

# Random Numbers

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### 1 UNIFORM RANDOM NUMBERS

Let  $U$  be a uniform random variable between 0 and 1.

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat **Solution:** Download the following files and execute the C program.

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/exrand.c
wget https://github.com/AP-51/random-
numbers/blob/main/code/coeffs.h
```

Use the below command in the terminal to run the code.

```
gcc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

1.3

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** The following code plots Fig. 1.3

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/plots/cdf_plot.py
```

- 1.4 **Solution:** As  $U$  is a Uniform Random Variable Distribution, CDF of  $P_U x = F_U x$

$$P_U x = 1 \quad F_U x = \int_0^1 P_U x dx \therefore F_U x = x \quad (1.2)$$

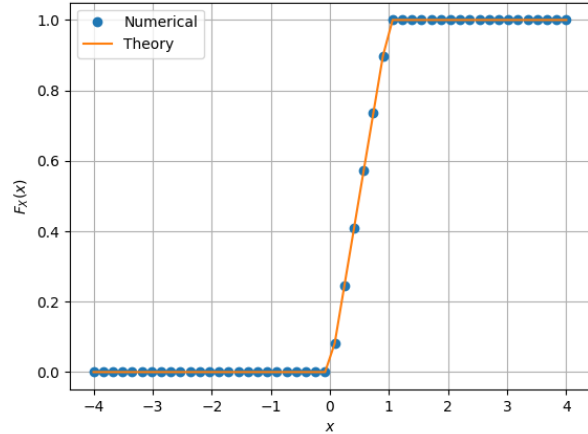


Fig. 1.3: The CDF of  $U$

- 1.5 The mean of  $U$  is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.3)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.4)$$

Write a C program to find the mean and variance of  $U$ . **Solution:**

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/mean.c
wget https://github.com/AP-51/random-
numbers/blob/main/code/coeffs.h
```

Use below command to run the code

```
gcc mean.c -lm
.\a.out
```

From this code we get Mean=0.500007 and Variance=0.083301

- 1.6 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.5)$$

**Solution:**

$$dF_U x = dx \quad (1.6)$$

$$\therefore E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

$$E[U] = \int_0^1 x dx \quad (1.8)$$

$$\text{as } P_U x = 0 \quad \forall x \in (-\infty, 0) \cup (1, \infty) \quad (1.9)$$

$$\therefore E[U] = \frac{1}{2} \quad (1.10)$$

$$E[U^2] = \int_0^1 x^2 dx \quad (1.11)$$

$$\therefore E[U^2] = \frac{1}{3} \quad (1.12)$$

$$\text{Var}(x) = E[U^2] - E[U]^2 \quad (1.13)$$

$$\Rightarrow \text{Var}(x) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \quad (1.14)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate  $10^6$  samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where  $U_i, i = 1, 2, \dots, 12$  are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat **Solution:**

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/exrand.c
wget https://github.com/AP-51/random-
numbers/blob/main/code/coeffs.h
```

Use the below command to run the above codes to generate the gau.dat file

```
gcc exrand.c -lm
.\a.out
```

2.2 Load gau.dat in python and plot the empirical CDF of  $X$  using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of  $X$  is plotted in Fig. 2.2 using the code below

```
wget https://github.com/gadepall/
probability/raw/master/manual/codes/
gau_cdf_plot.py
```

Properties of this CDF are:

- $\Phi(x) = P(Z \leq x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp\left\{-\frac{u^2}{2}\right\} du$
- $\lim_{x \rightarrow \infty} \Phi(x) = 1, \quad \lim_{x \rightarrow -\infty} \Phi(x) = 0$
- $\Phi(0) = \frac{1}{2}$
- $\Phi(-x) = 1 - \Phi(x)$

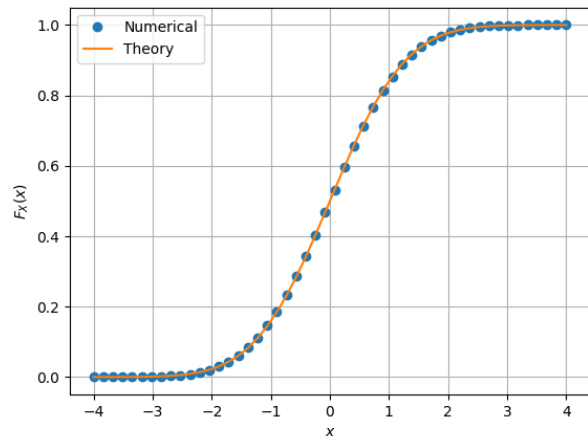


Fig. 2.2: The CDF of  $X$

2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

**Solution:** The PDF of  $X$  is plotted in Fig. 2.3 using the code below

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/plots/pdf_plot.py
```

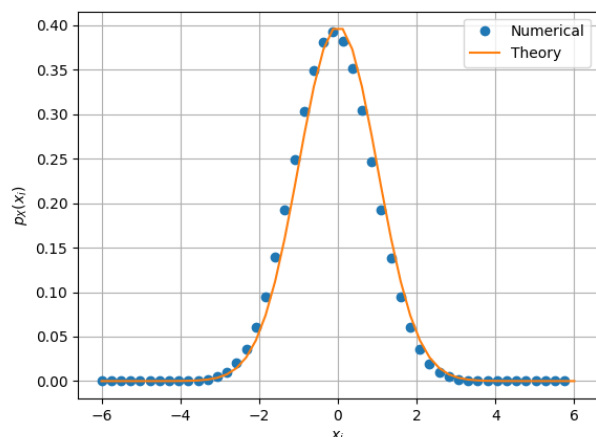


Fig. 2.3: The PDF of  $X$

Properties of this PDF are:

- PDF is symmetric about  $x = 0$
- graph is bell shaped
- mean of graph is situated at the apex point of the bell

2.4 Find the mean and variance of  $X$  by writing a C program. **Solution:**

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/gau_mean.c
wget https://github.com/AP-51/random-
numbers/blob/main/code/coeffs.h
```

Use below command to run the file

```
gcc gau_mean.c -lm
./a.out
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.

**Solution:** Given  $p_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x e^{-\frac{x^2}{2}} \quad (2.5)$$

$$\because x e^{-\frac{x^2}{2}} \text{ is a odd function,} \quad (2.6)$$

$$E[x] = 0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.7)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} x (x e^{-\frac{x^2}{2}}) dx \quad (2.8)$$

Using integration by parts:

$$= x \int x e^{-\frac{x^2}{2}} dx - \int \frac{d(x)}{dx} \int x e^{-\frac{x^2}{2}} dx \quad (2.9)$$

$$I = \int x e^{-\frac{x^2}{2}} \quad (2.10)$$

$$\text{Let } \frac{x^2}{2} = t \quad (2.11)$$

$$\Rightarrow x dx = dt \quad (2.12)$$

$$\Rightarrow \int e^{-t} dt = -e^{-t} + c \quad (2.13)$$

$$\therefore \int x e^{-\frac{x^2}{2}} = -e^{-\frac{x^2}{2}} + c \quad (2.14)$$

Using (2.14) in (2.9)

$$= -x e^{-\frac{x^2}{2}} + \int e^{-\frac{x^2}{2}} dx \quad (2.15)$$

$$\text{Also, } \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi} \quad (2.16)$$

$$\therefore \text{ substituting limits we get, } E[x^2] = 1 \quad (2.17)$$

$$\text{Var}(X) = E[x^2] - (E[x])^2 = 1 - 0 \quad (2.18)$$

### 3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF. **Solution:** Run the below code to generate samples of  $V$  in a file named `ln.dat`

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/ln.c
```

Run the below code to generate `ln.dat`

```
gcc ln.c -lm
./a.out
```

Now use the below code to plot the CDF

```
wget https://github.com/AP-51/random-
numbers/blob/main/code/plots/
ln_cdf_plot.py
```

Use the below command to run the plot code

```
python3 ln_cdf_plot.py
```

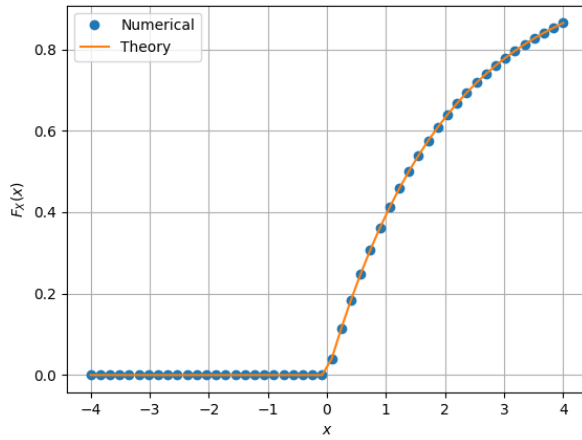


Fig. 3.1: The CDF of  $\log(1 - U)$

3.2 Find a theoretical expression for  $F_V(x)$ .

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2\ln(1 - U) \leq x) \quad (3.3)$$

$$= P(1 - e^{\frac{-x}{2}} \geq U) \quad (3.4)$$

$$P(U < x) = \int_0^x dx = x \quad (3.5)$$

$$\therefore P(1 - e^{\frac{-x}{2}} \geq U) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0 \quad (3.6)$$