

# Gödel’s God in Isabelle/HOL

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## 1 Introduction

A formalization and (partial) automation of Dana Scott’s version [10] of Goedel’s ontological argument [7] in quantified modal logic KB (QML KB) is presented. QML KB is in turn modeled as a fragment of classical higher-order logic (HOL). Thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benz Müller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (properties).

The formalization presented here has been carried and formally verified in the Isabelle/HOL proof assistant; for more information on this system see the textbook by Nipkow, Paulson, and Wenzel [9]. More recent tutorials on Isabelle can be found at the Isabelle homepage: <http://isabelle.in.tum.de>.

Some further notes:

1. This LaTeX text document has been produced automatically from the Isabelle source code document at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/Isabelle/GoedelGodSession> with the Isabelle build tool.
2. The formalization presented here is related to the THF [12] and Coq [4] formalizations available at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/>.
3. All reasoning gaps in Scott’s proof script have been automated with Sledgehammer [5] performing remote calls to the higher-order automated theorem prover LEO-II [3]. These calls then suggested respective Metis [8] calls as given below. The Metis proofs are then verified in Isabelle/HOL.
4. For consistency checking, the Nitpick model finder [6] has been employed.

## 2 An Embedding of QML KB in HOL

The types  $i$  for possible worlds (or states) and  $mu$  for individuals are introduced.

**typed**decl  $i$  — the type for possible worlds  
**typed**decl  $mu$  — the type for individuals

Possible worlds are connected by an accessibility relation .

**consts**  $r :: i \Rightarrow i \Rightarrow \text{bool}$  (**infixr**  $r$  70) — accessibility relation  $r$

The  $B$  axiom (symmetry) for relation  $r$  is stated.  $B$  is needed only for proving theorem T3.

**axiomatization where**  $\text{sym}: x \ r \ y \longrightarrow y \ r \ x$

QML formulas are identified with certain HOL terms of type  $i \Rightarrow \text{bool}$ . This type will be abbreviated in the remainder as  $\sigma$

**type-synonym**  $\sigma = (i \Rightarrow \text{bool})$

The classical connectives  $\neg, \wedge, \Rightarrow$ , and  $\forall$  (for individuals and over sets of individuals) and  $\exists$  (over individuals) are lifted to type  $\sigma$ . Further connectives could be introduced analogously. *definition* could be used instead of *abbreviation*; the latter are always fully expanded/rewritten, which is fine here, where the focus has been on proof automation, but which would lead to overly complex proof tasks in a purely interactive session.

**abbreviation**  $m\text{not} :: \sigma \Rightarrow \sigma \ (m\neg)$  **where**  $m\neg \varphi \equiv (\lambda w. \neg \varphi \ w)$

**abbreviation**  $m\text{and} :: \sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixr**  $m\wedge$  79) **where**  $\varphi \ m\wedge \psi \equiv (\lambda w. \varphi \ w \wedge \psi \ w)$

**abbreviation**  $m\text{implies} :: \sigma \Rightarrow \sigma \Rightarrow \sigma$  (**infixr**  $m\Rightarrow$  74) **where**  $\varphi \ m\Rightarrow \psi \equiv (\lambda w. \varphi \ w \longrightarrow \psi \ w)$

**abbreviation**  $m\text{forall-ind} :: (\mu \Rightarrow \sigma) \Rightarrow \sigma \ (\forall)$  **where**  $\forall \Phi \equiv (\lambda w. \forall x. \Phi \ x \ w)$

**abbreviation**  $m\text{exists-ind} :: (\mu \Rightarrow \sigma) \Rightarrow \sigma \ (\exists)$  **where**  $\exists \Phi \equiv (\lambda w. \exists x. \Phi \ x \ w)$

**abbreviation**  $m\text{forall-indset} :: ((\mu \Rightarrow \sigma) \Rightarrow \sigma) \Rightarrow \sigma \ (\Pi)$  **where**  $\Pi P \equiv (\lambda w. \forall x. P \ x \ w)$

**abbreviation**  $m\text{box} :: \sigma \Rightarrow \sigma \ (\Box)$  **where**  $\Box \varphi \equiv (\lambda w. \forall v. \neg w \ r \ v \vee \varphi \ v)$

**abbreviation**  $m\text{dia} :: \sigma \Rightarrow \sigma \ (\Diamond)$  **where**  $\Diamond \varphi \equiv (\lambda w. \exists v. w \ r \ v \wedge \varphi \ v)$

For the grounding of lifted formulas the meta-predicate *valid* is introduced.

**abbreviation**  $\text{valid} :: \sigma \Rightarrow \text{bool}$  ( $[-]$ ) **where**  $[p] \equiv \forall w. p \ w$

The model finder Nitpick confirms that the axioms and definitions above are consistent. Unfortunately, the respective command syntax for Nitpick is not very intuitive.

**lemma** *True* **nitpick** [*satisfy, user-axioms, expect = genuine*] **oops**

Constant symbol  $P$  (Gödel's "Positive") is introduced.

**consts**  $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of  $P$  is restricted by axioms  $A1(a/b): \forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$  (Either a property or its negation is positive, but not both.) and  $A2: \forall \phi \forall \psi [(P(\phi) \wedge \Box \forall x [\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$  (A property necessarily implied by a positive property is positive.).

**axiomatization where**

$A1a: [\Pi (\lambda \Phi. P (\lambda x. m\neg (\Phi \ x)) \ m\Rightarrow m\neg (P \ \Phi))] \text{ and}$

$A1b: [\Pi (\lambda \Phi. m\neg (P \ \Phi) \ m\Rightarrow P (\lambda x. m\neg (\Phi \ x)))] \text{ and}$

$A2: [\Pi (\lambda \Phi. \Pi (\lambda \psi. (P \ \Phi \ m\wedge \Box (\forall (\lambda x. \Phi \ x \ m\Rightarrow \psi \ x))) \ m\Rightarrow P \ \psi)]]$

We prove theorem T1:  $\forall \varphi [P(\varphi) \rightarrow \Diamond \exists x \varphi(x)]$  (Positive properties are possibly exemplified). T1 is proved directly by Sledgehammer with command *sledgehammer* [*provers = remote-leo2 remote-satallax*]. This successful attempt then suggest to instead try the Metis call in the line below. This Metis call generates a proof object that is verified in Isabelle/HOL's kernel.

**theorem** *T1*:  $[\Pi (\lambda \Phi. P \ \Phi \ m\Rightarrow \Diamond (\exists \ \Phi))]$

**sledgehammer** [*provers = remote-leo2*]

**using** *A2 A1a* **by** *metis*

Next, the symbol  $G$ , for "God-like", is introduced and defined as  $G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$  (A God-like being possesses all positive properties:).

**definition**  $G :: \mu \Rightarrow \sigma$  **where**  $G = (\lambda x. \Pi (\lambda \Phi. P \Phi \Rightarrow \Phi x))$

Axiom  $A3$  is added:  $P(G)$  (The property of being God-like is positive.). Sledgehammer and Metis then prove corollary  $C: \Diamond \exists x G(x)$  (Possibly, God exists.).

**axiomatization where**  $A3: [P G]$

**corollary**  $C: [\Diamond (\exists G)]$

**sledgehammer** [*provers = remote-leo2*]

**using**  $A3 T1$  **by** *metis*

We add axiom  $A4: \forall\phi[P(\phi) \rightarrow \Box P(\phi)]$  (Positive properties are necessarily positive).

**axiomatization where**  $A4: [\Pi (\lambda \Phi. P \Phi \Rightarrow \Box (P \Phi))]$

Symbol  $ess$ , for "Essence", is introduced and defined as  $\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$  (An *essence* of an individual is a property possessed by it and necessarily implying any of its properties.).

**definition**  $ess :: (\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$  (**infixr**  $ess$  85) **where**

$\Phi \text{ ess } x = \Phi x \wedge \Pi (\lambda \psi. \psi x \Rightarrow \Box (\forall (\lambda y. \Phi y \Rightarrow \psi y)))$

Next, Sledgehammer and Metis prove theorem  $T2: \forall x[G(x) \rightarrow G \text{ ess. } x]$  (Being God-like is an essence of any God-like being).

**theorem**  $T2: [\forall (\lambda x. G x \Rightarrow G \text{ ess } x)]$

**sledgehammer** [*provers = remote-leo2*]

**by** (*metis (lifting) A1b A4 G-def ess-def*)

Symbol  $NE$ , for "Necessary Existence", is introduced and defined as  $NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$  (Necessary existence of an individual is the necessary exemplification of all its essences.).

**definition**  $NE :: \mu \Rightarrow \sigma$  **where**  $NE = (\lambda x. \Pi (\lambda \Phi. \Phi \text{ ess } x \Rightarrow \Box (\exists \Phi)))$

Moreover, axiom  $A5$  is added:  $P(NE)$  (Necessary existence is a positive property.).

**axiomatization where**  $A5: [P NE]$

Finally, Sledgehammer and Metis prove the main theorem  $T3: \Box\exists x G(x)$  (Necessarily, God exists.).

**theorem**  $T3: [\Box (\exists G)]$

**sledgehammer** [*provers = remote-leo2*]

**using**  $A5 C T2 \text{ sym } G\text{-def } NE\text{-def}$  **by** *metis*

**corollary**  $T4: [\exists G]$

**sledgehammer** [*provers = remote-leo2*]

**using**  $T1 T3 \text{ sym } G\text{-def}$  **by** *metis*

Finally, the consistency of the entire theory is checked with Nitpick.

**lemma** *True nitpick* [*satisfy, user-axioms, expect = genuine*] **oops**

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