Mathematical Lecture 1

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Abstract

These series of documents aim to equip you with the tools needed for DSE M2, Physics, or advanced math and physics for those who are interested.

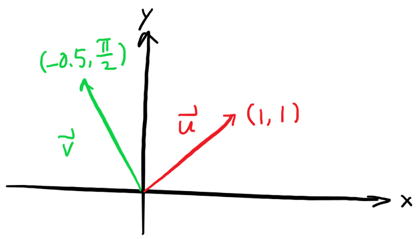
Please note that

1. The formatting and notation of this document might not be standard in public examinations.
2. Some tools are not allowed in internal and/or public exams.
3. Please don't kill me if I made any typos in this document. I will fix them upon request.
4. I would like to introduce the tools instead of drilling in the rigor. Rigorous proofs or definitions can be found at the end of the document. An asterisk (*) indicates that the rigorous construction is given at the end of the document.
5. The parts that are out-syllabus are marked with !!!.
6. The set notations will be explained at the end of the document.

1 Introduction

This document will mainly focus on vectors. Note that you might encounter different definitions from different sources. Physicists often define it as just an object with magnitude and orientation, while mathematicians define it as an element of a mathematical object called a vector space*. However, for the sake of visualization, for now, I would use the physicist definition. This document will be seen in both the Science Society and the Mathematics Society Google Classroom. I know this is nothing valuable or new but please notify me if you plan to share this to non-related people.

2 Vectors



Vectors are arrows (kind of). We notate them by their endpoints starting from the origin (e.g. $(1,1)$ for $\vec{u}$, which starts from $(0,0)$ and ends at $(1,1)$, and $(-0.5, \pi/2)$ for $\vec{v}$). They are also denoted like this: $\vec{OP}$. Note that vectors can be freely translated to any location in the 2D plane. That is, if you move $\vec{u}$ to start at $(3,3)$ and end at $(4,4)$, they are still the same vector.

Vectors in the 2D plane have been used as an example. Note that $(0,1,1)$ is a valid vector in 3D and $(3, e, \pi, \gamma)$ is a valid vector in 4D. (I hope you can deduce that these can be interpreted as arrows in 3D and 4D, respectively)

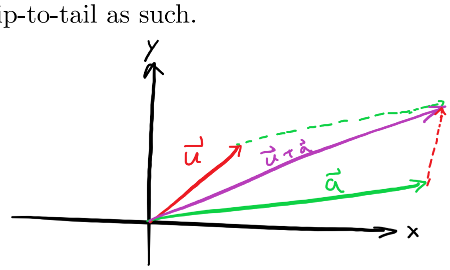
The *magnitudes (lengths)* of the vectors could easily be obtained by our favorite theorem, PT: $||\vec{v}|| = \sqrt{x_v^2 + y_v^2}$, where x_v, y_v are the components of $\vec{v}$ respectively.

Vectors are important because they are a natural way to represent stuff in physics that has magnitude and direction like:

1. Forces (Weight $W = mg$, friction $f = \mu N$ ...)
2. Position(displacement), velocity and acceleration
3. Torque
4. Electric, magnetic and gravitational fields

3 Vector addition and multiplication

A geometric interpretation of vector addition is to add arrows by joining them tip-to-tail as such.



The purple vector is the sum of the red and green vectors.
However, I have also mentioned that a vector could be written as components such as $(1, 2)$. Addition could hence be calculated by adding up components:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

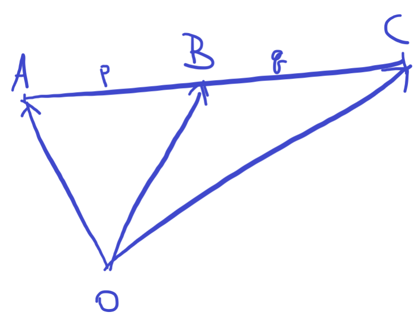
Scaling a vector is also intuitively defined as

$$a(x, y) = (ax, ay)$$

where a is a scalar (real number). This stretches the arrow by a factor of a without changing its direction.
This naturally extends to a question: what does vector subtraction mean? Analogous to the reals, $a - b = a + (-b) = a + (-1)b$. It could be shown that vector subtraction could be done the same way:

$$\vec{u} - \vec{v} = \vec{u} + (-1)\vec{v}$$

3.0.1 Example 1



Let's try to use the vector interpretation to prove some results from middle school geometry, starting with the section formula.
Let O be the origin and let A, B, C be collinear points with $AB : BC = p : q$.
Note that $\vec{AC} = (x_C, y_C) + (-1)(x_A, y_A) = (x_C - x_A, y_C - y_A)$.
Then, $\vec{AB} = \frac{p}{p+q}(x_C - x_A, y_C - y_A)$.
The coordinates of B could be seen as the end point of the vector $\vec{OB} = \vec{OA} + \vec{AB}$.
Hence,

$$\vec{OB} = (x_A + \frac{p}{p+q}(x_C - x_A), y_A + \frac{p}{p+q}(y_C - y_A)) = (\frac{px_C + qx_A}{p+q}, \frac{py_C + qy_A}{p+q})$$

as expected.

3.0.2 Example 2

Let O be the origin and let A, B be any two distinct points. Denote the mid-pts of OA, OB to be C, D respectively. (Try to draw it out!)

By letting $\vec{OA} = \vec{a}, \vec{OB} = \vec{b}$, it could be seen that

$$\vec{OC} = \frac{1}{2}\vec{a}, \vec{OD} = \frac{1}{2}\vec{b}$$

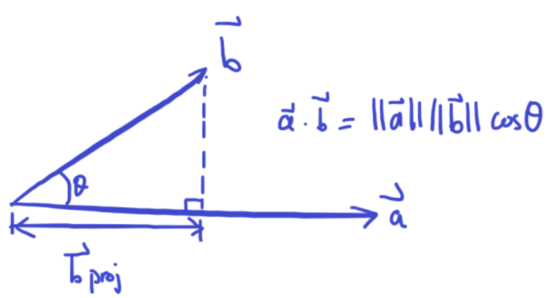
Note that by our definition of vector addition, $\vec{OA}+\vec{AB} = \vec{OB}, \vec{OC}+\vec{CD} = \vec{OD}$. So $\vec{AB} = \vec{b}-\vec{a}, \vec{CD} = \frac{1}{2}(\vec{b}-\vec{a})$ and hence $\vec{AB} = 2\vec{CD}$. As scaling a vector would not change its direction, $\vec{AB}, \vec{CD}$ have the same direction.
Therefore,

$$AB \parallel CD \text{ and } AB = 2CD. \text{ (mid-pt. theorem)}$$

Note that in our construction, the factor of 0.5 do not matter much, if OC, OD are a third of OA, OB, then the result still holds but with a slight difference: $AB = 3CD$.

3.1 Dot product

Unlike numbers, however, there is not a single unified definition for vector multiplication (except if you are dealing with multivectors in the study of Clifford Algebra). Actually, there are multiple ways that define a product for vectors that found common usage in math and physics.
There are also two ways to interpret the so-called *dot product* (or *inner product*) for vectors in $\mathbb{R}^n$, like addition:



Geometrically, it gives a sense in the angular difference of the directions of the two vectors. The dot product of $\vec{a} \cdot \vec{b}$ is defined as the product of the length of $\vec{a}$ multiplied with the length of the projection of $\vec{b}$ onto $\vec{a}$.
For those who have learnt some trigonometry, the cosine of the right-angled triangle showed in the figure is exactly expressed as follows:

$$\cos\theta = \frac{||\vec{b_p}||}{||\vec{b}||}$$

where $||\vec{b}||$ denotes the magnitude (length)

We can hence deduce that the length of the projection vector would be $||\vec{b_p}|| = ||\vec{b}|| \cos\theta$. So the dot product would be $\vec{a} \cdot \vec{b} = ||\vec{a}||||\vec{b}|| \cos\theta$.

The significance of this formula is that you could find the angle between two vectors if you have their lengths and dot product. But a problem naturally arises, how do we easily find the dot product?

Luckily, there is another way to find the dot product. Algebraically, the dot product is found by adding up the product of components. In symbols, this writes:

$\vec{a} \cdot \vec{b} = x_a x_b + y_a y_b$, where $\vec{a} = (x_a, y_b), \vec{b} = (x_b, y_b)$

So for all two vectors (in 2D), you could find the angle between two vectors as such: $x_a x_b + y_a y_b = \sqrt{x_a^2 + y_a^2}\sqrt{x_b^2 + y_b^2}\cos\theta$......(1)
Also, I hope that it would be fairly easy to see that the self-dot of a vector is its length squared: $\vec{a} \cdot \vec{a} = x_a^2 + y_a^2 = ||\vec{a}||^2$

A physical application could be seen through the work done by a force: $W = ||\vec{F}|| \cdot ||\vec{s}|| \cos\theta$

Corollary (Cauchy-Schwarz) By squaring both sides of (1), we have:

$(x_a x_b + y_a y_b)^2 = (x_a^2 + y_a^2)(x_b^2 + y_b^2)\cos^2\theta \leq (x_a^2 + y_a^2)(x_b^2 + y_b^2)$

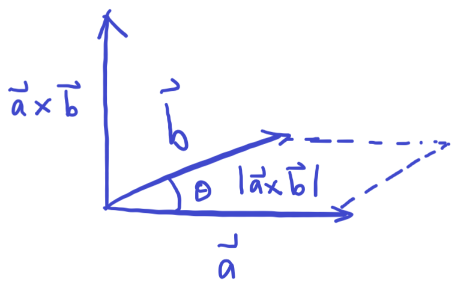
By considering the extreme left and right, a common inequality could be salvaged:
$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$
This is known as the Cauchy-Schwarz inequality.

Corollary (Orthogonality) If two vectors $\vec{u}, \vec{v}$ are $\perp$, then $\vec{u} \cdot \vec{v} = ||\vec{u}|| \cdot ||\vec{v}|| \cos(90°) = 0$.

In components, $x_u x_v + y_u y_v = 0$. Rearranging gives $\frac{y_u}{x_u} \cdot \frac{y_v}{x_v} = -1$, which fits the description of "perpendicular lines have slopes that multiply to -1" from F3 (I think).

3.2 Cross product

Compared to the dot product, the cross product is more annoying to deal with. The cross product $\vec{a} \times \vec{b}$ of vectors $\vec{a}, \vec{b}$ is defined as the vector perpendicular (orthogonal) to the plane formed by $\vec{a}, \vec{b}$ with magnitude $||\vec{a}|| ||\vec{b}|| \sin\theta$ (what a mouthful). I expect almost nobody could understand this from just reading this text, so another diagram is provided for your reference:



Note that reversing the order of multiplication will obtain a vector that points downwards, i.e. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$. The cross product vector has the magnitude equal to the area of the //gram formed by $\vec{a}, \vec{b}$, that is,

$$||\vec{a} \times \vec{b}|| = ||\vec{a}|| \cdot ||\vec{b}|| \sin \theta$$

There might be a lot of confusion arising from the weird definition of the cross product. In fact, it is actually just weird and only work in 3 or less dimensions. However, the generalization of the cross product is way out of the secondary syllabus. (For those who are interested, search up wedge product and how to convert it to the cross product). Just know that this product exists for now.

Despite the weirdness of the cross product, there are some physical applications of it, most notably being torque:

$$\vec{\tau} := \vec{F} \times \vec{d}$$

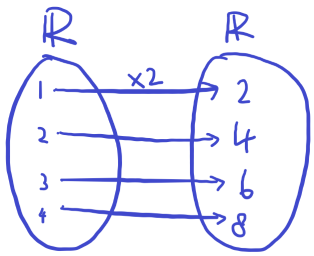
By considering the magnitude of τ, we can obtain the formula for torque in your physics textbooks: $||\vec{\tau}|| = ||\vec{F}|| \cdot ||\vec{d}|| \sin \theta$

4 Vector functions

!!!

I believe most of you have learnt the notion of a function. As this is a mathematical lecture, I would like to give a slightly more rigorous approach.

Functions that you are most familiar with "maps" a real number (a number on the number line such as $3, \pi$) to a real number as such:

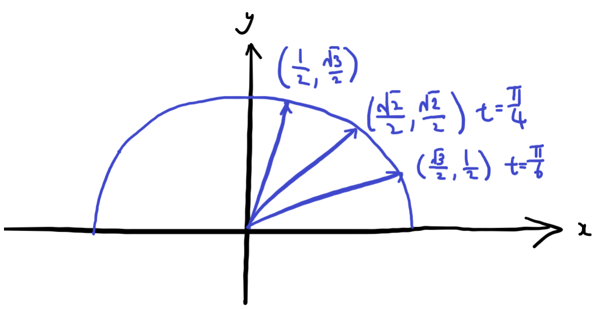


Here, $\mathbb{R}$ denotes the collection (set) of real numbers. A real number is sent by the function $\times 2$ to the number double its value. We denote that by $f : x \mapsto 2x$. I hope that this notation is not that hard to understand, and that it could be seen that this is equivalent to $f(x) = 2x$

However, limiting functions' input and output to just real numbers would hinder its potential.

As the topic is about vectors, this would naturally extend to a question, can a function take in a real-numbered input and output a vector?

It turns out that it absolutely can (ignoring the hundred details allowing it to work well). Consider the function $f : t \mapsto (\cos t, \sin t)$, which takes in a real number t and outputs a vector $(\cos t, \sin t)$, what would it look like for different values of t?



The endpoints of the output vector apparently traced out a unit circle. This is an example of a *vector-valued function*, where a curve is defined by endpoints of vectors instead of an formula $f(x, y) = 0$.

Also notice that we have traced out a circle only using one variable. This is called a *parametrization* of the circle.

(Question for fun: What does the function $g : t \mapsto (\cos 2t, \sin 2t)$ trace out? How does it differ from f?)

5 Vector fields

!!!

Notice in the previous section I've extended the idea of functions from being a mapping from a real number to a real number, to a real number to a vector. This could be further extended.

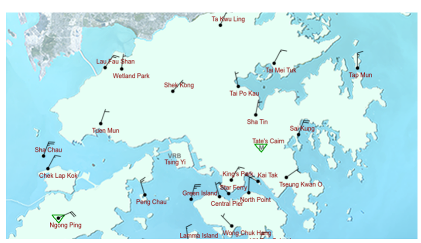
A *vector field* $\vec{F}$ is a function that assigns each point (x, y) in 2D space (or $(x_1, x_2, ...)$ in higher dimensional space with basis x_i) to a vector, i.e. $\vec{F} : (x, y) \to \vec{v}$

Common examples are the *electric field*, which assigns a point in space to a force (a vector) corresponding to the charge per unit charge:

$$\vec{E}(x, y) = \frac{-q}{4\pi\epsilon_0\sqrt{(x^2+y^2)^3}}(x, y)$$

This assigns each point in space to a vector pointing in the direction of the origin, as the vector $-(x, y)$ at the location (x, y) points in the direction of the center. The constants just affect the length of the vector.
Note: there might be some abuse of notation here as I'm referring to (x, y) as a point in 2D space in the left hand side and a vector at the right hand side. This is due to $\mathbb{R}^2$ itself being a vector space over $\mathbb{R}$. The definition of such is included at the end of the document and you can prove this if you want.

Another example is the map for showing a wind speed at different places in HK. (Credit to Mr. Mok for the idea):



This map assigns each point in 2D space to the local velocity of wind, which has magnitude and direction

I will not talk about this part much for now as this is way out of syllabus (for interested readers only)

6 Basis decomposition

Following the previous discussion on vectors, you might have noticed that the vector could be decomposed into two perpendicular components (common theme in physics questions).

Indeed, for an arbitrary vector $\vec{v} = (x_v, y_v)$, we can interpret it as the sum $\vec{v} = x_v(1,0) + y_x(0,1)$. The vectors $\vec{i} = (1,0)$ and $\vec{j} = (0,1)$ are called the unit vectors in the x and y direction and are used most often for decomposition of a vector. (For example, $(3,4) = 3\vec{i} + 4\vec{j}$). $\vec{i}$ and $\vec{j}$ are called the orthonormal bases of $\mathbb{R}^2$ (perpendicular vectors such that all vectors are linear combination of them)

However, the chosen basis need not be perpendicular. $(1,2)$ and $(3,1)$ could be valid bases for the 2D plane, in the sense that all vectors can be written in the form $a(1,2) + b(3,1)$. (Prove it!)

This is a very important skill to learn especially in physics

(Question for fun 2: When does 2 vectors in 2D space fail to become a bases for $\mathbb{R}^2$?)

7 Summary

This lecture hopefully covers some things you would like to know regarding vectors. You could expect more of these lectures to come, but please don't ask for it too frequently (I'm going to die lol).

8 Appendix

8.1 Vector Space

!!!
The following definitions are not expected to be understood by secondary school students. I just put this here for completion.
A *vector space* V equipped with an *addition operator* $+ : V \times V \to V$ over a *field* K with a *binary function* $\times : K \times V \to V$ is a set that satisfies the following properties $(u, v, w \in V, a, b \in K)$

1. $(u + v) + w = u + (v + w)$ (Associativity of vector addition)
2. $u + v = v + u$ (Commutativity of vector addition)
3. $\exists 0 \in V \mid \forall v \in V, v + 0 = v$ (Existence of zero vector)
4. $\forall v \in V, \exists(-v) \in V \mid v + (-v) = 0$ (Existence of additive inverse)

5. $(ab)v = a(bv)$
6. $a(u+v) = au + av$
7. $(a+b)u = au + bu$

Note that knowing this is again completely useless in the secondary school curriculum. The only thing I hope you can get from this is the fact that this definition has no mentioning of arrows, and did not specify what the elements we call "vectors" actually represent. This is what mathematicians refer to as "abstraction".

8.2 Notations

1. $\mathbb{R}$ means the set of real numbers, i.e. the collection of all numbers between $-\infty$ and ∞. Every number on the number line belongs to this collection.
2. $\in$ means "belongs to". So, writing $x \in \mathbb{R}$ literally means "x belongs to the set of real numbers", or in human language, x is a real number.
3. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is the *Cartesian product* of $\mathbb{R}$ with itself. It just means that the set $\mathbb{R} \times \mathbb{R}$ contains pair of real numbers, i.e. the 2D plane. (In set notation, $\mathbb{R}^2 = \{(a,b) \mid a, b \in \mathbb{R}\}$, so $(3,4)$ and (e,π) are elements of $\mathbb{R}^2$. Similarly, $\mathbb{R}^n$ refers to n-dimensional space.
4. $\exists$ reads "there exists".
5. $\mid$ means "such that"
6. $\forall$ means "for all"
7. $\mapsto$ means "maps to". This explicitly defines what a function is doing, e.g. $f : x \mapsto x^2$ is a function mapping an element x to its square. Similarly I have also extended this to functions like $g : t \mapsto (t, t^2)$, which upon checking traces out the same curve as the *graph* of f. Note that the function and its graph are different.

The End