Gödel's Ontological Proof of God's Existence

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Possible witnessing of positive properties

Axioms:

- **(1)** Properties necessarily entailed by *positive* properties are also positive:

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

- **(2)** A property's negation is positive iff the property is not positive:

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Theorem 1: Positive properties possibly have a witness:

$$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$$

Formal proof:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]}{\forall\psi.[(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \psi(x)]) \to P(\psi)]}}{(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \neg\varphi'(x)]) \to P(\neg\varphi')}}{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to P(\neg\varphi')} \qquad \cfrac{\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]}{P(\neg\varphi') \leftrightarrow \neg P(\varphi')}}{(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to \neg P(\varphi')}}{P(\varphi') \to \Diamond\exists x.\varphi'(x)}}{\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]}$$

2 Possible existence of a God

Axioms:

- **(3)** Being God is a positive property:

$$P(G)$$

Theorem 2: It is possible that a God exists:

$$\Diamond\exists x.G(x)$$

Formal proof:

$$\dfrac{P(G) \qquad \dfrac{\dfrac{}{\forall\varphi.[P(\varphi)\to\Diamond\exists x.\varphi(x)]}\ \text{Th. 1}}{P(G)\to\Diamond\exists x.G(x)}}{\Diamond\exists x.G(x)}$$

3 Essentiality of being God

Definitions:

- A property is *essential* for an individual if and only if it holds for that inidividual and necessarily entails every other property that holds for that individual: $\varphi\ ess\ x \leftrightarrow \varphi(x)\wedge\forall\psi.(\psi(x)\to\Box\forall x.(\varphi(x)\to\psi(x)))$

Axioms:

- **(4)** Positive properties are necessarily positive:

$$\forall\varphi.[P(\varphi)\to\Box\ P(\varphi)]$$

Theorem 3: If an individual is a God, then being God is an essential property for that individual:

$$\forall y.[G(y)\to G\ ess\ y]$$

Formal proof:

Let the following derivation with the open assumption $G(x)$ be Π_1:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(\psi)^1 \qquad \dfrac{\dfrac{}{\forall\varphi.(\neg P(\varphi)\to P(\neg\varphi))}\ \text{Ax. 2}}{\neg P(\psi)\to P(\neg\psi)}\ \forall\text{E}}{P(\neg\psi)}\ \to\text{E} \qquad \dfrac{\dfrac{G(x)}{\forall\varphi.(P(\varphi)\to\varphi(x))}\ \text{Definition of G}}{P(\varphi)\to\varphi(x)}\ \forall\text{E}}{\neg\psi(x)}\ \to\text{E} \qquad \psi(x)^2}{\bot}\ \to\text{E}}{P(\psi)}\ \text{RAA, 1}}{\psi(x)\to P(\psi)}\ \to\text{I, 2}$$

Let the following derivation with the open assumption $G(x)$ be Π_2:

$$\cfrac{\cfrac{\psi(x)^1 \qquad \begin{array}{c}\Pi_1\\ \hdashline \psi(x) \to P(\psi)\end{array}}{P(\psi)}\ \to\text{E} \qquad \cfrac{\cfrac{}{\forall\psi.(P(\psi) \to \Box P(\psi))}\ \text{Ax. 4}}{P(\psi) \to \Box P(\psi)}\ \forall\text{E}}{\cfrac{\Box P(\psi)}{\psi(x) \to \Box P(\psi)}\ \to\text{I, 1}}\ \to\text{E}$$

Let the following derivation without open assumptions be Π_3:

$$\cfrac{\cfrac{\cfrac{\cfrac{P(\psi)^1 \qquad \cfrac{\begin{array}{c}G(x)^2\\ \cdots\cdots\cdots\ \text{Definition of G}\\ \forall\varphi.(P(\varphi) \to \varphi(x))\end{array}}{P(\psi) \to \psi(x)}\ \forall\text{E}}{\psi(x)}\ \to\text{E}}{G(x) \to \psi(x)}\ \to\text{I, 2}}{\forall x.(G(x) \to \psi(x))}\ \forall\text{I}}{P(\psi) \to \forall x.(G(x) \to \psi(x))}\ \to\text{I, 1}$$

Let the following derivation with the open assumption $G(x)$ be Π_4:

$$\cfrac{\cfrac{\psi(x)^1 \qquad \begin{array}{c}\Pi_2\\ \hdashline \psi(x) \to \Box P(\psi)\end{array}}{\Box P(\psi)}\ \to\text{E} \qquad \cfrac{\cfrac{\cfrac{\cfrac{\Box P(\psi)^2}{P(\psi)}\ \Box\text{E} \qquad \begin{array}{c}\Pi_3\\ \hdashline P(\psi) \to \forall x.(G(x) \to \psi(x))\end{array}}{\forall x.(G(x) \to \psi(x))}\ \to\text{E}}{\Box\forall x.(G(x) \to \psi(x))}\ \text{Necessitation}}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))}\ \to\text{I, 2}}{\cfrac{\Box\forall x.(G(x) \to \psi(x)}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}\ \to\text{I, 1}}\ \to\text{E}$$

The use of the necessitation rule is valid, because the only open assumption $\Box P(\psi)$ is boxed.

We construct a derivation of theorem 3 with a subderivation $\Pi_4[G(x)^1]$, which means that the open assumption $G(x)$ in Π_4 is discharged with the rule labeled 1.

$$\cfrac{\cfrac{\begin{array}{c}\cfrac{G(x)^1 \qquad \cfrac{\begin{array}{c}\Pi_4[G(x)^1]\\ \hdashline \psi(x) \to \Box\forall x.(G(x) \to \psi(x))\end{array}}{\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\ \forall\text{I}}{G(x)\&\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}\ \&\text{I}\\ \cdots\cdots\cdots\cdots\ \text{Definition of ess}\\ G\ ess\ x\end{array}}{G(x) \to G\ ess\ x}\ \to\text{I, 1}}{\forall y.[G(y) \to G\ ess\ y]}$$

Note: the formal proof above uses the necessitation rule of the basic modal logic **K** and Axiom 4, which corresponds to a restricted form of necessitation.

4 Necessity of God's existence

Definitions:

- An individual is a *God* if and only if he possesses all positive properties:

$$G(x) \leftrightarrow \forall \varphi.[P(\varphi) \to \varphi(x)]$$

- An individual *necessarily exists* if and only if all its essential properties are necessarily witnessed:

$$E(x) \leftrightarrow \forall \varphi.[\varphi \ ess \ x \to \Box \exists x.\varphi(x)]$$

Axioms:

- **(5)** Necessary existence is a positive property:

$$P(E)$$

Theorem A: If there is a God, then there necessarily exists a God:

$$\exists z.G(z) \to \Box \exists x.G(x)$$

Formal proof:

$$\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\exists z.G(z)$}
\UnaryInfC{$G(g)$}
\end{prooftree}$$

$$\begin{prooftree}
\AxiomC{}
\dashedLine
\UnaryInfC{$G(g)$}
\AxiomC{}
\dashedLine
\RightLabel{Th. 3}
\UnaryInfC{$\forall y.[G(y) \to G \ ess \ y]$}
\UnaryInfC{$G(g) \to G \ ess \ g$}
\BinaryInfC{$G \ ess \ g$}
\AxiomC{$P(E)$}
\AxiomC{}
\dashedLine
\UnaryInfC{$G(g)$}
\dottedLine
\UnaryInfC{$\forall \varphi.[P(\varphi) \to \varphi(g)]$}
\UnaryInfC{$P(E) \to E(g)$}
\BinaryInfC{$E(g)$}
\dottedLine
\UnaryInfC{$\forall \varphi.[\varphi \ ess \ g \to \Box \exists x.\varphi(x)]$}
\UnaryInfC{$G \ ess \ g \to \Box \exists x.G(x)$}
\BinaryInfC{$\Box \exists x.G(x)$}
\RightLabel{1}
\UnaryInfC{$\exists z.G(z) \to \Box \exists x.G(x)$}
\end{prooftree}$$

Note: Theorem A could be proved more quickly using the necessitation rule. Interestingly, the proof above shows that, by using the given axioms and definitions of god and necessary existence, theorem A can be derived even without the necessitation rule.

5 Necessary existence of a God

Theorem 4: The existence of a God is necessary:

$$\Box\exists x.G(x)$$

Formal proof:

$$\dfrac{\dfrac{\overline{\forall\varphi.[\Diamond\ldots\Diamond\Box\varphi \leftrightarrow \Box\varphi]}\ \mathbf{S5}}{\Diamond\Box\exists x.G(x) \leftrightarrow \Box\exists x.G(x)} \qquad \dfrac{\overline{\Diamond\exists x.G(x)}\ \text{Th. 2} \qquad \overline{\exists z.G(z) \to \Box\exists x.G(x)}\ \text{Th. A}}{\Diamond\Box\exists x.G(x)}}{\Box\exists x.G(x)}$$

Note: The proof above relies on a theorem of the modal logic **S5**, which is a quite strong modal logic. It would be interesting to try to derive theorem 4 with weaker modal axioms.

6 God's existence

Axioms:

- **(M)** What is necessary is the case:

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Theorem: There exists a God:

$$\exists x.G(x)$$

Formal proof:

$$\dfrac{\overline{\Box\exists x.G(x)}\ \text{Th. 4} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$