

# 1 Local Redundancy

We first consider an example first posed by Postan:

$$\frac{\frac{L_2: P(A) \quad \eta_2: \neg P(x), \neg Q(x, B)}{\neg Q(x, B)} \quad \eta_1: \neg P(z), Q(z, y)}{\neg P(z)} \quad L_1: P(x) \quad \perp$$

Which is locally redundant; see the compressed version in his document.

## 2 “Example 2”

We consider example 2, from the LU/RPI paper, modified for first order predicates in a trivial way:

$$\frac{\frac{\eta_1: \neg P(A) \quad \eta_3: P(A), Q(B)}{\eta_4: Q(B)} \quad \eta_2: P(A), R(C), \neg Q(B)}{\eta_5: P(A), R(C)} \quad \eta_1: \neg P(A) \quad \frac{\eta_4: Q(B) \quad \eta_7: P(A), \neg Q(B), \neg R(C)}{\eta_8: P(A), \neg R(C)} \quad \eta_1: \neg P(A)}{\eta_6: R(C) \quad \eta_9: \neg R(C)} \quad \perp$$

### 2.1 Lower Units

Proceeds exactly the same as in the paper.

**TODO:** show exact steps?

### 2.2 RecyclePivots

Again, proceeds like in the paper.

### 3 Lower Units

#### 3.1 Research Notes

First, I consider the proofs 1-5 that were provided by Bruno on the Skeptik dev mailing list. In order to be explicit, I outline the case of compression from proof 1 to proof 2:

- Lower  $P(X)$  so that the terms using it were resolved against each other instead of with  $P(X)$
- Contract (trivially?); the unifier resulted in the duplicated terms
- Resolve the contracted formula against the lowered unit,  $P(X)$

The result is a trade of a resolution for a contraction, which is more compact (when we consider compactness as a count of the number of resolution rules).

In order to generalize, I think the best place to start was see under what conditions we can in fact make this contraction. It should not be required that contraction results in duplicated formulas; indeed, as long as a contraction is possible this seems to work. So in particular, I conjecture that we should lower a unit formula if and only if for all formulas which would be resolved against the unit clause of interest are pair-wise unifiable (disregarding the remainder of their premises), and unifiable with the unit. Further, the unit must be the most general form of the formula, as the following shows:

$$\begin{array}{c}
 \frac{\frac{\frac{\vdash P(y, x) \quad P(A, x) \vdash Q(A), R(B)}{\vdash Q(y), R(B)} \quad Q(y) \vdash Q(x)}{\vdash R(B), Q(x)} \quad \frac{Q(A), P(y, A) \vdash \quad \vdash P(y, x)}{Q(x) \vdash} \\
 \frac{\vdash R(B) \quad Q(x) \vdash}{\vdash R(B)} \quad R(B) \vdash \\
 \hline
 \perp
 \end{array}$$

but if we delay the resolution with  $P(y, x)$  we get

$$\begin{array}{c}
\frac{P(A, x) \vdash Q(A), R(B) \quad Q(y) \vdash Q(x)}{P(y, x) \vdash Q(x), R(B)} \quad \frac{Q(A), P(y, A) \vdash}{P(y, x), P(x, y) \vdash R(B)} \quad R(B) \vdash \\
\frac{P(y, x), P(y, A) \vdash}{P(y, x) \vdash} \quad \vdash P(y, x) \\
\frac{P(y, x) \vdash \quad \vdash P(y, x)}{\perp}
\end{array}$$

and now we actually the same number of resolution rules. No, we can still use a contraction, and reduce the proof.

The requirement for being pairwise unifiable is also seen in proof 1 and 2, but further, this is lacking the case of proof 3:  $P(a)$  and  $P(b)$  is not unifiable, and thus proof 5 is not actually compressed. But if  $P(b)$  had been  $P(B)$ , then we would have been fine. It also fails in the following example:

$$\begin{array}{c}
\frac{\vdash P(X) \quad P(a) \vdash Q(Y), R(Z)}{\vdash Q(Y), R(Z)} \quad R(X), P(b) \vdash S(Y) \\
\frac{P(b) \vdash S(Y), Q(Y) \quad S(Y), Q(Y) \vdash}{P(b) \vdash} \quad \vdash P(X) \\
\frac{}{\perp}
\end{array}$$

which is the 'potentially' globally reduction example from the original lower units paper.

### 3.2 Results

Let  $\lfloor x \rfloor$  denote a clause consisting of the formula  $x$ .

**Theorem 3.1.** *Let  $S$  be the set of premises being resolved against a unit clause  $u$ . Then  $u$  can be lowered if, for every distinct  $\eta_1, \eta_2 \in S$ ,  $\eta_1$  and  $\eta_2$  are unifiable.*

*Proof.* We proceed by induction. Base case:  $|S| = 1$ . In this case, the unit  $\lfloor u \rfloor$  is only involved in exactly one resolution; let  $\lfloor \eta \rfloor$  be the premise resolved against  $\lfloor u \rfloor$  so that we have  $P = \phi[\phi_1[\phi_\eta[\eta] \odot_\sigma \phi_u[u]] \odot_{\sigma_r} \phi_r[r]]$ . Note that  $\lfloor \eta \rfloor$  contains  $\lfloor \bar{u}' \rfloor$ , a negated version of  $\lfloor u \rfloor$ , which would be resolved out in conclusion of  $\phi$ , and let  $\sigma_u$  be a unifier of  $\lfloor \bar{u}' \rfloor$  and  $\lfloor u \rfloor$ . Consider instead  $P' = \phi[\phi']$ , the proof obtained by replacing the resolution  $\phi_1[\phi_\eta[\eta] \odot_\sigma \phi_u[u]] \odot_{\sigma_r} \phi_r[r]$  with just  $\phi' = \phi_\eta[\eta] \odot_{\sigma_{r'}} \phi_r[r]$  and maintaining the subsequent subproof of  $\phi$ . Note that all nodes of  $\phi'$  contain  $\lfloor \bar{u}' \rfloor$ . In particular, the final node in the proof  $P'$  is  $\lfloor \bar{u}' \rfloor$  instead of  $\lfloor \perp \rfloor$ . But then we can resolve against  $\phi_u[u]$  using  $\sigma_u$  to complete the proof.

Assume the result holds for all  $|S| \leq n$ , and consider  $|S| = n + 1$ . Assume that  $S$  is defined as above, and is pairwise unifiable. Order the elements from the top of the proof to the bottom (and break ties left-right), so that  $\lfloor \eta_1 \rfloor$  is the top-left-most premise resolved against  $\lfloor u \rfloor$ . In particular,  $\lfloor \eta_1 \rfloor$  contains  $\lfloor \bar{u}'_1 \rfloor$ , and we have that  $P = \phi[\phi_1[\phi_{\eta_1}[\eta_1] \odot_{\sigma_1} \phi_u[u]] \odot_{\sigma_{r_1}} \phi_{r_1}[r_1]]$ . Consider instead  $\phi[\phi'_1]$ , the proof obtained by replacing the resolution  $\phi_1[\phi_{\eta_1}[\eta_1] \odot_{\sigma_1} \phi_u[u]] \odot_{\sigma_{r_1}} \phi_{r_1}[r_1]$  with just  $\phi'_1 = \phi_{\eta_1}[\eta_1] \odot_{\sigma_{r'_1}} \phi_{r_1}[r_1]$  and maintaining the subproof of  $\phi$ . Note that all nodes of  $\phi'_1$  contain  $\bar{u}'_1$  still. In particular, the final node in the proof is  $\lfloor \bar{u}'_1 \rfloor$  instead of  $\lfloor \perp \rfloor$ .

Consider  $S' = S \setminus \lfloor \eta_1 \rfloor$ : since  $|S| = n + 1 > 1$ ,  $|S'| = |S| - 1 \leq n$ . Apply the induction hypothesis to the premises in  $S'$  to get a resolution  $\phi_2[\phi_{\eta_2}[\eta_2] \odot_{\sigma_2} \phi_u[u]] \odot_{\sigma_{r_2}} \phi_{r_2}[r_2]$  (where  $\lfloor \eta_2 \rfloor$  contains  $\lfloor \bar{u}'_2 \rfloor$ ); we can construct  $\phi'_2 = \phi_{\eta_2}[\eta_2] \odot_{\sigma_{r'_2}} \phi_{r_2}[r_2]$ . Consider  $\phi[\phi'_2]$  and follow an argument similar to that for  $\lfloor \eta_1 \rfloor$ ; the final proof node has  $\lfloor \bar{u}'_1 \rfloor \cup \lfloor \bar{u}'_2 \rfloor$  instead of  $\lfloor \bar{u}'_1 \rfloor$  where  $\lfloor \bar{u}'_2 \rfloor$  is the result of the lower, after contraction, but before the final resolution with  $\phi_u[u]$ , applied by the induction hypothesis. By assumption,  $\lfloor \bar{u}'_1 \rfloor$  and  $\lfloor \bar{u}'_2 \rfloor$  are pair-wise unifiable by some unifier  $\sigma_{1,2}$ . We can therefore contract  $\sigma_{1,2}(\lfloor \bar{u}'_1 \rfloor \cup \lfloor \bar{u}'_2 \rfloor)$  and call the result  $\lfloor \eta_{1,2} \rfloor$ . Now  $\lfloor \eta_{1,2} \rfloor$  and  $\lfloor u \rfloor$  must be unifiable by assumption with some unifier  $\sigma_u$ , so we can replace the last node in the proof with  $\phi_{\eta_{1,2}}[\eta_{1,2}] \odot_{\sigma_u} \phi_u[u]$  to complete the proof.  $\square$

## 4 Recycle Pivots

## 4.1 Research Notes

From the looks of some toy examples, I again conjecture we can recycle pivots when the formulas being moved are pairwise unifiable. Example from the video, trivially made first-order:

$$\frac{\frac{\frac{\vdash A(X)C(Y)D(Z)}{\vdash A(X)C(Y)}}{c: \vdash C(Y)} \quad \frac{\frac{A(X) \vdash C(Y)}{D(Z) \vdash} \quad \frac{C(Y) \vdash \quad D(Z) \vdash C(Y)}{A(X)C(Y) \vdash D(Z)}}{A(X)C(Y) \vdash}}{A(X) \vdash} \perp \vdash A(X)$$