Random Numbers

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1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat **Solution:** Download the following files and execute the C program.

```
wget https://github.com/AP−51/random−
    numbers/blob/main/code/exrand.c
wget https://github.com/AP−51/random−
    numbers/blob/main/code/coeffs.h
```

Use the below command in the terminal to run the code.

```
gcc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

1.3

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.3

```
wget https://github.com/AP−51/random−
    numbers/blob/main/code/plots/cdf_plot.py
```

1.4 **Solution:** As U is a Uniform Random Variable Distribution, CDF of $P_U x = F_U x$

$$P_U x = 1 \qquad F_U x = \int_0^1 P_U x dx \therefore F_U x = x \tag{1.2}$$

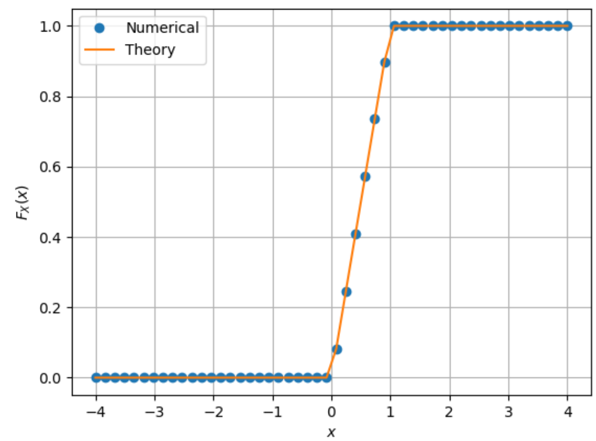

Fig. 1.3: The CDF of U

1.5 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.3}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.4}$$

Write a C program to find the mean and variance of U. **Solution:**

```
wget https://github.com/AP−51/random−
    numbers/blob/main/code/mean.c
wget https://github.com/AP−51/random−
    numbers/blob/main/code/coeffs.h
```

Use below command to run the code

```
gcc mean.c −lm
.\a.out
```

From this code we get Mean=0.500007 and Variance=0.083301

1.6 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.5}$$

Solution:

$$dF_U x = dx \tag{1.6}$$
$$\therefore E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$
$$E\left[U\right] = \int_0^1 x dx \tag{1.8}$$
$$\text{as } P_U x = 0 \quad \forall x \in (-\infty, 0) \cup (1, \infty) \tag{1.9}$$
$$\therefore E\left[U\right] = \frac{1}{2} \tag{1.10}$$
$$E\left[U^2\right] = \int_0^1 x^2 dx \tag{1.11}$$
$$\therefore E\left[U^2\right] = \frac{1}{3} \tag{1.12}$$
$$Var(x) = E\left[U^2\right] - E\left[U\right]^2 \tag{1.13}$$
$$\implies Var(x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.14}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat **Solution:**

```
wget https://github.com/AP−51/random−
    numbers/blob/main/code/exrand.c
wget https://github.com/AP−51/random−
    numbers/blob/main/code/coeffs.h
```

Use the below command to run the above codes to generate the gau.dat file

```
gcc exrand.c −lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2 using the code below

```
wget https://github.com/gadepall/
    probability/raw/master/manual/codes/
    gau_cdf_plot.py
```

Properties of this CDF are:

- $F_Z(x) = P(Z \leq x) = 1 - Q(x)$
- $\lim_{x\to\infty} F_Z(x) = 1, \quad \lim_{x\to-\infty} F_Z(x) = 0$
- $F_Z(0) = \frac{1}{2}$
- $F_Z(-x) = 1 - F_Z(x)$

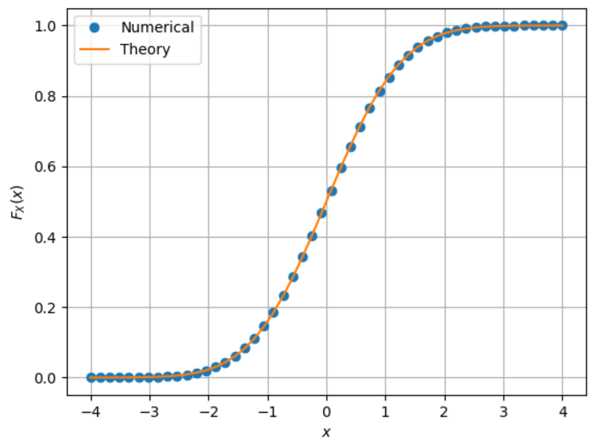

Fig. 2.2: The CDF of X

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
wget https://github.com/AP−51/random−
    numbers/blob/main/code/plots/pdf_plot.py
```

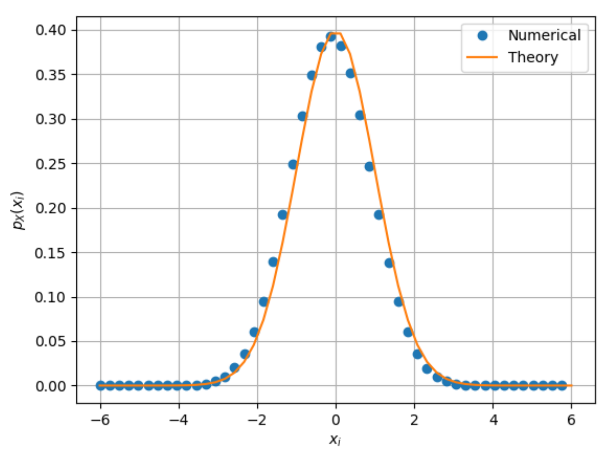

Fig. 2.3: The PDF of X

Properties of this PDF are:

- PDF is symmetric about $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of X by writing a C program. **Solution:**

```
wget https://github.com/AP-51/random-numbers/blob/main/code/gau_mean.c
wget https://github.com/AP-51/random-numbers/blob/main/code/coeffs.h
```

Ues below command to run the file

```
gcc gau_mean.c -lm
.\a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
Solution: Given ,$p_X(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{-x^2}{2}} \quad (2.5)$$

$$\because x e^{-\frac{-x^2}{2}} \text{is a odd function,} \quad (2.6)$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.7)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x(x e^{-\frac{x^2}{2}}) dx \quad (2.8)$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.9)$$

$$I = \int x e^{-\frac{x^2}{2}} \quad (2.10)$$

$$\text{Let} \frac{x^2}{2} = t \quad (2.11)$$

$$\implies x dx = dt \quad (2.12)$$

$$\implies = \int e^{-t} dt = -e^{-t} + c \quad (2.13)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.14)$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.15)$$

$$\text{Also,} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.16)$$

$$\therefore \text{substituting limits we get,} E[x^2] = 1 \quad (2.17)$$

$$Var(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.18)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF. **Solution:** Run the below code to generate samples of V in a file named ln.dat

```
wget https://github.com/AP-51/random-numbers/blob/main/code/ln.c
```

Run the below code to generate ln.dat

```
gcc ln.c -lm
./a.out
```

Now use the below code to plot the CDF

```
wget https://github.com/AP-51/random-numbers/blob/main/code/plots/ln_cdf_plot.py
```

Use the below command to run the plot code

```
python3 ln_cdf_plot.py
```

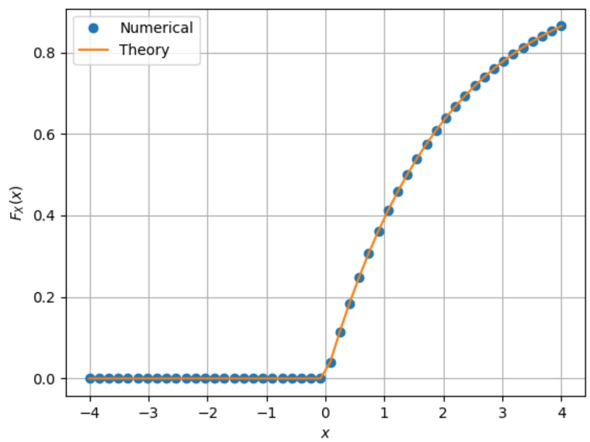

Fig. 3.1: The CDF of $log(1-U)$

3.2 Find a theoretical expression for $F_V(x)$.

$$F_V(x) = P(V \leq x) \tag{3.2}$$
$$= P(-2ln(1-U) \leq x) \tag{3.3}$$
$$= P(1 - e^{\frac{-x}{2}} \geq U) \tag{3.4}$$
$$P(U < x) = \int_0^x dx = x \tag{3.5}$$
$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \tag{3.6}$$

4 Triangular Distribution

4.1 Generate $T = U_1 + U_2$ **Solution:** Download and run the following programs:

```
wget https://github.com/AP−51/random−numbers/blob/main/code/coeffs.h
wget https://github.com/AP−51/random−numbers/blob/main/code/tri.c
```

Use the below command to run the program

```
gcc tri.c −lm
./a.out
```

4.2 Find the CDF of T: **Solution:** We know that

$$P_T(x) = P(U_1 + U_2 = x)$$
$$\implies P_T(x) = \int_{-\infty}^{\infty} P_{U_1}(\tau) P_{U_2}(x-\tau) d\tau \tag{4.1}$$
$$\text{As } P_{U_1} = 1 \ \forall \ x \in (0,1) \text{ and} \tag{4.2}$$
$$P_{U_1} = 0 \ \forall \ x \in (-\infty, 0] \cup [1, \infty) \tag{4.3}$$
$$\therefore P_T(x) = \int_0^1 P_{U_2}(x-\tau) d\tau \tag{4.4}$$
$$\text{Now } P_{U_2}(x-\tau) = 1 \text{ for } 0 < x - \tau < 1 \tag{4.5}$$
$$\implies x - 1 < \tau < x \tag{4.6}$$

If $x < 1, 0 < \tau < x$

$$\int_0^x 1 d\tau = x \tag{4.7}$$

If $x > 1, x - 1 < \tau < 1$

$$\int_{x-1}^1 1 d\tau = 2 - x \tag{4.8}$$

$$\therefore P_T(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \tag{4.9}$$

On Integrating $P_T(x)$ we get $F_T(x)$

$$F_T(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 < x < 1 \\ 1 - \frac{(2-x)^2}{2}, & 1 < x < 2 \\ 1, & x > 2 \end{cases} \tag{4.10}$$

4.3 Find the PDF of T:
Solution:

$$P_T(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \tag{4.11}$$

4.4 Find Theoretical expressions for CDF and PDF:
Solution: CDF:

$$F_T(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 < x < 1 \\ 1 - \frac{(2-x)^2}{2}, & 1 < x < 2 \\ 1, & x > 2 \end{cases} \tag{4.12}$$

PDF:

$$P_T(x) = \begin{cases} 0, & x < 0 \\ x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases} \tag{4.13}$$

4.5 Verify your results through a plot: **Solution:** Use the below code to plot CDF and PDF

```
wget https://github.com/AP-51/random-numbers/blob/main/code/plots/tri_cdf_plot.py
wget https://github.com/AP-51/random-numbers/blob/main/code/plots/tri_pdf_plot.py
```

Use the following command to run the code

```
python3 tri_cdf_plot.py
python3 tri_pdf_plot.py
```

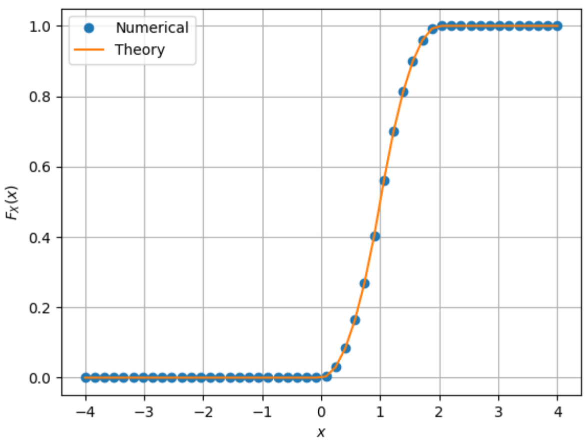

Fig. 4.5: The CDF of $T = U_1 + U_2$

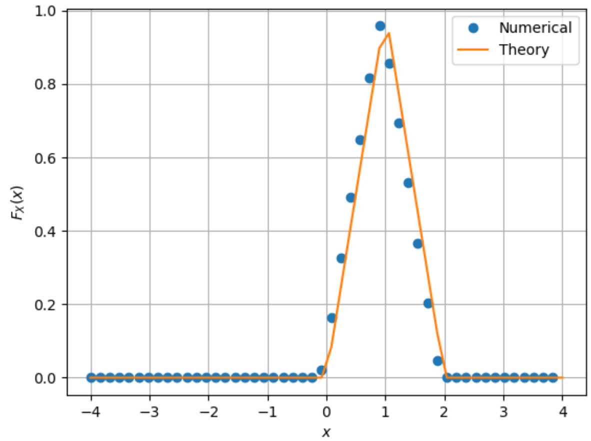

Fig. 4.5: The PDF of $T = U_1 + U_2$