1 Local Redundancy

We first consider an example first posed by Postan:

$$\dfrac{\dfrac{\dfrac{L_2\colon\ P(A) \qquad \eta_2\colon\ \neg P(x),\ \neg Q(x,B)}{\neg Q(x,B)} \qquad \eta_1\colon\ \neg P(z),\ Q(z,y)}{\neg P(z)} \qquad L_1\colon\ P(x)}{\bot}$$

Which is locally redundant; see the compressed version in his document.

2 “Example 2”

We consider example 2, from the LU/RPI paper, modified for first order predicates in a trivial way:

$$\dfrac{\dfrac{\dfrac{\dfrac{\eta_1\colon\ \neg P(A) \qquad \eta_3\colon\ P(A), Q(B)}{\eta_4\colon\ Q(B)} \qquad \eta_2\colon\ P(A), R(C), \neg Q(B)}{\eta_5\colon\ P(A), R(C)} \qquad \eta_1\colon\ \neg P(A)}{\eta_6\colon\ R(C)} \qquad \dfrac{\dfrac{\eta_4\colon\ Q(B) \qquad \eta_7\colon\ P(A), \neg Q(B), \neg R(C)}{\eta_8\colon\ P(A), \neg R(C)} \qquad \eta_1\colon\ \neg P(A)}{\eta_9\colon\ \neg R(C)}}{\bot}$$

2.1 Lower Units

Proceeds exactly the same as in the paper.
TODO: show exact steps?

2.2 RecyclePivots

Again, proceeds like in the paper.

3 Lower Units - Research Notes

First, I consider the proofs 1-5 that were provided by Bruno on the Skeptik dev mailing list. In order to be explicit, I outline the case of compression from proof 1 to proof 2:

- Lower $P(X)$ so that the terms using it were resolved against each other instead of with $P(X)$
- Contract (trivially?); the unifier resulted in the duplicated terms
- Resolve the contracted formula against the lowered unit, $P(X)$

The result is a trade of a resolution for a contraction, which is more compact (when we consider compactness as a count of the number of resolution rules).

In order to generalize, I think the best place to start was see under what conditions we can in fact make this contraction. It should not be required that contraction results in duplicated formulas; indeed, as long as a contraction is possible this seems to work. So in particular, I conjecture that we should lower a unit formula if and only if for all formulas which would be resolved against the unit clause of interest are pair-wise unifiable (disregarding the remainder of their premises), and unifiable with the unit. Further, the unit must be the most general form of the formula, as the following shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash P(y,x) \qquad P(A,x) \vdash Q(A), R(B)}{\vdash Q(y), R(B)} \qquad Q(y) \vdash Q(x)}{\vdash R(B), Q(x)} \qquad \dfrac{Q(A), P(y,A) \vdash \qquad \vdash P(y,x)}{Q(x) \vdash}}{\vdash R(B)} \qquad R(B) \vdash}{\bot}$$

but if we delay the resolution with $P(y,x)$ we get

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(A,x)\vdash Q(A),R(B) \qquad Q(y)\ \vdash Q(x)}{P(y,x)\vdash Q(x),R(B)} \qquad Q(A),P(y,A)\vdash}{P(y,x),P(x,y)\vdash R(B)} \qquad R(B)\vdash}{P(y,x),P(y,A)\vdash} \qquad \vdash P(y,x)}{P(y,x)} \qquad \vdash P(y,x)}{\bot}$$

and now we actually the same number of resolution rules. **TODO: -no, we can still use a contraction, and reduce the proof**

The requirement for being pairwise unifiable is also seen in proof 1 and 2, but further, this is lacking the case of proof 3: $P(a)$ and $P(b)$ is not unifiable, and thus proof 5 is not actually compressed. But if $P(b)$ had been $P(B)$, then we would have been fine. It also fails in the following example:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash P(X) \qquad P(a)\vdash Q(Y),R(Z)}{\vdash Q(Y),R(Z)} \qquad R(X),P(b)\vdash S(Y)}{P(b)\vdash S(Y),Q(Y)} \qquad S(Y),Q(Y)\vdash}{P(b)\vdash} \qquad \vdash P(X)}{\bot}$$

which is the 'potentially' globally reduction example from the original lower units paper.

Theorem 3.1. *Let S be the set of premises being resolved against a unit clause u. Then u can be lowered if, for every distinct $\eta_1, \eta_2 \in S$, η_1 and η_2 are unifiable.*

Proof. We proceed by induction. Base case: $|S| = 1$. In this case, the unit u is only involved in exactly one resolution; let η be the premise resolved against u so that we have $P = \phi[\phi_1[\eta \odot_\sigma u] \odot_{\sigma_r} r]$. Consider instead $\phi[\phi' \cup \overline{u}']$, the proof obtained by replacing the resolution $\phi_1[\eta \odot_\sigma u] \odot_{\sigma_r} r$ with just $\phi' = \phi_1[\eta] \odot_{\sigma_{r'}} r$ and maintaining the subproof of ϕ. Note that ϕ' contains $\overline{u}'$ still, and so the resulting subproof would have more occurrences of $\overline{u}'$. In particular, the final node in the proof is $\overline{u}'$ instead of $\bot$. But then we can resolve against the u to complete the proof.

Assume the result holds for all $|S| \leq n$, and consider $|S| = n+1$. Assume that S is defined as above, and is pairwise unifiable. Order the elements from the top of the proof to the bottom (and break ties left-right), so that η_1 is the top-left-most premise resolved against u. In particular, η_1 contains $\overline{u}'_1$, and we have that $P = \phi[\phi_1[\eta_1 \odot_{\sigma_1} u] \odot_{\sigma_{r_1}} r_1]$. Consider instead $\phi[\phi'_1 \cup \overline{u}'_1]$, the proof obtained by replacing the resolution $\phi_1[\eta_1 \odot_{\sigma_1} u] \odot_{\sigma_{r_1}} r_1$ with just $\phi'_1 = \phi_1[\eta_1] \odot_{\sigma_{r'_1}} r_1$ and maintaining the subproof of ϕ. Note that ϕ'_1 contains $\overline{u}'_1$ still, and so the resulting subproof would have more occurrences of $\overline{u}'_1$. In particular, the final node in the proof is $\overline{u}'_1$ instead of $\perp$. Consider $S' = S \setminus \eta_1$: since $|S| = n+1 > 1$, $|S'| \leq n$. Apply the induction hypothesis to the premises in S' to get a resolution $\phi_2[\eta_2 \odot_{\sigma_2} u] \odot_{\sigma_{r_2}} r_2$ (where η_2 contains $\overline{u}'_2$); we can construct $\phi'_2 = \phi_2[\eta_2] \odot_{\sigma_{r'_2}} r_2$. Consider $\phi[\phi'_2 \cup \overline{u}'_2]$ and follow an argument similar to that for η_1; the final proof node has $\overline{u}'_1 \cup \overline{u}'_2$ instead of $\overline{u}'_1$ where $\overline{u}'_2$ is the result of lower applied by the induction hypothesis. By assumption, $\overline{u}'_1$ and $\overline{u}'_2$ are pair-wise unifiable. So we can contract these terms, and then resolve against u, to complete the proof. $\square$

TODO: notation change in progress