# 1 Local Redundancy

We first consider an example first posed by Postan:

$$
\dfrac{\dfrac{\dfrac{L_2\colon\ P(A) \qquad \eta_2\colon\ \neg P(x),\ \neg Q(x,B)}{\neg Q(x,B)} \qquad \eta_1\colon\ \neg P(z),\ Q(z,y)}{\neg P(z)} \qquad L_1\colon\ P(x)}{\bot}
$$

Which is locally redundant; see the compressed version in his document.

# 2 "Example 2"

We consider example 2, from the LU/RPI paper, modified for first order predicates in a trivial way:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\eta_1\colon\ \neg P(A) \qquad \eta_3\colon\ P(A), Q(B)}{\eta_4\colon\ Q(B)} \qquad \eta_2\colon\ P(A), R(C), \neg Q(B)}{\eta_5\colon\ P(A), R(C)} \qquad \eta_1\colon\ \neg P(A)}{\eta_6\colon\ R(C)} \qquad \dfrac{\dfrac{\eta_4\colon\ Q(B) \qquad \eta_7\colon\ P(A), \neg Q(B), \neg R(C)}{\eta_8\colon\ P(A), \neg R(C)} \qquad \eta_1\colon\ \neg P(A)}{\eta_9\colon\ \neg R(C)}}{\bot}
$$

## 2.1 Lower Units

Proceeds exactly the same as in the paper.
**TODO:** show exact steps?

## 2.2 RecyclePivots

Again, proceeds like in the paper.

# 3 Lower Units

## 3.1 Research Notes

First, I consider the proofs 1-5 that were provided by Bruno on the Skeptik dev mailing list. In order to be explicit, I outline the case of compression from proof 1 to proof 2:

- Lower $P(X)$ so that the terms using it were resolved against each other instead of with $P(X)$
- Contract (trivially?); the unifier resulted in the duplicated terms
- Resolve the contracted formula against the lowered unit, $P(X)$

The result is a trade of a resolution for a contraction, which is more compact (when we consider compactness as a count of the number of resolution rules).

In order to generalize, I think the best place to start was see under what conditions we can in fact make this contraction. It should not be required that contraction results in duplicated formulas; indeed, as long as a contraction is possible this seems to work. So in particular, I conjecture that we should lower a unit formula if and only if for all formulas which would be resolved against the unit clause of interest are pair-wise unifiable (disregarding the remainder of their premises), and unifiable with the unit. Further, the unit must be the most general form of the formula, as the following shows:

$$\dfrac{\dfrac{\dfrac{\vdash P(y,x) \qquad P(A,x) \vdash Q(A), R(B)}{\vdash Q(y), R(B)} \qquad Q(y) \vdash Q(x)}{\vdash R(B), Q(x)} \qquad \dfrac{Q(A), P(y,A) \vdash \qquad \vdash P(y,x)}{Q(x) \vdash}}{\dfrac{\vdash R(B) \qquad\qquad R(B) \vdash}{\bot}}$$

but if we delay the resolution with $P(y,x)$ we get

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(A,x) \vdash Q(A), R(B) \qquad Q(y) \vdash Q(x)}{P(y,x) \vdash Q(x), R(B)} \qquad Q(A), P(y,A) \vdash}{P(y,x), P(x,y) \vdash R(B)} \qquad R(B) \vdash}{P(y,x), P(y,A) \vdash} \qquad \vdash P(y,x)}{P(y,x)} \qquad \vdash P(y,x)}{\bot}$$

and now we actually the same number of resolution rules. No, we can still use a contraction, and reduce the proof.

The requirement for being pairwise unifiable is also seen in proof 1 and 2, but further, this is lacking the case of proof 3: $P(a)$ and $P(b)$ is not unifiable, and thus proof 5 is not actually compressed. But if $P(b)$ had been $P(B)$, then we would have been fine. It also fails in the following example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash P(X) \qquad P(a) \vdash Q(Y), R(Z)}{\vdash Q(Y), R(Z)} \qquad R(X), P(b) \vdash S(Y)}{P(b) \vdash S(Y), Q(Y)} \qquad S(Y), Q(Y) \vdash}{P(b) \vdash} \qquad \vdash P(X)}{\bot}$$

which is the 'potentially' globally reduction example from the original lower units paper.

## 3.2 Results

Let $\lfloor x \rfloor$ denote a clause consisting of the formula $x$.

**Theorem 3.1.** *Let $S$ be the set of premises being resolved against a unit clause $u$. Then $u$ can be lowered if, for every distinct $\eta_1, \eta_2 \in S$, $\eta_1$ and $\eta_2$ are unifiable.*

*Proof.* We proceed by induction. Base case: $|S| = 1$. In this case, the unit $\lfloor u \rfloor$ is only involved in exactly one resolution; let $\lfloor \eta \rfloor$ be the premise resolved against $\lfloor u \rfloor$ so that we have $P = \phi[\phi_1[\phi_\eta[\eta] \odot_\sigma \phi_u[u]] \odot_{\sigma_r} \phi_r[r]]$. Note that $\lfloor \eta \rfloor$ contains $\lfloor \overline{u}' \rfloor$, a negated version of $\lfloor u \rfloor$, which would be resolved out in conclusion of $\phi$, and let $\sigma_u$ be a unifier of $\lfloor \overline{u}' \rfloor$ and $\lfloor u \rfloor$. Consider instead $P' = \phi[\phi']$, the proof obtained by replacing the resolution $\phi_1[\phi_\eta[\eta] \odot_\sigma \phi_u[u]] \odot_{\sigma_r} \phi_r[r]$ with just $\phi' = \phi_\eta[\eta] \odot_{\sigma_{r'}} \phi_r[r]$ and maintaining the subsequent subproof of $\phi$. Note that all nodes of $\phi'$ contain $\lfloor \overline{u}' \rfloor$. In particular, the final node in the proof $P'$ is $\lfloor \overline{u}' \rfloor$ instead of $\lfloor \perp \rfloor$. But then we can resolve against $\phi_u[u]$ using $\sigma_u$ to complete the proof.

Assume the result holds for all $|S| \leq n$, and consider $|S| = n + 1$. Assume that $S$ is defined as above, and is pairwise unifiable. Order the elements from the top of the proof to the bottom (and break ties left-right), so that $\lfloor \eta_1 \rfloor$ is the top-left-most premise resolved against $\lfloor u \rfloor$. In particular, $\lfloor \eta_1 \rfloor$ contains $\lfloor \overline{u}'_1 \rfloor$, and we have that $P = \phi[\phi_1[\phi_{\eta_1}[\eta_1] \odot_{\sigma_1} \phi_u[u]] \odot_{\sigma_{r_1}} \phi_{r_1}[r_1]]$. Consider instead $\phi[\phi'_1]$, the proof obtained by replacing the resolution $\phi_1[\phi_{\eta_1}[\eta_1] \odot_{\sigma_1} \phi_u[u]] \odot_{\sigma_{r_1}} \phi_{r_1}[r_1]$ with just $\phi'_1 = \phi_{\eta_1}[\eta_1] \odot_{\sigma_{r'_1}} \phi_{r_1}[r_1]$ and maintaining the subproof of $\phi$. Note that all nodes of $\phi'_1$ contain $\overline{u}'_1$ still. In particular, the final node in the proof is $\lfloor \overline{u}'_1 \rfloor$ instead of $\lfloor \perp \rfloor$.

Consider $S' = S \setminus \lfloor \eta_1 \rfloor$: since $|S| = n + 1 > 1$, $|S'| = |S| - 1 \leq n$. Apply the induction hypothesis to the premises in $S'$ to get a resolution $\phi_2[\phi_{\eta_2}[\eta_2] \odot_{\sigma_2} \phi_u[u]] \odot_{\sigma_{r_2}} \phi_{r_2}[r_2]$ (where $\lfloor \eta_2 \rfloor$ contains $\lfloor \overline{u}'_2 \rfloor$); we can construct $\phi'_2 = \phi_{\eta_2}[\eta_2] \odot_{\sigma_{r'_2}} \phi_{r_2}[r_2]$. Consider $\phi[\phi'_2]$ and follow an argument similar to that for $\lfloor \eta_1 \rfloor$; the final proof node has $\lfloor \overline{u}'_1 \rfloor \cup \lfloor \overline{u}'_2 \rfloor$ instead of $\lfloor \overline{u}'_1 \rfloor$ where $\lfloor \overline{u}'_2 \rfloor$ is the result of the lower, after contraction, but before the final resolution with $\phi_u[u]$, applied by the induction hypothesis. By assumption, $\lfloor \overline{u}'_1 \rfloor$ and $\lfloor \overline{u}'_2 \rfloor$ are pair-wise unifiable by some unifier $\sigma_{1,2}$. We can therefore contract $\sigma_{1,2}(\lfloor \overline{u}'_1 \rfloor \cup \lfloor \overline{u}'_2 \rfloor)$ and call the result $\lfloor \eta_{1,2} \rfloor$. Now $\lfloor \eta_{1,2} \rfloor$ and $\lfloor u \rfloor$ must be unifiable by assumption with some unifier $\sigma_u$, so we can replace the last node in the proof with $\phi_{\eta_{1,2}}[\eta_{1,2}] \odot_{\sigma_u} \phi_u[u]$ to complete the proof. $\square$

# 4 Recycle Pivots

## 4.1 Research Notes

From the looks of some toy examples, I again conjecture we can recycle pivots when the formulas being moved are pairwise unifiable. Example from the video, trivially made first-order (should there be a contraction at the "c: "?):

$$\dfrac{\dfrac{\dfrac{\vdash A(X)C(Y)D(Z) \qquad D(Z) \vdash A(X)C(Y)}{\vdash A(X)C(Y)} \qquad A(X) \vdash C(Y)}{\text{c: } \vdash C(Y)} \qquad \dfrac{\dfrac{C(Y) \vdash \qquad D(Z) \vdash C(Y)}{D(Z) \vdash} \qquad A(X)C(Y) \vdash D(Z)}{A(X)C(Y) \vdash}}{\dfrac{A(X) \vdash \qquad\qquad \vdash A(X)}{\bot}}$$

Which is, after the first (bottom-up) traversal:

$$\dfrac{\dfrac{\dfrac{\color{red}\vdash A(X)C(Y)D(Z) \qquad D(Z) \vdash A(X)C(Y)}{\color{red}\vdash A(X)C(Y)} \qquad A(X) \vdash C(Y)}{\text{c: } \vdash C(Y)\ \{A(X) \vdash C(Y)\}}\,{\scriptstyle A(X)} \qquad \dfrac{\dfrac{C(Y) \vdash \qquad {\color{red}D(Z) \vdash C(Y)}}{D(Z) \vdash \{A(X)C(Y)D(Z) \vdash\}}\,{\scriptstyle C(Y)} \qquad A(X)C(Y) \vdash D(Z)}{A(X)C(Y) \vdash \{A(X)C(Y) \vdash\}}\,{\scriptstyle D(Z)}}{{\scriptstyle A(X)}\,\dfrac{A(X) \vdash \{A(X) \vdash\} \qquad\qquad \vdash A(X)}{\bot\ \{\}}}\,{\scriptstyle C(Y)}$$

Now we start the second (top-down) traversal. We replace $D(Z)$ with $C(Y) \vdash$ since $C(Y) \vdash$ is in $D(Z)$'s safe formulas, and we replace with the left parent of $D(Z)$ since that is the one that contains the safe formula $C(Y) \vdash$.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\color{red}{\vdash A(X)C(Y)D(Z) \qquad D(Z) \vdash A(X)C(Y)}}{\color{red}{\vdash A(X)C(Y)}} \qquad A(X) \vdash C(Y)}{\text{c: } \vdash C(Y)\ \{A(X) \vdash C(Y)\}} \qquad \dfrac{C(Y) \vdash \qquad A(X)C(Y) \vdash D(Z)}{A(X)C(Y) \vdash \{A(X)C(Y) \vdash\ \}}}{A(X) \vdash \{A(X) \vdash\ \}} \qquad \vdash A(X)}{\bot\ \{\}}
$$

Now we lower $C(Y) \vdash$ again, because it is also in $A(X)C(Y) \vdash$ 's safe formulas, and we pick the left because the right parent might have unsafe formulas (e.g. $\vdash D(Z)$), but the left has only safe formulas.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\color{red}{\vdash A(X)C(Y)D(Z) \qquad D(Z) \vdash A(X)C(Y)}}{\color{red}{\vdash A(X)C(Y)}} \qquad A(X) \vdash C(Y)}{\text{c: } \vdash C(Y)\ \{A(X) \vdash C(Y)\}} \qquad C(Y) \vdash}{A(X) \vdash \{A(X) \vdash\ \}} \qquad \vdash A(X)}{\bot\ \{\}}
$$

Now we need to deal with the last remaining broken proof section (what is left in red). Since $A(X) \vdash C(Y)$ is safe with respect to the line under it, we lower it:

$$
\dfrac{\dfrac{A(X) \vdash C(Y) \qquad C(Y) \vdash}{A(X) \vdash} \qquad \vdash A(X)}{\bot}
$$

And we have the desired shorter proof.