Gödel's Ontological Proof of God's Existence (Draft)

Bruno Woltzenlogel Paleo, Annika Siders

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"There is a scientific (exact) philosophy and theology, which deals with concepts of the highest abstractness; and this is also most highly fruitful for science. [...] Religions are, for the most part, bad; but religion is not." - Kurt Gödel

1 Possible witnessing of positive properties

Axioms:

- **(A1)** Properties necessarily entailed by *positive* properties are also positive:
$$\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]$$

- **(A2)** A property's negation is positive iff the property is not positive:
$$\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]$$

Lemma 1: Positive properties possibly have a witness:
$$\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]$$

Formal proof:

$$
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\cfrac{
\forall\varphi.\forall\psi.[(P(\varphi) \wedge \Box\forall x.[\varphi(x) \to \psi(x)]) \to P(\psi)]
}{
\forall\psi.[(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \psi(x)]) \to P(\psi)]
}
}{
(P(\varphi') \wedge \Box\forall x.[\varphi'(x) \to \neg\varphi'(x)]) \to P(\neg\varphi')
}
}{
(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to P(\neg\varphi')
}
\qquad
\cfrac{
\forall\varphi.[P(\neg\varphi) \leftrightarrow \neg P(\varphi)]
}{
P(\neg\varphi') \leftrightarrow \neg P(\varphi')
}
}{
(P(\varphi') \wedge \Box\forall x.[\neg\varphi'(x)]) \to \neg P(\varphi')
}
}{
P(\varphi') \to \Diamond\exists x.\varphi'(x)
}
}{
\forall\varphi.[P(\varphi) \to \Diamond\exists x.\varphi(x)]
}
$$

2 Possible existence of a God

Axioms:

- **(A3)** Being God is a positive property:

$$P(G)$$

Lemma 2: It is possible that a God exists:

$$\Diamond \exists x. G(x)$$

Formal proof:

$$
\cfrac{P(G) \qquad \cfrac{\cfrac{}{\forall \varphi. [P(\varphi) \to \Diamond \exists x. \varphi(x)]} \ \text{Th. 1}}{P(G) \to \Diamond \exists x. G(x)}}{\Diamond \exists x. G(x)}
$$

3 Essentiality of being God

Definitions:

- **(D1)** An individual is a *God* if and only if he possesses all positive properties:

$$G(x) \leftrightarrow \forall \varphi. [P(\varphi) \to \varphi(x)]$$

- **(D2)** A property is *essential* for an individual if and only if it holds for that inidividual and necessarily entails every other property that holds for that individual: $\varphi \ ess \ x \leftrightarrow \varphi(x) \wedge \forall \psi. (\psi(x) \to \Box \forall x. (\varphi(x) \to \psi(x)))$

Axioms:

- **(A4)** Positive properties are necessarily positive:

$$\forall \varphi. [P(\varphi) \to \Box \ P(\varphi)]$$

Lemma 3: If an individual is a God, then being God is an essential property for that individual:

$$\forall y. [G(y) \to G \ ess \ y]$$

Formal proof:

Let the following derivation with the open assumption $G(x)$ be Π_1:

$$
\cfrac{\cfrac{\neg P(\psi)^1 \quad \cfrac{\cfrac{}{\forall \varphi.(\neg P(\varphi) \to P(\neg \varphi))} \ \text{Ax. 2}}{\neg P(\psi) \to P(\neg \psi)} \ \forall E}{P(\neg \psi)} \ \to E \qquad \cfrac{\cfrac{\cfrac{G(x)}{\forall \varphi.(P(\varphi) \to \varphi(x))} \ \text{Definition of G}}{P(\varphi) \to \varphi(x)} \ \forall E}{\neg \psi(x)} \to E \qquad \psi(x)^2}{\cfrac{\cfrac{\cfrac{\bot}{P(\psi)} \ \text{RAA, 1}}{\psi(x) \to P(\psi)} \ \to \text{I, 2}}{}} \ \to E
$$

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Let the following derivation with the open assumption $G(x)$ be Π_2:

$$\cfrac{\cfrac{\psi(x)^1 \quad \cfrac{\Pi_1}{\psi(x) \to P(\psi)}}{P(\psi)} \to\text{E} \quad \cfrac{\cfrac{\forall\psi.(P(\psi) \to \Box P(\psi))}{P(\psi) \to \Box P(\psi)} \text{Ax. } 4}{} \forall\text{E}}{\cfrac{\Box P(\psi)}{\psi(x) \to \Box P(\psi)} \to\text{I, 1}} \to\text{E}$$

Let the following derivation without open assumptions be Π_3:

$$\cfrac{\cfrac{P(\psi)^1 \quad \cfrac{\cfrac{\cfrac{G(x)^2}{\forall\varphi.(P(\varphi) \to \varphi(x))} \text{Definition of G}}{P(\psi) \to \psi(x)} \forall\text{E}}{}}{\cfrac{\cfrac{\psi(x)}{G(x) \to \psi(x)} \to\text{I, 2}}{\cfrac{\forall x.(G(x) \to \psi(x))}{} \forall\text{I}} } \to\text{E}}{P(\psi) \to \forall x.(G(x) \to \psi(x))} \to\text{I, 1}$$

Let the following derivation with the open assumption $G(x)$ be Π_4:

$$\cfrac{\cfrac{\psi(x)^1 \quad \cfrac{\Pi_2}{\psi(x) \to \Box P(\psi)}}{\Box P(\psi)} \to\text{E} \quad \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Box P(\psi)^2}{P(\psi)} \Box\text{E} \quad \cfrac{\Pi_3}{P(\psi) \to \forall x.(G(x) \to \psi(x))}}{\forall x.(G(x) \to \psi(x))} \to\text{E}}{\Box\forall x.(G(x) \to \psi(x))} \text{Necessitation}}{\Box P(\psi) \to \Box\forall x.(G(x) \to \psi(x))} \to\text{I, 2}}{}}{\cfrac{\Box\forall x.(G(x) \to \psi(x))}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))} \to\text{I, 1}} \to\text{E}$$

The use of the necessitation rule is valid, because the only open assumption $\Box P(\psi)$ is boxed.

We construct a derivation of theorem 3 with a subderivation $\Pi_4[G(x)^1]$, which means that the open assumption $G(x)$ in Π_4 is discharged with the rule labeled 1.

$$\cfrac{\cfrac{G(x)^1 \quad \cfrac{\cfrac{\Pi_4[G(x)^1]}{\psi(x) \to \Box\forall x.(G(x) \to \psi(x))}}{\forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))} \forall\text{I}}{\cfrac{G(x) \& \forall\psi.(\psi(x) \to \Box\forall x.(G(x) \to \psi(x)))}{\cfrac{G \ ess \ x}{G(x) \to G \ ess \ x} \to\text{I, 1}} \text{Definition of ess}}}{\forall y.[G(y) \to G \ ess \ y]}$$

4 Necessity of God's existence

Definitions:

- **(D3)** An individual *necessarily exists* if and only if all its essential properties are necessarily witnessed:

$$E(x) \leftrightarrow \forall\varphi.[\varphi \ ess \ x \rightarrow \Box\exists x.\varphi(x)]$$

Axioms:

- **(A5)** Necessary existence is a positive property:

$$P(E)$$

Auxiliary Lemma: If there is a God, then there necessarily exists a God:

$$\exists z.G(z) \rightarrow \Box\exists x.G(x)$$

Formal proof:

$$\cfrac{\cfrac{\overline{\exists z.G(z)} \ ^1}{G(g)}}{}$$

$$\cfrac{\cfrac{\overline{G(g)} \quad \cfrac{\cfrac{\overline{\forall y.[G(y) \rightarrow G \ ess \ y]}}{G(g) \rightarrow G \ ess \ g} \ \text{Th. 3}}{G \ ess \ g} \quad \cfrac{P(E) \quad \cfrac{\cfrac{\overline{G(g)}}{\forall\varphi.[P(\varphi) \rightarrow \varphi(g)]}}{P(E) \rightarrow E(g)}}{\cfrac{E(g)}{\cfrac{\forall\varphi.[\varphi \ ess \ g \rightarrow \Box\exists x.\varphi(x)]}{G \ ess \ g \rightarrow \Box\exists x.G(x)}}}}{\Box\exists x.G(x)}}{\exists z.G(z) \rightarrow \Box\exists x.G(x)} \ ^1$$

5 Necessary existence of a God

Theorem: The existence of a God is necessary:

$$\Box\exists x.G(x)$$

Formal proof:

$$\cfrac{\cfrac{\overline{\forall\varphi.[\Diamond\ldots\Diamond\Box\varphi \leftrightarrow \Box\varphi]} \ \textbf{S5}}{\Diamond\Box\exists x.G(x) \leftrightarrow \Box\exists x.G(x)} \quad \cfrac{\overline{\Diamond\exists x.G(x)} \ \text{Th. 2} \quad \overline{\exists z.G(z) \rightarrow \Box\exists x.G(x)} \ \text{Th. A}}{\Diamond\Box\exists x.G(x)}}{\Box\exists x.G(x)}$$

6 God's existence

Axioms:

- **(M)** What is necessary is the case:

$$\forall\varphi.[\Box\varphi \to \varphi]$$

Corollary: There exists a God:

$$\exists x.G(x)$$

Formal proof:

$$\frac{\dfrac{}{\Box\exists x.G(x)}\ \text{Th. 4} \qquad \dfrac{\forall\varphi.[\Box\varphi \to \varphi]}{\Box\exists x.G(x) \to \exists x.G(x)}}{\exists x.G(x)}$$