Circuits and Transforms

EE3900: Linear Systems and Signal Processing

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1. Definitions

1.1 The unit step function is defined as

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

1.2 The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\,\mathrm{d}t \tag{1.2}$$

2. Laplace Transform

2.1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes q_1 µC. Then S is switched to position Q. After a long time, the charge on the capacitor is q_2 µC

2.2. Draw the circuit using latex-tikz
Solution:

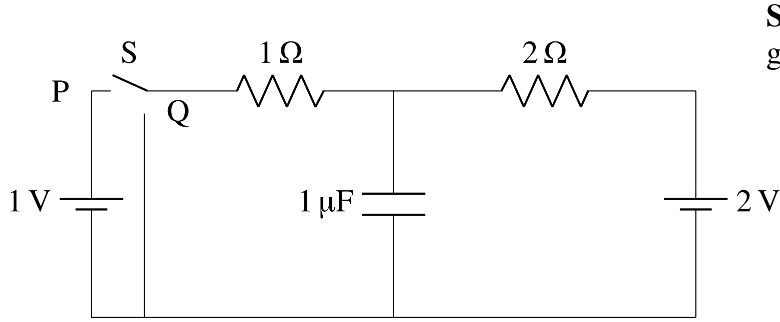

Fig. 2.2. Circuit diagram of the circuit in question

2.3. Find q_1
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero

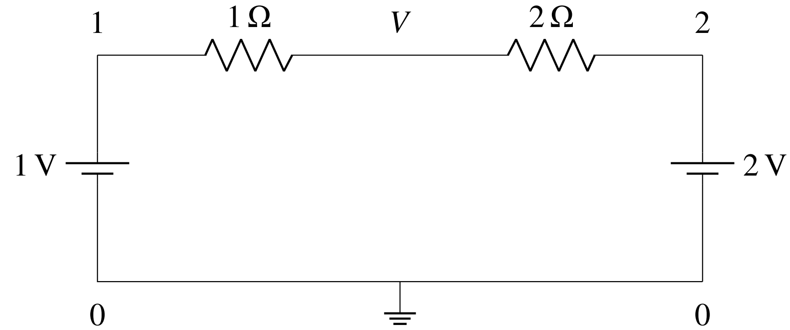

Fig. 2.3. Circuit diagram at steady state before flipping the switch

By Kirchoff's junction law, we get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \tag{2.1}$$

$$\implies V = \frac{4}{3}\,\mathrm{V} \tag{2.2}$$

$$\implies q_1 = CV = \frac{4}{3}\,\mu\mathrm{C} \tag{2.3}$$

2.4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC
Solution: The Laplace transform of $u(t)$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} u(t)e^{-st}\,\mathrm{d}t \tag{2.4}$$

$$= \int_{0}^{\infty} e^{-st}\,\mathrm{d}t \tag{2.5}$$

$$= \lim_{R\to\infty} \frac{1-e^{-sR}}{s} \tag{2.6}$$

This limit is finite only if $\Re(s) > 0$, which is going to be its ROC
Therefore

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \qquad \Re(s) > 0 \tag{2.7}$$

2.5. Show that

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad a > 0 \tag{2.8}$$

and find the ROC
Solution: The Laplace transform of $e^{-at}u(t)$ for $a > 0$ is given by

$$\mathcal{L}\{u(t)\} = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}\,\mathrm{d}t \tag{2.9}$$
$$= \int_{0}^{\infty} e^{-(s+a)t}\,\mathrm{d}t \tag{2.10}$$
$$= \lim_{R\to\infty} \frac{1 - e^{-(s+a)R}}{s+a} \tag{2.11}$$

This limit is finite only if $\Re(s+a) > 0$, which is going to be its ROC
Therefore

$$e^{-at}u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a} \qquad \Re(s) > -a \tag{2.12}$$

since a is real

2.6. Now consider the following resistive circuit transformed from Fig. 2.2

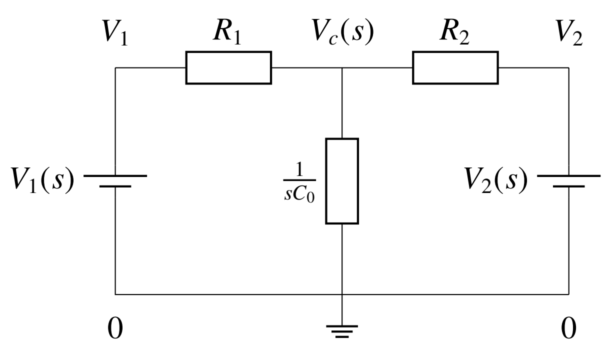

Fig. 2.6. Circuit diagram in s-domain before flipping the switch

where

$$u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.13}$$
$$2u(t) \stackrel{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.14}$$

Find the voltage across the capacitor $V_c(s)$
Solution:

$$V_1(s) = \frac{1}{s} \qquad \Re(s) > 0 \tag{2.15}$$
$$V_2(s) = \frac{2}{s} \qquad \Re(s) > 0 \tag{2.16}$$

By Kirchoff's junction law, we get

$$\frac{V_c - V_1}{R_1} + \frac{V_c - V_2}{R_2} + \frac{V_c - 0}{\frac{1}{sC_0}} = 0 \tag{2.17}$$
$$\implies V_c\left(\frac{1}{R_1} + \frac{1}{R_2} + sC_0\right) = \frac{V_1}{R_1} + \frac{V_2}{R_2} \tag{2.18}$$
$$\implies V_c(s) = \frac{\frac{1}{sR_1} + \frac{2}{sR_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \tag{2.19}$$
$$= \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{s\left(s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}\right)} \tag{2.20}$$

2.7. Find $v_c(t)$. Plot using Python.
Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{\frac{1}{R_1C_0} + \frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right), \Re(s) > 0 \tag{2.21}$$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{2R_1 + R_2}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{2.22}$$
$$= \frac{2R_1 + R_2}{R_1 + R_2}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{2.23}$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\text{F}$

$$v_c(t) = \frac{4}{3}\left(1 - e^{-\frac{3}{2}\times10^6 t}\right)u(t)\,\text{V} \tag{2.24}$$

2.8. Verify your result using ngspice
Solution: Download the following codes for simulation and plotting Fig. 2.8 respectively

```
wget https://github.com/AP−51/Signal−
    Processing/blob/main/circuit/codes/2.8.cir
wget https://github.com/AP−51/Signal−
    Processing/blob/main/circuit/codes/2.7.py
```

Run the codes by executing

```
ngspice 2.8.cir
python 2.7.py
```

2.9. Obtain Fig. 2.6 using the equivalent differential equation
Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{2.25}$$

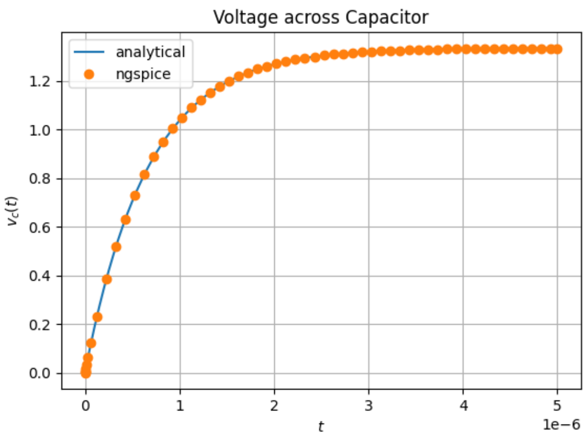

Fig. 2.8. Plot of $v_c(t)$ before flipping the switch

where $q(t)$ is the charge on the capacitor
On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)-V_1(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+\left(sQ(s)-q(0^-)\right)=0 \tag{2.26}$$

But $q(0^-)=0$ and

$$q(t)=C_0v_c(t) \tag{2.27}$$
$$\implies Q(s)=C_0V_c(s) \tag{2.28}$$

Thus

$$\frac{V_c(s)-V_1(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+sC_0V_c(s)=0 \tag{2.29}$$
$$\implies \frac{V_c(s)-V_1(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+\frac{V_c(s)-0}{\frac{1}{sC_0}}=0 \tag{2.30}$$

which is the same equation as the one we obtained from Fig. 2.6

3. Initial Conditions

3.1. Find q_2 in Fig. 2.2
Solution: After a long time, when steady state is achieved, a capacitor behaves like an open circuit, i.e., current passing through it is zero
By Kirchoff's junction law, we get

$$\frac{V-0}{1}+\frac{V-2}{2}=0 \tag{3.1}$$
$$\implies V=\frac{2}{3}\,\text{V} \tag{3.2}$$
$$\implies q_2=CV=\frac{2}{3}\,\mu\text{C} \tag{3.3}$$

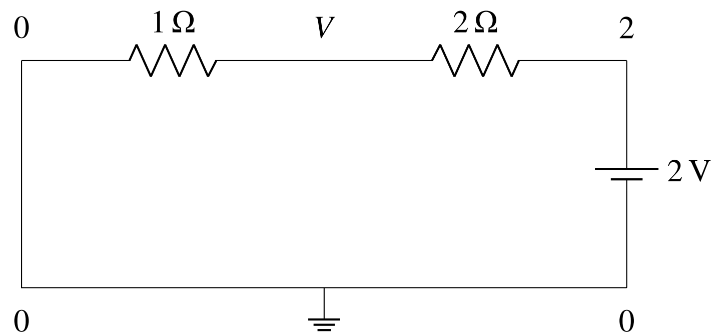

Fig. 3.1. Circuit diagram at steady state after flipping the switch

3.2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz
Solution:

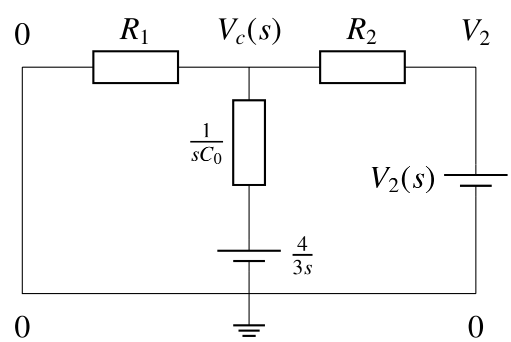

Fig. 3.2. Circuit diagram in s-domain after flipping the switch

The battery $\frac{4}{3s}$ corresponds to the intial potential difference of $\frac{4}{3}$ V across the capacitor just before switching it to Q

3.3. Find $V_c(s)$
Solution: By Kirchoff's junction law, we get

$$\frac{V_c-0}{R_1}+\frac{V_c-V_2}{R_2}+\frac{V_c-\frac{4}{3s}}{\frac{1}{sC_0}}=0 \tag{3.4}$$
$$\implies V_c\left(\frac{1}{R_1}+\frac{1}{R_2}+sC_0\right)=\frac{V_2}{R_2}+\frac{4}{3}C_0 \tag{3.5}$$
$$\implies V_c(s)=\frac{\frac{2}{sR_2}+\frac{4}{3}C_0}{\frac{1}{R_1}+\frac{1}{R_2}+sC_0} \tag{3.6}$$
$$=\frac{\frac{2}{R_2C_0}+\frac{4}{3}s}{s\left(s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}\right)} \tag{3.7}$$

3.4. Find $v_c(t)$. Plot using Python

Solution: On performing partial fraction decomposition

$$V_c(s) = \frac{4}{3}\left(\frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) + \frac{\frac{2}{R_2C_0}}{\frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\left(\frac{1}{s} - \frac{1}{s + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}}\right) \tag{3.8}$$

for $\Re(s) > 0$

On taking the inverse Laplace transform, we get

$$v_c(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2R_1}{R_1 + R_2}\left(u(t) - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t)\right) \tag{3.9}$$

Substitute the values $R_1 = 1\,\Omega, R_2 = 2\,\Omega, C_0 = 1\,\mu\text{F}$

$$v_c(t) = \frac{4}{3}e^{-\frac{3}{2}\times 10^6 t}u(t) + \frac{2}{3}\left(1 - e^{-\frac{3}{2}\times 10^6 t}\right)u(t) \tag{3.10}$$

$$= \frac{2}{3}\left(1 + e^{-\frac{3}{2}\times 10^6 t}\right)u(t)\,\text{V} \tag{3.11}$$

3.5. Verify your result using ngspice

Solution: Download the following codes for simulation and plotting Fig. 3.5 respectively

```
wget https://github.com/AP-51/Signal-
    Processing/blob/main/circuit/codes/3.5.cir
wget https://github.com/AP-51/Signal-
    Processing/blob/main/circuit/codes/3.4.py
```

Run the codes by executing

```
ngspice 3.5.cir
python 3.4.py
```

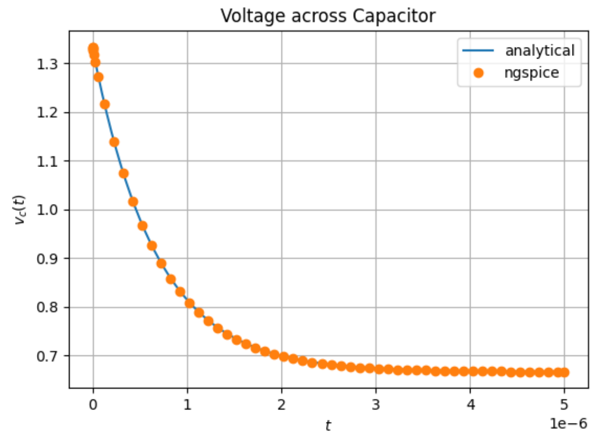

Fig. 3.5. Plot of $v_c(t)$ after flipping the switch

3.6. Find $v_c(0^-)$, $v_c(0^+)$ and $v_c(\infty)$

Solution: At $t = 0^-$, the switch still hasn't been switched to Q and the circuit is in steady state

$$v_c(0^-) = \frac{4}{3}\,\text{V} \tag{3.12}$$

For $t \geq 0$, we can use the above formula

$$v_c(0^+) = \lim_{t\to 0^+} v_c(t) = \frac{4}{3}\,\text{V} \tag{3.13}$$

$$v_c(\infty) = \lim_{t\to\infty} v_c(t) = \frac{2}{3}\,\text{V} \tag{3.14}$$

3.7. Obtain Fig. 3.2 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{\mathrm{d}q}{\mathrm{d}t} = 0 \tag{3.15}$$

where $q(t)$ is the charge on the capacitor

On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sQ(s) - q(0^-)\right) = 0 \tag{3.16}$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \tag{3.17}$$

$$\implies Q(s) = C_0 V_c(s) \tag{3.18}$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0V_c(s) - \frac{4}{3}C_0\right) = 0 \tag{3.19}$$

$$\implies \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \tag{3.20}$$

which is the same equation as the one we obtained from Fig. 3.2

4. Bilinear Transform

4.1. In Fig. 2.2, consider the case when S is switched to Q right in the beginning. Formulate the differential equation

Solution: The differential equation is the same as before

$$\frac{v_c(t)-0}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+\frac{dq}{dt}=0 \quad (4.1)$$

$$\text{i.e., } \frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{dv_c}{dt}=0 \quad (4.2)$$

but with a different initial condition

$$q(0^-)=q(0)=0 \quad (4.3)$$

4.2. Find $H(s)$ considering the ouput voltage at the capacitor
Solution: On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s)}{R_1}+\frac{V_c(s)-V_2(s)}{R_2}+sQ(s)-0=0 \quad (4.4)$$

$$\implies V_c(s)\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+sC_0V_c(s)=\frac{V_2(s)}{R_2} \quad (4.5)$$

$$\implies \frac{V_c(s)}{V_2(s)}=\frac{\frac{1}{R_2}}{\frac{1}{R_1}+\frac{1}{R_2}+sC_0} \quad (4.6)$$

The transfer function is thus

$$H(s)=\frac{\frac{1}{R_2C_0}}{s+\frac{1}{R_1C_0}+\frac{1}{R_2C_0}} \quad (4.7)$$

On substituting the values, we get

$$H(s)=\frac{5\times10^5}{s+1.5\times10^6} \quad (4.8)$$

4.3. Plot $H(s)$. What kind of filter is it?
Solution: Download the following Python code that plots Fig. 4.3

```
wget https://github.com/AP−51/Signal−
    Processing/blob/main/circuit/codes/4.3.py
```

Run the codes by executing

```
python 4.3.py
```

Consider the frequency-domain transfer function by putting $s=\jmath\omega$

$$H(\jmath\omega)=\frac{5\times10^5}{\jmath\omega+1.5\times10^6} \quad (4.9)$$

$$\implies \left|H(\jmath\omega)\right|=\frac{5\times10^5}{\sqrt{\omega^2+2.25\times10^{12}}} \quad (4.10)$$

As ω increases, $\left|H(\jmath\omega)\right|$ decreases

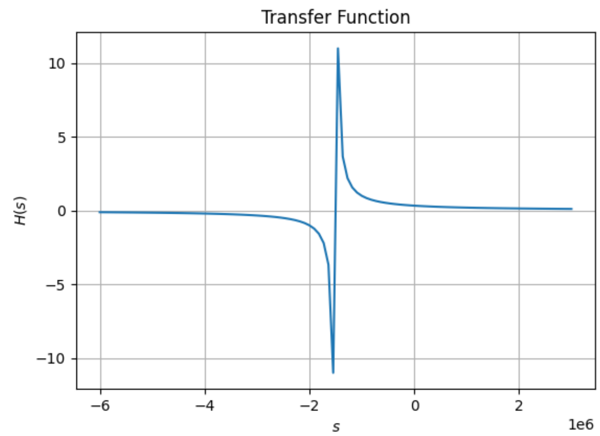

Fig. 4.3. Plot of $H(s)$

In other words, the amplitude of high-frequency signals gets diminished and they get filtered out
Therefore, this is a low-pass filter

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n)=y(t)|_{t=n} \quad (4.11)$$

Solution:

$$\frac{v_c(t)}{R_1}+\frac{v_c(t)-v_2(t)}{R_2}+C_0\frac{dv_c}{dt}=0 \quad (4.12)$$

$$\implies C_0\frac{dv_c}{dt}=\frac{2u(t)-v_c(t)}{R_2}-\frac{v_c(t)}{R_1} \quad (4.13)$$

$$\implies v_c(t)|_{t=n}^{n+1}=\int_n^{n+1}\left(\frac{2u(t)-v_c(t)}{R_2C_0}-\frac{v_c(t)}{R_1C_0}\right)dt \quad (4.14)$$

By the trapezoidal rule of integration

$$\int_a^b f(t)dt\approx\frac{b-a}{2}(f(a)+f(b)) \quad (4.15)$$

Consider $y(t)=v_c(t)$

$$y(n+1)-y(n)=\frac{1}{R_2C_0}(u(n)+u(n+1)) -\frac{1}{2}(y(n+1)+y(n))\left(\frac{1}{R_1C_0}+\frac{1}{R_2C_0}\right) \quad (4.16)$$

Thus, the difference equation is

$$y(n+1)\left(1+\frac{1}{2R_1C_0}+\frac{1}{2R_2C_0}\right) = y(n)\left(1-\frac{1}{2R_1C_0}-\frac{1}{2R_2C_0}\right) +\frac{1}{R_2C_0}(u(n)+u(n+1)) \quad (4.17)$$

4.5. Find $H(z)$

Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$

On taking the Z-transform on both sides of the difference equation

$$zY(z)\left(1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = Y(z)\left(1 - \frac{1}{2R_1C_0} - \frac{1}{2R_2C_0}\right) + \frac{1}{R_2C_0}\left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}}\right) \quad (4.18)$$

$$Y(z)\left(z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right) = \frac{1}{R_2C_0}\frac{1+z}{1-z^{-1}} \quad (4.19)$$

Also

$$v_2(t) = 2 \qquad \forall t \geq 0 \quad (4.20)$$
$$\implies x(n) = 2u(n) \quad (4.21)$$
$$\implies X(z) = \frac{2}{1-z^{-1}} \qquad |z| > 1 \quad (4.22)$$

Thus, the transfer function in z-domain is

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.23)$$
$$= \frac{\frac{1+z}{2R_2C_0}}{z + \frac{z}{2R_1C_0} + \frac{z}{2R_2C_0} - 1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}} \quad (4.24)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (4.25)$$

On substituting the values

$$H(z) = \frac{2.5 \times 10^5(1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.26)$$

with the ROC being

$$|z| > \max\left(1, \left|\frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1}\right|\right) \quad (4.27)$$
$$\implies |z| > 1 \quad (4.28)$$

4.6. How can you obtain $H(z)$ from $H(s)$?

Solution: The Z-transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \quad (4.29)$$

where T is the step size of the trapezoidal rule (1 in our case)

This is known as the bilinear transform

Thus

$$H(z) = \frac{\frac{1}{R_2C_0}}{2\frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1C_0} + \frac{1}{R_2C_0}} \quad (4.30)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1C_0} + \frac{1}{2R_2C_0}\right)(1+z^{-1})} \quad (4.31)$$
$$= \frac{\frac{1+z^{-1}}{2R_2C_0}}{1 + \frac{1}{2R_1C_0} + \frac{1}{2R_2C_0} - z^{-1} + \frac{z^{-1}}{2R_1C_0} + \frac{z^{-1}}{2R_2C_0}} \quad (4.32)$$
$$= \frac{2.5 \times 10^5(1+z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.33)$$

which is the same as what we obtained earlier

In order to find $h(n)$ we represent $H(z)$ as

$$H(z) = \frac{c(1+z^{-1})}{a+bz^{-1}} \quad (4.34)$$
$$= \frac{c}{b} + \frac{c(1-\frac{a}{b})}{a+bz^{-1}} \quad (4.35)$$

Taking the the inverse z transform

$$h(n) = \frac{c}{b}\delta(n) + \frac{c}{a}\left(1 - \frac{a}{b}\right)\left(\frac{a}{b}\right)^n u(n) \quad (4.36)$$