Gödel's God in Isabelle/HOL

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1 Introduction

A formalization and (partial) automation of Dana Scott's version [10] of Goedel's ontological argument [7] in quantified modal logic KB (QML KB) is presented. QML KB is in turn modeled as a fragment of classical higher-order logic (HOL). Thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benzmüller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (poperties).

The formalization presented here has been carried and formally verified in the Isabelle/HOL proof assistant; for more information on this system see the textbook by Nipkow, Paulson, and Wenzel [9]. More recent tutorials on Isabelle can be found at the Isabelle homepage: http://isabelle.in.tum.de.

Some further notes:

1. This LaTeX text document has been produced automatically from the Isabelle source code document at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Isabelle/GoedelGodSession with the Isabelle build tool.

2. The formalization presented here is related to the THF [12] and Coq [4] formalizations available at https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/.

3. All reasoning gaps in Scott's proof script have been automated with Sledgehammer [5] performing remote calls to the higher-order automated theorem prover LEO-II [3]. These calls then suggested respective Metis [8] calls as given below. The Metis proofs are then verified in Isabelle/HOL.

4. For consistency checking, the Nitpick model finder [6] has been employed.

2 An Embedding of QML KB in HOL

The types i for possible worlds (or states) and mu for individuals are introduced.

typedecl i — the type for possible worlds
typedecl mu — the type for indiviuals

Possible worlds are connected by an accessibility relation .

consts $r :: i \Rightarrow i \Rightarrow bool$ (**infixr** r *70*) — accessibility relation r

The B axiom (symmetry) for relation r is stated. B is needed only for proving theorem T3.

axiomatization where *sym*: $x\ r\ y \longrightarrow y\ r\ x$

QML formulas are identified with certain HOL terms of type $i \Rightarrow bool$. This type will be abbreviated in the remainder as σ

type-synonym $\sigma = (i \Rightarrow bool)$

The classical connectives $\neg, \wedge, \Rightarrow$, and $\forall$ (for individuals and over sets of individuals) and $\exists$ (over individuals) are lifted to type σ. Further connectives could be introduced analogously. *definition* could be used instead of *abbreviation*; the latter are always fully expanded/rewritten, which is fine here, where the focus has been on proof automation, but which would lead to overly complex proof tasks in a purely interactive session.

abbreviation *mnot* $:: \sigma \Rightarrow \sigma$ $(m\neg)$ **where** $m\neg\ \Phi \equiv (\lambda w.\ \neg\ \Phi\ w)$
abbreviation *mand* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\wedge$ *74*) **where** $\Phi\ m\wedge\ q \equiv (\lambda w.\ \Phi\ w \wedge q\ w)$
abbreviation *mimplies* $:: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\Rightarrow$ *79*) **where** $p\ m\Rightarrow\ q \equiv (\lambda w.\ p\ w \longrightarrow q\ w)$
abbreviation *mforall-ind* $:: (mu \Rightarrow \sigma) \Rightarrow \sigma$ $(\forall i)$ **where** $\forall i\ P \equiv (\lambda w.\ \forall x.\ P\ x\ w)$
abbreviation *mexists-ind* $:: (mu \Rightarrow \sigma) \Rightarrow \sigma$ $(\exists i)$ **where** $\exists i\ P \equiv (\lambda w.\ \exists x.\ P\ x\ w)$
abbreviation *mforall-indset* $:: ((mu \Rightarrow \sigma) \Rightarrow \sigma) \Rightarrow \sigma$ $(\forall p)$ **where** $\forall p\ P \equiv (\lambda w.\ \forall x.\ P\ x\ w)$
abbreviation *mbox* $:: \sigma \Rightarrow \sigma$ $(\Box)$ **where** $\Box\ p \equiv (\lambda w.\ \forall v.\ \neg\ w\ r\ v \vee p\ v)$
abbreviation *mdia* $:: \sigma \Rightarrow \sigma$ $(\Diamond)$ **where** $\Diamond\ p \equiv (\lambda w.\ \exists v.\ w\ r\ v \wedge p\ v)$

For the grounding of lifted formulas the meta-predicate *valid* is introduced.

abbreviation *valid* $:: \sigma \Rightarrow bool$ ([-]) **where** $[p] \equiv \forall w.\ p\ w$

The model finder Nitpick confirms that the axioms and definitions above are consistent. Unfortunately, the respective command syntax for Nitpick is not very intuitive.

lemma *True* **nitpick** [*satisfy*, *user-axioms*, *expect* = *genuine*] **oops**

Constant symbol P (Gödel's "Positive") is introduced.

consts $P :: (mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms *A1(a/b)*: $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$ (Either a property or its negation is positive, but not both.) and *A2*: $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \to \psi(x)]) \to P(\psi)]$ (A property necessarily implied by a positive property is positive.).

axiomatization where
A1a: $[\forall p\ (\lambda\Phi.\ P\ (\lambda x.\ m\neg\ (\Phi\ x))\ m\Rightarrow\ m\neg\ (P\ \Phi))]$ **and**
A1b: $[\forall p\ (\lambda\Phi.\ m\neg\ (P\ \Phi)\ m\Rightarrow\ P\ (\lambda x.\ m\neg\ (\Phi\ x)))]$ **and**
A2: $[\forall p\ (\lambda\Phi.\ \forall p\ (\lambda\psi.\ (P\ \Phi\ m\wedge\ \Box\ (\forall i\ (\lambda X.\ \Phi\ X\ m\Rightarrow\ \psi\ X)))\ m\Rightarrow\ P\ \psi))]$

We prove theorem T1: $\forall\varphi[P(\varphi) \to \Diamond\exists x\varphi(x)]$ (Positive properties are possibly exemplified). T1 is proved directly by Sledghammer with command *sledgehammer* [*provers* = *remote-leo2 remote-satallax*]. This successful attempt then suggest to instead try the Metis call in the line below. This Metis call generates a proof object that is verified in Isabelle/HOL's kernel.

theorem *T1*: $[\forall p\ (\lambda\Phi.\ P\ \Phi\ m\Rightarrow\ \Diamond\ (\exists i\ \Phi))]$
sledgehammer [*provers* = *remote-leo2*]
using *A2 A1a* **by** *metis*

Next, the symbol G, for "God-like", is introduced and defined as $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$ (A God-like being possesses all positive properties:).

definition $G :: mu \Rightarrow \sigma$ **where** $G = (\lambda x.\ \forall p\ (\lambda\Phi.\ P\ \Phi\ m\!\Rightarrow\ \Phi\ x))$

Axiom *A3* is added: $P(G)$ (The property of being God-like is positive.). Sledgehammer and Metis then prove corollary C: $\Diamond\exists x G(x)$ (Possibly, God exists.).

axiomatization where *A3*: $[P\ G]$

corollary C: $[\Diamond\ (\exists\ i\ G)]$
sledgehammer [*provers* = *remote-leo2*]
using *A3 T1* **by** *metis*

We add axiom *A4*: $\forall\phi[P(\phi) \rightarrow \Box\ P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where *A4*: $[\forall\ p\ (\lambda\Phi.\ P\ \Phi\ m\!\Rightarrow\ \Box\ (P\ \Phi))]$

Symbol *ess*, for "Essence", is introduced and defined as $\phi\ ess.\ x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$ (An *essence* of an individual is a property possessed by it and necessarily implying any of its properties.).

definition *ess* :: $(mu \Rightarrow \sigma) \Rightarrow mu \Rightarrow \sigma$ (**infixr** *ess 85*) **where**
$\Phi\ ess\ x = \Phi\ x\ m\!\wedge\ \forall\ p\ (\lambda\psi.\ \psi\ x\ m\!\Rightarrow\ \Box\ (\forall\ i\ (\lambda y.\ \Phi\ y\ m\!\Rightarrow\ \psi\ y)))$

Next, Sledgehammer and Metis prove theorem *T2*: $\forall x[G(x) \rightarrow G\ ess.\ x]$ (Being God-like is an essence of any God-like being).

theorem *T2*: $[\forall\ i\ (\lambda x.\ G\ x\ m\!\Rightarrow\ G\ ess\ x)]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis* (*lifting*) *A1b A4 G-def ess-def*)

Symbol *NE*, for "Necessary Existence", is introduced and defined as $NE(x) \leftrightarrow \forall\phi[\phi\ ess.\ x \rightarrow \Box\exists y\phi(y)]$ (Necessary existence of an individual is the necessary exemplification of all its essences.).

definition $NE :: mu \Rightarrow \sigma$ **where** $NE = (\lambda x.\ \forall\ p\ (\lambda\Phi.\ \Phi\ ess\ x\ m\!\Rightarrow\ \Box\ (\exists\ i\ \Phi)))$

Moreover, axiom *A5* is added: $P(NE)$ (Necessary existence is a positive property.).

axiomatization where *A5*: $[P\ NE]$

Finally, Sledgehammer and Metis prove the main theorem *T3*: $\Box\exists x G(x)$ (Necessarily, God exists).

theorem *T3*: $[\Box\ (\exists\ i\ G)]$
sledgehammer [*provers* = *remote-leo2*]
using *A5 C T2 sym G-def NE-def* **by** *metis*

corollary *T4*: $[\exists\ i\ G]$
sledgehammer [*provers* = *remote-leo2*]
using *T1 T3 sym G-def* **by** *metis*

Finally, the consistency of the entire theory is checked with Nitpick.

lemma *True* **nitpick** [*satisfy*, *user-axioms*, *expect* = *genuine*] **oops**

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