Definition of *Divine property*:

$D(F) =_D \Box\forall x(Gx \supset Fx)$

Comprehension principle for the existence of the property G of being *Godly*:

$\exists G\forall x(Hx \equiv \forall Y(D(Y) \supset \Box Yx))$

By substitution we have:

$\exists G\forall x(Gx \equiv \forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx))$

This is circular. As the occurence of G in the definiens is $\Diamond$-positive, i.e. positive but within the scope of a $\Diamond$-operator if written in disjunctive normal form, we may first seek to avoid this circularity by the impredicative definition:

$\exists G\forall x(Gx \equiv \forall K(\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset Kw) \supset Kx))$

Let Gx be an instantiation so that

$\forall x(Gx \equiv \forall K(\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset Kw) \supset Kx))$

We seek to prove

$\forall x(Gx \equiv \forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx))$

Suppose for arbritrary K:

$\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset Kw)$

Then

$\forall x(Gx \supset Kx)$

But here we do not achieve $\Box\forall x(Gx \supset Kx)$, which is needed here to utilize $\Diamond-$positiveness of the definition in order to substitute as desired.

This necessitates a change in comprehension:

$\forall x(Gx \equiv \forall K(\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw) \supset Kx))$

Let K be satiated iff

$\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw)$

Suppose K is satiated. Then $\Box\forall x(Gx \supset Kx)$, invoking also S5 principles in a conditional proof.

Since occurrences of K are $\Diamond-$positive in $\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw)$, $\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yw) \supset \forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw)$.

Since K is assumed to be satiated, $\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yw) \supset \Box Kw$. By the T-schema, $\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yw) \supset Kw$.

Again, as K was assumed to be arbitrary and satiated we discharge this assumption and generalize to get $\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx) \supset \forall K(\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw) \supset Kx)$. By the definition of G this is:

$\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx) \supset Gx$. By necessitation, $\Box(\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx) \supset Gx)$.

Introduce by comprehension:

$Jx \equiv \forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx)$.

We have shown that $\Box\forall x(Jx \supset Gx)$. But then again, since occurrences of G in $\forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yw)$ are $\Diamond-$positive, $\forall w(\forall Y(\Box\forall z(Jz \supset Yz) \supset \Box Yw) \supset \forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yw))$. By the definition of J, $\forall w(\forall Y(\Box\forall z(Jz \supset Yz) \supset \Box Yw) \supset Jw)$. By S5 principles and the second order Barcan fomula, $\forall w(\forall Y(\Box\forall z(Jz \supset Yz) \supset \Box Yw) \supset \Box Jw)$. So J is satiated, hence $\Box\forall x(Gx \supset Jx)$. By the definition of J, this is $\Box\forall x(Gx \supset \forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx))$.

Putting these together,

$\Box\forall x(Gx \equiv \forall Y(\Box\forall z(Gz \supset Yz) \supset \Box Yx))$.

q.e.d.

We inspect the comprehension scheme used, viz.

$\forall x(Gx \equiv \forall K(\forall w(\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw) \supset Kx))$.

To achieve prenex normal form from this we first have, equivalently,

$\forall x(Gx \equiv \forall K\exists w((\forall Y(\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw) \supset Kx))$. Next, equivalently

$\forall x(Gx \equiv \forall K\exists w(\exists Y((\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw) \supset Kx))$. Finally,

$\forall x(Gx \equiv \forall K\exists w\forall Y((\Box\forall z(Kz \supset Yz) \supset \Box Yw) \supset \Box Kw) \supset Kx)$.

Hence, $\prod\binom{1}{3}$-comprehension in the context of modal second order logic is sufficient for the introduction of G.

With this machinery we can show in second order modal S5, presupposing the definition of *Divine property* as given by

$D(F) =_D \Box\forall x(Gx \supset Fx)$, that

$\exists X(\backsim D(X)) \supset \Box\exists xGx$