1 Local Redundancy

We first consider an example first posed by Postan:

$$\dfrac{\dfrac{\dfrac{L_2\colon\ P(A) \qquad \eta_2\colon\ \neg P(x),\ \neg Q(x,B)}{\neg Q(x,B)} \qquad \eta_1\colon\ \neg P(z),\ Q(z,y)}{\neg P(z)} \qquad L_1\colon\ P(x)}{\bot}$$

Which is locally redundant; see the compressed version in his document.

2 "Example 2"

We consider example 2, from the LU/RPI paper, modified for first order predicates in a trivial way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\eta_1\colon\ \neg P(A) \qquad \eta_3\colon\ P(A), Q(B)}{\eta_4\colon\ Q(B)} \qquad \eta_2\colon\ P(A), R(C), \neg Q(B)}{\eta_5\colon\ P(A), R(C)} \qquad \eta_1\colon\ \neg P(A)}{\eta_6\colon\ R(C)} \qquad \dfrac{\dfrac{\eta_4\colon\ Q(B) \qquad \eta_7\colon\ P(A), \neg Q(B), \neg R(C)}{\eta_8\colon\ P(A), \neg R(C)} \qquad \eta_1\colon\ \neg P(A)}{\eta_9\colon\ \neg R(C)}}{\bot}$$

2.1 Lower Units

Proceeds exactly the same as in the paper.
TODO: show exact steps?

2.2 RecyclePivots

Again, proceeds like in the paper.

3 Lower Units - Research Notes

First, I consider the proofs 1-5 that were provided by Bruno on the Skeptik dev mailing list. In order to be explicit, I outline the case of compression from proof 1 to proof 2:

- Lower $P(X)$ so that the terms using it were resolved against each other instead of with $P(X)$
- Contract (trivially?); the unifier resulted in the duplicated terms
- Resolve the contracted formula against the lowered unit, $P(X)$

The result is a trade of a resolution for a contraction, which is more compact (when we consider compactness as a count of the number of resolution rules).

In order to generalize, I think the best place to start was see under what conditions we can in fact make this contraction. It should not be required that contraction results in duplicated formulas; indeed, as long as a contraction is possible this seems to work. So in particular, I conjecture that we should lower a unit formula if and only if for all formulas which would be resolved against the unit clause of interest are pair-wise unifiable (disregarding the remainder of their premises), and unifiable with the unit. Further, the unit must be the most general form of the formula, as the following shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash P(y,x) \qquad P(A,x) \vdash Q(A), R(B)}{\vdash Q(y), R(B)} \qquad Q(y) \vdash Q(x)}{\vdash R(B), Q(x)} \qquad \dfrac{Q(A), P(y,A) \vdash \qquad \vdash P(y,x)}{Q(x) \vdash}}{\vdash R(B)} \qquad R(B) \vdash}{\perp}$$

but if we delay the resolution with $P(y,x)$ we get

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(A,x) \vdash Q(A), R(B) \qquad Q(y) \ \vdash Q(x)}{P(y,x) \vdash Q(x), R(B)} \qquad Q(A), P(y,A) \vdash}{P(y,x), P(x,y) \vdash R(B)} \qquad R(B) \vdash}{P(y,x), P(y,A) \vdash} \qquad \vdash P(y,x)}{P(y,x)} \qquad \vdash P(y,x)}{\bot}
$$

and now we actually the same number of resolution rules. **TODO: -no, we can still use a contraction, and reduce the proof**

The requirement for being pairwise unifiable is also seen in proof 1 and 2, but further, this is lacking the case of proof 3: $P(a)$ and $P(b)$ is not unifiable, and thus proof 5 is not actually compressed. But if $P(b)$ had been $P(B)$, then we would have been fine. It also fails in the following example:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\vdash P(X) \qquad P(a) \vdash Q(Y), R(Z)}{\vdash Q(Y), R(Z)} \qquad R(X), P(b) \vdash S(Y)}{P(b) \vdash S(Y), Q(Y)} \qquad S(Y), Q(Y) \vdash}{P(b) \vdash} \qquad \vdash P(X)}{\bot}
$$

which is the 'potentially' globally reduction example from the original lower units paper.

Theorem 3.1. *Let S be the set of premises being resolved against a unit clause u. Then u can be lowered if, $|S| > 1$ and for every distinct $\eta, \eta' \in S$, η and η' are unifiable.*

Proof. Assume that S is defined as above, and is pairwise unifiable. Order the elements from the top of the proof to the bottom (and break ties left-right), so that η_1 is the top-left-most premise resolved against u. In particular, η_1 contains $\overline{u}_1'$, and we have that $P = \phi[\phi_1[\eta_1 \odot u] \odot r_1]$. Consider instead $\phi[r_1 \cup \overline{u}_1']$, the proof obtained by removing the resolution $\phi_1[\eta_1 \odot u] \odot r_1$ with just $\phi' = \eta_1 \odot r_1$ and then moving the subtree of ϕ to be the subtree of ϕ'. Note that ϕ' contains $\overline{u}_1'$ still, and so the resulting subtree would have more occurrences of $\overline{u}_1'$. In particular, the final node in the proof is $\overline{u}_1'$ instead of $\bot$. Since $|S| > 1$, at some other point, there exists η_2 such that η_2 is also resolved against u (with η_2 contains $\overline{u}'_2$. So consider $\phi[r_2 \cup \overline{u}'_2]$ and follow an argument similar to that for η_1; now the final proof node has $\overline{u}'_1 \cup \overline{u}'_2$ instead of $\overline{u}'_1$. By assumption, $\overline{u}'_1$ and $\overline{u}'_2$ are pair-wise unifiable. So we can contract these terms, and then resolve against u, to complete the proof. $\square$

TODO: use induction to show for any size $S > 1$? Also, the notation seems messy at best.