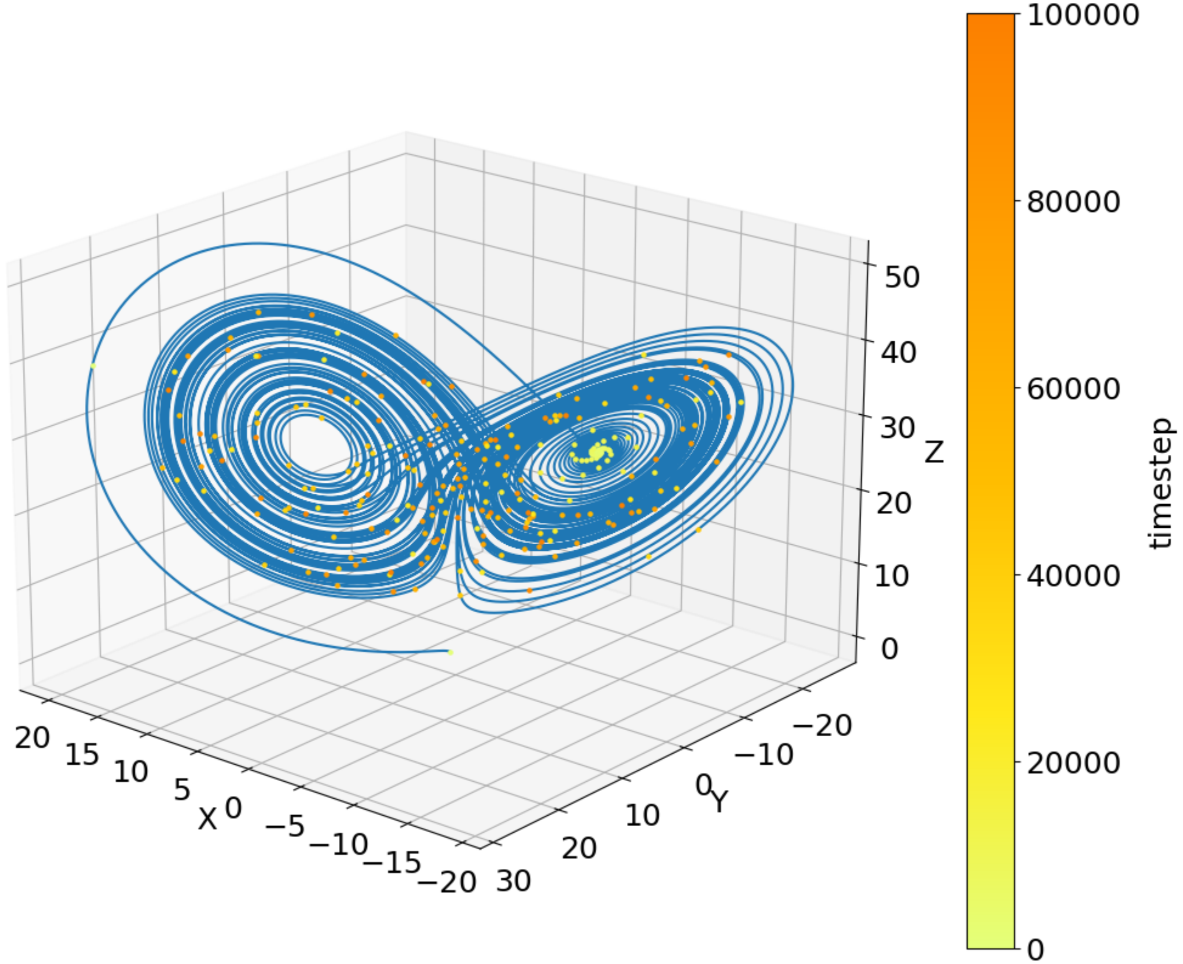
Final Project

The Lorenz Model and Numerical Methods



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1. Introduction

- Chaos is one of the important features of nonlinear systems
- Definition of chaos:

  Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions (Strogatz, 2015)
- Features of chaos: sensitive to initial condition / strange attractor
- Sensitive to initial condition
  - ✓In a chaotic system, small differences in initial conditions would be amplified as the system develops
  - ✓Whatever small the difference is, it would be magnified, and they have different states after sufficient time
- Strange attractor
  - ✓The trajectory of a chaotic system has a strange attractor, different from the point attractor/limit cycle (see appendix slide)

1. Introduction

- Lorenz model, proposed by Lorenz (1963), is one of the most famous chaotic systems
- It is a simplified model of a convective atmosphere
- Use the Lorenz model as an example of a chaotic system
- Goals of this project/questions to be answered
    - ➢Investigate the behavior of the Lorentz system and observe features of the chaotic system and its trajectory
    - ➢Investigate the time development of the Lorenz system with different numerical schemes. Do they have completely different results or similar to each other?
    - ➢How the system is sensitive to initial conditions and parameters?

2. Lorenz model

- The Lorenz model, proposed by Lorenz (1963)
- x, y, z: 3-dimensional variables
- σ, r, b: parameters: all positive
  - ➢σ: Prandtl number, kinematic viscosity/thermal conductivity
  - ➢r: Ratio of Rayleigh number/Reynolds number
- First, consider 25<r which is critical value
- Consider initial condition (x,y,z)=(0,1,0)

$$\begin{cases} \dfrac{dx}{dt} = \sigma(y-x) \\ \dfrac{dy}{dt} = rx - y - xz \\ \dfrac{dz}{dt} = xy - bz \end{cases}$$

2. Lorenz model: features

- Lorenz system is dissipative
  - Phase space volume decreases exponentially
  - Volume in phase space contract to 2D plane or 1D line or point
- Fixed points of the Lorenz system are surrounded by unstable limit circles
  - Trajectories fly away from them
  - Trajectories do not cross each other (never repeat the previous status)
  - Also, trajectories always stay in a finite region
- The above conditions look contradict each other:
  - Trajectories fly away from the limit circle but still stay in a finite region
  - Phase space volume decreases but infinitely transits to new status
- Strange attractor satisfies all these conditions (and that’s why it’s interesting!)

3. Numerical Methods

- Three methods were used
  - Predictor-Corrector
  - Runge-Kutta (RK4)
  - Backward Differentiation Formulas (BDF)

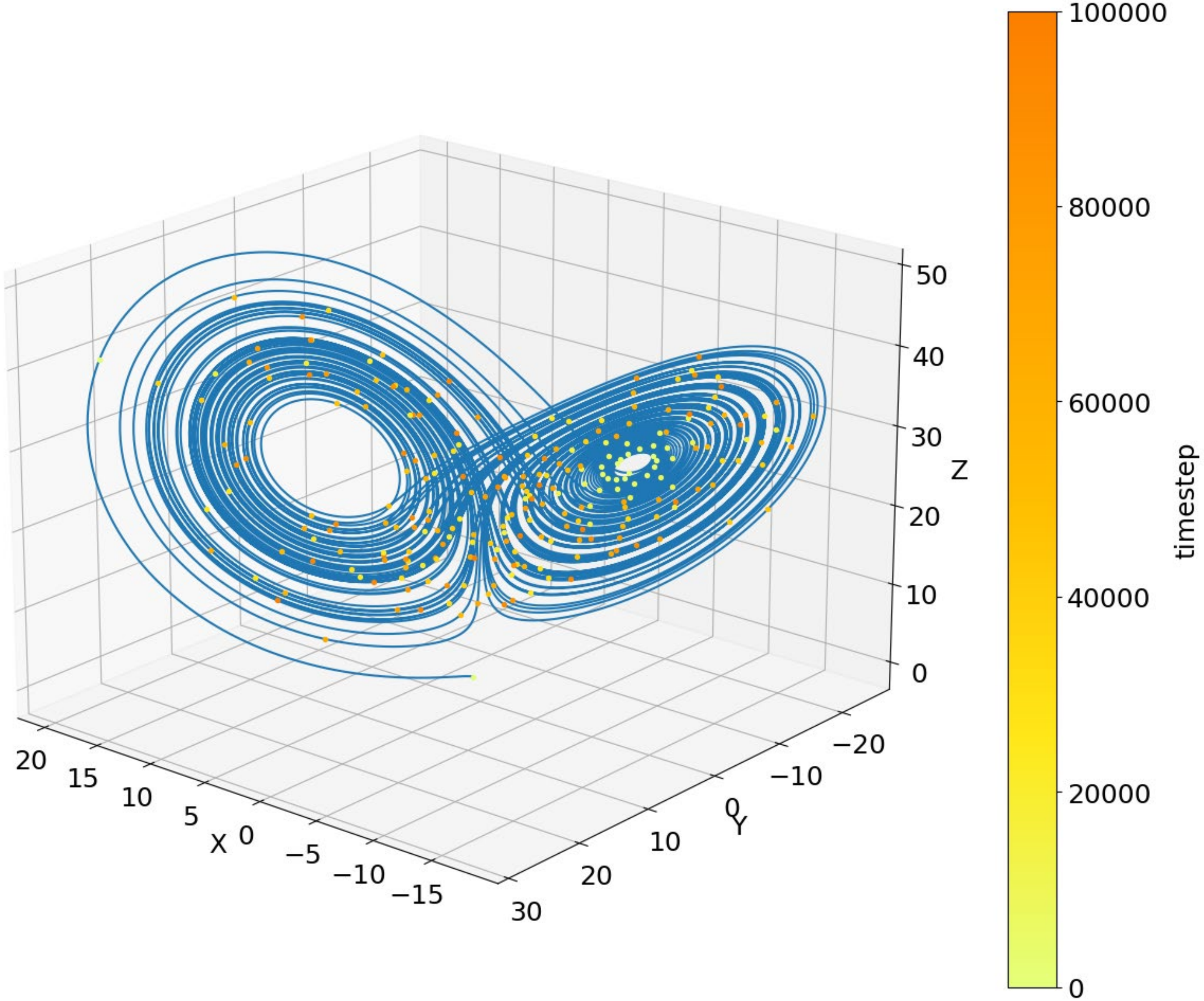
- For BDF, the Jacobi matrix with the initial condition was weak stiff
  - The ratio of eigenvalues

$$\frac{|\lambda_{max}|}{|\lambda_{min}|} \sim 8.58$$

- $\Delta \mathrm{t} = 0.001$ and total integration time=100000 was use (See Appendix B)
- Since the Lorenz system is dissipative, we cannot use Symplectic Integrators

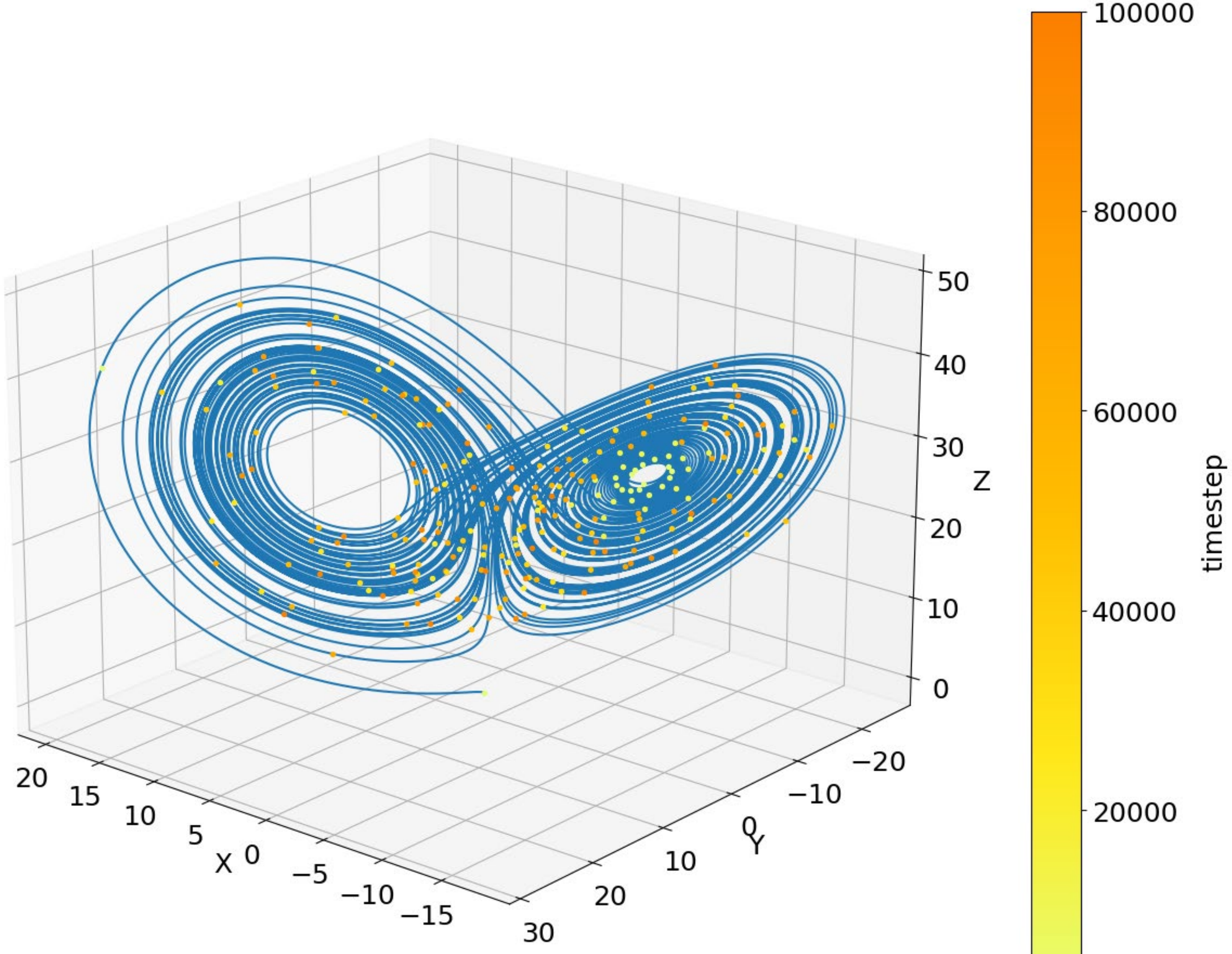
4. Results: Overview

- Result from RK4
- Δt=0.001
- 2 flat disks exist in XYZ space
- At first, round in a small region (see the right disk)
- Chaotic behavior starts from around 16000 steps
- Then, jump to another disk
- Trajectory contentiously jumps between the two disks after sufficient time



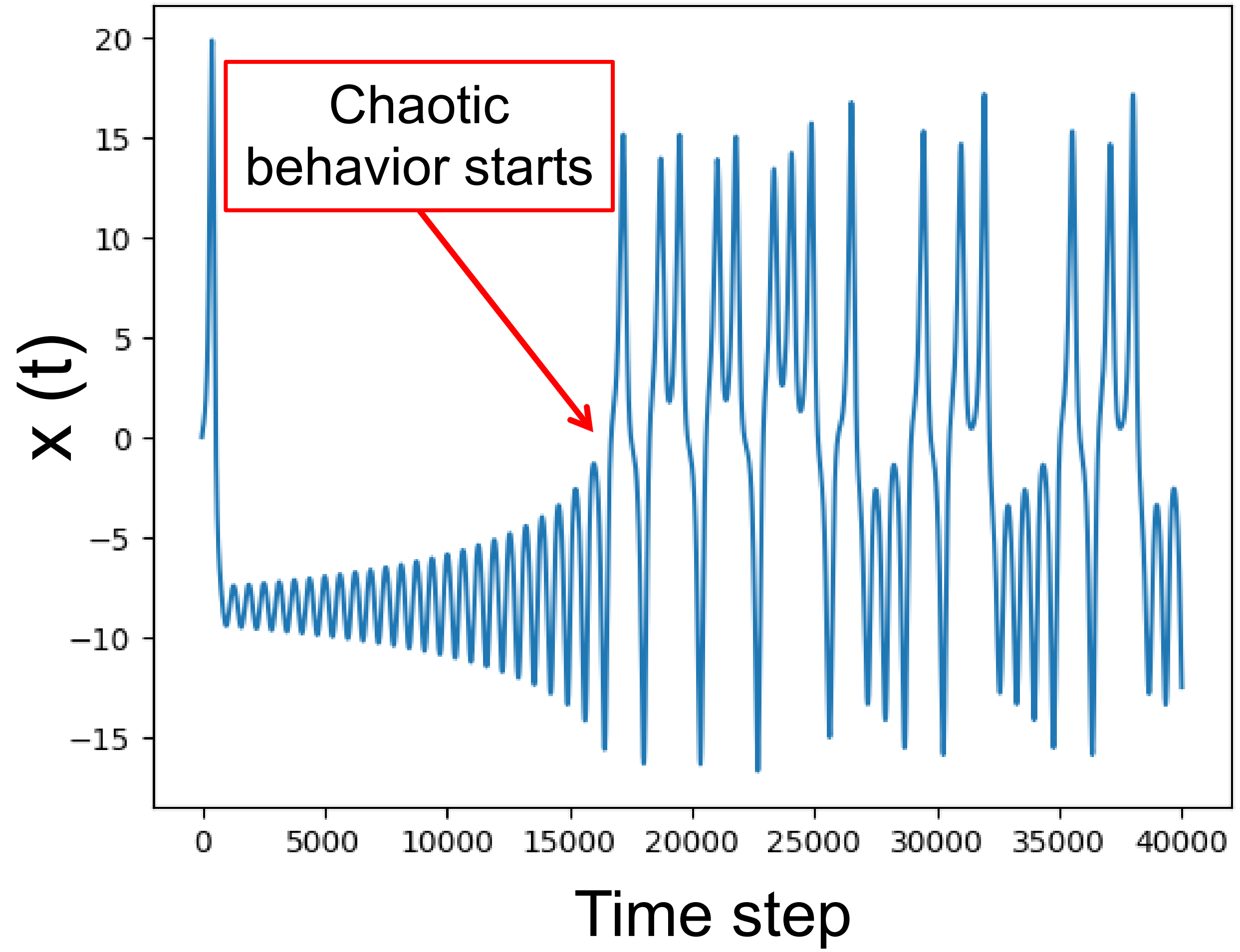
4. Results: Overview

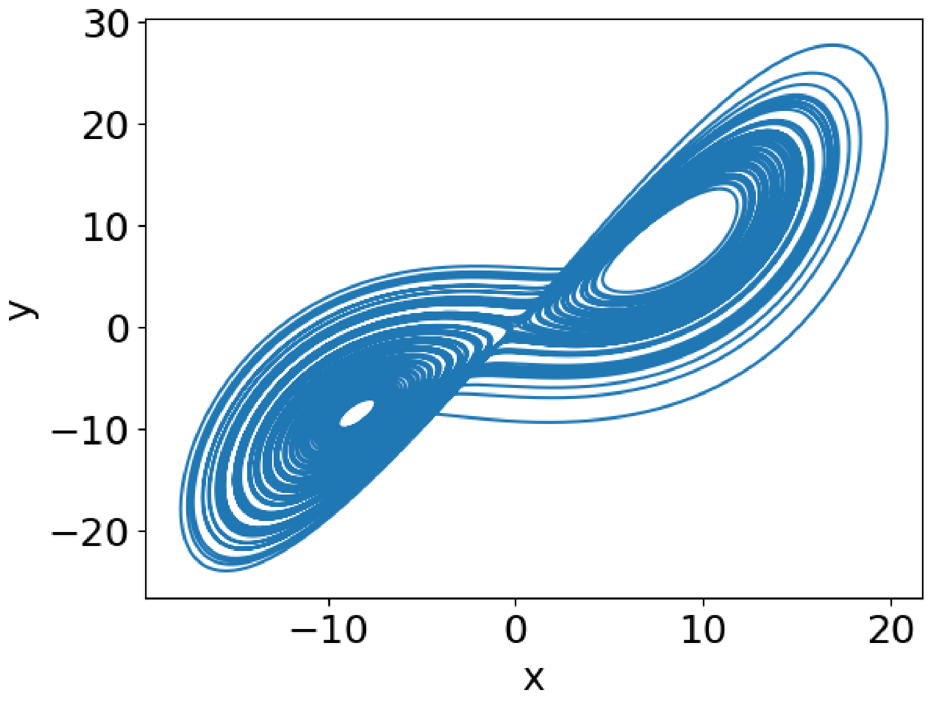
- This attractor satisfies the restriction we saw
  - ➢ <u>Volume construction</u>: trajectory is folded into 2-dimensional disks
  - ➢ <u>Unstable limit circle</u>: there are limit circles at the center of the disk, but they are unstable (apart from them)
  - ➢ <u>Stay in finite volume</u>: the starting point is the edge of the disk, and all trajectories stay inside of it

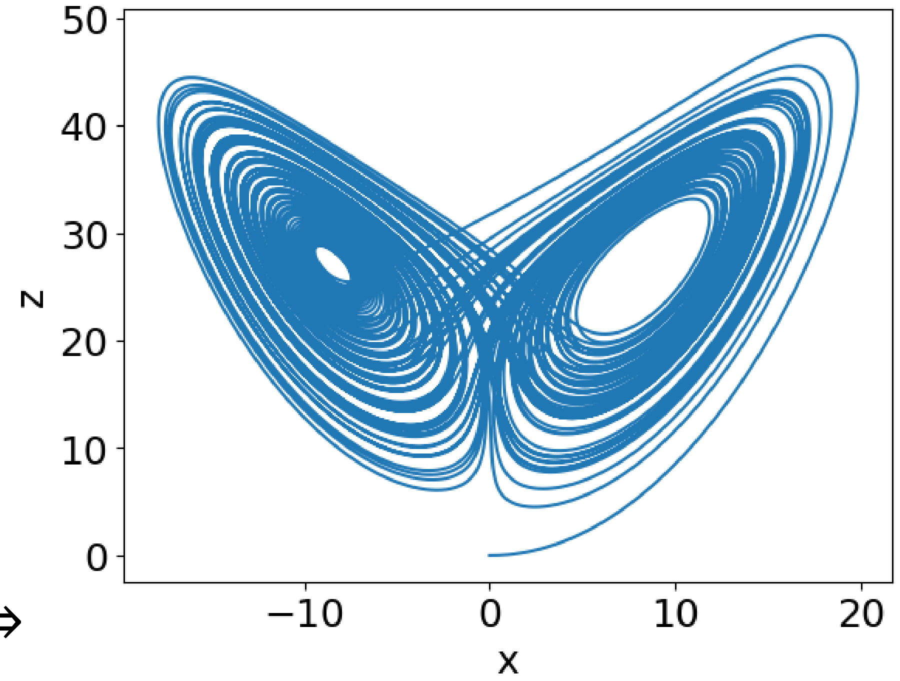


4. Results: Overview

Time development of strange attractor, x(t)



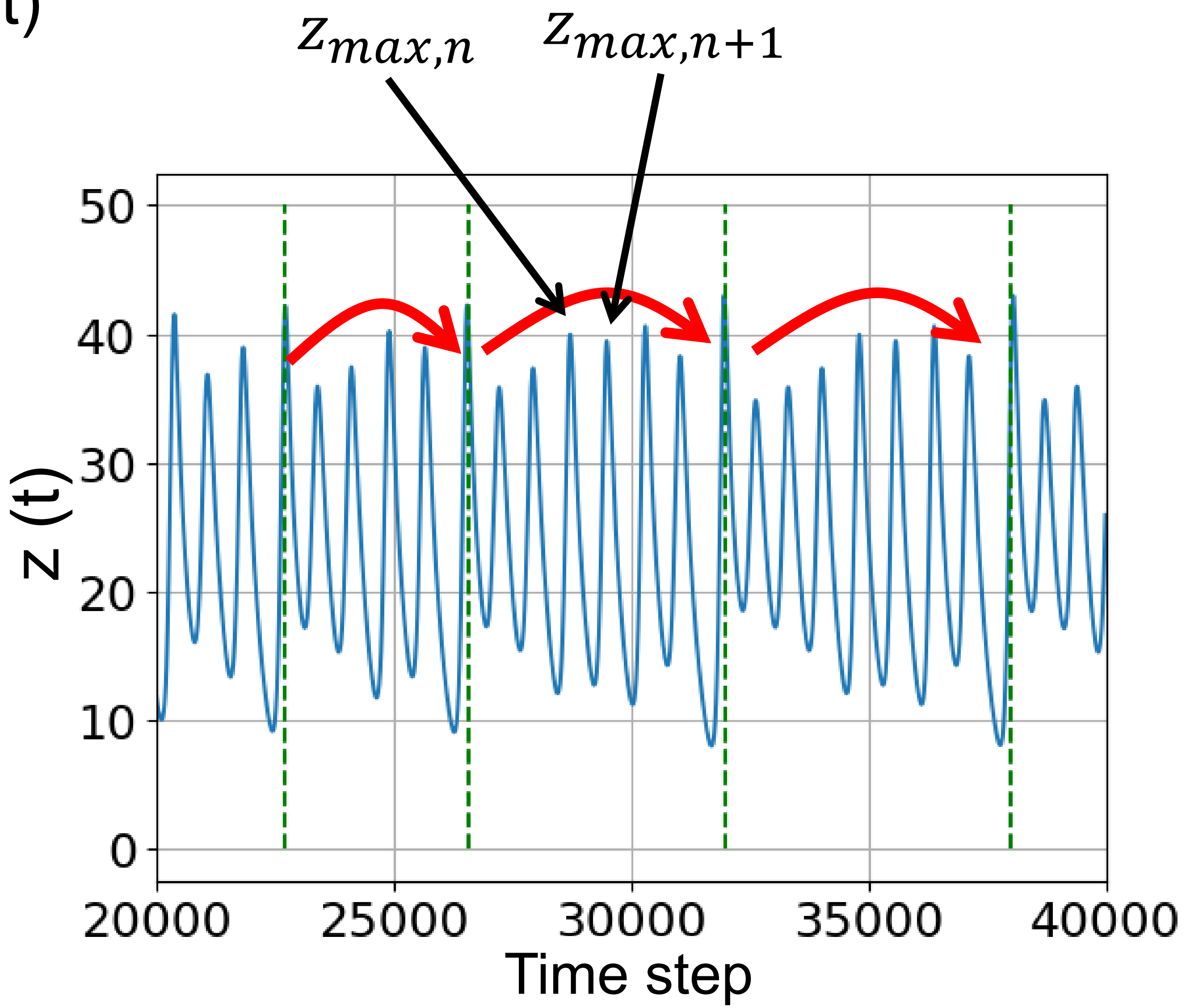




Projection of strange attractor to x-y, x-z plane⇒

4. Results: Lorenz mapping

- There is a pattern on peaks of z(t)
- Next to the largest peak is relatively small
  (see the red arrows)
  - Then increase
  - Decrease
  - Next largest peak
- Plot $z_{max,n}$ versus $z_{max,n+1}$
  - $z_{max,n}$: *n*-the peak of z(t)
  - $z_{max,n+1}$: the next peak of $z_{max,n}$
  - This plot is called Lorenz mapping

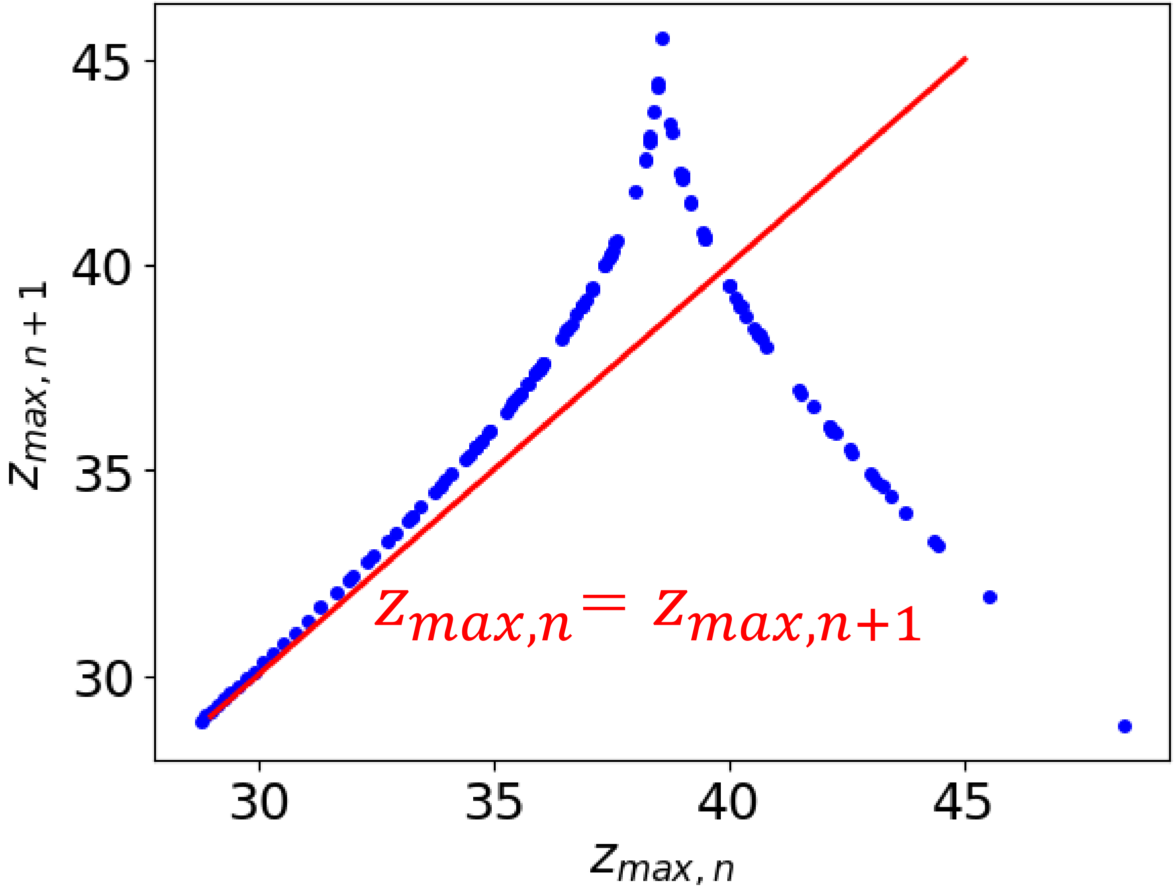


4. Results: Lorenz mapping

- Lorenz mapping:
  - ➢ Points fall into a line
  - ➢ Indicates there is a coherent structure
- At all points, the slope is always steeper than 1, i.e.

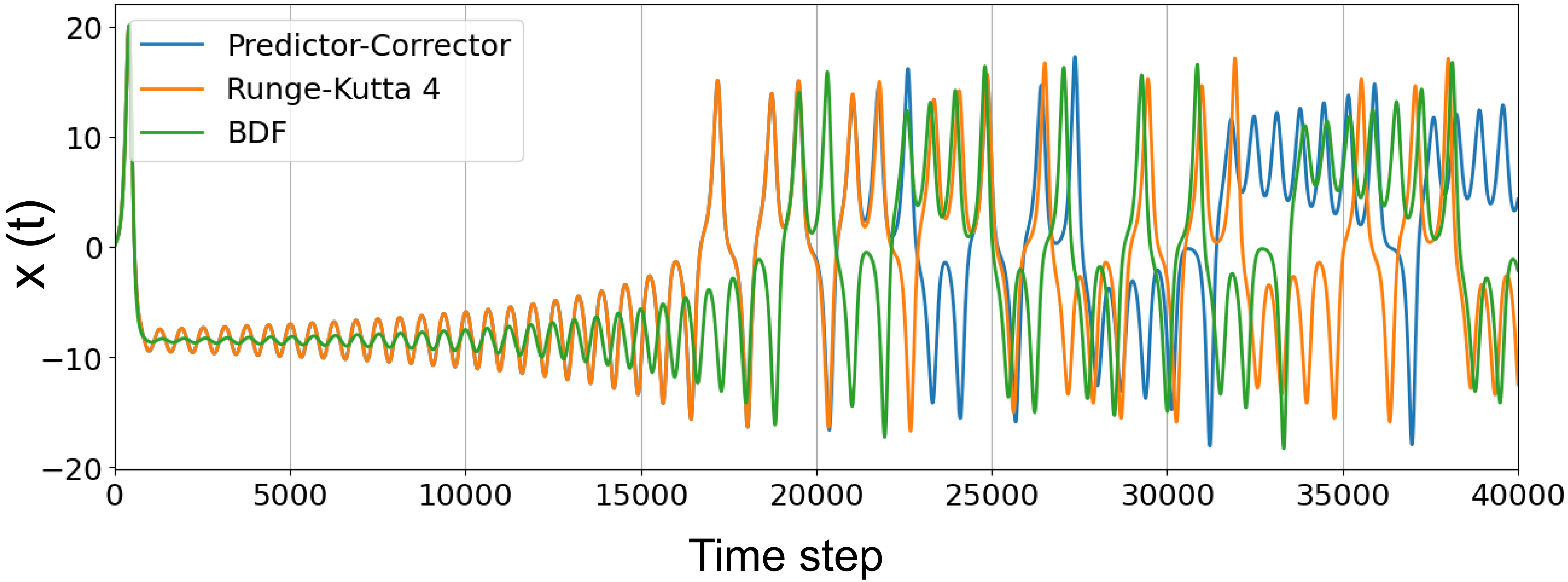
$$|f'(z_{max,n})| > 1$$

- This is used to prove that the fixed point of the system unstable
  - ➢ Small perturbation η, near fixed points, is amplitude
  - ➢ $\eta_{n+1} = |f'(z_{max,n})|\eta_n$
    $\Rightarrow |\eta_{n+1}| > |\eta_n|$

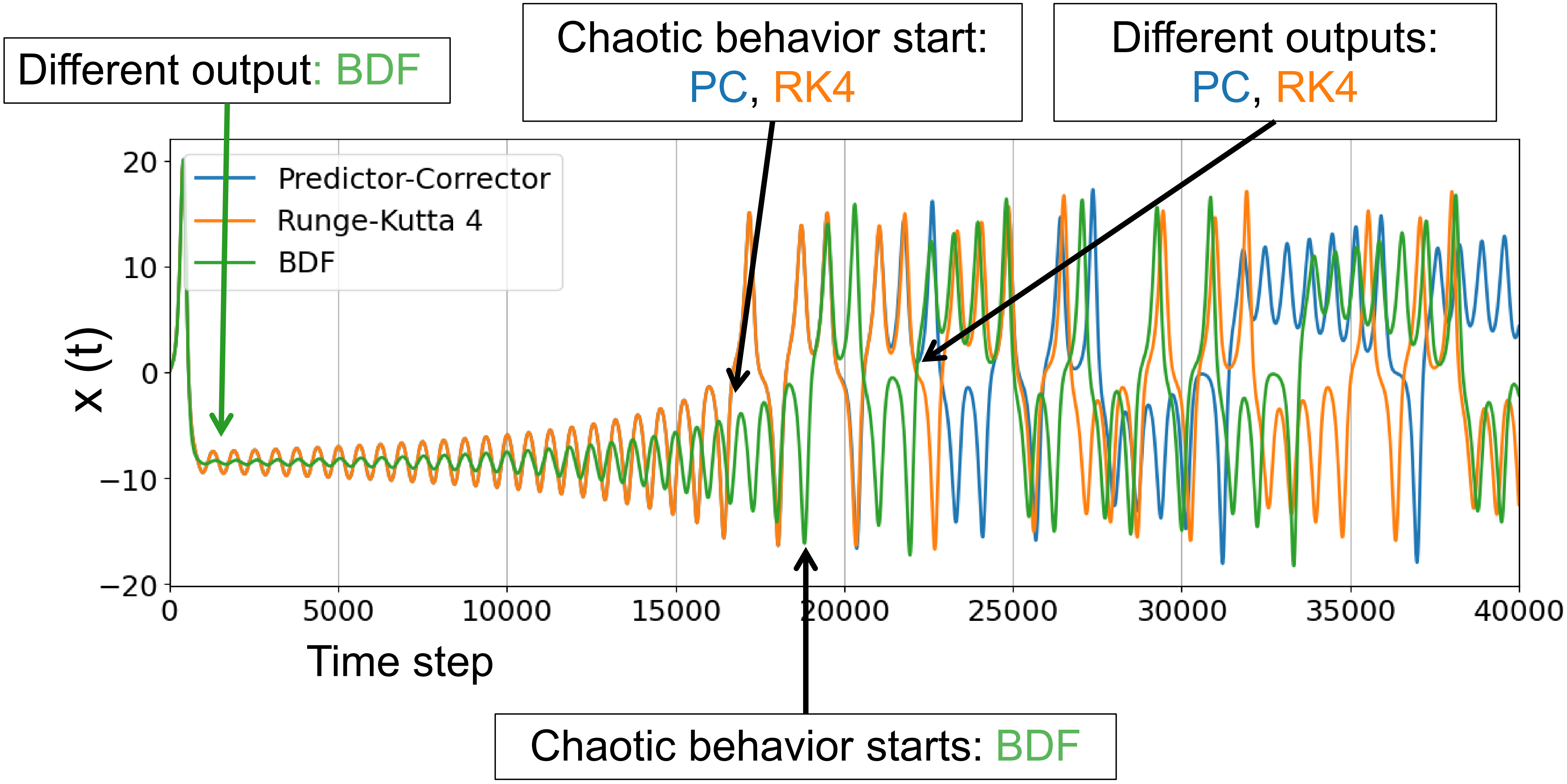


4. Results: Comparison of numerical methods

- Starting point of chaotic behavior
  - ➢Predictor-Corrector (PC) and RK4 start chaotic behavior earlier than BDF
  - ➢BDF shows different output since the early steps
  - ➢PC and RK4 shows the different outputs from around t~22500 timesteps

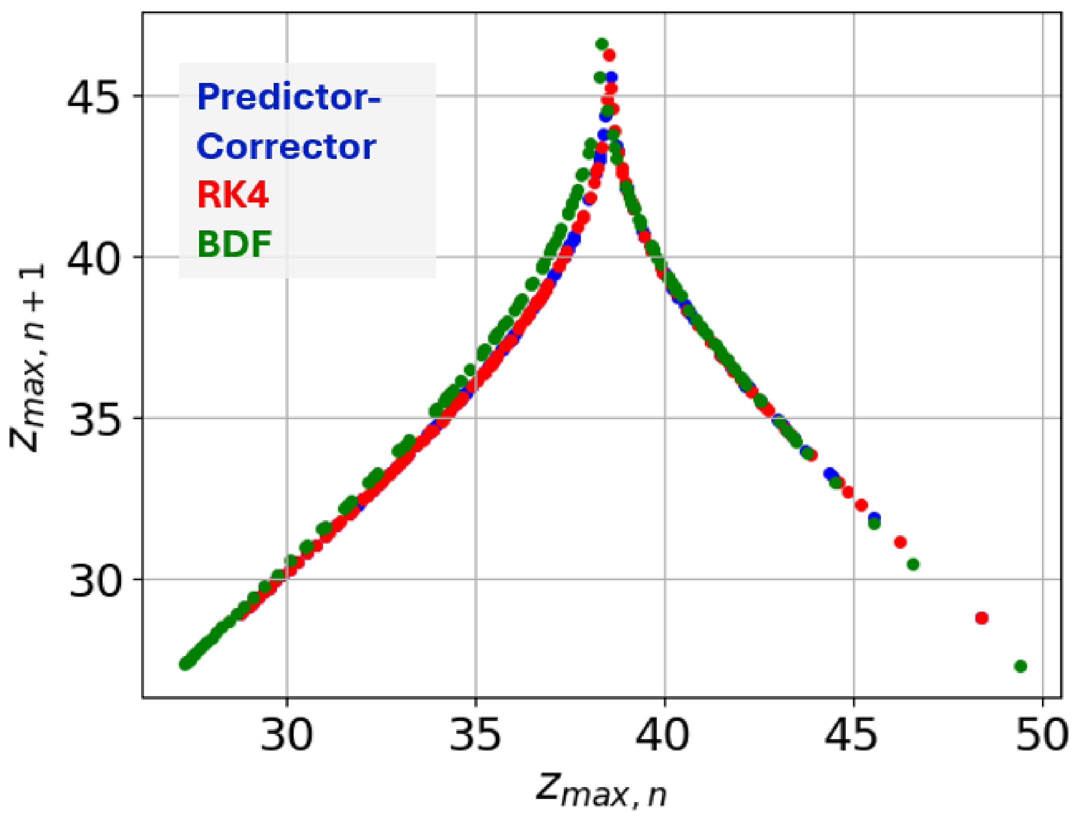


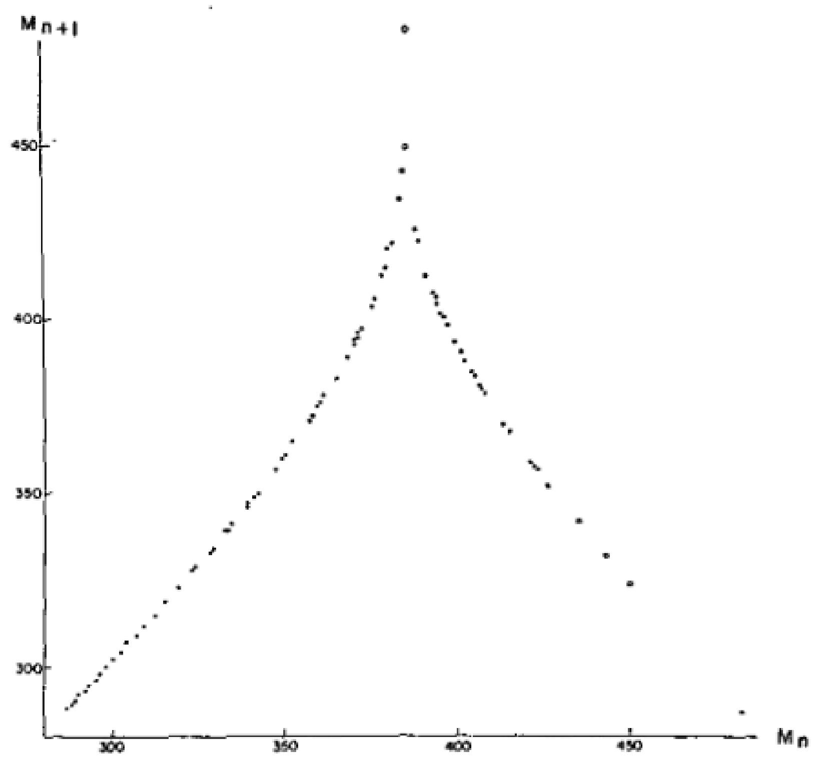
4. Results: Comparison of numerical methods



4. Results: Comparison of numerical methods

- Three methods shows almost the same Lorenz map
  - ➢Predictor-Corrector (PC) and RK4 gave almost the same output
  - ➢BDF method has the slightly smaller peak, but overall shape is identical to others
  - ➢The shape is the same as Lorenz (1963)





Lorenz (1963)

4. Results: Sensitivity on initial conditions

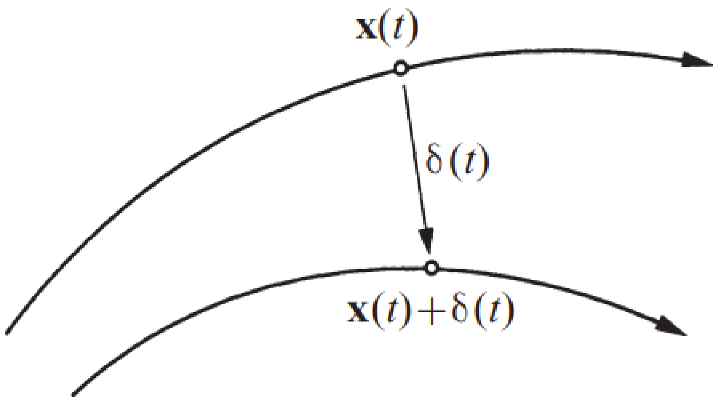
- The motion on the attractor exhibits sensitive dependence on initial condition

- How rapidly points apart from each other? Use Liapunov exponent
  - Consider two points $x(t)$ and $x(t)+\delta(t)$ initially close to each other
  - $\delta(t)$ is the distance of two point (see the figure)
  - $\delta(t)$ exponential increase, then we expect to have Liapunov exponent λ with

$$\delta(t) \sim \delta(0)e^{\lambda t}$$

  - $\delta(0)$ is the initial distance, say 10^{-9}

- Literature says, $\lambda \sim 0.9$

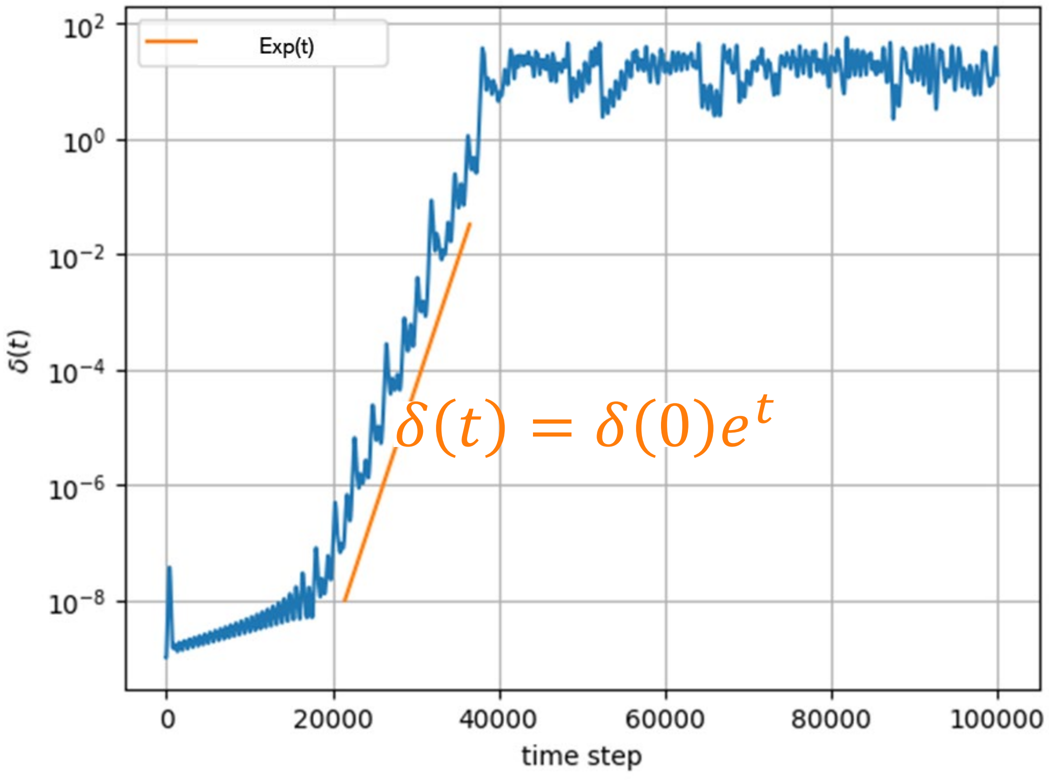


Strogatz (2015)

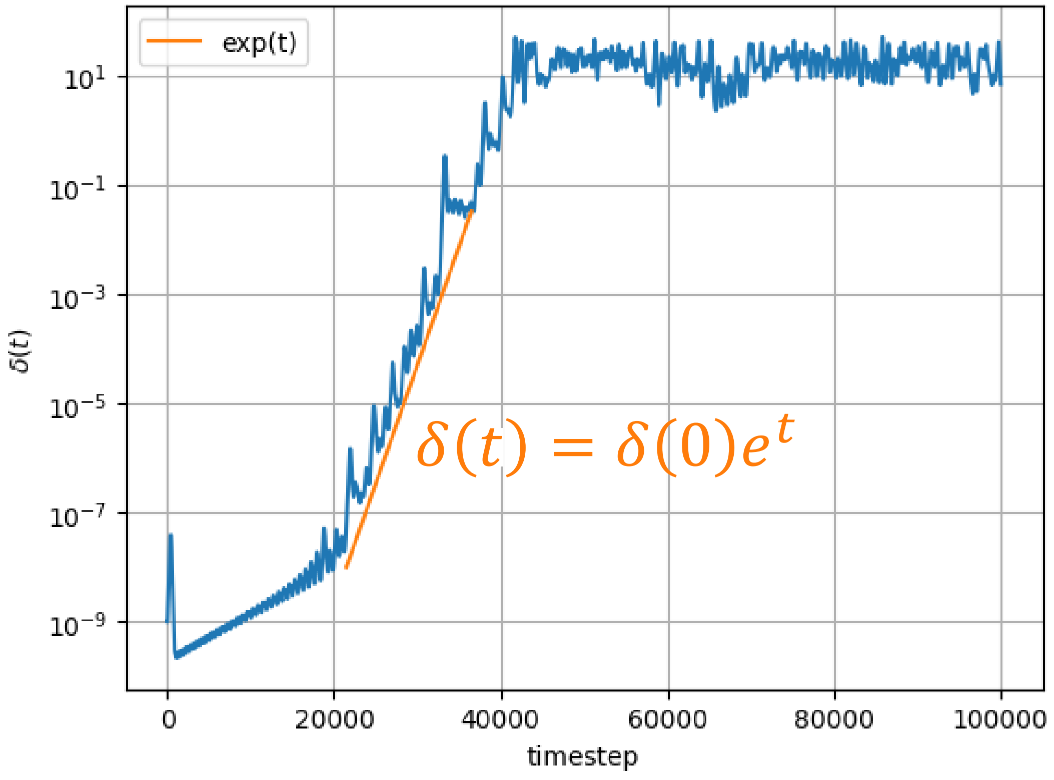
4. Results: Sensitivity on initial conditions

- Both RK4 and BDF method shows $\lambda \sim 1$, close to literatures
- $\delta(t)$ converge and constant, because the trajectory stay in a finite volume

$\delta(t)$



RK4

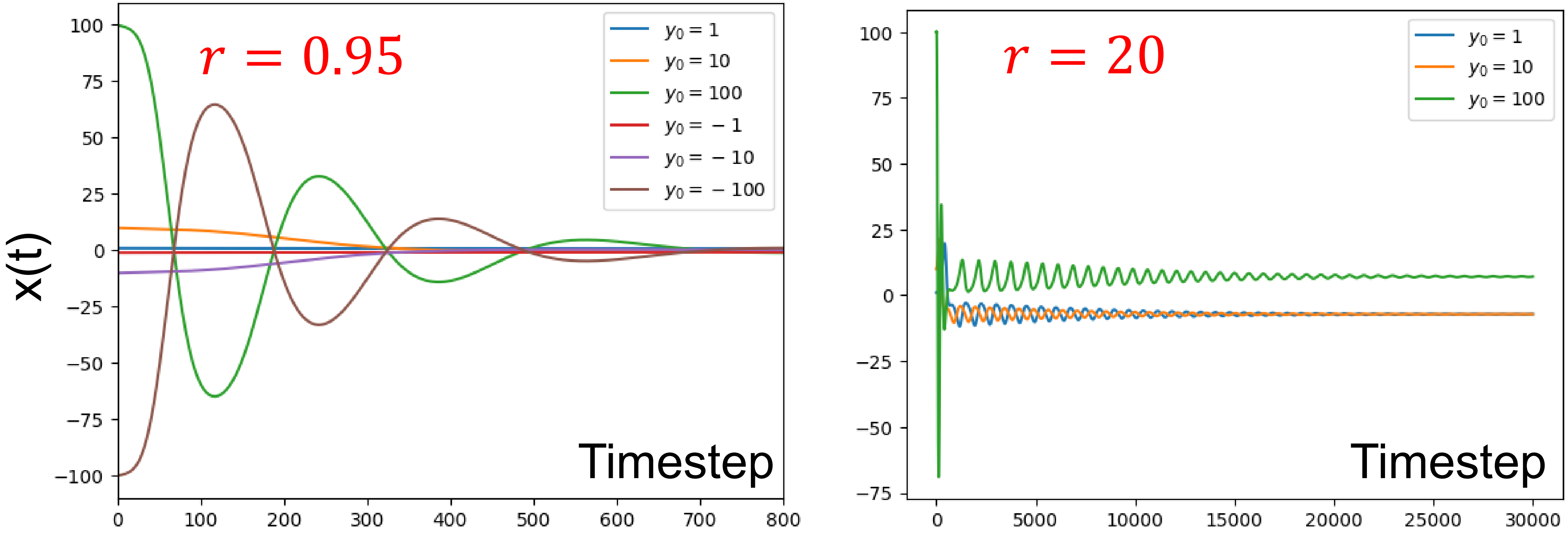


BDF

5. Parameter dependence

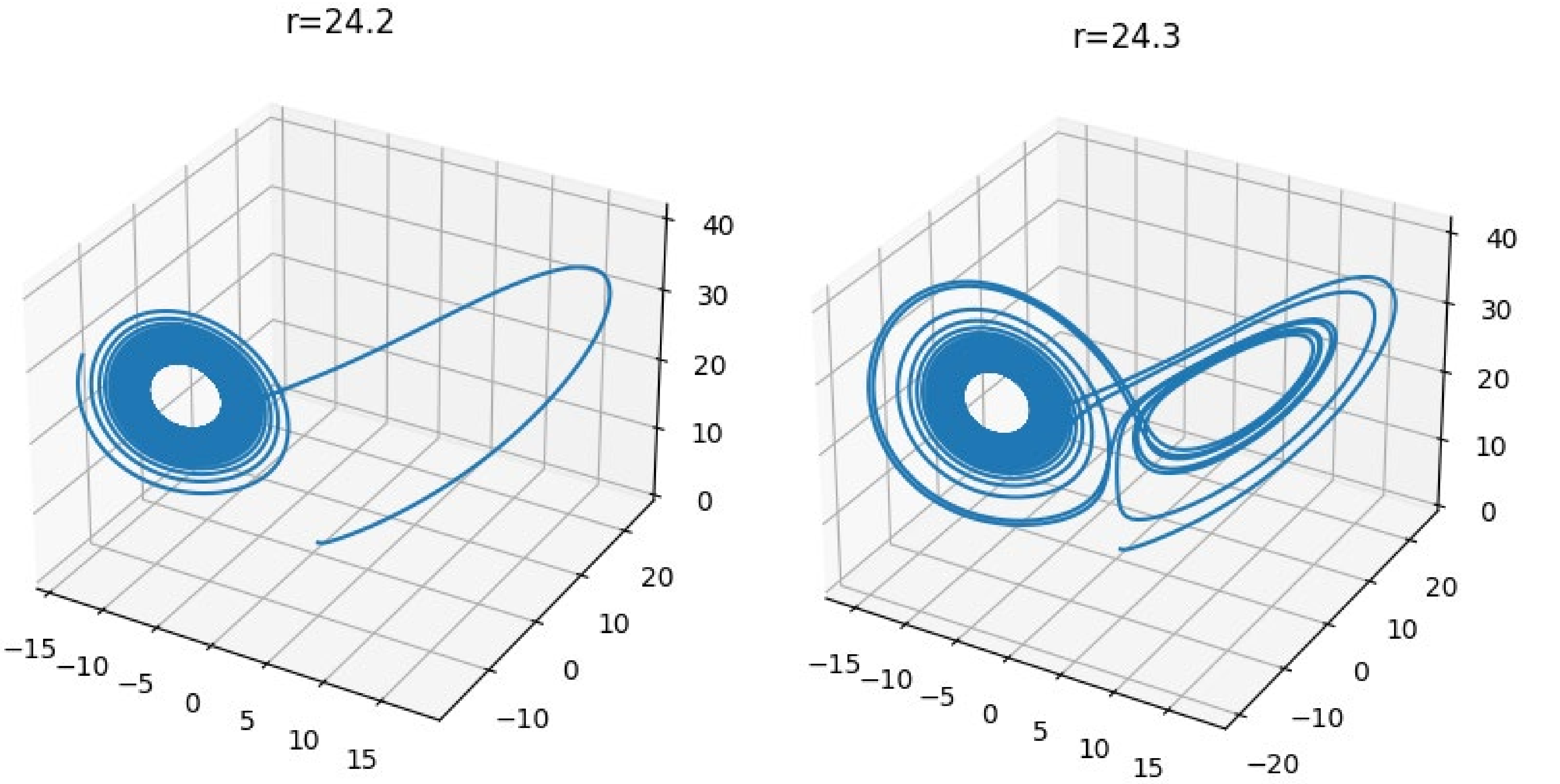
$$\begin{cases} \dfrac{dx}{dt} = \sigma(y - x) \\ \dfrac{dy}{dt} = rx - y - xz \\ \dfrac{dz}{dt} = xy - bz \end{cases}$$

- Change the parameter r
- When $r < 1$
  - ➢ Trajectories fall into a point (point attractor)
- When $1 < r < r_H$ (r_H critical value for behavior, ~25)
  - ➢ Two point attractors
  - ➢ Depends on the initial condition, trajectories fall into either two points



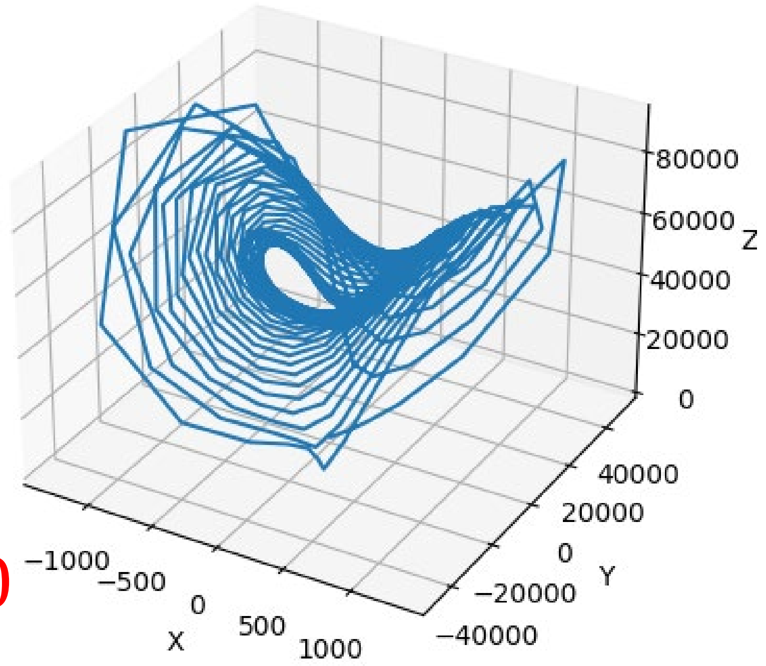
5. Parameter dependence

- When $r \approx< r_H$
  - ➢Transition from point attractor to strange attractor was observed around $r \sim 24.2$

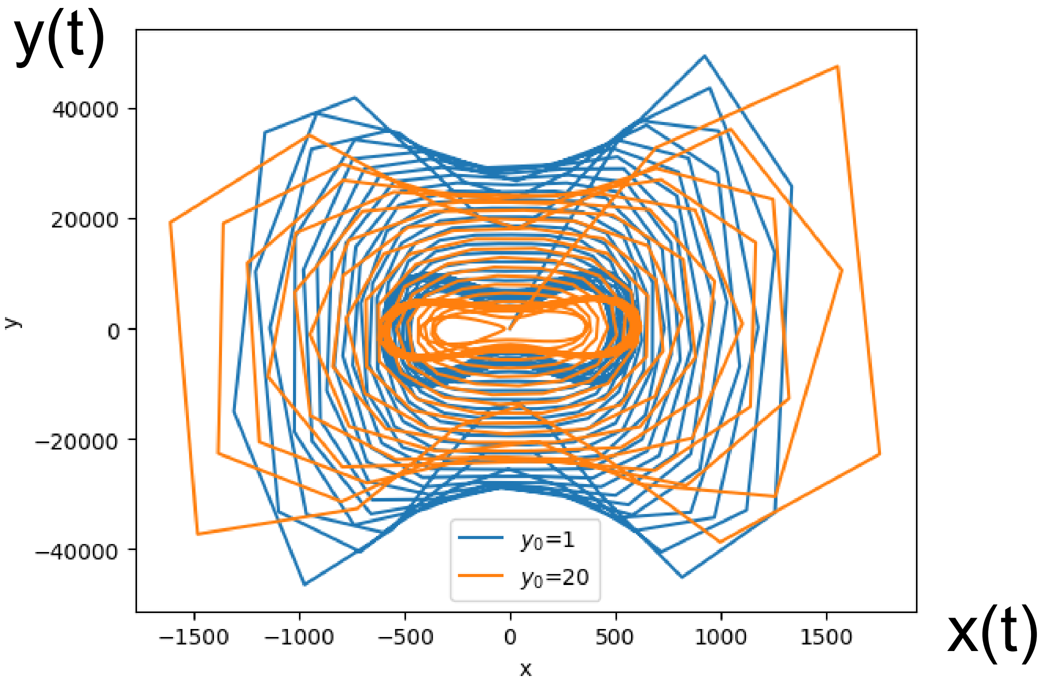


5. Parameter dependence

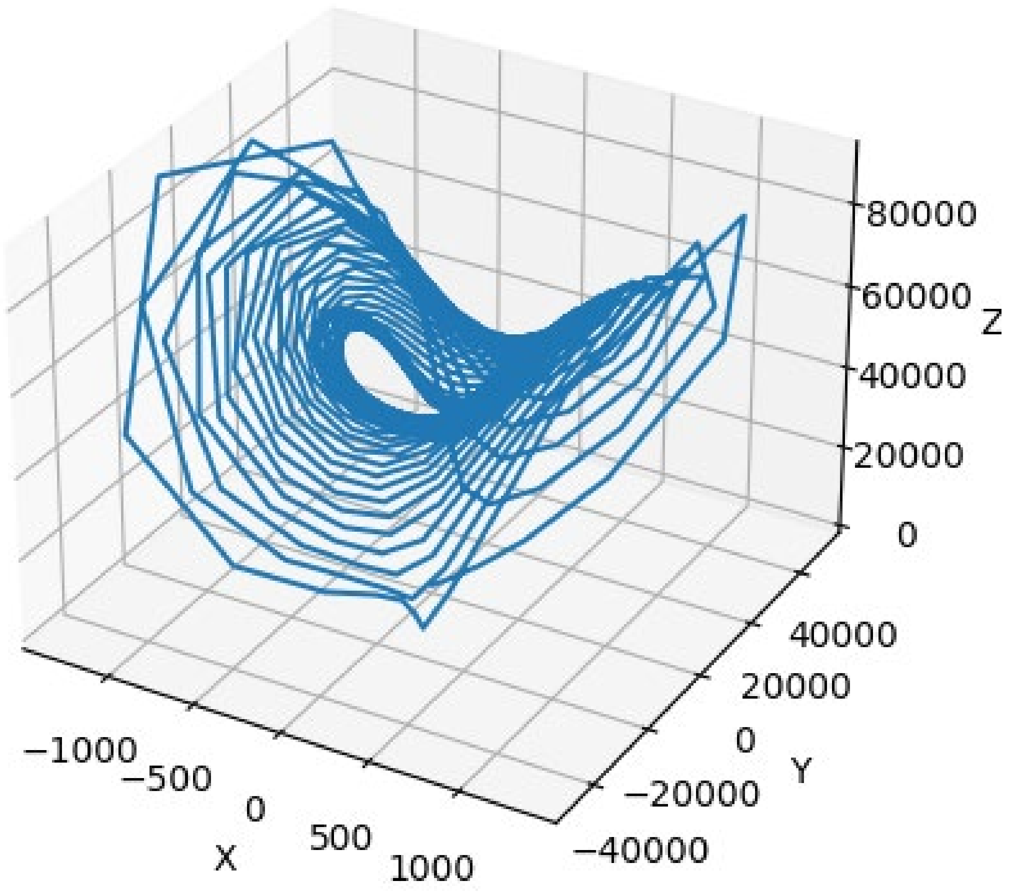
- When $r_H \ll r$
  - ➢Saddle shape trajectory was observed
  - ➢Plotting x(t)-y(t), there is limit circle-like attractor
  - ➢Trajectory falls to the circle after sufficient time
  - ➢However, the circle shape depends on initial condition, thus not a limit circle with strictly definition (it should be independent from initial condition)



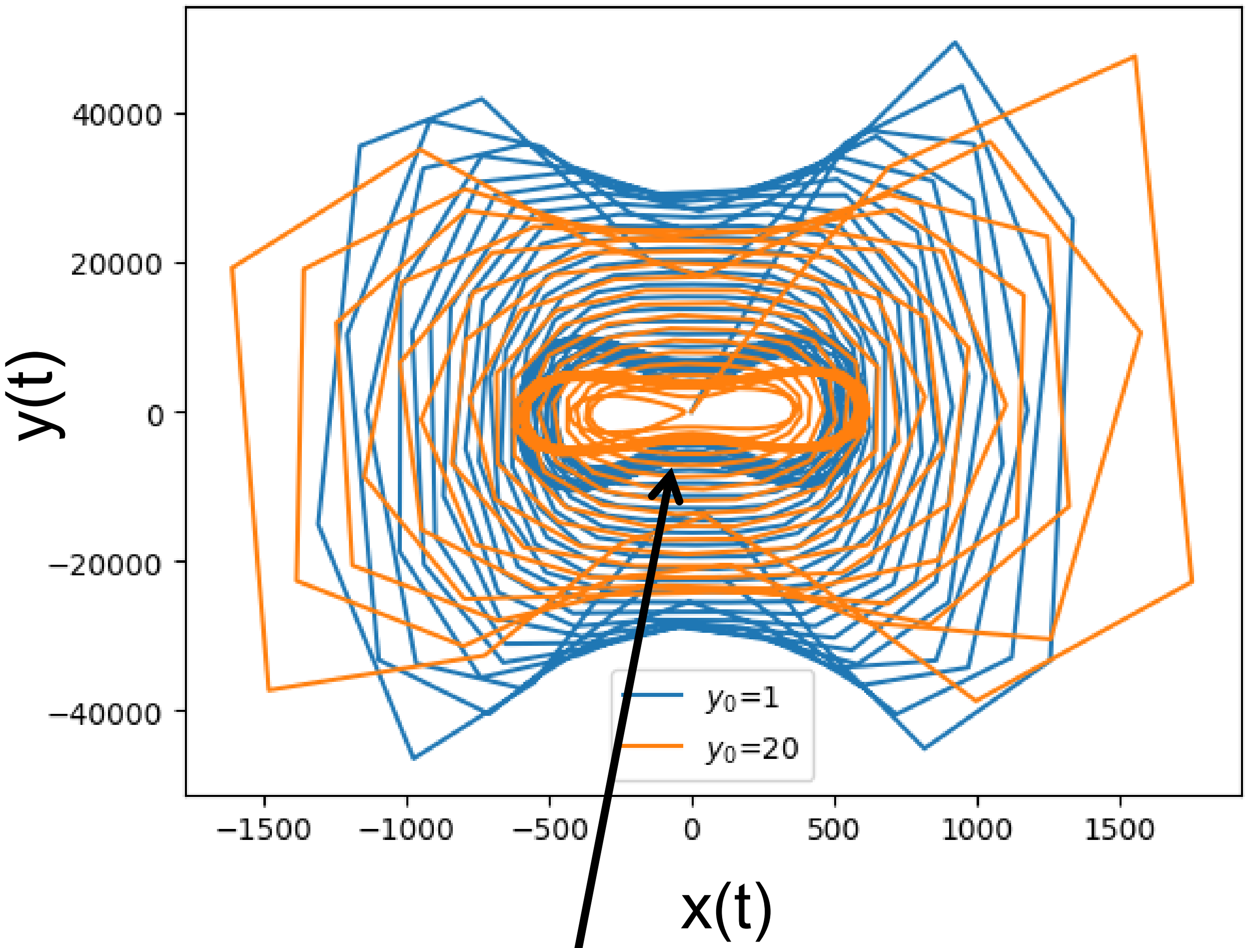
$r = 50000$



5. Parameter dependence



Saddle shape trajectory



Limit circle-like trajectory

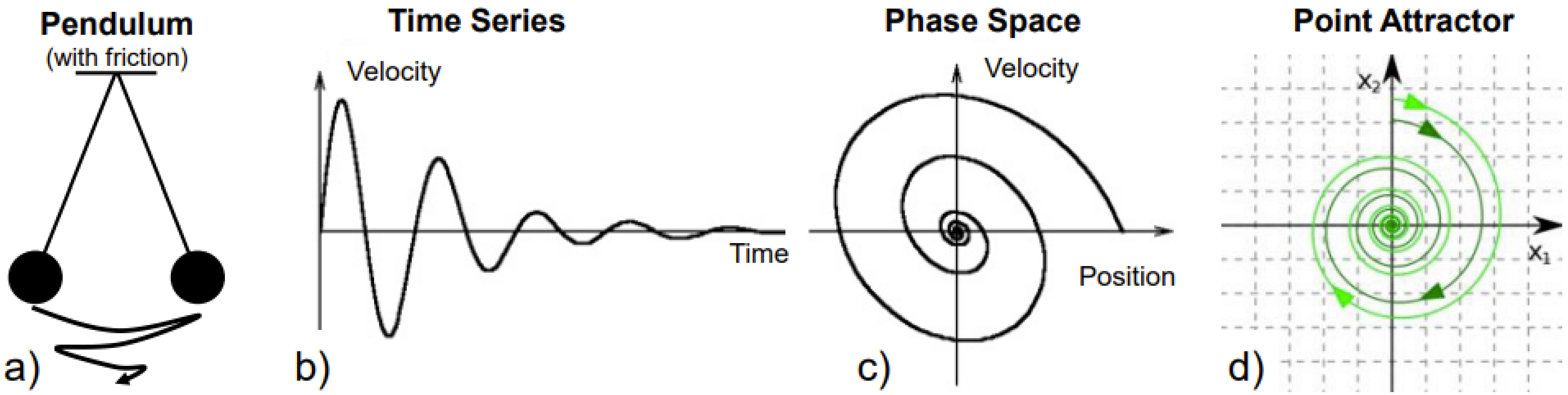
6. Summary

- Lorenz model feature was investigated with three numerical methods
- Statistical quantities (Lorenz map, Liapunov exponent) were same as other research/ literature
- There were difference where the chaos starts, or location (x(t), y(t),z(t)) after long time step
- With large *r*, saddle shape trajectory appeared
- There was limit circle-like trajectory, but the shape of the circle depends on the initial condition.

Appendix A: What is attractor?

- Attractor :
  - ➢ *an attractor is a set to which all neighboring trajectories converge. Stable fixed points and stable limit cycles are examples* (Strogatz, 2015, p331)

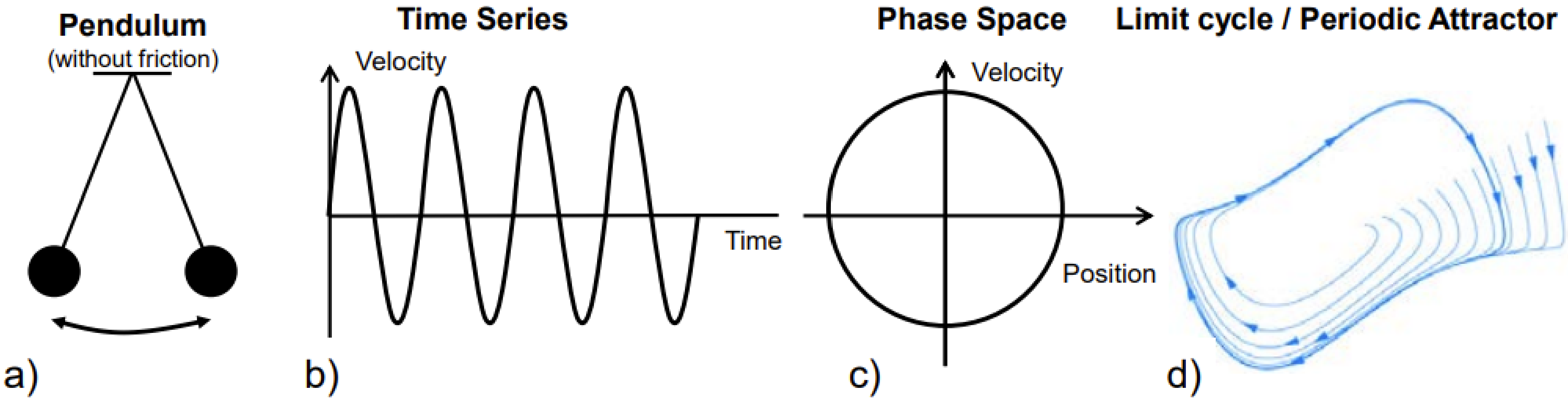
- Example 1: Pendulum with friction
  - The amplitude of the pendulum decreases over time and will stop at the end
  - In phase space (θ-v), Trajectory falls into a point = Point Attractor



Hasse and Bekker,2016

Appendix A: What is attractor?

- Example 2: Pendulum without friction
  - The amplitude of the pendulum is constant over time
  - In phase space (θ-v), the trajectory has a circle shape and rounds on it
    = Limit Circle

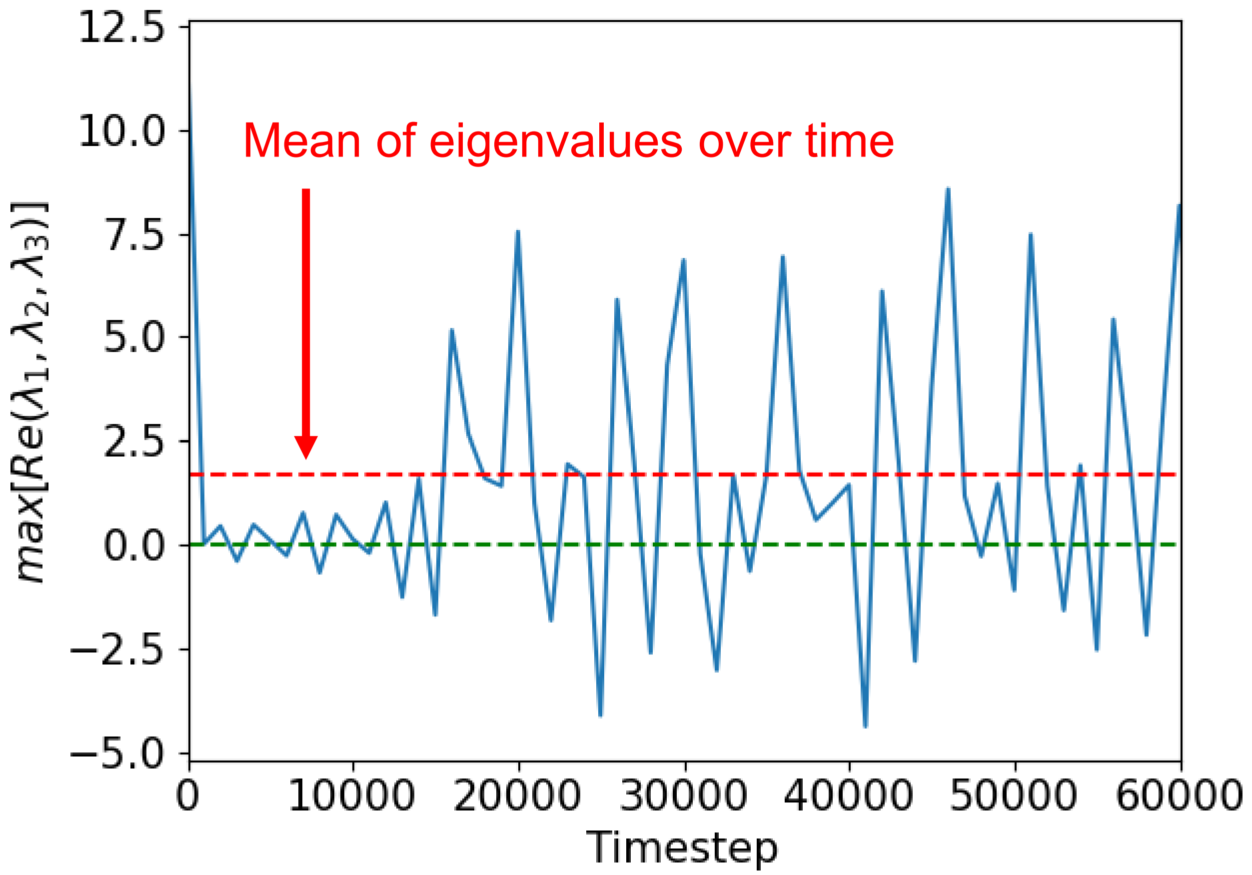


Hasse and Bekker,2016

The strange attractor of chaos is neither point (point attractor) nor periodic (limit circle), but still stays in a finite region and moves inside it forever

Appendix B: Numerical stability

- The eigenvalues of the initial condition are, 8/3, -11 $\pm\sqrt{1201}$
- Linear stability analyses: apply $\Omega = -11 + \sqrt{1201}$
$$\mathbf{u}(\mathrm{t}) = \mathbf{u}(0)e^{\Omega t} \rightarrow \infty$$
- Then, numerically unstable?
- The Jacobian (and its eigenvalues) changes over time, thus $\Delta t * 100000 = 100$ might not too big to explode
- The maximums eigenvalue (real part) changes over time positive to negative, not immediately go to infinity (see left figure)



Reference

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