alpha) \leftrightarrow \neg\mathbf{P}(-\alpha)$, the formula $\mathbf{P}(\mathsf{G})$ is added as an axiom, and the remaining axioms of Gödel's theory are kept.

Moving now to *Fitting's theory*, let us cite Gödel's footnote to axiom (1.4): "and for any number of summand". Our interpretation of this phrase is: for any non-zero natural number, which fits the use of Boolean operator $\cap$. Allowing here any cardinal number whatsoever one would need infinite intersections, and then cope with problems of infinitary languages. However, following C. A. Anderson and M. Gettings [2], also M. Fitting opted for this interpretation. He adopted an axiom saying that intersection of any collection of positive properties is positive and in consequence his theory needed a language of the third-order. The remaining axioms of Gödel's theory are kept by M. Fitting with the stipulation that quantifiers binding variables of the 1^{st} order are relativized to the existence predicate E which is a primitive symbol of the language (for example, $\forall x\boldsymbol{\phi}$ is replaced by $\forall x\big[\mathsf{E}(x) \rightarrow \boldsymbol{\phi}\big]$ and $\exists x\boldsymbol{\phi}$ by $\exists x\big[\mathsf{E}(x) \wedge \boldsymbol{\phi}\big]$). Of the three definitions employed in Gödel's theory the only new one is the definition of the essence, which is understood in the spirit of Dana Scott. Each of these definitions is augmented with a predicate abstraction axiom securing the existence of a corresponding predicate. A special feature of Fitting's theory is its tableau proof procedure.

Interesting results concerning the theories: of Dana Scott, with cautious comprehension, and of Anderson, with full comprehension, were presented by P. Hájek in [7], [8] and [9]. Briefly, in these papers it was established that the first of these extended theories is interpretable in the second and that they are free from a modal collapse. The result, meanwhile, of the second of these theories is the confirmation of the redundancy of its axiom system, from the point of view of the provability of the formula $\mathsf{L}\exists x\mathsf{G}(x)$, as well offering some modified fragments (in the words of the author) of Anderson's theory. For some of these theories completeness theorems with respect to fixed and/or variable domain semantics were formulated, although the proofs of these theorems were only sketched.

2. Technical preliminaries

We begin with describing a family of formal systems which we call *Anderson-like* theories. The formal language $\mathcal{L}$ of Anderson-like theories is equipped with a 2^{nd} order unary predicate $\mathbf{P}$, a necessity symbol L, two sorts of variables: $x, y, z, \ldots$ (1^{st} order), $\alpha, \beta, \gamma, \ldots$ (2^{nd} order), Boolean operator $-$ (complementation), logical symbols: $\wedge, \neg$ (conjunction, negation) and universal quantifier $\forall$ for both sorts of variables. The only terms of the 1^{st} sort are variables of the 1^{st} sort and terms of the 2^{nd} sort are formed from variables of the 2^{nd} sort by applying complementation any finite (possibly zero) number of times. Thus, the set of terms of the 2^{nd} sort and the set of formulas are given by the grammars:

$$A \overset{\mathrm{df}}{=} \alpha \;\big|\; -A$$

$$\boldsymbol{\phi} \overset{\mathrm{df}}{=} A(x) \;\big|\; \mathbf{P}(A) \;\big|\; \boldsymbol{\phi} \wedge \boldsymbol{\psi} \;\big|\; \neg\boldsymbol{\phi} \;\big|\; \mathsf{L}\boldsymbol{\phi} \;\big|\; \forall x\boldsymbol{\phi} \;\big|\; \forall\alpha\boldsymbol{\phi}$$

The remaining propositional connectives: $\vee, \rightarrow, \leftrightarrow$ as well as the existential quantifier, possibility operator and identity for terms of 1^{st} sort are introduced by usual definitions:

$$\boldsymbol{\phi} \vee \boldsymbol{\psi} \overset{\mathrm{df}}{=} \neg(\neg\boldsymbol{\phi} \wedge \neg\boldsymbol{\psi}), \quad \boldsymbol{\phi} \rightarrow \boldsymbol{\psi} \overset{\mathrm{df}}{=} \neg\boldsymbol{\phi} \vee \boldsymbol{\psi}, \quad \boldsymbol{\phi} \leftrightarrow \boldsymbol{\psi} \overset{\mathrm{df}}{=} (\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}) \wedge (\boldsymbol{\psi} \rightarrow \boldsymbol{\phi}),$$

$\exists\xi\boldsymbol{\phi} \overset{\mathrm{df}}{=} \neg\forall\xi\neg\boldsymbol{\phi}$ where ξ is a variable of any sort,

$\mathsf{M}\boldsymbol{\phi} \overset{\mathrm{df}}{=} \neg\mathsf{L}\neg\boldsymbol{\phi}$ and $(x \overset{1}{\approx} y) \overset{\mathrm{df}}{=} \forall\alpha[\alpha(x) \leftrightarrow \alpha(y)]$.

To increase readability, we will occasionally use symbols: $\forall$, $\exists$, $\Longleftarrow$, $\Longrightarrow$, $\Longleftrightarrow$ for quantifiers and propositional connectives of metalanguage.

The definitions of $\mathsf{G}(x)$, $A \preccurlyeq B$, $\alpha\ \mathsf{Ess}\ x$ and $\mathsf{NE}(x)$ adopted in Anderson-like theories are the same as proposed by Anderson [1]:

$$\mathsf{G}(x) \overset{\mathrm{df}}{=} \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)],$$

$$A \preccurlyeq B \overset{\mathrm{df}}{=} \mathsf{L}\forall x[A(x) \rightarrow B(x)],$$

$$A\ \mathsf{Ess}\ x \overset{\mathrm{df}}{=} \forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow (A \preccurlyeq \beta)\big],$$

$$\mathsf{NE}(x) \overset{\mathrm{df}}{=} \forall\alpha[\alpha\ \mathsf{Ess}\ x \rightarrow \mathsf{L}\exists y\alpha(y)].$$

Any Anderson-like theory will be determined by its axiomatic base whose content will depend on certain options. Options that have been chosen for a particular Anderson-like theory will determine its *acronym* by means of which a theory can be unequivocally identified. The list below contains *axioms and axiom schemas that are obligatory* in Anderson-like theories (we assume that $\boldsymbol{\phi}, \boldsymbol{\psi}$ are formulas, x, y are variables of 1^{st} sort, α, β are variables of 2^{nd} sort and ξ is a variable of any sort):

All what is needed for classical propositional logic, (2.15)

$$\forall\xi[\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}] \rightarrow [\forall\xi\boldsymbol{\phi} \rightarrow \forall\xi\boldsymbol{\psi}], \tag{2.16}$$

$$\forall\xi\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}(\xi/A) \text{ if } A \text{ is a term of the same sort as } \xi, \tag{2.17}$$

$$\boldsymbol{\phi} \rightarrow \forall\xi\boldsymbol{\phi} \text{ if } \xi \text{ is not free in } \boldsymbol{\phi}, \tag{2.18}$$

$$\mathsf{L}(\boldsymbol{\phi} \rightarrow \boldsymbol{\psi}) \rightarrow (\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{L}\boldsymbol{\psi}), \tag{2.19}$$

$$\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{M}\boldsymbol{\phi}, \tag{2.20}$$

$$\forall\xi\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{L}\forall\xi\boldsymbol{\phi} \quad \text{(Barcan formulas of both sorts)}, \tag{2.21}$$

$$\mathsf{L}\exists\alpha\boldsymbol{\phi} \rightarrow \exists\alpha\mathsf{L}\boldsymbol{\phi}, \tag{2.22}$$

$$\mathsf{L}(x \overset{1}{\approx} y) \rightarrow [\boldsymbol{\phi}(z/x) \leftrightarrow \boldsymbol{\phi}(z/y)] \text{ where } x \text{ and } y \text{ are free for } z \text{ in } \boldsymbol{\phi}(z), \tag{2.23}$$

$$\exists\alpha[\alpha(x) \wedge \neg\alpha(y)] \rightarrow \mathsf{L}\exists\alpha[\alpha(x) \wedge \neg\alpha(y)], \tag{2.24}$$

$$\mathbf{P}(\alpha) \rightarrow \neg\mathbf{P}(-\alpha), \tag{2.25}$$

$$[\mathbf{P}(\alpha) \wedge (\alpha \preccurlyeq \beta)] \rightarrow \mathbf{P}(\beta), \tag{2.26}$$

$$\mathsf{LG}(x) \leftrightarrow \mathsf{G}(x) \tag{2.27}$$

Every Anderson-like theory must be equipped with an axiom saying that the property of being God-like is positive and therefore it must be legitimate to treat the property of being God-like as a term of the 2^{nd} sort. Thus, the following axioms are obligatory for all Anderson-like theories:

$$\exists\alpha(\alpha \overset{2}{\approx} \mathsf{G}), \tag{2.28}$$

$$\exists\beta\big[\mathbf{P}(\beta) \wedge (\beta \overset{2}{\approx} \mathsf{G})\big] \text{ or shortly: } \mathbf{P}(\mathsf{G}). \tag{2.29}$$

where, of course, the symbol $\overset{2}{\approx}$ stands for the relation of identity of objects of 2^{nd} sort i.e. *properties*.

However, the relation $\overset{2}{\approx}$ can be introduced in Anderson-like theories in two different ways by the following *optional definitions*:

$$A \overset{2}{\approx} B \overset{\text{df}}{=} \forall x[A(x) \leftrightarrow B(x)], \tag{2.30}$$

$$A \overset{2}{\approx} B \overset{\text{df}}{=} \mathsf{L}\forall x[A(x) \leftrightarrow B(x)] \tag{2.31}$$

and it is clear that the translation of (2.28) and (2.29) to the original language depends on which optional definition of $\overset{2}{\approx}$ has been applied.

Moreover, the choice of definition of $\overset{2}{\approx}$ affects another obligatory axioms of Anderson-like theories. Those axioms take the form:

$$\forall x[(-\alpha)(x) \leftrightarrow \neg\alpha(x)] \text{ and} \tag{2.32}$$

$$(x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y) \tag{2.33}$$

or

$$\mathsf{L}\forall x[(-\alpha)(x) \leftrightarrow \neg\alpha(x)] \text{ and} \tag{2.34}$$

$$\mathsf{L}\big[(x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y)\big] \tag{2.35}$$

depending on which one of (2.30), (2.31) has been adopted.

Summing up, an axiomatic base of Anderson-like theory contains:

$$\exists\beta\Big[\mathbf{P}(\beta) \wedge \forall x\big[\beta(x) \leftrightarrow \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)]\big]\Big] \text{ and}$$
$$\forall x[(-\alpha)(x) \leftrightarrow \neg\alpha(x)] \text{ and}$$
$$(x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y),$$

or:

$$\exists\beta\Big[\mathbf{P}(\beta) \wedge \mathsf{L}\forall x\big[\beta(x) \leftrightarrow \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)]\big]\Big] \text{ and}$$
$$\mathsf{L}\forall x[(-\alpha)(x) \leftrightarrow \neg\alpha(x)] \text{ and}$$
$$\mathsf{L}\big[(x \overset{1}{\approx} y) \rightarrow \mathsf{L}(x \overset{1}{\approx} y)\big]$$

depending on whether the definition (2.30) or (2.31) has been opted for.

The reader should also be aware of the fact, that neither of two optional definitions of $\overset{2}{\approx}$ provides what one might have expected of an identity relation. Indeed, the formula: $(\alpha \overset{2}{\approx} \beta) \rightarrow [\alpha(x) \rightarrow \beta(x)]$ is unprovable on the basis of the definition (2.31), however, it can be proved if (2.30) is applied. On the other hand, the formula: $(\alpha \overset{2}{\approx} \beta) \rightarrow [\mathbf{P}(\alpha) \rightarrow \mathbf{P}(\beta)]$ is unprovable on the basis of (2.30) but it can be proved if (2.31) is applied.

Any Anderson-like theory employing the definition (2.31) will be given an acronym ending with the symbol $\star$ and thus, theories employing (2.30) can be easily recognized by their $\star$-less acronyms.

Optional axioms of Anderson-like theories are chosen according to the following criteria:

(i) treatment of the property of *necessary existence*,
(ii) treatment of so called *singletons*,
(iii) characterization of modal operators.

As to (i), if we intend to treat the property of necessary existence as a term of the 2^{nd} sort we should adopt an optional axiom:

$$\exists\alpha(\alpha \overset{2}{\approx} \mathsf{NE}) \tag{2.36}$$

and augment the acronym of theory with the symbol **n**.

The option (ii) concerns co called *singletons*, i.e. properties abstracted from expressions of the form $\mathsf{I}_x(y)$ defined by:

$$\mathsf{I}_x(y) \overset{\text{df}}{=} (x \overset{1}{\approx} y). \tag{2.37}$$

If we intend to treat singletons as terms of 2^{nd} sort we should adopt an optional axiom:

$$\exists\alpha(\alpha \overset{2}{\approx} \mathsf{I}_x) \tag{2.38}$$

and augment the acronym of theory with the symbol **s**.

Once again we wish to emphasize that the form of optional axioms (2.36) and (2.38) depends on which definition ((2.30) or (2.31)) of the relation $\overset{2}{\approx}$ has been chosen.

As to (iii), we choose one of the following:

(5) $\mathsf{ML}\boldsymbol{\phi} \rightarrow \mathsf{L}\boldsymbol{\phi}$

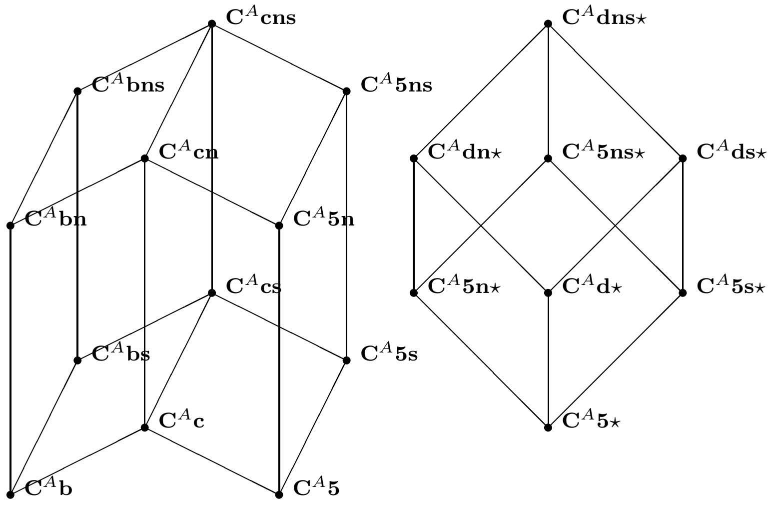
(b) $\mathsf{ML}\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}$

(c) $\mathsf{ML}\boldsymbol{\phi} \rightarrow \mathsf{L}\boldsymbol{\phi}$, $\mathsf{ML}\boldsymbol{\phi} \rightarrow \boldsymbol{\phi}$ and $\mathbf{P}(\alpha) \rightarrow \mathsf{L}\mathbf{P}(\alpha)$

(d) $\mathsf{ML}\boldsymbol{\phi} \rightarrow \mathsf{L}\boldsymbol{\phi}$, $\mathsf{L}\boldsymbol{\phi} \rightarrow \mathsf{LL}\boldsymbol{\phi}$ and $\mathbf{P}(\alpha) \rightarrow \mathsf{L}\mathbf{P}(\alpha)$

and augment the acronym of theory by symbol **5, b, c** or **d** indicating the choice that has been made.

The following diagram illustrates inclusion ordering of axiomatic bases of all Anderson-like theories that will be considered in this paper. Each node of the diagram is labeled with an acronym of the corresponding theory. After inspecting the diagram the reader is invited to perform a test. If we succeeded in explaining our — somewhat complicated — naming system, it should be easy to recover the axiomatic base of any Anderson-like theory from its acronym assuming that the symbol $\mathbf{C}^A$ stands for all obligatory axioms.



The only inference rule of Anderson-like theories is the *modus ponens*:

$$\text{RO:} \qquad \frac{\phi, \phi \to \psi}{\psi}$$

For any finite (including the empty) sequence Q of symbols each of which has the form $\forall\xi$ or L:

$$\boxed{Q\phi \text{ is } \textit{clothed axiom} \text{ of the Anderson-like theory } \mathbf{Th} \text{ if } \phi \text{ is its axiom.}} \tag{2.39}$$

By $\vdash_{\mathbf{Th}}$ we denote the *inference relation* determined by clothed axioms of an Anderson-like theory **Th** and the rule inference RO. Thus, for a set of formulas X and a formula ϕ we write: $X \vdash_{\mathbf{Th}} \phi$ to mean that there exists a **Th**–*derivation* of ϕ from X. Such a derivation is a finite sequence of formulas (*derivation steps*) each of which has to be *justified* in an appropriate manner. Each step of a derivation is therefore required to be a clothed axiom of **Th** or an element of X or the result of applying the inference rule RO to preceding steps.

One more small comment: Maybe the reader will wish to know, why do we use only modus ponens and not generalization and necessitation inference rules. What is invariable in the definition of a **Th**–*derivation* of ϕ from X that to formulas of the set X only modus ponens may be applied. If generalization and necessitation inference rules would have been introduced, then their use must be restricted to theorems of a **Th** theory. But then, the axiomatic basis obtained thus could easily be proved to be equivalent to the one we are using.

An easy proof of the following useful fact will be left to the reader:

PROPOSITION 2.1. (i) $X \cup \{\phi\} \vdash_{\mathbf{T}} \psi$ *iff* $X \vdash_{\mathbf{T}} \phi \to \psi$,
(ii) *if* $X \vdash_{\mathbf{T}} \phi$ *then* $\{\mathsf{L}\psi \ : \ \psi \in X\} \vdash_{\mathbf{T}} \mathsf{L}\phi$.

By a *model structure* we mean a quintuple of the form $\mathfrak{W} = \langle W, R, D_1, D_2, g\rangle$, where: $W \neq \emptyset$ is the set of *possible worlds*; $R \subseteq W^2$ is the *relation of accessibility*; D_1 is the domain of the 1^{st} sort, i.e. the set of *existing objects*; $D_2 = (D_w)_{w\in W}$ is the family of the 2^{nd} sort domains, i.e. $\bigcup_{w\in W} D_w$ is the set of *existing properties*, where $D_w \subseteq 2^{D_1}$ for every $w \in W$. Apart from existing properties we also consider so called *conceptual properties* of the structure, by which we mean functions $f \in W \mapsto \bigcup_{w\in W} D_w$ such that $f(w) \in D_w$ for every $w \in W$. The set of all conceptual properties of the structure $\mathfrak{W}$ will be denoted by $C_{\mathfrak{W}}$. In addition, we assume that the following conditions are satisfied:

$$g \in D_1, \tag{2.40}$$

$$\{g\} \in \bigcap_{w \in W} D_w, \tag{2.41}$$

$$\forall w \in W \big[X \in D_w \Longrightarrow D_1 - X \in D_w \big], \tag{2.42}$$

$$\forall a, b \in D_1 \; \forall w \in W \; \exists X \in D_w \;\; |\{a, b\} \cap X| = 1, \tag{2.43}$$

$$R \text{ is } \textit{serial} \text{ i.e. } \forall w \in W \; \exists v \in W \; wRv. \tag{2.44}$$

If $\mathfrak{W}$ is a model structure then by an *assignment* in $\mathfrak{W}$ we mean a function $\mathfrak{a}$ which maps variables of the 1^{st} sort to existing objects (i.e members of D_1) and variables of the 2^{nd} sort to conceptual properties of the structure (i.e. members of $C_{\mathfrak{W}}$). An assignment $\mathfrak{a}$ is extended to all terms of the 2^{nd} order by putting: $(\mathfrak{a}(-A))(w) \overset{\text{df}}{=} D_1 - (\mathfrak{a}(A))(w)$, for every $w \in W$ and every term A of the 2^{nd} sort. If $\mathfrak{a}$ is an assignment, then the symbol $\mathfrak{a}^o_\xi$ denotes the assignment defined by:

$$\mathfrak{a}^o_\xi(\zeta) \overset{\text{df}}{=} \begin{cases} o & \text{if } \zeta = \xi, \\ \mathfrak{a}(\zeta) & \text{if } \zeta \neq \xi. \end{cases}$$

Of course, o is tacitly assumed to be an entity suitable for the variable ξ depending on its sort and both $\mathfrak{a}$ and $\mathfrak{a}^o_\xi$ are assumed to be assignments in the same model structure. We say that assignments $\mathfrak{a}, \mathfrak{b}$ *agree apart from* ξ (symbolically: $\mathfrak{a} \equiv^?_\xi \mathfrak{b}$) if for some o, $\mathfrak{a}^o_\xi = \mathfrak{b}$. Note that $\equiv^?_\xi$ is an equivalence relation on the set of all assignments of a model structure. The equivalence class of $\mathfrak{a}$ with respect to $\equiv^?_\xi$ will be further denoted by $\{\mathfrak{a}^?_\xi\}$.

A pair of the form $\langle \mathfrak{W}, \mathfrak{a} \rangle$ will be called *model* and the symbol $\models$ will be used for the *satisfiability relation* — the expression $\mathfrak{W}, \mathfrak{a}, w \models \phi$, where $w \in W$ reads: *the formula* ϕ *is satisfied in the world* w *of model* $\langle \mathfrak{W}, \mathfrak{a} \rangle$. If no misunderstanding is likely as to the particular model structure $\mathfrak{W}$ in which an assignment $\mathfrak{a}$ has been chosen, we simplify the notation by writing: $\mathfrak{a}, w \models \phi$ instead of $\mathfrak{W}, \mathfrak{a}, w \models \phi$. Given a model $\langle \mathfrak{W}, \mathfrak{a} \rangle$, the satisfiability relation $\models$ is defined as usual, for any possible world $w \in W$ by the following conditions, where x is a variable of the 1^{st} sort, A is a term of the 2^{nd} sort, ξ is a variable of arbitrary sort and ϕ, ψ are a formulas:

(i) $\mathfrak{a}, w \models A(x)$ iff $\mathfrak{a}(x) \in (\mathfrak{a}(A))(w)$,

(ii) $\mathfrak{a}, w \models \phi \wedge \psi$ iff $\mathfrak{a}, w \models \phi$ and $\mathfrak{a}, w \models \psi$,

(iii) $\mathfrak{a}, w \models \neg\phi$ iff not $\mathfrak{a}, w \models \phi$ (symbolically: $\mathfrak{a}, w \not\models \phi$),
(iv) $\mathfrak{a}, w \models \forall\xi\phi$ iff $\mathfrak{b}, w \models \phi$ for every $\mathfrak{b} \in \{\mathfrak{a}^?_\xi\}$,
(v) $\mathfrak{a}, w \models \mathsf{L}\phi$ iff $\mathfrak{a}, v \models \phi$ for every $v \in W$ such that wRv,
(vi) $\mathfrak{a}, w \models \mathbf{P}(A)$ iff $g \in (\mathfrak{a}(A))(v)$ for every $v \in W$ such that wRv.

We omit conditions corresponding to $\vee, \rightarrow, \leftrightarrow$ and $\exists$ as they are defined in terms of $\wedge, \neg$ and $\forall$.

The set of all formulas satisfied in a world w of a model $\langle\mathfrak{W}, \mathfrak{a}\rangle$ will be denoted by $\mathsf{Sat}(\mathfrak{W}, \mathfrak{a}, w)$ or simply by $\mathsf{Sat}(\mathfrak{a}, w)$, if the model structure in question is clear from the context.

As customary, we say that a formula ϕ *is true* in a model structure $\mathfrak{W}$ (symbolically: $\mathfrak{W} \models \phi$) iff $\mathfrak{a}, w \models \phi$, for every assignment $\mathfrak{a}$ in $\mathfrak{W}$ and every world $w \in W$. The set of all formulas true in $\mathfrak{W}$ will be denoted by $\mathsf{Th}(\mathfrak{W})$. We also put $\mathsf{Th}(\mathbb{K}) \stackrel{\text{df}}{=} \bigcap\{\mathsf{Th}(\mathfrak{W}) : \mathfrak{W} \in \mathbb{K}\}$, for an arbitrary class of model structures $\mathbb{K}$. If X is a set of formulas then we write $\mathfrak{W} \models X$, $\mathbb{K} \models X$ if $X \subseteq \mathsf{Th}(\mathfrak{W})$, $X \subseteq \mathsf{Th}(\mathbb{K})$ respectively. We write $X \models_{\mathbb{K}} \phi$ to express that for every assignment $\mathfrak{a}$ in a model structure $\mathfrak{W} \in \mathbb{K}$ and for every $w \in W$, if $X \subseteq \mathsf{Sat}(\mathfrak{W}, \mathfrak{a}, w)$ then $\phi \in \mathsf{Sat}(\mathfrak{W}, \mathfrak{a}, w)$.

The following fact is sometimes called *substitution lemma*. Its proof — a routine induction on the degree of complexity of ϕ — will be omitted.

Proposition 2.2. *If A is a term of the same sort as a variable ξ then $\mathfrak{a}, w \models \phi(\xi/A)$ iff $\mathfrak{a}^{\mathfrak{a}(A)}_\xi, w \models \phi$ for every $w \in W$.*

It will be useful to distinguish a certain subset $W^{acc} \subseteq W$. Members of W^{acc} are called *accessible worlds* and W^{acc} is defined as the R-image of W. We also define *inaccessible worlds* putting $W^{inacc} \stackrel{\text{df}}{=} W - W^{acc}$. We will define a class of so called *special model structures* in which inaccessible worlds will be treated in a special way – they will be provided with a separate family E_2 of the 2^{nd} sort domains. Thus, by a *special model structure* we shall mean a sextuple of the form $\mathfrak{W} = \langle W, R, D_1, D_2, E_2, g\rangle$ where $\langle W, R, D_1, D_2, g\rangle$ is an ordinary model structure and $E_2 = (E_w)_{w\in W^{inacc}}$, where $\emptyset \neq E_w \subseteq D_w$ for every $w \in W^{inacc}$. By conceptual properties of a special model structure we shall mean those functions $f \in W \mapsto \bigcup_{w\in W}(D_w \cup E_w)$ such that for every $w \in W$: $f(w) \in D_w$ if $w \in W^{acc}$, and $f(w) \in E_w$ if $w \in W^{inacc}$. The above restriction on the set of conceptual properties of a special model structure forces a revision of treatment of terms of the 2^{nd} sort. Indeed, if $\mathfrak{a}$ is an assignment in a special model structure $\mathfrak{W}$ and $w \in W^{inacc}$ then we can no longer put: $(\mathfrak{a}(-A))(w) \stackrel{\text{df}}{=}$

$D_1 - (\mathfrak{a}(A))(w)$ because the value $(\mathfrak{a}(-A))(w)$ has to belong to E_w which has not been assumed to be closed under complementation. Thus, for $w \in W^{inacc}$, we allow $(\mathfrak{a}(-A))(w)$ to be an arbitrary element of E_w and in effect, *in inaccessible worlds of special model structures, the complementation operator is deprived of its usual sense.*

The behaviour of various defined concepts in ordinary and in special model structures will be clarified in the following series of lemmas. Despite of their simplicity, lemmas will be proved in detail.

LEMMA 2.3. *For every assignment $\mathfrak{a}$ in a special model structure the following conditions hold:*

(i) $\mathfrak{a}, w \models x \overset{1}{\approx} y$ *iff* $\begin{cases} \forall X \in E_w[\mathfrak{a}(x) \in X \iff \mathfrak{a}(y) \in X] & \textit{if } w \in W^{inacc} \\ \mathfrak{a}(x) = \mathfrak{a}(y) & \textit{otherwise}, \end{cases}$

(ii) $\mathfrak{a}, w \models A \overset{2}{\approx} B$ *iff* $\forall v \in R(w)\ (\mathfrak{a}(A))(v) = (\mathfrak{a}(B))(v)$.

For every assignment $\mathfrak{a}$ in an ordinary model structure the following conditions hold:

(iii) $\mathfrak{a}, w \models x \overset{1}{\approx} y$ *iff* $\mathfrak{a}(x) = \mathfrak{a}(y)$,

(iv) $\mathfrak{a}, w \models A \overset{2}{\approx} B$ *iff* $(\mathfrak{a}(A))(w) = (\mathfrak{a}(B))(w)$.

PROOF. To prove (i) let us compute:

$$\begin{aligned}
&\mathfrak{a}, w \models x \overset{1}{\approx} y \text{ iff } \mathfrak{a}, w \models \forall\alpha[\alpha(x) \leftrightarrow \alpha(y)] \\
&\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{b}, w \models \alpha(x) \leftrightarrow \alpha(y)\big] \\
&\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{b}, w \models \alpha(x) \iff \mathfrak{b}, w \models \alpha(y)\big] \\
&\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{b}(x) \in (\mathfrak{b}(\alpha))(w) \iff \mathfrak{b}(y) \in (\mathfrak{b}(\alpha))(w)\big] \\
&\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{a}(x) \in (\mathfrak{b}(\alpha))(w) \iff \mathfrak{a}(y) \in (\mathfrak{b}(\alpha))(w)\big]
\end{aligned}$$

Now, $\Longleftarrow$ is obvious and applying 2.43 one gets $\Longrightarrow$.

To prove (ii) let us compute again:

$$\begin{aligned}
&\mathfrak{a}, w \models A \overset{2}{\approx} B \text{ iff } \mathfrak{a}, w \models \mathsf{L}\forall x[A(x) \leftrightarrow B(x)] \\
&\text{iff } \forall v \in R(w)\big[\mathfrak{a}, v \models \forall x[A(x) \leftrightarrow B(x)]\big] \\
&\text{iff } \forall v \in R(w)\, \forall \mathfrak{b} \in \{\mathfrak{a}^?_x\}\big[\mathfrak{b}, v \models A(x) \leftrightarrow B(x)\big] \\
&\text{iff } \forall v \in R(w)\, \forall \mathfrak{b} \in \{\mathfrak{a}^?_x\}\big[\mathfrak{b}, v \models A(x) \iff \mathfrak{b}, v \models B(x)\big] \\
&\text{iff } \forall v \in R(w)\, \forall \mathfrak{b} \in \{\mathfrak{a}^?_x\}\big[\mathfrak{b}(x) \in (\mathfrak{b}(A))(v) \iff \mathfrak{b}(x) \in (\mathfrak{b}(B))(v)\big]
\end{aligned}$$

$$\text{iff } \forall v \in R(w)\, \forall o \in D_1 \big[o \in (\mathfrak{b}(A))(v) \iff o \in (\mathfrak{b}(B))(v) \big]$$
$$\text{iff } \forall v \in R(w)\ (\mathfrak{b}(A))(v) = (\mathfrak{b}(B))(v).$$

The proofs of (iii) and (iv) are left to the reader ■

LEMMA 2.4. *For every assignment* $\mathfrak{a}$ *in a model structure of any kind (ordinary or special) and for every world* $w \in W$ *(no matter whether accessible or not) the following conditions hold:*

(i) $\mathfrak{a}, w \models \mathsf{G}(x)$ *iff* $\mathfrak{a}(x) = g$,

(ii) $\mathfrak{a}, w \models A \ \mathsf{Ess}\ x$ *iff* $\forall v \in R(w) \big[(\mathfrak{a}(A))(v) = \{\mathfrak{a}(x)\} \big]$,

(iii) $\mathfrak{a}, w \models \mathsf{NE}(x)$.

PROOF. To prove (i) let us compute:

$$\mathfrak{a}, w \models \mathsf{G}(x) \text{ iff } \mathfrak{a}, w \models \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\} \big[\mathfrak{b}, w \models \mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x) \big]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\} \big[\mathfrak{b}, w \models \mathbf{P}(\alpha) \iff \mathfrak{b}, w \models \mathsf{L}\alpha(x) \big]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\} \big[\mathfrak{b}, w \models \mathbf{P}(\alpha) \iff \forall v \in R(w)[\mathfrak{b}, v \models \alpha(x)] \big]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\} \big[\forall v \in R(w)[g \in (\mathfrak{b}(\alpha))(v)] \iff \forall v \in R(w)[\mathfrak{b}(x) \in (\mathfrak{b}(\alpha))(v)] \big]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\alpha\} \big[\forall v \in R(w)[g \in (\mathfrak{b}(\alpha))(v)] \iff \forall v \in R(w)[\mathfrak{a}(x) \in (\mathfrak{b}(\alpha))(v)] \big].$$

Since R is serial (see (2.44)) then $R(w) \neq \emptyset$ and thus, (2.41) yields $\Longrightarrow$. The converse implication is obvious.

To prove (ii) we compute again:

$$\mathfrak{a}, w \models A \ \mathsf{Ess}\ x$$
$$\text{iff } \mathfrak{a}, w \models \forall\beta[\mathsf{L}\beta(x) \leftrightarrow (A \preccurlyeq \beta)]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\beta\} \big[\mathfrak{b}, w \models \mathsf{L}\beta(x) \leftrightarrow (A \preccurlyeq \beta) \big]$$
$$\text{iff } \forall \mathfrak{b} \in \{\mathfrak{a}^?_\beta\} \big[\mathfrak{b}, w \models \mathsf{L}\beta(x) \iff \mathfrak{b}, w \models (A \preccurlyeq \beta) \big]$$

Since

$$\mathfrak{b}, w \models \mathsf{L}\beta(x)$$
$$\text{iff } \forall v \in R(w)[\mathfrak{b}, v \models \beta(x)]$$
$$\text{iff } \forall v \in R(w)[\mathfrak{b}(x) \in (\mathfrak{b}(\beta))(v)]$$
$$\text{iff } \mathfrak{a}(x) \in \bigcap_{v \in R(w)} (\mathfrak{b}(\beta))(v)$$

and

$$
\begin{aligned}
\mathfrak{b}, w \models (A \preccurlyeq \beta)& \\
&\text{iff } \ \mathfrak{b}, w \models \mathsf{L}\forall y[A(y) \to \beta(y)] \\
&\text{iff } \ \forall v \in R(w)\big[\mathfrak{b}, v \models \forall y[A(y) \to \beta(y)]\big] \\
&\text{iff } \ \forall v \in R(w)\, \forall \mathfrak{c} \in \{\mathfrak{b}^?_y\}\big[\mathfrak{c}, v \models A(y) \to \beta(y)\big] \\
&\text{iff } \ \forall v \in R(w)\, \forall \mathfrak{c} \in \{\mathfrak{b}^?_y\}\big[\mathfrak{c}, v \models A(y) \Longrightarrow \mathfrak{c}, v \models \beta(y)\big] \\
&\text{iff } \ \forall v \in R(w)\, \forall \mathfrak{c} \in \{\mathfrak{b}^?_y\}\big[\mathfrak{c}(y) \in (\mathfrak{c}(A))(v) \Longrightarrow \mathfrak{c}(y) \in (\mathfrak{c}(\beta))(v)\big] \\
&\text{iff } \ \forall v \in R(w)\, \forall o \in D_1\big[o \in (\mathfrak{b}(A))(v) \Longrightarrow o \in (\mathfrak{b}(\beta))(v)\big] \\
&\text{iff } \ \forall v \in R(w)\ [(\mathfrak{b}(A))(v) \subseteq (\mathfrak{b}(\beta))(v)] \\
&\text{iff } \ \forall v \in R(w)\ [(\mathfrak{a}(A))(v) \subseteq (\mathfrak{b}(\beta))(v)]
\end{aligned}
$$

then we finally get the following useful equivalence:

$\mathfrak{a}, w \models A \ \mathsf{Ess}\ x$ iff

$$\forall \mathfrak{b} \in \{\mathfrak{a}^?_\beta\}\Big[\mathfrak{a}(x) \in \bigcap_{v \in R(w)} (\mathfrak{b}(\beta))(v) \Longleftrightarrow \forall v \in R(w)\ [(\mathfrak{a}(A))(v) \subseteq (\mathfrak{b}(\beta))(v)]\Big]$$

which makes $\Longleftarrow$ completely obvious. It may also be used to prove:

$$\text{if } \ \mathfrak{a}, w \models A \ \mathsf{Ess}\ x \ \text{ then } \ \mathfrak{a}(x) \in \bigcap_{v \in R(w)} (\mathfrak{a}(A))(v).$$

Indeed, suppose that $\mathfrak{a}, w \models A \ \mathsf{Ess}\ x$ and take $\mathfrak{b} \in \{\mathfrak{a}^?_\beta\}$ defined by:

$$(\mathfrak{b}(\beta))(v) \stackrel{\text{df}}{=} \begin{cases} (\mathfrak{a}(A))(v) & \text{if } v \in R(w), \\ (\mathfrak{a}(\beta))(v) & \text{otherwise.} \end{cases}$$

Clearly, $\forall v \in R(w)\ [(\mathfrak{a}(A))(v) \subseteq (\mathfrak{b}(\beta))(v)]$ and our equivalence yields that:

$$\mathfrak{a}(x) \in \bigcap_{v \in R(w)} (\mathfrak{b}(\beta))(v) = \bigcap_{v \in R(w)} (\mathfrak{a}(A))(v)$$

as required. Next, we will prove:

$$\text{if } \ \mathfrak{a}, w \models A \ \mathsf{Ess}\ x \ \text{ then } \ \forall v \in R(w)\ [(\mathfrak{a}(A))(v) = \{\mathfrak{a}(x)\}].$$

We already know that $\forall v \in R(w)\ [\{\mathfrak{a}(x)\} \subseteq (\mathfrak{a}(A))(v)]$ and thus it suffices to prove that $\subsetneq$ may not happen. Now, suppose that $\varnothing \neq V = \{s \in R(w) \mid \{\mathfrak{a}(x)\} \subsetneq (\mathfrak{a}(A))(s)\}$ and for every $s \in V$ take any $o_s \in (\mathfrak{a}(A))(s) - \{\mathfrak{a}(x)\}$. From (2.42) and (2.43) it follows that for every $s \in V$ there exists $X_s \in D_s$ such that $\mathfrak{a}(x) \in X_s$ and $o_s \notin X_s$. Take $\mathfrak{b} \in \{\mathfrak{a}^?_\beta\}$ defined by:

$$(\mathfrak{b}(\beta))(v) \stackrel{\text{df}}{=} \begin{cases} X_v & \text{if } v \in V, \\ (\mathfrak{a}(\beta))(v) & \text{otherwise.} \end{cases}$$

Note that $\mathfrak{a}(x) \in \bigcap_{v \in R(w)} (\mathfrak{b}(\beta))(v)$ and $\exists v \in R(w)\,[(\mathfrak{a}(A))(v) \nsubseteq (\mathfrak{b}(\beta))(v)]$. Thus $\mathfrak{a}, w \nvDash A \ \mathsf{Ess}\ x$, by virtue of our useful equivalence and this finishes the proof of (ii).

Our proof of (iii) also starts with a computation:

$$\begin{aligned} \mathfrak{a}, w \models \mathsf{NE}(x) \ \text{ iff } & \ \mathfrak{a}, w \models \forall\alpha[\alpha \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)] \\ \text{iff } & \ \forall\mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{b}, w \models \alpha \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)\big] \\ \text{iff } & \ \forall\mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{b}, w \models \alpha \ \mathsf{Ess}\ x \Longrightarrow \mathfrak{b}, w \models \mathsf{L}\exists y\alpha(y)\big] \end{aligned}$$

Note that

$$\begin{aligned} \mathfrak{b}, w \models \mathsf{L}\exists y\alpha(y) \ \text{ iff } & \ \forall v \in R(w)[\mathfrak{b}, v \models \exists y\alpha(y)] \\ \text{iff } & \ \forall v \in R(w)\ \exists\mathfrak{c} \in \{\mathfrak{b}^?_y\}[\mathfrak{c}, v \models \alpha(y)] \\ \text{iff } & \ \forall v \in R(w)\ \exists\mathfrak{c} \in \{\mathfrak{b}^?_y\}[\mathfrak{c}(y) \in (\mathfrak{c}(\alpha))(v)] \\ \text{iff } & \ \forall v \in R(w)\ \exists o \in D_1[o \in (\mathfrak{b}(\alpha))(v)] \\ \text{iff } & \ \forall v \in R(w)\ (\mathfrak{b}(\alpha))(v) \neq \emptyset \end{aligned}$$

Thus we obtain:

$$\begin{aligned} \mathfrak{a}, w \models \mathsf{NE}(x) \ \text{ iff } & \ \forall\mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\mathfrak{b}, w \models \alpha \ \mathsf{Ess}\ x \Longrightarrow \forall v \in R(w)\ (\mathfrak{b}(\alpha))(v) \neq \emptyset\big] \\ \text{iff } & \ \forall\mathfrak{b} \in \{\mathfrak{a}^?_\alpha\}\big[\forall v \in R(w)\ (\mathfrak{b}(\alpha))(v) = \{\mathfrak{b}(x)\} \Longrightarrow \forall v \in R(w)\ (\mathfrak{b}(\alpha))(v) \neq \emptyset\big] \end{aligned}$$

which proves (iii) because the condition above is true. ■

The following corollary reveals an unexpected feature of Anderson's definition of necessary existence.

Corollary 2.5. *For every model structure $\mathfrak{W}$ of any kind (ordinary or special) $\mathfrak{W} \models \forall x\mathsf{NE}(x)$.*

Another peculiarity of Anderson's definition of necessary existence ($\mathsf{NE}(x) \stackrel{\text{df}}{=} \forall\alpha[\alpha \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)]$) is caused by implicational form of the defining phrase. Indeed, the definition yields that *every object having no essence at all necessarily exists*. This is quite harmless if one is willing to adopt a global assumption that *every object of the 1^{st} sort has an essence* (symb. $\forall x\exists\alpha[\alpha \ \mathsf{Ess}\ x]$) but without this global assumption the implication $\forall x\big[\neg\exists\alpha[\alpha \ \mathsf{Ess}\ x] \to \mathsf{NE}(x)\big]$ sounds strange — why should lack of essence be rewarded with necessary existence? To make the concept of necessary existence independent of global assumptions of that kind one could try to modify Anderson's definition, for example by putting: $\mathsf{NE}^+(x) \stackrel{\text{df}}{=} \exists\alpha[\alpha \ \mathsf{Ess}\ x] \wedge \forall\alpha[\alpha \ \mathsf{Ess}\ x \rightarrow \mathsf{L}\exists y\alpha(y)]$. In this paper, however, we will not pursue this this topic any further.

Now, we will define certain classes of model structures which will play the role of semantical counterparts of Anderson-like theories. Each class will be affixed the same acronym as its corresponding Anderson-like theory, however, the symbols: $\mathbf{C}^A$, **5**, **b**, **c**, **d**, **n**, **s** and $\star$ will be interpreted in a different manner according to the following simple rules:

(**A**) The first symbol of an acronym i.e. $\mathbf{C}^A$ stands for the class of all model structures which subsequently undergo restrictions forced by successive symbols of acronym;

(**5**) The symbol **5** in an acronym indicates that model structures in the class are *Euclidean* i.e. they obey the condition: if wRv and wRv_1 then vRv_1, for every $w, v, v_1 \in W$;

(**b**) The symbol **b** indicates that model structures in the class are *symmetric* i.e. if wRv then vRw, for every $w, v \in W$;

(**c**) The symbol **c** indicates that model structures in the class are *Euclidean* and *symmetric*;

(**d**) The symbol **d** indicates that model structures in the class are *Euclidean* and *transitive* i.e. if wRv and vRv_1 then wRv_1, for every $w, v, v_1 \in W$;

(**n**) The symbol **n** indicates that model structures in the class obey the following condition: $D_1 \in D_w$ for every $w \in W$;

(**s**) The symbol **s** indicates that model structures in the class *have singletons* i.e: $\{a\} \in D_w$, for every $a \in D_1$ and every $w \in W$;

($\star$) If an acronym ends with $\star$ then all model structures in the class are required to be *special*.

3. Completeness theorems

Let X be a set of formulas and **Th** be of one of the Anderson-like theories. We shall say that X is **Th**-*consistent* if there exists no formula $\boldsymbol{\phi}$ such that both $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ and $X \vdash_{\mathbf{Th}} \neg\boldsymbol{\phi}$, **Th**-*inconsistent* otherwise. We declare X *maximally* **Th**-*consistent* if it is **Th**-consistent and for any formula $\boldsymbol{\phi}$ that does not belong to X, $X \cup \{\boldsymbol{\phi}\}$ is **Th**-inconsistent. X is $\boldsymbol{\omega}$-**Th**-*complete* if for all formulas $\boldsymbol{\phi}(\xi)$: if $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}(\xi/A)$ for all terms A of the same sort as ξ, then $X \vdash_{\mathbf{Th}} \forall\xi\boldsymbol{\phi}(\xi)$.

Lemma 3.1. *Let X be a maximally **Th**-consistent set of formulas. Then:*

(i) $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ *iff* $\boldsymbol{\phi}$ *belongs to X,*

(ii) $\neg\boldsymbol{\phi}$ *belongs to X iff* $\boldsymbol{\phi}$ *does not belong to X,*

(iii) $\boldsymbol{\phi} \wedge \boldsymbol{\psi}$ *belongs to X iff both* $\boldsymbol{\phi}$ *and* $\boldsymbol{\psi}$ *belong to X,*

(iv) *If* $\mathsf{L}(x \overset{1}{\approx} y)$ *belongs to X, then* $\boldsymbol{\phi}(z/x)$ *belongs to X iff* $\boldsymbol{\phi}(z/y)$ *belongs to X.*

Proof. By an easy verification. ■

Lemma 3.2. *Let X be a maximally **Th**-consistent and $\boldsymbol{\omega}$-**Th**-complete set of formulas. Then,* $\forall\xi\boldsymbol{\phi}(\xi)$ *belongs to X iff* $\boldsymbol{\phi}(\xi/A)$ *does for every term A of the same sort as* ξ*,*

Proof. By an easy verification. ■

Lemma 3.3. *Let X be a maximally **Th**-consistent set of formulas. Then, X is $\boldsymbol{\omega}$-**Th**-complete iff for all formulas* $\boldsymbol{\phi}$*: if* $\exists\xi\boldsymbol{\phi}(\xi)$ *belongs to X, then for some term A of the same sort as* ξ *also* $\boldsymbol{\phi}(\xi/A)$ *does.*

Proof. Let X be a maximally **Th**-consistent set of formulas. In addition suppose that $\exists\xi\boldsymbol{\phi}(\xi)$ belongs to X. Hence, by maximal **Th**-consistency of X, $\forall\xi\neg\boldsymbol{\phi}(\xi)$ does not belong to X. Therefore, by Lemma 3.2, there exists a term A of the same sort as ξ such that $\neg\boldsymbol{\phi}(\xi/A)$ does not belong to X. Consequently, again on the strength of the maximal **Th**-consistency of X, there exists a term A such that $\boldsymbol{\phi}(\xi/A)$ belongs to X . Conversely, let $\boldsymbol{\phi}(\xi/A)$ belongs to X for some term A. It is obvious that $X \vdash_{\mathbf{Th}} \boldsymbol{\phi}(\xi/A) \to \exists\boldsymbol{\phi}(\xi)$. So, by Lemma 3.1(i) and (iii), $\exists\boldsymbol{\phi}(\xi)$ belongs to X. ■

Lemma 3.4. *Let X be an $\boldsymbol{\omega}$-**Th**-complete set of formulas. Then, for any formula* $\boldsymbol{\phi}$, $X \cup \{\boldsymbol{\phi}\}$ *is also $\boldsymbol{\omega}$-**Th**-complete.*

Proof. Suppose that $\boldsymbol{\phi}$ and $\forall\xi\boldsymbol{\psi}(\xi)$ are arbitrary formulas that such $X \cup \{\boldsymbol{\phi}\} \vdash_{\mathbf{Th}} \boldsymbol{\psi}(\xi/A)$ for every term A of the same sort as ξ and $X \cup \{\boldsymbol{\phi}\} \nvdash_{\mathbf{Th}} \forall\xi\boldsymbol{\psi}(\xi)$. And let ρ be a variable foreign to both $\boldsymbol{\phi}$ and $\forall\xi\boldsymbol{\psi}(\xi)$. So, by Proposition 2.1(i), $X \vdash_{\mathbf{Th}} \boldsymbol{\phi} \to \boldsymbol{\psi}(\xi/A)$ for every term A of the same sort as ξ. Because of the hypothesis on ρ, $\boldsymbol{\phi}(\rho/A)$ is the same as $\boldsymbol{\phi}$ and $\boldsymbol{\psi}(\xi/\rho)(\rho/A)$ is the same as $\boldsymbol{\psi}(\xi/A)$. And obviously, $(\boldsymbol{\phi} \to \boldsymbol{\psi}(\xi/\rho))(\rho/A)$ is the same as $\boldsymbol{\phi} \to \boldsymbol{\psi}(\xi/A)$. Thus, $X \vdash_{\mathbf{Th}} (\boldsymbol{\phi} \to \boldsymbol{\psi}(\xi/\rho))(\rho/A)$ for every term A . From this latter, by the $\boldsymbol{\omega}$-**Th**-completeness of X, we obtain that $X \vdash_{\mathbf{Th}} \forall\rho(\boldsymbol{\phi} \to \boldsymbol{\psi}(\xi/\rho))$, therefore also $X \vdash_{\mathbf{Th}} \boldsymbol{\phi} \to \forall\rho\boldsymbol{\psi}(\xi/\rho)$. By applying to the latter Proposition 2.1(i) we have $X \cup \{\boldsymbol{\phi}\} \vdash_{\mathbf{Th}} \forall\rho\boldsymbol{\psi}(\xi/\rho)$, and consequently, $X \cup \{\boldsymbol{\phi}\} \vdash_{\mathbf{Th}} \forall\xi\boldsymbol{\psi}(\xi)$ — a contradiction. ■

LEMMA 3.5. *Let X be an* $\boldsymbol{\omega}$-**Th**-*complete set of formulas. Then, for any formula* $\exists\xi\boldsymbol{\phi}(\xi)$, *if* $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$ *is* **Th**-*consistent, then there exists a term* A *of the same sort as* ξ *such that* $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \nvdash_{\mathbf{Th}} \neg\boldsymbol{\phi}(\xi/A)$

PROOF. Suppose that $\exists\xi\boldsymbol{\phi}(\xi)$ is a formula such that $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$ is **Th**-consistent, where X is $\boldsymbol{\omega}$-**Th**-complete. So, by Lemma 3.4, $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$ is also $\boldsymbol{\omega}$-**Th**-complete. Hence, if it were the case that $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \neg\boldsymbol{\phi}(\xi/A)$ for every term A of the same sort as ξ, then we would have $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \forall\xi\neg\boldsymbol{\phi}(\xi)$, i.e. $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \neg\exists\xi\boldsymbol{\phi}(\xi)$ So, $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$ would be **Th**-inconsistent — a contradiction. Therefore, there exists a term A of the same sort as ξ such that $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \nvdash_{\mathbf{Th}} \neg\boldsymbol{\phi}(\xi/A)$. ∎

LEMMA 3.6. *Let X be a set of formulas. If* $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$ *is* **Th**-*consistent, then so is* $X \cup \{\exists\xi\boldsymbol{\phi}(\xi), \boldsymbol{\phi}(\xi/A)\}$ *for any term A of the same sort as* ξ *foreign to* $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$.

PROOF. Suppose that $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$ is **Th**-consistent and for some term A of the same sort as ξ foreign to $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$, $X \cup \{\exists\xi\boldsymbol{\phi}(\xi), \boldsymbol{\phi}(\xi/A)\}$ is **Th**-inconsistent. So, $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \neg\boldsymbol{\phi}(\xi/A)$ and owing to Proposition 2.1(i), $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \exists\xi\boldsymbol{\phi}(\xi) \to \neg\boldsymbol{\phi}(\xi/A)$. Because A is foreign to $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\}$, then $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \exists\xi\boldsymbol{\phi}(\xi) \to \forall\xi\neg\boldsymbol{\phi}(\xi/A)$. Finally, $X \cup \{\exists\xi\boldsymbol{\phi}(\xi)\} \vdash_{\mathbf{Th}} \neg\exists\xi\boldsymbol{\phi}(\xi)$, which proves that X is **Th**-inconsistent — a contradiction. ∎

We shall say that a set X of formulas is *infinitely extendable* if infinitely many variables of the 1^{st} sort and infinitely many variables of the 2^{nd} sort are foreign to X.

In the preamble to the next two definitions we assume that all formulas have been arranged in some denumerable sequence: $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_i, \ldots$. We suppose also that some particular enumerations are fixed on so that we may speak of the 1^{st}, $2^{nd}, \ldots, i^{th}, \ldots$ variable of the 1^{st} or 2^{nd} sort, respectively.

Let X be a infinitely extendable set of formulas. By *the normal* **Th** *-extension of* X we shall understand the union $\bigcup_{i \geq 0} X_i$, where X_0 is equal to X and for every $i \geq 1$:

(i) X_i is equal to X_{i-1}, when $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$ is **Th**-inconsistent,

(ii) X_i is equal to $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$, when $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$ is **Th**-consistent and $\boldsymbol{\phi}_i$ is not of the sort $\exists\xi\boldsymbol{\psi}$,

(iii) X_i is equal to $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}, \boldsymbol{\psi}(\xi/\zeta\}$ where ζ is the alphabetically earliest variable of the same sort as ξ foreign to $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}\}$, when $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$ is **Th**-consistent and $\boldsymbol{\phi}_i$ is $\exists\xi\boldsymbol{\psi}$.

Now, let X be an $\boldsymbol{\omega}$-**Th**-complete set of formulas. By the *special* **Th** *-extension of* X we shall understand the union $\bigcup_{i\geq 0} X_i$, where X_0 is equal to X and for every $i \geq 1$ the sets X_i are defined as before by (i) and (ii) but with (iii) as follows:

(iii) X_i is equal to $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}, \boldsymbol{\psi}(\xi/\zeta)\}$ where ζ is the alphabetically earliest variable of the same sort as ξ such that $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}\} \nvdash_{\mathbf{Th}} \neg\boldsymbol{\psi}(\xi/\zeta)$, when $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$ is **Th**-consistent and $\boldsymbol{\phi}_i$ is $\exists\xi\boldsymbol{\psi}$.

LEMMA 3.7. *Let X be an infinitely extendable set of formulas. If X is* ***Th****-consistent, then the normal* ***Th****-extension of X is both maximally* ***Th****-consistent and* $\boldsymbol{\omega}$*-****Th****-complete.*

PROOF. Let $X_\infty = \bigcup_{i\geq 0} X_i$ be the normal **Th**-extension of X. We shall only prove that:

(•) For each i, $i > 0$, the set X_i is **Th**- consistent; and

(••) X_∞ is $\boldsymbol{\omega}$-**Th**-complete.

The maximal **Th**-consistency of X_∞ is the easily provable result.

The proof of (•) proceeds by induction with respect to i, $i \geq 0$. In view of the assumptions the set X_0 is **Th**-consistent, according to the induction step assume that X_{i-1}, $i > 0$, is **Th**-consistent. If X_i were defined as in (i), then it would be the same as X_{i-1}, and hence — **Th**-consistent. If X_i were defined as in (ii),then it would be the same as $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$, and $X_{i-1} \cup \{\boldsymbol{\phi}_i\}$ is **Th**-consistent. Finally, suppose that X_i is defined as in (iii). Then X_i is the same as $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}, \boldsymbol{\psi}(\xi/\zeta)\}$. If X_i were **Th**-inconsistent, then we would have $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}\} \vdash_{\mathbf{Th}} \neg\boldsymbol{\psi}(\xi/\zeta)$. Since ζ is foreign to $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}\}$, then also $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}\} \vdash_{\mathbf{Th}} \forall\xi\neg\boldsymbol{\psi}(\xi)$ would hold. Thus the set $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}\}$ would be **Th**-inconsistent, but this is not possible by the instructions for choosing $\exists\xi\boldsymbol{\psi}$. Therefore, the set X_i is **Th**-consistent, which completes the proof of (•).

For proof of (••), suppose that $X_\infty \nvdash_{\mathbf{Th}} \forall\xi\boldsymbol{\psi}(\xi)$. Hence, by Lemma 3.1(i), it follows that $\forall\xi\boldsymbol{\psi}(\xi)$ does not belong to X_∞. Therefore, on the basis of Lemma 3.1(ii), $\exists\xi\neg\boldsymbol{\psi}(\xi)$ belongs to X_∞. Let $\exists\xi\neg\boldsymbol{\psi}(\xi)$ be the i^{th} formula in the denumerable sequence: $\boldsymbol{\phi}_1, \boldsymbol{\phi}_2, \ldots, \boldsymbol{\phi}_i, \ldots$. Suppose now to the contrary that the set $X_{i-1} \cup \{\exists\xi\neg\boldsymbol{\psi}(\xi)\}$ is **Th**-inconsistent. Then $X_{i-1} \vdash_{\mathbf{Th}} \forall\xi\boldsymbol{\psi}(\xi)$, and further $X_\infty \vdash_{\mathbf{Th}} \forall\xi\boldsymbol{\psi}(\xi)$ — contradiction. But, if $X_{i-1} \cup \{\exists\xi\neg\boldsymbol{\psi}(\xi)\}$ is **Th**-consistent, then X_i is the same as $X_{i-1} \cup \{\exists\xi\neg\boldsymbol{\psi}(\xi), \neg\boldsymbol{\psi}(\xi/\zeta)\}$, where ζ is the alphabetically earliest variable of the same sort as ξ foreign to $X_{i-1} \cup \{\exists\xi\neg\boldsymbol{\psi}(\xi)\}$. So, $\neg\boldsymbol{\psi}(\xi/\zeta)$ belongs to X_i, and consequently — $\neg\boldsymbol{\psi}(\xi/\zeta)$ belongs to X_∞. From this, by the maximal **Th**-consistency of X_∞, it follows that $X_\infty \vdash_{\mathbf{Th}} \neg\boldsymbol{\psi}(\xi/\zeta)$, and obviously — $X_\infty \nvdash_{\mathbf{Th}} \boldsymbol{\psi}(\xi/\zeta)$, which completes the proof of (••). ■

LEMMA 3.8. *Let X be a* ***Th****-consistent and* $\boldsymbol{\omega}$*-****Th****-complete set of formulas. Then the special* ***Th****-extension of X is both maximally* ***Th****-consistent and* $\boldsymbol{\omega}$*-****Th****-complete.*

PROOF. Suppose this time that $X_\infty = \bigcup_{i\geq 0} X_i$ is the special **Th**-extension of X. We will omit as well here the proof that X_∞ is maximally **Th**-consistent. Returning to our previous reasoning under Lemma 3.7, in the proofs of (•) and (••) we shall consider only the points in which they differ.

For the proof of (•), suppose that X_i is defined as in (iii). Then X_i is the same as $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}(\xi), \boldsymbol{\psi}(\xi/\zeta)\}$. If X_i were **Th**-inconsistent, then we would have $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}(\xi)\} \vdash_{\mathbf{Th}} \neg\boldsymbol{\psi}(\xi/\zeta)$, but this is not possible by the instructions for choosing ζ. Therefore, X_i is **Th**-consistent.

For the proof of (••), suppose in the induction step that we have already shown the **Th**-consistency of $X_{i-1} \cup \{\exists\xi\neg\boldsymbol{\psi}(\xi)\}$. Thus X_i is the same as $X_{i-1} \cup \{\exists\xi\neg\boldsymbol{\psi}(\xi), \neg\boldsymbol{\psi}(\xi/\zeta)\}$, where ζ is the alphabetically earliest variable such that $X_{i-1} \cup \{\exists\xi\boldsymbol{\psi}(\xi)\} \nvdash_{\mathbf{Th}} \boldsymbol{\psi}(\xi/\zeta)$. Therefore, X_i is **Th**-consistent, and trivially $\neg\boldsymbol{\psi}(\xi/\zeta)$ belongs to X_∞ . The further reasoning is as in the proof of Lemma 3.7. ■

LEMMA 3.9. *Let X be a maximally* ***Th****-consistent and* $\boldsymbol{\omega}$*-****Th****-complete set of formulas. Then the set $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\}$, where $\mathsf{M}\boldsymbol{\psi} \in X$, is* $\boldsymbol{\omega}$*-****Th****-complete.*

PROOF. Let the assumptions of the lemma be satisfied. It can be proven, by applying the axiom (2.20), Proposition 2.1(i) and Lemma 3.1(i),that for any formula $\boldsymbol{\psi}$, $\mathsf{M}\boldsymbol{\psi} \in X$ or $\mathsf{M}\neg\boldsymbol{\psi} \in X$. Let $\gamma(\xi)$ be a formula such that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \vdash_{\mathbf{Th}} \gamma(\xi/\zeta)$ for every variable ζ. After applying Proposition 2.1(ii), this gives $X \vdash_{\mathbf{Th}} \mathsf{L}\gamma(\xi/\zeta)$ for every variable ζ. So, by the $\boldsymbol{\omega}$-**Th**-completeness of X, we obtain that $X \vdash_{\mathbf{Th}} \forall\xi\mathsf{L}\gamma(\xi)$. Now, in view of the Barcan formula of the 1^{st} or of the 2^{nd} sort — according to the order of the variable ξ — it follows that $X \vdash_{\mathbf{Th}} \mathsf{L}\forall\xi\gamma(\xi)$. Further, on the strength of the maximal **Th**-consistency of X and Lemma 3.1(i), we have $\mathsf{L}\forall\xi\gamma(\xi) \in X$, and consequently, $\forall\xi\gamma(\xi) \in \{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\}$. Thus we have shown that the set $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\}$ is $\boldsymbol{\omega}$-**Th**-complete. Finally, on the basis of Lemma 3.4, the set $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\}$ is also $\boldsymbol{\omega}$-**Th**-complete. ■

By a *order-preserving function on the variables* we shall understand any function $\Re$ which pairs with each variable of the 1^{st} sort a variable of the 1^{st} sort and with each variable of the 2^{nd} sort a variable of the 2^{nd} sort.

For any formula $\boldsymbol{\phi}$, by $\Re(\boldsymbol{\phi})$ we shall understand the formula $\boldsymbol{\phi}(x_1/\Re(x_1), x_2/\Re(x_2), \ldots, x_i/\Re(x_i))$, where $\Re$ is a order-preserving function on the variables and $x_1, x_2, \ldots, x_i$ are all variables of the 1^{st} and 2^{nd} sorts occurring free in $\boldsymbol{\phi}$. And for any set X of formulas, $\Re(X) = \{\Re(\boldsymbol{\phi}) \mid \boldsymbol{\phi} \in X\}$. We shall say that the sets of formulas X,Y are *isomorphic* if for some one-one and order-preserving function on the variables $\Re$, $\Re(X) = Y$.

Lemma 3.10. *Let $\Re$ be one-one and order-preserving function on the variables. Let X be a set of formulas. Then:*

(i) *If X is a* ***Th****-consistent, then so is $\Re(X)$,*

(ii) *For each $\boldsymbol{\phi} \in X$, each $w \in W$ and every model $\langle \mathfrak{W}, \mathfrak{a} \rangle$, if $\mathfrak{a}, w \models \Re(\boldsymbol{\phi})$, then $\mathfrak{a}(\Re), w \models \boldsymbol{\phi}$.*

Proof. For the proof of (i), suppose that X is **Th**-consistent and $\Re(X)$ is **Th**-inconsistent. Therefore, there exists a formula $\boldsymbol{\phi}$ such that both $\Re(X) \vdash_{\mathbf{Th}} \boldsymbol{\phi}$ and $\Re(X) \vdash_{\mathbf{Th}} \neg\boldsymbol{\phi}$ hold. Let $\Re(\boldsymbol{\phi})$ be some member of $\Re(X)$. It can easily be proven that both $\Re(X) \vdash_{\mathbf{Th}} \Re(\boldsymbol{\phi})$ and $\Re(X) \vdash_{\mathbf{Th}} \neg\Re(\boldsymbol{\phi})$ hold. Hence it follows that X is **Th**-inconsistent — a contradiction.

Next, (ii) can be verified by an easy induction with respect to the length of $\boldsymbol{\phi}$. ■

Let X be a maximally **Th**-consistent and $\boldsymbol{\omega}$-**Th**-complete set of formulas. By a *subordinate of* X we shall mean any special **Th**-extension of $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\}$, for some $\mathsf{M}\boldsymbol{\psi} \in X$.

Lemma 3.11. *Let X be a maximally* ***Th****-consistent and $\boldsymbol{\omega}$-****Th****-complete set of formulas. Then every subordinate of X is both maximally* ***Th****-consistent and $\boldsymbol{\omega}$-****Th****-complete.*

Proof. Let the assumptions of the lemma be satisfied and let $\boldsymbol{w}$ be the subordinate of X such that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\} \subseteq \boldsymbol{w}$ for some $\mathsf{M}\boldsymbol{\psi} \in X$. Moreover, suppose that the set $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\}$ is **Th**-inconsistent. Hence, it can be easily proved that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \vdash_{\mathbf{Th}} \neg\boldsymbol{\psi}$. But then, on the basis of Proposition 2.1(ii), $X \vdash_{\mathbf{Th}} \mathsf{L}\neg\boldsymbol{\psi}$, i.e. $X \vdash_{\mathbf{Th}} \neg\mathsf{M}\boldsymbol{\psi}$. Since X is maximally **Th**-consistent, then according to Lemma 3.1(i), $\neg\mathsf{M}\boldsymbol{\psi} \in X$ — a contradiction. Therefore, we conclude that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\}$ is **Th**-consistent. Further, since the assumptions of Lemma 3.9 are satisfied, the set $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in X\} \cup \{\boldsymbol{\psi}\}$ is $\boldsymbol{\omega}$-**Th**-complete. Consequently, by applying to latter Lemma 3.8 to the latter, $\boldsymbol{w}$ is both maximally **Th**-consistent and $\boldsymbol{\omega}$-**Th**-complete. ■

We shall work with a family of sets, say $\boldsymbol{W}$, consisting of the normal **Th**-extension $\boldsymbol{w_1}$ of a certain **Th**-consistent set X, the subordinates (if any) of $\boldsymbol{w_1}$, the subordinates (if any) of the subordinates of $\boldsymbol{w_1}$, etc.

The members of $\boldsymbol{W}$ will be ordered in four following steps:

Step 1 First, we assign a $r$ank to each $\boldsymbol{w} \in \boldsymbol{W}$ (*rank* ($\boldsymbol{w}$) for short). This done, we declare $r$ank ($\boldsymbol{w_1}$) $= 1$; $r$ank ($\boldsymbol{v}$) $= r$ank ($\boldsymbol{w}$) $+ 1$ if $\boldsymbol{v}$ is a subordinate of $\boldsymbol{w}$ and $\boldsymbol{v}$ has not yet got a *rank* ($\boldsymbol{v}$) $\leq r$ank ($\boldsymbol{w}$).

Step 2 Second, we order the various subordinates of each $\boldsymbol{w} \in \boldsymbol{W}$ that has at least two attendants. So, suppose $\boldsymbol{w''}$ and $\boldsymbol{w'''}$ are distinct subordinates of $\boldsymbol{w'}$. Let $\boldsymbol{w''}$ be the special **Th**-extension of $\{\phi \mid \mathsf{L}\phi \in \boldsymbol{w'}\} \cup \{\psi''\}$ and $\boldsymbol{w'''}$ be that of $\{\phi \mid \mathsf{L}\phi \in \boldsymbol{w'}\} \cup \{\psi'''\}$, where ψ'' and ψ''' are distinct formulas such that both $\mathsf{M}\psi''$ and $\mathsf{M}\psi'''$ belong to $\boldsymbol{w'}$. Then $\boldsymbol{w''}$ is to *precede* or *follow* $\boldsymbol{w'''}$ according as to whether ψ'' precedes or follows ψ'''.

Step 3 We partition $\boldsymbol{W}$ into cells $\boldsymbol{W^1}$, $\boldsymbol{W^2}$,..., $\boldsymbol{W^r}$,..., consisting for each r, $r \geq 1$, of the members of $\boldsymbol{W}$ of rank r, and next we order the members of each cell. If $\boldsymbol{W^r}$ has exactly one member, we declare it *the first member of* $\boldsymbol{W^r}$. Otherwise, we employ the following inductive procedure:

Case 1 $r = 2$. Then the members of

Case 2 $r > 2$. Given any two members of $\boldsymbol{W^r}$, one — call it $\boldsymbol{w'}$, — is sure to be for some j' and k', the k'^{th} subordinates of the j'^{th} member of $\boldsymbol{W^{r-1}}$, and the other — call it $\boldsymbol{w''}$ — is sure to be for some j'' and k'', the k''^{th} subordinates of the j''^{th} member of $\boldsymbol{W^{r-1}}$. Then $\boldsymbol{w'}$ *will precede* $\boldsymbol{w''}$ in $\boldsymbol{W^r}$ if $j'+k' < j''+k''$ or, when $j' + k' = j'' + k''$ and $j' < j''$; otherwise, $\boldsymbol{w'}$ *will follow* $\boldsymbol{w''}$ in $\boldsymbol{W^r}$.

Step 4 We now order the members of $\boldsymbol{W}$ in a single run:

(i) $\boldsymbol{w_1}$, the one member of $\boldsymbol{W^1}$, is to $p$recede all other members of $\boldsymbol{W}$;

(ii) $\boldsymbol{w'}$ being the j'^{th} member of $\boldsymbol{W^{r'}}$ ($r' > 1$), and $\boldsymbol{w''}$ the j''^{th} member of $\boldsymbol{W^{r''}}$ ($r'' > 1$), $\boldsymbol{w'}$ is to *precede* $\boldsymbol{w''}$ if $j'+r' < j''+r''$ or, when $j' + r' = j'' + r''$ and $r' < r''$; otherwise, $\boldsymbol{w'}$ *follows* $\boldsymbol{w''}$.

We are now in a position to prove strong completeness theorems.

THEOREM 3.12 (Strong completeness). *Let X be a set of formulas. Then:*

(i) $X \models_{\boldsymbol{C^A 5}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5}} \boldsymbol{\phi}$,
(ii) $X \models_{\boldsymbol{C^A b}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A b}} \boldsymbol{\phi}$,
(iii) $X \models_{\boldsymbol{C^A c}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A c}} \boldsymbol{\phi}$,
(iv) $X \models_{\boldsymbol{C^A 5s}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5s}} \boldsymbol{\phi}$,
(v) $X \models_{\boldsymbol{C^A bs}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A bs}} \boldsymbol{\phi}$,
(vi) $X \models_{\boldsymbol{C^A cs}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A cs}} \boldsymbol{\phi}$,
(vii) $X \models_{\boldsymbol{C^A 5n}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5n}} \boldsymbol{\phi}$,
(viii) $X \models_{\boldsymbol{C^A bn}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A bn}} \boldsymbol{\phi}$,
(ix) $X \models_{\boldsymbol{C^A cn}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A cn}} \boldsymbol{\phi}$,
(x) $X \models_{\boldsymbol{C^A 5ns}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5ns}} \boldsymbol{\phi}$,
(xi) $X \models_{\boldsymbol{C^A bns}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A bns}} \boldsymbol{\phi}$,
(xii) $X \models_{\boldsymbol{C^A cns}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A cns}} \boldsymbol{\phi}$.

PROOF.
PROOF OF (i):
We consider only the non-trivial case, when $X \nvdash_{\mathbf{C^A 5}} \boldsymbol{\phi}$. Hence, $X \cup \{\neg\boldsymbol{\phi}\}$ is $\mathbf{C}^A\mathbf{5}$-consistent. Our aim is to find a model $\langle \mathfrak{W}, \mathfrak{a} \rangle$, where $\mathfrak{W} \in \mathbf{C}^A\mathbf{5}$, such that for every $\boldsymbol{\psi} \in X \cup \{\neg\boldsymbol{\phi}\}$ and for each $w \in W$, $\mathfrak{W}, \mathfrak{a}, w \models \boldsymbol{\psi}$. The proof is organized in three parts:

A. Construction of the frame $\langle \boldsymbol{W}, \boldsymbol{R} \rangle$,

B. Introduction of the 1^{st} and 2^{nd} sort domains,

C. Proof of the *Truth Lemma*.

Step A:

Having the instrument of subordinates we define by induction with respect to n, $n \geqslant 1$, the members $\boldsymbol{w_n}$ of a set $\boldsymbol{W}$:

Basis: n = 1.
In the case $X \cup \{\neg\boldsymbol{\phi}\}$ is infinitely extendable, we put $\boldsymbol{w_1}$ to be the normal $\mathbf{C}^A\mathbf{5}$-extension $\bigcup_{i \geq 0} X_i$, where X_0 is the set $X \cup \{\neg\boldsymbol{\phi}\}$. On the strength of Lemma 3.7, $\boldsymbol{w_1}$ is both maximally $\mathbf{C}^A\mathbf{5}$-consistent and $\boldsymbol{\omega}$-$\mathbf{C}^A\mathbf{5}$-complete.

In the case $X \cup \{\neg\boldsymbol{\phi}\}$ is not infinitely extendable, let $\Re$ be the function on the variables of the 1^{st} and 2^{nd} sort such that, for each $i \geq 1$, $\Re(x_i) = x_{2i}$ and $\Re(\alpha_i) = \alpha_{2i}$. Clearly, $\Re$ is one-one and order-preserving function on the variables, and the sets $X \cup \{\neg\boldsymbol{\phi}\}$ and $\Re(X \cup \{\neg\boldsymbol{\phi}\})$ are isomorphic. But, because all 1^{st} sort variables $x_1, x_3, \ldots, x_{2i-1}, \ldots$ and all 2^{nd} sort variables $\alpha_1, \alpha_3, \ldots, \alpha_{2i-1}, \ldots$ are foreign to $\Re(X \cup \{\neg\boldsymbol{\phi}\})$, therefore $\Re(X \cup \{\neg\boldsymbol{\phi}\})$ is infinitely extendable. Now, we put $\boldsymbol{w_1}$ to be the normal $\mathbf{C}^A\mathbf{5}$-extension of $\Re(X \cup \{\neg\boldsymbol{\phi}\})$. It can easily be seen that $\Re(X \cup \{\neg\boldsymbol{\phi}\})$ is $\mathbf{C}^A\mathbf{5}$-consistent, therefore and here $\boldsymbol{w_1}$ is maximally $\mathbf{C}^A\mathbf{5}$-consistent and $\boldsymbol{\omega}$-$\mathbf{C}^A\mathbf{5}$-complete.

Inductive step: $n > 1$.
Let us suppose for induction that the set $\boldsymbol{w_n}$ is already defined. Thus, there exist parameters $j \geq 1$ and $r \geq 2$ such that $\boldsymbol{w_n}$ is the j^{th} member of the $\boldsymbol{W^r}$. For each i, $2 \leq i < r + j$, we next put

$$\boldsymbol{V^i} = \boldsymbol{W^i} - \{\boldsymbol{v} \mid \boldsymbol{v} \in \boldsymbol{W^i} \text{ and } \boldsymbol{v} \text{ precedes or equals } \boldsymbol{w_n}\}, \textit{ and}$$
$$\boldsymbol{V} = \{\boldsymbol{v} \mid \boldsymbol{v} \text{ is the first member of some } \boldsymbol{V^i},\ 2 \leq i < r + j\}.$$

In the case $\boldsymbol{V} = \emptyset$, $\boldsymbol{w_n}$ is the last member of $\boldsymbol{W}$. Supposing then that $\boldsymbol{V} \neq \emptyset$, we define $\boldsymbol{w_{n+1}}$ to be the first member of $\boldsymbol{V}$. It is easily shown, when $\boldsymbol{w_n}$ is not the last member of $\boldsymbol{W}$, that there not exist a member of $\boldsymbol{W}$ which follows $\boldsymbol{w_n}$ and precedes $\boldsymbol{w_{n+1}}$.

We define now the accessibility relation $\boldsymbol{R}$ on $\boldsymbol{W}$:

(**R**) For every $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W}$, $\boldsymbol{wRv}$ if and only $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \cup \{\boldsymbol{\psi}\} \subseteq \boldsymbol{v}$ for some $\mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w}$.

And we can then prove that

(•) For every formula $\boldsymbol{\phi}$ and all $\boldsymbol{w} \in \boldsymbol{W}$, $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$ if and only if $\boldsymbol{\phi} \in \boldsymbol{v}$ for each $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$.

Let $\boldsymbol{\phi}$ be any formula and $\boldsymbol{w}$ any member of $\boldsymbol{W}$. We leave it to the reader to verify that for every axiom $\boldsymbol{\phi}$ of $\mathbf{C}^A\mathbf{5}$, $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{M}\boldsymbol{\phi}$. Hence, trivially, by the construction of $\boldsymbol{w}$, $\boldsymbol{w}$ has members of the sort $\mathsf{M}\boldsymbol{\psi}$. And therefore, if $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$, then by (**R**), $\boldsymbol{\phi} \in \boldsymbol{v}$ for each $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$. Suppose, on the other hand, that $\boldsymbol{\phi} \in \boldsymbol{v}$ for each $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$, and let $\mathsf{L}\boldsymbol{\phi} \notin \boldsymbol{w}$. Because $\boldsymbol{w}$ is maximally $\mathbf{C}^A\mathbf{5}$-consistent and $\boldsymbol{\omega}$-$\mathbf{C}^A\mathbf{5}$-complete, then with respect to Lemma 3.1(ii), $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{w}$. Hence, by the construction of members of $\boldsymbol{W}$, there exists $\boldsymbol{v} \in \boldsymbol{W}$ such that $\neg\boldsymbol{\phi} \in \boldsymbol{v}$, which contradicts the assumption that $\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{v}$ is $\mathbf{C}^A\mathbf{5}$-consistent.

To prove that the relation $\boldsymbol{R}$ is serial, let us again note that each $\boldsymbol{w} \in \boldsymbol{W}$ has members of the sort $\mathsf{M}\boldsymbol{\psi}$. Consequently, by our construction of members of $\boldsymbol{W}$, for each $\boldsymbol{w} \in \boldsymbol{W}$ there exists $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$.

Finally, we must also require that $\boldsymbol{R}$ is Euclidean. Let us assume to the contrary that for some members $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{v_1} \in \boldsymbol{W}$: $\boldsymbol{wRv}$, $\boldsymbol{wRv_1}$ and not $\boldsymbol{vRv_1}$. Hence, by the definition (**R**), $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \cup \{\boldsymbol{\psi}\} \subseteq \boldsymbol{v}$, $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in$

$\boldsymbol{w}\} \cup \{\boldsymbol{\psi}_1\} \subseteq \boldsymbol{v_1}$, where $\mathsf{M}\boldsymbol{\psi}$ and $\mathsf{M}\boldsymbol{\psi}_1$ belong to $\boldsymbol{w}$, and there is no formula $\mathsf{M}\chi$ in $\boldsymbol{v}$ such that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}\} \cup \{\chi\} \subseteq \boldsymbol{v_1}$. Because $\mathsf{M}(\boldsymbol{\psi} \vee \boldsymbol{\psi}_1) \in \boldsymbol{v}$ and $\boldsymbol{\psi} \vee \boldsymbol{\psi}_1 \in \boldsymbol{v_1}$, therefore there must exist a formula $\boldsymbol{\phi}$ such that $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{\phi} \notin \boldsymbol{v_1}$. Hence, $\mathsf{L}\boldsymbol{\phi} \notin \boldsymbol{w}$ and, by Lemma 3.1(ii), $\neg\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$. Thus $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{w}$, and since $\mathsf{M}\neg\boldsymbol{\phi} \to \mathsf{LM}\neg\boldsymbol{\phi} \in \boldsymbol{w}$, then in view of Proposition 2.1(ii) and Lemma 3.1(i), $\mathsf{LM}\neg\boldsymbol{\phi} \in \boldsymbol{w}$. The last implies $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{v}$, which is equivalent to $\neg\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ — a contradiction. So, $\boldsymbol{R}$ is Euclidean.

Step B:

For each 1^{st} sort variable x, let f(x) be the least element in the set $\{y \mid (x \overset{1}{\approx} y) \in \boldsymbol{w_1}\}$ with respect to our fixed order. Then, we put

$$\boldsymbol{D_1} = \{f(x) \mid \text{ x is variable of the } 1^{st} \text{ sort}\}.$$

This is what we are going to establish.

(●●) If $(x \overset{1}{\approx} y) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$, then $(x \overset{1}{\approx} y) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$.

It is obvious that (●●) holds, when $\boldsymbol{W} = \{\boldsymbol{w_1}\}$. Therefore, suppose that there exists a $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{v} \neq \boldsymbol{w_1}$ and $(x \overset{1}{\approx} y) \in \boldsymbol{v}$. Employing **Step 3** of the definition of the order on $\boldsymbol{W}$, we may suppose that $\boldsymbol{v} \in \boldsymbol{W^r}$ for some $r \geq 2$. Therefore there exists a finite sequence $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_r}$ such that $\boldsymbol{w_r} = \boldsymbol{v}$ and for each i, $1 < i \leq r$, $\boldsymbol{w_i}$ is a subordinate of $\boldsymbol{w_{i-1}}$.
For(i): $r = 2$. If $(x \overset{1}{\approx} y) \in \boldsymbol{w_1}$, then by Lemma 3.1(i) and Proposition 2.1(ii), $\mathsf{L}(x \overset{1}{\approx} y) \in \boldsymbol{w_1}$. Hence, $(x \overset{1}{\approx} y) \in \boldsymbol{v}$. On the other hand, if $(x \overset{1}{\approx} y) \notin \boldsymbol{w_1}$, then Lemma 3.1(ii), $\exists\alpha(\alpha(x) \wedge \neg\alpha(x)) \in \boldsymbol{w_1}$. Therefore, on the strength of axiom (2.24), Lemma 3.1(i) and Proposition 2.1(ii), $\mathsf{L}\exists\alpha(\alpha(x) \wedge \neg\alpha(x)) \in \boldsymbol{w_1}$. Thus, $\exists\alpha(\alpha(x) \wedge \neg\alpha(x)) \in \boldsymbol{v}$, which by Lemma 3.1(ii) is equivalent to $(x \overset{1}{\approx} y) \notin \boldsymbol{v}$.
For(ii): $r > 2$. Suppose now that for any two members $\boldsymbol{w_i}, \boldsymbol{w_j}$ of the sequence $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_p}$, $p < r$, $(x \overset{1}{\approx} y) \in \boldsymbol{w_i}$ if and only $(x \overset{1}{\approx} y) \in \boldsymbol{w_j}$. In a way similar to (i), it can be shown that $(x \overset{1}{\approx} y) \in \boldsymbol{w_p}$ if and only if $(x \overset{1}{\approx} y) \in \boldsymbol{w_{p+1}}$. Hence, for any two members $\boldsymbol{w_i}$, $\boldsymbol{w_j}$, $1 \leq i < j \leq r$, $(x \overset{1}{\approx} y) \in \boldsymbol{w_i}$ if and only if $(x \overset{1}{\approx} y) \in \boldsymbol{w_j}$. And finally, it is not hard to see that for any $\boldsymbol{w} \in \boldsymbol{W}$, $(x \overset{1}{\approx} y) \in \boldsymbol{w_1}$ if and only if $(x \overset{1}{\approx} y) \in \boldsymbol{w}$, which completes the proof of (●●).

So, by (●●), it should be clear that $\boldsymbol{D_1}$ is the constant objectual domain.

Our next aim is to prove that

(● ● ●) If $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$, then $\mathsf{G}(x) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W}$.

Suppose that $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W}$. And, for the non-trivial case, suppose that the family $\boldsymbol{W}$ is at least two-element.
Notice first that $\vdash_{\mathbf{C}^A\mathbf{5}} \exists y\mathsf{G}(y)$. Hence, with respect to Lemma 3.1(i), $\exists y\mathsf{G}(y) \in \boldsymbol{w}$ for all $\boldsymbol{w} \in \boldsymbol{W}$. And because each $\boldsymbol{w} \in \boldsymbol{W}$ is $\boldsymbol{\omega}$-$\mathbf{C}^A\mathbf{5}$-complete, therefore for every $\boldsymbol{w} \in \boldsymbol{W}$ there exists 1^{st} sort variable x_w such that $\mathsf{G}(x_w) \in \boldsymbol{w}$.
In order to show that $\mathsf{G}(x_{w_1}) \in \boldsymbol{w}$ for any $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{w} \neq \boldsymbol{w_1}$, let $\boldsymbol{w}$ be the any choice element of $\boldsymbol{W}$. We may suppose that $\boldsymbol{w}$ is member of $\boldsymbol{W^r}$. So, by the **Step 3** of the definition of the order on $\boldsymbol{W}$, there exists a finite sequence $\boldsymbol{w_1}$, $\boldsymbol{w_2}, \dots, \boldsymbol{w_r}$, $r \geq 2$ such that $\boldsymbol{w_r} = \boldsymbol{w}$ and for each i, $1 < i \leq r$, $\boldsymbol{w_i}$ is a subordinate of $\boldsymbol{w_{i-1}}$. *F*or (i): $r = 2$. Since $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(x_{w_1}) \to \mathsf{LG}(x_{w_1})$, therefore by Lemma 3.1(i) and Proposition 2.1(ii), $\mathsf{LG}(x_{w_1}) \in \boldsymbol{w_1}$. And, by applying to the latter the definition ($\mathbf{R}$), we obtain that $\mathsf{G}(x_{w_1}) \in \boldsymbol{w}$. *F*or (ii): $r > 2$. Suppose now that $\mathsf{G}(x_{w_1}) \in \boldsymbol{w_p}$, $p < r$. Similar to (i), it can be shown that $\mathsf{G}(x_{w_p}) \in \boldsymbol{w_{p+1}}$. Since, for any 1^{st} sort variables y, z, $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(y) \wedge \mathsf{G}(z) \to (y \overset{1}{\approx} z)$ and $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(y) \wedge (y \overset{1}{\approx} z) \to \mathsf{G}(z)$, then by ($\bullet\bullet$), Lemma 3.1(i) and Proposition 2.1(ii),$(x_{w_1} \overset{1}{\approx} x_{w_{p+1}}) \in \boldsymbol{w_{p+1}}$, and consequently — $\mathsf{G}(x_{w_1}) \in \boldsymbol{w_{p+1}}$. Therefore, $\mathsf{G}(x_{w_1}) \in \boldsymbol{w}$.
Furthermore, by the same argument, we obtain $(x_{w_1} \overset{1}{\approx} x) \in \boldsymbol{w}$, and consequently, $\mathsf{G}(x) \in \boldsymbol{w}$, which finishes the proof of ($\bullet\bullet\bullet$).

By ($\bullet\bullet\bullet$), it can easy be seen that the following definition:

$$\boldsymbol{g} \text{ is the least element in the set } \{x \mid \mathsf{G}(x) \in \boldsymbol{w_1}\},$$

is correct.

With each 2^{nd} term A and $\boldsymbol{w} \in \boldsymbol{W}$ we associate now the set

$$F(A, \boldsymbol{w}) = \{a \in \boldsymbol{D_1} \mid A(a) \in \boldsymbol{w}\},$$

and we put

$$\text{for every } \boldsymbol{w} \in \boldsymbol{W},\ \boldsymbol{D_w} \text{ to be the family of all sets } F(A, \boldsymbol{w}) \in 2^{\boldsymbol{D_1}},$$
$$\boldsymbol{D_2} = (\boldsymbol{D_w})_{\boldsymbol{w} \in \boldsymbol{W}}$$
$$\boldsymbol{C_{\mathfrak{M}}} = \{\boldsymbol{f} \in \boldsymbol{W} \mapsto \bigcup_{w \in W} \boldsymbol{D_w} \mid \boldsymbol{f} \in \boldsymbol{D_w} \text{ for every } \boldsymbol{w} \in \boldsymbol{W}\}.$$

For the proof that $\{\boldsymbol{g}\} \in \bigcap_{\boldsymbol{w} \in \boldsymbol{W}} \boldsymbol{D_w}$ let us assume $a \in F(\mathsf{G}, \boldsymbol{w_1})$. So, $a \in \boldsymbol{D_1}$ and $\mathsf{G}(a) \in \boldsymbol{w_1}$. Clearly, $a = f(a)$, i.e. a is the least element in the set $\{x \mid (a \overset{1}{\approx} x) \in \boldsymbol{w_1}\}$. Since $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(a) \wedge \mathsf{G}(x) \to (a \overset{1}{\approx} x)$ and

$\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(a) \wedge (a \overset{1}{\approx} x) \rightarrow \mathsf{G}(x)$, then by Proposition 2.1(ii), Lemma 3.1(i) and Lemma 3.1(iii) we obtain that: $(a \overset{1}{\approx} x) \in \boldsymbol{w_1}$ if and only if $\mathsf{G}(x) \in \boldsymbol{w_1}$. Therefore, $\{x \mid (a \overset{1}{\approx} x) \in \boldsymbol{w_1}\} = \{x \mid \mathsf{G}(x) \in \boldsymbol{w_1}\}$, which proves that $a = \boldsymbol{g}$. Consequently, $F(\mathsf{G}, \boldsymbol{w_1}) = \{\boldsymbol{g}\}$. Hence, by $(\bullet\bullet\bullet)$, $\{\boldsymbol{g}\} = F(\mathsf{G}, \boldsymbol{w})$ for any $\boldsymbol{w} \in \boldsymbol{W}$. And finally, $\{\boldsymbol{g}\} \in \bigcap_{\boldsymbol{w}\in\boldsymbol{W}} \boldsymbol{D_w}$.

To prove that for each $a, b \in \boldsymbol{D_1}$ and every $\boldsymbol{w} \in \boldsymbol{W}$ there exists $X \in \boldsymbol{D_w}$ such that $|\{a, b\} \cap X| = 1$ let us suppose that $a, b \in \boldsymbol{D_1}$. Let $a = f(x)$ be the least element in the set $\{z \mid (x \overset{1}{\approx} z) \in \boldsymbol{w}\}$ and $b = f(y)$ be the least element in the set $\{z \mid (y \overset{1}{\approx} z) \in \boldsymbol{w}\}$. If $(x \overset{1}{\approx} y) \in \boldsymbol{w}$ were true, then a and b would be the same element. And trivially, there would exist a $X \in \boldsymbol{D_w}$ such that $|\{a, b\} \cap X| = 1$. If $(x \overset{1}{\approx} y) \notin \boldsymbol{w}$ were true, then we would have $\exists\alpha(\alpha(x) \wedge \neg\alpha(y)) \in \boldsymbol{w}$. Because of the $\boldsymbol{\omega}$-$\mathbf{C}^A\mathbf{5}$-completeness of $\boldsymbol{w}$ there would exist a variable of the 2^{nd} sort β such that $(\beta(x) \wedge \neg\beta(y)) \in \boldsymbol{w}$ would be the case. Hence, with respect to (2.23), (2.33), Lemma 3.1(i) and Proposition 2.1(ii), $(\beta(a) \wedge \neg\beta(b)) \in \boldsymbol{w}$ would be obtained. So, and in this case we have showed that there exists $X \in \boldsymbol{D_w}$ such that $|\{a, b\} \cap X| = 1$.

In this way we have finished our construction of the model structure $\mathfrak{W} = \langle \boldsymbol{W}, \boldsymbol{R}, \boldsymbol{D_1}, \boldsymbol{D_2}, \boldsymbol{g}\rangle$, called a *canonical model structure*.

Step C:

The assignment $\mathfrak{a}$ in the canonical model structure such that for any 1^{st} order variable x, $\mathfrak{a}(x) = f(x)$, and for any 2^{nd} sort term A and each $\boldsymbol{w} \in \boldsymbol{W}$, $\mathfrak{a}(A)(\boldsymbol{w}) = F(A, \boldsymbol{w})$, will be called a *canonical assignment*.

One can show that

(**TL**) Given the canonical model structure $\mathfrak{W} = \langle \boldsymbol{W}, \boldsymbol{R}, \boldsymbol{D_1}, \boldsymbol{D_2}, \boldsymbol{g}\rangle$ and the canonical assignment $\mathfrak{a}$ in it; for any formula $\boldsymbol{\phi}$ and each $\boldsymbol{w} \in \boldsymbol{W}$, $\mathfrak{a}, \boldsymbol{w} \models \boldsymbol{\phi}$ if and only if $\boldsymbol{\phi} \in \boldsymbol{w}$.

The proof of (**TL**) proceeds by simultaneous induction on the complexity of $\boldsymbol{\phi}$.

$\boldsymbol{\phi}$ is of the form $A(x)$:
Then, $\mathfrak{a}, \boldsymbol{w} \models A(x)$ iff $\mathfrak{a}(x) \in \mathfrak{a}(A)(\boldsymbol{w})$ iff $f(x) \in F(A, \boldsymbol{w})$, by the definition of $F(A, \boldsymbol{w})$), this last iff $A(f(x)) \in \boldsymbol{w}$, and with respect to $(\bullet\bullet)$, (2.23), (2.33), Lemma 3.1(i) and Proposition 2.1(ii), iff $A(x) \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\boldsymbol{\psi} \wedge \chi$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \boldsymbol{\psi} \wedge \chi$ iff $\mathfrak{a}, \boldsymbol{w} \models \boldsymbol{\psi}$ and $\mathfrak{a}, \boldsymbol{w} \models \chi$, by the inductive hypothesis, this last iff $\boldsymbol{\psi} \in \boldsymbol{w}$ and $\chi \in \boldsymbol{w}$, so on the strength of Lemma 3.1(iii), this last iff $\boldsymbol{\psi} \wedge \chi \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\neg\boldsymbol{\psi}$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \neg\boldsymbol{\psi}$ iff $\mathfrak{a}, \boldsymbol{w} \not\models \boldsymbol{\psi}$, and by the inductive hypothesis, this last iff $\boldsymbol{\psi} \notin \boldsymbol{w}$, and owing to Lemma 3.1(i), this last iff $\neg\boldsymbol{\psi} \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\forall\xi\boldsymbol{\psi}$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \forall\xi\boldsymbol{\phi}$ iff $\mathfrak{b}, \boldsymbol{w} \models \boldsymbol{\phi}$ for every $\mathfrak{b} \in \{\mathfrak{a}^?_\xi\}$, and further on the strength of Proposition 2.2, this last iff $\mathfrak{b}, \boldsymbol{w} \models \boldsymbol{\phi}(\xi/\mathfrak{b}(\xi))$ for every $\mathfrak{b} \in \{\mathfrak{a}^?_\xi\}$, and by the inductive hypothesis, iff $\boldsymbol{\phi}(\xi/\mathfrak{b}(\xi)) \in \boldsymbol{w}$ for every assignment $\mathfrak{b} \in \{\mathfrak{a}^?_\xi\}$, which on the basis of Lemma 3.2, is equivalent to $\forall\xi\boldsymbol{\psi} \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\mathsf{L}\boldsymbol{\psi}$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \mathsf{L}\boldsymbol{\psi}$ iff $\mathfrak{a}, \boldsymbol{v} \models \boldsymbol{\psi}$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$, therefore by the inductive hypothesis, this last iff $\boldsymbol{\psi} \in \boldsymbol{v}$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$, and further on the strength of the condition ($\bullet$), this last iff $\mathsf{L}\boldsymbol{\psi} \in \boldsymbol{w}$.

$\boldsymbol{\phi}$ is of the form $\mathbf{P}(A)$:
Then, $\mathfrak{a}, \boldsymbol{w} \models \mathbf{P}(A)$ iff $\boldsymbol{g} \in \mathfrak{a}(A)(\boldsymbol{v})$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff $\boldsymbol{g} \in F(A, \boldsymbol{v})$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff $A(\boldsymbol{g}) \in \boldsymbol{v}$ for every $\boldsymbol{v} \in \boldsymbol{W}$ such that $\boldsymbol{wRv}$ iff, on the strength of the condition ($\bullet$), $\mathsf{L}A(\boldsymbol{g}) \in \boldsymbol{w}$. We have already demonstrated that $\mathsf{G}(\boldsymbol{g}) \in \boldsymbol{w}$. Hence, by definition (1.10), $\forall\alpha(\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(\boldsymbol{g})) \in \boldsymbol{w}$. But then, by Lemma 3.2, $\mathbf{P}(A) \leftrightarrow \mathsf{L}A(\boldsymbol{g}) \in \boldsymbol{w}$. Therefore, $\mathsf{L}A(\boldsymbol{g}) \in \boldsymbol{w}$ iff $\mathbf{P}(A) \in \boldsymbol{w}$.

This concludes our proof of (**TL**).

Reminding ourselves of the assumption $X \nvdash_{\mathbf{C}^{A}\mathbf{5}} \boldsymbol{\phi}$ we apply now the semantic instrument, which we have here introduced. So, let $\mathfrak{W} = \langle \boldsymbol{W}, \boldsymbol{R}, \boldsymbol{D_1}, \boldsymbol{D_2}, \boldsymbol{g} \rangle$ be the canonical model structure and $\mathfrak{a}$ be the canonical assignment in it. Firstly, if the set $X \cup \{\neg\boldsymbol{\phi}\}$ is infinitely extendable, then for every $\boldsymbol{\psi} \in X$, $\mathfrak{W}, \mathfrak{a}, \boldsymbol{w_1} \models \boldsymbol{\psi}$ and $\boldsymbol{\psi} \in X$, $\mathfrak{W}, \mathfrak{a}, \boldsymbol{w_1} \not\models \boldsymbol{\phi}$.
Secondly, if the set $X \cup \{\neg\boldsymbol{\phi}\}$ is not infinitely extendable, then for every $\Re(\boldsymbol{\psi}) \in \Re(X \cup \{\neg\boldsymbol{\phi}\})$, $\mathfrak{W}, \mathfrak{a}, \boldsymbol{w_1} \models \Re(\boldsymbol{\psi})$. Hence, by Lemma 3.10(ii), for every $\boldsymbol{\psi} \in X \cup \{\neg\boldsymbol{\phi}\}$, $\mathfrak{W}, \Re(\mathfrak{a}), \boldsymbol{w_1} \models \boldsymbol{\psi}$. And consequently, for every $\boldsymbol{\psi} \in X$, $\mathfrak{W}, \Re(\mathfrak{a}), \boldsymbol{w_1} \models \boldsymbol{\psi}$ and $\mathfrak{W}, \Re(\mathfrak{a}), \boldsymbol{w_1} \not\models \boldsymbol{\phi}$.

Proof of (ii):
By the proof of (i), we must here only show that the relation $\boldsymbol{R}$ is symmetric. Let us assume to the contrary that for some members $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W}$: $\boldsymbol{wRv}$ and

not $\boldsymbol{vRw}$. So, by definition $(\mathbf{R})$, $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \cup \{\boldsymbol{\psi}\} \subseteq \boldsymbol{v}$ for some $\mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w}$ and there is no formula $\mathsf{M}\chi \in \boldsymbol{v}$ such that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}\} \cup \{\chi\} \subseteq \boldsymbol{w}$. If $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}\} \nsubseteq \boldsymbol{w}$ were true, then there would exist a formula $\boldsymbol{\phi}$ such that $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ and $\boldsymbol{\phi} \notin \boldsymbol{w}$. Since, $\mathsf{ML}\boldsymbol{\phi} \to \boldsymbol{\phi} \in \boldsymbol{w}$, then Proposition 2.1(ii) and Lemma 3.1(i), would still guarantee that $\mathsf{ML}\boldsymbol{\phi} \notin \boldsymbol{w}$, and hence, $\mathsf{LM}\neg\boldsymbol{\phi} \in \boldsymbol{w}$. But it is not possible, because then we would obtain $\mathsf{M}\neg\boldsymbol{\phi} \in \boldsymbol{v}$, and thus $\neg\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ — a contradiction. And further, for any formula $\boldsymbol{\phi}$, $\mathsf{M}(\boldsymbol{\phi} \to \boldsymbol{\phi}) \in \boldsymbol{v}$ and $\boldsymbol{\phi} \to \boldsymbol{\phi} \in \boldsymbol{w}$. Therefore, there exists a formula $\boldsymbol{\psi}$ such that $\mathsf{L}\boldsymbol{\psi} \in \boldsymbol{v}$ and $\boldsymbol{\psi} \in \boldsymbol{w}$. So, $\boldsymbol{R}$ is symmetric.

PROOF OF (iii):
By dint of (i) and (ii).

PROOFS OF (iv)–(vi):
Relying on these results (i)–(iii), it suffices here to show that $\{a\} \in \boldsymbol{D_w}$ for every $a \in \boldsymbol{D_1}$ and every $\boldsymbol{w} \in \boldsymbol{W}$. And so, using definition (2.37) and the axiom (2.38), we obtain that $\vdash_{\mathbf{Th}} \mathsf{I}_x(x)$ for each 1^{st} sort variable x and every theory $\mathbf{Th} \in \{\mathbf{C}^A\mathbf{5s}, \mathbf{C}^A\mathbf{bs}, \mathbf{C}^A\mathbf{cs}\}$. Hence, by Lemma 3.1(i), $\mathsf{I}_x(x) \in \boldsymbol{w}$ for each 1^{st} sort variable x and every $\boldsymbol{w} \in \boldsymbol{W}$. And from this latter it easily follows that for every $a \in \boldsymbol{D_1}$ and every $\boldsymbol{w} \in \boldsymbol{W}$, $\{a\} \in \boldsymbol{D_w}$.

PROOFS OF (vii)–(ix):
Banking on these results (i)–(iii), it suffices here to prove that $\boldsymbol{D_1} \in \boldsymbol{D_w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$. And so, it is not hard to verify that for every theory $\mathbf{Th} \in \{\mathbf{C}^A\mathbf{5n}, \mathbf{C}^A\mathbf{bn}, \mathbf{C}^A\mathbf{cn}\}$, $\vdash_{\mathbf{Th}} \forall x \mathsf{NE}(x)$. Hence, with respect to Lemma 3.1(i), $\forall x\mathsf{NE}(x) \in \boldsymbol{w_1}$. And finally, $\boldsymbol{D_1} = F(\mathsf{NE}, \boldsymbol{w}) \in \boldsymbol{D_w}$ for every $\boldsymbol{w} \in \boldsymbol{W}$.

PROOFS OF (x)–(xii):
By dint of (iv)–(vi) and (vii)–(ix), respectively.

In this way we have finished the proof of Theorem 3.12. ■

THEOREM 3.13 (Strong completeness). *Let X be a set of formulas. Then:*

(i) $X \models_{\boldsymbol{C^A 5\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5\star}} \boldsymbol{\phi}$,
(ii) $X \models_{\boldsymbol{C^A 5s\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5s\star}} \boldsymbol{\phi}$,
(iii) $X \models_{\boldsymbol{C^A d\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A d\star}} \boldsymbol{\phi}$,
(iv) $X \models_{\boldsymbol{C^A 5n\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5n\star}} \boldsymbol{\phi}$,
(v) $X \models_{\boldsymbol{C^A ds\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A ds\star}} \boldsymbol{\phi}$,
(vi) $X \models_{\boldsymbol{C^A 5ns\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A 5ns\star}} \boldsymbol{\phi}$,
(vii) $X \models_{\boldsymbol{C^A dn\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A dn\star}} \boldsymbol{\phi}$,
(viii) $X \models_{\boldsymbol{C^A dns\star}} \boldsymbol{\phi}$ *implies* $X \vdash_{\boldsymbol{C^A dns\star}} \boldsymbol{\phi}$.

PROOF.
PROOF OF (i):
Like that of Theorem 3.12(i), but by using of several distinct or new points in **Step B**.

And so, the conditions (••) and (• • •) are now made to read:

(••) If $(x \overset{1}{\approx} y) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W^{acc}} = \boldsymbol{W} - \boldsymbol{w_1}$, then $(x \overset{1}{\approx} y) \in \boldsymbol{w}$ for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$.

Suppose, for the non-trivial case, that the family $\boldsymbol{W}$ is at least two-element. We shall first prove that for any $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W^{acc}}$, $\boldsymbol{wRv}$. So, let $\boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{W^{acc}}$. Employing **Step 3** of the definition of the order on $\boldsymbol{W}$, we may suppose that $\boldsymbol{w} \in \boldsymbol{W^r}$ and $\boldsymbol{v} \in \boldsymbol{W^{r'}}$ for some $r, r' \geq 2$. Therefore there exists a finite sequences $\boldsymbol{w_1}, \boldsymbol{w_2}, \ldots, \boldsymbol{w_r}$ and $\boldsymbol{w_1}, \boldsymbol{w'_2}, \ldots, \boldsymbol{w'_{r'}}$ such that $\boldsymbol{w_r} = \boldsymbol{w}$, $\boldsymbol{w'_{r'}} = \boldsymbol{v}$ and for each i, $1 < i \leq r$, $\boldsymbol{w_i}$ is a subordinate of $\boldsymbol{w_{i-1}}$, and for each j, $1 < j \leq r'$, $\boldsymbol{w'_j}$ is a subordinate of $\boldsymbol{w'_{j-1}}$. The fact that $a\boldsymbol{R}b$, where $a, b \in \{\boldsymbol{w_2}, \boldsymbol{w'_2}\}$, follows directly from the Euclideanness of $\boldsymbol{R}$. Assuming next that $\boldsymbol{w_2Rw_p}$ and $\boldsymbol{w_pRw_2}$, where $r > 2$ and $r > p$, we obtain that $\boldsymbol{w_2Rw_{p+1}}$ and $\boldsymbol{w_{p+1}Rw_2}$. Consequently, $\boldsymbol{w_2Rw}$ and $\boldsymbol{wRw_2}$. Similarly, we can prove that $\boldsymbol{w'_2Rv}$ and $\boldsymbol{vRw'_2}$. Finally, from $\boldsymbol{w_2Rw}$ and $\boldsymbol{w_2Rw'_2}$ follows $\boldsymbol{w'_2Rw}$, and from $\boldsymbol{w'_2Rw}$ and $\boldsymbol{w'_2Rv}$ follows $\boldsymbol{wRv}$.

Suppose now that $(x \overset{1}{\approx} y) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W^{acc}}$. It is obvious that (••) holds, when $\boldsymbol{w} = \boldsymbol{v}$. Therefore, suppose that there exists $\boldsymbol{w} \in \boldsymbol{W^{acc}}$ such that $\boldsymbol{w} \neq \boldsymbol{v}$. Hence, on the strength of the axiom (2.35), Lemma 3.1(i) and Proposition 2.1(ii), $\mathsf{L}(x \overset{1}{\approx} y) \in \boldsymbol{v}$. And finally, by our construction, $(x \overset{1}{\approx} y) \in \boldsymbol{w}$, which completes the proof of (••).

(• • •) $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W^{acc}}$, then $\mathsf{G}(x) \in \boldsymbol{w}$ for each $\boldsymbol{w} \in \boldsymbol{W^{acc}}$.

Suppose that $\mathsf{G}(x) \in \boldsymbol{v}$ for some $\boldsymbol{v} \in \boldsymbol{W^{acc}}$. Moreover, for the non-trivial case, suppose that the family $\boldsymbol{W}$ is at least two-element.
Since $\vdash_{\mathbf{C}^{A}\mathbf{5}_\star} \mathsf{L}\exists y\mathsf{G}(y)$, then, from the construction of the family $\boldsymbol{W}$ and Lemma 3.1(i), it follows that $\exists y\mathsf{G}(y) \in \boldsymbol{w}$ for all $\boldsymbol{w} \in \boldsymbol{W^{acc}}$. And by the useful moves, we obtain that for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$ there exists a 1^{st} sort variable x_w such that $\mathsf{G}(x_w) \in \boldsymbol{w}$.
As we have already established, for any $\boldsymbol{w} \in \boldsymbol{W}$, $\boldsymbol{wRv}$ and $\boldsymbol{vRw}$. It is obvious that (•••) holds, when $\boldsymbol{w} = \boldsymbol{v}$. Therefore, suppose that there exists $\boldsymbol{w} \in \boldsymbol{W^{acc}}$ such that $\boldsymbol{w} \neq \boldsymbol{v}$. Since, for any 1^{st} sort variables y, $\vdash_{\mathbf{C}^{A}\mathbf{5}_\star} \mathsf{L}\big[\mathsf{G}(y) \rightarrow \mathsf{LG}(y)\big]$, then after applying Lemma 3.1(i) and Proposition 2.1(ii), we obtain that

$\mathsf{G}(x_w) \in \boldsymbol{v}$. And since, for any 1^{st} sort variables y, z, $\vdash_{\mathbf{C}^A\mathbf{5}\star} \mathsf{L}\big[\mathsf{G}(y) \wedge \mathsf{G}(z) \to (y \stackrel{1}{\approx} z)\big]$ and $\vdash\vdash_{\mathbf{C}^A\mathbf{5}\star} \mathsf{L}\big[\mathsf{G}(y) \wedge (y \stackrel{1}{\approx} z) \to \mathsf{G}(z)\big]$, then by (••), Lemma 3.1(i) and Proposition 2.1(ii), $(x \stackrel{1}{\approx} x_w) \in \boldsymbol{v}$, and further — $\mathsf{G}(x) \in \boldsymbol{w}$, which finishes the proof of (• • •).

Given some (any chosen) member $\boldsymbol{w}$ of $\boldsymbol{W^{acc}}$, for each 1^{st} sort variable x, $f(x)$ denotes the least element in the set $\{y \mid (x \stackrel{1}{\approx} y) \in \boldsymbol{w}\}$ with respect to our fixed order. Then, we put

$$\boldsymbol{D_1} = \{f(x) \mid \quad x \text{ is variable of the } 1^{st} \text{ sort}\},$$

and

$$\boldsymbol{g} \text{ is the least element in the set } \{x \mid \mathsf{G}(x) \in \boldsymbol{w}\}.$$

Conditions (••) and (• • •) guarantee that both definitions are correct.

Again, with each 2^{nd} term A and $\boldsymbol{w} \in \boldsymbol{W}$ we associate the set

$$F(A, \boldsymbol{w}) = \{a \in \boldsymbol{D_1} \mid A(a) \in \boldsymbol{w}\},$$

and we put

$\boldsymbol{E_{w_1}}$ to be the family of all sets $F(A, \boldsymbol{w_1}) \in 2^{\boldsymbol{D_1}}$,

$\boldsymbol{D_{w_1}} = 2^{\boldsymbol{D_1}}$,

for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}$, $\boldsymbol{D_w}$ to be the family of all sets $F(A, \boldsymbol{w}) \in 2^{\boldsymbol{D_1}}$,

$\boldsymbol{D_2} = (\boldsymbol{D_w})_{\boldsymbol{w} \in \boldsymbol{W}}$,

$\boldsymbol{C_{\mathfrak{W}}} = \{\boldsymbol{f} \in \boldsymbol{W} \mapsto \bigcup\limits_{w \in W} \boldsymbol{D_w} \mid \boldsymbol{f}(\boldsymbol{w_1}) \in \boldsymbol{E_{w_1}} \text{ and } \boldsymbol{f}(\boldsymbol{w}) \in \boldsymbol{D_w}$
for every $\boldsymbol{w} \in \boldsymbol{W^{acc}}\}$.

Proof of (ii):
By the proof of **(i)**, we must here only show that $\{a\} \in \boldsymbol{D_w}$ for every $a \in \boldsymbol{D_1}$ and every $\boldsymbol{v} \in \boldsymbol{W^{acc}}$. It follows by the same argument as in the proof of Theorem 3.12(iv), but using $\vdash_{\mathbf{C}^A\mathbf{5s}\star} \mathsf{LI}_x(x)$ in place $\vdash_{\mathbf{C}^A\mathbf{5s}} \mathsf{I}_x(x)$.

Proof of (iii):
Relying on **(i)**, it suffices here to show that $\boldsymbol{R}$ is transitive. So, suppose that $\boldsymbol{wRv}$ and $\boldsymbol{vRv_1}$. Therefore, by definition (**R**), $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \cup \{\boldsymbol{\psi}\} \subseteq \boldsymbol{v}$ and $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}\} \cup \{\boldsymbol{\psi}\} \subseteq \boldsymbol{v_1}$, where $\mathsf{M}\boldsymbol{\psi} \in \boldsymbol{w}$ and $\mathsf{M}\boldsymbol{\psi}_1 \in \boldsymbol{v}$. In order to show that $\{\boldsymbol{\phi} \mid \mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}\} \subseteq \boldsymbol{v_1}$ let us assume that $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{w}$. Because $\mathsf{L}\boldsymbol{\phi} \to \mathsf{LL}\boldsymbol{\phi} \in \boldsymbol{w}$, then by Lemma 3.1(i) and Proposition 2.1(ii), $\mathsf{LL}\boldsymbol{\phi} \in \boldsymbol{w}$. Hence, $\mathsf{L}\boldsymbol{\phi} \in \boldsymbol{v}$ and

$\boldsymbol{\phi} \in \boldsymbol{v_1}$. But then, $\mathsf{M}(\boldsymbol{\psi} \vee \boldsymbol{\psi}_1) \in \boldsymbol{w}$ and $\boldsymbol{\psi} \vee \boldsymbol{\psi}_1 \in \boldsymbol{v_1}$. Thus $\boldsymbol{wRv_1}$, i.e. $\boldsymbol{R}$ is transitive.

PROOF OF (iv):
By the proof of **(i)**, we must here only show that $\boldsymbol{D_1} \in \boldsymbol{D_w}$ for every $\boldsymbol{v} \in \boldsymbol{W^{acc}}$. It follows by the same argument as in the proof of Theorem 3.12(vii), but using $\vdash_{\mathbf{C}^A\mathbf{5n\star}} \mathsf{L}\forall x \mathsf{NE}(x)$ in place $\vdash_{\mathbf{C}^A\mathbf{5n}} \forall x \mathsf{NE}(x)$.

PROOF OF (v)–(viii):
Adapting the arguments of (i)-(iv) to suit the proofs of (v)–(viii) are a bit more work. ■

4. Appendix

We shall prove here a number of interesting theorems.

T1: $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(x) \to \mathsf{L}\exists y \mathsf{G}(y)$

PROOF. **1.** $\forall\alpha\big[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)\big] \to \forall\alpha\big[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)\big]$ (2.15), (2.39), 2.16), RO
2. $\big[\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)]\big] \to \big[\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)]\big]$ (2.15), **1**, RO
3. $\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)]$ (2.15), (1.10), RO
4. $\forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)] \to [\mathbf{P}(\mathsf{G}) \to \mathsf{LG}(x)]$ (2.16)
5. $\mathsf{G}(x) \to [\mathbf{P}(\mathsf{G}) \to \mathsf{LG}(x)]$ (2.15), **3**, **4**, RO
6. $\mathsf{G}(x) \to \mathsf{LG}(x)$ (2.15), **5**, (2.29), RO
7. $\mathsf{L}[\mathsf{G}(x) \to \exists y \mathsf{G}(y)]$ (2.15), (2.39)
8. $\mathsf{LG}(x) \to \mathsf{L}\exists y \mathsf{G}(y)$ (2.19), **7**, RO
9. $\mathsf{G}(x) \to \mathsf{L}\exists y \mathsf{G}(y)$ (2.15), **6**, **8**, RO
■

T2: $\vdash_{\mathbf{C}^A\mathbf{5\star}} \mathsf{L}\big[\mathsf{G}(x) \to \mathsf{L}\exists y \mathsf{G}(y)\big]$

PROOF. **1.** $\mathsf{L}\forall x\big[\forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)] \to \forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)]\big]$
(2.15), (2.39), (2.16), (2.19), RO
2. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)]\big] \to \mathsf{L}\forall x\big[\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)]\big]$
(2.15), (2.39), (2.16), (2.19), **1**, RO
3. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)]\big]$ **2**, (1.10), (2.39), RO
4. $\mathsf{L}\forall x\big[\forall\alpha[\mathbf{P}(\alpha) \to \mathsf{L}\alpha(x)] \to [\mathbf{P}(\mathsf{G}) \to \mathsf{LG}(x)]\big]$ (2.17), (2.39)
5. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to [\mathbf{P}(\mathsf{G}) \to \mathsf{LG}(x)]\big]$ (2.15), (2.39), (2.16), (2.19), **3**, **4**, RO
6. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to [\mathbf{P}(\mathsf{G}) \to \mathsf{LG}(x)]\big] \to \mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{LG}(x)\big]$
(2.15), (2.39), (2.16), (2.19), (2.29), RO
7. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{LG}(x)\big]$ **6**, **5**, RO
8. $\mathsf{L}\forall x\big[\forall y \neg\mathsf{LG}(y) \to \neg\mathsf{LG}(x)\big] \to \mathsf{L}\forall x\big[\mathsf{LG}(x) \to \exists y \mathsf{LG}(y)\big]$
(2.15), (2.39), (2.16), (2.19), RO
9. $\mathsf{L}\forall x\big[\forall y \neg\mathsf{LG}(y) \to \neg\mathsf{LG}(x)\big]$ (2.16), (2.39)
10. $\mathsf{L}\forall x\big[\mathsf{LG}(x) \to \exists y \mathsf{LG}(y)\big]$ **8**, **9**, RO
11. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{LG}(x)\big] \to \big[\mathsf{L}\forall x[\mathsf{LG}(x) \to \exists y \mathsf{LG}(y)] \to \mathsf{L}\forall x[\mathsf{G}(x) \to \mathsf{L}\exists y \mathsf{G}(y)]\big]$
(2.15), (2.39), (2.16), (2.19), RO

12. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{L}\exists y\mathsf{G}(y)\big]$ **11**, **7**, **10**, RO

■

T3: $\vdash_{\mathbf{C}^{A}\mathbf{5}} \mathbf{P}(\alpha) \to \mathsf{M}\exists x\alpha(x)$

PROOF.
1. $\forall x[\neg\beta(x) \leftrightarrow -\beta(x)]$ — (2.32), (2.39), (2.15), RO
2. $\forall x[\alpha(x) \to \neg\beta(x)] \to \forall x[\alpha(x) \to -\beta(x)]$ — (2.15), (2.39), (2.16), **1**, RO
3. $\mathbf{P}(\alpha) \wedge \neg\mathsf{M}\exists x\alpha(x) \to \mathbf{P}(\alpha) \wedge \mathsf{L}\forall x\neg\alpha(x)$ — (2.15), the definition of M
4. $\mathbf{P}(\alpha) \wedge \mathsf{L}\forall x\neg\alpha(x) \to \mathbf{P}(\alpha) \wedge \mathsf{L}\forall x[\alpha(x) \to \beta(x)]$ — (2.15), (2.39), (2.16), (2.19), **3**, RO
5. $\mathbf{P}(\alpha) \wedge \mathsf{L}\forall x\neg\alpha(x) \to \mathbf{P}(\alpha) \wedge \mathsf{L}\forall x[\alpha(x) \to -\beta(x)]$ — (2.15), (2.39), (2.16), (2.19), **3**, RO
6. $\mathbf{P}(\alpha) \wedge \mathsf{L}\forall x\neg\alpha(x) \to \mathbf{P}(\beta)$ — (2.15), **4**, (2.26), RO
7. $\mathbf{P}(\alpha) \wedge \mathsf{L}\forall x\neg\alpha(x) \to \mathbf{P}(-\beta)$ — (2.15), **5**, (2.26), RO
8. $\mathbf{P}(\alpha) \wedge \neg\mathsf{M}\exists x\alpha(x) \to \mathbf{P}(\beta))$ — **6**, the definition of M
9. $\mathbf{P}(\alpha) \wedge \neg\mathsf{M}\exists x\alpha(x) \to \neg\mathbf{P}(\beta))$ — **7**, the definition of M
10. $\mathbf{P}(\alpha) \wedge \neg\mathsf{M}\exists x\alpha(x) \to \mathbf{P}(\beta) \wedge \neg\mathbf{P}(\beta)$ — (2.15), **8**, **9**, RO
11. $\mathbf{P}(\alpha) \to \mathsf{M}\exists x\alpha(x)$ — (2.15), **10**, RO

■

T4: $\vdash_{\mathbf{C}^{A}\mathbf{5}_{\star}} \mathbf{P}(\alpha) \to \mathsf{M}\exists x\alpha(x)$

PROOF.
1. $\mathsf{L}\forall x[\neg\beta(x) \to -\beta(x)]$ — (2.32), (2.39)
2. $\mathsf{L}\forall x[\alpha(x) \to \neg\beta(x)] \to \mathsf{L}\forall x[\alpha(x) \to -\beta(x)]$ — (2.15), (2.39), (2.16), (2.19), RO

The steps **3–11** are the same as for **T3** ■

T5: $\vdash_{\mathbf{C}^{A}\mathbf{5}} \mathsf{L}\exists x\mathsf{G}(x)$

PROOF. The steps **1-12** are the same as for **T2**

13. $\mathsf{L}\forall x[\mathsf{G}(x) \to \mathsf{L}\exists y\mathsf{G}(y)] \to \mathsf{L}[\exists x\mathsf{G}(x) \to \mathsf{L}\exists y\mathsf{G}(y)]$ — (2.15), (2.39), (2.19), RO
14. $\mathsf{L}[\exists x\mathsf{G}(x) \to \mathsf{L}\exists y\mathsf{G}(y)] \to \mathsf{L}[\neg\mathsf{L}\exists y\mathsf{G}(y) \to \neg\exists x\mathsf{G}(x)]$ — (2.15), (2.39), (2.19), RO
15. $\mathsf{L}\forall x[\mathsf{G}(x) \to \mathsf{L}\exists y\mathsf{G}(y)] \to \mathsf{L}[\neg\mathsf{L}\exists y\mathsf{G}(y) \to \neg\exists x\mathsf{G}(x)]$ — (2.15), **13**, **14**, RO
16. $\mathsf{L}[\neg\mathsf{L}\exists y\mathsf{G}(y) \to \neg\exists x\mathsf{G}(x)]$ — **15**, **T2**, RO
17. $\mathsf{L}\neg\mathsf{L}\exists y\mathsf{G}(y) \to \mathsf{L}\neg\exists x\mathsf{G}(x)$ — (2.19), **16**, RO
18. $\mathsf{M}\exists x\mathsf{G}(x) \to \mathsf{M}\mathsf{L}\exists y\mathsf{G}(y)$ — (2.15), **17**, the definition of M, RO
19. $\mathsf{M}\mathsf{L}\exists y\mathsf{G}(y) \to \mathsf{L}\exists y\mathsf{G}(y)$ — (**5**)
20. $\mathsf{M}\exists x\mathsf{G}(x) \to \mathsf{L}\exists y\mathsf{G}(y)$ — (2.15), **18**, **19**, RO
21. $[\mathbf{P}(\mathsf{G}) \to \mathsf{M}\exists y\mathsf{G}(y)] \to [\mathbf{P}(\mathsf{G}) \to \mathsf{L}\exists y\mathsf{G}(y)]$ — (2.15), **T20**, RO
22. $\mathsf{L}\exists y\mathsf{G}(y)$ — **21**, **T3**, (2.29), RO
23. $\mathsf{L}\exists x\mathsf{G}(x)$ — (2.15), (2.39), (2.19), **22**, RO

■

T6: $\vdash_{\mathbf{C}^{A}\mathbf{5}_{\star}} \mathsf{L}\exists x\mathsf{G}(x)$

PROOF. The steps **1-12** of **T2** and **13-23** of **T5** ■

T7: $\vdash_{\mathbf{C}^{A}\mathbf{b}} \mathsf{L}\exists x\mathsf{G}(x)$

PROOF. The steps **1-16** are the same as for **T5**

17. $\mathsf{L}\mathsf{L}[\neg\mathsf{L}\exists y\mathsf{G}(y) \to \neg\exists x\mathsf{G}(x)]$ — **b**, **16**, RO
18. $\mathsf{L}[\mathsf{L}\neg\mathsf{L}\exists y\mathsf{G}(y) \to \mathsf{L}\neg\exists x\mathsf{G}(x)]$ — **17**, (2.19), RO
19. $\mathsf{L}[\mathsf{M}\exists x\mathsf{G}(x) \to \mathsf{M}\mathsf{L}\exists y\mathsf{G}(y)]$ — (2.15), (2.39), (2.19), **18**, the definition of M, RO

20. $\mathsf{L}\big[\mathsf{ML}\exists y\mathsf{G}(y) \rightarrow \exists y\mathsf{G}(y)\big]$ — (2.15), (**b**), (2.39), (2.19), **19**, RO
21. $\mathsf{L}[\mathbf{P}(\mathsf{G}) \rightarrow \mathsf{M}\exists y\mathsf{G}(y)] \rightarrow \mathsf{L}[\mathbf{P}(\mathsf{G}) \rightarrow \exists y\mathsf{G}(y)]$ — (2.15), (2.39), (2.19), **T20**, RO
22. $\mathsf{L}[\mathbf{P}(\mathsf{G}) \rightarrow \mathsf{M}\exists y\mathsf{G}(y)] \rightarrow \exists y\mathsf{G}(y)$ — (2.15), (2.39), (2.19), (2.29), **21**, RO
23. $\mathsf{L}[\mathbf{P}(\mathsf{G}) \rightarrow \mathsf{M}\exists y\mathsf{G}(y)]$ — **T4**, (2.39), RO
24. $\mathsf{L}\exists y\mathsf{G}(y)$ — **22**, **23**, RO
25. $\mathsf{L}\exists x\mathsf{G}(x)$ — (2.15), (2.39), (2.19), **24**, RO

■

T8: $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{L}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{L}\mathsf{G}(x)$

PROOF. **1.** $\forall x\big[\mathsf{MLG}(x) \rightarrow \mathsf{LG}(x)\big]$ — (**5**), (2.39)
2. $\forall x\big[\mathsf{MLG}(x) \rightarrow \mathsf{G}(x)\big]$ — (2.15), (2.27), (2.39), (2.16), **1**, RO
3. $\forall x\big[\mathsf{MG}(x) \rightarrow \mathsf{MLG}(x)\big]$ — (2.15), (2.27), (2.19), (2.39), RO
4. $\forall x\big[\mathsf{MG}(x) \rightarrow \mathsf{G}(x)\big]$ — (2.15), (2.16), **3**, **2**, RO
5. $\forall x\big[\mathsf{MG}(x) \rightarrow \mathsf{LG}(x)\big]$ — (2.15), (2.27), (2.39), **4**, RO
6. $\exists x\mathsf{MG}(x) \rightarrow \exists x\mathsf{LG}(x)$ — (2.15), (2.16), **5**, RO
7. $\forall x\mathsf{L}\neg\mathsf{G}(x) \rightarrow \mathsf{L}\forall x\neg\mathsf{G}(x)$ — (2.21)
8. $\neg\mathsf{L}\forall x\neg\mathsf{G}(x) \rightarrow \neg\forall x\mathsf{L}\neg\mathsf{G}(x)$ — (2.15), **7**, RO
9. $\mathsf{M}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{MG}(x)$ — **8**, the definitions of M and $\exists$
10. $\mathsf{M}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{LG}(x)$ — (2.15), **9**, **6**, RO
11. $\mathsf{L}\exists x\mathsf{G}(x) \rightarrow \mathsf{M}\exists x\mathsf{G}(x)$ — (2.20)
12. $\mathsf{L}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{LG}(x)$ — (2.15), **11**, **10**, RO

■

T9: $\vdash_{\mathbf{C}^A\mathbf{5}_\star} \mathsf{L}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{LG}(x)$

PROOF. The same as for **T8**. ■

T10: $\vdash_{\mathbf{C}^A\mathbf{b}} \mathsf{L}\exists x\mathsf{G}(x) \rightarrow \exists x\mathsf{LG}(x)$

PROOF. Like that of **T8**, but using (**b**) instead of (**5**). ■

T11: $\vdash_{\mathbf{C}^A\mathbf{5}} \mathsf{G}(x) \wedge \mathsf{G}(y) \rightarrow (x \overset{1}{\approx} y)$

PROOF. **1.** $\exists\alpha\big[\alpha(x)\wedge\neg\alpha(y)\big] \rightarrow \mathsf{L}\exists\alpha\big[\alpha(x)\wedge\neg\alpha(y)\big]$ — (2.24)
2. $\exists\alpha\big[\alpha(x)\wedge\neg\alpha(y)\big] \rightarrow \exists\alpha\mathsf{L}\big[\alpha(x)\wedge\neg\alpha(y)\big]$ — (2.15), **1**, (2.22), RO
3. $\exists\alpha\big[\alpha(x)\wedge\neg\alpha(y)\big] \rightarrow \exists\alpha\big[\mathsf{L}\alpha(x)\wedge\mathsf{L}\neg\alpha(y)\big]$ — (2.15), (2.16), (2.19), **2**, RO
4. $\neg\exists\alpha\big[\mathsf{L}\alpha(x)\wedge\mathsf{L}\neg\alpha(y)\big] \rightarrow \neg\exists\alpha\big[\alpha(x)\wedge\neg\alpha(y)\big]$ — (2.15), **3**, RO
5. $\forall\alpha\big[\mathsf{L}\alpha(x) \rightarrow \mathsf{M}\alpha(y)\big] \rightarrow \forall\alpha\big[\alpha(x) \rightarrow \alpha(y)\big]$ — (2.15), **4**, the definitions of M and $\exists$, RO
6. $\exists\alpha\big[\alpha(y)\wedge\neg\alpha(x)\big] \rightarrow \mathsf{L}\exists\alpha\big[\alpha(y)\wedge\neg\alpha(x)\big]$ — (2.24)
7. $\exists\alpha\big[\alpha(y)\wedge\neg\alpha(x)\big] \rightarrow \exists\alpha\mathsf{L}\big[\alpha(y)\wedge\neg\alpha(x)\big]$ — (2.15), **6**, **T8**, RO
8. $\exists\alpha\big[\alpha(y)\wedge\neg\alpha(x)\big] \rightarrow \exists\alpha\big[\mathsf{L}\alpha(y)\wedge\mathsf{L}\neg\alpha(x)\big]$ — (2.15), (2.16), (2.19), **7**, RO
9. $\neg\exists\alpha\big[\mathsf{L}\alpha(y)\wedge\mathsf{L}\neg\alpha(x)\big] \rightarrow \neg\exists\alpha\big[\alpha(y)\wedge\neg\alpha(x)\big]$ — (2.15), **8**, RO
10. $\forall\alpha\big[\mathsf{L}\alpha(y) \rightarrow \mathsf{M}\alpha(x)\big] \rightarrow \forall\alpha\big[\alpha(y) \rightarrow \alpha(x)\big]$ — (2.15), **9**, the definitions of M and $\exists$, RO
11. $\forall\alpha\big[\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\alpha(y)\big] \rightarrow \forall\alpha\big[\mathsf{L}\alpha(x) \rightarrow \mathsf{M}\alpha(y)\big] \wedge \forall\alpha\big[\mathsf{L}\alpha(y) \rightarrow \mathsf{M}\alpha(x)\big]$ — (2.15),(2.20),RO
12. $\forall\alpha\big[\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\alpha(y)\big] \rightarrow \forall\alpha\big[\alpha(x) \leftrightarrow \alpha(y)\big]$ — (2.15), **11**, **5**, **10**, RO
13. $\mathsf{G}(x)\wedge\mathsf{G}(y) \rightarrow \forall\alpha\big[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)\big]\wedge\forall\alpha\big[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(y)\big]$ — (2.15), (1.10), RO
14. $\mathsf{G}(x)\wedge\mathsf{G}(y) \rightarrow \forall\alpha\big[\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\alpha(y)\big]$ — (2.15), (2.16), **13**, RO
15. $\mathsf{G}(x)\wedge\mathsf{G}(y) \rightarrow \forall\alpha\big[\alpha(x) \leftrightarrow \alpha(y)\big]$ — (2.15), **14**, **12**, RO

16. $\mathsf{G}(x) \wedge \mathsf{G}(y) \rightarrow (x \overset{1}{\approx} y)$ **15**, the definitions of $\overset{1}{\approx}$

■

T12: $\vdash_{\mathbf{C}^A\mathbf{5}_\star} \mathsf{G}(x) \wedge \mathsf{G}(y) \rightarrow (x \overset{1}{\approx} y)$

PROOF. The same as for **T11**. ■

T13: $\vdash_{\mathbf{C}^A\mathbf{b}} \mathsf{G}(x) \wedge \mathsf{G}(y) \rightarrow (x \overset{1}{\approx} y)$

PROOF. The same as for **T11**. ■

T14: $\vdash_{\mathbf{C}^A\mathbf{5}} A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}(A \overset{2}{\approx} B)$

PROOF.
1. $A \mathsf{\ Ess\ } x \rightarrow \forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[A(y) \rightarrow \beta(y)]\big]$ (1.12)
2. $\forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[A(y) \rightarrow \beta(y)]\big] \rightarrow \big[\mathsf{L}\forall y[A(y) \rightarrow A(y)] \rightarrow \mathsf{L}A(x)\big]$ (2.16), (2.15), RO
3. $A \mathsf{\ Ess\ } x \rightarrow \mathsf{L}A(x)$ (2.15), **1**, **2**, RO
4. $B \mathsf{\ Ess\ } x \rightarrow \forall\alpha\big[\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y[B(y) \rightarrow \alpha(y)]\big]$ (1.12)
5. $\forall\alpha\big[\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y[B(y) \rightarrow \alpha(y)]\big] \rightarrow \big[\mathsf{L}\forall y[B(y) \rightarrow B(y)] \rightarrow \mathsf{L}B(x)\big]$ (2.16), (2.15), RO
6. $B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}B(x)$ (2.15), **4**, **5**, RO
7. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \big[\mathsf{L}B(x) \leftrightarrow \mathsf{L}\forall y[A(y) \rightarrow B(y)]\big] \wedge \mathsf{L}B(x)$
(2.15),(1.12),(2.16),**6**,RO
8. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}\forall y\big[A(y) \rightarrow B(y)\big]$ (2.15), **7**, RO
9. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \big[\mathsf{L}A(x) \leftrightarrow \mathsf{L}\forall y[B(y) \rightarrow A(y)]\big] \wedge \mathsf{L}A(x)$
(2.15),(1.12),(2.16),**3**,RO
10. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}\forall y\big[B(y) \rightarrow A(y)\big]$ (2.15), **9**, RO
11. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}\forall y\big[A(y) \leftrightarrow B(y)\big]$ (2.15), (2.19), **8**, **10**, RO
12. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}(A \overset{2}{\approx} B)$ **11**, (2.30)

■

T15: $\vdash_{\mathbf{C}^A\mathbf{5}_\star} A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow (A \overset{2}{\approx} B)$

PROOF. The steps **1-11** are the same as for **T14**

12. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow (A \overset{2}{\approx} B)$ (2.15), (2.31), RO

■

T16: $\vdash_{\mathbf{C}^A\mathbf{b}} A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}(A \overset{2}{\approx} B)$

PROOF. The same as for **T14** . ■

T17: $\vdash_{\mathbf{C}^A\mathbf{d}_\star} A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}(A \overset{2}{\approx} B)$

PROOF. The steps **1-11** are the same as for **T14**

12. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{LL}\forall y\big[A(y) \leftrightarrow B(y)\big]$ (2.15), **11**, **d**, RO
13. $A \mathsf{\ Ess\ } x \wedge B \mathsf{\ Ess\ } x \rightarrow \mathsf{L}(A \overset{2}{\approx} B)$ **12**, (2.31)

■

T18: $\vdash_{\mathbf{C}^A\mathbf{5}} \mathbf{P}(\mathsf{NE})$

PROOF.
1. $\mathsf{L}\forall x\forall\alpha\mathsf{L}\big[\alpha(x) \rightarrow \exists y\alpha(y)\big]$ (2.15), (2.39)

2. $\mathsf{L}\forall x\forall\alpha\big[\mathsf{L}\alpha(x) \to \mathsf{L}\exists y\alpha(y)\big]$ (2.19), (2.16), (2.19), **1**, RO

3. $\mathsf{L}\forall x\forall\alpha\Big[\forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[\alpha(y) \to \beta(y)]\big] \to \big[\mathsf{L}\alpha(x) \leftrightarrow \mathsf{L}\forall y[\alpha(y) \to \alpha(y)]\big]\Big]$ (2.17), (2.39)

4. $\mathsf{L}\forall x\forall\alpha\Big[\forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[\alpha(y) \to \beta(y)]\big] \to \mathsf{L}\alpha(x)\Big]$ (2.15), (2.39), (2.16), (2.19), **3**, RO

5. $\mathsf{L}\forall x\forall\alpha\Big[\forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[\alpha(y) \to \beta(y)]\big] \to \mathsf{L}\exists y\alpha(y)\Big]$ (2.15), (2.39), (2.16), (2.19), **4**, **2**, RO

6. $\mathsf{L}\forall x\forall\alpha\Big[\mathsf{L}\exists y\alpha(y) \to \big[[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)] \to \mathsf{L}\exists y\alpha(y)\big]\Big]$ (2.15), (2.39)

7. $\mathsf{L}\forall x\forall\alpha\Big[\forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[\alpha(y) \to \beta(y)]\big] \to \big[[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)] \to \mathsf{L}\exists y\alpha(y)\big]\Big]$ (2.15), (2.39), (2.16), (2.19), **5**, **6**, RO

8. $\mathsf{L}\forall x\forall\alpha\Big[A \ \mathsf{Ess}\ x \to \big[[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)] \to \mathsf{L}\exists y\alpha(y)\big]\Big]$ **7**, (1.12)

9. $\mathsf{L}\forall x\big[\forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)] \to \forall\alpha[A \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)]\big]$ (2.15), (2.39), (2.16), (2.19), **8**, RO

10. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \forall\alpha[\mathbf{P}(\alpha) \leftrightarrow \mathsf{L}\alpha(x)]\big] \to \mathsf{L}\forall x\big[\mathsf{G}(x) \to \forall\alpha[A \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)]\big]$ (2.15), (2.39), (2.16), (2.19), **9**, RO

11. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \forall\alpha[A \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)]\big]$ **10**, (1.10), (2.39), RO

12. $\mathsf{L}\forall x\big[\forall\alpha[A \ \mathsf{Ess}\ x \to \mathsf{L}\exists y\alpha(y)]\mathsf{NE}(x)\big] \to \mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{NE}(x)\big]$ (2.15), (2.39), (2.16), (2.19), **11**, RO

13. $\mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{NE}(x)\big]$ **12**, (1.13), (2.39), RO

14. $\mathbf{P}(\mathsf{G})\wedge\mathsf{L}\forall x\big[\mathsf{G}(x) \to \mathsf{NE}(x)\big] \to \mathbf{P}(\mathsf{NE})$ (2.26)

15. $\mathbf{P}(\mathsf{NE})$ **14**, (2.29), **13**, RO

■

T19: $\vdash_{\mathbf{C}^{A}\mathbf{n}\star} \mathbf{P}(\mathsf{NE})$

PROOF. The same as for **T18**. ■

T20: $\vdash_{\mathbf{C}^{A}\mathbf{bn}} \mathbf{P}(\mathsf{NE})$

PROOF. The same as for **T18**. ■

T21: $\vdash_{\mathbf{C}^{A}\mathbf{5s}} A \ \mathsf{Ess}\ x \to \mathsf{L}\forall y\big[A(y) \to (x \overset{1}{\approx} y)\big]$

PROOF. **1.** $\forall\beta\big[\mathsf{L}\beta(x) \leftrightarrow \mathsf{L}\forall y[A(y) \to \beta(y)]\big] \to \big[\mathsf{L}\mathsf{I}_x(x) \leftrightarrow \mathsf{L}\forall y[A(y) \to \mathsf{I}_x(y)]\big]$ (2.17)

2. $\mathsf{L}\mathsf{I}_x(x)$ (2.15), (2.23), (2.33), (2.37), RO

3. $A \ \mathsf{Ess}\ x \to \mathsf{L}\forall y\big[A(y) \to (x \overset{1}{\approx} y)\big]$ (2.15), **1**, **2**, (1.12), RO

■

T22: $\vdash_{\mathbf{C}^{A}\mathbf{5s}\star} A \ \mathsf{Ess}\ x \to \mathsf{L}\forall y\big[A(y) \to (x \overset{1}{\approx} y)\big]$

PROOF. The same as for **T21**. ■

T23: $\vdash_{\mathbf{C}^{A}\mathbf{bs}} A \ \mathsf{Ess}\ x \to \mathsf{L}\forall y\big[A(y) \to (x \overset{1}{\approx} y)\big]$

PROOF. The same as for **T21**. ■

T24: $\vdash_{\mathbf{C}^{A}\mathbf{5s}} A \ \mathsf{Ess}\ x \wedge A \ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y)$

PROOF. **1.** $\forall\beta\big[\mathsf{L}\beta(y) \leftrightarrow \mathsf{L}\forall x[A(x) \to \beta(x)]\big] \to \big[\mathsf{L}A(y) \leftrightarrow \mathsf{L}\forall x[A(x) \to A(x)]\big]$ (2.17)

2. $A \ \mathsf{Ess}\ x \to \mathsf{L}A(y)$ (2.15), (1.12), **1**, RO

3. $A \ \mathsf{Ess}\ x \to \mathsf{L}\forall\big[A(y) \to (x \overset{1}{\approx} y)\big]$ **T21**

4. $A \ \mathsf{Ess}\ x \wedge A \ \mathsf{Ess}\ y \to \mathsf{L}A(y) \wedge \mathsf{L}\forall y\big[A(y) \to (x \overset{1}{\approx} y)\big]$ (2.15), **2**, **3**, RO

5. $A \ \mathsf{Ess}\ x \wedge A \ \mathsf{Ess}\ y \to (x \overset{1}{\approx} y)$ (2.15), (2.19), **4**, RO

■

T25: $\vdash_{\mathbf{C}^A\mathbf{5s}\star} A \ \mathsf{Ess}\ x \wedge A \ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y)$

PROOF. The same as for **T24**. ■

T26: $\vdash_{\mathbf{C}^A\mathbf{bs}} A \ \mathsf{Ess}\ x \wedge A \ \mathsf{Ess}\ y \to \mathsf{L}(x \overset{1}{\approx} y)$

PROOF. The same as for **T24**. ■

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