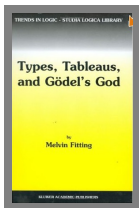


Formalization, Mechanization and Automation of Gödel's Proof of God's Existence

Christoph Benz Müller and Bruno Woltzenlogel Paleo

November 1, 2013



$$\frac{\text{Axiom 3} \quad \frac{\overline{\forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]}}{P(G) \rightarrow \Diamond \exists x. G(x)} \forall_E}{\Diamond \exists x. G(x)} \rightarrow_E$$

A gift to **Priest Edvaldo** and his church in Piracicaba, Brazil

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Nachrichten > Wissenschaft > Mensch > Mathematik > Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Formel von Kurt Gödel: Mathematiker bestätigen Gottesbeweis

Von Tobias Hürter



Kurt Gödel (um das Jahr 1935): Der Mathematiker hielt seinen Gottesbeweis jahrzehntlang geheim

Ein Wesen existiert, das alle positiven Eigenschaften in sich vereint. Das bewies der legendäre Mathematiker Kurt Gödel mit einem komplizierten Formelgebilde. Zwei Wissenschaftler haben diesen Gottesbeweis nun überprüft - und für gültig befunden.

Jetzt sind die letzten Zweifel ausgeräumt: Gott existiert tatsächlich. Ein Computer hat es mit kalter Logik bewiesen - das MacBook des Computerwissenschaftlers Christoph Benzmüller von der Freien Universität Berlin.

Montag, 09.09.2013 - 12:03 Uhr

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Researchers say they used MacBook to prove Goedel's God theorem

Oct. 23, 2013 | 8:14 PM | [1 comments](#)

Are we in contact with Steve Jobs?

No

Do you really need a MacBook to obtain the results?

No

Is Apple sending us money?

No

(but maybe they should)

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Def: Ontological Argument/Proof

- * deductive argument
- * for the existence of God
- * starting from premises, which are justified by pure reasoning, i.e. they do not depend on observation in the world.

Existence of God: different types of arguments/proofs

- a posteriori (use experience/observation in the world)
 - teleological
 - cosmological
 - moral
 - ...
- a priori (based on pure reasoning, independent)
 - ontological argument
 - definitional
 - modal
 - ...
 - other a priori arguments

Def: **Ontological Argument/Proof**

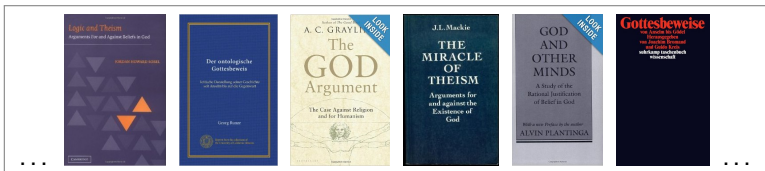
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Wohl eine jede Philosophie kreist um den ontologischen Gottesbeweis

(Adorno, Th. W.: Negative Dialektik. Frankfurt a. M. 1966, p.378)



Rich history on ontological arguments (**pros** and **cons**)

... Anselm v. G.
Gaunilo Th. Aquinas Descartes
Spinoza Leibniz Hume
Kant Hegel Frege Hartshorne
Malcolm Lewis Plantinga
Gödel ...

Anselm's notion of God:

"God is that, than which nothing greater can be conceived."

Gödel's notion of God:

"A God-like being possesses all 'positive' properties."

To show by logical reasoning:

"(Necessarily) God exists."

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Different Interests in Ontological Arguments:

- **Philosophical:** Boundaries of Metaphysics & Epistemology
 - We talk about a metaphysical concept (God),
● but we want to draw a conclusion for the real world.
 - Necessary Existence (NE): metaphysical NE vs. logical NE vs. modal NE
- **Theistic:** Successful argument should convince atheists
- **Our:** Can computers (theorem provers) be used ...
 - ... to formalize the definitions, axioms and theorems?
 - ... to verify the arguments step-by-step?
 - ... to fully automate (sub-)arguments?

“Computer-assisted Theoretical Philosophy”

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

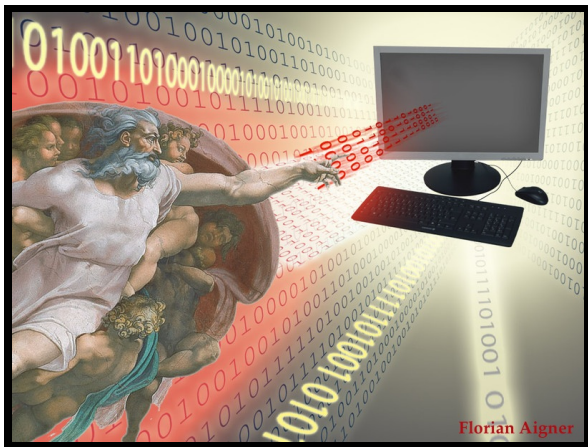
[BenzmüllerPaulson, Logica Universalis, 2013]

What we did (rough outline for remaining presentation!):

- A: Pen and paper: detailed natural deduction proof
- B: Formalization: in classical higher-order logic (HOL)
- Automation: theorem provers LEO-II and SATALLAX
- Consistency: model finder NITPICK (NITROX)
- C: Step-by-step verification: proof assistant Coq
- D: Automation & verification: proof assistant ISABELLE

Did we get any new results?

Yes — let's discuss this later!



Part A: Informal Proof and Natural Deduction Proof

Gödel's Manuscript (1970)

Ontologische Beweis

Feb. 10, 1970

P(φ) φ is positive (i.e. $\varphi \in P$)

At. 1 $P(\varphi) \cdot P(\psi) \supset P(\varphi \cdot \psi)$ At. 2 $P(\varphi) \supset P(\sim \varphi)$

P1 $G(x) \equiv (\varphi) [P(\varphi) \supset \varphi(x)]$ (God)

P2 $\varphi \text{ Ess } x \equiv (\psi) [\psi(x) \supset N(\psi) \supset \varphi(\psi)]$ (Essence of x)

P3 $p \supset_N q = N(p \supset q)$ Necessity

At. 2 $P(\varphi) \supset NP(\varphi)$
 $\sim P(\varphi) \supset N \sim P(\varphi)$ } because it follows from the nature of the property

Th. $G(x) \supset G \text{ Ess } x$

Df. $E(x) \equiv (\varphi) [\varphi \text{ Ess } x \supset N \exists x \varphi(x)]$ necessary Existence

Ax 3 $P(E)$

Th. $G(x) \supset N(\exists y) G(y)$

hence $(\exists x) G(x) \supset N(\exists y) G(y)$

" $M(\exists x) G(x) \supset MN(\exists y) G(y)$

" $\supset N(\exists y) G(y)$

$M = possibility$

any two essences of x are rec. equivalent

exclusive or * and for any number of humanoids

$M(\exists x) G(x)$ means ^{the system of} all pos. props. is compatible
 This is true because of:

At. 4: $P(\varphi) \cdot \varphi \supset_N \psi \supset P(\psi)$ which impl.

~~then~~ $\begin{cases} x=x & \text{is positive} \\ x \neq x & \text{is negative} \end{cases}$

But if a system S of pos. props. were inconsistent it would mean that the sum prop. S (which is positive) would be $x \neq x$

Positive means positive in the moral aesthetic sense (independently of the accidental structure of the world). Only ^{at the} ~~then~~ ^{at the} time, it means "attribution" as opposed to "privation" (or containing privation). - This is the crucial point

If φ is positive $(x) N \varphi(x) \supset N \varphi(x)$ ~~then~~ $\varphi(x) \supset_N x \neq x$

hence $x \neq x$ positive ~~not~~ $x=x$ ~~is~~ necessary At.

on the existence of pos. props.

dag
 x i.e. the normal form in terms of elem. prop. contains a member without negation.

- A1 Either a property is positive or its negation is (never both):
 $\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
- A2 A property necessarily implied by a positive property is positive:
 $\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
- T1 Positive properties are possibly exemplified:
 $\forall\varphi[P(\varphi) \rightarrow \Diamond\exists x\varphi(x)]$
- D1 A *God-like* being possesses all positive properties:
 $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
- A3 The property of being God-like is positive: $P(G)$
- C Possibly, God exists: $\Diamond\exists xG(x)$
- A4 Positive properties are necessarily positive:
 $\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$
- D2 An *essence* of an individual is a property possessed by it and necessarily implying any of its properties:
 $\phi \text{ ess } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
- T2 Being God-like is an essence of any God-like being:
 $\forall x[G(x) \rightarrow G \text{ ess } x]$
- D3 *Necessary existence* of an individual is the necessary exemplification of all its essences:
 $E(x) \leftrightarrow \forall\phi[\phi \text{ ess } x \rightarrow \Box\exists y\phi(y)]$
- A5 Necessary existence is a positive property: $P(E)$
- T3 Necessarily, God exists: $\Box\exists xG(x)$

$$\mathbf{D1:} \ G(x) \equiv \forall \varphi. [P(\varphi) \rightarrow \varphi(x)]$$

$$\mathbf{D2:} \ \varphi \text{ ess } x \equiv \varphi(x) \wedge \forall \psi. (\psi(x) \rightarrow \Box \forall x. (\varphi(x) \rightarrow \psi(x)))$$

$$\mathbf{D3:} \ E(x) \equiv \forall \varphi. [\varphi \text{ ess } x \rightarrow \Box \exists y. \varphi(y)]$$

$$\begin{array}{c}
 \begin{array}{c}
 \mathbf{A3} \\
 \hline \overline{P(G)}
 \end{array}
 \quad
 \frac{
 \frac{
 \mathbf{A2} \\
 \hline \overline{\forall \varphi. \forall \psi. [(P(\varphi) \wedge \Box \forall x. [\varphi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]}
 }{
 \mathbf{T1:} \ \forall \varphi. [P(\varphi) \rightarrow \Diamond \exists x. \varphi(x)]
 }
 }{
 \mathbf{C1:} \ \Diamond \exists x. G(x)
 }
 \\
 \\
 \frac{
 \frac{
 \mathbf{A1b} \\
 \hline \overline{\forall \varphi. [\neg P(\varphi) \rightarrow P(\neg \varphi)]}
 }{
 \mathbf{T2:} \ \forall y. [G(y) \rightarrow G \text{ ess } y]
 }
 \quad
 \frac{
 \mathbf{A4} \\
 \hline \overline{\forall \varphi. [P(\varphi) \rightarrow \Box P(\varphi)]}
 }{
 \mathbf{A5} \\
 \hline \overline{P(E)}
 }
 }{
 \frac{
 \mathbf{L1:} \ \exists z. G(z) \rightarrow \Box \exists x. G(x) \\
 \hline \Diamond \exists z. G(z) \rightarrow \Diamond \Box \exists x. G(x)
 }{
 \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)
 }
 }
 \\
 \\
 \frac{
 \mathbf{C1:} \ \Diamond \exists x. G(x) \quad \mathbf{L2:} \ \Diamond \exists z. G(z) \rightarrow \Box \exists x. G(x)
 }{
 \mathbf{T3:} \ \Box \exists x. G(x)
 }
 \end{array}$$

$$\frac{A \vee B \quad \begin{array}{c} \overline{A} \\ \vdots \\ C \end{array} \quad \begin{array}{c} \overline{B} \\ \vdots \\ C \end{array}}{C} \vee_E$$

$$\frac{A \quad B}{A \wedge B} \wedge_I$$

$$\frac{\begin{array}{c} \overline{A} \\ \vdots \\ B \end{array}}{A \rightarrow B} \rightarrow_I^n$$

$$\frac{A}{A \vee B} \vee_{I_1}$$

$$\frac{A \wedge B}{A} \wedge_{E_1}$$

$$\frac{B}{A \rightarrow B} \rightarrow_I$$

$$\frac{B}{A \vee B} \vee_{I_2}$$

$$\frac{A \wedge B}{B} \wedge_{E_2}$$

$$\frac{A \quad A \rightarrow B}{B} \rightarrow_E$$

$$\frac{A[\alpha]}{\forall x.A[x]} \forall_I$$

$$\frac{\forall x.A[x]}{A[t]} \forall_E$$

$$\frac{A[t]}{\exists x.A[x]} \exists_I$$

$$\frac{\exists x.A[x]}{A[\beta]} \exists_E$$

$$\neg A \equiv A \rightarrow \perp$$

$$\frac{\neg\neg A}{A} \neg\neg_E$$

$$\frac{\alpha : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Box A} \Box_I$$

$$\frac{\Box A}{t : \boxed{\begin{array}{c} A \\ \vdots \end{array}}} \Box_E$$

$$\frac{t : \boxed{\begin{array}{c} \vdots \\ A \end{array}}}{\Diamond A} \Diamond_I$$

$$\frac{\Diamond A}{\beta : \boxed{\begin{array}{c} A \\ \vdots \end{array}}} \Diamond_E$$

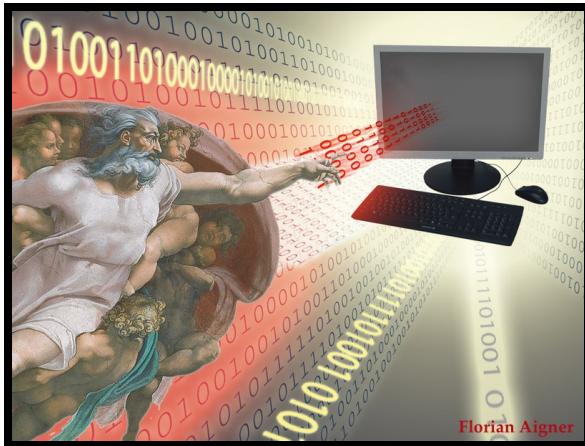
$$\Diamond A \equiv \neg \Box \neg A$$

A set of small navigation icons typically found in Beamer presentations, including symbols for back, forward, search, and other slide controls.

Natural Deduction Proofs

T2 (Partial)

$$\begin{array}{c}
 \frac{\psi(x)^6 \quad \frac{\psi(x) \rightarrow \Box P(\psi)}{\Box P(\psi)} \Pi_2}{\Box P(\psi)} \rightarrow E \\
 \frac{\Box P(\psi) \quad \frac{\frac{\frac{\Box P(\psi)^7 \quad P(\psi)}{P(\psi)} \Box E \quad \frac{\frac{P(\psi) \rightarrow \forall x.(G(x) \rightarrow \psi(x))}{\forall x.(G(x) \rightarrow \psi(x))} \Pi_3}{\Box \forall x.(G(x) \rightarrow \psi(x))} \Box I}{\Box P(\psi) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow I}{\psi(x) \rightarrow \Box \forall x.(G(x) \rightarrow \psi(x))} \rightarrow E
 \end{array}$$



Part B:

Formalization:
Automation:
Consistency:

in classical higher-order logic (HOL)
theorem provers LEO-II and SATALLAX
model finder NITPICK (NITROX)

Challenge: No provers for *Higher-order Quantified Modal Logic* (QML)

Our solution: Embedding in *Higher-order Classical Logic* (HOL)

Then use existing HOL theorem provers for reasoning in QML

[BenzmüllerPaulson, Logica Universalis, 2013]

Previous empiricial findings:

Embedding of *First-order Modal Logic* in HOL works well

[BenzmüllerOttenRaths, ECAI, 2012]

[Benzmüller, LPAR, 2013]

QML $\varphi, \psi ::= \dots \mid \neg\varphi \mid \varphi \wedge \psi \mid \varphi \rightarrow \psi \mid \Box\varphi \mid \Diamond\varphi \mid \forall x\varphi \mid \exists x\varphi \mid \forall P\varphi$

- Kripke style semantics (possible world semantics)

HOL $s, t ::= C \mid x \mid \lambda x s \mid s t \mid \neg s \mid s \vee t \mid \forall x t$

- meanwhile very well understood
- **Henkin semantics** vs. standard semantics
- various theorem provers do exist

interactive: Isabelle/HOL, HOL4, Hol Light, Coq/HOL, PVS, ...

automated: TPS, LEO-II, Satallax, Nitpick, Isabelle/HOL, ...

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QML in **HOL**: **QML** formulas φ are mapped to **HOL** predicates $\varphi_{t \rightarrow o}$

\neg	=	$\lambda\varphi_{t \rightarrow o} \lambda s_t \neg\varphi s$	Ax
\wedge	=	$\lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\varphi s \wedge \psi s)$	
\rightarrow	=	$\lambda\varphi_{t \rightarrow o} \lambda\psi_{t \rightarrow o} \lambda s_t (\neg\varphi s \vee \psi s)$	
\Box	=	$\lambda\varphi_{t \rightarrow o} \lambda s_t \forall u_t (\neg r s u \vee \varphi u)$	
\Diamond	=	$\lambda\varphi_{t \rightarrow o} \lambda s_t \exists u_t (r s u \wedge \varphi u)$	
\forall	=	$\lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu h d s$	
\exists	=	$\lambda h_{\mu \rightarrow (t \rightarrow o)} \lambda s_t \exists d_\mu h d s$	
\forall	=	$\lambda H_{(\mu \rightarrow (t \rightarrow o)) \rightarrow (t \rightarrow o)} \lambda s_t \forall d_\mu H d s$	
valid	=	$\lambda\varphi_{t \rightarrow o} \forall w_t \varphi w$	

The equations in **Ax** are given as axioms to the **HOL** provers!

(Remark: Note that we are here dealing with constant domain quantification)

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Example

QML formula

$$\Diamond \exists x G(x)$$

QML formula in **HOL**

$$\text{valid } (\Diamond \exists x G(x))_{i \rightarrow o}$$

expansion, $\beta\eta$ -conversion

$$\forall w_i (\Diamond \exists x G(x))_{i \rightarrow o} w$$

expansion, $\beta\eta$ -conversion

$$\forall w_i \exists u_i (rwu \wedge (\exists x G(x))_{i \rightarrow o} u)$$

expansion, $\beta\eta$ -conversion

$$\forall w_i \exists u_i (rwu \wedge \exists x Gxu)$$

What are we doing?

In order to prove that φ is valid in **QML**,

\rightarrow we instead prove that $\text{valid } \varphi_{i \rightarrow o}$ can be derived from **Ax** in **HOL**.

This can be done with interactive or automated **HOL** theorem provers.

Expansion: user or prover may flexibly choose expansion depth

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$$\forall w_l \exists u_l (rwu \wedge \exists x Gxu)$$

What are we doing?

In order to prove that φ is valid in **QML**,

\rightarrow we instead prove that **valid** $\varphi_{l \rightarrow o}$ can be derived from **Ax** in **HOL**.

This can be done with interactive or automated **HOL** theorem provers.

Expansion: user or prover may flexibly choose expansion depth

Example

QML formula

$$\Diamond \exists x G(x)$$

QML formula in **HOL**

$$\text{valid } (\Diamond \exists x G(x))_{l \rightarrow o}$$

expansion, $\beta\eta$ -conversion

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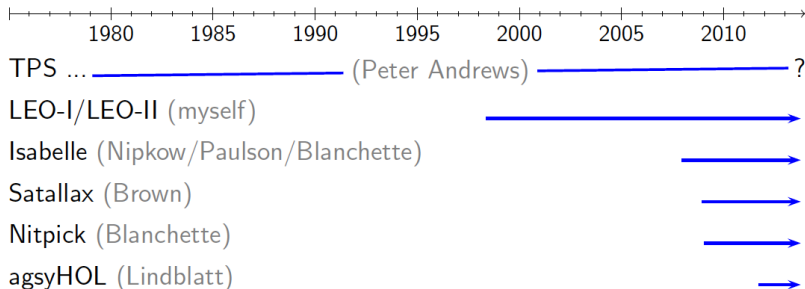
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Automated Theorem Provers and Model Finders for HOL



- all accept TPTP THF Syntax [SutcliffeBenzmüller, J.Form.Reas, 2009]
 - can be called remotely via SystemOnTPTP at Miami
 - they significantly gained in strength over the last years
 - they can be bundled into a combined prover **HOL-P**

Exploit HOL with Henkin semantics as metalogic
Automate other logics (& combinations) via semantic embeddings
— **HOL-P** becomes a **Universal Reasoner** —

Proof Automation and Consistency Checking: Demo!

```
Terminal — bash — 125x32

MacBook-Chris %
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh T3.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyH0L---1.0 : T3.p +++++ RESULT: S0T_7L4x_Y - agsyH0L---1.0 says Unknown - CPU = 0.00 WC = 0.02
LE0-II---1.6.0 : T3.p +++++ RESULT: S0T_E4SCha - LE0-II---1.6.0 says Theorem - CPU = 0.03 WC = 0.09
Satallax---2.7 : T3.p +++++ RESULT: S0T_kVZ1cB - Satallax---2.7 says Theorem - CPU = 0.00 WC = 0.14
Isabelle---2013 : T3.p +++++ RESULT: S0T_xa0gEp - Isabelle---2013 says Theorem - CPU = 14.06 WC = 17.73 SolvedBy = auto
TPS---3.120601S1b : T3.p +++++ RESULT: S0T_R0Egsq - TPS---3.120601S1b says Unknown - CPU = 33.56 WC = 41.57
Nitrox---2013 : T3.p +++++ RESULT: S0T_WGY1Tx - Nitrox---2013 says Unknown - CPU = 75.55 WC = 49.24

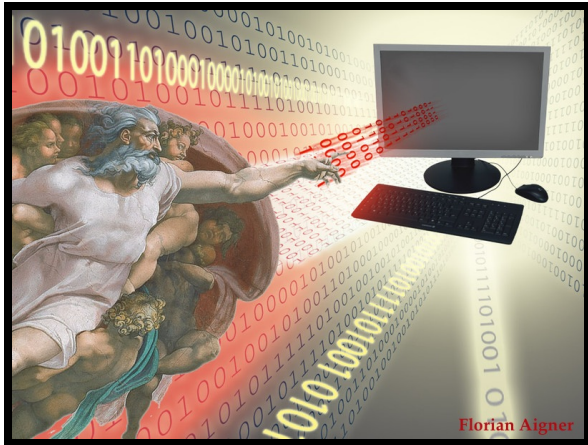
MacBook-Chris %
MacBook-Chris % ./call_tptp.sh Consistency.p

Asking various HOL-ATPs in Miami remotely (thanks to Geoff Sutcliffe)

MacBook-Chris % agsyH0L---1.0 : Consistency.p +++++ RESULT: S0T_ZtY_7o - agsyH0L---1.0 says Unknown - CPU = 0.00 WC = 0.00
Nitrox---2013 : Consistency.p +++++ RESULT: S0T_HUz10C - Nitrox---2013 says Satisfiable - CPU = 6.56 WC = 8.50
TPS---3.120601S1b : Consistency.p +++++ RESULT: S0T_fpJxTM - TPS---3.120601S1b says Unknown - CPU = 43.00 WC = 49.42
Isabelle---2013 : Consistency.p +++++ RESULT: S0T_6Tpp9i - Isabelle---2013 says Unknown - CPU = 69.96 WC = 72.62
LE0-II---1.6.0 : Consistency.p +++++ RESULT: S0T_dY10sj - LE0-II---1.6.0 says Timeout - CPU = 90 WC = 89.86
Satallax---2.7 : Consistency.p +++++ RESULT: S0T_Q9WSLf - Satallax---2.7 says Timeout - CPU = 90 WC = 90.50

MacBook-Chris %
```

Provers are called remotely in Miami — no local installation needed!



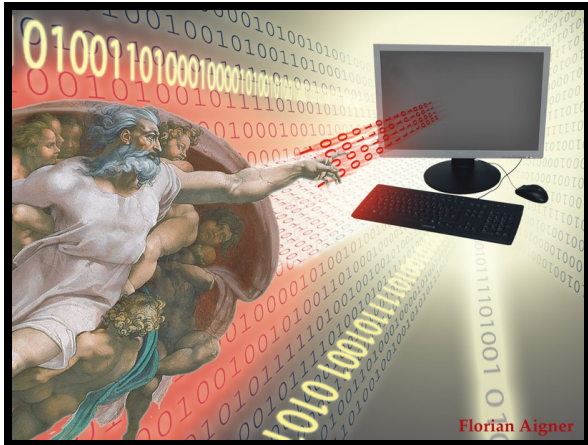
Part C: Formalization and Verification in Coq

- Goal: verification of the natural deduction proof
 - Step-by-step formalization
 - Almost no automation (intentionally!)
- Interesting facts to note:
 - Embedding is transparent to the user
 - Embedding gives labeled calculus for free

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Part D:

automation & verification: proof assistant ISABELLE

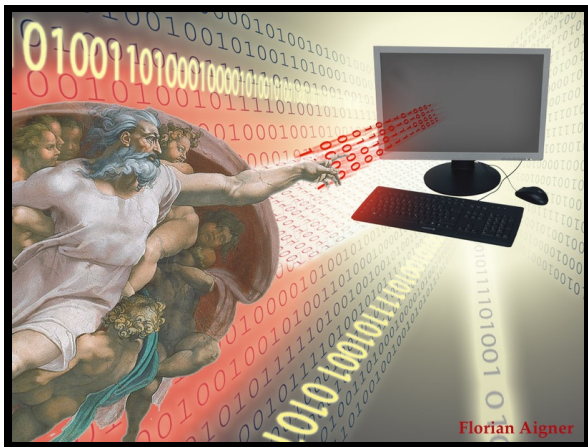
Isabelle/HOL (Cambridge University/TU Munich)

- HOL instance of the generic Isabelle proof assistant
- User interaction and proof automation
- Automation is supported by SLEDGEHAMMER tool
- Verification of the proofs in Isabelle/HOL's small proof kernel

What have we done?

- Proof automation of Gödel's proof script (Scott version)
- SLEDGEHAMMER makes calls to remote THF provers in Miami
- These calls suggest respective calls to the METIS prover
- METIS proofs are verified in Isabelle/HOL's proof kernel

See the handout (generated from the Isabelle source file).



Part E: Criticisms

$$\forall P. [\Diamond \Box P \rightarrow \Box P]$$

If something is possibly necessary, then it is necessary.

$$\Diamond \Box (A \vee \neg A) \quad \Box (A \vee \neg A)$$

logical necessity ~ validity

for all $M, M \models F \longrightarrow \Box F$

logical possibility ~ satisfiability

exists $M, M \models F \longrightarrow \Diamond F$

What about iterations?

$$\Diamond \Box \Diamond F$$

Weak intuitions \Rightarrow dozens of modal logics

S5 is considered adequate

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Everything that is the case is so necessarily.

Follows from T2, T3 and D2.

There are no contingent “truths”.

Everything is determined.

There is no free will.

God’s existence makes no difference.

Many proposed solutions: Anderson, Fitting, Hájek, ...

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Either a property is positive or its negation is (never both)

Are the following properties positive or negative?

$\lambda x.G(x)$ $\lambda x.human(x)$ $\lambda x.foreigner(x)$ $\lambda x.\neg foreigner(x), \dots$

Solution:

“... positive in the moral aesthetic sense (independently of the accidental structure of the world). Only then the ax. true. [sic] ...”

- Gödel, 1970

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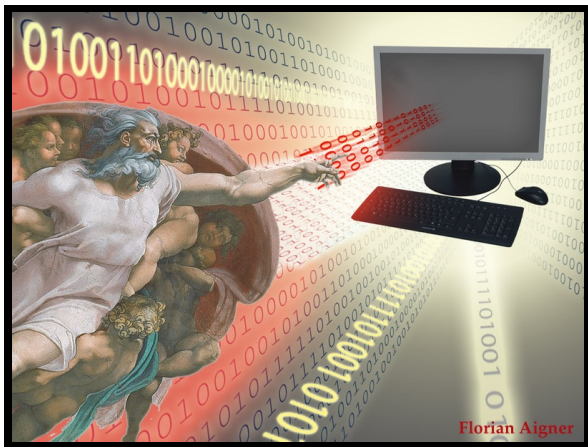
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Part F: Conclusions

- K sufficient for T1, C1 and T2
- S5 not needed for T3
- KB sufficient for T3
- A simpler new proof of C1
- Gödel's original axioms (without conjunct $\phi(x)$ in D2) are inconsistent
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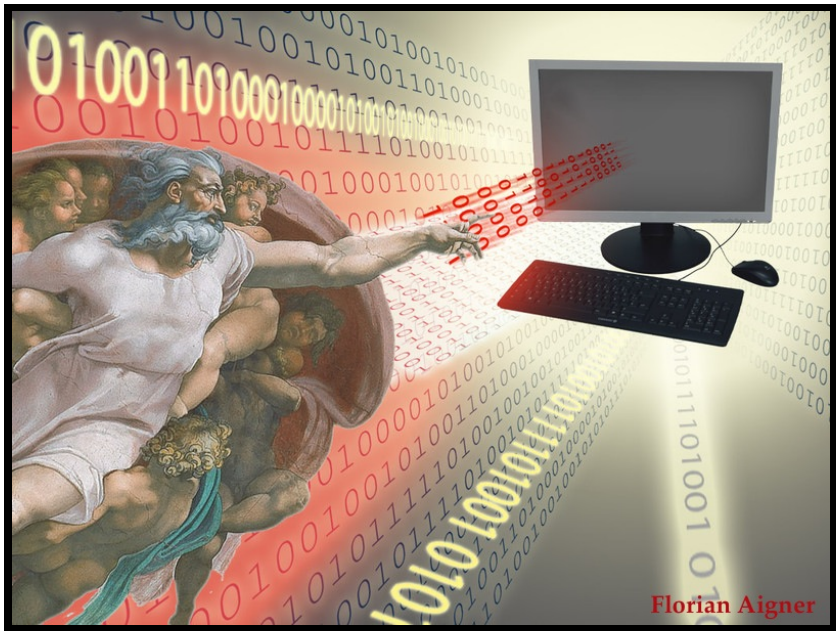
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Florian Aigner

What have we achieved

- Verification of Gödel's ontological argument with HOL provers
 - exact parameters known: constant domain quantification, Henkin Semantics
 - parameters can be varied and experiments can be repeated
- Gained some novel results and insights
- Major step towards **Computer-assisted Theoretical Philosophy**
 - see also Ed Zalta's *Computational Metaphysics* project at Stanford University
 - remember Leibniz' dictum — *Calculemus!*
- Highly fascinating bridge between CS, Philosophy and Theology
- Major public interest

Future Work

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