Final Project

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Abstract

Lorentz attractor and its features were investigated as an example of a chaotic system with different numerical schemes (Predictor-Corrector, 4th order classical Runge-Kutta, 3-point Backward Differentiation Formulas). All schemes showed similar trajectories consistent with theoretical analysis. However, the starting point of chaotic behavior and exact status after a long time was different. Lorenz map and Liapunov exponent were the same for all methods and matched with other works. It was also observed the trajectory shape changes as parameter r, representing Ra/Re changes. As mathematically expected, all trajectories fall to point(s) when $r < r_H$, while there was the transition from point attractor to a strange attractor around $r \sim r_H$. When $r_H << r$, the trajectory becomes saddle shape and limit cycle-like attractor as observed.

1 Introduction

Chaos is one of the most characteristic behaviors of nonlinear systems, and it is widely researched in many fields. The definition of chaos is, based on Strogatz (2015);

> Chaos is aperiodic long-term behavior in a deterministic system that exhibits sensitive dependence on initial conditions (p. 331)

Chaotic systems have several features; sensitive dependence on initial conditions and the strange attractor. In a chaotic system, small differences in initial conditions would be amplified as the system develops. Whatever small the difference is, it would be magnified, and they have different states after sufficient time. This property, sensitive dependence on initial conditions, is captured by Lyapunov exponents (Chapter 4.3). The strange attractor is another unique character of chaotic systems. "Attractor" means points or regions in phase space that attract trajectories nearby over time (Pryor and Bright, 2007). For example, an oscillating pendulum with friction would gradually decrease its amplitude, and stop at center. Thus, its trajectory in phase space ($\theta - \dot{\theta}$ space) falls into a point since energy dissipates over time (See Fig. 1(a)). This is called a point attractor. If we ignore the dissipation by the air, the pendulum repeats its motion, and the trajectory falls to a circle in phase space (limit circle). On the other hand, strange attractors, never fall into a single attractor (which means the systems do not repeat previous status again) but stay in a certain range.

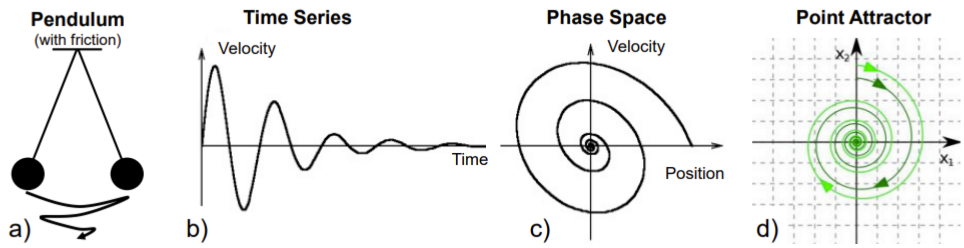
The Lorenz system, proposed by Lorenz (1963), is one of the most famous 3-dimensional chaotic systems. In this report, the above features are discussed using the Lorenz system. Furthermore, different numerical methods are compared.

The goals and questions to be answered of this report is the following;

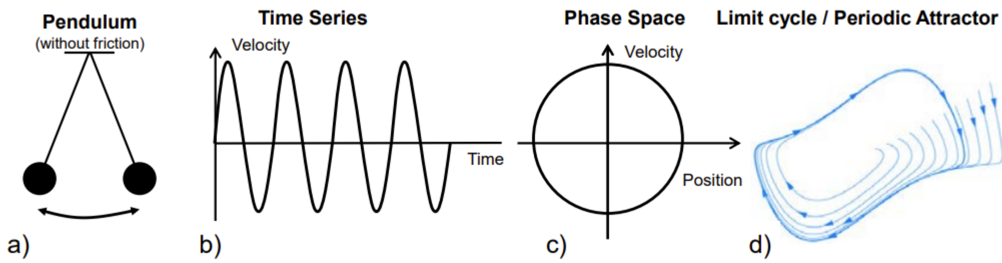
1. Investigate the behavior of the Lorentz system and observe features of the chaotic system and its trajectory
2. Investigate time development of the Lorenz system with different numerical schemes. Do they have completely different results or similar to each other?

3. How the system is sensitive to initial conditions and parameters?

The definition of the Lorenz system and its mathematical analysis is given in Chapter 2. Numerical methods are explained in chapter 3. The result is displayed in chapter 4, and the dependence of parameter is discussed in chapter 5.



(a) Example of point attractor: a pendulum with friction



(b) Example of limit cycle: pendulum without friction

Figure 1: Point attractor and limit cycle schematic with pendulum example (Hasse and Bekker,2016)

2 Lorenz model

The Lorenz equations are

$$\begin{cases} \dfrac{dx}{dt} = \sigma(y-x) \\ \dfrac{dy}{dt} = rx - y - xz \\ \dfrac{dz}{dt} = xy - bz \end{cases} \tag{1}$$

Here, σ, r, and b are all positive parameters. This Lorenz model is a simplified model of atmospheric convection proposed by Saltzman (1962). In this chapter, the behavior of the Lorenz equations trajectory is discussed mathematically. This discussion provides frameworks to analyze the trajectory. The following discussion is mainly based on Strogatz (2015).

2.1 Volume Contraction

The Lorenz system is dissipative; volumes in phase space decrease under the flow. Consider the volume $V(t)$ changes over time. The volume change equals the total volume in/outflow through the surface, S. Then,

$$\frac{dV}{dt} = \int_S \mathbf{f} \cdot \mathbf{n} dA \tag{2}$$

Here, $\mathbf{f}(x, y, z)$ is the velocity on the surface, n is a vector normal to the surface, and dA is the infinitesimal area on the surface. Using the Gauss' theorem,

$$\frac{dV}{dt} = \int_V \nabla \cdot \boldsymbol{f} dV \tag{3}$$

Apply Eq. 3 to the Lorenz model, we obtain;

$$\begin{aligned} \nabla \cdot \boldsymbol{f} &= \frac{\partial}{\partial x}[\sigma(y-x)] + \frac{\partial}{\partial x}[rx - y - xz] + \frac{\partial}{\partial z}[xy - bz] \\ &= -\sigma - 1 - b < 0 \end{aligned} \tag{4}$$

σ and b are positive constants, then

$$\begin{aligned} &\frac{dV}{dt} = -(\sigma + 1 + b)V \\ &V = V(0)e^{-(\sigma+1+b)t} \end{aligned} \tag{5}$$

Thus, volumes in phase space shrink exponentially fast. Even if we begin with a large solid mass of initial conditions, it eventually contracts to a limiting set with zero volume. Using this property, we can also prove that the Lorenz equations do not have quasiperiodic solutions (See Strogatz 2015, p321).

2.2 Fixed Points and stability

Consider the fixed point of the Lorenz system, where $dx/dt = dy/dt = dz/dt = 0$. Obviously, $(x_*, y_*, z_*) = (0, 0, 0)$ is a fixed point regardless of the parameters. For $r > 1$, there are two fixed point;$(x_*, y_*, z_*) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$. When we take $r \to 1^+$, two fixed points merge at 0.

When $r < 1$, it is known that every trajectory approaches the origin as $t \to \infty$. Thus, the origin is the stable point for $r < 1$.

Now, consider $r > 1$. As we saw, there are two fixed points when $r > 1$. These two points are linearly stable for

$$1 < r < r_H = \frac{\sigma(\sigma + b + 3)}{\sigma - b - 1}. \tag{6}$$

Here, $\sigma - b - 1 > 0$ is assumed. Then, if r is in the range, trajectories with different initial conditions will fall into either of the two fixed points. Then, what happens if $r_H < r$?

When $r_H < r$, it is shown that the fixed points are surrounded by **unstable** limit cycles; trajectories do not fall into a circle, but apart from it. However, at the same time, it is also shown that all trajectories eventually enter and remain in a certain large ellipsoid (Strogatz, 2015). Thus, when $r_H < r$, the trajectories do not fall into a point/ circle (i.e. trajectories do not intersect themselves or others), but still stay in a certain region. Also as we saw in section 2.1, the volume of the Lorenz system in phase space decreases to zero exponentially, This is the motivation for investigating the Lorenz system with $r_H < r$, and we expect to see different types of trajectory from point attractor or limit cycle.

2.3 Comment on this chapter

Here, let me emphasize the above discussion again since they tell us restriction of the trajectory.

It seems that sections 2.1 and 2.2 contradict: the volume in phase space exponentially decreases, thus it goes to some 2D plane, 1D line, or point. However, at the same time, there are unstable limit circles, so trajectories increase the distance from it. Furthermore, they stay in a finite region but never repeat their previous status (trajectories do not cross themselves: there is no periodic limit circle). Interestingly, as we see in Chapter 4, the trajectory of the Lorenz model satisfies these conditions and actually, this is a new type of attractor different from others.

3 Method

By default, the Lorenz system with $\sigma = 10, b = 8/3, r = 28$ and $x_0 = z_0 = 0, y_0 = 1$ was calculated. Based on the above discussion, $r = 28 > r_H$ is set. Three numerical methods were used to calculate the Lorenz system: Predictor-Corrector scheme, 4th-order classical Runge-Kutta (RK4), and 3-point Backward Differentiation Formulas (BDF). Consider $\mathbf{u} = (x, y, z)$ and $\mathbf{f} = d\mathbf{u}/dt$, each method described in next sections.

3.1 Predictor-Corrector

The Predictor-Corrector scheme is;

$$\begin{aligned} \mathbf{u}^{n+1/2} &= \mathbf{u}^n + \mathbf{f}(\mathbf{u}^n)\frac{\Delta t}{2} \\ \mathbf{u}^{n+1} &= \mathbf{u}^n + \mathbf{f}(\mathbf{u}^{n+1/2})\Delta t \end{aligned} \tag{7}$$

3.2 Runge-Kutta (RK4)

The 4th order classical Runge-Kutta (RK4) is;

$$
\begin{aligned}
&\mathbf{u}^{n+1} = \mathbf{u}^n + \frac{\Delta t}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
&k_1 = \mathbf{f}(\mathbf{u}^n) \\
&k_2 = \mathbf{f}(\mathbf{u}^n + k_1 \Delta t/2) \\
&k_3 = \mathbf{f}(\mathbf{u}^n + k_2 \Delta t/2) \\
&k_4 = \mathbf{f}(\mathbf{u}^n + k_3 \Delta t)
\end{aligned}
\tag{8}
$$

3.3 Backward Differentiation Formulas (BDF)

The Jacobi matrix with the initial condition, $x = z = 0, y = 0$, and parameters $\sigma = 10$, $b = 8/3$, and $r - 28$, is.

$$
\begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix} \tag{9}
$$

The eigenvalues of the matrix are $\lambda = 8/3, (-11 \pm \sqrt{1201})$. Then,

$$
\frac{|\lambda_{max}|}{|\lambda_{min}|} \sim 8.581 > 1 \tag{10}
$$

Then, the Lorenz system is not a strong stiff system. A 3-point BDF scheme is used and compared with other schemes.

$$
\frac{d\mathbf{u}}{dt} = \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n + \mathbf{u}^{n-1}}{2\Delta t} \tag{11}
$$

3.4 Time step estimation

In all simulations, $\Delta t = 0.001$ was used, and 100000 time steps were calculated. As we saw in Section 3.3, the eigenvalue with the initial condition is $8/3, -11 \pm \sqrt{1201}$. We have a positive eigenvalue, so the semi-discrete scheme is basically unstable.

Following the linear stable analysis, the solution at time t is

$$
\mathbf{u}(t) = \mathbf{u}_0 e^{\Omega t} \tag{12}
$$

If we apply $\Omega = -11 + \sqrt{1201}$, $t = 0.001 \times 100000$, $\mathbf{u}(t)$ is much bigger value than Python3 can handle. However, we expect that the eigenvalue of the transition matrix changes over time. Figure 2 shows the maximum of the real part of the Jacobi matrix 3 eigenvalues over time steps, $max[Real(\lambda_1(t), \lambda_2(t), \lambda_3(t))]$. The largest eigenvalue oscillates between positive and negative, thus after $t = 0.001 \times 100000$, $\mathbf{u}(t)$ is still in the range python3 can handle (note Fig. 2 shows the maximum *Re.* for all eigenvalues, the direction of largest eigenvalue is not always the same). Although, the average of maximum eigenvalues is positive, then if Δt is too big or integration time is large, the system would give infinity at some point ($\Delta t = 1$ was tried, and it went to infinity during the calculation).

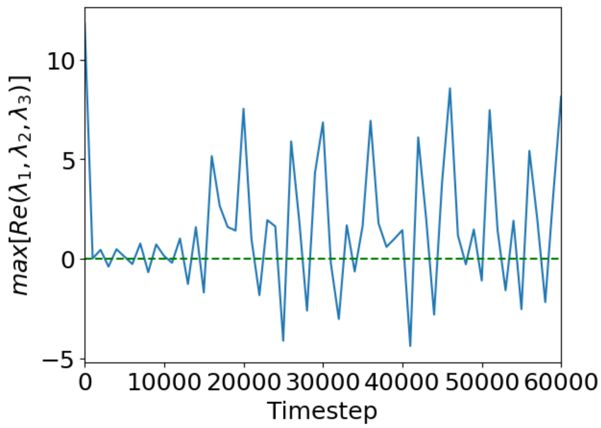


Figure 2: Largest real part of eigenvalues

3.5 Symplectic Integrators

Symplectic Integrator schemes were also used (Symplectic Euler and Verlet Scheme). The Lorenz system was converted to Eq. 13 and the following 2nd derivative system;

$$\begin{cases} \dfrac{d^2x}{dt^2} = \sigma(\dfrac{dy}{dt} - \dfrac{dx}{dt}) \\ \dfrac{d^2y}{dt^2} = r\dfrac{dx}{dt} - \dfrac{dy}{dt} - x\dfrac{dz}{dt} - z\dfrac{dx}{dt} \\ \dfrac{d^2z}{dt^2} = x\dfrac{dy}{dt} + y\dfrac{dx}{dt} - b\dfrac{dz}{dt} \end{cases} \tag{13}$$

However, all the Symplectic Integrator gave infinite at some point and calculations stopped. This is because the Lorenz system is dissipative, thus not Symplectic.

4 Result and discussion

4.1 Overview of Lorenz attractor

Figure 4(a) shows the time development of the Lorenz system plotted in x-y-z space calculated by RK4. Points are plotted for every 400 steps, gradually getting darker as time steps increase (At time step=0, the points are while and at 100000, it is orange). We can see there are two "disks" in the space. Starting from the initial condition ($x = z = 0, y = 1$), the trajectory circles around the right disk, and then jump to the left disk. The same trajectory was obtained from other numerical methods (not shown here). The trajectory being flat disks corresponds to the phase volume contraction we saw in section 2.1. Whatever initial condition we start, the same process happens and the trajectory falls into this two-disk shape (thus, this is an attractor since it "attracts" all tragedies). In addition, the trajectory stays in a finite volume as we saw in section 2.2. This is a new type of attractor different from the point attractor or limit cycle, then called the "Strange attractor" (Strogatz, 2015). Fig 3a plots $x(t)$ against $y(t)$, and $x(t)$ against $z(t)$ (b). Trajectories circle around two regions and two disks are tilted in 3-dimensional space.

Figure 5 shows the time development of $x(t)$ and $z(t)$ (Result of RK4). Until around 16000 time steps, x oscillates around -10 with increasing amplitude. Then, it starts to jump between positive and negative regions. The z profile shows a similar behavior until 16000 time steps. Peaks of z, different from peaks of x, look like have a pattern. Detailed time development of z is displayed in Figure 6 (20000 to 40000 time steps). Green dashed lines are plotted on the largest peaks. We can see a similar pattern between each green line; after the largest peaks, the next peak is smaller than the others. Then, the amplitude of peaks gradually increases and decreases, reaching the next biggest peak.

This feature is more obvious in Figure 7. The abscissa is $z_{max,n}$, the nth maximum value of z. Then, the y-axis, $z_{max,n+1}$ is the next maximum value (the next peak value of $z_{max,n}$). This plotting is called the Lorenz map. We can see there is a two-to-one relation between $z_{max,n}$ and $z_{max,n+1}$. For instance, when $z_{max,n} = 43$, $z_{max,n+1}$ is around 35. This is corresponding to Fig.6. We saw the biggest peak values in the 20000-400000 time step was around 43, and the next peak value was around 35. The red line in Fig. 7 is the line $y = x$, indicating that the $z_{max,n} - z_{max,n+1}$ has bigger slope than 1 everywhere, both positive and negative. (Around $z_{max,n}, z_{max,n+1} \sim 30$ in Fig7, slope is almost 1. This region corresponds to the 0 to 10000 steps, where the peak value almost linearly increases. We can see this on 5(b))

Using this observation, we can explain that there is no stable limit cycle with the Lorenz system (i.e. even though we take long enough time steps, limit cycles would not appear). The slope of the Lorenz map is always

$$|f'(z_{max,n})| > 1 \tag{14}$$

Now, consider a fixed point z_* of the map f. z_* satisfies

$$z_* = f(z_*) \tag{15}$$

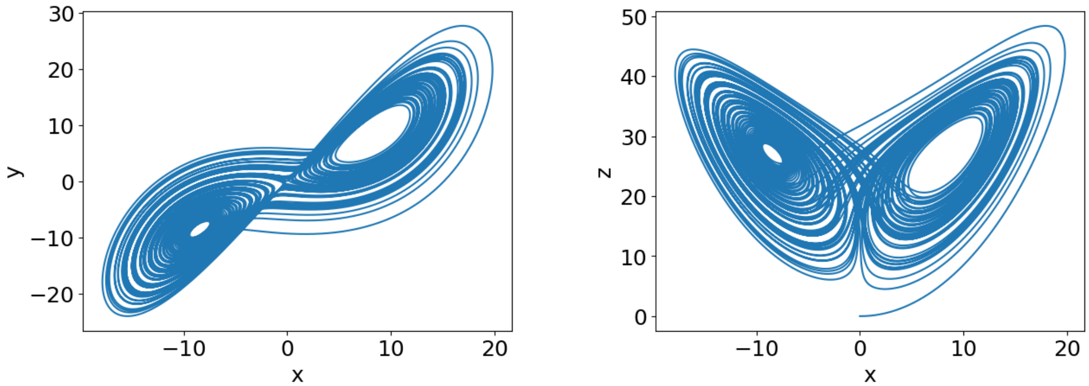
Fig. 7 shows there is one fixed point, where the 45° diagonal ($z_{max,n} = z_{max,n+1}$) intersects the graph. It represents a closed orbit trajectory. Consider a slightly perturbed trajectory with $z_n = z_* + \eta_n$, where η_n is a small perturbation. Using the linearization,

$$\eta_{n+1} \sim f'(z_*)\eta_n \tag{16}$$

Using Eq.14, we see

$$|\eta_{n+1}| > |\eta_n| \tag{17}$$

Then, small perturbations around the η grow with each step, the closed orbit is unstable.



(a) x-y plot

(b) x-z plot

Figure 3: Time development of the Lorenz system

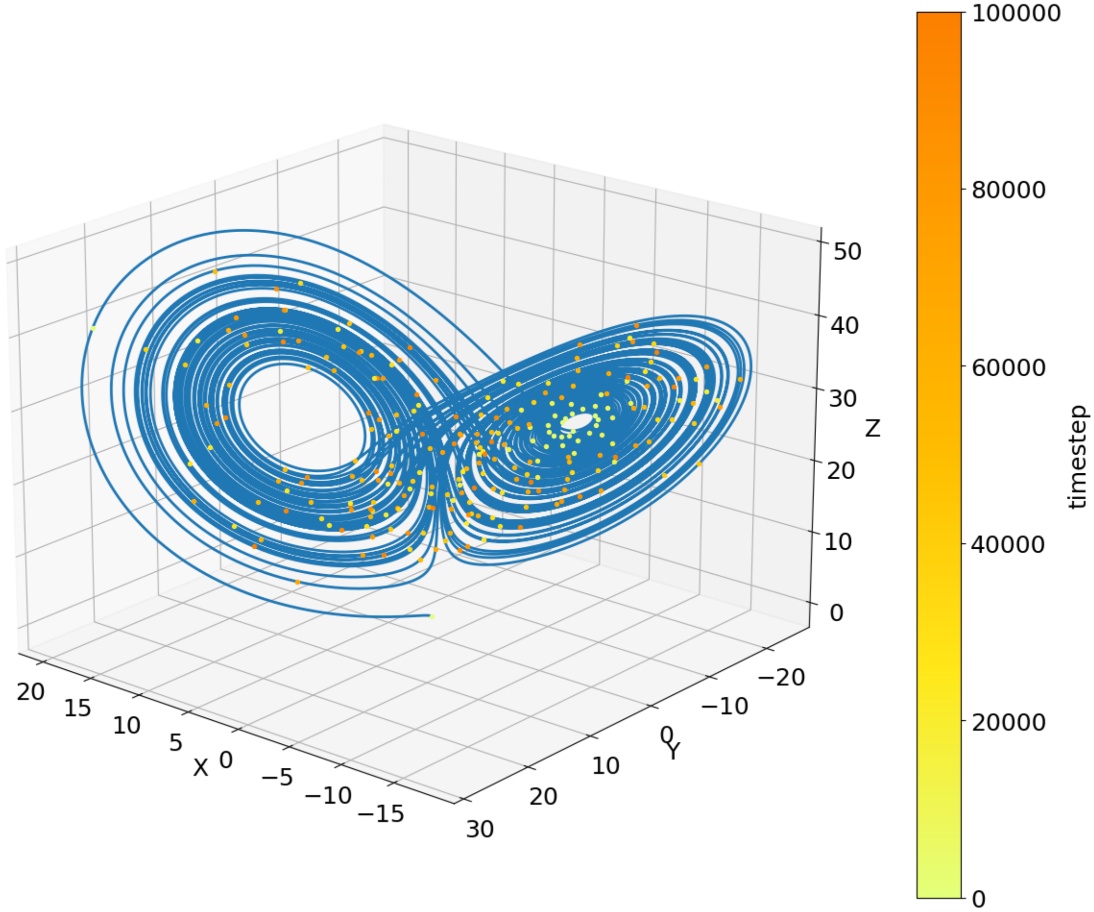
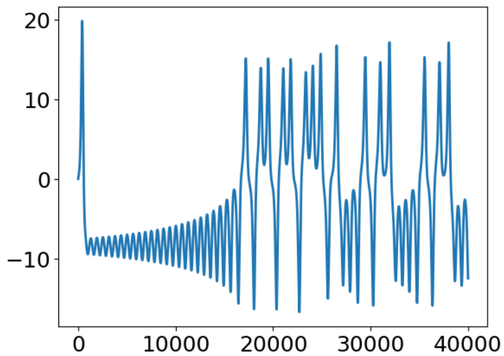
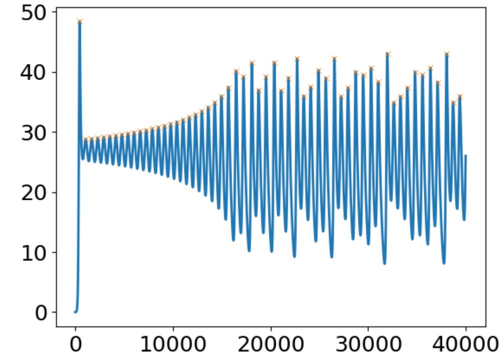


Figure 4: Trajectory of the Lorenz system. Points are plotted for every 400 steps, the color changes from white to orange as time steps.



(a) time development of x



(b) time development of z

Figure 5: Time development of x (a) and z (b) for the first 40000 times steps

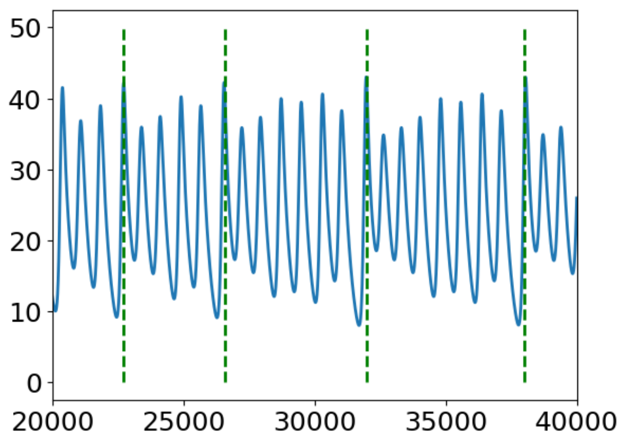


Figure 6: Detail of z development

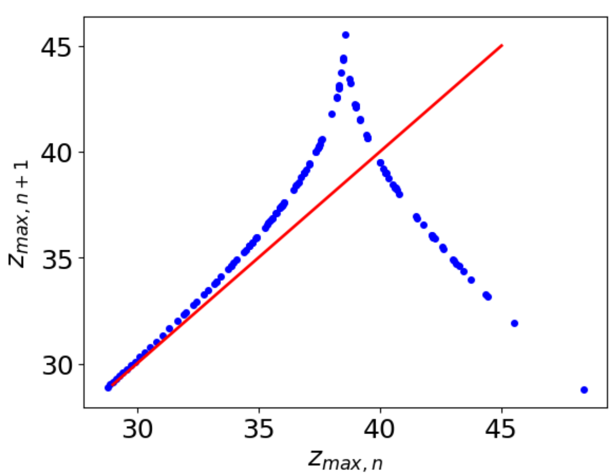


Figure 7: Peak transition

4.2 Comparison of numerical methods

This section compares results by the three numerical methods–Predictor-Corrector, Runge-Kutta 4 (RK4), and 3-point Backward Differentiation Formulas (BDF). Figure 8 shows the time development of x by each method. In the first few hundred, all methods give the same result. Around time step$\sim$1000, the BDF method gives a different output from the other methods. Predictor-Corrector and RK4 start to output different results around the 22000 time step. The time step when the chaotic behavior starts is around 19000 steps with the BDF method, while that of the other two methods is around 160000. After a long time of integration, three methods have different statuses, $(x(t), y(t), z(t))$, even though we start with the same parameters and initial condition. This is because each method have a different error order.

While each method has a different chaotic output, the Lorenz map profile is almost the same for all cases (See Fig. 9). They have the same slope and peak (the BDF method gives a slightly smaller peak position).

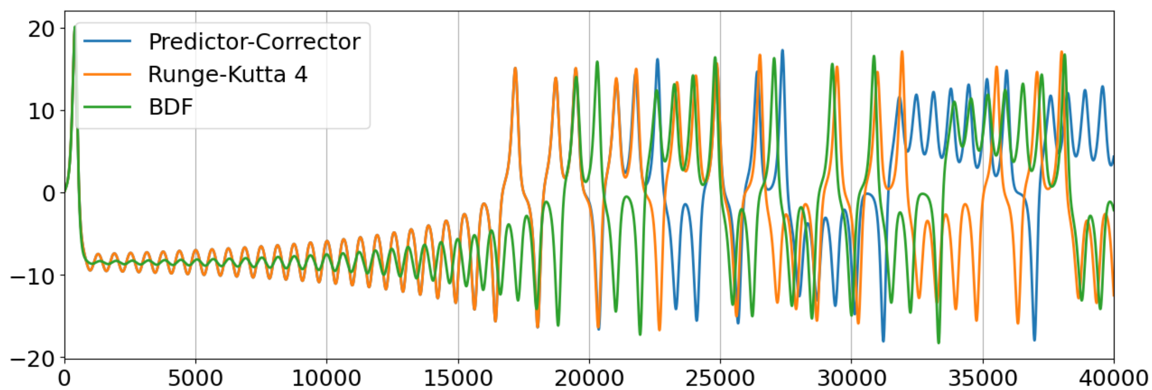


Figure 8: Time development of x with different schemes

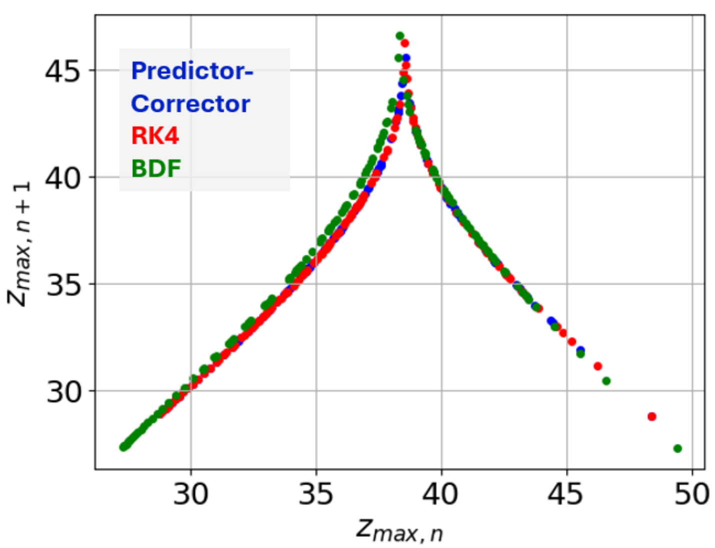


Figure 9: Peak transition of 3 methods

4.3 Sensitive dependence on initial conditions

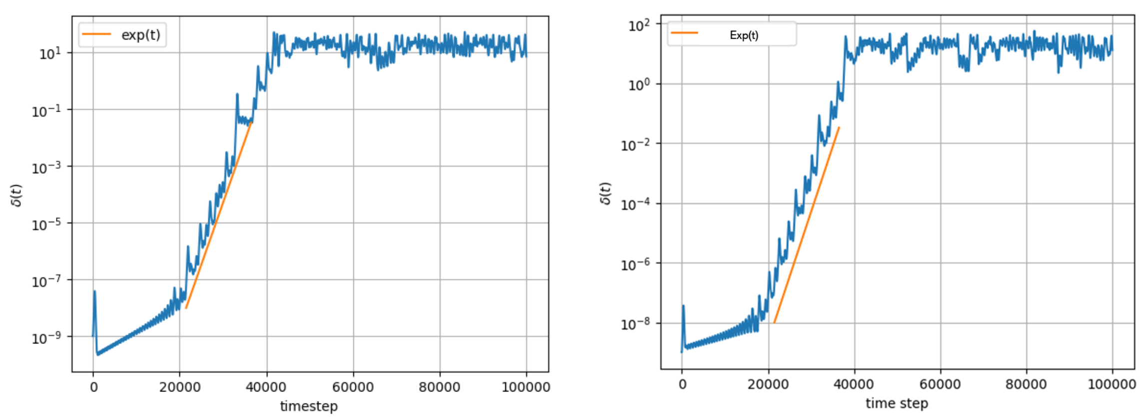
The motion on the attractor exhibits sensitive dependence on initial conditions. This implies that two trajectories that begin in close proximity will quickly move apart from one another, leading to entirely distinct outcomes. Fig 11 demonstrates this divergence by tracing the development of 4 points close to each other at initial condition (they are 0.002 different from each other). Eventually, the points spread across the entire attractor. Consequently, nearby trajectories can land anywhere on the attractor. The practical consequence is that making long-term predictions becomes unfeasible in such a system, where minor uncertainties escalate rapidly. This is a characteristic feature of chaos called sensitive dependence on initial conditions.

We can quantitatively measure how fast each point spreads by using the Liapunov exponent. Consider $\mathbf{x}(t)$ is a point on the attractor at time t, and there is a nearby point $\mathbf{x}(t) + \delta(t)$. $\delta(t)$ is a tiny separation length between 2 points. Now, $\delta(t)$ grows over time. We expect to have a Liapunov exponent λ, with

$$||\delta(t)|| \sim ||\delta(0)||e^{\lambda t} \tag{18}$$

Here, $\delta(0)$ is the first distance. Fig10 shows $||\delta(t)||$ obtained by the BDF and RK4 method. $\delta(0)$ is set to 10^{-9}. The Liapunov exponent is approximately $\lambda = 1$ in both cases, close to $\lambda = 0.9$ obtained by other research (Strogatz,2015).

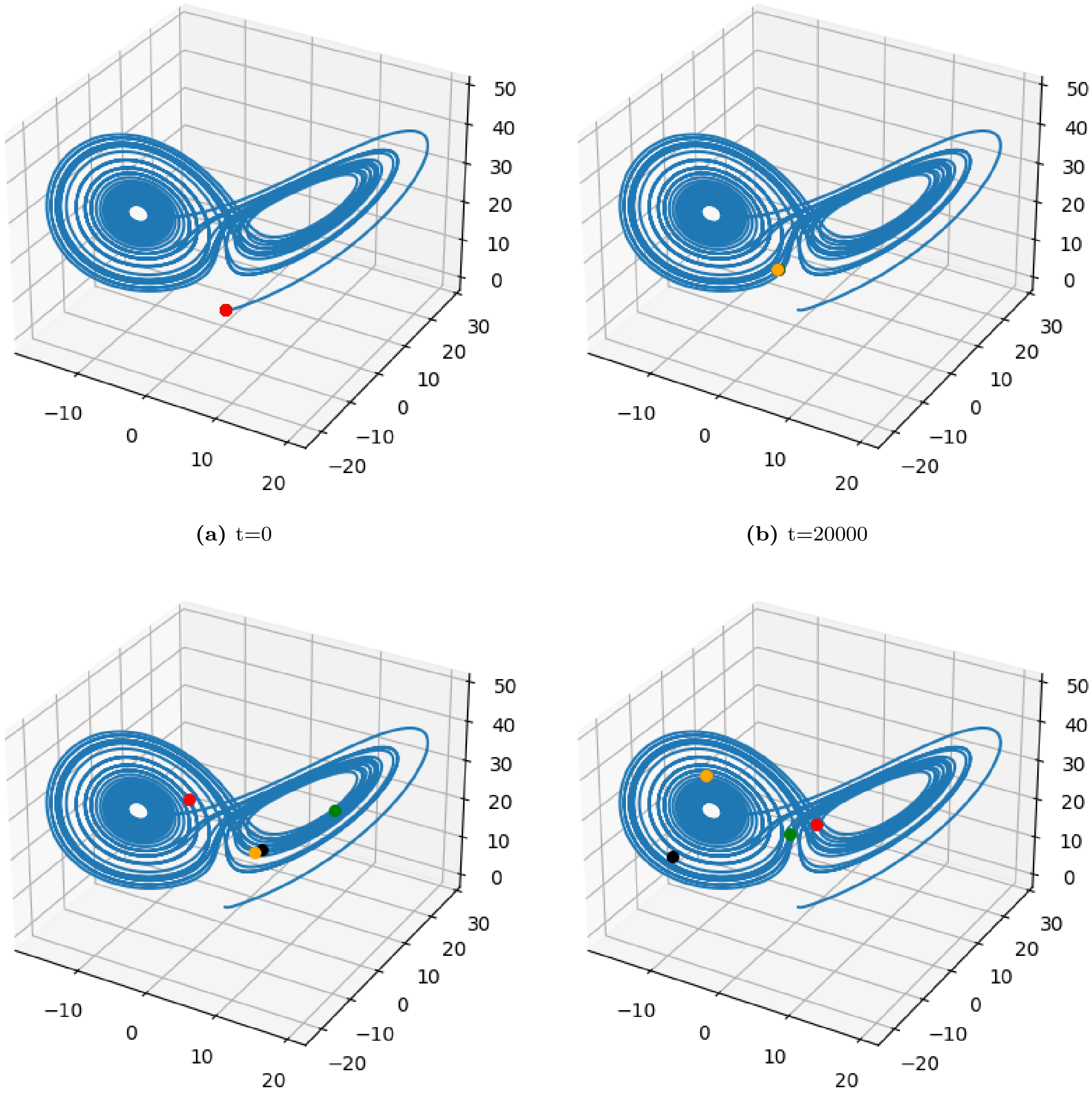
Another point we should focus on in Fig. 10 is that $||\delta(t)||$ increases exponentially, but converge at some point and almost be constant around that. This is because there is an unstable limit cycle and trajectories fly away from it, but the whole trajectory is bounded in a finite region (see Section 2.2). The BDF method has a smaller fluctuation of δ after convergence, which might indicate that the size of the whole trajectory is a bit smaller with the BDF method.



(a) $||\delta(t)||$ obtained by BDF and $e^{1\cdot t}$

(b) $||\delta(t)||$ obtained by RK4 and $e^{1\cdot t}$

Figure 10: Liapunov exponent



(a) t=0

(b) t=20000

(c) t=30000

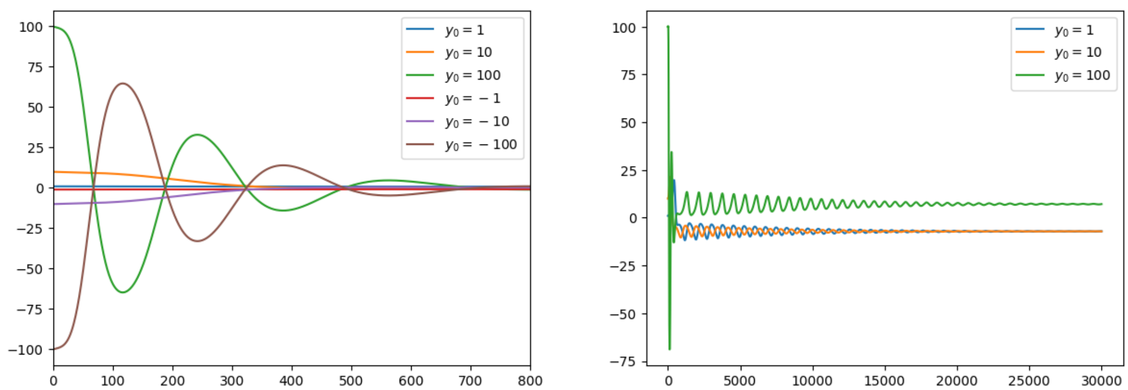
(d) t=40000

Figure 11: 4 points spread across the attractor over time (Red, orange, black, and green points)

5 Parameter dependence of the Lorenz system

So far, we considered the Lorenz system with $\sigma = 10, b = 8/3$, and $r = 28$. What happens if we change the parameters? Existing research often fixes σ and b, and changes r. This is physically reasonable since σ represents the Prandtl number, which depends on fluid, not on flow (thus, we expect to have fixed value through time). While r represents the ratio between Re and Ra, depending on flow field or temperature difference.

As we saw in Section 2.2, there are stable fixed point(s) when $r < r_H$. Fig. 12 shows $y(t)$ when $r < r_H$. Fig.12a is the case $r = 0.95$, all the $y(t)$ converge to 0 regardless of the initial condition. When $1 < r < r_H$, Fig. 12b, there are points to converge depending on the initial condition.



(a) y(t) when $r = 0.95$

(b) y(t) when $r = 20$

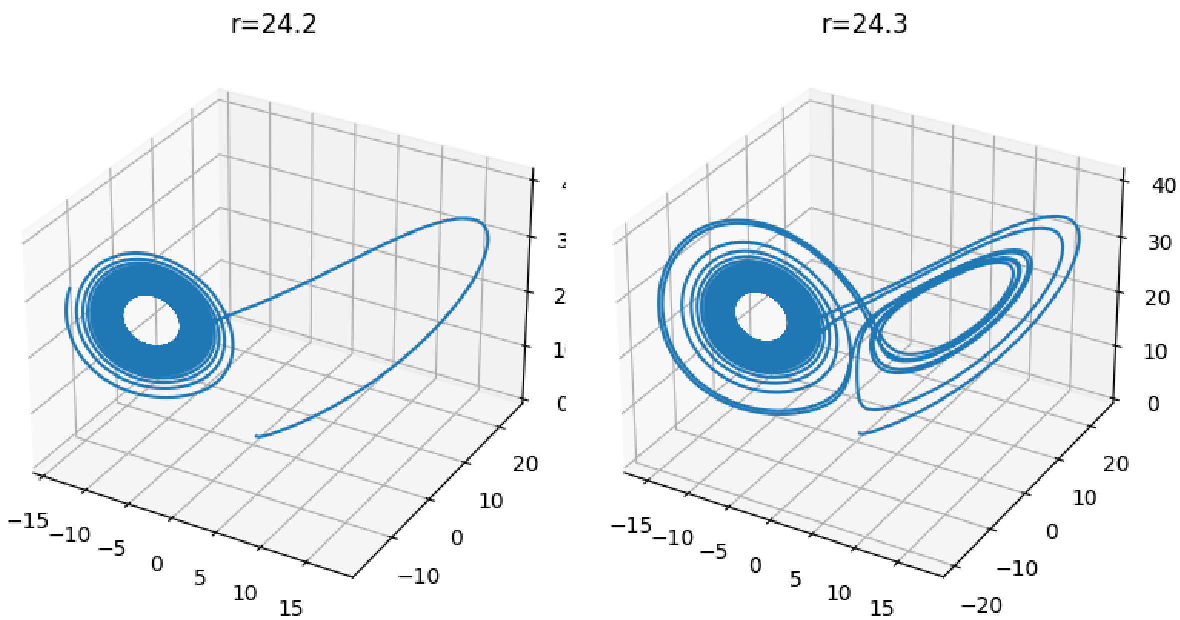
Figure 12: y(t) when $r < r_H$. y_0 indicates initial condition $(x, y, z) = (0, y_0, 0)$

When $r < r_H$ but r is close to r_H, we can see the transition to a chaotic system. Fig 13 show trajectory when $r = 24.2$ and 24.3 (by RK4). When $r = 24.2$, the trajectory circles around a region (say 1 "disk"), but does not jump to another disk. On the other hand, $r = 24.3$ clearly shows two disks and trajectories jump to each other. Strogatz (2015) points out that chaos and a stable point co-exist at $r = 24.5$, but this result indicates the co-existing r is slightly smaller than $r = 24.5$.

When r is much larger, i.e. temperature difference is much larger and viscosity is dominant, the trajectory becomes saddle-shape. Fig 14 shows the trajectory when $r = 50000$. Plot them in the x-y plane, it looks like there is a limit cycle near the center. Although, the limit cycle shape would change by initial conditions, since $y_0 = 1, 20$ (y_0 indicates initial condition, $x(0) = z(0) = 0$) cases have different limit circles.

6 Summary

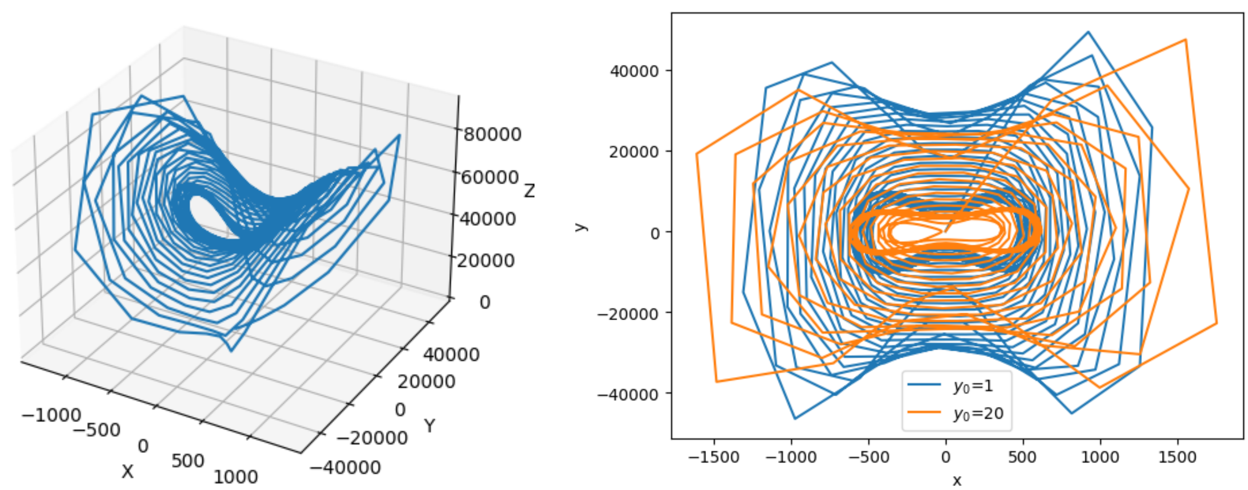
The Lorenz model was investigated using three numerical methods. Although the three methods give different statuses, $(x(t), y(t), z(t))$ after sufficient time, they basically show the same trajectory and Lorenz map, Liapunov exponent. Changing the parameter r gave the different shapes of the trajectory, especially when r is much larger, there is a limit cycle-like trajectory.



(a) Trajectory when $r = 24.2$

(b) Trajectory when $r = 24.3$

Figure 13: Transition to chaos



(a) x-y-z space trajectory

(b) x-y plane

Figure 14: Trajectory with $r = 50000$

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