Formalization and Automated Verification of Gödel's Proof of God's Existence*

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Attempts to prove the existence (or non-existence) of God by means of abstract ontological arguments are an old tradition in philosophy and theology. Gödel's proof [1, 2] is a modern culmination of this tradition, following particularly the footsteps of Leibniz. Gödel defines God as a being who possesses all *positive* properties. He does not extensively discuss what positive properties are, but instead he states a few reasonable (but debatable) axioms that they should satisfy. Various slightly different versions of axioms and definitions have been considered by Gödel and by several philosophers who commented on his proof. We have analyzed Scott's version of Gödel's proof [3] for the first-time with an unprecedent degree of detail and formality with the help of theorem provers; cf. https://github.com/FormalTheology/GoedelGod.

The following has been done (and in this order):[3] (i) a detailed natural deduction proof; (ii) a formalization in TPTP THF syntax; (iii) an automatic verification of the consistency of the axioms and definitions with Nitpick; (iv) an automatic demonstration of the theorems with the provers LEO-II and Satallax; (v) a step-by-step formalization using the Coq proof assistant; (vi) a formalization using the Isabelle proof assistant, where the theorems (and some additional lemmata) have been automated with the Isabelle tools Sledgehammer and Metis.

Gödel's proof is challenging to formalize and verify because it requires an expressive logical language with modal operators (*possibly* and *necessarily*) and with quantifiers for individuals and properties. Our computer-assisted formalizations rely on an embedding of the modal logic into classical higher-order logic with Henkin semantics. The formalization is thus essentially done in classical higher-order logic where quantified modal logic is emulated.

In our ongoing computer-assisted study of Gödel's proof our proof tools have made some interesting observations, including: (a) The basic modal logic K is sufficient for proving the first three theorems (T1, Coro and T2) as outlined in Scott's notes. (b) For proving the final theorem (T3), logic KB is sufficient. (c) Gödel's original version of the proof [2], which omits conjunct $\phi(x)$ in definition of *essence*, seems inconsistent.

This work attests the maturity of contemporary interactive and automated deduction tools for classical higher-order logic and demonstrates the elegance and practical relevance of the embeddings-based approach. Most importantly, our work opens new perspectives for a computer-assisted theoretical philosophy. The critical discussion of the underlying concepts, definitions and axioms remains a human responsibility, but the computer can assist in building and checking rigorously correct logical arguments. In case of logico-philosophical disputes, the computer can check the disputing arguments and partially fulfill Leibniz' dictum: Calculemus — Let us calculate!

References

1. K. Gödel. Ontological proof. In *Kurt Gödel: Collected Works Vol. 3: Unpublished Essays and Letters*. Oxford University Press, 1970.
2. K. Gödel. *Appendix A. Notes in Kurt Gödel's Hand*, pages 144–145. In [4], 2004.
3. D. Scott. *Appendix B. Notes in Dana Scott's Hand*, pages 145–146. In [4], 2004.
4. J.H. Sobel. *Logic and Theism: Arguments for and Against Beliefs in God.* Cambridge U. Press, 2004.

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[3] System URLs: TPTP—http://tptp.org; Nitpick—http://www4.in.tum.de/~blanchet/nitpick.html; LEO-II—http://leoprover.org; Satallax—http://www.ps.uni-saarland.de/~cebrown/satallax; Coq—http://coq.inria.fr; Isabelle—http://isabelle.in.tum.de.