Fourier Series

EE3900: Linear Systems and Signal Processing

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1. Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$
Solution: Download the following Python code that plots Fig. 1.1.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/1.1.py
```

Run the code by executing

```
python 1.1.py
```

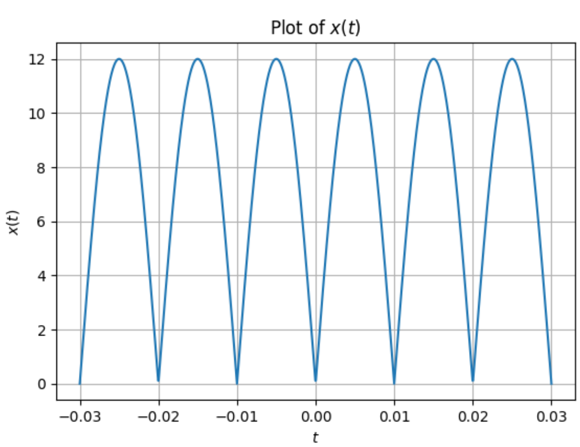

Fig. 1.1. Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period
Solution: Since $x(t)$ is the absolute value of a sinusoidal function, it is periodic, which is also evident from the plot

Consider $x(t + \frac{1}{2f_0})$

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right)\right| \tag{1.2}$$
$$= A_0 \left|\sin\left(2\pi f_0 t + \pi f_0\right)\right| \tag{1.3}$$
$$= A_0 \left|(-1)^{f_0} \sin\left(2\pi f_0 t\right)\right| \tag{1.4}$$
$$= A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.5}$$
$$= x(t) \tag{1.6}$$

Therefore, $x(t)$ is periodic with period $\frac{1}{2f_0}$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:

$$x(t) e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi (n-k) f_0 t} \tag{2.3}$$

$$\implies \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi n f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t}\, dt \tag{2.4}$$

But

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t}\, dt = \begin{cases} \frac{1}{f_0} & k = n \\ 0 & k \neq n \end{cases} \tag{2.5}$$
$$= \frac{1}{f_0} \delta(n-k) \tag{2.6}$$

$$\sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\mathrm{j}2\pi(n-k)f_0 t}\, \mathrm{d}t = \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0}\delta(n-k) \tag{2.7}$$

$$= \frac{1}{f_0} c_n * \delta(n) \tag{2.8}$$

$$= \frac{1}{f_0} c_n \tag{2.9}$$

Therefore

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\mathrm{j}2\pi k f_0 t}\, \mathrm{d}t \tag{2.10}$$

2.2 Find c_k for (1.1)
Solution:

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin\left(2\pi f_0 t\right)\right| e^{-\mathrm{j}2\pi k f_0 t}\, \mathrm{d}t \tag{2.11}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{0} A_0 \left(-\sin\left(2\pi f_0 t\right)\right) e^{-\mathrm{j}2\pi k f_0 t}\, \mathrm{d}t + f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\sin\left(2\pi f_0 t\right)\right) e^{-\mathrm{j}2\pi k f_0 t}\, \mathrm{d}t \tag{2.12}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 u\right) e^{\mathrm{j}2\pi k f_0 u}\, \mathrm{d}t + f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) e^{-\mathrm{j}2\pi k f_0 t}\, \mathrm{d}t \tag{2.13}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin\left(2\pi f_0 t\right) \left(e^{\mathrm{j}2\pi k f_0 t} + e^{-\mathrm{j}2\pi k f_0 t}\right) \mathrm{d}t \tag{2.14}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} 2\sin\left(2\pi f_0 t\right)\cos\left(2\pi k f_0 t\right)\, \mathrm{d}t \tag{2.15}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left\{\sin\left(2\pi(1+k)f_0 t\right) + \sin\left(2\pi(1-k)f_0 t\right)\right\} \mathrm{d}t \tag{2.16}$$

$$= f_0 A_0 \left[-\frac{\cos\left(2\pi(1+k)f_0 t\right)}{2\pi(1+k)f_0} - \frac{\cos\left(2\pi(1-k)f_0 t\right)}{2\pi(1-k)f_0}\right]_0^{\frac{1}{2f_0}} \tag{2.17}$$

$$= \frac{f_0 A_0}{2\pi f_0}\left[\frac{1-(-1)^{1+k}}{1+k} + \frac{1-(-1)^{1-k}}{1-k}\right] \tag{2.18}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{2\pi}\left[\frac{1}{1+k} + \frac{1}{1-k}\right] \tag{2.19}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{\pi(1-k^2)} \tag{2.20}$$

Therefore

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases} \tag{2.21}$$

2.3 Verify (1.1) using Python
Solution: Download the following Python code that plots Fig. 3.8.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/2.3.py
```

Run the code by executing

```
python 2.3.py
```

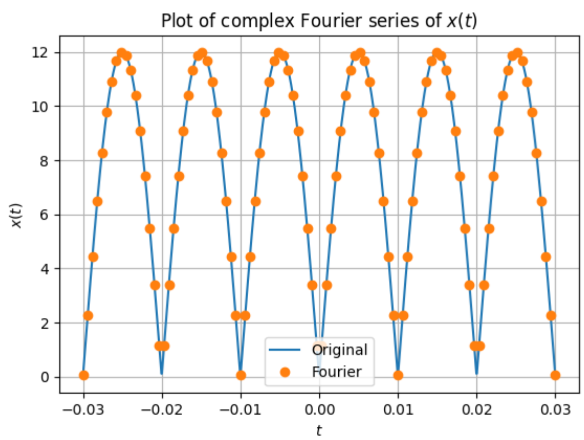

Fig. 2.3. Plot of $x(t)$ along with its complex Fourier series expansion

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty}\left(a_k \cos\left(2\pi k f_0 t\right) + b_k \sin\left(2\pi k f_0 t\right)\right) \tag{2.22}$$

and obtain the formulae for a_k and b_k
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\mathrm{j}2\pi k f_0 t} \tag{2.23}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\mathrm{j}2\pi k f_0 t} + c_{-k} e^{-\mathrm{j}2\pi k f_0 t} \tag{2.24}$$

Thus

$$x(t) = c_0 + \sum_{k=1}^{\infty}\left(c_k + c_{-k}\right)\cos\left(2\pi k f_0 t\right) + \sum_{k=1}^{\infty} \mathrm{j}\left(c_k - c_{-k}\right)\sin\left(2\pi k f_0 t\right) \tag{2.25}$$

Therefore

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.26}$$

$$b_k = \jmath(c_k - c_{-k}) \quad k \geq 0 \tag{2.27}$$

2.5 Find a_k and b_k for (1.1)
Solution:

$$a_0 = c_0 = \frac{2A_0}{\pi} \tag{2.28}$$

For $k > 0$, if k is odd

$$a_k = 0 + 0 = 0 \tag{2.29}$$

and if k is even

$$a_k = \frac{2A_0}{\pi(1-k^2)} + \frac{2A_0}{\pi(1-k^2)} = \frac{4A_0}{\pi(1-k^2)} \tag{2.30}$$

For odd or even k, $c_k = c_{-k}$ always

$$b_k = 0 \quad \forall k \geq 0 \tag{2.31}$$

Therefore

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k = 2m, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{2.32}$$

$$b_k = 0 \qquad k \geq 0 \tag{2.33}$$

2.6 Verify (2.22) using Python
Solution: Download the following Python code that plots Fig. 2.6.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/2.6.py
```

Run the code by executing

```
python 2.6.py
```

3. Fourier Transform

3.1

$$\delta(t) = 0 \qquad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t) \mathrm{d}t = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-\jmath 2\pi f t} \, \mathrm{d}t \tag{3.3}$$

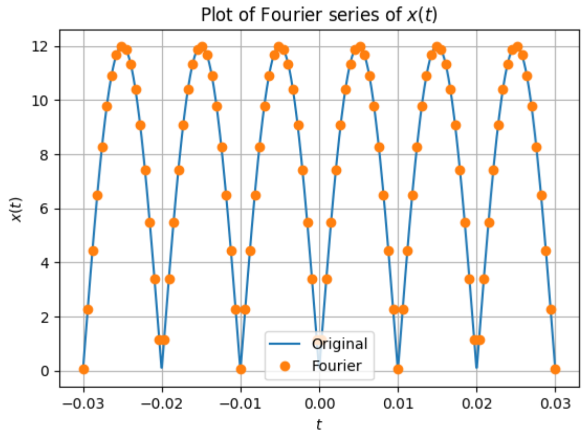

Fig. 2.6. Plot of $x(t)$ along with its Fourier series expansion

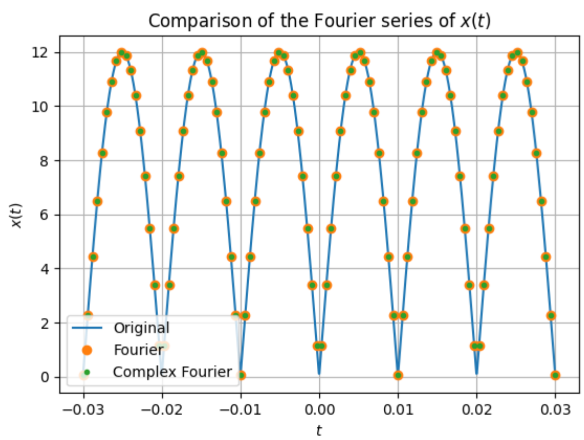

Fig. 2.6. Comparison of the Fourier series of $x(t)$

3.3 Show that

$$g(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} G(f) e^{-\jmath 2\pi f t_0} \tag{3.4}$$

Solution:

$$g(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t - t_0) e^{-\jmath 2\pi f t} \, \mathrm{d}t \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f (u + t_0)} \, \mathrm{d}u \tag{3.6}$$

$$= e^{-\jmath 2\pi f t_0} \int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f u} \, \mathrm{d}u \tag{3.7}$$

$$= G(f) e^{-\jmath 2\pi f t_0} \tag{3.8}$$

3.4 Show that

$$G(t) \stackrel{\mathcal{F}}{\longleftrightarrow} g(-f) \tag{3.9}$$

Solution:

$$G(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} G(t) e^{-\jmath 2\pi f t} \, \mathrm{d}t \tag{3.10}$$

But

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{\jmath 2\pi f t} \, df \tag{3.11}$$

$$= \int_{-\infty}^{\infty} G(u) e^{\jmath 2\pi u t} \, du \tag{3.12}$$

$$\implies g(-f) = \int_{-\infty}^{\infty} G(u) e^{-\jmath 2\pi u f} \, du \tag{3.13}$$

$$= \mathcal{F}\{G(t)\} \tag{3.14}$$

3.5 Find the Fourier transform of $\delta(t)$
Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} \, dt \tag{3.15}$$

$$= e^{-j2\pi f t}\Big|_{t=0} \tag{3.16}$$

$$=1 \tag{3.17}$$

3.6 Find the Fourier transform of $e^{-\jmath 2\pi f_0 t}$
Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \tag{3.18}$$

$$\implies \delta(t - f_0) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f f_0} \tag{3.19}$$

$$\implies e^{-j2\pi t f_0} \xleftrightarrow{\mathcal{F}} \delta(-f - f_0) \tag{3.20}$$

$$\therefore e^{-j2\pi t f_0} \xleftrightarrow{\mathcal{F}} \delta(f + f_0) \tag{3.21}$$

3.7 Find the Fourier transform of $\cos(2\pi f_0 t)$
Solution:

$$\cos(2\pi f_0 t) = \frac{e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}}{2} \tag{3.22}$$

$$\implies \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{\delta(f - f_0) + \delta(f + f_0)}{2} \tag{3.23}$$

3.8 Find the Fourier transform of $x(t)$ and plot it. Verify using Python
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{3.24}$$

$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left\{e^{\jmath 2\pi k f_0 t}\right\} \tag{3.25}$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \tag{3.26}$$

$$= \frac{2A_0}{\pi} \sum_{k \text{ is even}} \frac{\delta(f - k f_0)}{1 - k^2} \tag{3.27}$$

Download the following Python code that plots Fig. **??**.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/3.8.py
```

Run the code by executing

```
python 3.8.py
```

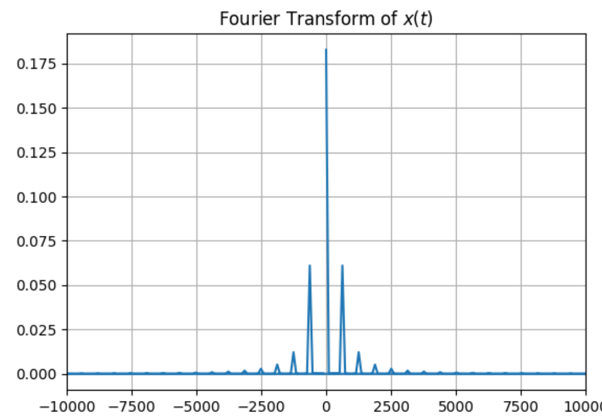

Fig. 3.8. Plot of the Fourier transform of $x(t)$

3.9 Show that

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \text{sinc}(f) \tag{3.28}$$

Verify using Python
Solution:

$$\text{rect}(t) = \begin{cases} 1 & |t| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \tag{3.29}$$

Its Fourier transform is given by

$$\text{rect}(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \text{rect}(t) \, e^{-j2\pi f t} \, dt \tag{3.30}$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi f t} dt \tag{3.31}$$

$$= \frac{e^{-j\pi f} - e^{j\pi f}}{-\jmath 2\pi f} \tag{3.32}$$

$$= \frac{\sin \pi f}{\pi f} \tag{3.33}$$

$$= \text{sinc}(f) \tag{3.34}$$

Download the following Python code that plots Fig. 3.9.

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Fourier/codes/3.9.py
```

Run the code by executing

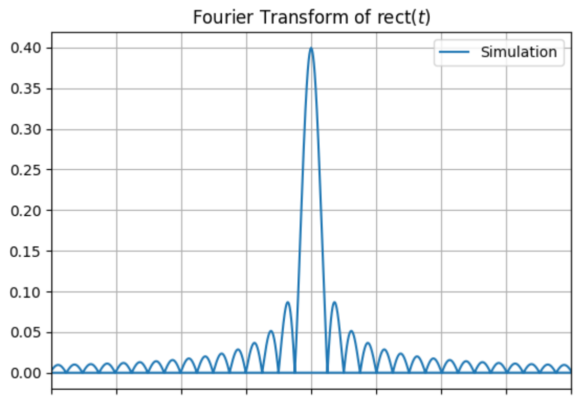

Fig. 3.9. Plot of the Fourier transform of rect (t)

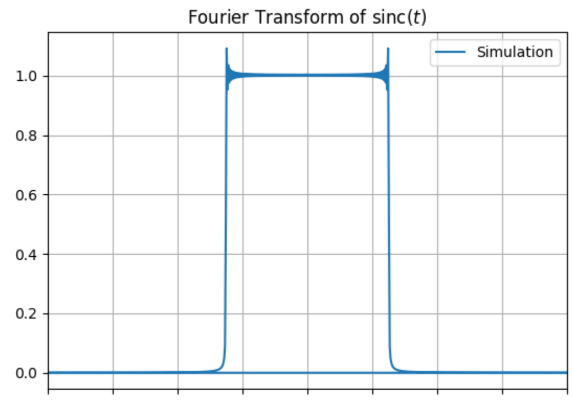

Fig. 3.10. Plot of the Fourier transform of sinc (t)

```
python 3.9.py
```

3.10 Find the Fourier transform of sinc (t). Verify using Python
Solution:

$$\operatorname{rect}(t) \xleftrightarrow{\mathcal{F}} \operatorname{sinc}(f) \tag{3.35}$$

$$\implies \operatorname{sinc}(t) \xleftrightarrow{\mathcal{F}} \operatorname{rect}(-f) \tag{3.36}$$

$$= \operatorname{rect}(f) \tag{3.37}$$

Download the following Python code that plots Fig. 3.10.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/3.10.py
```

Run the code by executing

```
python 3.10.py
```

4. Filter

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5 V
Solution: Since we want a DC output, the filter we need is a low-pass filter that only lets the zero frequency component pass through, i.e., the amplitude of a frequency components with a frequency higher than the cutoff frequency f_c has to be zero
We can use a rectangular filter for this purpose

$$H(f) = k \operatorname{rect}\left(\frac{f}{2f_c}\right) = \begin{cases} k & |f| \le f_c \\ 0 & \text{otherwise} \end{cases} \tag{4.1}$$

Now

$$H(0) = \frac{Y(0)}{X(0)} \tag{4.2}$$

where $Y(k)$ and $X(k)$ are the Fourier transforms of the output 5 V DC and the input signal respectively

$$k = \frac{5}{\frac{2A_0}{\pi}} = \frac{5\pi}{2A_0} \tag{4.3}$$

$$\therefore H(f) = \frac{5\pi}{2A_0} \operatorname{rect}\left(\frac{f}{2f_c}\right) \tag{4.4}$$

4.2 Find $h(t)$
Solution:

$$\operatorname{sinc}(t) \xleftrightarrow{\mathcal{F}} \operatorname{rect}(f) \tag{4.5}$$

$$\implies \operatorname{sinc}(2f_c t) \xleftrightarrow{\mathcal{F}} \frac{1}{2f_c} \operatorname{rect}\left(\frac{f}{2f_c}\right) \tag{4.6}$$

$$\implies \frac{5\pi}{2A_0} 2f_c \operatorname{sinc}(2f_c t) \xleftrightarrow{\mathcal{F}} \frac{5\pi}{2A_0} \operatorname{rect}\left(\frac{f}{2f_c}\right) \tag{4.7}$$

$$\therefore h(t) = \frac{5\pi f_c}{A_0} \operatorname{sinc}(2f_c t) \tag{4.8}$$

4.3 Verify your result using through convolution
Solution: Download the following Python code that plots Fig. 4.3.

```
wget https://github.com/Ankit-Saha-2003/
    EE3900/raw/main/Fourier/codes/4.3.py
```

Run the code by executing

```
python 4.3.py
```

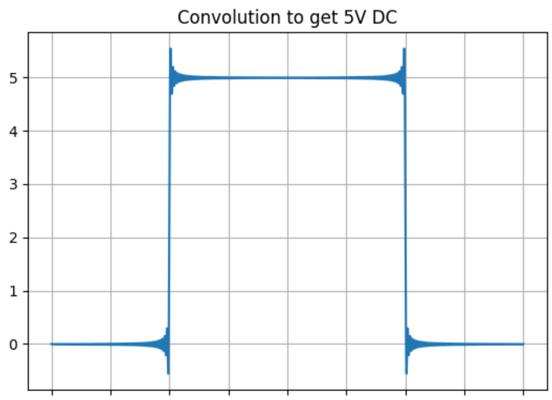

Fig. 4.3. Plot of the convolution of $x(t)$ and $h(t)$

5. Filter Design

5.1 Design a Butterworth filter for $H(f)$

Solution: The transfer function of a Butterworth filter is given by

$$|H_n(f)| = \frac{1}{\sqrt{1+\left(\frac{f}{f_c}\right)^{2n}}} \tag{5.1}$$

where n is the order of the filter and f_c is the cutoff frequency

Let the passband and stopband frequency thresholds be 50 Hz and 100 Hz and their corresponding attenuations be -1 dB and -5 dB respectively

$$A_p = 10\log_{10}\left|H_n(f_p)\right|^2 \tag{5.2}$$

$$= -10\log_{10}\left(1+\left(\frac{f_p}{f_c}\right)^{2n}\right) \tag{5.3}$$

$$A_s = -10\log_{10}\left(1+\left(\frac{f_s}{f_c}\right)^{2n}\right) \tag{5.4}$$

$$\implies n = \frac{\log\left(\frac{10^{-\frac{A_p}{10}}-1}{10^{-\frac{A_s}{10}}-1}\right)}{2\log\left(\frac{f_p}{f_s}\right)} \approx 1.53 \tag{5.5}$$

Hence, we choose a 2nd order Butterworth filter with

$$f_c = \frac{f_p}{\left(10^{-\frac{A_p}{10}}-1\right)^{\frac{1}{2n}}} \approx 77.74\,\text{Hz} \tag{5.6}$$

5.2 Design a Chebyschev filter for $H(f)$

Solution: The transfer function of a Chebyshev filter is given by

$$|H_n(f)| = \frac{1}{\sqrt{1+\epsilon^2 T_n^2\left(\frac{f}{f_c}\right)}} \tag{5.7}$$

where ϵ is the ripple factor, f_c is the cutoff frequency and T_n is a Chebyshev polynomial of the n^{th} order

Assuming the same parameters as before along with a ripple of 0.1 dB, we get

$$\epsilon = \sqrt{10^{\frac{\delta}{10}}-1} \approx 0.15 \tag{5.8}$$

Also, assume that $f_c = f_p \implies \frac{f_s}{f_c} > 1$

$$A_s = -10\log_{10}\left(1+\epsilon^2 T_n^2\left(\frac{f_s}{f_c}\right)\right) \tag{5.9}$$

$$\implies T_n\left(\frac{f_s}{f_c}\right) = \frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon} \tag{5.10}$$

$$\implies \cosh\left(n\cosh^{-1}\left(\frac{f_s}{f_c}\right)\right) = \frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon} \tag{5.11}$$

Thus

$$n = \frac{\cosh^{-1}\left(\frac{\sqrt{10^{-\frac{A_s}{10}}-1}}{\epsilon}\right)}{\cosh^{-1}\left(\frac{f_s}{f_c}\right)} \approx 2.26 \tag{5.12}$$

Hence, we choose a 3^{rd} order Chebyshev filter

5.3 Design a circuit for your Butterworth filter

Solution: Using the table of normalized Butterworth coefficients, we can see that for a 2^{nd} order Butterworth filter

$$C_1 = 1.4142\,\text{F} \tag{5.13}$$

$$L_2 = 1.4142\,\text{H} \tag{5.14}$$

On denormalizing these values, we get

$$C_1' = \frac{C_1}{2\pi f_c} = 2.89\,\text{mF} \tag{5.15}$$

$$L_2' = \frac{L_2}{2\pi f_c} = 2.89\,\text{mH} \tag{5.16}$$

5.4 Design a circuit for your Chebyschev filter

Solution: Using the table of normalized Chebyshev coefficients, we can see that for a 3^{rd} order Chebyshev filter

$$C_1 = 1.4328\,\text{F} \tag{5.17}$$

$$L_2 = 1.5937\,\text{H} \tag{5.18}$$

$$C_3 = 1.4328\,\text{F} \tag{5.19}$$

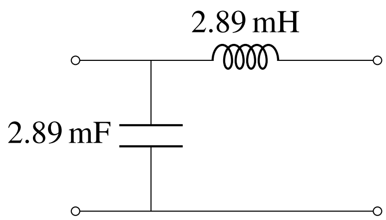

Fig. 5.3. 2nd order Butterworth filter circuit

On denormalizing these values, we get

$$C_1' = \frac{C_1}{2\pi f_c} = 4.56\,\mathrm{mF} \tag{5.20}$$

$$L_2' = \frac{L_2}{2\pi f_c} = 5.07\,\mathrm{mH} \tag{5.21}$$

$$C_3' = \frac{C_3}{2\pi f_c} = 4.56\,\mathrm{mF} \tag{5.22}$$

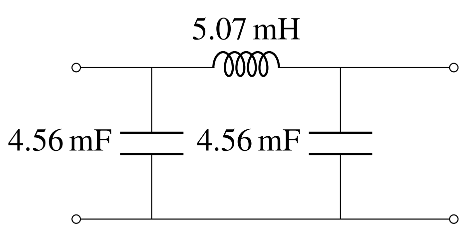

Fig. 5.4. 3rd order Chebyshev filter circuit