

Gödel's God in Isabelle/HOL

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A1	Either a property or its negation is positive, but not both:	$\forall\phi[P(\neg\phi) \leftrightarrow \neg P(\phi)]$
A2	A property necessarily implied by a positive property is positive:	$\forall\phi\forall\psi[(P(\phi) \wedge \Box\forall x[\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$
T1	Positive properties are possibly exemplified:	$\forall\phi[P(\phi) \rightarrow \Diamond\exists x\phi(x)]$
D1	A <i>God-like</i> being possesses all positive properties:	$G(x) \leftrightarrow \forall\phi[P(\phi) \rightarrow \phi(x)]$
A3	The property of being God-like is positive:	$P(G)$
C	Possibly, God exists:	$\Diamond\exists xG(x)$
A4	Positive properties are necessarily positive:	$\forall\phi[P(\phi) \rightarrow \Box P(\phi)]$
D2	An <i>essence</i> of an individual is a property possessed by it and necessarily implying any of its properties:	$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall\psi(\psi(x) \rightarrow \Box\forall y(\phi(y) \rightarrow \psi(y)))$
T2	Being God-like is an essence of any God-like being:	$\forall x[G(x) \rightarrow G \text{ ess. } x]$
D3	<i>Necessary existence</i> of an individual is the necessary exemplification of all its essences:	$NE(x) \leftrightarrow \forall\phi[\phi \text{ ess. } x \rightarrow \Box\exists y\phi(y)]$
A5	Necessary existence is a positive property:	$P(NE)$
T3	Necessarily, God exists:	$\Box\exists xG(x)$

Figure 1: Scott's version of Gödel's ontological argument [12].

1 Introduction

Dana Scott's version [12] (cf. Fig. 1) of Gödel's ontological argument for God's Existence [8] is formalized in quantified modal logic KB (QML KB) within the proof assistant Isabelle/HOL. QML KB is modeled as a fragment of classical higher-order logic (HOL); thus, the formalization is essentially a formalization in HOL. The employed embedding of QML KB in HOL is adapting the work of Benz Müller and Paulson [2, 1]. Note that the QML KB formalization employs quantification over individuals and quantification over sets of individuals (properties). The gaps in Scott's proof have been automated with Sledgehammer [5], performing remote calls to the higher-order automated theorem prover LEO-II [3]. Sledgehammer suggests the Metis [9] calls, which result in proofs that are verified by Isabelle/HOL. For consistency checking, the model finder Nitpick [6] has been employed. The successful calls to Sledgehammer are deliberately kept in the file for demonstration purposes (normally, they are automatically eliminated by Isabelle/HOL).

Isabelle is described in the textbook by Nipkow, Paulson, and Wenzel [10] and in tutorials available at: <http://isabelle.in.tum.de>.

1.1 Related Work

The formalization presented here is related to the THF [14] and Coq [4] formalizations at <https://github.com/FormalTheology/GoedelGod/tree/master/Formalizations/>.

An older ontological argument by Anselm was formalized in PVS by John Rushby [11].

2 An Embedding of QML KB in HOL

The types i for possible worlds and μ for individuals are introduced.

typeddecl i — the type for possible worlds
typeddecl μ — the type for individuals

Possible worlds are connected by an accessibility relation r .

consts $r :: i \Rightarrow i \Rightarrow \text{bool}$ (**infixr** r 70) — accessibility relation r

QML formulas are translated as HOL terms of type $i \Rightarrow \text{bool}$. This type is abbreviated as σ .

type-synonym $\sigma = (i \Rightarrow \text{bool})$

The classical connectives $\neg, \wedge, \rightarrow$, and \forall (over individuals and over sets of individuals) and \exists (over individuals) are lifted to type σ . The lifted connectives are $m\neg$, $m\wedge$, $m\rightarrow$, $m\forall$, and $m\exists$ (the latter two are modeled as constant symbols). Other connectives can be introduced analogously. We exemplarily do this for $m\vee$, $m\leftrightarrow$, and $m=$. Moreover, the modal operators \Box and \Diamond are introduced. Definitions could be used instead of abbreviations.

abbreviation $mnot :: \sigma \Rightarrow \sigma$ (**where** $m\neg \varphi \equiv (\lambda w. \neg \varphi w)$)
abbreviation $mand :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\wedge$ 65) **where** $m\wedge \varphi \psi \equiv (\lambda w. \varphi w \wedge \psi w)$
abbreviation $mor :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\vee$ 70) **where** $m\vee \varphi \psi \equiv (\lambda w. \varphi w \vee \psi w)$
abbreviation $mimplies :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\rightarrow$ 74) **where** $m\rightarrow \varphi \psi \equiv (\lambda w. \varphi w \longrightarrow \psi w)$
abbreviation $mequiv :: \sigma \Rightarrow \sigma \Rightarrow \sigma$ (**infixr** $m\equiv$ 76) **where** $m\equiv \varphi \psi \equiv (\lambda w. (\varphi w \longleftrightarrow \psi w))$
abbreviation $meq :: 'a \Rightarrow 'a \Rightarrow \sigma$ (**infixr** $m=$ 50) **where** $x m= y \equiv (\lambda w. x = y)$
abbreviation $mforall :: ('a \Rightarrow \sigma) \Rightarrow \sigma$ (\forall) **where** $\forall \Phi \equiv (\lambda w. \forall x. \Phi x w)$
abbreviation $mexists :: ('a \Rightarrow \sigma) \Rightarrow \sigma$ (\exists) **where** $\exists \Phi \equiv (\lambda w. \exists x. \Phi x w)$
abbreviation $mbox :: \sigma \Rightarrow \sigma$ (\Box) **where** $\Box \varphi \equiv (\lambda w. \forall v. w r v \longrightarrow \varphi v)$
abbreviation $mdia :: \sigma \Rightarrow \sigma$ (\Diamond) **where** $\Diamond \varphi \equiv (\lambda w. \exists v. w r v \wedge \varphi v)$

For grounding lifted formulas, the meta-predicate *valid* is introduced.

abbreviation $valid :: \sigma \Rightarrow \text{bool}$ ($[p]$) **where** $[p] \equiv \forall w. p w$

3 Gödel's Ontological Argument

Constant symbol P (Gödel's 'Positive') is declared.

consts $P :: (\mu \Rightarrow \sigma) \Rightarrow \sigma$

The meaning of P is restricted by axioms $A1(a/b)$: $\forall \phi [P(\neg \phi) \leftrightarrow \neg P(\phi)]$ (Either a property or its negation is positive, but not both.) and $A2$: $\forall \phi \forall \psi [(P(\phi) \wedge \Box \forall x [\phi(x) \rightarrow \psi(x)]) \rightarrow P(\psi)]$ (A property necessarily implied by a positive property is positive).

axiomatization where

A1a: $[\forall (\lambda \Phi. P (\lambda x. m \neg (\Phi x)) m \rightarrow m \neg (P \Phi))] \text{ and}$
A1b: $[\forall (\lambda \Phi. m \neg (P \Phi) m \rightarrow P (\lambda x. m \neg (\Phi x)))] \text{ and}$
A2: $[\forall (\lambda \Phi. \forall (\lambda \Psi. (P \Phi m \wedge \Box (\forall (\lambda x. \Phi x m \rightarrow \Psi x))) m \rightarrow P \Psi))]$

We prove theorem *T1*: $\forall \phi [P(\phi) \rightarrow \Diamond \exists x \phi(x)]$ (Positive properties are possibly exemplified). *T1* is proved directly by Sledgehammer with command *sledgehammer [provers = remote-leo2]*. Sledgehammer suggests to call Metis with axioms *A1a* and *A2*. Metis sucesfully generates a proof object that is verified in Isabelle/HOL's kernel.

theorem *T1*: $[\forall (\lambda \Phi. P \Phi m \rightarrow \Diamond (\exists \Phi))]$
sledgehammer *[provers = remote-leo2]*
by (*metis A1a A2*)

Next, the symbol *G* for ‘God-like’ is introduced and defined as $G(x) \leftrightarrow \forall \phi [P(\phi) \rightarrow \phi(x)]$ (A God-like being possesses all positive properties).

definition *G* :: $\mu \Rightarrow \sigma$ **where** $G = (\lambda x. \forall (\lambda \Phi. P \Phi m \rightarrow \Phi x))$

Axiom *A3* is added: $P(G)$ (The property of being God-like is positive). Sledgehammer and Metis then prove corollary *C*: $\Diamond \exists x G(x)$ (Possibly, God exists).

axiomatization where *A3*: $[P G]$

corollary *C*: $[\Diamond (\exists G)]$
sledgehammer *[provers = remote-leo2]*
by (*metis A3 T1*)

Axiom *A4* is added: $\forall \phi [P(\phi) \rightarrow \Box P(\phi)]$ (Positive properties are necessarily positive).

axiomatization where *A4*: $[\forall (\lambda \Phi. P \Phi m \rightarrow \Box (P \Phi))]$

Symbol *ess* for ‘Essence’ is introduced and defined as

$$\phi \text{ ess. } x \leftrightarrow \phi(x) \wedge \forall \psi (\psi(x) \rightarrow \Box \forall y (\phi(y) \rightarrow \psi(y)))$$

(An *essence* of an individual is a property possessed by it and necessarily implying any of its properties).

definition *ess* :: $(\mu \Rightarrow \sigma) \Rightarrow \mu \Rightarrow \sigma$ (**infixr** *ess* 85) **where**
 $\Phi \text{ ess } x = \Phi x m \wedge \forall (\lambda \Psi. \Psi x m \rightarrow \Box (\forall (\lambda y. \Phi y m \rightarrow \Psi y)))$

Next, Sledgehammer and Metis prove theorem *T2*: $\forall x [G(x) \rightarrow G \text{ ess. } x]$ (Being God-like is an essence of any God-like being).

theorem *T2*: $[\forall (\lambda x. G x m \rightarrow G \text{ ess } x)]$
sledgehammer *[provers = remote-leo2]*
by (*metis A1b A4 G-def ess-def*)

Symbol *NE*, for ‘Necessary Existence’, is introduced and defined as

$$NE(x) \leftrightarrow \forall \phi [\phi \text{ ess. } x \rightarrow \Box \exists y \phi(y)]$$

(Necessary existence of an individual is the necessary exemplification of all its essences).

definition *NE* :: $\mu \Rightarrow \sigma$ **where** $NE = (\lambda x. \forall (\lambda \Phi. \Phi \text{ ess } x m \rightarrow \Box (\exists \Phi)))$

Moreover, axiom *A5* is added: $P(NE)$ (Necessary existence is a positive property).

axiomatization where *A5*: $[P\ NE]$

The *B* axiom (symmetry) for relation *r* is stated. *B* is needed only for proving theorem *T3* and for corollary *C2*.

axiomatization where *sym*: $x\ r\ y \longrightarrow y\ r\ x$

Finally, Sledgehammer and Metis prove the main theorem *T3*: $\Box\exists xG(x)$ (Necessarily, God exists).

theorem *T3*: $[\Box\ (\exists\ G)]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis A5 C T2 sym G-def NE-def*)

Surprisingly, the following corollary can be derived even without the *T* axiom (reflexivity).

corollary *C2*: $[\exists\ G]$
sledgehammer [*provers* = *remote-leo2*](*T1 T3 G-def sym*)
by (*metis T1 T3 G-def sym*)

The consistency of the entire theory is checked with Nitpick.

lemma *True nitpick* [*satisfy, user-axioms, expect* = *genuine*] **oops**

4 Additional Results on Gödel's God.

Gödel's God is flawless: (s)he does not have non-positive properties.

theorem *Flawlessness*: $[\forall (\lambda\Phi. \forall (\lambda x. (G\ x\ m \rightarrow (m \neg (P\ \Phi)\ m \rightarrow m \neg (\Phi\ x)))))]$
sledgehammer [*provers* = *remote-leo2*]
by (*metis A1b G-def*)

There is only one God: any two God-like beings are equal.

theorem *Monotheism*: $[\forall (\lambda x. \forall (\lambda y. (G\ x\ m \rightarrow (G\ y\ m \rightarrow (x\ m = y)))))]$
sledgehammer [*provers* = *remote-satallax*]
sledgehammer [*provers* = *remote-satallax*](*Flawlessness G-def*) **oops**

Unfortunately, Metis is too weak to reconstruct the proof. (*metis Flawlessness G-def*) fails.

5 Modal Collapse

Gödel's axioms have been criticized for entailing the so-called modal collapse. The prover Satallax [7] confirms this. However, sledgehammer is not able to determine which axioms, definitions and previous theorems are used by Satallax; hence it suggests to call Metis using everything, but this (unsurprisingly) fails. Attempting to use 'Sledgehammer min' to minimize Sledgehammer's suggestion does not work. Calling Metis with *T2*, *T3* and *ess-def* also does not work.

lemma *MC*: $[\forall (\lambda\Phi. (\Phi\ m \rightarrow (\Box\ \Phi)))]$
sledgehammer [*provers* = *remote-satallax*] **oops**

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