A Variant of Gödel's Ontological Proof in a Natural Deduction Calculus

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not."
- Kurt Gödel [10][p. 316]

1 Introduction

Ontological arguments for the existence of God can be traced back at least to St. Anselm (1033-1109). His argument considers a greatest conceivable being, who must exist, because if it did not have the property of existence, then we could conceive of a greater being that, apart from the other properties, also has the property of existence.

A major critique of this argument is that we do not know whether the concept of greatest conceivable being in fact designates anything or if it is inconsistent, like a round square [3][p.134]. As Bertrand Russell pointed out, the definition of greatest conceivable being allows us to define properties, like having boots, which the being then also must have [**?**].

Kant argued against the ontological argument on the basis that existence is not an analytic property [5]. This means that existence cannot be contained in the definition of a concept, because it is generally synthetic. The strongest claim that can be proven by the ontological argument is a conditional claim of necessary existence: if God exists, then God necessarily exists [**?**].

St. Anselm's argument was further elaborated by Descartes and Leibniz. Leibniz identified the possible existence of God as a critical missing step in St. Anselm's argument. To fill this gap, he argued that the properties of God, the perfections, are compatible. This means that it is possible to satisfy all perfections at once which implies that the existence of a greatest conceivable being with all these properties is possible.

Gödel studied Leibniz's work [**?**] and brought his ontological argument to a modern form using a modal logic with higher-order quantification over properties. In this setting, he gave precise axioms describing the notion of *positive* property and defined God as a being having all positive properties.

Gödel's notion of positive property and Leibniz's notion of perfection differ [**?**]. The main formal distinction seems to be that Gödel's positive properties are not just atomic properties, like Leibniz's perfections, but can also consist of complex combinations of atomic properties [3][p.139.] (TODO: Check this claim). In particular, one of Gödel's axioms states that any conjunction of positive properties is itself positive. From this axiom, it is immediately deduced that the property of being God-like is positive. While this step is intuitively and informally clear, it is not easily formalizable in a standard logical calculus, because it requires inferring that being God-like is a (possibly infinite) conjunction of positive properties, from the universal definition of God-likeness as the property satisfied by individuals who have all positive properties. This interplay between universal quantification (in the definition of a God-like being) and infinite conjunctions (in Gödel's axiom for positive properties) is a technical difficulty that probably explains why, starting with Scott [7], to whom Gödel confided his manuscript, this axiom of Gödel's has been replaced by another that simply assumes the positivity of the property of being God-like.

The main criticism against Gödel's ontological argument is an undesirable consequence of the stipulated axioms, called *modal collapse*. Many recent works [**?**] on the ontological argument have focused on proposing modifications of the argument that would not entail a modal collapse. This is discussed in greater detail in Section 6, but possible solutions are not implemented in this paper.

The aim of this paper is to present two detailed formal proofs of Gödel's ontological argument [4] in a natural deduction calculus (as defined in Section 2). The natural deduction style [**? ?**] was chosen for three reasons. Firstly, presentations of Gödel's proof are typically either informal or formalized in other styles of calculi (e.g. Fitting's tableaux [3] or Sobel's sentential modal calculus [9]). Therefore, a formalization in natural deduction is a valuable complement to the existing presentations. Secondly, it makes the ontological proof accessible to people who are familiarized with a natural deduction style. Thirdly, as natural deduction is the style used by proof assistants such as Coq [**?**] and Isabelle [**?**], the natural deduction formalizations have been easily checked step-by-step in Coq [**?**].

The first contribution of this paper is a detailed formalization of Scott's version [7] of Gödel's ontological argument [4] (as shown in Section 4) in a natural deduction calculus. The second contribution of the paper is a new proof (also in natural deduction style) of the lemmas, theorems and corollaries in Gödel's manuscript. In contrast to Scott's proofs [7], the proofs presented here are simpler and shorter, as discussed in Section 7.

2 Natural Deduction

The language of higher-order modal logic used here is inspired by that of Church's simple type theory [**?**].

Definition 1. Simple types *are given by the following grammar:*

$$\theta, \tau \quad ::= \quad \mu \ \mid \ o \ \mid \ \theta \to \tau$$

where μ is the atomic type for individuals, o is the atomic type for propositions and $\theta \to \tau$ is the type for functions taking an argument of type θ and returning something of type τ. '$\to$' is assumed to be right associative.

ToDo: decide which connectives and corresponding inference rules will be primitive.

Definition 2. Terms and formulas *are given by the following grammar:*

$$\begin{array}{rcl} s,t & ::= & p_\tau \ \mid \ X_\tau \ \mid \ (\lambda X_\theta . s_\tau)_{\theta\to\tau} \ \mid \ (s_{\theta\to\tau}\, t_\theta)_\tau \ \mid \\ & & \bot_o \ \mid \to_{o\to o\to o} \ \mid \ \wedge_{o\to o\to o} \ \mid \ \vee_{o\to o\to o} \ \mid \\ & & \forall_{(\tau\to o)\to o} \ \mid \ \exists_{(\tau\to o)\to o} \ \mid \ \Box_{o\to o} \ \mid \ \Diamond_{o\to o} \end{array}$$

where p_τ and X_τ range over, respectively, constants and variables of type τ. Parenthesis conventions, infix notation for propositional connectives and binding notation for quantifiers are assumed. Furthermore, subscript types are omitted when they are clear from the context. Negation ($\neg_{o\to o}$) and equivalence ($\leftrightarrow_{o\to o\to o}$) are defined by $\neg A \equiv A \to \bot$ and $(A \leftrightarrow B) \equiv (A \to B) \wedge (B \to A)$.

ToDo: decide if rules for diamond should be added.

The natural deduction calculus used here has standard rules for propositional connectives and quantifiers, as shown in Figures 1 and 3. The extension to classical logic is achieved by adding a rule for double negation elimination, shown in Figure 2. Finally, modal operators are handled by special rules that insert or remove formulas from boxes, as shown in Figure 4. Apart from the use of labels and the dual rules for '$\Diamond$', these rules are essentially the modal rules from [**?**]. Beta-reduction is implicit; all rules are assumed modulo beta-reduction. A *derivation* is a directed acyclic graph whose nodes are formulas and whose edges correspond to applications of the inference rules. A *proof* of a formula F is a derivation without open assumptions and having F as root not inside any box.

Double lines are used to abbreviate tedious propositional reasoning steps in the derivations. Dashed lines are used to refer to an axiom or theorem with proof shown elsewhere. Dotted lines are used to indicate folding and unfolding of definitions. Furthermore, as it is inconvenient to draw boxes around large derivations in LaTeX, formulas inside boxes are labeled with the names of the boxes surrounding them. Therefore, the boxes can be omitted without loss of information.
The calculus having only the rules shown in Figures 1, 2 and 3 is named **ND**. The calculus with the additional rules shown in Figure 4 is called $\mathbf{ND_K}$.

2.1 Suitability for Rigid Higher-Order Modal Logic K

Adding the modal rules results in a calculus that is suitable for the basic modal logic **K**. In other words, $\mathbf{ND_K}$ is sound and complete relative to **ND** extended with axiom K ($\Box(A \to B) \to (\Box A \to \Box B)$) and the necessitation rule (which establishes that $\Box A$ is a theorem if A is a theorem).

Fig. 1. propositional rules

$$\frac{\bot}{A}\ \bot_E \qquad \frac{B}{A \to B}\ \to_I \qquad \frac{\begin{array}{c}\overline{A}^{\;n}\\ \vdots \\ B\end{array}}{A \to B}\ \to_I^n \qquad \frac{A \quad A \to B}{B}\ \to_E$$

$$\frac{A \quad B}{A \wedge B}\ \wedge_I \qquad \frac{A \wedge B}{A}\ \wedge_{E_1} \qquad \frac{A \wedge B}{B}\ \wedge_{E_2}$$

Fig. 2. double negation elimination

$$\frac{\neg\neg A}{A}\ \neg\neg_E$$

Fig. 3. quantifier rules

$$\frac{A[\alpha]}{\forall x_\tau . A[x]}\ \forall_I \qquad \frac{\forall x_\tau . A[x]}{A[t]}\ \forall_E \qquad \frac{A[t]}{\exists x_\tau . A[x]}\ \exists_I \qquad \frac{\exists x_\tau . A[x]}{A[\beta]}\ \exists_E$$

eigen-variable conditions:
if ρ is a $\forall_I$ inference eliminating a variable α, then any occurrence of α in the proof should be an ancestor of the occurrence of α eliminated by ρ;
if ρ is a $\exists_E$ inference introducing a variable β, then any occurrence of β in the proof should be a descendant of the occurrence of β introduced by ρ.

Theorem 1. $\mathbf{ND_K}$ *is complete, relative to* $\mathbf{ND}$ *extended with axiom K and the necessitation rule.*

Proof. The necessitation rule can be immediately simulated with the $\Box_I$ rule. Axiom K can be derived in $\mathbf{ND_K}$ as shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\Box(A \to B)}^{\,2}}{\omega : A \to B}\ \Box_E \qquad \dfrac{\overline{\Box A}^{\,1}}{\omega : A}\ \Box_E}{\omega : B}\ \to_E}{\Box B}\ \Box_I}{\Box A \to \Box B}\ \to_I^1}{\Box(A \to B) \to (\Box A \to \Box B)}\ \to_I^2$$

Fig. 4. Rules for Modal Operators

$$\dfrac{\omega : \boxed{\begin{matrix}\vdots\\ A\end{matrix}}}{\Box A}\,\Box_I \qquad\qquad \dfrac{\Box A}{w : \boxed{\begin{matrix}A\\ \vdots\end{matrix}}}\,\Box_E$$

$$\dfrac{w : \boxed{\begin{matrix}\vdots\\ A\end{matrix}}}{\Diamond A}\,\Diamond_I \qquad\qquad \dfrac{\Diamond A}{\omega : \boxed{\begin{matrix}A\\ \vdots\end{matrix}}}\,\Diamond_E$$

eigen-box condition:
$\Box_I$ and $\Diamond_E$ are *strong* modal rules:
ω must be a fresh name for the box they access
(in analogy to the eigen-variable condition for strong quantifier rules).
Every box must be accessed by *exactly one* strong modal inference.

boxed assumption condition:
assumptions should be discharged within the box where they are created.

Theorem 2. $\mathbf{ND_K}$ *is sound, relative to* **ND** *extended with axiom K and the necessitation rule.*

Proof. It is necessary to show that $\mathbf{ND_K}$ proofs of the following form can be translated to proofs in **ND** extended with the axiom K and the necessitation rule.

$$\dfrac{\begin{matrix}\dfrac{\Box A_1}{\omega : A_1}\,\Box_E \\ \vdots\end{matrix} \quad \dots \quad \begin{matrix}\dfrac{\Box A_n}{\omega : A_n}\,\Box_E\\ \vdots\end{matrix}}{\dfrac{\omega : B}{\Box B}\,\Box_I}$$

A translation to **ND** extended with axiom K and necessitation is shown below for the case when $n = 1$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}\dfrac{}{A_1}\,1\\ \vdots\\ B\end{matrix}}{A_1 \to B}\,\to_I^1}{\Box(A_1 \to B)}\,\text{necessitation} \qquad \dfrac{}{\Box(A_1 \to B) \to (\Box A_1 \to \Box B)}\,\text{Axiom K}}{\Box A_1 \to \Box B}\,\to_E \qquad \Box A_1}{\Box B}\,\to_E$$

For $n > 1$, the translation is a straightforward generalization:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{A_1}
\noLine
\UnaryInfC{$\vdots$}
\AxiomC{$\ldots$}
\AxiomC{}
\RightLabel{n}
\UnaryInfC{A_n}
\noLine
\UnaryInfC{$\vdots$}
\TrinaryInfC{B}
\doubleLine
\RightLabel{$\rightarrow_I^*$}
\UnaryInfC{$A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B$}
\RightLabel{nec.}
\UnaryInfC{$\Box(A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B)$}
\AxiomC{Axiom K, iterated}
\doubleLine
\dashedLine
\UnaryInfC{$\Box(A_1 \rightarrow \ldots \rightarrow A_n \rightarrow B) \rightarrow (\Box A_1 \rightarrow \ldots \rightarrow \Box A_n \rightarrow \Box B)$}
\RightLabel{$\rightarrow_E$}
\BinaryInfC{$\Box A_1 \rightarrow \ldots \rightarrow \Box A_n \rightarrow \Box B$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\dashedLine
\UnaryInfC{$\Box A_1 \rightarrow \ldots \rightarrow \Box A_n \rightarrow \Box B$}
\AxiomC{$\Box A_1$}
\AxiomC{$\ldots$}
\AxiomC{$\Box A_n$}
\doubleLine
\RightLabel{$\rightarrow_E$}
\QuaternaryInfC{$\Box B$}
\end{prooftree}
$$

Without the restriction that every box must be accessed by exactly one strong modal inference, the calculus would be unsound for the modal logic **K**, because the formula $\forall\psi.(\Box\psi \rightarrow \Diamond\psi)$ would be derivable although it is not valid in **K**.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\Box\psi$}
\RightLabel{$\Box_E$}
\UnaryInfC{$\omega : \psi$}
\RightLabel{$\Diamond_I$}
\UnaryInfC{$\Diamond\psi$}
\RightLabel{$\rightarrow_I^1$}
\UnaryInfC{$\Box\psi \rightarrow \Diamond\psi$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\forall\psi.(\Box\psi \rightarrow \Diamond\psi)$}
\end{prooftree}
$$

This example of an unsound derivation is prevented by the eigen-box condition because the box labelled by ω is not accessed by any strong inference.

The straightforward combinations of the quantifier rules of **ND** either with the modal rules of $\mathbf{ND_K}$ or with axiom K and the necessitation rule are suitable for a higher-order modal logic where constants and variables are *rigid*. From the point of view of a *possible worlds* semantics, rigidity means that their interpretation is independent of the world in which they are being interpreted. Rigidity is silently assumed by most works investigating the ontological argument, and is explicitly assumed here. Nevertheless, it should be noted that its adequacy has already been contested [3].

ToDo: Talk about constant domains and varying domains, mention Conchiarella's semantics and Anderson's footnote 14.

3 Some Derivable Modal Principles

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As mentioned in [1][Footnote 5], there are sources that discuss what modal principles are needed for the ontological argument. The standard system for the argument is S5. Through an analysis of the proofs weaker systems, in particular KB, can be shown to be sufficient for versions of the ontological argument [9][p. 152]. The proof presented in Section 5 is such a version and explicitly shows a single use of Brouwer Reduction Principle, which, in turn, is provable from axiom B. All other proof steps are formalizable in a basic second-order modal logic K.

In standard first-order modal logics the modal axioms correspond to geometric conditions on the frames of Kripke models [6]. The modal axiom **T** corresponds to reflexivity of the frame and **B** corresponds to symmetry of the frame. The frames of the modal logic **S5** are in addition transitive.

ToDo: Sobel Anderson on KB: Sobel page 152 and Anderson's footnote 5 on KB

3.1 A modal distribution principle in K

To prove Gödel's ontological proof in natural deduction we use a modal distribution principle derivable in K and Brouwer's reduction theorem derivable from the modal Axiom B.

Lemma 1. *The distribution principle*

$$\Box(A \to B) \to (\Diamond A \to \Diamond B)$$

is provable in our system K of modal logic.

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\Box(A \to B)$}
\RightLabel{$\Box_E$}
\UnaryInfC{$\omega : A \to B$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\Diamond A$}
\RightLabel{$\Diamond_E$}
\UnaryInfC{$\omega : A$}
\RightLabel{$\to_E$}
\BinaryInfC{$\omega : B$}
\RightLabel{$\Diamond_I$}
\UnaryInfC{$\Diamond B$}
\RightLabel{$\to_I^1$}
\UnaryInfC{$\Diamond A \to \Diamond B$}
\RightLabel{$\to_I^2$}
\UnaryInfC{$\Box(A \to B) \to (\Diamond A \to \Diamond B)$}
\end{prooftree}
$$

3.2 Axiom B and Brouwer's Reduction Principle

Brouwer's reduction theorem is a version of axiom B for modal logic and a property that enforces symmetry as a geometric property for the frame of a Kripke sematics.

Lemma 2. *The Brouwer Reduction Principle $\Diamond\Box A \to A$ is derivable from axiom B, $A \to \Box\Diamond A$.*

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\neg\Box\neg\Box A$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\neg A$}
\AxiomC{Axiom B}
\dashedLine
\UnaryInfC{$\neg A \to \Box\neg\Box\neg\neg A$}
\RightLabel{$\to_E$}
\BinaryInfC{$\Box\neg\Box\neg\neg A$}
\doubleLine
\UnaryInfC{$\Box\neg\Box A$}
\BinaryInfC{$(\neg\Box\neg\Box A)\&(\Box\neg\Box A)$}
\RightLabel{$\neg_I^1$}
\UnaryInfC{$\neg\neg A$}
\RightLabel{$\neg\neg_E$}
\UnaryInfC{A}
\RightLabel{$\to_I^2$}
\UnaryInfC{$\neg\Box\neg\Box A \to A$}
\end{prooftree}
$$

3.3 S5 and its Iteration Principle

ToDo: Is this principle really commonly called “S5 Iteration Principle”?

Lemma 3. *What is possibly necessary is necessary:*

$$\Diamond\Box A \to \Box A$$

Proof. ToDo

4 Scott’s Proof in Natural Deduction

ToDo: natural language explanation of the proof in Anderson.

The acceptance of the correctness of the ontological argument by Gödel’s work boils down to the intuitive correctness of the axioms and definitions and the belief in the soundness of the deductive system. The formal argument of Gödel is based on Leibniz proof, which in turn is based on Descartes proof. These proofs have two parts; a proof that if the exemplification of the god-likeness property is possible, then it is necessary and a proof that the exemplification is in fact possible.

Axiom 1 *Either a property or its negation is positive, but not both:*

$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Axiom 2 *A property necessarily implied by a positive property is positive:*

$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

Theorem 1. *Positive properties are possibly exemplified:*

$$P(\varphi) \to \Diamond\exists x.\varphi(x)$$

Proof.

$$\dfrac{\dfrac{\dfrac{\ }{P(\rho)}\,3 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\text{Reflexivity}}{\omega : \gamma = \gamma}}{\omega : \rho\gamma \to \gamma = \gamma}\,\to_I}{\omega : \forall x.\rho(x) \to x = x}\,\forall_I}{\Box(\forall x.\rho(x) \to x = x)}\,\Box_I}{P(\rho) \wedge \Box(\forall x.\rho(x) \to x = x)}\,\wedge_I \qquad \dfrac{\text{Axiom 2 for } \rho \text{ and } \lambda x.x = x}{P(\rho) \wedge \Box(\forall x.\rho(x) \to x = x) \to P(\lambda x.x = x)}}{P(\lambda x.x = x)}$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$P(\rho)$}
\AxiomC{}
\RightLabel{8}
\UnaryInfC{$\omega : \rho(\beta)$}
\AxiomC{}
\RightLabel{5}
\UnaryInfC{$\Box\forall x.\neg\rho(x)$}
\RightLabel{$\Box_E$}
\UnaryInfC{$\omega : \forall x.\neg\rho(x)$}
\RightLabel{$\forall_E$}
\UnaryInfC{$\omega : \neg\rho(\beta)$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\omega : \bot$}
\RightLabel{$\neg_I^9$}
\UnaryInfC{$\omega : \neg(\beta = \beta)$}
\dottedLine
\UnaryInfC{$\omega : (\beta \neq \beta)$}
\RightLabel{$\rightarrow_I^8$}
\UnaryInfC{$\omega : (\rho(\beta) \rightarrow (\beta \neq \beta))$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\omega : (\forall x.\rho(x) \rightarrow (x \neq x))$}
\RightLabel{$\Box_I$}
\UnaryInfC{$\Box(\forall x.\rho(x) \rightarrow (x \neq x))$}
\RightLabel{$\wedge_I$}
\BinaryInfC{$P(\rho) \wedge \Box(\forall x.\rho(x) \rightarrow x \neq x)$}
\AxiomC{Axiom 2 for ρ and $\lambda x.x \neq x$}
\dashedLine
\UnaryInfC{$P(\rho) \wedge \Box(\forall x.\rho(x) \rightarrow x \neq x) \rightarrow P(\lambda x.x \neq x)$}
\BinaryInfC{$P(\lambda x.x \neq x)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\dashedLine
\UnaryInfC{$P(\lambda x.x \neq x)$}
\AxiomC{}
\dashedLine
\UnaryInfC{$P(\lambda x.x = x)$}
\AxiomC{Half of Axiom 1}
\dashedLine
\UnaryInfC{$P(\lambda x.x = x) \rightarrow \neg P(\lambda x.x \neq x)$}
\RightLabel{$\rightarrow_E$}
\BinaryInfC{$\neg P(\lambda x.x \neq x)$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\bot$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\exists x.\rho(x)$}
\RightLabel{$\Box_E$}
\UnaryInfC{$\omega : \neg\exists x.\rho(x)$}
\AxiomC{}
\RightLabel{4}
\UnaryInfC{$\rho(\alpha)$}
\RightLabel{$\exists_I$}
\UnaryInfC{$\omega : \exists x.\rho(x)$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\omega : \bot$}
\RightLabel{$\neg_I^4$}
\UnaryInfC{$\omega : \neg\rho(\alpha)$}
\UnaryInfC{$\omega : \forall x.\neg\rho(x)$}
\UnaryInfC{$\Box\forall x.\neg\rho(x)$}
\AxiomC{}
\dashedLine
\UnaryInfC{$\bot$}
\RightLabel{$\neg_I^5$}
\UnaryInfC{$\neg\Box\forall x.\neg\rho(x)$}
\RightLabel{$\neg_E$}
\BinaryInfC{$\bot$}
\RightLabel{$\neg_I^2$}
\UnaryInfC{$\neg\Box\neg\exists x.\rho(x)$}
\doubleLine
\UnaryInfC{$\Diamond\exists x.\rho(x)$}
\RightLabel{$\rightarrow_I^3$}
\UnaryInfC{$P(\rho) \rightarrow \Diamond\exists x.\rho(x)$}
\end{prooftree}
$$

Definition 1. *A* god-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

Axiom 3 *The property of being god-like is positive:*

$$P(G)$$

Corollary 1. *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\text{Theorem 1 for G}}{P(G) \to \Diamond\exists x.G(x)}}{\Diamond\exists x.G(x)}\to_E$$

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi) \to \Box\ P(\varphi)]$$

Definition 2. *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi\ ess\ x \leftrightarrow \varphi(x) \wedge \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

Theorem 2. *Being God-like is an essence of any God-like being:*

$$\forall x.G(x) \to G\ ess\ x$$

Proof.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{G(\gamma)}1 \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\rho(\gamma)}2 \qquad \dfrac{\dfrac{\dfrac{}{\neg P(\rho)}4 \quad \text{Axiom 1 for } \rho}{P(\lambda x.\neg\rho(x))}\to_E \qquad \dfrac{\dfrac{\dfrac{}{G(\gamma)}1}{\forall\varphi.P(\varphi) \to \varphi(\gamma)}}{P(\lambda x.\neg\rho(x)) \to \neg\rho(\gamma)}\forall_E}{\neg\rho(\gamma)}\to_E}{\bot}\neg_E}{\neg\neg P(\rho)}\neg_I^4}{P(\rho)}\neg\neg_E \qquad \dfrac{\text{Axiom 4}}{P(\rho) \to \Box P(\dots}}{\Box P(\rho)}\ \dots}{\Box(\forall\dots}}{\rho(\gamma) \to \dots}}{\forall q.q(\gamma) - \dots}}{G(\gamma) \wedge (\forall q.q(\gamma) \to \Box(\forall y.G(y) \to q(y)))}}{\dfrac{\dfrac{G\ ess\ \gamma}{G(\gamma) \to G\ ess\ \gamma}\to_I^1}{\forall x.G(x) \to G\ ess\ x}\forall_I}$$

Definition 3. Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1. *Exemplification of the god-likeness property implies the exemplification is necessary:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof.

$$\dfrac{\overline{\exists z.G(z)}^{\,1}}{G(\gamma)}\,\exists_E$$

$$\dfrac{\dfrac{\dfrac{G(\gamma) \qquad \dfrac{\overset{\text{Theorem 2}}{\forall x.G(x) \to G\ ess\ x}}{G(\gamma) \to G\ ess\ \gamma}\,\forall_E}{G\ ess\ \gamma} \qquad \dfrac{\dfrac{\dfrac{\overset{\text{Axiom 5}}{P(E)} \qquad \dfrac{\dfrac{G(\gamma)}{\forall \varphi.P(\varphi) \to \varphi(\gamma)}}{P(E) \to E(\gamma)}\,\forall_E}{E(\gamma)}}{\forall \varphi.\varphi\ ess\ \gamma \to \Box\exists x.\varphi(x)}}{G\ ess\ \gamma \to \Box\exists x.G(x)}\,\forall_E}{\Box\exists x.G(x)}}{\exists z.G(z) \to \Box\exists x.G(x)}\,\to_I^1$$

Lemma 2. *If the existence of a God-like being is possible, then it is necessary:*

$$\Diamond\exists z.G(z) \to \Box\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{\dfrac{\overline{\Diamond\exists x.G(x)}^{\,1} \qquad \dfrac{\dfrac{\overset{\text{Lemma 1}}{\omega : \exists x.G(x) \to \Diamond\Box\exists x.G(x)}}{\Box(\exists x.G(x) \to \Box\exists x.G(x))}\,\Box_I \qquad \text{Distribution Principle}}{\Diamond\exists x.G(x) \to \Diamond\Box\exists x.G(x)}}{\Diamond\Box\exists x.G(x)}\,\to_E \qquad \overset{\text{S5 Iteration Principle}}{\Diamond\Box\exists x.G(x) \to \Box\exists x.G(x)}}{\dfrac{\Box\exists x.G(x)}{\Diamond\exists z.G(z) \to \Box\exists x.G(x)}\,\to_I^1}$$

Theorem 3. *Necessarily, there exists a God-like being:*

$$\Box\exists x.G(x)$$

Proof.

$$\dfrac{\overset{\text{Corollary 1}}{\Diamond\exists x.G(x)} \qquad \overset{\text{Lamma 2}}{\Diamond\exists x.G(x) \to \Box\exists x.G(x)}}{\Box\exists x.G(x)}\,\to_E$$

5 New Proof

A outline of the proof is that firstly all positive properties are proven to be possibly exemplified (Theorem 1). Then this theorem is applied to the property of god-likeness, thus showing the possible existence of a god-like being (Corollary 1). In Lemma 1 it is classically proven that the hypothetically assumed existence of a god-like being implies that the existence is necessary. Lastly, the necessary existence of a god-like being is proven (Theorem 2) from Corollary 1 and Lemma 1 by properties of the modal system. The properties required for the proof are the distribution principle and Brouwer's reduction theorem mentioned in Section 3. The proof of Theorem 2 follows the proof of Sobel [9][p. 126–127].

5.1 Possibly, God Exists

Axiom 1 *Either a property or its negation is positive, but not both: Axiom schema, for all φ:*

$$P(\neg\varphi) \leftrightarrow \neg P(\varphi)$$

Axiom 2 *A property necessarily implied by a positive property is positive: Axiom schema, for all φ and ψ:*

$$(P(\varphi) \wedge \Box\forall x.[\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)$$

Theorem 1. *Positive properties are possibly exemplified: Axiom schema, for all φ:*

$$P(\varphi) \rightarrow \Diamond\exists x.\varphi(x)$$

Proof.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$P(\rho)$}
\AxiomC{}
\RightLabel{2}
\UnaryInfC{$\Box\neg\exists x.\rho(x)$}
\RightLabel{$\Box_E$}
\UnaryInfC{$\omega : \neg\exists x.\rho(x)$}
\AxiomC{}
\RightLabel{1}
\UnaryInfC{$\omega : \rho(x)$}
\RightLabel{$\exists_I$}
\UnaryInfC{$\omega : \exists x.\rho(x)$}
\RightLabel{$\rightarrow_E$}
\BinaryInfC{$\omega : \bot$}
\RightLabel{$\rightarrow_I^1$}
\UnaryInfC{$\omega : \neg\rho(x)$}
\RightLabel{$\rightarrow_I$}
\UnaryInfC{$\omega : \rho(x) \rightarrow \neg\rho(x)$}
\RightLabel{$\forall_I$}
\UnaryInfC{$\omega : \forall x.(\rho(x) \rightarrow \neg\rho(x))$}
\RightLabel{$\Box_I$}
\UnaryInfC{$\Box\forall x.(\rho(x) \rightarrow \neg\rho(x))$}
\RightLabel{$\wedge_I$}
\BinaryInfC{$P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \neg\rho(x)]$}
\AxiomC{Axiom 2 for ρ and $\neg\rho$}
\dashedLine
\UnaryInfC{$(P(\rho) \wedge \Box\forall x.[\rho(x) \rightarrow \neg\rho(x)]) \rightarrow P(\neg\rho)$}
\RightLabel{$\rightarrow_E$}
\BinaryInfC{$P(\neg\rho)$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\dashedLine
\UnaryInfC{$P(\neg\rho)$}
\AxiomC{Half of Axiom 1}
\dashedLine
\RightLabel{$\rightarrow_E$}
\UnaryInfC{$P(\neg\rho) \rightarrow \neg P(\rho)$}
\RightLabel{$\rightarrow_E$}
\BinaryInfC{$\neg P(\rho)$}
\AxiomC{}
\RightLabel{3}
\UnaryInfC{$P(\rho)$}
\BinaryInfC{$\bot$}
\RightLabel{$\rightarrow_I^2$}
\UnaryInfC{$\neg\Box\neg\exists x.\rho(x)$}
\doubleLine
\UnaryInfC{$\Diamond\exists x.\rho(x)$}
\RightLabel{$\rightarrow_I^3$}
\UnaryInfC{$P(\rho) \rightarrow \Diamond\exists x.\rho(x)$}
\end{prooftree}
$$

Definition 1. *A* god-like *being possesses all positive properties:*

$$G(x) \leftrightarrow \forall\varphi.[P(\varphi) \rightarrow \varphi(x)]$$

Axiom 3 *The property of being god-like is positive:*

$$P(G)$$

Corollary 1. *Possibly, God exists:*

$$\Diamond\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{\text{Axiom 3}}{P(G)} \qquad \dfrac{\text{Theorem 1 for G}}{P(G) \to \Diamond\exists x.G(x)}}{\Diamond\exists x.G(x)}\ \to_E$$

5.2 If God exists, then God necessarily exists

Axiom 4 *Positive properties are necessarily positive:*

$$\forall\varphi.[P(\varphi) \to \Box\ P(\varphi)]$$

Definition 2. *An* essence *of an individual is a property possessed by it and necessarily implying any of its properties:*

$$\varphi\ ess\ x \leftrightarrow \varphi(x) \land \forall\psi.(\psi(x) \to \Box\forall x.(\varphi(x) \to \psi(x)))$$

Definition 3. Necessary existence *of an individual is the necessary exemplification of all its essences:*

$$E(x) \leftrightarrow \forall\varphi.[\varphi\ ess\ x \to \Box\exists y.\varphi(y)]$$

Axiom 5 *Necessary existence is a positive property:*

$$P(E)$$

Lemma 1. *Exemplification of the god-likeness property implies the exemplification is necessary:*

$$\exists z.G(z) \to \Box\exists x.G(x)$$

Proof.

$$\dfrac{\dfrac{}{\exists z.G(z)}\ 1}{G(g)}\ \exists_E$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P(\psi)}\ 2 \quad \dfrac{\text{Axiom 1}}{\neg P(\psi) \to P(\neg\psi)}}{P(\neg\psi)}\ \to_E \quad \dfrac{\dfrac{G(g)}{\forall\varphi.(P(\varphi) \to \varphi(g))}\ \text{D1}}{P(\neg\psi) \to \neg\psi(g)}\ \forall_E}{\neg\psi(g)}\ \to_E \quad \dfrac{}{\psi(g)}\ 3}{\bot}\ \to_E}{\neg\neg P(\psi)}\ \to_I^2}{P(\psi)}\ \neg\neg_E \quad \dfrac{\dfrac{\text{Axiom 4}}{\forall\varphi.[P(\psi) \to \Box\ P(\psi)]}}{P(\psi) \to \Box\ P(\psi)}\ \forall_E}{\Box P(\psi)}\ \to_E}{\psi(g) \to \Box\ P(\psi)}\ \to_I^3$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Box P(\psi)}\,5}{\omega : P(\psi)}\,\Box_E \quad \dfrac{\dfrac{\dfrac{}{\omega : G(y)}\,4}{\omega : \forall\varphi.(P(\varphi)\to\varphi(y))}\,\text{D1}}{\omega : P(\psi)\to\psi(y)}\,\forall_E}{\omega : \psi(y)}\,\to_E}{\omega : G(y)\to\psi(y)}\,\to_I^4}{\omega : \forall y.(G(y)\to\psi(y))}\,\forall_I}{\Box\forall y.(G(y)\to\psi(y))}\,\Box_I}{\Box P(\psi)\to\Box\forall y.(G(y)\to\psi(y))}\,\to_I^5 \quad \dfrac{\dfrac{}{\psi(g)}\,6 \quad \psi(g)\to\Box P(\psi)}{\Box P(\psi)}\,\to_E}{\dfrac{\dfrac{\Box\forall y.(G(y)\to\psi(y))}{\psi(g)\to\Box\forall y.(G(y)\to\psi(y))}\,\to_I^6}{\forall\psi.(\psi(g)\to\Box\forall y.(G(y)\to\psi(y)))}\,\forall_I}\,\to_E \quad G(g)}{\dfrac{G(g)\land\forall\psi.(\psi(g)\to\Box\forall y.(G(y)\to\psi(y)))}{G\ ess\ g}\,\text{D2}}\,\land_I
$$

$$
\dfrac{\dfrac{G\ ess\ g \quad \dfrac{\dfrac{\dfrac{\dfrac{\text{Axiom 5}}{P(E)} \quad \dfrac{\dfrac{G(g)}{\forall\varphi.(P(\varphi)\to\varphi(g))}\,\text{D1}}{P(E)\to E(g)}\,\forall_E}{E(g)}\,\to_E}{\forall\varphi.[\varphi\ ess\ g\to\Box\exists x.\varphi(x)]}\,\text{D3}}{G\ ess\ g\to\Box\exists x.G(x)}\,\forall_E}{\Box\exists x.G(x)}\,\to_E}{\exists z.G(z)\to\Box\exists x.G(x)}\,\to_I^1
$$

5.3 God necessarily exists

The exemplification of the god-likeness property is provable from its possible exemplification by the modal distribution principle 1 and Brouwer's reduction principle 2. The necessary exemplification of the god-likeness property can then be deduced from lemma 1. This single use of the modal distribution principle 1 and Brouwer's reduction principle 2 follows the presentation of in Sobel [9][p.126–127].

Theorem 3. *The property of god-likeness is necessarily exemplified:*

$$\Box\exists x.G(x)$$

Proof.

$$
\dfrac{\dfrac{\text{Distribution principle 1}}{\Box[\exists x.G(x)\to\Box\exists x.G(x)]\to[\Diamond\exists x.G(x)\to\Diamond\Box\exists x.G(x)]} \quad \dfrac{\dfrac{\text{Lemma 1}}{\exists x.G(x)\to\Box\exists x.G(x)}}{\Box[\exists x.G(x)\to\Box\exists x.G(x)]}\,\Box_I}{\Diamond\exists x.G(x)\to\Diamond\Box\exists x.G(x)}\,\to_E
$$

$$\dfrac{\dfrac{\dfrac{[\Diamond\exists x.G(x) \to \Diamond\Box\exists x.G(x)] \qquad \dfrac{\text{Corollary 1}}{\Diamond\exists x.G(x)}}{\Diamond\Box\exists x.G(x)}\to_E \qquad \dfrac{\text{Brouwer's reduction principle 2}}{\Diamond\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}\to_E}{\Box\exists x.G(x)}\Box_I$$

6 Modal collapse

A major criticism against Gödel's proof is that its axioms lead to the so-called *modal collapse* [8]: it is possible to prove that everything that is the case is so necessarily, and hence actuality, possibility and necessity coincide [9][Ch. 4, section 6, theorems 9 and 10]. That is: for all properties, φ,

$$\varphi \leftrightarrow \Diamond\varphi \leftrightarrow \Box\varphi$$

If Gödel's ontological proof is abstractly analysed, then we are proving a restricted modal collapse, which applies to one specific formula, the exemplification of god-likeness. In addition we prove and that the definition of god-likeness is sound, which means that it is possibly exemplified. Thus, the necessity of the formula follows. The interest in the proof naturally decreases if a consequence of the axiomatization is a modal collapse for all formulas. Therefore, an improvement would be obtained if the modal collapse was limited to one property, namely the property of god-likeness. A number of solutions to the problem of the modal collapse have been proposed.

Anderson's solution [**?**] modifies the definitions of god-like being and essence, and eliminates half of an axiom. This not only avoids the modal collapse, but also makes two of Gödel's five axioms derivable from the others [**?**] under some implicit additional assumptions [**?**]. Another solution involving more substantial modifications is that of Bjørdal [**? ?**].

On another track, Fitting has argued that greater care has to be taken with the semantics of higher-order modal logics. Quantified variables may be rigid or flexible; and properties may be treated as intensional or extensional. Making the right choices may prevent the modal collapse [3][Sections 11.9 and 11.10].

Anderson [1][p. 292] and Sobel [9][p. 133] also discuss the idea that the notion of property over which quantification is allowed might be too general and restrictions might be appropriate.

It is beyond the scope of this paper to analyze these solutions in detail or propose new solutions. The purpose of this section is simply to show natural deduction derivations of the modal collapse, thus confirming that it holds for the axioms used in the previous sections.

Theorem 4. *For all constant formulas, A, (without free variables) a modal collapse*

$$A \to \Box A$$

is provable.

$$
\begin{array}{cl}
\text{Theorem } \mathbf{??} & \\ \hdashline
\forall y.[G(y) \to G\ ess\ y] & \\ \hline\hline
\forall y.[G(y) \to G(y) \wedge \forall \psi.(\psi(y) \to \Box \forall x.(G(x) \to \psi(x)))] & \text{D2} \\ \hline
\forall y.[G(y) \to \forall \psi.(\psi(y) \to \Box \forall x.(G(x) \to \psi(x)))] & \\ \hline\hline
\forall y.[G(y) \to (A(y) \to \Box \forall x.(G(x) \to A(x)))] & \\ \hline
\forall y.[G(y) \to (A \to \Box \forall x.(G(x) \to A))] & A \text{ is constant} \\ \hline\hline
\exists y.G(y) \to (A \to \Box \forall x.(G(x) \to A)) &
\end{array}
$$

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{\;}{\exists y.G(y) \to (A \to \Box \forall x.(G(x) \to A))}
          \qquad
          \dfrac{\dfrac{\text{Theorem 2}}{\Box \exists y.G(y)}}{\exists y.G(y)}\,\Box_E
        }{A \to \Box \forall x.(G(x) \to A)}\,\to_E
      }{A \to \Box(\exists x.G(x) \to A)}
      \qquad
      \dfrac{\;}{A}\,1
    }{\Box(\exists x.G(x) \to A)}\,\to_E
  }{\omega : \exists x.G(x) \to A}\,\Box_E
  \qquad
  \dfrac{\dfrac{\text{Theorem 2}}{\Box \exists y.G(y)}\,\to_E}{\omega : \exists x.G(x)}\,\Box_E
}{
  \dfrac{\dfrac{\omega : A}{\Box A}\,\Box_I}{A \to \Box A}\,\to_I^1
}\,\to_E
$$

7 Conclusions

The proofs of theorem T1 and corollary C1 ($\Diamond \exists x.G(x)$) do not rely on equality and use axiom A2 ($\forall \varphi.\forall \psi.[(P(\varphi) \wedge \Box \forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$) only once. Corollary C2 ($\exists x.G(x)$), which is usually regarded as a trivial corollary of main theorem T3 ($\Box \exists x.G(x)$) using the modal logic axiom T, is here derived directly from lemma L1 and corollary C1, not relying on T. The main theorem T3 becomes derivable from C2 by a single application of the necessitation rule. Furthermore, all proofs are done in the modal logic K, except for the proof of corollary C2, which requires one use of the axiom B of modal logic KB.

ToDo: Sobel Anderson on KB: Sobel page 152

Anderson's footnote 5 on KB

ToDo: talk about anderson's footnote 2, where he argues that an anonymous referee knew the simpler proof of T1.

Hajek (magari and others) on non-provability of ex x. G(x) in KD45.

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***Gentzen and Prawitz: references to Natural Deduction. Adams, Fuhrmann, Hajek and Bjordal: References to amendations. Church: Reference to simple type theory.