An interesting temporalization of Gödel's ontological proof

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Recent theologies concerning God's death after Auschwitz are mathematically formalized through a suitable temporalization of Gödel's Ontological Proof.

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II. MODAL LOGIC

Let us introduce briefly Charles Lewis' formal systems of Modal Logic [1].

Introduced the *necessity operator* $\Box$ and the *possibility operator* $\Diamond$ let us introduce the following:

Definition II.1

T formal system:

the formal system obtained adding to the propositional logic the modal operators $\Box$ and $\Diamond$ and the following axioms:

$$\Box\phi \ \rightarrow \ \phi \tag{2.1}$$

$$\phi \ \rightarrow \ \Diamond\phi \tag{2.2}$$

$$\Box\phi \ \leftrightarrow \ \neg\Diamond\neg\phi \tag{2.3}$$

$$\Diamond\phi \ \leftrightarrow \ \neg\Box\neg\phi \tag{2.4}$$

$$\Box(\phi \wedge \psi) \ \leftrightarrow \ \Box\phi \wedge \Box\psi \tag{2.5}$$

$$\Diamond(\phi \vee \psi) \ \leftrightarrow \ \Diamond\phi \vee \Diamond\psi \tag{2.6}$$

$$\Box\phi \vee \Box\psi \ \rightarrow \ \Box(\phi \vee \psi) \tag{2.7}$$

$$\Diamond(\phi \wedge \psi) \ \rightarrow \ \Diamond\phi \wedge \Diamond\psi \tag{2.8}$$

$$\Box(\phi \rightarrow \psi) \ \rightarrow \ (\Box\phi \rightarrow \Box\psi) \tag{2.9}$$

$$(\Diamond\phi \rightarrow \Diamond\psi) \ \rightarrow \ \Diamond(\phi \rightarrow \psi) \tag{2.10}$$

Definition II.2

S4 formal system:

the formal system obtained adding to the *T formal system* the following axioms:

$$\Box\Box\phi \ \leftrightarrow \ \Box\phi \tag{2.11}$$

$$\Diamond\Diamond\phi \ \leftrightarrow \ \Diamond\phi \tag{2.12}$$

Definition II.3

S5 formal system:

the formal system obtained adding to the *S4 formal system* the following axiom:

$$\Diamond\Box\phi \ \rightarrow \ \Box\phi \tag{2.13}$$

III. GÖDEL'S ONTOLOGICAL PROOF

Kurt Gödel formalized (Leibniz's elaboration of Descartes' elaboration of) the ontological proof of the existence of God furnished by *Anselmo da Aosta* in his *Proslogion* as a theorem of a suitable formal system of Modal Logic though he didn't published his result since, according to Morgenstern, he was afraid that his purely logical investigation could be seen as a religious affair.

Gödel ontological proof has survived his death as many of his unpublished works [2] and owing to the fact that in 1970 Gödel showed his proof to Dana Scott and allowed him to make a copy of his handwriting pages photocopies of which began to circulate in the early eighties, making his first public appearance in [3].

The idea of Anselm's ontological proof is that, defined God as the entity having every perfection (i.e. every ethically and aesthetically positive property), it must exist since otherwise one could think a more perfect being having also the perfection of existing.

Descartes remarked how the essence of such an ontological proof lies in the fact that God is defined as en entity possessing the property of *necessary existence*, i.e. the property that if it is possible than it is necessary.

Leibniz remarked how Descartes' reformulation of Anselm's argument as a proof of the fact that, by definition, God possesses the property of necessary existence, had to be augmented with the proof that the existence of God is possible.

In the framework of the *S5 formal system* let us introduce a *positivity predicate* $P(\phi)$ and let us assume the following:

AXIOM III.1

$$P(\neg\phi) \;\leftrightarrow\; \neg P(\phi) \tag{3.1}$$

AXIOM III.2

$$P(\phi) \wedge \forall x[\phi(x) \rightarrow \psi(x)] \;\rightarrow\; P(\psi) \tag{3.2}$$

Then:

Theorem III.1

$$P(\phi) \;\rightarrow\; \Diamond\exists x\, \phi(x) \tag{3.3}$$

PROOF:

Let as assume the *ad absurdum hypothesis* $P(\phi) \wedge \neg\Diamond\exists x\phi(x)$.

By the Duns Scoto's principle *ex absurdo quodlibet sequitur*:

$$\neg\Diamond\exists x\, \phi(x\phi) \;\rightarrow\; \forall\psi\Box\forall x[\phi(x) \rightarrow \psi(x)] \tag{3.4}$$

Choosing in particular $\psi := \neg\phi$ the equation 3.4 becomes:

$$\neg\Diamond\exists x\, \phi(x\phi) \;\rightarrow\; \forall x[\phi(x) \rightarrow \neg\phi(x)] \tag{3.5}$$

By the axiom III.2 it follows that:

$$P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow (\neg\phi)x \;\rightarrow\; P(\neg\phi) \tag{3.6}$$

By the axiom III.1:

$$P(\neg\phi) \rightarrow \neg P(\phi) \tag{3.7}$$

so that we obtain that $\neg P(\phi)$ contradicting the hypothesis $\blacksquare$

Let us now define the predicate of *being God-like* as the condition of having all the positive properties:

Definition III.1

$$G(x) \ := \ \forall\phi[P(\phi) \to \phi(x)] \tag{3.8}$$

Let us now assume that to be God-like is positive:

AXIOM III.3

$$P(G) \tag{3.9}$$

Let us assume furthermore that to be positive cannot be a contingent property:

AXIOM III.4

$$P(\phi) \ \to \ \Box P(\phi) \tag{3.10}$$

Let us now define the essence of a entity as a property of that entity implying any other property of its:

Definition III.2

$$\phi \, ess \, x \ := \ \phi(x) \wedge \forall\psi\{\psi(x) \to \Box\forall y[\phi(y) \to \psi(y)]\} \tag{3.11}$$

Let us now prove that if an entity is God-like to be God-like is its essence:

Theorem III.2

$$G(x) \ \to \ G \, ess \, x \tag{3.12}$$

PROOF:

Applying the following:

Lemma III.1

$$\psi(x) \ \to \ P(\psi) \tag{3.13}$$

it follows by the axiom III.4 that:

$$P(\psi) \ \to \ \Box P(\psi) \tag{3.14}$$

By the following:

Lemma III.2

$$\forall x\{G(x) \leftrightarrow \forall\phi[P(\phi) \to \phi(x)]\} \ \to \ \Box\{P(\psi) \to \forall x[G(x) \to \psi(x)]\} \tag{3.15}$$

and by the definition III.1 and the definition III.2 it follows that:

$$\Box\forall x\{G(x) \leftrightarrow \forall\phi[P(\phi) \to \psi(x)]\} \tag{3.16}$$

By the lemma III.2:

$$\Box\{P(\phi) \to \forall x[G(x) \to \psi(x)]\} \tag{3.17}$$

and hence, by the definition II.3:

$$\Box P(\psi) \ \to \ \Box\forall x[G(x) \to \psi(x)] \tag{3.18}$$

By *modus ponens*:

$$\Box\forall x[G(x) \to \psi(x)] \tag{3.19}$$

Hence:

$$G(x) \wedge \forall\psi\{\psi(x) \to \Box\forall y[G(y) \to \psi(y)]\} \tag{3.20}$$

and hence $G \, ess \, x$ ■

Let us now introduce the notion of *necessary existence*:

Definition III.3

$$NE(x) \;:=\; \forall \phi [\phi \; ess \, x \; \rightarrow \Box \exists \phi(x)] \tag{3.21}$$

Assuming that having the property of necessary existence is positive:

AXIOM III.5

$$P(NE) \tag{3.22}$$

we are at last able to prove the following::

Theorem III.3

Gödel's Ontological Theorem

$$\Box \exists x G(x) \tag{3.23}$$

PROOF:

Lemma III.3

$$G(x) \;\rightarrow\; NE(x) \wedge G \; ess \, x \tag{3.24}$$

Lemma III.4

$$\exists x G(x) \;\rightarrow\; \Box \exists x G(x) \tag{3.25}$$

Lemma III.5

$$\Diamond \exists x G(x) \;\rightarrow\; \Diamond \Box \exists x G(x) \tag{3.26}$$

Lemma III.6

$$\Diamond \exists x G(x) \;\rightarrow\; \Box \exists x G(x) \tag{3.27}$$

Lemma III.7

$$\Diamond \exists x G(x) \tag{3.28}$$

■

In order to discuss briefly the literature concerning Gödel's ontological proof, let us introduce the following:

Definition III.4

Gödel's ontological formal system:

the formal system $\mathcal{O}$ consisting in the *S5 formal system* of the definition II.3 augmented with the axiom III.1 ,the axiom III.2, the axiom III.3, the axiom III.4 and the axiom III.5

A disturbing property of Gödel's ontological formal system has been remarked by Jordan Howard Sobel [4]:

Theorem III.4

Sobel's Theorem on modal collapse:
HP:

$$\mathcal{O}$$

TH:

$$\Diamond\phi \;\Rightarrow\; \Box\phi \tag{3.29}$$

Modifications of Gödel's ontological proof aimed to bypass the problems related to the theorem III.4 have been proposed by C. Anthony Anderson [5].

IV. TEMPORALIZED ONTOLOGICAL PROOF

Let us now introduce the temporal operators of Arthur Prior's Temporal Logic [6]:

- $\forall_-$:= "it has been always true that"
- $\forall_+$:= "it will be always true that"
- $\exists_-$:= "it has been true that"
- $\exists_+$:= "it will be true that"

Let us then introduce the following:

Definition IV.1

temporal logic:
the modal logic obtained from the propositional logic introducing the temporal operators $\forall_-, \forall_+, \exists_-, \exists_+$ satisfying the axioms:

$$\forall_- \phi \ \rightarrow \ \exists_- \phi \tag{4.1}$$

$$\forall_+ \phi \ \rightarrow \ \exists_+ \phi \tag{4.2}$$

Let us now introduce the following:

Definition IV.2

temporalization operator:

$$\forall \phi \ \stackrel{\mathcal{T}}{\rightarrow} \ \forall_- \phi \wedge \forall_+ \phi \tag{4.3}$$

$$\exists \phi \ \stackrel{\mathcal{T}}{\rightarrow} \ \exists_- \phi \wedge \exists_+ \phi \tag{4.4}$$

Let us now introduce the following:

Definition IV.3

time reversal operator:
the operator acting on the temporal labels of the temporal quantificators in the following way:

$$\forall_- \phi \ \stackrel{T}{\mapsto} \ \forall_+ \phi \tag{4.5}$$

$$\exists_- \phi \ \stackrel{T}{\mapsto} \ \exists_+ \phi \tag{4.6}$$

$$\forall_+ \phi \ \stackrel{T}{\mapsto} \ \forall_- \phi \tag{4.7}$$

$$\exists_+ \phi \ \stackrel{T}{\mapsto} \ \exists_- \phi \tag{4.8}$$

By construction:

Theorem IV.1

time-reversal invariance of temporalized formal systems:

$$T \mathcal{T} S \ = \ \mathcal{T} S \ \ \forall S \tag{4.9}$$

In particular let us introduce the following:

Definition IV.4

temporalized Gödel's ontological formal system:

$$\mathcal{O}_{\mathcal{T}} \ := \ \mathcal{T} \mathcal{O} \tag{4.10}$$

V. GOD'S DEATH AS A BREAKING OF TIME-REVERSAL SYMMETRY IN THE TEMPORALIZED ONTOLOGICAL PROOF

The logical problem of *theodicy*, namely the problem of bypassing the logical incompatibility between the definition of God as the maximally good entity and the existence of evil in the world was first raised by Epicurus and has been at the center of Rational Theology from Leibniz's treatise to recent time.

Taking literally Theodor Wiesengrund Adorno's remark that every metaphysical notion becomes impotent in front and after Auschwitz, we can think that it affects the same concept of perfection and hence the same Anselm's definition of God as an entity having every perfection.

This lead to think radically about the title of a celebrated book by Hans Jonas, i.e. the same concept of God after Auschwitz loses its meaning.

These philosophical remarks much in the spirit of the contemporary *theologies of God's death* may be mathematically formalized as a time-reversal symmetry breaking within the formal system IV.4.

With this regard let us first of all introduce the temporal analogue of Godel's ontological proof:

Theorem V.1

Temporalized ontological proof:

HP:

$$\mathcal{O}_{\mathcal{T}}$$

TH:

$$\Box \exists_{-} x G(x) \tag{5.1}$$

$$\Box \exists_{+} x G(x) \tag{5.2}$$

PROOF:

It is sufficient to combine the definition IV.2 and the theorem III.3 ■

Let us now formalize the breaking of time-reversal symmetry through the following:

Definition V.1

time-reversal breaking temporalization's operator:

$$\forall \phi \;\stackrel{\mathcal{T}_{\mathcal{B}}}{\rightarrow}\; (\forall_{-} \phi \wedge \neg \forall_{+} \phi) \vee (\forall_{+} \phi \wedge \neg \forall_{-} \phi) \tag{5.3}$$

$$\exists \phi \;\stackrel{\mathcal{T}_{\mathcal{B}}}{\rightarrow}\; (\exists_{-} \phi \wedge \neg \exists_{+} \phi) \vee (\exists_{+} \phi \wedge \neg \exists_{-} \phi) \tag{5.4}$$

Let us now introduce the following:

Definition V.2

time-reversal breaking temporalized Gödel's ontological formal system:

$$\mathcal{O}_{\mathcal{T}_{\mathcal{B}}} \;:=\; \mathcal{T}_{\mathcal{B}} \mathcal{O} \tag{5.5}$$

Then:

Theorem V.2

Time-reversal breaking temporalized ontological proof:

HP:

$$\mathcal{O}_{\mathcal{T}_B}$$

TH:

$$(\Box\exists_{-}xG(x) \land \Box\neg\exists_{+}xG(x)) \lor (\Box\exists_{+}xG(x) \land \Box\neg\exists_{-}xG(x)) \tag{5.6}$$

PROOF:

It is sufficient to combine the definition V.2 and the theorem III.3 ■

The impossibility of speaking of any kind of perfection after Auschwitz may be formalized as the following:

AXIOM V.1

Impossibility of positivity in the future:

$$\Box\forall_{+}\neg P(\phi) \tag{5.7}$$

Then:

Theorem V.3

Theorem about God's death:

HP:

$$\mathcal{O}_{\mathcal{T}_B} \land \text{ axiom V.1}$$

TH:

$$\Box\exists_{-}xG(x) \land \Box\neg\exists_{+}xG(x) \tag{5.8}$$

PROOF:

The thesis trivially follows combining the theorem V.2 and the axiom V.1 ■

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