The Readable Calculus

AB Version

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Chapter 1 Overview: Review of Derivatives

The purpose of this chapter is to review the “how” of differentiation. We will review all the derivative rules learned last year in PreCalculus. In the next several chapters, we will be exploring the “why” of differentiation as well. As a quick reference, here are those rules:

The Power Rule: $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

The Product Rule: $\frac{d}{dx}(u \cdot v) = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$

The Quotient Rule: $\frac{d}{dx}\left(\frac{u(x)}{v(x)}\right) = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$

The Chain Rule**:** $\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$

$\frac{d}{dx}\left[\sin u\right] = (\cos u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\csc u\right] = (-\csc u \cot u)\frac{du}{dx}$

$\frac{d}{dx}\left[\cos u\right] = (-\sin u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\sec u\right] = (\sec u \tan u)\frac{du}{dx}$

$\frac{d}{dx}\left[\tan u\right] = (\sec^2 u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\cot u\right] = (-\csc^2 u)\frac{du}{dx}$

$\frac{d}{dx}\left[e^u\right] = (e^u)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\ln u\right] = \left(\frac{1}{u}\right)\frac{du}{dx}$

$\frac{d}{dx}\left[a^u\right] = (a^u \cdot \ln a)\frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\log_a u\right] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$

$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

$\frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-1}{\sqrt{1-u^2}} \cdot \frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}} \cdot \frac{du}{dx}$

$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{u^2+1} \cdot \frac{du}{dx}$ $\qquad$ $\frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-1}{u^2+1} \cdot \frac{du}{dx}$

Here is a quick review of from last year:

Identities While all will eventually be used somewhere in Calculus, the ones that occur most often early are the Reciprocals and Quotients, the Pythagoreans, and the Double Angle Identities.

$$\tan x = \frac{\sin x}{\cos x}; \qquad \cot x = \frac{\cos x}{\sin x}; \qquad \sec x = \frac{1}{\cos x}; \qquad \csc x = \frac{1}{\sin x}$$

$$\sin^2 x + \cos^2 x = 1; \qquad \tan^2 x + 1 = \sec^2 x; \qquad \cot^2 x + 1 = \csc^2 x$$

$$\sin 2x = 2\sin x \ \cos x; \qquad \cos 2x = \cos^2 x - \sin^2 x$$

Inverses Because of the quadrants, taking an inverse yields two answers, only one of which your calculator can show. How the second answer is found depends on the kind of inverse:

$$\cos^{-1} x = \begin{Bmatrix} calculator \ \pm 2\pi n \\ -calculator \ \pm 2\pi n \end{Bmatrix} \qquad \sin^{-1} x = \begin{Bmatrix} calculator \ \pm 2\pi n \\ \pi - calculator \ \pm 2\pi n \end{Bmatrix}$$

$$\tan^{-1} x = \begin{Bmatrix} calculator \ \pm 2\pi n \\ \pi + calculator \ \pm 2\pi n \end{Bmatrix} = calculator \ \pm \pi n$$

$$\log_a x + \log_a y = \log_a (xy)$$
$$\log_a x - \log_a y = \log_a \frac{x}{y}$$
$$\log_a x^n = n \log_a x$$

1.1: The Power Rule and the Exponential Rules

In PreCalculus, we developed the idea of the Derivative geometrically. That is, the derivative initially arose from our need to find the slope of the tangent line. In Chapter 2 and 3, that meaning, its link to limits, and other conceptualizations of the Derivative will be Explored. In this Chapter, we are primarily interested in how to find the Derivative and what it is used for.

Derivative—Def'n: $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

—Means: The function that yields the slope of the tangent line.

Numerical Derivative—Def'n: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

—Means: The numerical value of the slope of the tangent line at x = a.

Symbols for the Derivative

$\frac{dy}{dx}$ = "d-y-d-x" $f'(x)$ = "f prime of x" y' = "y prime"

$\frac{d}{dx}$ = "d-d-x" D_X = "d sub x"

OBJECTIVES

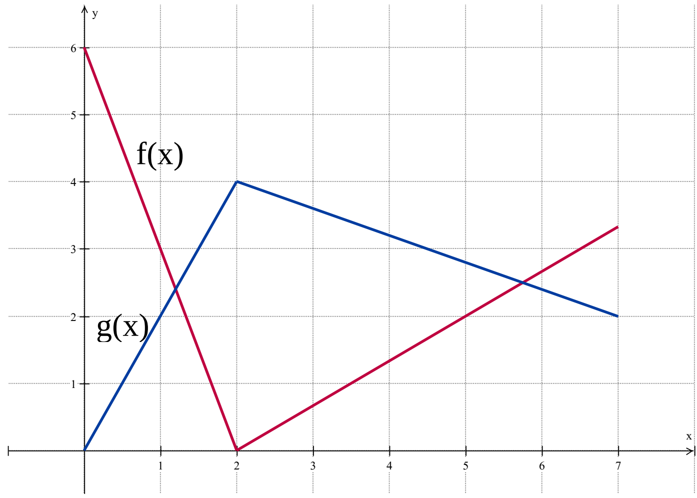
Use the Power Rule and Exponential Rules to find Derivatives.
Find the equations of tangent and normal lines.
Use the equation of a tangent line to approximate a function value.

Key Idea from PreCalc: The derivative yields the slope of the tangent line. [But there is more to it than that.]

Key Concept: The derivative of a function is the slope of the function.

The definitions of the derivative, above, both come from the expression for the slope of a line, combined with the idea of a limit. The numerical derivative of a function is the slope of that function at a point.

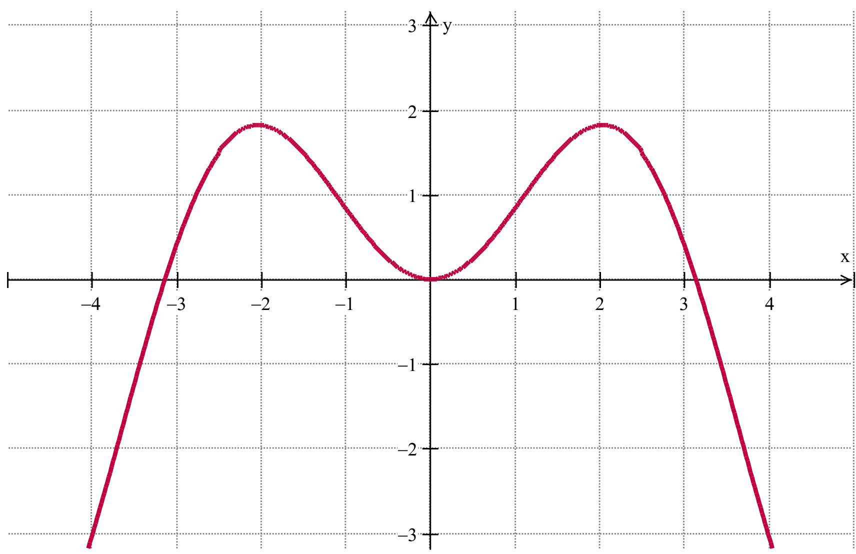
Ex. 1 Find the values of $f'(1)$, $f'(3)$, $g'(1)$, and $g'(4)$

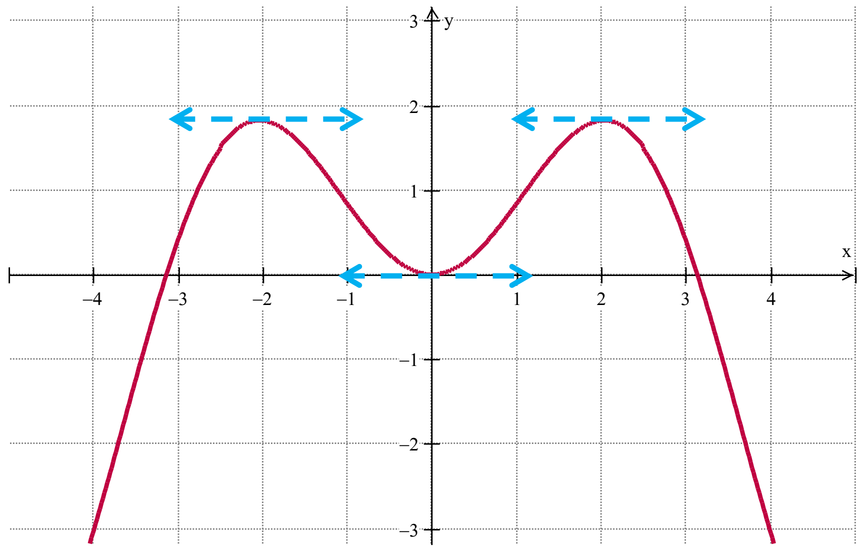


$$f'(1) = -3, \quad f'(3) = \frac{2}{3}, \quad g'(1) = 2, \quad g'(4) = -\frac{2}{5}$$

Note that at $x = 2$ that the derivative of both f and g does not exist because there is not a single tangent line at this point. We will discuss this further in the chapter on limits and differentiability.

Ex. 2 Find the values of x where the function whose graph is below has a derivative value of 0.





A slope of 0 is a horizontal line, so this function has tangent lines with slope of 0 at $x = -2$, $x = 0$, and $x = 2$.

The first and most basic derivative rule is the Power Rule. Among the last rules we learned in PreCalculus were the Exponential Rules. They look similar to one another, therefore it would be a good idea to view them together.

The Power Rule:	The Exponential Rules:
$\frac{d}{dx} x^n = n\, x^{n-1}$	$\frac{d}{dx}\left[e^x\right] = e^x$ $\frac{d}{dx}\left[a^x\right] = a^x \cdot \ln a$

The difference between these is where the variable is. The Power Rule applies when the variable is in the base, while the Exponential Rules apply when the variable is in the Exponent. The difference between the two Exponential rules is what the base is. $e = 2.718281828459\ldots$, while a is any positive number other than 1.

Ex 3 Find a) $\frac{d}{dx}\left[x^5\right]$ and b) $\frac{d}{dx}\left[5^x\right]$

The first is a case of the Power Rule while the second is a case of the second Exponential Rule. Therefore,

a) $\dfrac{d}{dx}\left[x^5\right] = 5x^4$ b) $\dfrac{d}{dx}\left[5^x\right] = 5^x \ln 5$

There were a few other basic rules that we need to remember.

D_x [constant] is always 0
$D_x\left[cx^n\right] = (cn)x^{n-1}$
$D_x\left[f(x)+g(x)\right] = D_x\left[f(x)\right] + D_x\left[g(x)\right]$

These rules allow us to easily differentiate a polynomial--term by term.

Ex 4 $y = 3x^2 + 5x + 1$; find $\dfrac{dy}{dx}$

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[3x^2 + 5x + 1] \\ &= 3(2)x^{2-1} + 5(1)x^{1-1} + 1(0) \\ &= 6x + 5\end{aligned}$$

Ex 5 $f(x) = x^2 + 4x - 3 + e^x$; find $f\,'(x)$.

$f\,'(x) = 2x + 4 + e^x$

Ex 6 $y=\sqrt{x^3}+\dfrac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4$; find $\dfrac{dy}{dx}$.

$$y=\sqrt{x^3}+\frac{4}{\sqrt{x}}-\sqrt[4]{x^3}+e^4$$
$$y=x^{3/2}+4x^{-1/2}-x^{3/4}+e^4$$

$$\frac{dy}{dx}=\frac{3}{2}x^{1/2}-2x^{-3/2}-\frac{3}{4}x^{-1/4}$$

Note in Ex 6 that e^4 is a constant, therefore, its derivative is 0.

Ex 7 Find the equations of the lines tangent and normal to

$$f(x)=x^4-x^3-2x^2+1 \text{ at } x=-1$$

The slope of the tangent line will be $f'(-1)$

$$f'(x)=4x^3-3x^2-4x$$
$$f'(-1)=-3$$

[Note that we could have gotten this more easily with the nDeriv function on our calculator.]

$f(-1)=1$, so the tangent line will be

$$y-1=-3(x+1)$$

or

$$y=-3x-2$$

The normal line is perpendicular to the tangent line and, therefore, has the negative reciprocal slope $=\dfrac{1}{3}$. The normal line is

$$y-1=\frac{1}{3}(x+1)$$

One of the uses of the tangent line is based on the idea of Local Linearity. This means that in small areas, algebraic curves act like lines—namely their tangent lines. Therefore, one can get an approximate *y*-value for points near the point of tangency by plugging *x*-values into the equation of the tangent line. We will explore this more in a later section.

As we have seen, when the variable is in the Exponent, we use the Exponential Rules. When the variable was in the base, we used the Power Rule. But what if the variable is in both places, such as $\frac{d}{dx}\left[(2x-1)^{x^2}\right]$? It is definitely an Exponential problem, but the base is not a constant as the rules above have. The Change of Base Rule allows us to clarify the problem:

$$\frac{d}{dx}\left[(2x-1)^{x^2}\right]=\frac{d}{dx}\left[e^{x^2\ \ln\,(2x-1)}\right]$$

but we will need the Product Rule for this derivative. Therefore, we will save this for section 1-4.

1.1 Homework Set A

Differentiate.

1. $f(x) = x^2 + 3x - 4$

2. $f(t) = \frac{1}{4}\left(t^4 + 8\right)$

3. $y = x^{-2/3}$

4. $y = 5e^x + 3$

5. $v(r) = \frac{4}{3}\pi r^3$

6. $g(x) = x^2 + \frac{1}{x^2}$

7. $y = \frac{x^2 + 4x + 3}{\sqrt{x}}$

8. $u = \sqrt[3]{t^2} + 2\sqrt{t^3}$

9. $z(y) = \frac{A}{y^{10}} + Be^y$

10. $y = e^{x+1} + 1$

11. Find an equation of the tangent to the curve $y = x^4 + 2e^x$ at the point $(0,2)$.

12. Find the points on the curve $y = 2x^3 + 3x^2 - 12x + 1$ where the tangent is horizontal.

13. Find the equation of the tangent line to $f(x) = x^5 - 5x + 1$ at $x = -2$.

14. Find the tangent line equation to $F(x) = \frac{5}{x^2} - \sqrt{x}$ at $x = 1$.

15. Find the equation of all tangent lines that have a slope of 5 for the function $y = \frac{x^2}{2} + 4x$

16. Find the equation of the tangent line for $f(x) = x^2 - 5x + 6$ that has a **<u>slope</u>** of 4.

17. Find the equations of the lines tangent and normal to the function $f\left(x\right)=5-x^3+\sqrt{x}$ at the point (1,5)

Answers: 1.1 Homework Set A

1. $f(x)=x^2+3x-4$

$f'(x)=2x+3$

2. $f(t)=\frac{1}{4}\left(t^4+8\right)$

$f'(t)=t^3$

3. $y=x^{-2/3}$

$\frac{dy}{dx}=-\frac{2}{3}x^{-5/3}$

4. $y=5e^x+3$

$\frac{dy}{dx}=5e^x$

5. $v(r)=\frac{4}{3}\pi r^3$

$v'(r)=4\pi r^2$

6. $g(x)=x^2+\frac{1}{x^2}$

$g'(x)=2x-2x^{-3}$

7. $y=\frac{x^2+4x+3}{\sqrt{x}}$

$\frac{dy}{dx}=\frac{3}{2}x^{1/2}+2x^{-1/2}-\frac{3}{2}x^{-3/2}$

8. $u=\sqrt[3]{t^2}+2\sqrt{t^3}$

$\frac{du}{dt}=\frac{2}{3}t^{-1/3}+3t^{1/2}$

9. $z(y)=\frac{A}{y^{10}}+Be^y$

$z'(y)=-10Ay^{-11}+Be^y$

10. $y=e^{x+1}+1$

$\frac{dy}{dx}=e^{x+1}$

11. Find an equation of the tangent to the curve $y=x^4+2e^x$ at the point $(0,2)$.

$y=2x+2$

12. Find the points on the curve $y=2x^3+3x^2-12x+1$ where the tangent is horizontal.

(1, –6) (–2, 21)

13. Find the equation of the tangent line to $f(x)=x^5-5x+1$ at $x=-2$.

$y+21=75(x+2)$

14. Find the tangent line equation to $F(x)=\frac{5}{x^2}-\sqrt{x}$ at $x=1$.

$y-4=\frac{-21}{2}(x-1)$

15. Find the equation of all tangent lines that have a slope of 5 for the function $y = \frac{x^2}{2} + 4x$

$y - \frac{9}{2} = 5(x-1)$

16. Find the equation of the tangent line for $f(x) = x^2 - 5x + 6$ that has a **slope** of 4.

$y - \frac{15}{4} = 4\left(x - \frac{9}{2}\right)$

17. Find the equations of the lines tangent and normal to the function $f(x) = 5 - x^3 + \sqrt{x}$ at the point (1,5)

$y - 5 = -\frac{5}{2}(x-1)$

1.2: Composite Functions and the Chain Rule

Composite Function--A function made of two other functions, one within the other. For Example, $y=\sqrt{16x-x^3}$, $y=\sin\ x^3$, $y=\cos^3 x$, and $y=(x^2+2x-5)^3$. The general symbols are $f(g(x))$ or $[f\circ g](x)$.

Ex 1 Given $f(x)=\cos^{-1}x$, $g(x)=x^2-1$, and $h(x)=\sqrt{1+x^2}$, find (a) $f\left(g\left(\sqrt{2}\right)\right)$, (b) $h\left(g(1)\right)$, and (c) $f\left(h\left(g(1)\right)\right)$.

(a) $g\left(\sqrt{2}\right)=\left(\sqrt{2}\right)^2-1=1$, so $f\left(g\left(\sqrt{2}\right)\right)=f(1)=\cos^{-1}(1)=0$.

(b) $g(1)=0$, so $h\left(g(1)\right)=h(0)=\sqrt{1+0^2}=1$.

(c) $g(1)=0$ and $h\left(g(1)\right)=h(0)=\sqrt{1+0^2}=1$, so
$f\left(h\left(g(1)\right)\right)=f(1)=\cos^{-1}(1)=0$

So how do we take the derivative of a composite function? There are two (or more) functions that must be differentiated, but, since one is inside the other, the derivatives cannot be taken at the same time. Just as a radical cannot be distributed over addition, a derivative cannot be distributed concentrically. The composite function is like a matryoshka (Russian doll) that has a doll inside a doll. The derivative is akin to opening them. They cannot both be opened at the same time and, when one is opened, there is an unopened one within. You end up with two open dolls next to each other.

The Chain Rule: $\frac{d}{dx}\left[f(g(x))\right]=f'(g(x))\cdot g'(x)$

If you think of the inside function (the $g(x)$) as equaling *u*, we could write The Chain Rule like this: $\frac{d}{dx}\left[f(u)\right]=f'(u)\cdot\frac{du}{dx}$. This is the way that most of the derivatives are written with The Chain Rule.

The Chain Rule is one of the cornerstones of Calculus. It can be embedded within each of the other Rules, as it was in the introduction to this chapter. So the Power Rule and Exponential Rules in the last section really should have been stated as:

The Power Rule:	The Exponential Rules:
$\frac{d}{dx}\left[u^n\right] = n\,u^{n\text{-}1} \cdot \frac{du}{dx}$	$\frac{d}{dx}\left[e^u\right]=\left(e^u\right)\frac{du}{dx}$ $\frac{d}{dx}\left[a^u\right]=\left(a^u \cdot \ln a\right)\frac{du}{dx}$

where u is a function of x.

OBJECTIVE

Find the Derivative of Composite Functions.

Ex 2 If $y=\sqrt{16x-x^3}$, find $\frac{dy}{dx}$.

$$y=\sqrt{16x-x^3}=\left(16x-x^3\right)^{1/2}$$

$$\frac{dy}{dx}=\frac{1}{2}\left(16x-x^3\right)^{-1/2}\left(16-3x^2\right)$$

$$=\frac{\left(16-3x^2\right)}{2\left(16x-x^3\right)^{1/2}}$$

In this case, the $\sqrt{\ }$ is the *f* function and the polynomial $16x-x^3$ is the *g*. Each derivative is found by the Power Rule, but, as $16x-x^3$ is inside the $\sqrt{\ }$, it is inside the derivative of the $\sqrt{\ }$.

Ex 3 $\dfrac{d}{dx}\left[\left(4x^2-2x-1\right)^{10}\right]$

$$\frac{d}{dx}\left[\left(4x^2-2x-1\right)^{10}\right]=10\left(4x^2-2x-1\right)^9\ (8x-2)$$
$$=20\left(4x^2-2x-1\right)^9\ (4x-1)$$

Ex 4 $\dfrac{d}{dx}\left[e^{4x^2}\right]$

$$\frac{d}{dx}\left[e^{4x^2}\right]=e^{4x^2}\cdot 8x\ =\ 8xe^{4x^2}$$

Ex 5 Given this table of values, find $\dfrac{d}{dx}\left[f\left(g(x)\right)\right]$ and $\dfrac{d}{dx}\left[g\left(f(x)\right)\right]$ at $x=1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

$\dfrac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(x)\right)g'(x)$ and $\dfrac{d}{dx}\left[g\left(f(x)\right)\right]=g'\left(f(x)\right)f'(x)$.

At $x=1$,
$$\frac{d}{dx}\left[f\left(g(x)\right)\right]=f'\left(g(1)\right)g'(1)$$
$$=f'(2)(6)$$
$$=(5)(6)$$
$$=30$$

At $x=1$,
$$\frac{d}{dx}\left[g\left(f(x)\right)\right]=g'\left(f(1)\right)f'(1)$$
$$=g'(3)(4)$$
$$=(9)(4)$$
$$=36$$

Ex 6 Using the table from example 5, find $\left.\frac{d}{dx}\right|_{x=2}\left[f\left(f(x)\right)\right]$.

$$\left.\frac{d}{dx}\right|_{x=2}\left[f\left(f(x)\right)\right]=20$$

Of course, this kind of problem can extend to where we don't have a table of values, but where we still only have a function that we can derive symbolically.

Ex 7 If $g(2)=-5$ and $g'(2)=4$, find $f'(2)$ if $f(x)=e^{g(x)}+g\left(x^3-6\right)+\left(g(x)\right)^3$

Notice that while we do not actually know the function that *g* represents, we still can take its derivative, because we know the derivative of *g* is *g*'. Of course, the Chain Rule is still essential in this process.

$$f(x)=e^{g(x)}+g\left(x^3-6\right)+\left(g(x)\right)^3$$
$$f'(x)=e^{g(x)}\cdot g'(x)+g'\left(x^3-6\right)\cdot\left(3x^2\right)+3\left(g(x)\right)^2\cdot g'(x)$$

$$f'(2)=e^{g(2)}\cdot g'(2)+g'\left(2^3-6\right)\cdot\left(3(2)^2\right)+3\left(g(2)\right)^2\cdot g'(2)$$
$$f'(2)=e^{-5}\cdot 4+g'(2)\cdot(12)+3(-5)^2\cdot 4$$

$$f'(2)=\frac{4}{e^5}+4\cdot(12)+300=\frac{4}{e^5}+348$$

1.2 Homework Set A

1. $\frac{d}{dx}\left[x^3+4x-\pi\right]^{-7}$

2. $y=e^{\sqrt{x}}$, find $\frac{dy}{dx}$.

3. $f(x)=\sqrt[4]{1+2x+x^3}$, find $f\,'(x)$.

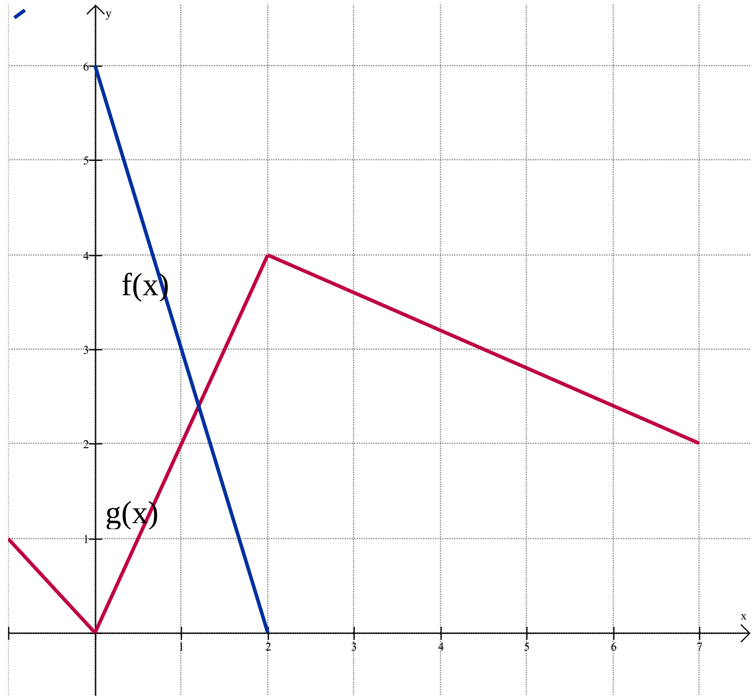
4. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$
2	1	7
8	5	-3

a. $g'(2)$, where $g(x)=\left[f(x)\right]^3$ b. $h'(2)$, where $h(x)=f(x^3)$

5. Given this graph and $u = f\left(g(x)\right)$, $v = g\left(f(x)\right)$, and $w = f\left(f(x)\right)$, find

(a) $u'(4.5)$ (b) $v'(1)$ (c) $w'(1)$



6. If $f(x) = \left(x^3 + 2x\right)^{37}$, find $f'(x)$.

7. If $g(2)=3$ and $g'(2)=-4$, find $f'(2)$ if $f(x)=e^{(g(x))}$.

8. $f(x)=\sqrt{4-\frac{4}{9}x^2}$; find $f'(\sqrt{5})$

9. $\frac{d}{dx}\left[\sqrt{3x^2-4x+9}\right]$

10. $y=\sqrt[7]{x^3-2x}$; find $\frac{dy}{dx}$

1.2 Homework Set B

1. Given the table of values below, find $g'(2)$ if $g(x)=e^{f(h(x))}$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\ln 4$	1	3	-7
3	$\ln 9$	2	-9	11
-7	$\ln 49$	3	8	-1

2. If $g(2)=\pi$ and $g'(2)=-2\pi$, find $f'(2)$ if $f(x)=e^{(g(x))}+g\left(x^2-2x+2\right)$

3. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=\left(h(x)\right)^3+e^{h(x)}$

4. Find $\frac{d}{dx}\left[f\left(g\left(h\left(x^3\right)\right)\right)\right]$

5. If $g(0)=5$ and $g'(0)=-3$, find $f'(0)$ if $f(x)=g(x)+g\left(x^3-6\right)+\left(g(x)\right)^2$

1.2 Homework Set A

1. $\frac{d}{dx}\left[x^3+4x-\pi\right]^{-7}$

   $=-7\left(x^3+4x-\pi\right)^{-8}\left(3x^2+4\right)$

2. $y=e^{\sqrt{x}}$, find $\frac{dy}{dx}$.

   $\frac{dy}{dx}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

3. $f(x)=\sqrt[4]{1+2x+x^3}$, find $f'(x)$.

   $f'(x)=\frac{1}{4}\left(1+2x+x^3\right)^{-3/4}\left(2+3x^2\right)$

4. Given the following table of values, find the indicated derivatives.

x	$f(x)$	$f'(x)$
2	1	7
8	5	-3

a. $g'(2)$, where $g(x)=\left[f(x)\right]^3$

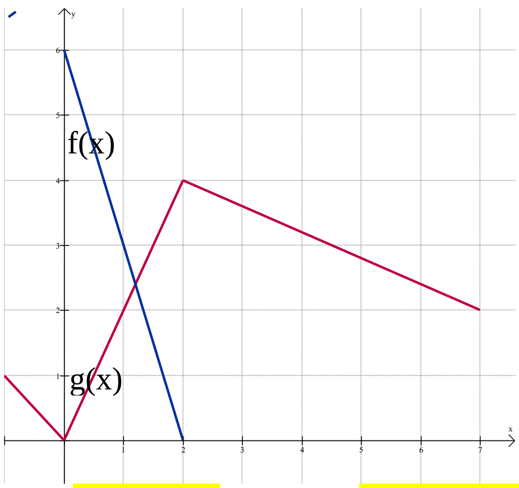
$g'(2)=21$

b. $h'(2)$, where $h(x)=f(x^3)$

$h'(2)=-36$

5. Given this graph and $u=f\left(g(x)\right)$, $v=g\left(f(x)\right)$, and $w=f\left(f(x)\right)$, find

(a) $u'(4.5)$ (b) $v'(1)$ (c) $w'(1)$



(a) $u'(4.5)=\frac{-4}{15}$ (b) $v'(1)=\frac{2}{15}$ (c) $w'(1)=\frac{-1}{9}$

6. If $f(x)=\left(x^3+2x\right)^{37}$, find $f'(x)$.

$f'(x)=37\left(x^3+2x\right)^{36}\left(3x^2+2\right)$

7. If $g(2)=3$ and $g'(2)=-4$, find $f'(2)$ if $f(x)=e^{(g(x))}$.

$f'(2)=-4e^3$

8. $f(x)=\sqrt{4-\frac{4}{9}x^2}$; find $f'\left(\sqrt{5}\right)$

$f'\left(\sqrt{5}\right)=-\frac{16}{27}\sqrt{5}$

9. $\frac{d}{dx}\left[\sqrt{3x^2-4x+9}\right]$

$=\frac{1}{2}\left(3x^2-4x+9\right)^{-1/2}\left(6x-4\right)$

10. $y=\sqrt[7]{x^3-2x}$; find $\frac{dy}{dx}$

$\frac{dy}{dx}=\frac{1}{7}\left(x^3-2x\right)^{-6/7}\left(3x^2-2\right)$

1.2 Homework Set B

1. Given the table of values below, find $g'(2)$ if $g(x)=e^{f\left(h(x)\right)}$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\ln 4$	1	3	-7
3	$\ln 9$	2	-9	11
-7	$\ln 49$	3	8	-1

$g'(2)=-126$

2. If $g(2)=\pi$ and $g'(2)=-2\pi$, find $f'(2)$ if $f(x)=e^{(g(x))}+g\left(x^2-2x+2\right)$

$f'(2)=-2\pi e^{\pi}-4\pi$

3. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=\left(h(x)\right)^3+e^{h(x)}$

$f'(1)=225+3e^5$

4. Find $\frac{d}{dx}\left[f\left(g\left(h\left(x^3\right)\right)\right)\right]$

$=3x^2\cdot f'\left(g\left(h\left(x^3\right)\right)\right)\cdot g'\left(h\left(x^3\right)\right)\cdot h'\left(x^3\right)$

5. If $g(0)=5$ and $g'(0)=-3$, find $f'(0)$ if $f(x)=g(x)+g\left(x^3-6\right)+\left(g(x)\right)^2$

$f'(0)=-33$

1.3: Introduction to Implicit Differentiation, an Advanced Use of the Chain Rule

One of the most common applications of the Chain Rule in Calculus is called implicit differentiation. Recall that differentiation just means "to take the derivative".

All of what we did in precalculus was explicit differentiation – that is, the functions were explicitly solved for the dependent variable (usually *y*). But we can actually use the Chain Rule to find the derivative of functions and relations where we do not have the dependent variable isolated.

Ex. 1 Find the $\frac{dy}{dx}$ for the function $y = 5x^4 - 22x$

$$\frac{d}{dx}\left[y = 5x^4 - 22x\right]$$

$$\frac{dy}{dx} = 20x^3 - 22$$

Now, this is an obvious and easy example, but notice what we have on the left side of the equation – in the process of taking the derivative, the *y* became a $\frac{dy}{dx}$. That is because the derivative of *y* is $\frac{dy}{dx}$!

OBJECTIVES
Find Derivatives of functions and relations that are not explicitly solved for *y*.

As you saw in example 1, the derivative of *y* is $\frac{dy}{dx}$. This applies wherever the *y* is located in a problem. And if the *y* is located within another function – like y^4 – which is actually just $(y)^4$, then the Chain Rule applies, as it has in every other aspect of the derivative we have ever worked with.

Ex. 2 Find $\frac{dy}{dx}$ for $16 = x - x^2 + y^4$

$$\frac{d}{dx}\left[16 = x - x^2 + y^4\right]$$

$$0 = 1 - 2x + 4y^3 \frac{dy}{dx}$$

Simply take the derivative, using the Chain Rule on the y^4

$$4y^3 \frac{dy}{dx} = 2x - 1$$

Then just use basic algebra to solve for $\frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{2x-1}{4y^3}$$

So the derivative of $16 = x - x^2 + y^4$ is $\frac{dy}{dx} = \frac{2x-1}{4y^3}$.

Note that just like the original equation, there is still a *y* in the solution of the problem. This is a very common occurrence in implicit differentiation.

In the case above, we could've solved it explicitly for *y* by using algebra, and done the derivative that way. But in many cases, this is not a possibility.

Ex. 3 Find $\frac{dy}{dx}$ for $x^2 + y^2 + e^y = 17$

$$\frac{d}{dx}\left[x^2 + y^2 + e^y = 17\right]$$

$$2x + 2y\frac{dy}{dx} + e^y \frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} + e^y \frac{dy}{dx} = -2x$$

$$\frac{dy}{dx}\left(2y + e^y\right) = -2x$$

$$\frac{dy}{dx} = \frac{-2x}{2y + e^y}$$

Ex. 4 Find the equations of the lines tangent to the relation $x^2 + y^2 + e^y = 17$ at $y = 0$.

Note that we were given a *y*-value, so we can plug that in and find an *x*-value. We need a point to use to find the equation of the tangent line.

$$x^2 + 0^2 + e^0 = 17$$
$$x^2 = 16$$
$$x = \pm 4$$

So we have two tangent lines, one passing through (4,0) and the other passing through (–4,0).

From example 3, we know that $\frac{dy}{dx} = \frac{-2x}{2y + e^y}$, so we find the values for $\frac{dy}{dx}$ at the two points.

$$\left.\frac{dy}{dx}\right|_{(4,0)} = \frac{-2(4)}{2(0) + e^0} = -8 \qquad \left.\frac{dy}{dx}\right|_{(-4,0)} = \frac{-2(-4)}{2(0) + e^0} = 8$$

Now we use the points and slopes and get the tangent lines:

$y = -8(x - 4)$ and $y = 8(x + 4)$

1.3 Homework Set A

Find $\frac{dy}{dx}$ for each of the following relations.

1. $x^2 + 4y^2 = 16$

2. $e^x + y^3 - 2y = 25$

3. $x + 5y^2 = \sqrt{y - 16}$

4. $e^{y^2} + e^{x^2} = 20y$

5. Find the equations for the lines tangent and normal to the relation $x^2 - 2x + y^2 + 4y = 20$ passing through the point (5,1).

6. Find the equations for the lines tangent to the relation $x^3 + y^2 - 8y - 21 = 0$ when $x = 1$

1.3 Homework Set A

Find $\frac{dy}{dx}$ for each of the following relations.

1. $x^2+4y^2=16$

$\frac{dy}{dx}=\frac{-x}{4y}$

2. $e^x+y^3-2y=25$

$\frac{dy}{dx}=\frac{-e^x}{3y^2-2}$

3. $x+5y^2=\sqrt{y-16}$

$\frac{dy}{dx}=\frac{1}{\frac{1}{2}(y-16)^{-1/2}-10y}$

or

$\frac{dy}{dx}=\frac{2\sqrt{y-16}}{1-20y\sqrt{y-16}}$

4. $e^{y^2}+e^{x^2}=20y$

$\frac{dy}{dx}=\frac{2xe^{x^2}}{20-2ye^{y^2}}$

5. Find the equations for the lines tangent and normal to the relation $x^2-2x+y^2+4y=20$ passing through the point (5,1).

Tangent: $y-1=-\frac{4}{3}(x-5)$ Normal: $y-1=\frac{3}{4}(x-5)$

6. Find the equations for the lines tangent to the relation $x^3+y^2-8y-21=0$ when $x = 1$

Tangent to (1,10): $y-10=-\frac{1}{4}(x-1)$

Tangent to (1,–2): $y+2=\frac{1}{4}(x-1)$

1.4: Trig and Log Rules

Trigonometric--Defn: "A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure."
Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

Logarithmic--Defn: "The inverse of an Exponential function."
Means: there is a Log or Ln in the equation.

$$\frac{d}{dx}[\sin u] = (\cos u)\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = (-\csc u \cot u)\frac{du}{dx}$$
$$\frac{d}{dx}[\cos u] = (-\sin u)\frac{du}{dx} \qquad \frac{d}{dx}[\sec u] = (\sec u \tan u)\frac{du}{dx}$$
$$\frac{d}{dx}[\tan u] = (\sec^2 u)\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = (-\csc^2 u)\frac{du}{dx}$$

$$\frac{d}{dx}[\ln u] = \left(\frac{1}{u}\right)\frac{du}{dx}$$
$$\frac{d}{dx}[\log_a u] = \left(\frac{1}{u \cdot \ln a}\right)\frac{du}{dx}$$

Note that all these Rules are expressed in terms of the Chain Rule.

OBJECTIVES

Find Derivatives involving Trig and Logarithmic Functions.

Ex 1 $\dfrac{d}{dx}\left(\sin^3 x\right)$

$$\frac{d}{dx}\left(\sin^3 x\right) = 3\sin^2 x \cos x$$

Ex 2 $\dfrac{d}{dx}\left[\sin\left(x^3\right)\right]$

$$\frac{d}{dx}\left[\sin\left(x^3\right)\right] = \cos x^3\left(3x^2\right)$$

$$= 3x^2 \cos x^3$$

Ex 3 $\dfrac{d}{dx}\left[\ln\left(4x^3\right)\right]$

$$\frac{d}{dx}\left[\ln\left(4x^3\right)\right] = \frac{1}{4x^3}\cdot 12x^2$$

$$= \frac{3}{x}$$

We could have also simplified algebraically before taking the derivative:

$$\frac{d}{dx}\left[\ln\left(4x^3\right)\right] = \frac{d}{dx}\left[\ln 4 + 3\ln x\right]$$
$$= \frac{3}{x}$$

Oftentimes it is significantly easier to take a derivative if we simplify the function first – in the above case, it eliminated the need for The Chain Rule.

Of course, composites can involve more than two functions. The Chain Rule has as many derivatives in the chain as there are functions.

Ex 4 $\frac{d}{dx}\left(\sec^5 3x^4\right)$

$$\frac{d}{dx}\left(\sec^5 3x^4\right) = 5\sec^4 3x^4 \left(\sec 3x^4 \tan 3x^4\right)\left(12x^3\right)$$

$$= 60x^3 \sec^5 3x^4 \tan 3x^4$$

Ex 5 $\frac{d}{dx}\ln\left(\cos\sqrt{x}\right)$

$$\frac{d}{dx}\ln\left(\cos\sqrt{x}\right) = \frac{1}{\cos x^{1/2}} \cdot \left(-\sin x^{1/2}\right)\left(\frac{1}{2}x^{-1/2}\right)$$

$$= -\tan x^{1/2}\left(\frac{1}{2}x^{-1/2}\right)$$

$$= \frac{-\tan x^{1/2}}{2x^{1/2}}$$

1.4 Homework Set A

For #1-12, find the derivatives of the given functions.

1. $y = \sin 4x$

2. $y = 4\sec x^5$

3. $y = a^3 + \cos^3 x$

4. $y = \cot^2(\sin\theta)$

5. $f(t) = \sqrt[3]{1+\tan t}$

6. $f(\theta) = \ln\left(\cos\theta\right)$

7. $y = \cos\left(a^3 + x^3\right)$

8. $y = \tan^2\left(3\theta\right)$

9. $f(x)=\cos(\ln x)$

10. $f(x)=\sqrt[5]{\ln x}$

11. $f(x)=\log_{10}(2+\sin x)$

12. $f(x)=\log_2(1-3x)$

13. Find the equation of the tangent line to $y=x+\cos x$ at the point $(0,1)$.

14. Find the equation of the tangent line to $y=\sec x-2\cos x$ at the point $\left(\pi/3,1\right)$.

15. Find all points on the graph of $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

16. Find the equations of the lines tangent and normal to the function $f(x) = e^{x^2} + \ln\left(x^2 + 4x + 5\right)$ at the point (0,1+ln5)

17. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

18. Find the equation of the line tangent to $y = \sec(2x) + \cot(2x)$ through the point $\left(\frac{\pi}{8}, 1+\sqrt{2}\right)$. Use exact values in your answers.

1.4 Homework Set B

1. Given the table of values below, find $g'(3)$ if $g(x) = f(h(x))$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	-7
3	$\frac{\pi}{4}$	2	$\frac{\pi}{3}$	1
$\frac{\pi}{3}$	2	3	$\frac{\pi}{4}$	-1

2. $\frac{d}{d\theta}\left[e^{\csc\theta}+\ln\left(\cot\theta^{2}\right)-\sec\theta\right]$

3. If $g(3)=\frac{\pi}{2}$, $g'(3)=\frac{\pi}{4}$, and $f(x)=g(x)+g\left(-3\cos\left(\frac{\pi}{3}x\right)\right)-e^{\sin\left(g(x)\right)}$, find $f'(3)$

4. If $z=\ln\left(\cot\theta\right)+\sec\left(\ln\theta\right)$, find $\frac{dz}{d\theta}$

5. $\frac{d}{dx}\left[\ln\left(\sec\left(x^3+5\ln x+7\right)^3\right)\right]$

6. If $z=\ln(\cos t)+\sec\left(e^t\right)+7\pi^2$, find $\frac{dz}{dt}$

7. If $z=\ln(\tan t)+\sin\left(e^t\right)+7\pi^2$, find $\frac{dz}{dt}$

8. If $z=\ln(\cos\theta)+\sin(\ln\theta)$, find $\frac{dz}{d\theta}$

9. Find $f'\left(\frac{\pi}{6}\right)$ when $f(x)=\cos^3(3x)$

10. Find $g'(2)$ when $g(x)=\ln\left(x^2-3\right)$

11. $\frac{d}{dx}\left[\ln\left(\sqrt{x^2+4x-5}\right)\right]$

12. $\frac{d}{dt}\left[\sin^5\left(\ln(7t+3)\right)\right]$

13. $\frac{d}{dx}\left[\csc\left(\ln\left(7x^2+x\right)\right)\right]$

14. $\frac{d}{dt}\left[\ln\left(\sqrt{e^{4t^2+6}}\right)\right]$

15. $\frac{d}{dx}\left[\sec(5x)+\cot\left(e^{x}\right)-10\ln x\right]$

For problems 16 to 18, find the first derivative for the following functions

16. $z(y)=\frac{A}{y^{10}}+Be^{y}$

17. $f(r)=\frac{A}{r}+Be^{\sin r}$

18. $y=\frac{x^{3}+3x^{2}-12}{x^{2}}$

19. $\frac{d}{dx}\left[\frac{d}{dx}\left[\sqrt{9x-27x^{2}+\frac{5}{x^{3}}}\right]\right]$

20. $\frac{d}{dx}\left[\frac{d}{dx}\left[9x-27x^{2}+\frac{5}{x^{3}}\right]\right]$

21. $\frac{d}{dx}\left[\ln\left(\tan\left(x^{2}+5e^{x}+7\right)^{3}\right)\right]$

22. $\frac{d}{dx}\left[\frac{\cos\left(\ln\left(5x^{2}\right)\right)}{\sin\left(\ln\left(5x^{2}\right)\right)}\right]$

1.4 Homework Set A

1. $y=\sin 4x$

$\dfrac{dy}{dx}=4\cos 4x$

2. $y=4\sec x^5$

$y'=20x^4\sec x^5\tan x^5$

3. $y=a^3+\cos^3 x$

$\dfrac{dy}{dx}=-3\cos^2 x\sin x$

4. $y=\cot^2(\sin\theta)$

$\dfrac{dy}{dx}=-2\cos\theta\cot(\sin\theta)\csc^2(\sin\theta)$

5. $f(t)=\sqrt[3]{1+\tan t}$

$f'(t)=\dfrac{\sec^2 t}{3(1+\tan t)^{2/3}}$

6. $f(\theta)=\ln(\cos\theta)$

$f'(\theta)=-\tan\theta$

7. $y=\cos\left(a^3+x^3\right)$

$y=-3x^2\sin\left(a^3+x^3\right)$

8. $y=\tan^2(3\theta)$

$\dfrac{dy}{dx}=6\tan(3\theta)\sec^2(3\theta)$

9. $f(x)=\cos(\ln x)$

$f'(x)=\dfrac{-\sin(\ln x)}{x}$

10. $f(x)=\sqrt[5]{\ln x}$

$f'(x)=\dfrac{1}{5x\ln^{4/5} x}$

11. $f(x)=\log_{10}(2+\sin x)$

$f'(x)=\dfrac{\cos x}{\ln 10(2+\sin x)}$

12. $f(x)=\log_2(1-3x)$

$f'(x)=\dfrac{-3}{\ln 2(1-3x)}$

13. Find the equation of the tangent line to $y=x+\cos x$ at the point $(0,1)$.

$y-1=1(x-0)$

14. Find the equation of the tangent line to $y=\sec x-2\cos x$ at the point $\left(\pi/3,1\right)$.

$y-1=3\sqrt{3}\left(x-\pi/3\right)$

15. Find all points on the graph of $f(x) = 2\sin x + \sin^2 x$ at which the tangent line is horizontal.

$\left(\frac{\pi}{2} \pm 2\pi n, 3\right), \left(-\frac{\pi}{2} \pm 2\pi n, 1\right)$

16. Find the equations of the lines tangent and normal to the function $f(x) = e^{x^2} + \ln\left(x^2 + 4x + 5\right)$ at the point (0,1+ln5)

$y - 1 - \ln 5 = \frac{4}{5}(x - 0)$

17. Find the equation of the line tangent to $y = \frac{2}{\pi}x + \cos(4x)$ when $x = \frac{\pi}{2}$.

$y - 2 = \frac{2}{\pi}\left(x - \pi/2\right)$

18. Find the equation of the line tangent to $y = \sec(2x) + \cot(2x)$ through the point $\left(\frac{\pi}{8}, 1 + \sqrt{2}\right)$. Use exact values in your answers.

$y - 1 - \sqrt{2} = \left(2\sqrt{2} - 4\right)\left(x - \frac{\pi}{8}\right)$

1.4 Homework Set B

1. Given the table of values below, find $g'(3)$ if $g(x) = f\left(h(x)\right)$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	-7
3	$\frac{\pi}{4}$	2	$\frac{\pi}{3}$	1
$\frac{\pi}{3}$	2	3	$\frac{\pi}{4}$	-1

$g'(3) = 3$

2. $\dfrac{d}{d\theta}\left[e^{\csc\theta}+\ln\left(\cot\theta^2\right)-\sec\theta\right]$

$=-\csc\theta\cot\theta e^{\csc\theta}-\dfrac{2\theta\csc^2\theta^2}{\cot\theta^2}-\sec\theta\tan\theta$

3. If $g(3)=\dfrac{\pi}{2}$, $g'(3)=\dfrac{\pi}{4}$, and $f(x)=g(x)+g\left(-3\cos\left(\dfrac{\pi}{3}x\right)\right)-e^{\sin(g(x))}$, find $f'(3)$

$f'(3)=\dfrac{\pi}{4}$

4. If $z=\ln(\cot\theta)+\sec(\ln\theta)$, find $\dfrac{dz}{d\theta}$

$\dfrac{dz}{d\theta}=\dfrac{-\csc^2\theta}{\cot\theta}+\dfrac{\sec(\ln\theta)\tan(\ln\theta)}{\theta}$

5. $\dfrac{d}{dx}\left[\ln\left(\sec\left(x^3+5\ln x+7\right)^3\right)\right]$

$=3\left(x^3+5\ln x+7\right)^2\left(3x^2+\dfrac{5}{x}\right)\tan\left(x^3+5\ln x+7\right)^3$

6. If $z=\ln(\cos t)+\sec\left(e^t\right)+7\pi^2$, find $\dfrac{dz}{dt}$

$\dfrac{dz}{dt}=-\tan t+e^t\sec\left(e^t\right)\tan\left(e^t\right)$

7. If $z=\ln(\tan t)+\sin\left(e^t\right)+7\pi^2$, find $\dfrac{dz}{dt}$

$\dfrac{dz}{dt}=\dfrac{\sec^2 t}{\tan t}+e^t\cos\left(e^t\right)$

8. If $z=\ln(\cos\theta)+\sin(\ln\theta)$, find $\dfrac{dz}{d\theta}$

$\dfrac{dz}{d\theta}=-\tan\theta+\dfrac{\cos(\ln\theta)}{\theta}$

9. Find $f'\left(\dfrac{\pi}{6}\right)$ when $f(x)=\cos^3(3x)$

$f'\left(\dfrac{\pi}{6}\right)=0$

10. Find $g'(2)$ when $g(x)=\ln\left(x^2-3\right)$

$g'(2)=4$

11. $\frac{d}{dx}\left[\ln\left(\sqrt{x^2+4x-5}\right)\right]$

$=\frac{x+2}{x^2+4x-5}$

12. $\frac{d}{dt}\left[\sin^5\left(\ln(7t+3)\right)\right]$

$=\frac{35\sin^4\left(\ln(7t+3)\right)\cos\left(\ln(7t+3)\right)}{7t+3}$

13. $\frac{d}{dx}\left[\csc\left(\ln\left(7x^2+x\right)\right)\right]$

$=\frac{-(14x+1)\csc\left(\ln\left(7x^2+x\right)\right)\cot\left(\ln\left(7x^2+x\right)\right)}{7x^2+x}$

14. $\frac{d}{dt}\left[\ln\left(\sqrt{e^{4t^2+6}}\right)\right]$

$=6t$

15. $\frac{d}{dx}\left[\sec(5x)+\cot\left(e^x\right)-10\ln x\right]$

$=5\sec(5x)\tan(5x)-\left(e^x\right)\csc^2\left(e^x\right)-\frac{10}{x}$

16. $z(y)=\frac{A}{y^{10}}+Be^y$

$z'(y)=-10Ay^{-11}+Be^y$

17. $f(r)=\frac{A}{r}+Be^{\sin r}$

$f(r)=-Ar^{-2}+B\cos r\cdot e^{\sin r}$

18. $y=\frac{x^3+3x^2-12}{x^2}$

$\frac{dy}{dx}=1+24x^{-3}$

19. $\frac{d}{dx}\left[\frac{d}{dx}\left[\sqrt{9x-27x^2+\frac{5}{x^3}}\right]\right]$

$=\frac{\left(18x-54x^2+10x^{-3}\right)\left(60x^{-5}+54\right)-\left(9-54x-15x^{-4}\right)\left(18-108x-30x^{-4}\right)}{\left(18x-54x^2+10x^{-3}\right)^2}$

20. $\frac{d}{dx}\left[\frac{d}{dx}\left[9x-27x^2+\frac{5}{x^3}\right]\right]$

$=54-60x^{-5}$

21. $\frac{d}{dx}\left[\ln\left(\tan\left(x^2+5e^x+7\right)^3\right)\right]$

$= \frac{3\left(x^2+5e^x+7\right)^2\sec^2\left(x^2+5e^x+7\right)^3}{\tan\left(x^2+5e^x+7\right)^3}$

22. $\frac{d}{dx}\left[\frac{\cos\left(\ln\left(5x^2\right)\right)}{\sin\left(\ln\left(5x^2\right)\right)}\right]$

$= -\frac{2\csc^2\left(\ln\left(5x^2\right)\right)}{x}$

1.5: Product and Quotient Rules

Remember:

$$\textbf{The Product Rule}: \frac{d}{dx}\left[u \cdot v\right] = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\textbf{The Quotient Rule}: \frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u \cdot \frac{du}{dx} - v \cdot \frac{dv}{dx}}{v^2}$$

OBJECTIVE

Find the Derivative of a product or quotient of two functions.

Ex 1 $\frac{d}{dx}\left(x^2 \sin x\right)$

$$\frac{d}{dx}\left(x^2 \sin x\right) = x^2 \cos x + \sin x\ (2x)$$

$$= x^2 \cos x + 2x \sin x$$

Ex 2 $\frac{d}{dx}\left(\frac{x^2 + 2x - 3}{x-4}\right)$

$$U = x^2 + 2x - 3, \text{ so } \frac{du}{dx} = 2x + 2$$

$$V = x - 4, \text{ so } \frac{dv}{dx} = 1$$

$$f'(x) = \frac{V \cdot \frac{du}{dx} - U \cdot \frac{dv}{dx}}{V^2} = \frac{(x-4)\cdot(2x+2) - (x^2+2x-3)\cdot 1}{(x-4)^2}$$

$$= \frac{2x^2 - 6x - 8 - x^2 - 2x + 3}{(x-4)^2}$$

$$= \frac{x^2 - 8x - 5}{(x-4)^2}$$

Ex 3 $\frac{d}{dx}\left(\frac{x^2-4x+3}{2x^2-5x-3}\right)$

Notice that this problem becomes much easier if we simplify before applying the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\left(\frac{x^2-4x+3}{2x^2-5x-3}\right) &= \frac{d}{dx}\left(\frac{(x-1)(x-3)}{(2x+1)(x-3)}\right)\\ &= \frac{d}{dx}\left(\frac{x-1}{2x+1}\right)\\ &= \frac{(2x+1)(1)-(x-1)(2)}{(2x+1)^2}\\ &= \frac{3}{(2x+1)^2}\end{aligned}$$

Ex 4 $\frac{d}{dx}\left(\frac{\cot\ 3x}{x^2+1}\right)$

$$\begin{aligned}\frac{d}{dx}\left(\frac{\cot\ 3x}{x^2+1}\right) &= \frac{\left(x^2+1\right)\left(-\csc^2 3x\right)(3)-\left(\cot\ 3x\right)2x}{\left(x^2+1\right)^2}\\ &= \frac{-3x^2\csc^2 3x-3\csc^2 3x-2x\ \cot\ 3x}{\left(x^2+1\right)^2}\end{aligned}$$

Ex 5 $\frac{d}{dx}\left(5^x\cos\ x\right)$

$$\begin{aligned}\frac{d}{dx}\left(5^x\cos\ x\right) &= 5^x\left(-\sin\ x\right)+\cos\ x\left(5^x\ln 5\right)\\ &= 5^x\left(\left(\ln 5\right)\cos\ x-\sin\ x\right)\end{aligned}$$

Remember that, in section 1.1, we said we would need the Product Rule to deal with the derivative of a function where the variable is in both the base and the Exponent. We can now address that situation.

Ex 6 $\frac{d}{dx}\left[(\cos x)^{x^2}\right]$

$$\begin{aligned}\frac{d}{dx}\left[(\cos x)^{x^2}\right] &= \frac{d}{dx}\left(e^{x^2\ln(\cos x)}\right)\\ &= e^{x^2\ln(\cos x)}\left(x^2\frac{1}{\cos x}(-\sin x)+(\ln(\cos x))\ 2x\right)\\ &= (\cos x)^{x^2}\left(2x\ \ln(\cos x)-x^2\ \tan x\right)\end{aligned}$$

1.5 Homework Set A

Find the derivative of the following functions.

1. $y = t^3 \cos t$

2. $g(x) = (1+4x)^5 \left(3+x-x^2\right)^8$

3. $y = \dfrac{-1+\tan x}{\sec x}$

4. $y = (2x-5)^4 \left(8x^2-5\right)^{-3}$

5. $y = xe^{-x^2}$

6. $y = \dfrac{\sin x}{x^2}$

7. $y = e^{x\cos x}$

8. $y = \dfrac{r}{\sqrt{r^2+1}}$

9. $y = x\sin\dfrac{1}{x}$

10. $y = e^{-5x}\cos 3x$

11. $f(x) = x\sqrt{\ln x}$

12. $y = \ln\left(e^{-x} + xe^{-x}\right)$

13. Find the equation of the line tangent to $y = x^2e^{-x}$ at the point $\left(1, \frac{1}{e}\right)$.

14. If $f(x)=\dfrac{x}{\ln x}$, find $f'(e)$.

15. $h(t)=\left(\dfrac{1+x^2}{1-x^2}\right)^{17}$, find $h'(t)$

16. $y=x^2\sqrt{5-x^2}$, find $y'(1)$

17. $f(x)=\left[x\sin(2x)+\tan^4\left(x^7\right)\right]^5$, find $f'(x)$.

18. Find the equation of the lines tangent and normal to $y = x\sin\left(\frac{\pi}{2}\ln x\right)$ when $x = e$

19. Find the equation of the line tangent to $y = e^{x\sin(4x)} + 2$ when $x = 0$

20. Find the equation of the lines tangent and normal to $y = x\sin\left(\frac{1}{x}\right)$ when $x = \frac{4}{\pi}$.

1.5 Homework Set B

1. Find the first derivative for the following function: $x(t)=e^{t^2}\sin\left(t^2-5t^4\right)$

2. Find the first derivative for the following function: $x(t)=e^{5t}\tan\left(3t^4\right)$

3. Find the first derivative for the following function: $y=\dfrac{x^2+2x-15}{x-3}$

4. Find the first derivative for the following function: $x(t)=e^{t}\left(t^2-5t^4\right)$

5. $$\frac{d}{dx}\left[\frac{e^x+7x^2+5}{\sin x^3}\right]$$

6. $$\frac{d}{dy}\left[e^{(\sin y)}\ln\left(\cot\left(e^y\right)\right)\right]$$

7. $$\frac{d}{dx}\left[x^2\sin x^2+\frac{x+1}{\ln x}\right]$$

8. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=\left(h(x)\right)^4+x\ln\left(h(x)\right)$

9. Find $g'(z)$ if $g(z)=\left(\frac{e^{5z}}{1+\ln z}\right)^{118}$

10. $$\frac{d}{dx}\left[x^2\cos x^2+\frac{e^x}{x}\right]$$

11. $\dfrac{d}{dx}\left[\dfrac{x^2+2x-3}{x-4}\right]$

12. $\dfrac{d}{dx}\left[\dfrac{\cos\left(x^2-3\right)}{e^{-5x}}\right]$

13. $\dfrac{d}{dx}\left[x^5\ln\left(5x+4\right)+\dfrac{x}{\ln x}\right]$

14. Find $g'(t)$ if $g(t)=\left(\dfrac{t^2-4}{1-t^2}\right)^{15}$

15. $\dfrac{d}{dx}\left[e^{x^2}\cos x\right]$

16. $\dfrac{d}{dx}\left[\dfrac{1+\tan x}{\ln 4x}\right]$

17. $\frac{d}{dt}\left[\sin t \tan t\right]$

18. $\frac{d}{dx}\left[\frac{1+\ln x}{\csc x}\right]$

19. $\frac{d}{dx}\left[e^{5x^4}\ln(\sin x)\right]$

20. $\frac{d}{dp}\left[5p\sin\left(p+e^{2p}\right)-\ln\left(3p^2+1\right)+\frac{p}{p^2+1}\right]$

21. $\frac{d}{dx}\left[\tan\left(e^x\right)\left(x^4-5x^3+x\right)\right]$

22. $\frac{d}{dx}\left[\frac{5x+2}{\ln(3x+7)}\right]$

23. $\dfrac{d}{dx}\left[\dfrac{x-4}{x^2-9x+20}\right]$

24. $\dfrac{d}{dx}\left[\dfrac{d}{dx}\left[\sin^2(4x+2)\right]\right]$

25. $\dfrac{d}{dc}\left[\dfrac{c^5-12c^3-19c}{3c^3}\right]$

26. Given the table of values below, find $g'(3)$ if $g(x)=f(h(x))\sin(h(x))$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	-7
3	$\frac{\pi}{4}$	2	$\frac{\pi}{3}$	1
$\frac{\pi}{3}$	2	3	$\frac{\pi}{4}$	-1

27. If $g(3)=\frac{\pi}{2}$, $g'(3)=\frac{\pi}{4}$, and $f(x)=x^3g(x)+g\left(-3\cos\left(\frac{\pi}{3}x\right)\right)-e^{\sin(g(x))}$, find $f'(3)$

1.5 Homework Set A

1. $y = t^3 \cos t$

$y' = t^2(3\cos t - t\sin t)$

2. $g(x) = (1+4x)^5(3+x-x^2)^8$

$g'(x) = 4(1+4x)^4(3+x-x^2)^7(17+9x-21x^2)$

3. $y = \dfrac{-1+\tan x}{\sec x}$

$y' = \dfrac{\tan x + 1}{\sec x}$

4. $y = (2x-5)^4(8x^2-5)^{-3}$

$y' = 8(2x-5)^3(8x^2-5)^{-4}(-4x^2+30x-5)$

5. $y = xe^{-x^2}$

$y' = e^{-x^2}(1-2x^2)$

6. $y = \dfrac{\sin x}{x^2}$

$y' = \dfrac{x\cos x - 2\sin x}{x^3}$

7. $y = e^{x\cos x}$

$y' = e^{x\cos x}(\cos x - x\sin x)$

8. $y = \dfrac{r}{\sqrt{r^2+1}}$

$y' = \dfrac{1}{(r^2+1)^{3/2}}$

9. $y = x\sin\dfrac{1}{x}$

$\dfrac{dy}{dx} = -\dfrac{1}{x}\cos\dfrac{1}{x} + \sin\dfrac{1}{x}$

10. $y = e^{-5x}\cos 3x$

$\dfrac{dy}{dx} = e^{-5x}(5\cos 3x - 3\sin 3x)$

11. $f(x) = x\sqrt{\ln x}$

$f'(x) = \dfrac{1+2\ln x}{2\sqrt{\ln x}}$

12. $y = \ln(e^{-x} + xe^{-x})$

$\dfrac{dy}{dx} = \dfrac{-x}{1+x}$

13. Find the equation of the line tangent to $y = x^2e^{-x}$ at the point $\left(1, \frac{1}{e}\right)$.

$y - \dfrac{1}{e} = \dfrac{1}{e}(x-1)$

14. If $f(x)=\frac{x}{\ln x}$, find $f'(e)$.

$f'(e)=0$

15. $h(t)=\left(\frac{1+x^2}{1-x^2}\right)^{17}$, find $h'(t)$

$h'(t)=\frac{68t\left(1+t^2\right)^{16}}{\left(1-t^2\right)^{18}}$

16. $y=x^2\sqrt{5-x^2}$, find $y'(1)$

$y'(1)=\frac{7}{2}$

17. $f(x)=\left[x\sin(2x)+\tan^4\left(x^7\right)\right]^5$, find $f'(x)$.

$f'(x)=5\left[x\sin(2x)+\tan^4\left(x^7\right)\right]^4\left[\sin(2x)+2x\cos 2x+28x^6\tan^3\left(x^7\right)\sec^2\left(x^7\right)\right]$

18. Find the equation of the lines tangent and normal to $y=x\sin\left(\frac{\pi}{2}\ln x\right)$ when $x=e$

Tangent: $y-e=1(x-e)$ Normal: $y-e=-1(x-e)$

19. Find the equation of the line tangent to $y=e^{x\sin(4x)}+2$ when $x=0$

$y=3$

20. Find the equation of the lines tangent and normal to $y=x\sin\left(\frac{1}{x}\right)$ when $x=\frac{4}{\pi}$.

Tan: $y-2\pi\sqrt{2}=\frac{\sqrt{2}(-\pi+4)}{8}\left(x-\frac{4}{\pi}\right)$

Normal: $y-2\pi\sqrt{2}=\frac{4\sqrt{2}}{(-\pi+4)}\left(x-\frac{4}{\pi}\right)$

1.5 Homework Set B

1. Find the first derivative for the following function: $x(t)=e^{t^2}\sin\left(t^2-5t^4\right)$

$x'(t)=te^{t^2}\left[\left(2-5t^2\right)\cos\left(t^2-5t^4\right)+2\sin\left(t^2-5t^4\right)\right]$

2. Find the first derivative for the following function: $x(t)=e^{5t}\tan(3t^4)$

$$x'(t)=e^{5t}\left[12t^3\sec^2(3t^4)+5\tan(3t^4)\right]$$

3. Find the first derivative for the following function: $y=\dfrac{x^2+2x-15}{x-3}$

$$\frac{dy}{dx}=1$$

4. Find the first derivative for the following function: $x(t)=e^t(t^2-5t^4)$

$$x'(t)=te^t(5t^3-20t^2-t+2)$$

5. $\dfrac{d}{dx}\left[\dfrac{e^x+7x^2+5}{\sin x^3}\right]$

$$=\frac{(e^x+14x)\sin x^3-(e^x+7x^2+5)\cos x^3}{\sin^2 x^3}$$

6. $\dfrac{d}{dy}\left[e^{(\sin y)}\ln(\cot(e^y))\right]$

$$=\frac{-e^y\csc^2(e^y)e^{(\sin y)}}{\cot(e^y)}+\cos y\cdot e^{(\sin y)}\ln(\cot(e^y))$$

7. $\dfrac{d}{dx}\left[x^2\sin x^2+\dfrac{x+1}{\ln x}\right]$

$$=2x^3\cos x^2+2x\sin x^2+\frac{(x^2+x)\ln(x)-1}{x\ln^2 x}$$

8. If $h(1)=5$ and $h'(1)=3$, find $f'(1)$ if $f(x)=(h(x))^4+x\ln(h(x))$

$$f'(1)=\frac{7503}{5}+\ln 5$$

9. Find $g'(z)$ if $g(z)=\left(\dfrac{e^{5z}}{1+\ln z}\right)^{118}$

$$g'(z)=\frac{118e^{590z}(5z+5z\ln z-1)}{z(1+\ln z)^{119}}$$

10. $\dfrac{d}{dx}\left[x^2\cos x^2+\dfrac{e^x}{x}\right]$

$$=-2x^3\sin x^2+2x\cos x^2+\frac{e^x(x+1)}{x^2}$$

11. $\dfrac{d}{dx}\left[\dfrac{x^2+2x-3}{x-4}\right]$

$=\dfrac{-x^2+8x+5}{(x-4)^2}$

12. $\dfrac{d}{dx}\left[\dfrac{\cos(x^2-3)}{e^{-5x}}\right]$

$=\dfrac{-5\cos(x^2-3)+2x\sin(x^2-3)}{e^{-5x}}$

13. $\dfrac{d}{dx}\left[x^5\ln(5x+4)+\dfrac{x}{\ln x}\right]$

$=\dfrac{x^5}{5x+4}+5x^4\ln(5x+4)+\dfrac{-1+\ln x}{\ln^2 x}$

14. Find $g'(t)$ if $g(t)=\left(\dfrac{t^2-4}{1-t^2}\right)^{15}$

$g'(t)=\dfrac{150t(t^2-4)^{14}}{(1-t^2)^{16}}$

15. $\dfrac{d}{dx}\left[e^{x^2}\cos x\right]$

$=e^{x^2}(-\sin x+2x\cos x)$

16. $\dfrac{d}{dx}\left[\dfrac{1+\tan x}{\ln 4x}\right]$

$=\dfrac{-1+x\sec^2 x\ln 4x-\tan x}{x\ln^2 4x}$

17. $\dfrac{d}{dt}\left[\sin t\tan t\right]$

$=\sin t\sec^2 t+\cos t\tan t$

18. $\dfrac{d}{dx}\left[\dfrac{1+\ln x}{\csc x}\right]$

$=\dfrac{1+\cot x+\cot x\ln x}{x\csc x}$

19. $\dfrac{d}{dx}\left[e^{5x^4}\ln(\sin x)\right]$

$=e^{5x^4}\left[\cot x+20x^3\ln(\sin x)\right]$

20. $\dfrac{d}{dp}\left[5p\sin\left(p+e^{2p}\right)-\ln\left(3p^2+1\right)+\dfrac{p}{p^2+1}\right]$

$=5p\left(1+2e^{2p}\right)\cos\left(p+e^{2p}\right)+5\sin\left(p+e^{2p}\right)-\dfrac{6p}{3p^2+1}+\dfrac{1-p^2}{\left(p^2+1\right)^2}$

21. $\frac{d}{dx}\left[\tan\left(e^x\right)\left(x^4-5x^3+x\right)\right]$

$=\tan\left(e^x\right)\left(4x^3-15x^2+1\right)+\sec^2\left(e^x\right)\left(x^4-5x^3+x\right)$

22. $\frac{d}{dx}\left[\frac{5x+2}{\ln(3x+7)}\right]$

$=\frac{5(3x+7)\ln(3x+7)-5x-2}{(3x+7)\ln^2(3x+7)}$

23. $\frac{d}{dx}\left[\frac{x-4}{x^2-9x+20}\right]$

$=-(x-5)^{-2}$

24. $\frac{d}{dx}\left[\frac{d}{dx}\left[\sin^2(4x+2)\right]\right]$

$=32\cos(8x+4)$

25. $\frac{d}{dc}\left[\frac{c^5-12c^3-19c}{3c^3}\right]$

$=\frac{2}{3}c+38c^{-3}$

26. Given the table of values below, find $g'(3)$ if $g(x)=f(h(x))\sin(h(x))$

x	$f(x)$	$f'(x)$	$h(x)$	$h'(x)$
2	$\frac{\pi}{2}$	1	$\frac{\pi}{2}$	-7
3	$\frac{\pi}{4}$	2	$\frac{\pi}{3}$	1
$\frac{\pi}{3}$	2	3	$\frac{\pi}{4}$	-1

$g'(3)=1+\frac{3\sqrt{3}}{2}$

27. If $g(3)=\frac{\pi}{2}$, $g'(3)=\frac{\pi}{4}$, and $f(x)=x^3g(x)+g\left(-3\cos\left(\frac{\pi}{3}x\right)\right)-e^{\sin(g(x))}$, find $f'(3)$

$f'(3)=\frac{81\pi}{4}$

1.6: Higher Order Derivatives

What we have been calling the Derivative is actually the First Derivative. There can be successive uses of the derivative rules, and they have meanings other than the slope of the tangent line. In this section, we will explore the process of finding the higher order derivatives.

Second Derivative--Defn: The derivative of the derivative.

Just as with the First Derivative, there are several symbols for the 2nd Derivative:

<u>Higher Order Derivative Symbols</u>

Liebnitz: $\frac{d^2y}{dx^2}$ = d squared y, d x squared; $\frac{d^3y}{dx^3}$;… $\frac{d^ny}{dx^n}$

Function: $f''(x)$ = f double prime of x; $f'''(x)$; $f^{IV}(x)$;… $f^n(x)$

Combination: y'' = y double prime

OBJECTIVE

Find higher order derivatives.

Ex 1 $\frac{d^2}{dx^2}\left[x^4-7x^3-3x^2+2x-5\right]$

$$\frac{d^2}{dx^2}\left[x^4-7x^3-3x^2+2x-5\right]=\frac{d}{dx}\left[\frac{d}{dx}\left[x^4-7x^3-3x^2+2x-5\right]\right]$$

$$=\frac{d}{dx}\left[4x^3-21x^2-6x+2\right]$$

$$=12x^2-42x-6$$

Ex 2 Find $\frac{d^3y}{dx^3}$ if $y = \sin 3x$

$$y = \sin 3x$$
$$\frac{dy}{dx} = \cos 3x \cdot 3 = 3\cos 3x$$
$$\frac{d^2y}{dx^2} = 3\left(-\sin 3x\right)\cdot 3 = -9 \sin 3x$$
$$\frac{d^3y}{dx^3} = -9 \cos 3x \cdot 3 = -27 \cos 3x$$

More complicated functions, in particular Composite Functions, have a complicated process. When the Chain Rule is applied, the answer often becomes a product or quotient. Therefore, the 2nd Derivative will require the Product or Quotient Rules as well as, possibly, the Chain Rule again.

Ex 3 $y = e^{3x^2}$, find y".

$$\frac{dy}{dx} = e^{3x^2} \cdot 6x = 6xe^{3x^2}$$
$$\frac{d^2y}{dx^2} = 6x\left(e^{3x^2} \cdot 6x\right) + e^{3x^2} \cdot 6$$
$$= 36x^2e^{3x^2} + 6e^{3x^2}$$
$$= 6e^{3x^2}\left(6x^2 + 1\right)$$

Ex 4 $y = \sin^3 x$, find y"

$$y' = 3\sin^2 x \cdot \cos x$$
$$y'' = 3\sin^2 x\left(-\sin x\right) + \cos x\left(6\sin x \cdot \cos x\right)$$
$$= 3\sin x \left(2\cos^2 x - \sin^2 x\right)$$

Ex 5 $f(x)=\ln\left(x^2+3x-1\right)$, find $f''(x)$.

$$f'(x)=\frac{1}{x^2+3x-1}(2x+3)=\frac{2x+3}{x^2+3x-1}$$

$$f''(x)=\frac{\left(x^2+3x-1\right)(2)-(2x+3)(2x+3)}{\left(x^2+3x-1\right)^2}$$

$$=\frac{\left(2x^2+6x-2\right)-\left(4x^2+12x+9\right)}{\left(x^2+3x-1\right)^2}$$

$$=\frac{-2x^2-6x-11}{\left(x^2+3x-1\right)^2}$$

Ex 6 $g(x)=\sqrt{4x^2+1}$, find $g''(x)$.

$$g'(x)=\frac{1}{2}\left(4x^2+1\right)^{-1/2}(8x)=\frac{4x}{\left(4x^2+1\right)^{1/2}}$$

$$g''(x)=\frac{\left(4x^2+1\right)^{1/2}(4)-(4x)\left[\frac{1}{2}\left(4x^2+1\right)^{-1/2}(8x)\right]}{\left[\left(4x^2+1\right)^{1/2}\right]^2}$$

$$=\frac{\left(4x^2+1\right)^{1/2}(4)-\dfrac{16x^2}{\left(4x^2+1\right)^{1/2}}}{\left(4x^2+1\right)}$$

$$=\frac{\left(4x^2+1\right)(4)-16x^2}{\left(4x^2+1\right)^{3/2}}$$

$$=\frac{4}{\left(4x^2+1\right)^{3/2}}$$

1.6 Homework Set A

In #1-5, find the first and second derivatives of the given function.

1. $f(x) = x^5 + 6x^2 - 7x$

2. $h(x) = \sqrt{x^2 + 1}$

3. $y = \left(x^3 + 1\right)^{2/3}$

4. $H(t) = \tan 3t$

5. $g(t) = t^3 e^{5t}$

6. If $y = e^{3x^2}$, find y''.

7. If $y = \sin^3 x$, find y''.

8. If $f(t) = t\cos t$, find $f'''(0)$.

1.6 Homework Set B

1. Find $f'(x)$, $f''(x)$, and $f'''(x)$ if $f(x) = \ln(\sec x)$.

2. Find $\frac{dy}{d\theta}$ and $\frac{d^2y}{d\theta^2}$ for $y = 10\cot(2\theta + 1)$

3. For the function $y = \tan 2x$, show that $\frac{d^2y}{dx^2} = 8\sec^2 2x \tan 2x$.

4. Find $f''(x)$ for $f(x) = \ln\left(x^2 + 3x - 1\right)$

5. Find the first, second, and third derivative for $f(x) = x^2 - 5x + 6 + e^x$

6. Find the first, second, and third derivative for $f(\theta) = \tan\left(3\theta + 2\right)$

7. A fourth differentiable function is defined for all real numbers and satisfies each of the following:

$g(2)=5$, $g'(2)=-2$, and $g''(2)=3$

If the function f is given by $f(x)=e^{k(x-1)}+g(2x)$, where k is a constant.

a. Find $f(1)$, $f'(1)$, $f''(1)$

b. Show that the fourth derivative of f is $k^4e^{k(x-1)}+16g''''(2x)$

1.6 Homework Set A

1. $f(x)=x^5+6x^2-7x$

$f'(x)=5x^4+12x-7$

$f''(x)=20x^3+12$

2. $h(x)=\sqrt{x^2+1}$

$h'(x)=\dfrac{x}{\sqrt{x^2+1}}$

$h''(x)=\dfrac{1+x^2}{\left(x^2+1\right)^{3/2}}$

3. $y=\left(x^3+1\right)^{2/3}$

$\dfrac{dy}{dx}=\dfrac{2x^2}{\left(x^3+1\right)^{1/3}}$

$\dfrac{d^2y}{dx^2}=\dfrac{2x\left(x^3+2\right)}{\left(x^3+1\right)^{4/3}}$

4. $H(t)=\tan 3t$

$H'(t)=3\sec^2 3t$

$H''(t)=18\sec^2 3t\tan 3t$

5. $g(t)=t^3e^{5t}$

$g'(t)=t^2e^{5t}(5t+3)$

$g''(t)=te^{5t}\left(25t^2+30t+6\right)$

6. If $y=e^{3x^2}$, find y''.

$y''=6e^{3x^2}\left(6x^2+1\right)$

7. If $y=\sin^3 x$, find y''.

$y''=3\sin x\left(2\cos^2 x-\sin^2 x\right)$

8. If $f(t)=t\cos t$, find $f'''(0)$.

$f'''(0)=-3$

1.6 Homework Set B

1. Find $f'(x)$, $f''(x)$, and $f'''(x)$ if $f(x)=\ln(\sec x)$.

$f'(x)=\tan x$

$f''(x)=\sec^2 x$

$f'''(x)=2\sec^2 x\tan x$

2. Find $\frac{dy}{d\theta}$ and $\frac{d^2y}{d\theta^2}$ for $y = 10\cot(2\theta+1)$

$$\frac{dy}{d\theta} = -20\csc^2(2\theta+1)$$

$$\frac{d^2y}{d\theta^2} = 80\csc^2(2\theta+1)\cot(2\theta+1)$$

3. For the function $y = \tan 2x$, show that $\frac{d^2y}{dx^2} = 8\sec^2 2x \tan 2x$.

$$\frac{dy}{dx} = 2\sec^2 2x$$

$$\frac{d^2y}{dx^2} = 4\sec 2x \cdot \sec 2x \tan 2x \cdot 2$$

$$\frac{d^2y}{dx^2} = 8\sec^2 2x \tan 2x$$

4. Find $f''(x)$ for $f(x) = \ln(x^2+3x-1)$

$$f''(x) = \frac{-2x^2-6x-11}{(x^2+3x-1)^2}$$

5. Find the first, second, and third derivative for $f(x) = x^2-5x+6+e^x$

$$f'(x) = 2x-5+e^x$$

$$f''(x) = 2+e^x$$

$$f'''(x) = e^x$$

6. Find the first, second, and third derivative for $f(\theta) = \tan(3\theta+2)$

$$f'(\theta) = 3\sec^2(3\theta+2)$$

$$f''(\theta) = 18\sec^2(3\theta+2)\tan(3\theta+2)$$

$$f'''(\theta) = 54\sec^2(3\theta+2)\left(\sec^2(3\theta+2)+2\tan^2(3\theta+2)\right)$$

7. A fourth differentiable function is defined for all real numbers and satisfies each of the following:

$g(2)=5$, $g'(2)=-2$, and $g''(2)=3$

If the function f is given by $f(x)=e^{k(x-1)}+g(2x)$, where k is a constant.

a. Find $f(1)$, $f'(1)$, $f''(1)$

b. Show that the fourth derivative of f is $k^4e^{k(x-1)}+16g''''(2x)$

a. $f(1)=6$, $f'(1)=k-4$, $f''(1)=k^2+12$

b. $f'(x)=ke^{k(x-1)}+2g'(2x)$

$f''(x)=k^2e^{k(x-1)}+4g''(2x)$

$f'''(x)=k^3e^{k(x-1)}+8g'''(2x)$

$f''''(x)=k^4e^{k(x-1)}+16g''''(2x)$

1.7: Derivatives of Inverse Functions

We already know something about inverse functions. Exponential and Logarithmic functions are inverses of each other, as are Radicals and Powers.

Inverse Functions—Defn: Two functions wherein the domain of one serves as the range of the other and vice versa.

--Means: Two functions which cancel (or undo) each other.

The definition gives us a way to find the inverse for any function, the symbol for which is f^{-1}. You just switch the *x* and *y* variables and isolate *y*.

Ex 1 Find f^{-1} if $f(x)=x^3$

$$f(x)=x^3 \to y=x^3$$

So, for f^{-1},

$$\begin{aligned} x &= y^3 \\ y &= \sqrt[3]{x} \\ f^{-1} &= \sqrt[3]{x} \end{aligned}$$

This is little more interesting with a Rational Function.

Ex 2 Find f^{-1} if $f(x)=\dfrac{x-1}{2-x}$

$$f(x)=\frac{x-1}{2-x} \to y=\frac{x-1}{2-x}$$

So, for f^{-1},

$$\begin{aligned} x &= \frac{y-1}{2-y} \\ (2-y)x &= y-1 \\ 2x-xy &= y-1 \\ 2x+1 &= y+xy \\ 2x+1 &= y(1+x) \\ \frac{2x+1}{1+x} &= y \\ f^{-1}(x) &= \frac{2x+1}{1+x} \end{aligned}$$

Note that $f^{-1}\left(f\left(x\right)\right) = x$. For Example, $f\left(0\right) = \frac{1}{2}$ and $f^{-1}\left(\frac{1}{2}\right) = 0$.

General inverses are not all that interesting. We are more interested in particular inverse functions, like the ***natural logarithm*** (which is the inverse of the exponential function). Another particular kind of inverse function that bears more study is the Trig Inverse Function. Interestingly, as with the Logarithmic Functions, the derivatives of these Transcendental Functions become Algebraic Functions.

Inverse Trig Derivative Rules

$$\frac{d}{dx}\left[\sin^{-1} u\right] = \frac{1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\csc^{-1} u\right] = \frac{-1}{|u|\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\cos^{-1} u\right] = \frac{-1}{\sqrt{1-u^2}} \cdot D_u \qquad \frac{d}{dx}\left[\sec^{-1} u\right] = \frac{1}{|u|\sqrt{u^2-1}} \cdot D_u$$

$$\frac{d}{dx}\left[\tan^{-1} u\right] = \frac{1}{u^2+1} \cdot D_u \qquad \frac{d}{dx}\left[\cot^{-1} u\right] = \frac{-1}{u^2+1} \cdot D_u$$

OBJECTIVE

Find the derivatives of inverse trig functions.

Proof that $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$.

$$y = \sin^{-1} x \rightarrow \sin y = x$$

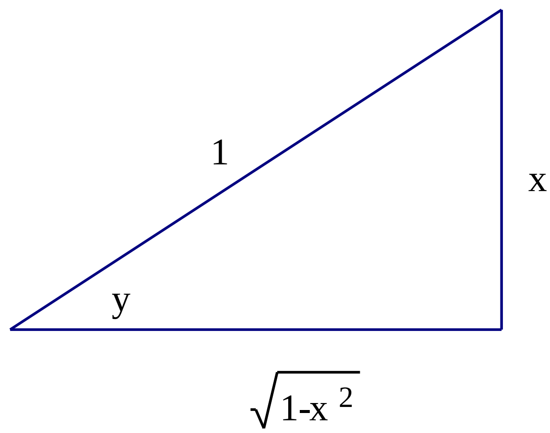
$$D_x\left(\sin y = x\right)$$

$$\left(\cos y\right)\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

Note the use of implicit differentiation when we took the derivative of $\left(\sin y\right)$

But this derivative is not in terms of x, so we are not done. Consider the right triangle that would yield this SOHCAHTOA relationship:



Note that for sin y to equal x, x must be the opposite leg and the hypotenuse is 1 (SOH). The Pythagorean Theorem gives us the adjacent leg. By CAH,

$$\cos\ y = \sqrt{1-x^2}$$

Therefore,

$$\begin{aligned}\frac{d}{dx}\left(\sin^{-1}\ x\right) &= \frac{1}{\cos\ y}\\ &= \frac{1}{\sqrt{1-x^2}}\end{aligned}$$

Note that in the proof above, when we wanted to take the derivative of sin y, we had to use The Chain Rule (since the y is a function other than x, and the derivative of y is $\frac{dy}{dx}$.

Ex 4 $\frac{d}{dx}\left[\tan^{-1} 3x^4\right]$

$$\begin{aligned}\frac{d}{dx}\left[\tan^{-1} 3x^4\right] &= \frac{1}{\left(3x^4\right)^2+1}\cdot\left(12x^3\right)\\ &= \frac{12x^3}{9x^8+1}\end{aligned}$$

Ex 5 $\frac{d}{dx}\left[\sec^{-1} x^2\right]$

$$\frac{d}{dx}\left[\sec^{-1} x^2\right] = \frac{1}{|x^2|\sqrt{\left(x^2\right)^2 - 1}} \cdot 2x$$

$$= \frac{2x}{\left(x^2\right)\sqrt{\left(x^2\right)^2 - 1}}$$

$$= \frac{2}{x\sqrt{x^4 - 1}}$$

1.7 Homework Set A

Find the derivative of the function. Simplify where possible.

1. $y = \sin^{-1}\left(e^{x}\right)$

2. $y = \tan^{-1}\left(\sqrt{x}\right)$

3. $y = \sin^{-1}\left(2x+1\right)$

4. $H(x) = \left(1+x^{2}\right)\tan^{-1}(x)$

5. $y = x\cos^{-1}x - \sqrt{1-x^{2}}$

6. $f(x) = e^{x} - x^{2}\arctan x$

7. $y = \sin^{-1}\left(\sqrt{2}\,(x)\right)$

8. $y = \csc^{-1}\left(x^{2}+1\right)$

9. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x$

10. $y = \cos^{-1} x + x\sqrt{1-x^2}$

11. $y = \frac{\sec^{-1} x}{x}$ *for* $x > 0$

12. $y = \ln\left(x^2+4\right) - x \tan^{-1}\left(\frac{x}{2}\right)$

1.7 Homework Set B

1. $y = \cos^{-1}\left(e^{3z}\right)$

2. $y = \tan^{-1}\left(\sqrt{x^2-1}\right)$

3. $y = 4\sin^{-1}\left(\frac{1}{2}x\right) + x\sqrt{4-x^2}$

4. $y = \cos^{-1}\left(\frac{x-1}{x+1}\right)$

5. $y = \sec^{-1} 4x + \csc^{-1} 4x$

6. $f(t) = c\sin^{-1}\frac{t}{c} - \sqrt{c^2 - t^2}$

7. $f(x) = x^2 \arccos x$

8. $f(x) = \ln\left(\tan^{-1} 5x\right)$

9. $g(w) = \sin^{-1}(5w) + \cos^{-1}(5w)$

10. $f(t) = \sec^{-1}\sqrt{9+t^2}$

11. $y=\ln\left(u^2+1\right)-u\cot^{-1}(u)$

12. $y=\tan^{-1}\left(\dfrac{2e^x}{1-e^{2x}}\right)$

1.7 Homework Set A

1. $y=\sin^{-1}\left(e^{x}\right)$

$$\frac{dy}{dx}=\frac{e^{x}}{\sqrt{1-e^{2x}}}$$

2. $y=\tan^{-1}\left(\sqrt{x}\right)$

$$\frac{dy}{dx}=\frac{1}{2\left(x^{1/2}+x^{3/2}\right)}$$

3. $y=\sin^{-1}\left(2x+1\right)$

$$\frac{dy}{dx}=\frac{1}{\left(-x^{2}-x\right)^{1/2}}$$

4. $H(x)=\left(1+x^{2}\right)\tan^{-1}(x)$

$$H'(x)=1+2x\tan^{-1}(x)$$

5. $y=x\cos^{-1}x-\sqrt{1-x^{2}}$

$$\frac{dy}{dx}=\cos^{-1}x$$

6. $f(x)=e^{x}-x^{2}\arctan x$

$$f'(x)=e^{x}-\frac{x^{2}}{1+x^{2}}-2x\arctan x$$

7. $y=\sin^{-1}\left(\sqrt{2}(x)\right)$

$$\frac{dy}{dx}=\frac{\sqrt{2}}{\sqrt{1-2x^{2}}}$$

8. $y=\csc^{-1}\left(x^{2}+1\right)$

$$\frac{dy}{dx}=\frac{-2x}{\left(x^{2}+1\right)|x|\sqrt{x^{2}+2}}$$

9. $y=\cot^{-1}\left(\frac{1}{x}\right)-\tan^{-1}x$

$$\frac{dy}{dx}=0$$

10. $y=\cos^{-1}x+x\sqrt{1-x^{2}}$

$$\frac{dy}{dx}=\frac{-3x^{2}}{\sqrt{1-x^{2}}}$$

11. $y=\frac{\sec^{-1}x}{x}$ *for* $x>0$

$$\frac{dy}{dx}=\frac{1-\sqrt{x^{2}-1}\sec^{-1}x}{x^{2}\sqrt{x^{2}-1}}$$

12. $y=\ln\left(x^{2}+4\right)-x\tan^{-1}\left(\frac{x}{2}\right)$

$$\frac{dy}{dx}=-\tan^{-1}\left(\frac{x}{2}\right)$$

1.7 Homework Set B

1. $y=\cos^{-1}\left(e^{3z}\right)$

$$\frac{dy}{dx}=\frac{3e^{3z}}{\sqrt{1-e^{6z}}}$$

2. $y=\tan^{-1}\left(\sqrt{x^{2}-1}\right)$

$$\frac{dy}{dx}=\frac{1}{x\sqrt{x^{2}-1}}$$

3. $y=4\sin^{-1}\left(\frac{1}{2}x\right)+x\sqrt{4-x^2}$

$\frac{dy}{dx}=\frac{12-2x^2}{\sqrt{4-x^2}}$

4. $y=\cos^{-1}\left(\frac{x-1}{x+1}\right)$

$\frac{dy}{dx}=\frac{-1}{\sqrt{x}\,(x+1)}$

5. $y=\sec^{-1}4x+\csc^{-1}4x$

$\frac{dy}{dx}=0$

6. $f(t)=c\sin^{-1}\frac{t}{c}-\sqrt{c^2-t^2}$

$f'(t)=\frac{c^2+t}{\sqrt{c^2-t^2}}$

7. $f(x)=x^2\arccos x$

$f'(x)=\frac{-x^2}{\sqrt{1-x^2}}+2x\arccos x$

8. $f(x)=\ln\left(\tan^{-1}5x\right)$

$f'(x)=\frac{5}{\left(1+25x^2\right)\tan^{-1}5x}$

9. $g(w)=\sin^{-1}(5w)+\cos^{-1}(5w)$

$g'(w)=0$

10. $f(t)=\sec^{-1}\sqrt{9+t^2}$

$f'(t)=\frac{t}{\left(9+t^2\right)\sqrt{8+t^2}}$

11. $y=\ln\left(u^2+1\right)-u\cot^{-1}(u)$

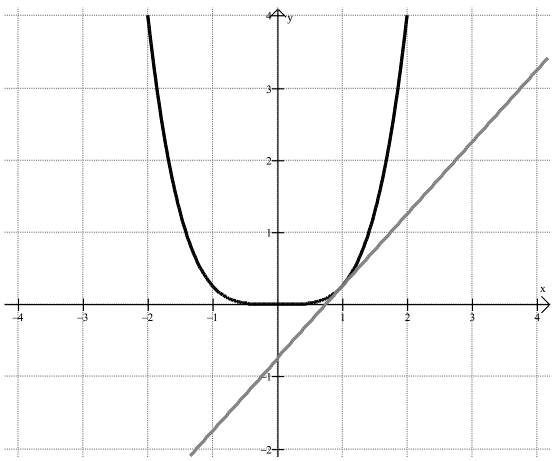
$\frac{dy}{du}=\frac{3u}{u^2+1}-\cot^{-1}(u)$

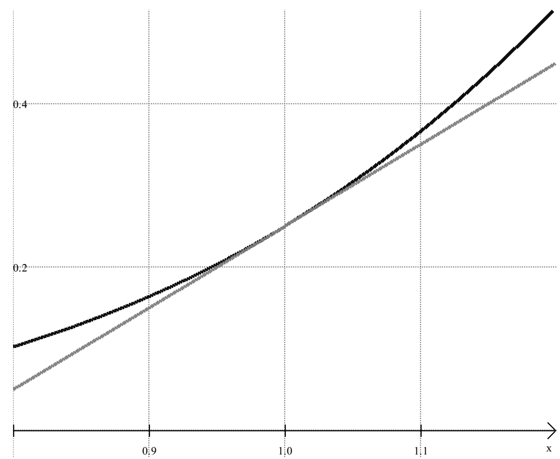
12. $y=\tan^{-1}\left(\frac{2e^x}{1-e^{2x}}\right)$

$\frac{dy}{dx}=\frac{2e^x}{1+e^{2x}}$

1.8: Local Linearity and Approximations

Before calculators, one of the most valuable uses of the derivative was to find approximate function values from a tangent line. Since the tangent line only shares one point on the function, *y*-values on the line are very close to *y*-values on the function. This idea is called **local linearity**—near the point of tangency, the function curve appears to be a line. This can be easily demonstrated with the graphing calculator by zooming in on the point of tangency. Consider the graphs of $y = .25x^4$ and its tangent line at $x = 1$, $y = x + .75$.





The closer you zoom in, the more the line and the curve become one. The *y*-values on the line are good approximations of the *y*-values on the curve. For a good animation of this concept, see

http://www.ima.umn.edu/~arnold/tangent/tangent.mpg

Since it is easier to find the *y*-value of a line arithmetically than for other functions—especially transcendental functions—the tangent line approximation is useful if you have no calculator.

OBJECTIVES
Use the equation of a tangent line to approximate function values.

Ex 1 Find the tangent line equation to $f(x) = x^4 - x^3 - 2x^2 + 1$ at $x = -1$ and use it to approximate value of $f(-0.9)$.

The slope of the tangent line will be $f'(-1)$

$$f'(x) = 4x^3 - 3x^2 - 4x$$
$$f'(-1) = -3$$

[Note that we could have gotten this more easily with the nDeriv function on our calculator.]

$f(-1) = 1$, so the tangent line will be

$$y - 1 = -3(x+1)$$

or

$$y = -3x - 2$$

While we can find the exact value of $f(-0.9)$ with a calculator, we can get a quick approximation from the tangent line. If $x = -0.9$ on the tangent line, then

$$f(-0.9) \approx y(-0.9) = -3(-0.9) - 2 = .7$$

This last example is somewhat trite in that we could have just plugged –0.9 into $f(x) = x^4 - x^3 - 2x^2 + 1$ and figured out the exact value even without a calculator. It would have been a pain, but a person could actually do it by hand because of the operations involved (there is only basic arithmetic involved). Consider the next example, though.

Ex 2 Find the tangent line equation to $f(x)=e^{2x}$ at $x=0$ and use it to approximate value of $e^{0.2}$.

Without a calculator, we could not find the exact value of $e^{0.2}$. In fact, even the calculator only gives an approximate value.

$$f'(x)=2e^{2x} \text{ and } f'(0)=2e^{2x}=2$$
$$f(0)=e^{0}=1$$

So the tangent line equation is $y-1=2(x-0)$ or $y=2x+1$

In order to find $e^{0.2}$, we are looking for $f(0.1)$. Therefore, we plug in $x = 0.1$

$$e^{0.2}\approx 2(0.1)+1=1.2$$

Note that the value that you get from a calculator for $e^{0.2}$ is 1.221403…
Our approximation of 1.2 seems very reasonable.

1.8 Homework Set A

1. Find the equation of the tangent line to $f(x) = x^5 - 5x + 1$ at $x = -2$ and use it to get an approximate value of $f(-1.9)$.

2. Find the tangent line equation to $F(x) = \frac{5}{x^2} - \sqrt{x}$ at $x = 1$ and use it to get an approximate value of $F(1.1)$.

3. Find all points on the graph of $y = 2\sin x + \sin^2 x$ where the tangent line is horizontal.

4. Find the equation of the tangent line at $x = 2$ for $g(x) = \ln(x^2 - 3)$. Use this to approximate $g(2.1)$.

5. Find the equation of the line tangent to $g(x) = 5 + 2x + \tan(x^2 - 1)$ when $x = 1$. Use this tangent line to find an approximation for $g(1.1)$.

6. Use a tangent line to find the approximate the value of $f(.02)$ if $f(x) = 2e^{x\sin x} + x$. Then use your calculator to find an actual value for $f(.02)$.

7. Find the equation of the tangent line for $f(x) = \ln x \sin\left(\frac{\pi}{2}x^2\right)$ for $x = 1$.

Plug in $x = 1.1$ into both the original function and the tangent line. Explain why the values are so similar.

8. Given the relation, $x^4 + 3y - y^2 = 3$, which passes through the point (1,2), find each of the following:

a. $\frac{dy}{dx}$

b. $\left.\frac{dy}{dx}\right|_{(1,2)}$

c. The equation of the line tangent to the relation, passing through (1,2)

d. An approximate value for y when $x = 1.01$

1.8 Homework Set A

1. Find the equation of the tangent line to $f(x)=x^5-5x+1$ at $x=-2$ and use it to get an approximate value of $f(-1.9)$.

   $y+21=75(x+2)$ $\quad\quad$ $f(-1.9)\approx-13.5$

2. Find the tangent line equation to $F(x)=\dfrac{5}{x^2}-\sqrt{x}$ at $x=1$ and use it to get an approximate value of $F(1.1)$.

   $y-4=-\dfrac{21}{2}(x-1)$ $\quad\quad$ $F(1.1)\approx2.95$

3. Find all points on the graph of $y=2\sin x+\sin^2 x$ where the tangent line is horizontal.

   $\left(\dfrac{\pi}{2}\pm2\pi n,3\right),\left(-\dfrac{\pi}{2}\pm2\pi n,1\right)$

4. Find the equation of the tangent line at $x = 2$ for $g(x)=\ln\left(x^2-3\right)$. Use this to approximate $g(2.1)$.

   $y=4(x-2)$ $\quad\quad$ $g(2.1)\approx.4$

5. Find the equation of the line tangent to $g(x)=5+2x+\tan\left(x^2-1\right)$ when $x=1$. Use this tangent line to find an approximation for $g(1.1)$.

   $y-7=4(x-1)$ $\quad\quad$ $g(1.1)\approx7.4$

6. Use a tangent line to find the approximate the value of $f(.02)$ if $f(x)=2e^{x\sin x}+x$. Then use your calculator to find an actual value for $f(.02)$.

   $f(.02)\approx2.04;\ f(.02)=2.021...$

7. Find the equation of the tangent line for $f(x) = \ln x \sin\left(\frac{\pi}{2}x^2\right)$ for $x = 1$.

Plug in $x = 1.1$ into both the original function and the tangent line. Explain why the values are so similar.

$y = x - 1$ $\quad\quad$ $f(1.1) \approx .1;\ f(1.1) = .090$

The results are so similar because of *local linearity*. The tangent line is extremely close to the curve near the point of tangency, so values of y for the tangent line serve as good approximations for values of f near the point of tangency.

8. Given the relation, $x^4 + 3y - y^2 = 3$, which passes through the point (1,2), find each of the following:

a. $\frac{dy}{dx} = \frac{-4x^3}{3-2y}$

b. $\left.\frac{dy}{dx}\right|_{(1,2)} = 4$

c. The equation of the line tangent to the relation, passing through (1,2)

$y - 2 = 4(x-1)$

d. An approximate value for y when $x = 1.01$

$y - 2 = 4(x-1)$

Chapter 1 Test

1. $\frac{d}{dx}\left[3x^5 - 2\sqrt[4]{x^3} + 3^x - \frac{1}{6\sqrt[3]{x^4}} + \frac{1}{5x^3}\right]$

2. $y = \tan^{-1}\left(\frac{x}{3}\right) + \cot^{-1}\left(\frac{3}{x}\right)$; find y'.

3. $D_x\left[\left(7x^4 - 5\right)^3\left(2x^7 + 1\right)^5\right]$

4. Find the equation of the line tangent to $f(x)=\left(2x^3-1\right)^4$ at x = 1 and use it to estimate f (0.98).

5. $g(z)=\ln\sqrt[3]{\dfrac{z-2}{z^2+2}}$; Find $g'(w)$ and $g'(4)$.

6. Find the equation of the line tangent to $y=\ln\left(4x^2-3\right)^5$ at $x=1$.

7. Find $f''\left(\frac{\pi}{6}\right)$ if $f(t)=7\sin^2 3t$.

8. $\frac{d^2}{dx^2}\left[x^7-2^3+e^{5x}+\cos\left(\frac{1}{x}\right)\right]$

9. Find $\frac{dy}{dx}$ and the equation for the tangent line for the relation $x^3-5x+y^2-6y=7$ passing through the point (3,1).

Chapter 1 Test

1. $\frac{d}{dx}\left[3x^5 - 2\sqrt[4]{x^3} + 3^x - \frac{1}{6\sqrt[3]{x^4}} + \frac{1}{5x^3}\right]$

$= 15x^4 - \frac{3}{2}x^{-1/4} + 3^x \ln 3 + \frac{4}{9}x^{-7/3} - \frac{3}{5}x^{-4}$

2. $y = \tan^{-1}\left(\frac{x}{3}\right) + \cot^{-1}\left(\frac{3}{x}\right)$; find y'.

$\frac{dy}{dx} = \frac{18}{9+x^2}$

3. $D_x\left[\left(7x^4-5\right)^3\left(2x^7+1\right)^5\right]$

$= 14x^3\left(7x^4-5\right)^2\left(2x^7+1\right)^4\left(77x^7-25x^3+6\right)$

4. Find the equation of the line tangent to $f(x) = \left(2x^3-1\right)^4$ at $x = 1$ and use it to estimate $f(0.98)$.

$y - 1 = 24(x-1)$ $\qquad$ $f(0.98) \approx 0.52$

5. $g(z) = \ln\sqrt[3]{\frac{z-2}{z^2+2}}$; Find $g'(w)$ and $g'(4)$.

$g'(w) = \frac{2-4w-w^2}{3(w-2)\left(w^2+2\right)}$ $\qquad$ $g'(4) = \frac{-5}{36}$

6. Find the equation of the line tangent to $y = \ln\left(4x^2-3\right)^5$ at $x = 1$.

$y = 80(x-1)$

7. Find $f''\left(\frac{\pi}{6}\right)$ if $f(t) = 7\sin^2 3t$.

$f''\left(\frac{\pi}{6}\right) = -126$

8. $\dfrac{d^2}{dx^2}\left[x^7 - 2^3 + e^{5x} + \cos\left(\dfrac{1}{x}\right)\right]$

$$= \frac{42x^9 + 25x^4 e^{5x} - \cos\frac{1}{x} - \sin\frac{1}{x}}{x^4}$$

9. Find $\dfrac{dy}{dx}$ and the equation for the tangent line for the relation $x^3 - 5x + y^2 - 6y = 7$ passing through the point (3,1).

$\dfrac{dy}{dx} = \dfrac{5-3x^2}{2y-6}$

Tangent Line: $y-1=1(x-3)$

Chapter 2 Overview: Indefinite Integrals

As noted in the introduction, Calculus is essentially comprised of four operations.

- Limits
- Derivatives
- Indefinite integrals (or Anti-Derivatives)
- Definite Integrals

There are two kinds of Integrals--the Definite Integral and the Indefinite Integral. For a long time in the mathematical world we did not know that integrals and derivatives were connected. But in the mid-1600s Scottish mathematician James Gregory published the first proof of what is now called the Fundamental Theorem of Calculus, changing the math world forever. The Indefinite Integral is often referred to as the Anti-Derivative, because, as an operation, it and the Derivative are inverse operations (just as squares and square roots, or exponential and log functions). In this chapter, we will consider how to reverse the differentiation process we have been employing.

Anti-derivatives will also be used to handle differential equations. Differential equations--equations that include $\frac{dy}{dx}$ as well as x and y--form a large subfield of study in Calculus. We will explore the basics of differential equations through two lenses of analysis—algebraic and graphic. (There is a numerical lens, but we will not consider it in this class.)

Algebraically, we will solve separable differential equations. These are relatively simple equations that will provide more practice for integration. Non-separable equations are beyond the scope of this class, but we will see what they look like and why they are so much more difficult to solve.

The graphical approach to differential equations use a tool called a slope field. This is a visual representation of the tangent line slopes to a family of curves at a variety of points in the Cartesian plane. It looks, in many ways, like a magnetic field, and we may explore some of the applications of slope fields to physics, if time allows.

2.1: Anti-Derivatives--the Anti-Power Rule

As we have seen, we can deduce things about a function if its derivative is know. It would be valuable to have a formal process to determine the original function from its derivative accurately. The process is called Anti-differentiation, or Integration.

Symbol: $\int (f(x))\, dx$ ="the integral of f of x, d-x"

The *dx* is called the differential. For now, we will just treat it as part of the integral symbol. It tells us the independent variable of the function (usually, but not always, *x*) and, in a sense, is where the increase in the exponent comes from. It does have meaning on its own, but we will explore that later.

Looking at the integral as an anti-derivative, that is, as an operation that reverses the derivative, we should be able to figure out the basic process.

Remember:

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

and D_X [constant] is always 0

(or, multiply the power in front and subtract one from the power). If we are starting with the derivative and want to reverse the process, the power must increase by one and we should divide by the new power. Also, we do not know, from the derivative, if the original function had a constant that became zero, let alone what the constant was.

The Anti-Power Rule
$\int (x^n)\, dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$

The "+ *C*" is to account for any constant that might have been there before the derivative was taken.

NB. This Rule will not work if n = -1, because it would require that we divide by zero. But we know from the Derivative Rules what yields $x^{-1}\left(or\ {}^{1}\!/\!_{x}\right)$ as the derivative--Ln x. So we can complete the anti-Power Rule as:

The Anti-Power Rule

$$\int\left(x^{n}\right)dx=\frac{x^{n+1}}{n+1}+C \text{ if } n\neq-1$$
$$\int\frac{1}{x}\,dx=\ln\left|\,x\,\right|+C^{*}$$

*NB. The absolute values are necessary to maintain the domain, since negative numbers cannot be logged. This is because $\frac{1}{x}$ can have negative values for x, the result of the antidifferentiation must be able to have negative values for x.

Since $D_x\left[f(x)+g(x)\right]=D_x\left[f(x)\right]+D_x\left[g(x)\right]$ and $D_x\left[c\cdot f(x)\right]=c\cdot D_x\left[f(x)\right]$, then

$$\int\bigl(f(x)+g(x)\bigr)dx=\int f(x)\,dx+\int g(x)\,dx$$
$$\int c\bigl(f(x)\bigr)dx=c\int f(x)\,dx$$

These allows us to integrate a polynomial by integrating each term separately.

OBJECTIVES

Find the Indefinite Integral of a polynomial.
Use Integration to solve rectilinear motion problems.

Ex 1 $\int (3x^2 + 4x + 5)dx$

$$\int (3x^2 + 4x + 5)\, dx = 3\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 5\frac{x^{0+1}}{0+1} + C$$
$$= \frac{3x^3}{3} + \frac{4x^2}{2} + \frac{5x^1}{1} + C$$

$$= x^3 + 2x^2 + 5x + C$$

Ex 2 $\int \left(x^4 + 4x^2 + 5 + \frac{1}{x} \right) dx$

$$\int \left(x^4 + 4x^2 + 5 + \frac{1}{x} \right) dx = \frac{x^{4+1}}{4+1} + \frac{4x^{2+1}}{2+1} + \frac{5x^{0+1}}{0+1} + \ln|x| + C$$

$$= \frac{1}{5}x^5 + \frac{4}{3}x^3 + 5x + \ln|\, x\,| + C$$

Ex 3 $\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x} \right) dx$

$$\int \left(x^2 + \sqrt[3]{x} - \frac{4}{x} \right) dx = \int \left(x^2 + x^{1/3} - \frac{4}{x} \right) dx$$
$$= \frac{x^{2+1}}{2+1} + \frac{x^{1/3+1}}{1/3+1} - 4\ln|\, x\,| + C$$

$$= \frac{1}{3}x^3 + \frac{3}{4}x^{4/3} - 4\ln|\, x\,| + C$$

Integrals of products and quotients can be done easily IF they can be turned into a polynomial.

Ex 4 $\int \left(x^2+\sqrt[3]{x}\right)(2x+1)\,dx$

$$\int \left(x^2+\sqrt[3]{x}\right)(2x+1)\,dx = \int \left(2x^3+2x^{4/3}+x^2+x^{1/3}\right)dx$$
$$= \frac{2x^4}{4}+\frac{2x^{7/3}}{7/3}+\frac{x^3}{3}+\frac{x^{4/3}}{4/3}+C$$
$$= \frac{1}{2}x^4+\frac{6}{7}x^{7/3}+\frac{1}{3}x^3+\frac{3}{4}x^{4/3}+C$$

Example 5 is called an initial value problem. It has an ordered pair (or initial value pair) that allows us to solve for *C*.

Ex 5 $f'(x)=4x^3-6x+3$. Find $f(x)$ if $f(0)=13$

$$f(x)=\int \left(4x^3-6x+3\right)dx$$
$$=x^4-3x^2+3x+C$$
$$f(0)=0^4-3(0)^2+3(0)+C=13$$
$$\therefore\ C=13$$

$$f(x)=x^4-3x^2+3x+13$$

Ex 6 The acceleration of a particle is described by $a(t) = 3t^2 + 8t + 1$. Find the distance equation for $x(t)$ if $v(0) = 3$ and $x(0) = 1$.

$$\begin{aligned}
v(t) &= \int \left(a(t)\right) dt = \int \left(3t^2 + 8t + 1\right) dt \\
&= t^3 + 4t^2 + t + C_1 \\
3 &= (0)^3 + 4(0)^2 + (0) + C_1 \\
3 &= C_1 \\
v(t) &= t^3 + 4t^2 + t + 3
\end{aligned}$$

$$\begin{aligned}
x(t) &= \int \left(v(t)\right) dt = \int \left(t^3 + 4t^2 + t + 3\right) dt \\
&= \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + C_2 \\
1 &= \frac{1}{4}(0)^4 + \frac{4}{3}(0)^3 + \frac{1}{2}(0)^2 + 3(0) + C_2 \\
1 &= C_2
\end{aligned}$$

$$x(t) = \frac{1}{4}t^4 + \frac{4}{3}t^3 + \frac{1}{2}t^2 + 3t + 1$$

It is important to note that the value of the C is not always equal to the value given at the beginning of the problem. In the problems above, $t = 0$, and we had polynomial functions – if this were not the case, C could have very different values.

Ex 7   The acceleration of a particle is described by $a(t) = 12t^2 - 6t + 4$. Find the distance equation for $x(t)$ if $v(1) = 0$ and $x(1) = 3$.

$$
\begin{aligned}
v(t) &= \int \left(a(t)\right)\, dt = \int \left(12t^2 - 6t + 4\right)\, dt \\
&= 4t^3 - 3t^2 + 4t + C_1 \\
0 &= 4(1)^3 - 3(1)^2 + 4(1) + C_1 \\
-5 &= C_1 \\
v(t) &= 4t^3 - 3t^2 + 4t - 5
\end{aligned}
$$

$$
\begin{aligned}
x(t) &= \int \left(v(t)\right)\, dt = \int \left(4t^3 - 3t^2 + 4t - 5\right)\, dt \\
&= t^4 - t^3 + 2t^2 - 5t + C_2 \\
3 &= (1)^4 - (1)^3 + 2(1)^2 - 5(1) + C_2 \\
6 &= C_2
\end{aligned}
$$

$$x(t) = t^4 - t^3 + 2t^2 - 5t + 6$$

2.1 Homework Set A

Perform the Anti-differentiation.

1. $\int\left(6x^2-2x+3\right)dx$

2. $\int\left(x^3+3x^2-2x+4\right)dx$

3. $\int\frac{2}{\sqrt[3]{x}}\,dx$

4. $\int\left(8x^4-4x^3+9x^2+2x+1\right)dx$

5. $\int x^3\left(4x^2+5\right)dx$

6. $\int(4x-1)(3x+8)\,dx$

7. $\int\left(\sqrt{x}-\frac{6}{\sqrt{x}}\right)dx$

8. $\int\left(\frac{x^2+\sqrt{x}+3}{x}\right)dx$

9. $\int (x+1)^3 \, dx$

10. $\int (4x-3)^2 \, dx$

11. $\int \left(\sqrt{x} + 3\sqrt[2]{x^3} - \frac{6}{\sqrt{x}} \right) dx$

12. $\int \left(\frac{4x^3 + \sqrt{x} + 3}{x^2} \right) dx$

Solve the initial value problems.

13. $f'(x) = 3x^2 - 6x + 3$. Find *f(x)* if f(0) = 2.

14. $f'(x) = x^3 + x^2 - x + 3$. Find *f(x)* if f(1) = 0.

15. $f'(x) = (\sqrt{x} - 2)(3\sqrt{x} + 1)$. Find *f(x)* if f(4) = 1.

16. The acceleration of a particle is described by $a(t) = 36t^2 - 12t + 8$. Find the distance equation for *x(t)* if *v*(1) = 1 and *x*(1) = 3.

17. The acceleration of a particle is described by $a(t) = t^2 - 2t + 4$. Find the distance equation for *x(t)* if *v*(0) = 2 and *x*(0) = 4.

2.1 Homework Set B

Perform the Anti-differentiation.

1. $\int \left(x^2 + 5x + 6\right) dx$

2. $\int \left(\dfrac{x^2 - 4x + 7}{x}\right) dx$

3. $\int \dfrac{x^5 - 7x^3 + 2x - 9}{2x} dx$

4. $\int \dfrac{x^3 + 3x^2 + 3x + 1}{x + 1} dx$

5. $\int \left(y^2 + 5\right)^2 dy$

6. $\int \dfrac{t^2 + 7t + 9}{t + 1} dt$

7. $\int \left(4t^2 + 1\right)\left(3t^3 + 7\right) dt$

8. $\int \dfrac{x^5 - 3x^3 - x + 7}{2x} dx$

Answers: 3.1 Homework Set A

1. $\int\left(6x^2-2x+3\right)dx$

$=2x^3-x^2+3x+C$

2. $\int\left(x^3+3x^2-2x+4\right)dx$

$=\frac{1}{4}x^4+x^3-x^2+4x+C$

3. $\int\frac{2}{\sqrt[3]{x}}\,dx$

$=2\sqrt[3]{x^2}+C$

4. $\int\left(8x^4-4x^3+9x^2+2x+1\right)dx$

$=\frac{8}{5}x^5-x^4+3x^3+x^2+x+C$

5. $\int x^3\left(4x^2+5\right)dx$

$=\frac{2}{3}x^6+\frac{5}{4}x^4+C$

6. $\int(4x-1)(3x+8)\,dx$

$=4x^3+\frac{29}{2}x^2-8x+C$

7. $\int\left(\sqrt{x}-\frac{6}{\sqrt{x}}\right)dx$

$=\frac{2}{3}x^{3/2}-12x^{1/2}+C$

8. $\int\left(\frac{x^2+\sqrt{x}+3}{x}\right)dx$

$=\frac{1}{2}x^2+2x^{1/2}+3\ln|x|+C$

9. $\int(x+1)^3\,dx$

$=\frac{1}{4}x^4+x^3+\frac{3}{2}x^2+x+C$

10. $\int(4x-3)^2\,dx$

$=\frac{16}{3}x^3-12x^2+9x+C$

11. $\int\left(\sqrt{x}+3\sqrt[2]{x^3}-\frac{6}{\sqrt{x}}\right)dx$

$=\frac{2}{3}x^{3/2}+\frac{6}{5}x^{5/2}-12x^{1/2}+C$

12. $\int\left(\frac{4x^3+\sqrt{x}+3}{x^2}\right)dx$

$=2x^2-2x^{-1/2}-3x^{-1}+C$

13. $f'(x)=3x^2-6x+3$. Find $f(x)$ if $f(0)=2$.

$f(x)=x^3-3x^2+3x+2$

14. $f'(x)=x^3+x^2-x+3$. Find $f(x)$ if $f(1)=0$.

$f(x)=\frac{1}{4}x^4+\frac{1}{3}x^3-\frac{1}{2}x^2+3x-\frac{37}{12}$

15. $f'(x)=\left(\sqrt{x}-2\right)\left(3\sqrt{x}+1\right)$. Find $f(x)$ if $f(4) = 1$.

$f(x)=\frac{3}{2}x^2-\frac{10}{3}x^{3/2}-2x+\frac{35}{3}$

16. The acceleration of a particle is described by $a(t)=36t^2-12t+8$. Find the distance equation for $x(t)$ if $v(1) = 1$ and $x(1) = 3$.

$x(t)=3t^4-2t^3+4t^2-13t+11$

17. The acceleration of a particle is described by $a(t)=t^2-2t+4$. Find the distance equation for $x(t)$ if $v(0) = 2$ and $x(0) = 4$.

$x(t)=\frac{1}{12}t^4-\frac{1}{3}t^3+2t^2+2t+4$

3.1 Homework Set B

1. $\int\left(x^2+5x+6\right)dx$

$=\frac{1}{3}x^3+\frac{5}{2}x^2+6x+C$

2. $\int\left(\frac{x^2-4x+7}{x}\right)dx$

$=\frac{1}{2}x^2-4x+7\ln|x|+C$

3. $\int\frac{x^5-7x^3+2x-9}{2x}dx$

$=\frac{1}{10}x^5-\frac{7}{6}x^3+x-9\ln|x|+C$

4. $\int\frac{x^3+3x^2+3x+1}{x+1}dx$

$=\frac{1}{3}x^3+x^2+x+C$

5. $\int\left(y^2+5\right)^2dy$

$=\frac{1}{5}y^5+\frac{10}{3}y^3+25y+C$

6. $\int\frac{t^2+7t+9}{t+1}dt$

$=\frac{1}{2}t^2+6t+3\ln|t+1|+C$

7. $\int\left(4t^2+1\right)\left(3t^3+7\right)dt$

$=\frac{12}{7}t^7+\frac{3}{4}t^4+\frac{28}{3}t^3+7t+C$

8. $\int\frac{x^5-3x^3-x+7}{2x}dx$

$=\frac{1}{10}x^5-\frac{1}{2}x^3-\frac{1}{2}x+\frac{7}{2}\ln|x|+C$

2.2: Integration by Substitution--the Chain Rule

The other three basic derivative rules--The Product, Quotient and Chain Rules--are a little more complicated to reverse than the Power Rule. This is because they yield a more complicated function as a derivative, one which usually has several algebraic simplifications. The Integral of a Rational Function is particularly difficult to unravel because, as we saw, a Rational derivative can be obtained by differentiating a composite function with a Log or a radical, or by differentiating another rational function. Reversing the Product Rule is as complicated, though for other reasons. We will leave both these subjects for a more advanced Calculus Class. The Chain Rule is another matter.

Composite functions are among the most pervasive situations in math. Though not as simple at reverse as the Power Rule, the overwhelming importance of this rule makes it imperative that we address it here.

Remember:

The Chain Rule: $$\frac{d}{dx}\left[f\left(g(x)\right)\right] = f'\left(g(x)\right) \cdot g'(x)$$

The derivative of a composite turns into a product of a composite and a non-composite. So if we have a product to integrate, it might be that the product came from the Chain Rule. The integration is not done by a formula so much as a process that might or might not work. We make an educated guess and hope it works out. You will learn other processes in Calculus for when it does not work.

<u>Integration by Substitution (The Unchain Rule)</u>

0) Notice that you are trying to integrate a product.
1) Identify the inside function of the composite and call it *u*.
2) Find *du* from *u*.
3) If necessary, multiply a constant inside the integral to create *du*, and balance it by multiplying the reciprocal of that constant outside the integral. (See EX 2)
4) Substitute *u* and *du* into the equation.
5) Perform the integration by Anti-Power (or Transcendental Rules, in next section.)
6) Substitute your initial expression back in for the *u*.

This is one of those mathematical processes that makes little sense when first seen. But after seeing several examples, the meaning suddenly becomes clear. **Be patient.**

OBJECTIVE
Use the Unchain Rule to integrate composite, product expressions.

Ex 1 $\int\left(3x^2\left(x^3+5\right)^{10}\right)dx$

$\left(x^3+5\right)^{10}$ is the composite function.

$u = x^3 + 5$
$du = 3x^2\ dx$

$$\int\left(3x^2\left(x^3+5\right)^{10}\right)dx = \int\left(u^{10}\right)du$$

$$= \frac{u^{11}}{11} + C$$

$$= \frac{1}{11}\left(x^3+5\right)^{11} + C$$

Ex 2 $\int\left(x\left(x^2+5\right)^3\right)dx$

$\left(x^2+5\right)^3$ is the composite function. So

$u = x^2 + 5$
$du = 2x\ dx$

$$\int\left(x\left(x^2+5\right)^3\right)dx = \frac{1}{2}\int\left(x^2+5\right)^3\left(2x\ dx\right)$$

$$= \frac{1}{2}\int\left(u^3\right)du$$

$$= \frac{1}{2}\cdot\frac{u^4}{4} + C$$

$$= \frac{1}{8}\left(x^2+5\right)^4 + C$$

Notice that our *du* had a constant in it that was not in our initial problem. We simply introduced the constant inside our integral, and multiplied by the reciprocal outside the integral. The problem hasn't changed because we have just multiplied by 1 (the product of a number and its reciprocal is 1).

We can only employ that trick with **constants**. This is very important. If you feel like you need to introduce a variable, either you have picked your *u* incorrectly, or this technique simply won't work, and we need more advanced techniques to perform the integration. The reason we can do this with constants is because a constant coefficient has no effect on integration – it can be moved in and out of the integral sign freely (just as a constant coefficient had no effect on our derivative rules).

Ex 3 $\int \left(\left(x^3 + x \right) \sqrt[4]{x^4 + 2x^2 - 5} \right) dx$

$\sqrt[4]{x^4 + 2x^2 - 5}$ is the composite function. So

$$\boxed{\begin{aligned} &u = x^4 + 2x^2 - 5 \\ &du = \left(4x^3 + 4x\right) dx = 4\left(x^3 + x\right) dx \end{aligned}}$$

$$\begin{aligned}
\int \left(\left(x^3 + x \right) \sqrt[4]{x^4 + 2x^2 - 5} \right) dx &= \frac{1}{4} \int \left(\sqrt[4]{x^4 + 2x^2 - 5} \right) \, 4\left(x^3 + x\right) dx \\
&= \frac{1}{4} \int \left(\sqrt[4]{u} \right) du \\
&= \frac{1}{4} \int \left(u^{1/4} \right) du \\
&= \frac{1}{4} \frac{u^{5/4}}{5/4} + C \\
&= \frac{1}{5} \left(x^4 + 2x^2 - 5 \right)^{5/4} + C
\end{aligned}$$

Ex 4 $\displaystyle\int\left(\frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\right)dx$

$$u = x^3+2x^2-5x+2$$
$$du = \left(3x^2+4x-5\right)dx$$

$$\int\left(\frac{3x^2+4x-5}{\left(x^3+2x^2-5x+2\right)^3}\right)dx = \int\left(x^3+2x^2-5x+2\right)^{-3}\left(\left(3x^2+4x-5\right)\,dx\right)$$
$$= \int\left(u^{-3}\right)\,du$$
$$= \frac{u^{-2}}{-2}+C$$
$$= \frac{-1}{2\left(x^3+2x^2-5x+2\right)^2}+C$$

Sometimes, the other factor is not the *du*, or there is an extra *x* that must be replaced with some form of *u*.

Ex 5 $\int(x+1)\sqrt{x-1}dx$

$$u = x-1$$
$$x = u+1$$
$$du = dx$$

$$\int(x+1)\sqrt{x-1}\,dx = \int\left((u+1)+1\right)\sqrt{u}\,du$$
$$= \int(u+2)u^{1/2}du$$
$$= \int\left(u^{3/2}+2u^{1/2}\right)du$$
$$= \frac{u^{5/2}}{5/2}+\frac{2u^{3/2}}{3/2}+C$$
$$= \frac{2}{5}(x-1)^{5/2}+\frac{4}{3}(x-1)^{3/2}+C$$

Ex 6 $\int \left(x^3 (x^2+4)^{3/2} \right) dx$

$$\boxed{\begin{aligned} u &= x^2+4 \\ x^2 &= u-4 \\ du &= 2x\ dx \end{aligned}}$$

$$\begin{aligned}
\int \left(x^3 \left(x^2+4\right)^{3/2} \right) dx &= \frac{1}{2}\int \left(x^2 \left(x^2+4\right)^{3/2} \right)(2x\ dx) \\
&= \frac{1}{2}\int \left((u-4)u^{3/2} \right) du \\
&= \frac{1}{2}\int \left(u^{5/2} - 4u^{3/2} \right) du \\
&= \frac{1}{2}\left(\frac{u^{7/2}}{7/2} - \frac{4u^{5/2}}{5/2} \right) + C \\
&= \frac{1}{7}\left(x^2+4\right)^{7/2} - \frac{4}{5}\left(x^2+4\right)^{5/2} + C
\end{aligned}$$

2.2 Homework Set A

Perform the Anti-differentiation.

1. $\int (5x+3)^3 \, dx$

2. $\int \left(x^3 (x^4+5)^{24} \right) dx$

3. $\int (1+x^3)^2 \, dx$

4. $\int (2-x)^{2/3} \, dx$

5. $\int \left(x\sqrt{2x^2+3} \right) dx$

6. $\int \frac{dx}{(5x+2)^3}$

7. $\displaystyle\int \frac{x^3}{\sqrt{1+x^4}}\, dx$

8. $\displaystyle\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\, dx$

9. $\displaystyle\int \left(x^5 \left(x^2+4\right)^2 \right) dx$

10. $\displaystyle\int \sqrt{x+3}\, \left(x+1\right)^2\, dx$

2.2 Homework Set B

1. $\displaystyle\int (2x+5)\left(x^2+5x+6\right)^6 dx$

2. $\displaystyle\int 3t^2 \left(t^3+1\right)^5 dt$

3. $\displaystyle\int \frac{10m+15}{\sqrt[4]{m^2+3m+1}}\, dm$

4. $\displaystyle\int \frac{3x^2}{\left(1+x^3\right)^5}\, dx$

5. $\int (4s+1)^5 \, ds$

6. $\int \frac{5t}{t^2+1} \, dt$

7. $\int \frac{3m^2}{m^3+8} \, dm$

8. $\int (181x+1)^5 \, dx$

9. $\int \frac{v^2}{5-v^3} \, dv$

Answers: 2.2 Homework Set A

1. $\int (5x+3)^3\,dx$

$= \frac{1}{20}(5x+3)^4 + C$

2. $\int \left(x^3\left(x^4+5\right)^{24}\right)dx$

$= \frac{1}{100}\left(x^4+5\right)^{25} + C$

3. $\int \left(1+x^3\right)^2 dx$

$= \frac{1}{7}x^7 + \frac{1}{2}x^4 + x + C$

4. $\int (2-x)^{2/3}\,dx$

$= -\frac{3}{5}(2-x)^{5/3} + C$

5. $\int \left(x\sqrt{2x^2+3}\right)dx$

$= \frac{1}{6}\left(2x^2+3\right)^{3/2} + C$

6. $\int \frac{dx}{(5x+2)^3}$

$= \frac{1}{-10(5x+2)^2} + C$

7. $\int \frac{x^3}{\sqrt{1+x^4}}\,dx$

$= \frac{1}{2}\sqrt{1+x^4} + C$

8. $\int \frac{x+1}{\sqrt[3]{x^2+2x+3}}\,dx$

$= \frac{3}{4}\left(x^2+2x+3\right)^{2/3} + C$

9. $\int \left(x^5\left(x^2+4\right)^2\right)dx$

$= \frac{1}{10}\left(x^2+4\right)^5 - \left(x^2+4\right)^4 + \frac{8}{3}\left(x^2+4\right)^3 + C$

10. $\int \sqrt{x+3}\,(x+1)^2\,dx$

$= \frac{2}{7}(x+3)^{7/2} - \frac{8}{5}(x+3)^{5/2} + \frac{8}{3}(x+3)^{3/2} + C$

2.2 Homework Set B

1. $\int (2x+5)\left(x^2+5x+6\right)^6 dx$

$= \frac{1}{7}\left(x^2+5x+6\right)^7 + C$

2. $\int 3t^2\left(t^3+1\right)^5 dt$

$= \frac{1}{6}\left(t^3+1\right)^6 + C$

3. $\int \frac{10m+15}{\sqrt[4]{m^2+3m+1}}\,dm$

$= \frac{20}{3}\left(m^2+3m+1\right)^{3/4} + C$

4. $\int \frac{3x^2}{\left(1+x^3\right)^5}\,dx$

$= -\frac{1}{4}\left(1+x^3\right)^{-4} + C$

5. $\int (4s+1)^5\, ds$

$= \frac{1}{24}(4s+1)^6 + C$

6. $\int \frac{5t}{t^2+1}\, dt$

$= \frac{5}{2}\ln(t^2+1) + C$

7. $\int \frac{3m^2}{m^3+8}\, dm$

$= \ln\left|m^3+8\right| + C$

8. $\int (181x+1)^5\, dx$

$= \frac{1}{1086}(181x+1)^6 + C$

9. $\int \frac{v^2}{5-v^3}\, dv$

$= -\frac{1}{3}\ln\left|5-v^3\right| + C$

2.3: Anti-derivatives--the Transcendental Rules

The proof of the Transcendental Integral Rules can be left to a more formal Calculus course. But since the integral is the inverse of the derivative, the discovery of the rules should be obvious from looking at the comparable derivative rules.

Derivative Rules

$$\frac{d}{dx}\left[\sin u\right]=\left(\cos u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\csc u\right]=\left(-\csc u\ \cot u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\cos u\right]=\left(-\sin u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\sec u\right]=\left(\sec u\ \tan u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\tan u\right]=\left(\sec^2 u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[\cot u\right]=\left(-\csc^2 u\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[e^u\right]=\left(e^u\right)\frac{du}{dx} \qquad \frac{d}{dx}\left[Ln\ u\right]=\left(\frac{1}{u}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[a^u\right]=a^u\cdot\ln(a)\frac{du}{dx} \qquad \frac{d}{dx}\left[Log_a\ u\right]=\left(\frac{1}{u\cdot\ln a}\right)\frac{du}{dx}$$

$$\frac{d}{dx}\left[\sin^{-1} u\right]=\frac{1}{\sqrt{1-u^2}}\cdot D_u \qquad \frac{d}{dx}\left[\csc^{-1} u\right]=\frac{-1}{|\ u\ |\sqrt{u^2-1}}\cdot D_u$$

$$\frac{d}{dx}\left[\cos^{-1} u\right]=\frac{-1}{\sqrt{1-u^2}}\cdot D_u \qquad \frac{d}{dx}\left[\sec^{-1} u\right]=\frac{1}{|\ u\ |\sqrt{u^2-1}}\cdot D_u$$

$$\frac{d}{dx}\left[\tan^{-1} u\right]=\frac{1}{u^2+1}\cdot D_u \qquad \frac{d}{dx}\left[\cot^{-1} u\right]=\frac{-1}{u^2+1}\cdot D_u$$

Transcendental Integral Rules

$$\int (\cos u)\, du = \sin u + C$$

$$\int (\sin u)\, du = -\cos u + C$$

$$\int \left(\sec^2 u\right) du = \tan u + C$$

$$\int \left(e^u\right) du = e^u + C$$

$$\int \left(a^u\right) du = \frac{a^u}{\ln a} + C$$

$$\int (\csc u \cot u)\, du = -\csc u + C$$

$$\int (\sec u \tan u)\, du = \sec u + C$$

$$\int \left(\csc^2 u\right) du = -\cot u + C$$

$$\int \left(\frac{1}{u}\right) du = \ln|u| + C$$

$$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C$$

$$\int \frac{du}{1+u^2} = \tan^{-1} u + C$$

$$\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1} u + C$$

Note that there are only three integrals that yield inverse trig functions where there were six inverse trig derivatives. This is because the other three rules derivative rules are just the negatives of the first three. As we will see later, these three rules are simplified versions of more general rules, but for now we will stick with the three.

OBJECTIVE

Integrate functions involving Transcendental operations.

Ex 1 $\int (\sin x + 3\cos x)\, dx$

$$\begin{aligned}\int (\sin x + 3\cos x)\, dx &= \int (\sin x)\, dx + 3\int (\cos x)\, dx \\ &= -\cos x + 3\sin x + C\end{aligned}$$

Ex 2 $\int\left(e^x+4+3\csc^2 x\right)dx$

$$\int\left(e^x+4+3\csc^2 x\right)dx=\int\left(e^x\right)dx+4\int dx+3\int\left(\csc^2 x\right)dx$$

$$=e^x+4x-3\cot x+C$$

Ex 3 $\int\left(\sec x\left(\sec x+\tan x\right)\right)dx$

$$\int\left(\sec x\left(\sec x+\tan x\right)\right)dx=\int\left(\sec^2 x\right)dx+\int\left(\sec x\ \tan x\right)dx$$

$$=\tan x+\sec x+C$$

Of course, the Unchain Rule will apply to the transcendental functions quite well. We need to remember all of the ideas from the previous section, because integrals of this type are very common in Calculus. In addition, **you must have all of the integral rules memorized –** these problems become virtually impossible if you have to keep looking up rules.

Ex 4 $\int\left(\sin 5x\right)dx$

$$\boxed{\begin{aligned}u&=5x\\du&=5dx\end{aligned}}$$

$$\begin{aligned}\int\left(\sin 5x\right)dx&=\frac{1}{5}\int\left(\sin 5x\right)\ 5\ dx\\&=\frac{1}{5}\int\left(\sin u\right)du\\&=\frac{1}{5}\left(-\cos u\right)+C\\&=-\frac{1}{5}\cos\ 5x\ +C\end{aligned}$$

Ex 5 $\int\left(\sin^6 x\ \cos x\right)dx$

$$\boxed{\begin{aligned}u&=\sin x\\du&=\cos\ x\ dx\end{aligned}}$$

$$\begin{aligned}\int\left(\sin^6 x\ \cos\ x\right)dx&=\int\left(u^6\right)du\\&=\frac{1}{7}u^7+C\\&=\frac{1}{7}\sin^7\ x\ +C\end{aligned}$$

Ex 6 $\int \left(x^5 \sin x^6\right) dx$

$$\boxed{\begin{aligned} u &= x^6 \\ du &= 6x^5 dx \end{aligned}}$$

$$\int \left(x^5 \sin x^6\right) dx = \frac{1}{6}\int \left(\sin x^6\right)\left(6x^5 dx\right)$$

$$= \frac{1}{6}\int \left(\sin u\right) du$$

$$= -\frac{1}{6}\cos u + C$$

$$= -\frac{1}{6}\cos x^6 + C$$

Ex 7 $\int \left(\cot^3 x \ \csc^2 x\right) dx$

$$\boxed{\begin{aligned} u &= \cot x \\ du &= -\csc^2 x \, dx \end{aligned}}$$

$$\int \left(\cot^3 x \ \csc^2 x\right) dx = -\int \left(\cot^3 x\right)\left(-\csc^2 x \, dx\right)$$

$$= -\int \left(u^3\right) du$$

$$= -\frac{1}{4}u^4 + C$$

$$= -\frac{1}{4}\cot^4 x + C$$

Ex 8 $\int \left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right) dx$

$$\boxed{\begin{aligned} u &= \sqrt{x} = x^{1/2} \\ du &= \frac{1}{2}x^{-1/2}\ dx = \frac{1}{2x^{1/2}} dx \end{aligned}}$$

$$\int \left(\frac{\cos\sqrt{x}}{\sqrt{x}}\right) dx = 2\int \left(\cos\sqrt{x}\right)\left(\frac{1}{2\sqrt{x}} dx\right)$$

$$= 2\int \left(\cos u\right) du$$

$$= 2\sin u + C$$

$$= 2\sin\sqrt{x} + C$$

Ex 9 $\int \left(xe^{x^2+1}\right)dx$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\ dx \end{aligned}}$$

$$\int \left(xe^{x^2+1}\right)d\ x = \frac{1}{2}\int \left(e^{x^2+1}\right)(2x\ dx)$$

$$= \frac{1}{2}\int \left(e^{u}\right) du$$

$$= \frac{1}{2}e^{u} + C$$

$$= \frac{1}{2}e^{x^2+1} + C$$

Ex 10 $\int \frac{x}{\sqrt{1-x^4}}\ dx$

$$\boxed{\begin{aligned} u &= x^2 \\ du &= 2x\ dx \end{aligned}}$$

$$\int \frac{x}{\sqrt{1-x^4}}\ dx = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(x^2\right)^2}}(2x\ dx)$$

$$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}\ du$$

$$= \frac{1}{2}\sin^{-1} u + C$$

$$= \frac{1}{2}\sin^{-1} x^2 + C$$

2.3 Homework Set A

Perform the Anti-differentiation.

1. $\int \left(x^4 \cos x^5\right) dx$

2. $\int \left(\sin\left(7x+1\right)\right) dx$

3. $\int \left(\sec^2\left(3x-1\right)\right) dx$

4. $\int \left(\frac{\sin\sqrt{x}}{\sqrt{x}} \right) dx$

5. $\int \left(\tan^4 x \sec^2 x\right) dx$

6. $\int \frac{\ln x}{x}\, dx$

7. $\int \left(e^{6x}\right) dx$

8. $\int \frac{\cos 2x}{\sin^3 2x}\, dx$

9. $\displaystyle\int \frac{x \ln\left(x^2+1\right)}{x^2+1}\,dx$

10. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

11. $\displaystyle\int \left(\sqrt{\cot x}\ \csc^2 x\right)dx$

12. $\displaystyle\int \frac{1}{x^2}\left(\sin\frac{1}{x}\right)\left(\cos\frac{1}{x}\right)dx$

13. $\displaystyle\int \frac{x}{1+x^4}\,dx$

14. $\displaystyle\int \frac{\cos x}{\sqrt{1-\sin^2 x}}\,dx$

2.3 Homework Set B

1. $\int \left(x^5 - \sin(3x) + xe^{x^2}\right)dx$

2. $\int \frac{\cos x}{1+\sin x}\,dx$

3. $\int \sec^2(2x)\,dx$

4. $\int \frac{\csc^2\left(e^{-x}\right)}{e^x}\,dx$

5. $\int \frac{\sec(\ln x)\tan(\ln x)}{3x}\,dx$

6. $\int \left(x^5 + \frac{7}{x^2} - e^{2x} + \sec^2 x\right)dx$

7. $\int e^x \csc e^x \cot e^x\,dx$

8. $\int \left(e^x - 2\right)\left(e^x - 1\right)dx$

9. $\int x^2 \sec^2(x^3) + 2xe^{x^2}\, dx$

10. $\int \left(\sec^2 y \tan^5 y\right) dy$. Verify that your integration is correct by taking the derivative of your answer.

11. $\int \left(\cos\theta e^{\sin\theta} + \dfrac{\theta}{\theta^2+1}\right) d\theta$. Verify that your integration is correct by taking the derivative of your answer.

12. $\int t \sec^2\left(4t^2\right)\sqrt{\tan\left(4t^2\right)}\, dt$. Verify that your integration is correct by taking the derivative of your answer.

13. $\int x^2 \sin x^3\, dx$

14. $\int te^{5t^2+1}\, dt$

15. $\int \left(e^y + 1\right)^2 dy$

16. $\int x \sec^2 x^2 \sqrt{\tan x^2}\, dx$

17. $\int \sin(3t)\cos^5(3t)\, dt$

18. $\int x \cos x^2 e^{\sin x^2}\, dx$

19. $\int \tan\theta \ln(\sec\theta)\, d\theta$

20. $\int \left(e^{4y} + 2y^2 - 7\cos 3y\right) dy$

21. $\int \frac{\sin(x+4)}{\cos^7(x+4)} dx$

22. $\int \left(\frac{2x}{x^2+5} - \sec^2(3x) + xe^{x^2} - \pi \right) dx$

23. $\int e^{2t} \sec^2 e^{2t} \, dt$

24. $\int \frac{18 \ln m}{m} \, dm$

25. $\int \frac{2y \cos\left(y^2\right)}{\sin^4\left(y^2\right)} dy$

2.3 Homework Set A

1. $\int \left(x^4 \cos x^5\right) dx$

$= \frac{1}{5}\sin x^5 + C$

2. $\int \left(\sin(7x+1)\right) dx$

$= -\frac{1}{7}\cos(7x+1) + C$

3. $\int \left(\sec^2(3x-1)\right) dx$

$= \frac{1}{3}\tan(3x-1) + C$

4. $\int \left(\frac{\sin\sqrt{x}}{\sqrt{x}}\right) dx$

$= -2\cos\sqrt{x} + C$

5. $\int \left(\tan^4 x \ \sec^2 x\right) dx$

$= \frac{1}{5}\tan^5 x + C$

6. $\int \frac{\ln x}{x}\, dx$

$= \frac{1}{2}\ln^2 x + C$

7. $\int \left(e^{6x}\right) dx$

$= \frac{1}{6}e^{6x} + C$

8. $\int \frac{\cos 2x}{\sin^3 2x}\, dx$

$= -\frac{1}{4}\csc^2 2x + C$

9. $\int \frac{x \ln\left(x^2+1\right)}{x^2+1}\, dx$

$= \frac{1}{4}\ln^2\left(x^2+1\right) + C$

10. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$

$= 2e^{\sqrt{x}} + C$

11. $\int \left(\sqrt{\cot x} \ \csc^2 x\right) dx$

$= -\frac{2}{3}\cot^{3/2} x + C$

12. $\int \frac{1}{x^2}\left(\sin\frac{1}{x}\right)\left(\cos\frac{1}{x}\right) dx$

$= -\frac{1}{2}\sin^2\frac{1}{x} + C$

13. $\int \frac{x}{1+x^4}\, dx$

$= \frac{1}{2}\tan^{-1} x^2 + C$

14. $\int \frac{\cos x}{\sqrt{1-\sin^2 x}}\, dx$

$= x + C$, $x \neq \frac{\pi}{2} \pm 2\pi n$

2.3 Homework Set B

1. $\int\left(x^5-\sin(3x)+xe^{x^2}\right)dx$

   $=\frac{1}{6}x^6+\frac{1}{3}\cos(3x)+\frac{1}{2}e^{x^2}+C$

2. $\int\frac{\cos x}{1+\sin x}dx$

   $=\ln(1+\sin x)+C$

3. $\int\sec^2(2x)dx$

   $=\frac{1}{2}\tan(2x)+C$

4. $\int\frac{\csc^2\left(e^{-x}\right)}{e^x}dx$

   $=\cot\left(e^{-x}\right)+C$

5. $\int\frac{\sec(\ln x)\tan(\ln x)}{3x}dx$

   $=\frac{1}{3}\sec(\ln x)+C$

6. $\int\left(x^5+\frac{7}{x^2}-e^{2x}+\sec^2 x\right)dx$

   $=\frac{1}{6}x^6-7x^{-1}-\frac{1}{2}e^{2x}+\tan x+C$

7. $\int e^x\csc e^x\cot e^x\,dx$

   $=-\csc\left(e^x\right)+C$

8. $\int\left(e^x-2\right)\left(e^x-1\right)dx$

   $=\frac{1}{2}e^{2x}-3e^x+2x+C$

9. $\int x^2\sec^2\left(x^3\right)+2xe^{x^2}\,dx$

   $=\frac{1}{3}\tan\left(x^3\right)+e^{x^2}+C$

10. $\int\left(\sec^2 y\tan^5 y\right)dy$. Verify that your integration is correct by taking the derivative of your answer.

   $=\frac{1}{6}\tan^6(y)+C$

   $\frac{d}{dy}\left[\frac{1}{6}\tan^6(y)+C\right]$

   $=\sec^2 y\tan^5 y$

11. $\int\left(\cos\theta e^{\sin\theta}+\frac{\theta}{\theta^2+1}\right)d\theta$. Verify that your integration is correct by taking the derivative of your answer.

   $=e^{\sin\theta}+\frac{1}{2}\ln\left(\theta^2+1\right)+C$

   $\frac{d}{d\theta}\left[e^{\sin\theta}+\frac{1}{2}\ln\left(\theta^2+1\right)+C\right]$

   $=\cos\theta e^{\sin\theta}+\frac{\theta}{\theta^2+1}$

12. $\int t\sec^2\left(4t^2\right)\sqrt{\tan\left(4t^2\right)}\,dt$. Verify that your integration is correct by taking the derivative of your answer.

$= \frac{1}{12}\tan^{3/2}\left(4t^2\right)+C$

$\frac{d}{dt}\left[\frac{1}{12}\tan^{3/2}\left(4t^2\right)+C\right]$

$= t\sec^2\left(4t^2\right)\sqrt{\tan\left(4t^2\right)}$

13. $\int x^2\sin x^3\,dx$

$= -\frac{1}{3}\cos x^3 + C$

14. $\int te^{5t^2+1}dt$

$= \frac{1}{10}e^{5t^2+1}+C$

15. $\int\left(e^y+1\right)^2 dy$

$= \frac{1}{2}e^{2y}+2e^y+y+C$

16. $\int x\sec^2 x^2\sqrt{\tan x^2}\,dx$

$= \frac{1}{3}\tan^{3/2}\left(x^2\right)+C$

17. $\int \sin\left(3t\right)\cos^5\left(3t\right)dt$

$= \frac{1}{18}\cos^6\left(3t\right)+C$

18. $\int x\cos x^2 e^{\sin x^2}\,dx$

$= \frac{1}{2}e^{\sin x^2}+C$

19. $\int \tan\theta\ln\left(\sec\theta\right)d\theta$

$= \frac{1}{2}\ln^2\left(\sec\theta\right)+C$

20. $\int\left(e^{4y}+2y^2-7\cos 3y\right)dy$

$= \frac{1}{4}e^{4y}+\frac{2}{3}y^3-\frac{7}{3}\sin 3y+C$

21. $\int \frac{\sin\left(x+4\right)}{\cos^7\left(x+4\right)}dx$

$= \frac{1}{6}\sec^6\left(x+4\right)+C$

22. $\int\left(\frac{2x}{x^2+5}-\sec^2\left(3x\right)+xe^{x^2}-\pi\right)dx$

$= \ln\left(x^2+5\right)-\frac{1}{3}\tan\left(3x\right)+\frac{1}{2}e^{x^2}-\pi x+C$

23. $\int e^{2t}\sec^2 e^{2t}\,dt$

$= \frac{1}{2}\tan e^{2t}+C$

24. $\int \frac{18\ln m}{m}\,dm$

$= 9\ln^2 m + C$

25. $\int \frac{2y\cos\left(y^2\right)}{\sin^4\left(y^2\right)}dy$

$$= -\frac{1}{3}\csc^3\left(y^2\right) + C$$

2.4: Separable Differential Equations

Vocabulary:

Differential Equation (differential equation) – an equation that contains an unknown function and one or more of its derivatives.
General Solution – The solution obtained from solving a differential equation. It still has the $+C$ in it.
Initial Condition – Constraint placed on a differential equation; sometimes called an initial value.
Particular Solution – Solution obtained from solving a differential equation when an initial condition allows you to solve for C.
Separable Differential Equation – A differential equation in which all terms with y's can be moved to the left side of an equals sign (=), and in which all terms with x's can be moved to the right side of an equals sign (=), by multiplication and division only.

OBJECTIVES
Given a separable differential equation, find the general solution. Given a separable differential equation and an initial condition, find a particular solution.

Ex 1 Find the general solution to the differential equation $\frac{dy}{dx} = -\frac{x}{y}$.

$\frac{dy}{dx} = -\frac{x}{y}$	Start here.
$ydy = -xdx$	Separate all the y terms to the left side of the equation and all of the x terms to the right side of the equation.
$\int ydy = \int -xdx$	Integrate both sides.
$\frac{1}{2}y^2 = -\frac{1}{2}x^2 + C$	You only need C on one side of the equation and we put it on the side containing the x.

$y^2 = -x^2 + C$	Multiply both sides by 2. Note: 2*C* is still a constant, so we'll continue to note it just by *C*.
$x^2 + y^2 = C$	This equation should seem familiar. It's the family of circles centered at the origin with radius $\sqrt{C}$.

Usually, if it's possible, we will solve our equation for *y* – so our solution can be written as $y = \pm\sqrt{C - x^2}$.

Note that we could check our solution by taking the derivative of our solution.

$$x^2 + y^2 = C \Rightarrow 2x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}.$$

This process is called implicit differentiation because we are not explicitly solved for *y*. We introduced this idea in Chapter 1, and we will revisit it more in a later chapter. The key idea is that the derivative of *y* is $\frac{dy}{dx}$ because of the Chain Rule.

Steps to Solving a Differential Equation:

1. Separate the variables. Note: Leave constants on the right side of the equation.
2. Integrate both sides of the equation. Note: only write the +*C* on the right side of the equation. Here's why – you will have a constant on either of your equation when you integrate both sides of your differential Equation, but you would wind up subtracting one from the other eventually, and two constants subtracted from one another is still a constant, so we only write +*C* on one side of the equation.
3. Solve for *y*, if possible. If you integrate and get a natural log in your result, solve for *y*. If there is no natural log, solve for *C*. Note: $e^{\ln|y|} = y$ because *e* raised to any power is automatically positive, so the absolute values are not necessary.
4. Plug in the initial condition, if you are given one, and solve for *C*.

Ex 2 Find the particular solution to $y' = 2xy - 3y$, given $y(3) = 2$.

$$\frac{dy}{dx} = (2x-3)y$$
$$\frac{dy}{y} = (2x-3)dx$$
$$\ln|y| = x^2 - 3x + C$$
$$|y| = e^{x^2-3x+C}$$
$$|y| = e^{x^2-3x}e^C$$

Remember that $x^a \cdot x^b = x^{a+b}$

$$y = Ke^{x^2-3x}$$

Since K could be positive or negative, we can get rid of the absolute values here.

$$y(3) = 2 \to 2 = Ke^0$$
$$K = 2$$
$$y = 2e^{x^2-3x}$$

What if $\frac{dy}{dx} = y + x$? Since this differential equation can't be separated, we can't use this technique. If we multiplied both sides by *dx*, we would end up with a *y* with the *dx*. We couldn't just subtract the *y* because we have only dealt with *dy* or *dx* multiplied by the function we are integrating. This requires a more advanced technique that is usually introduced in a class specifically on differential equations in college.

How about $\frac{dy}{dx} = e^{-x^2}$? While we can separate the variables, we get an integral that we cannot solve with the Unchain Rule. For $\int dy = \int e^{-x^2} dx$ we cannot choose a *u* that will make this integral work with the Unchain Rule.

Ex 3   Find the general solution to the differential equation $\frac{dm}{dt} = mt$.

$\frac{dm}{dt} = mt$	Start here.
$\frac{dm}{m} = tdt$	Separate all the *m* terms to the left side of the equation and all of the *t* terms to the right side of the equation.
$\int \frac{dm}{m} = \int tdt$	Integrate both sides.
$\ln\lvert m\rvert = \frac{1}{2}t^2 + C$	You only need *C* on one side of the equation and we put it on the side containing the *x*.
$e^{\ln\lvert m\rvert} = e^{\frac{1}{2}t^2 + C}$	*e* both sides of the equation to solve for *y*.
$\lvert m\rvert = e^{\frac{1}{2}t^2} e^C$	
$m = Ke^{\frac{1}{2}t^2}$	e^C is still a constant, so we will just note it as *K*.

Again, we could check our solution by taking the derivative of our solution. Also notice that the variables we used did not matter – what is important is making sure that the *dm* is in the numerator on the *m* side, and the *dt* is on the numerator on the *t* side.

Ex 5   Solve the differential equation $\frac{dy}{dx} = \frac{9x^2 + 2x + 5}{2y + e^y}$

$$\frac{dy}{dx} = \frac{9x^2 + 2x + 5}{2y + e^y}$$

$$\left(2y + e^y\right)dy = \left(9x^2 + 2x + 5\right)dx$$

$$\int\left(2y + e^y\right)dy = \int\left(9x^2 + 2x + 5\right)dx$$

$$y^2 + e^y = 3x^3 + x^2 + 5x + C$$

$$y^2 + e^y - 3x^3 - x^2 - 5x = C$$

This is the general solution. Since it is not possible to explicitly solve this for *y*, we have to leave it in this form.

Ex 6 Find the particular solution to $\frac{dr}{dt}=\frac{3t^2-\sin t}{4r}$ given that $r(0)=3$.

$$\begin{aligned}
\frac{dr}{dt}&=\frac{3t^2-\sin t}{4r}\\
4r\,dr&=\left(3t^2-\sin t\right)dt\\
\int 4r\,dr&=\int\left(3t^2-\sin t\right)dt\\
2r^2&=t^3+\cos t+C\\
2\cdot 3^2&=0^3+\cos 0+C\Rightarrow C=17\\
2r^2&=t^3+\cos t+17\\
r&=\pm\sqrt{\frac{t^3}{3}+\frac{1}{2}\cos t+\frac{17}{2}}\\
r&=\sqrt{\frac{t^3}{3}+\frac{1}{2}\cos t+\frac{17}{2}}
\end{aligned}$$

Since our initial value, $r(0)=3$, was positive, we only need the positive portion of the radical. It is implied in these processes that we are dealing with functions for our final equation (by the function notation on the intial value), so we have to pick a single function, either the + or the –, for our final answer.
What would have happened if you had solved for *r* <u>before</u> plugging in the initial condition?

$$\begin{aligned}
r^2&=\frac{t^3}{2}+\frac{1}{2}\cos t+C\\
r&=\pm\sqrt{\frac{t^3}{3}+\frac{1}{2}\cos t+C}\\
3&=\pm\sqrt{\frac{0^3}{3}+\frac{1}{2}\cos 0+C}\Rightarrow C=\frac{17}{2}\\
r&=\sqrt{\frac{t^3}{3}+\frac{1}{2}\cos t+\frac{17}{2}}
\end{aligned}$$

Again, you can check your solution by taking the derivative of your solution.

2.4 Homework Set A

Solve the differential equation.

1. $\frac{dy}{dx} = \frac{y}{x}$

2. $(x^2 + 1)y' = xy$

3. $(1 + \sec y \tan y)y' = x^2 + 1$

4. $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

5. $\frac{dy}{dx} = \frac{x^2\sqrt{x^3 - 3}}{y^2}$

Find the solution of the differential equation that satisfies the given initial condition.

6. $\frac{dy}{dx} = \frac{2x^3}{3y^2};\ y(\sqrt{2}) = 0$

7. $\frac{dy}{dx} = (y^2 + 1); y(1) = 0$

8. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u};\ u(0) = -5$

9. Solve the initial-value problem $y' = \frac{\sin x}{\sin y}$; $y(0) = \frac{\pi}{2}$, and graph the solution.

10. Solve the equation $e^{-y} y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant C varies?

2.4 Homework – Set B

Solve the differential equation.

1. $\dfrac{dy}{dx} = 4xy^3$

2. $y' = y^2 \sin x$

3. $\dfrac{dv}{dt} = 2 + 2v + t + tv$

4. $\dfrac{dy}{dt} = \dfrac{t}{y\sqrt{y^2+1}}$

5. $\dfrac{d\theta}{dr} = \dfrac{1+\sqrt{r}}{\sqrt{\theta}}$

Find the solution of the differential equation that satisfies the given initial condition.

6. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1+y^2};\ y(0) = 1$

7. Find an equation of the curve that satisfies $\dfrac{dy}{dx} = 4x^3 y$ and whose y-intercept is 7.

8. Solve the initial-value problem $y' = \dfrac{\cos 3x}{\sin 2y};\ y\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{2}$, and graph the solution.

Answers: 2.4 Homework Set A

1. $\frac{dy}{dx} = \frac{y}{x}$

   $y = kx$

2. $\left(x^2+1\right)y' = xy$

   $y = C\sqrt{x^2+1}$

3. $\left(1+\sec y \tan y\right)y' = x^2+1$

   $y = \sec^{-1}\left(\frac{x^3}{3}+x+C\right)$

4. $\frac{dy}{dx} = \frac{e^{2x}}{4y^3}$

   $y = \pm\left(\frac{1}{2}e^{2x}+C\right)^{1/4}$

5. $\frac{dy}{dx} = \frac{x^2\sqrt{x^3-3}}{y^2}$

   $y = \left(\frac{2}{3}\left(x^3-3\right)^{3/2}+C\right)^{1/3}$

6. $\frac{dy}{dx} = \frac{2x^3}{3y^2};\ y\left(\sqrt{2}\right) = 0$

   $y = \left(\frac{1}{2}x^4-2\right)^{1/3}$

7. $\frac{dy}{dx} = \left(y^2+1\right);\ y(1) = 0$

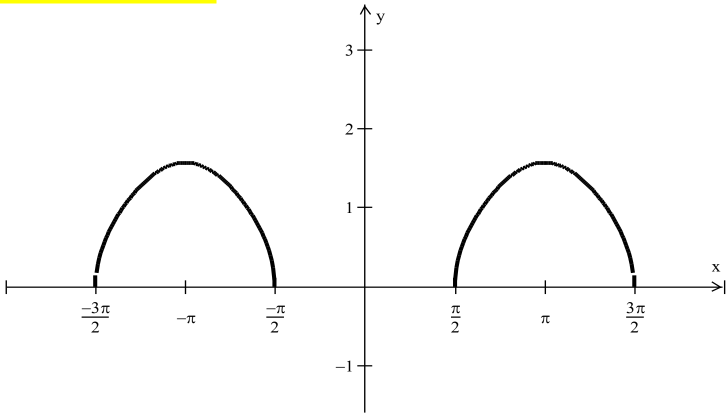
   $y = \tan\left(x-1\right)$

8. $\frac{du}{dt} = \frac{2t+\sec^2 t}{2u};\ u(0) = -5$

   $u = -\sqrt{t^2+\tan t+25}$

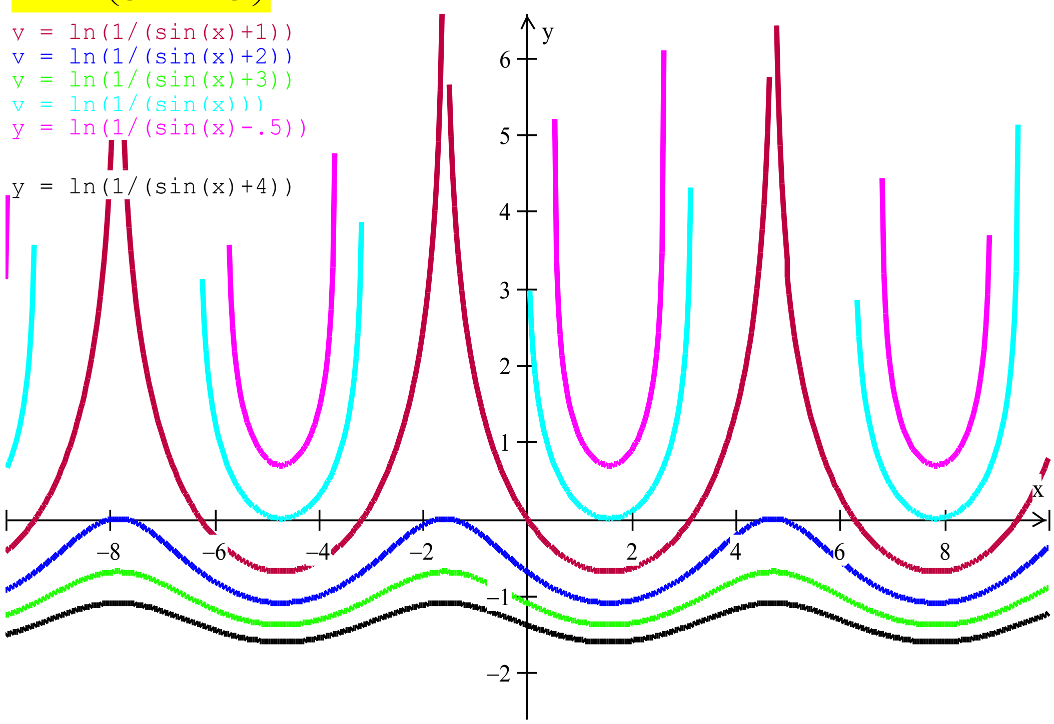
9. Solve the initial-value problem $y' = \dfrac{\sin x}{\sin y}$; $y(0) = \dfrac{\pi}{2}$, and graph the solution.

$y = \cos^{-1}\left(\cos(x) - 1\right)$



10. Solve the equation $e^{-y}y' + \cos x = 0$ and graph several members of the family of solutions. How does the solution curve change as the constant C varies?

$y = \ln\left(\dfrac{1}{\sin x + C}\right)$



2.4 Homework – Set B

1. $\dfrac{dy}{dx} = 4xy^3$

   $y = \pm\sqrt{\dfrac{1}{-4x^2 + C}}$

2. $y' = y^2 \sin x$

   $y = \dfrac{1}{\cos(x) + C}$

3. $\dfrac{dv}{dt} = 2 + 2v + t + tv$

   $v = Ke^{2t + \frac{t^2}{2}} - 1$

4. $\dfrac{dy}{dt} = \dfrac{t}{y\sqrt{y^2 + 1}}$

   $y = \pm\sqrt{\left(\dfrac{3t^2}{2} + C\right)^{2/3} - 1}$

5. $\dfrac{d\theta}{dr} = \dfrac{1 + \sqrt{r}}{\sqrt{\theta}}$

   $\theta = \left(\dfrac{3}{2}r + r^{3/2} + C\right)^{2/3}$

6. $\dfrac{dy}{dx} = \dfrac{y\cos x}{1 + y^2};\ \ y(0) = 1$

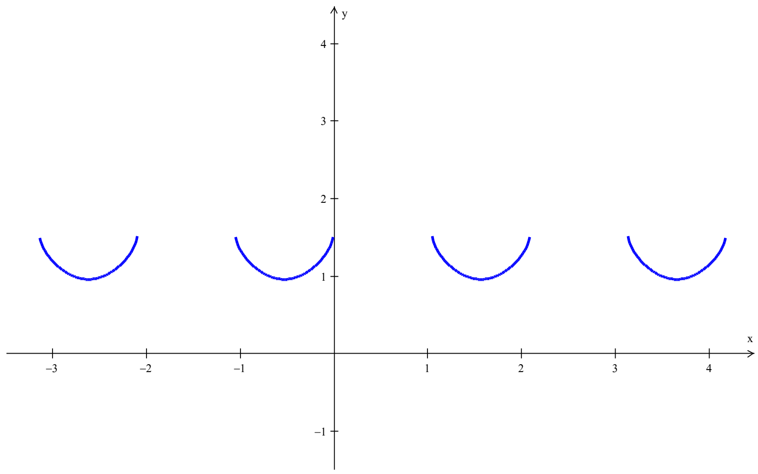
   $\ln|y| + \dfrac{y^2}{2} = \sin(x) + \dfrac{1}{2}$

7. Find an equation of the curve that satisfies $\dfrac{dy}{dx} = 4x^3 y$ and whose y-intercept is 7.

   $y = 7e^{x^4}$

8. Solve the initial-value problem $y' = \dfrac{\cos 3x}{\sin 2y}$; $y\left(\dfrac{\pi}{3}\right) = \dfrac{\pi}{2}$, and graph the solution.

$$y = \frac{1}{2}\cos^{-1}\left(-1 - \frac{2}{3}\sin 3x\right)$$



2.5 Slope Fields

Vocabulary:

Slope field – Given any function or relation, a slope field is drawn by taking evenly spaced points on the Cartesian coordinate system (usually points having integer coordinates) and, at each point, drawing a small line with the slope of the function.

Objectives:

Given a differential equation, sketch its slope field.
Given a slope field, sketch a particular solution curve.
Given a slope field, determine the family of functions to which the solution curves belong.
Given a slope field, determine the differential equation that it represents.

We use slope fields for several reasons, some of which are important mathematically, and others that are more pragmatic.

- They provide for a way to visually see an antiderivative as a family of functions. When we find an indefinite integral, there is always the "+*C*". A slope field shows us a picture of the variety of solution curves.
- Some differential equations do not have an antiderivative that can be represented as a known function (like $\frac{dy}{dx} = e^{-x^2}$). Slope fields allow us to represent the antiderivative of functions that are impossible to integrate.
- The most important reason, however, is that the AP Calculus test asks these questions. Whether or not there is any practical value to the mathematics of it, because we must answer questions on an AP test, we have to learn it.

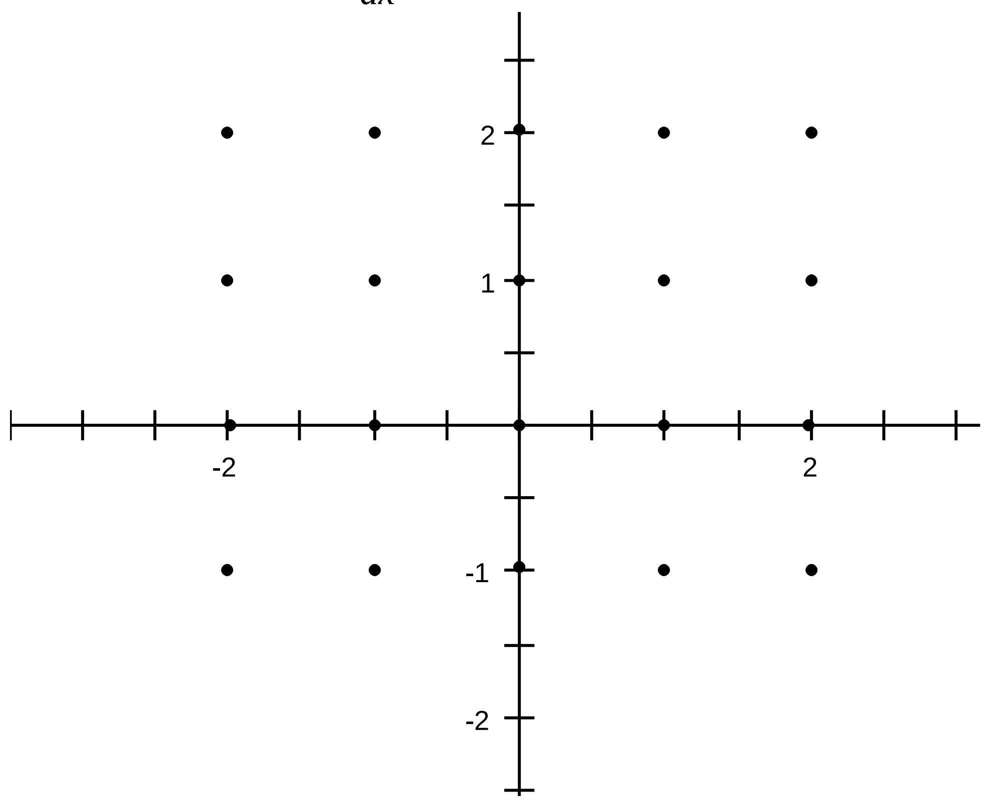
Slope fields give us a graphical solution to a differential equation even is we cannot find an analytical solution. This just means that even though we don't have an actual equation for a solution, we still know what the solution looks like.

<u>Key Idea:</u> Remember that $\frac{dy}{dx}$ is the slope of a tangent line. Slope fields are represented the way they are because of this fact.

Steps to Sketching a Slope Field:

1. Determine the grid of points for which you need to sketch (many times the points are given).
2. Pick your first point. Note its *x* and *y* coordinate. Plug these numbers into the differential equation. This is the slope at that point.
3. Find that point on the graph. Make a little line (or dash) at that point whose slope represents the slope that you found in Step 2. Positive slopes point up as you look left to right, and negative slopes point down. The bigger the numeric value, the steeper the slope.
4. Repeat this process for all the points needed.

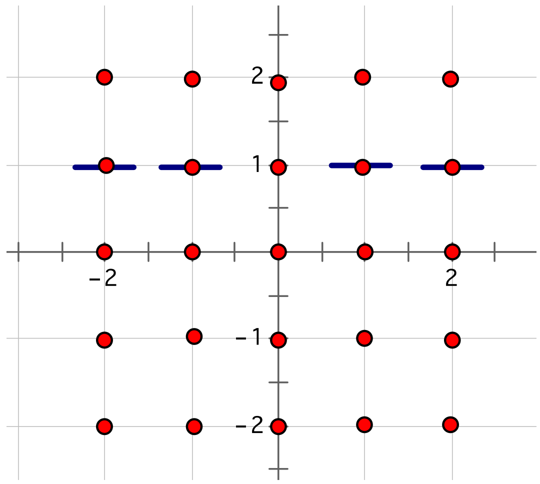
Ex 1 Sketch the slope field for $\frac{dy}{dx} = y + x$ at the points indicated.



Ex 2 Sketch the slope field for $\frac{dy}{dx}=\frac{1-y}{x}$ at the points indicated.

Note that, wherever $y=1$, $\frac{dy}{dx}=0$. So, the segments at $y=1$ will be horizontal.

Also, where $x=0$, $\frac{dy}{dx}=\text{dne}$. So, there are not segments on the *x*-axis.



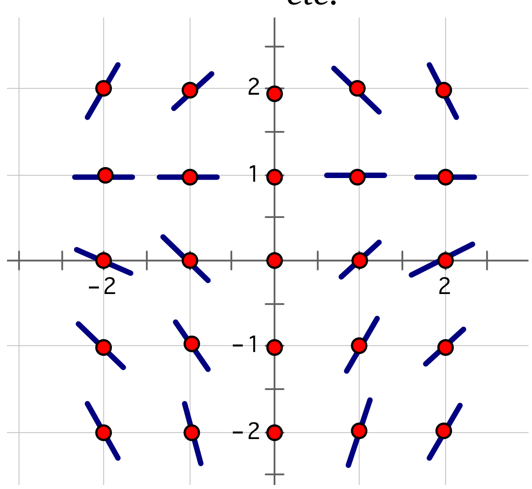
We can then plug the numerical values of the points into $\frac{dy}{dx}=\frac{1-y}{x}$ to determine the slant of the line segments.

$$(-2, 2)\rightarrow\frac{dy}{dx}=\frac{1}{2} \qquad (2, 2)\rightarrow\frac{dy}{dx}=-\frac{1}{2}$$

$$(-2, 0)\rightarrow\frac{dy}{dx}=-\frac{1}{2} \qquad (2, 0)\rightarrow\frac{dy}{dx}=\frac{1}{2}$$

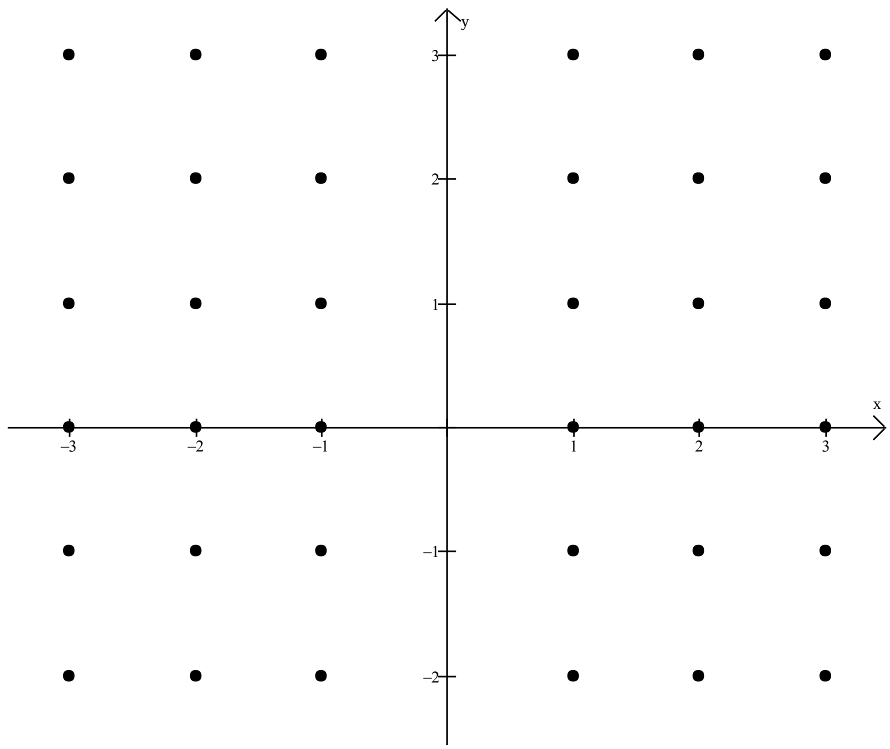
$$(-2, -2)\rightarrow\frac{dy}{dx}=-\frac{3}{2} \qquad (2, -2)\rightarrow\frac{dy}{dx}=\frac{3}{2}$$

etc.



Ex 3 Sketch the slope field for $x\dfrac{dy}{dx}=1$ at the points indicated.

We first need to solve this equation for $\dfrac{dy}{dx}$, so $\dfrac{dy}{dx}=\dfrac{1}{x}$. Then we find the slope values for each of the (x, y) points indicated. Notice that there is no y in the equation, so y does not matter in finding the slopes. That should mean that the slope for any individual x is identical regardless of the y.



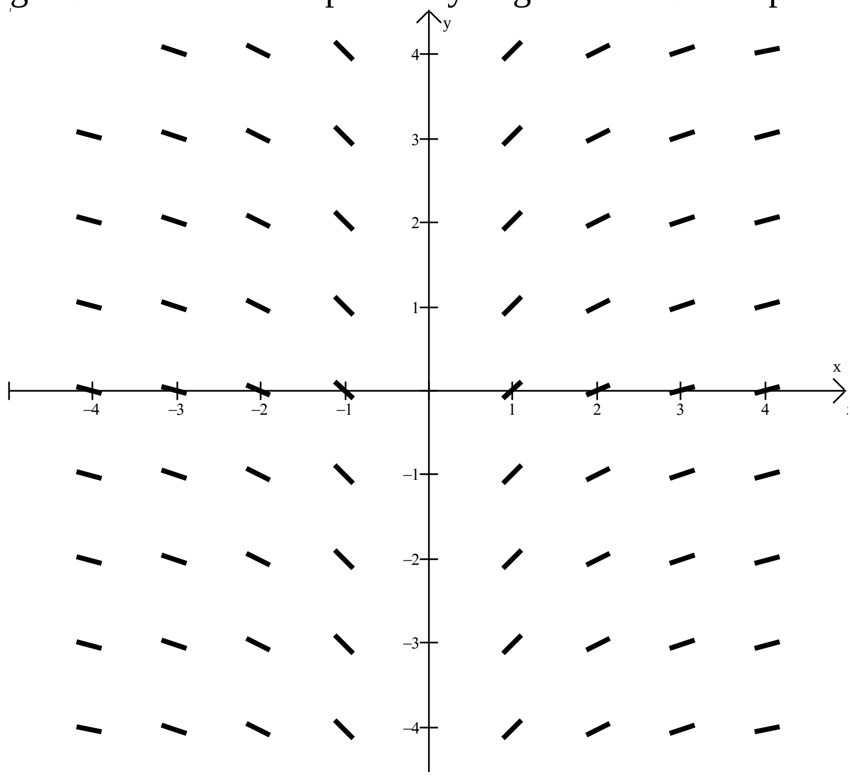
Notice that there are no points for the values where x = 0. This is because there are no slope values for these points (we would be dividing by zero).

There are two schools of thought on this:

1. Since there are no numeric values for $\dfrac{dy}{dx}$ at this point, we should not draw in a slope.
2. Since an undefined value on $\dfrac{dy}{dx}$ often indicates a vertical tangent line, we should show the slopes as vertical.

The AP Calculus test simply avoids this issue and never asks you to draw a slope field at a point where it would be undefined. We will consider that these undefined values will not be drawn in.

Below is a larger version of the slope field you generated on the previous page:



If we were to solve the differential equation by separation of variables, this is what we would get:

$$\frac{dy}{dx} = \frac{1}{x}$$
$$dy = \frac{1}{x}dx$$
$$\int dy = \int \frac{1}{x}dx$$
$$y = \ln|x| + C$$

If you look at the slope field, it does look like our graph of $y = \ln x$ from last year. Their appears to be a vertical asymptote at $x = 0$, but the slope field shows a graph on the left side of the axis as well – this is because of the absolute value. The negative x values become positive.

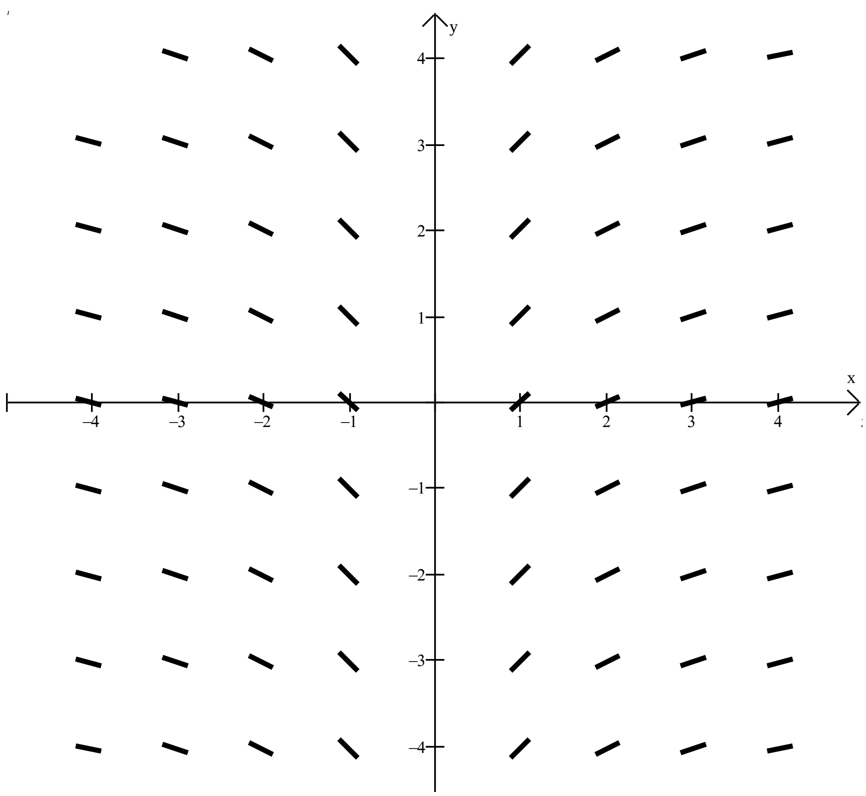
Suppose we want the solution curve that passes through the point (1,1). Analytically, we would just do what we did in the section on separation of variables.

$$y = \ln|x| + C$$
$$1 = \ln(1) + C$$
$$C = 1$$

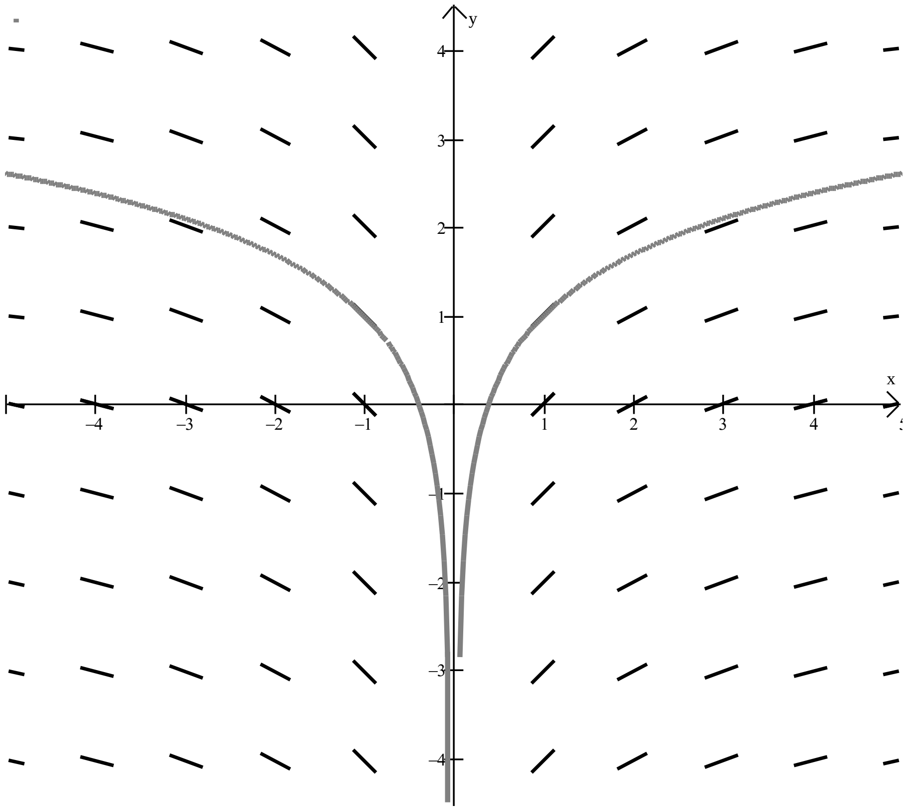
$$y = \ln|x| + 1$$

To sketch the solution curve, however, we don't need to do that at all. All we need is to find the point (1,1) on our slope field and start sketching our curve. At that point, our sketch must match the slope perfectly (since that is the exact slope of the curve at that point). Then we just bend the curve in the way that the slopes are pointing – we just follow the trend given by our field.

Ex 4: Sketch the solution curve for the slope field below that passes through (1,1).



Below is what your solution curve should have looked like.

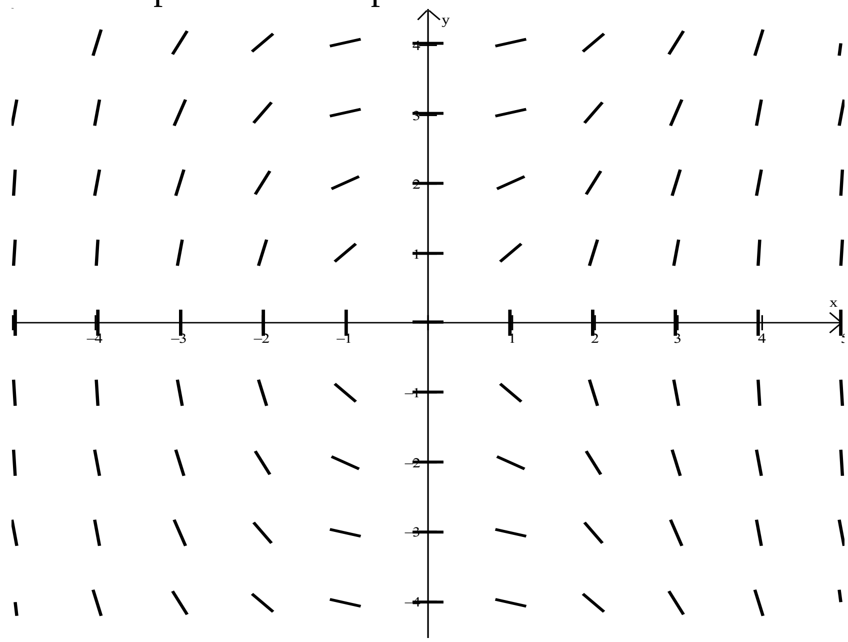


Notice that we needed our solution curve on both sides of the axis, even though there appears to be a vertical asymptote at the line $x = 0$. If we were just given this picture, however, and not the differential equation that corresponds to it, we could not assume that the original curve was discontinuous at $x = 0$ (a curve can be continuous and not differentiable, if you recall from last year).

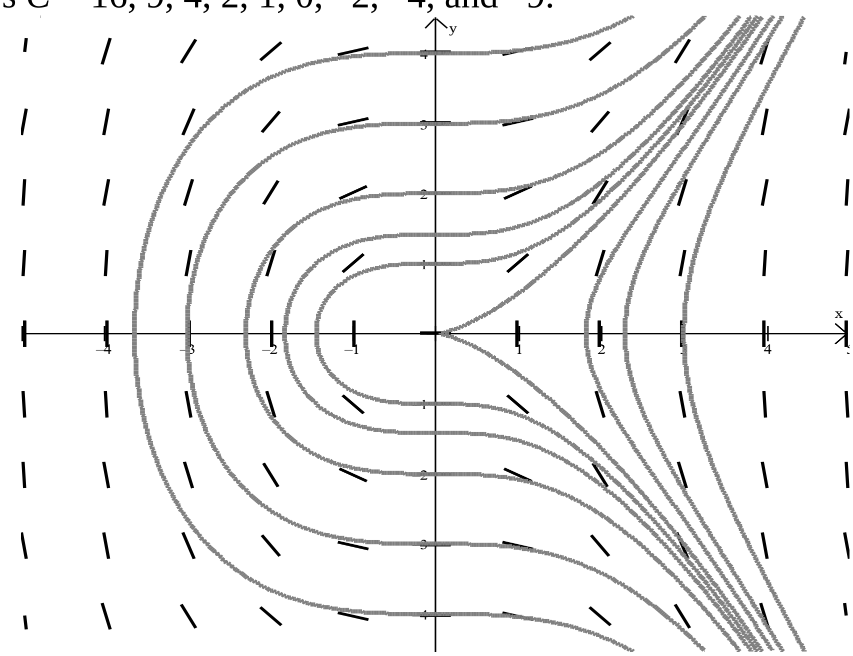
<u>Steps to Sketching a Solution Curve:</u>

1. Find the initial condition point on the graph.
2. Sketch the beginning of your curve so that it matches the slope at that point exactly.
3. Sketch the rest of the curve by using the slopes as **guidelines**. The curve does not necessarily go through any slopes other than the initial condition point.

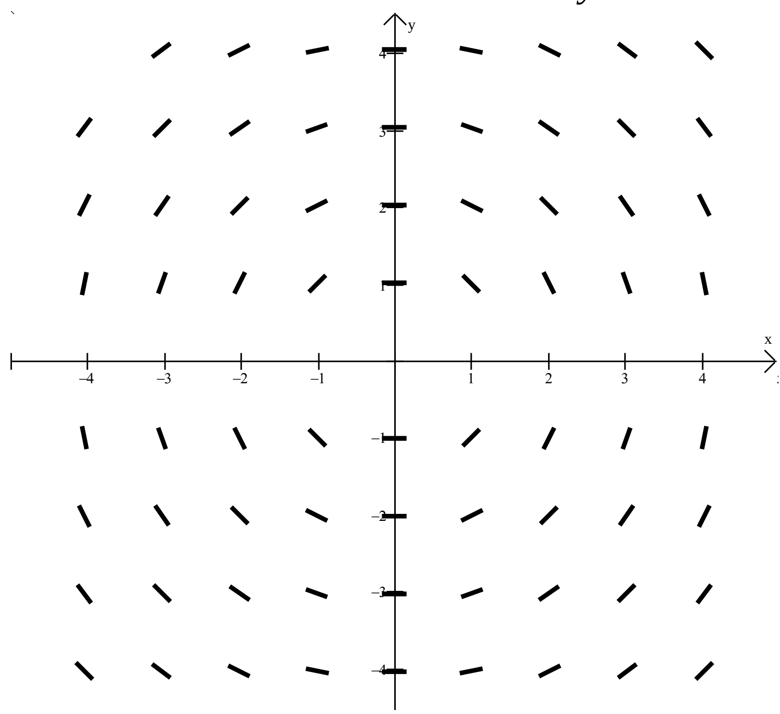
Below is the slope field for $\frac{dy}{dx} = \frac{x^2}{y}$. You may notice that the slopes trace out a very unusual looking curve. While we could separate the variables and solve, it is useful to look at the slope field as a representation of **all** of the solutions.



The solution to the differential ends up being $y = \pm\sqrt{\frac{1}{3}x^3 + C}$. Below are several of the solution curves that illustrate various different values of *C* (from left to right are the values *C* = 16, 9, 4, 2, 1, 0, –2, –4, and –9.



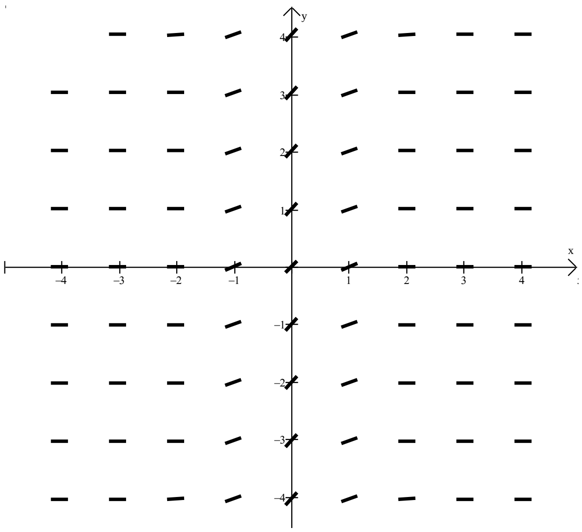
Ex 5 Recall from the last section we looked at $\dfrac{dy}{dx} = -\dfrac{x}{y}$. Below is its slope field.



If you remember from the previous section, the analytic solution was the equation of a circle. If you look at the slope field, the field pretty obviously traces out circular solutions.

Ex 6 Match the slope fields with their differential equations.

(a) $\dfrac{dy}{dx} = 3y - 4x$ (b) $\dfrac{dy}{dx} = e^{-x^2}$ (c) $\dfrac{dy}{dx} = \cos x$ (d) $\dfrac{dy}{dx} = \dfrac{x}{y}$



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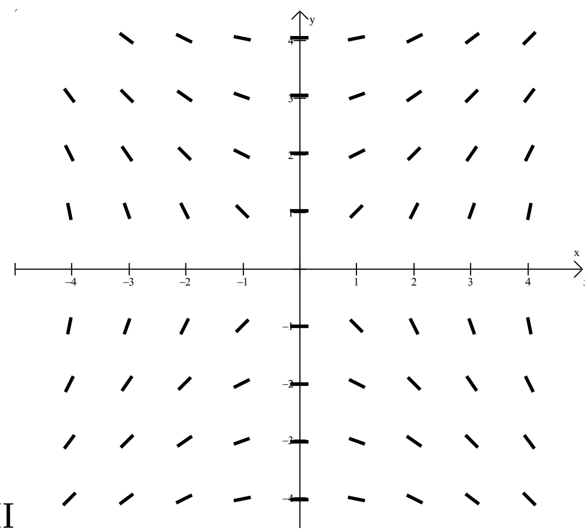

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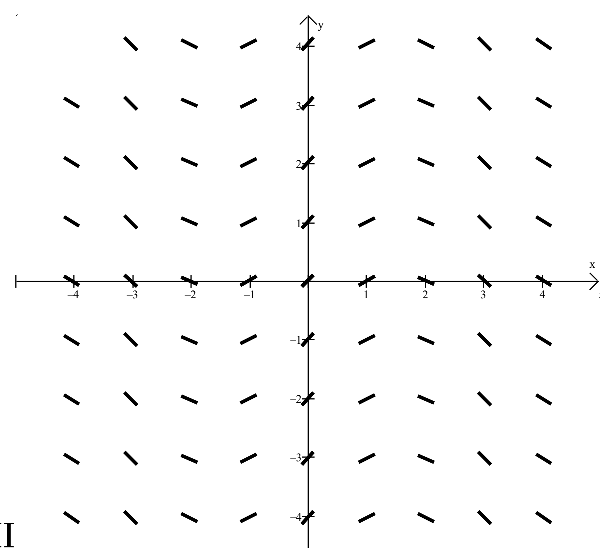

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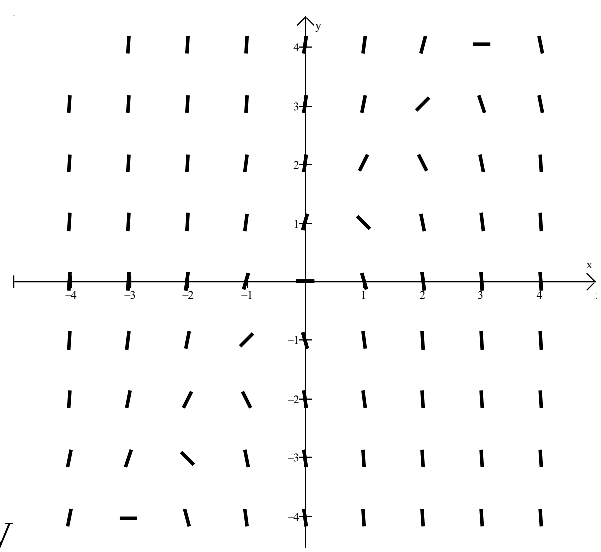
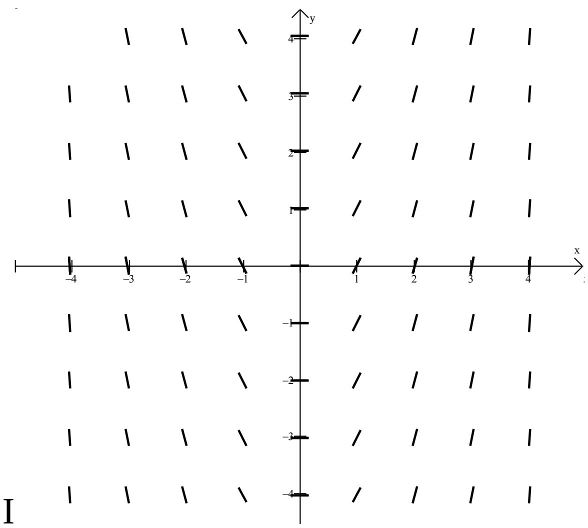


IV

Ex 5 Match the slope fields with their solution curve.

(a) $y=3$ (b) $y=x+2$ (c) $y=x^2-4$ (d) $y=\dfrac{1}{x^2}$ (e) $y=x^3$



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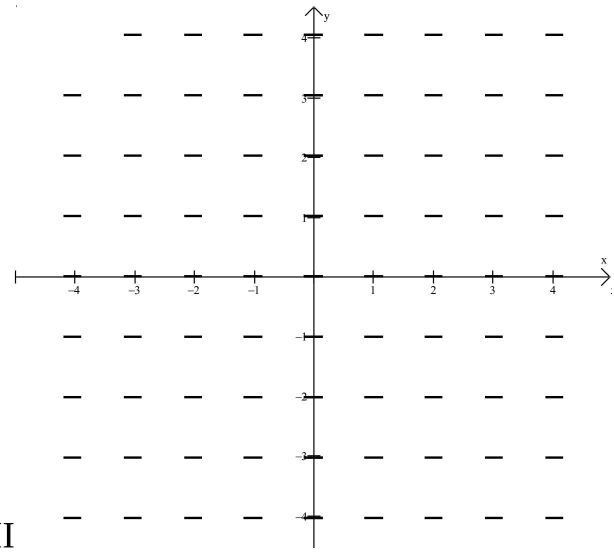

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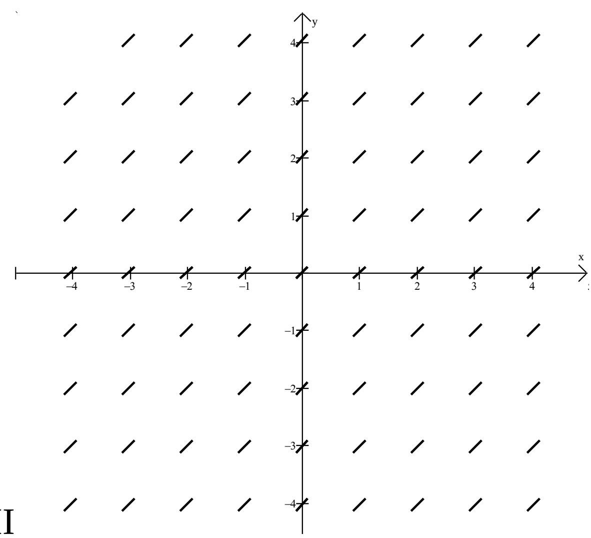

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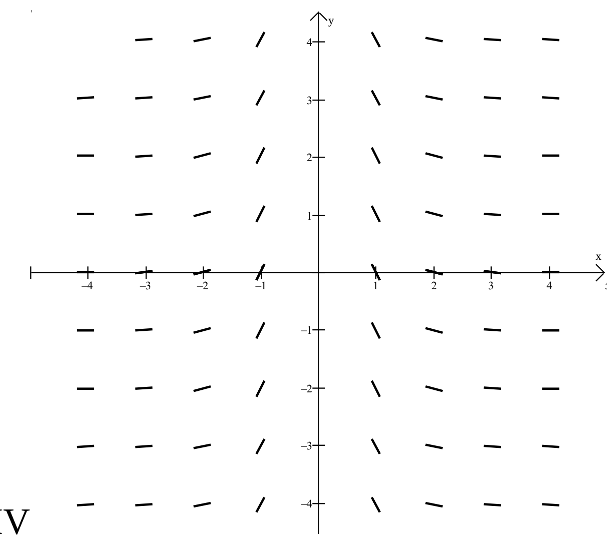

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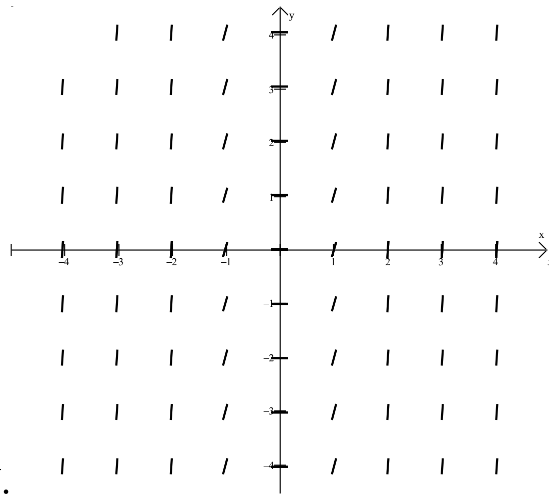
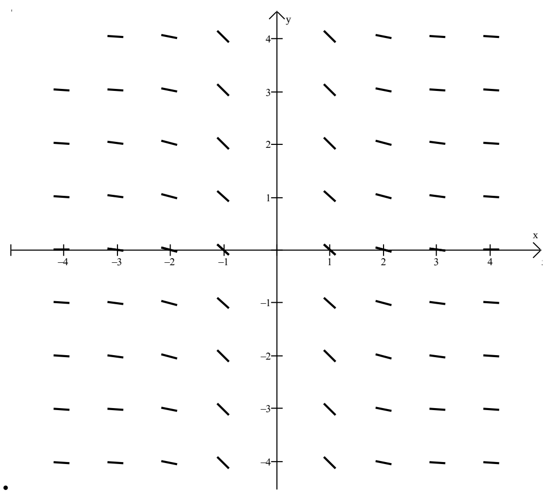


V.

Ex 6   Match the slope fields with their solution curve.

(a) $y=\dfrac{1}{x}$ (b) $y=\ln|x|$ (c) $y=\sin x$ (d) $y=\cos x$



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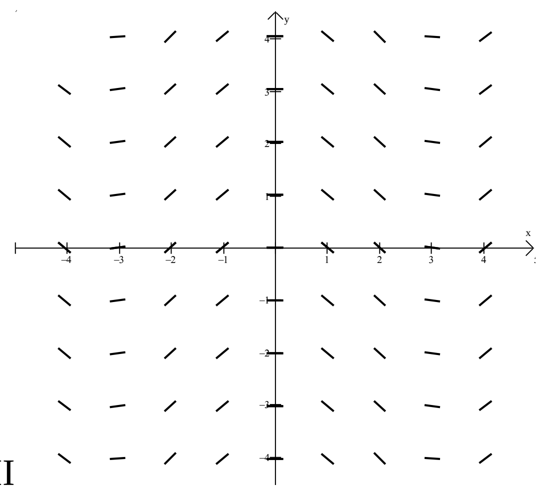

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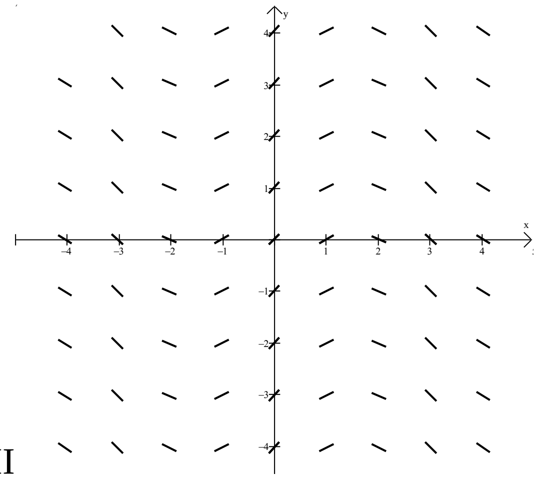

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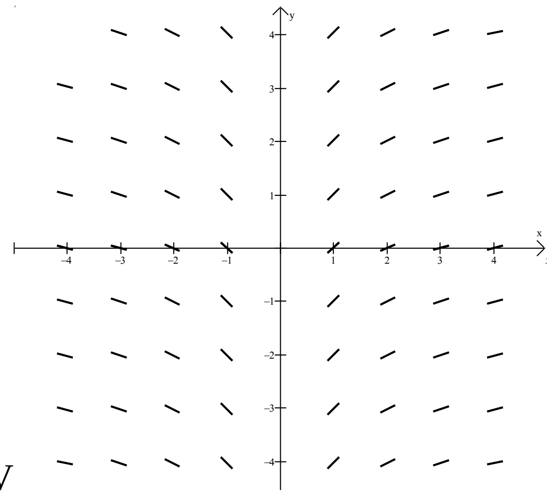


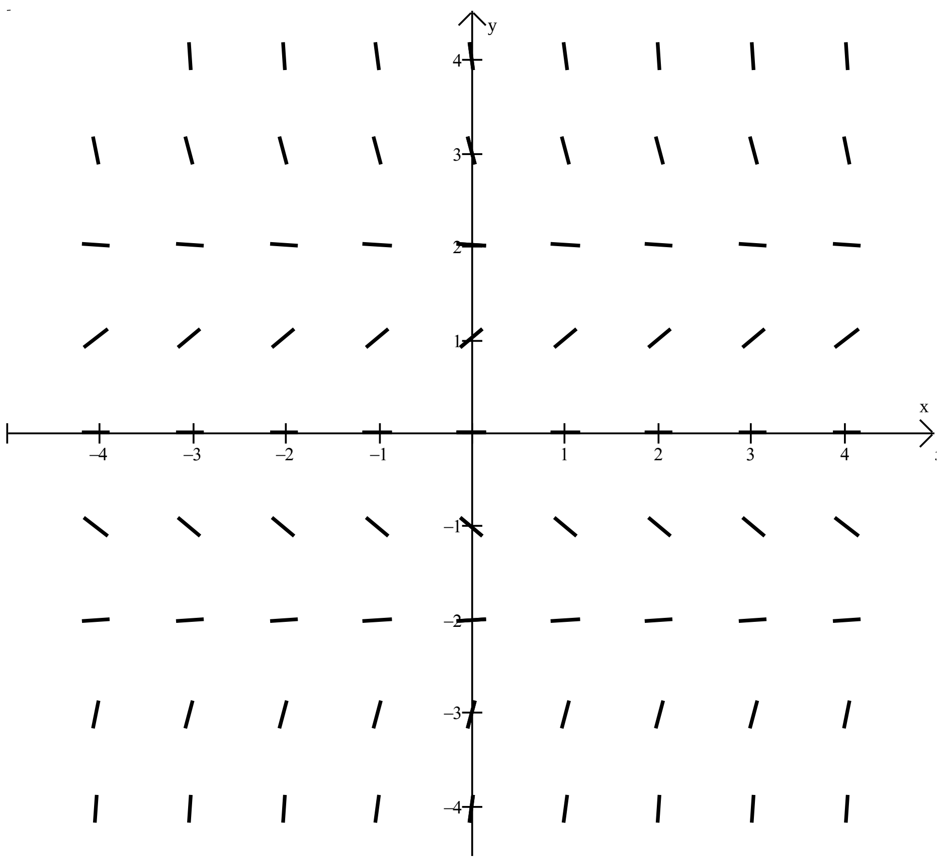
IV

2.5 Homework Set A

1. A slope field for the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$ is shown.

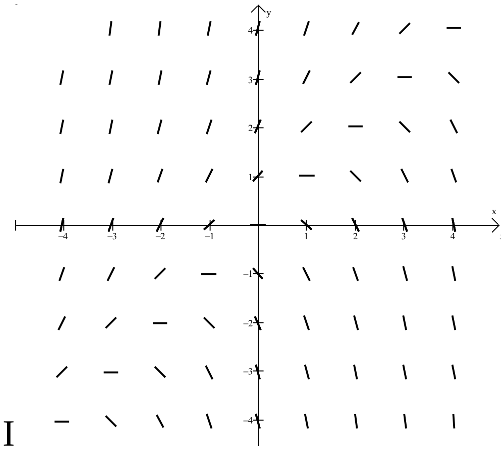
Sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0)=1$ (b) $y(0)=-1$ (c) $y(0)=-3$ (d) $y(0)=3$



Match the differential equation with its slope field (labeled I-IV). Give reasons for your answer.

2. $y' = y - 1$ 3. $y' = y - x$ 4. $y' = y^2 - x^2$ 5. $y' = y^3 - x^3$



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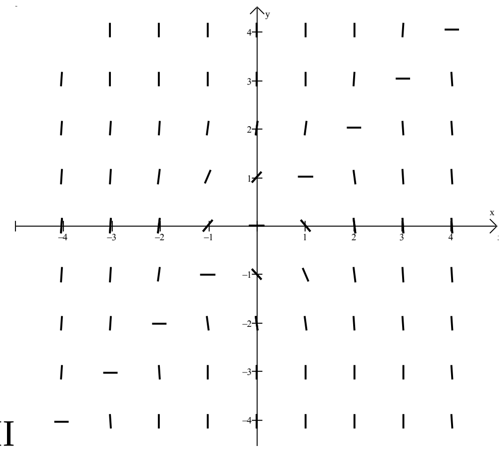

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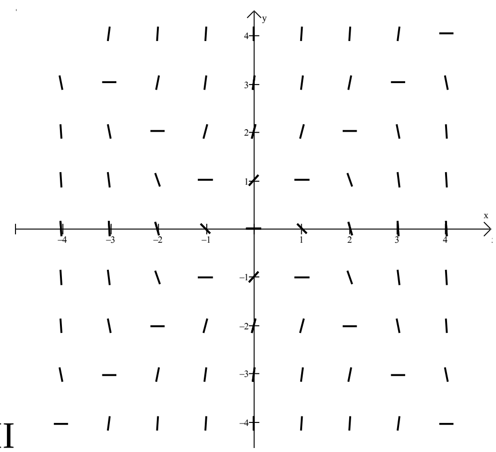

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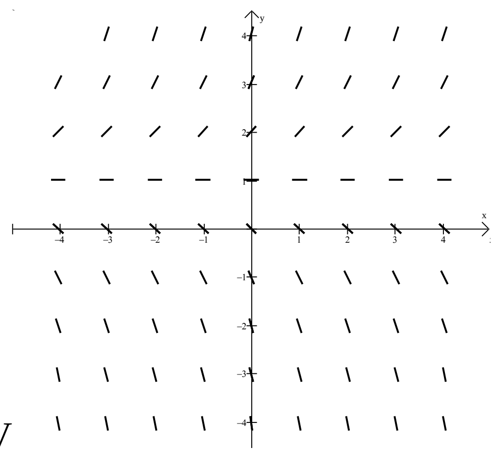
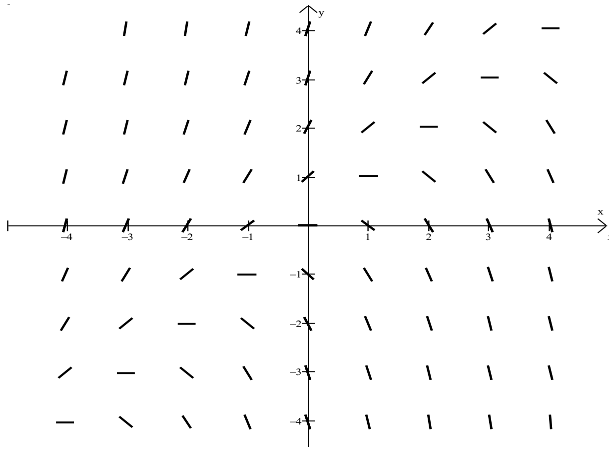


IV

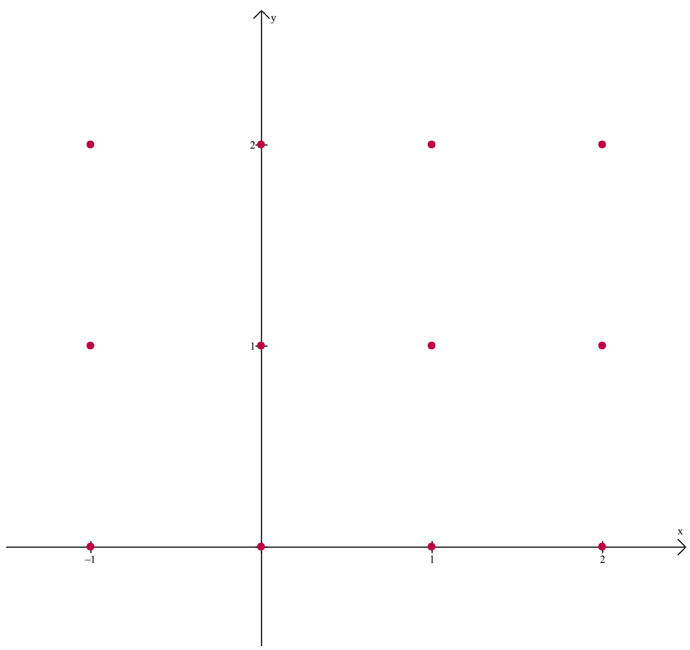
6. Use the slope field labeled below to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0)=1$ (b) $y(0)=0$ (c) $y(0)=-1$

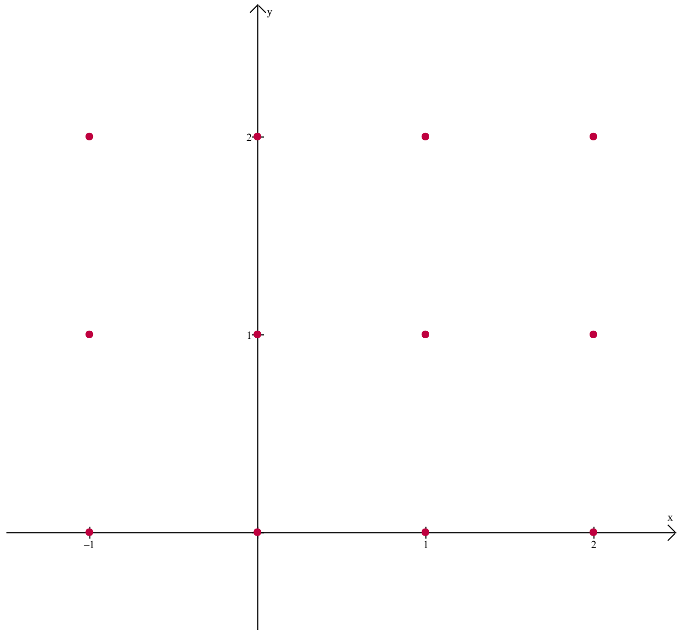


7. Sketch a slope field for the differential equation on the axes provided. Then use it to sketch three solution curves.

$y'=1+y$

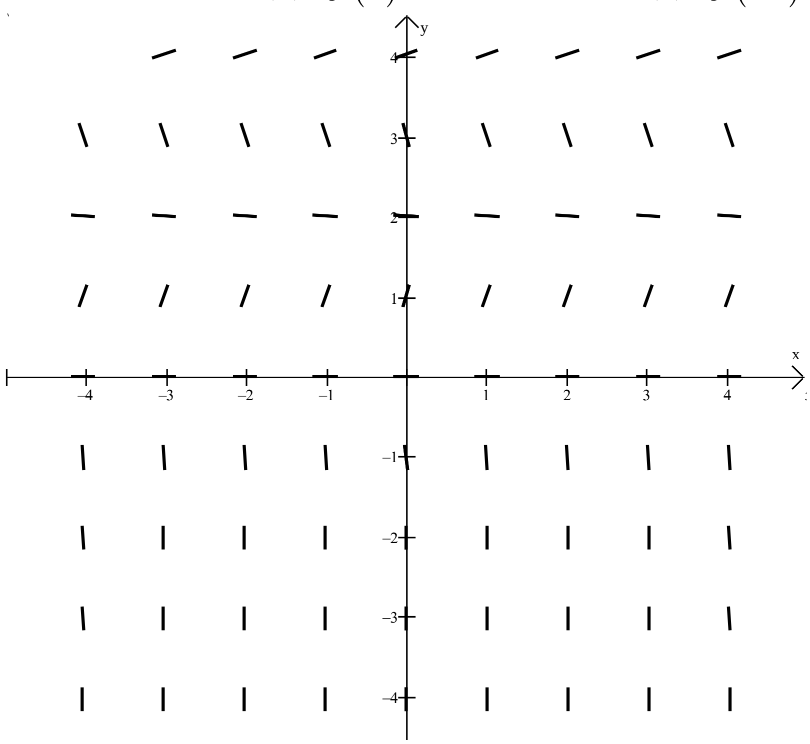


8. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point.
$y' = y - 2x$; $(1,0)$



9. A slope field for the differential equation $y' = y(y-2)(y-4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0)=1$ (b) $y(2)=3$ (c) $y(-2)=1$



2.5 Homework Set B

Draw the slope fields for the intervals $x \in [-2,2]$ and $y \in [-1,1]$

1. $\dfrac{dy}{dx} = x - 2y$

2. $\dfrac{dy}{dx} = x\cos\left(\dfrac{\pi}{2}y\right)$

3. $\dfrac{dy}{dx} = y^2$

4. $\dfrac{dy}{dx} = x^2$

5. $\dfrac{dy}{dx} = xy^2$

6. $\dfrac{dy}{dx} = x\ln y$

7. $\frac{dy}{dx} = \sqrt{xy}$

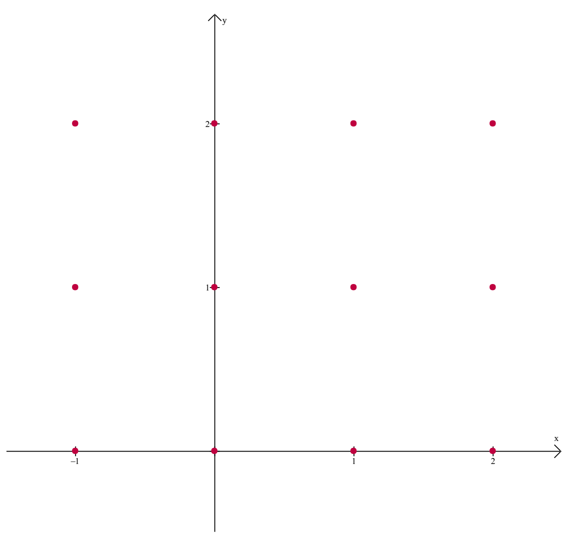
8. $\frac{dy}{dx} = \frac{xe^y}{2}$

9. Given the differential equation, $\frac{dy}{dx} = 2x + y$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.

b. If the solution curve passes through the point (0, 1), sketch the solution curve on the same set of axes as your slope field.

c. Find the equation for the solution curve of $\frac{dy}{dx} = 2xy$, given that $y(0) = 1$

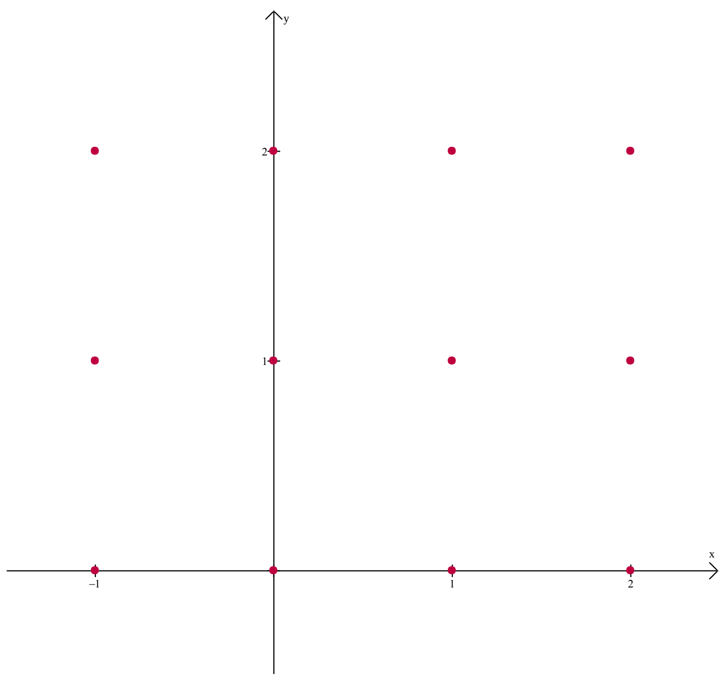


10. Given the differential equation, $\frac{dy}{dx} = y^2 \sin\left(\frac{\pi}{2}x\right)$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.

b. There is some value of c such that the horizontal line $y = c$ satisfies the differential equation. Find the value of c.

c. Find the equation for the solution curve, given that $y(1) = 1$

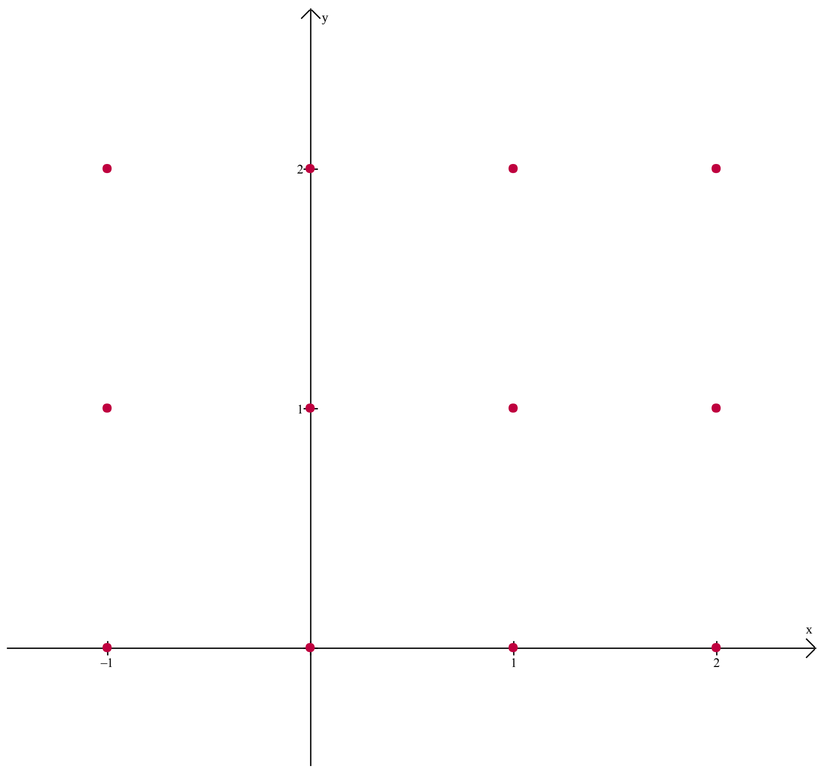


11. Given the differential equation, $\frac{dy}{dx} = x^2 y$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.

b. If the solution curve passes through the point (0, 1), sketch the solution curve on the same set of axes as your slope field.

c. Find the equation for the solution curve, given that $y(0) = 1$

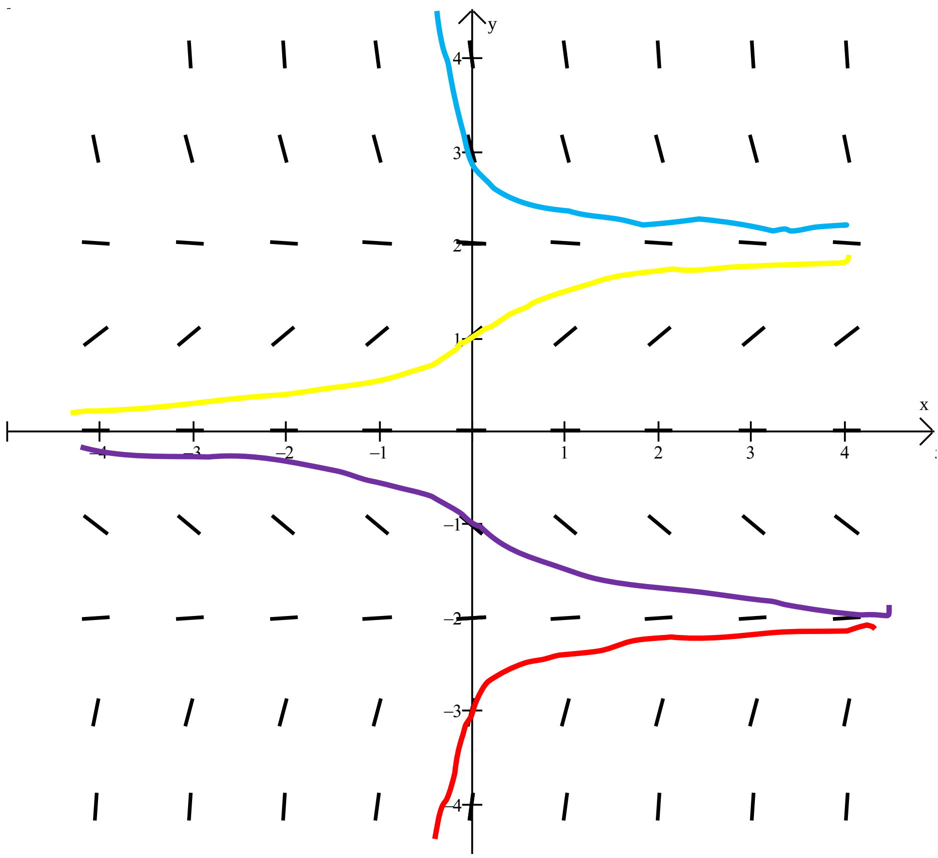


Answers: 2.5 Homework Set A

1. A slope field for the differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$ is shown.

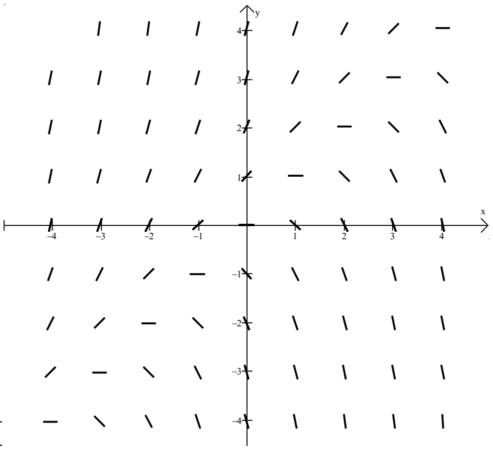
Sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0) = 1$ (b) $y(0) = -1$ (c) $y(0) = -3$ (d) $y(0) = 3$



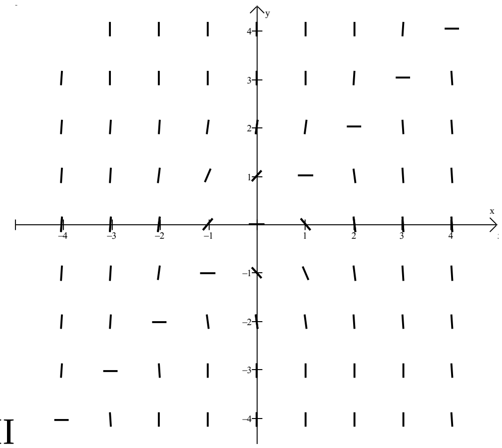
Match the differential equation with its slope field (labeled I-IV). Give reasons for your answer.

2. $y' = y - 1$ 3. $y' = y - x$ 4. $y' = y^2 - x^2$ 5. $y' = y^3 - x^3$



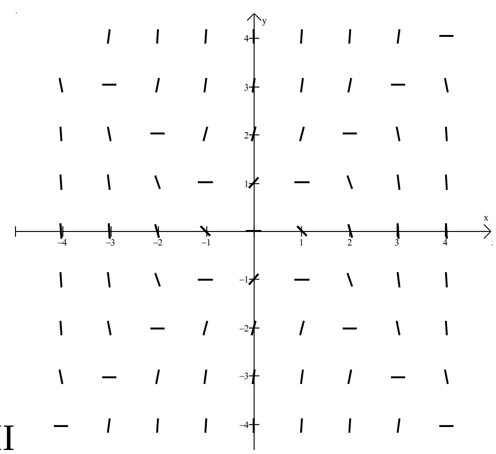
I

3. $y' = y - x$, plug in points, slopes match



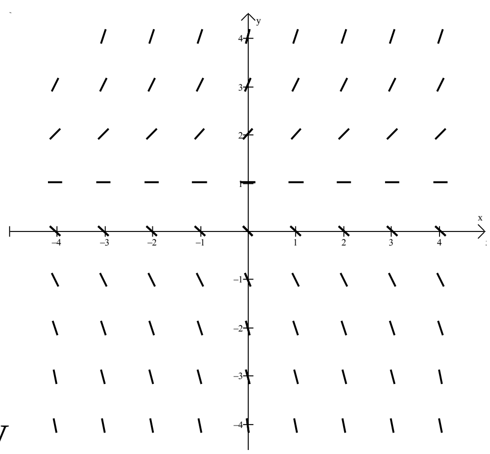
II

5. $y' = y^3 - x^3$, slopes very steep, cubed numbers are large, so slopes should be as well.



III

4. $y' = y^2 - x^2$, quadrants are identical, which would happen with the squaring.

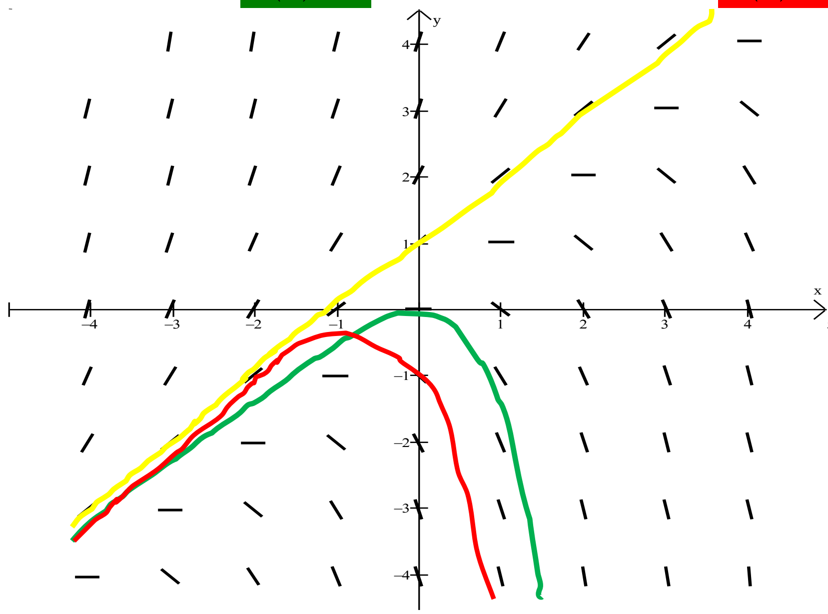


IV

2. $y' = y - 1$, no x so rows all the same

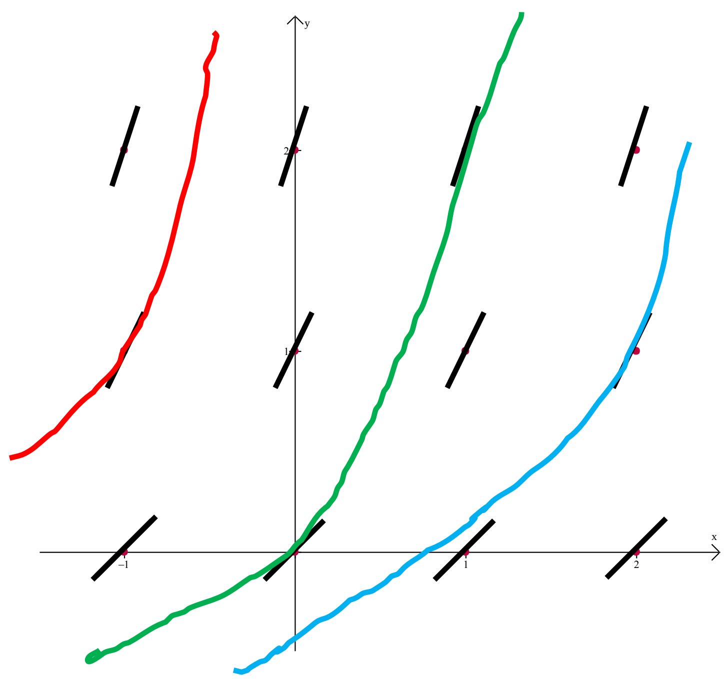
6. Use the slope field labeled below to sketch the graphs of the solutions that satisfy the given initial conditions.

(a) $y(0)=1$ (b) $y(0)=0$ (c) $y(0)=-1$

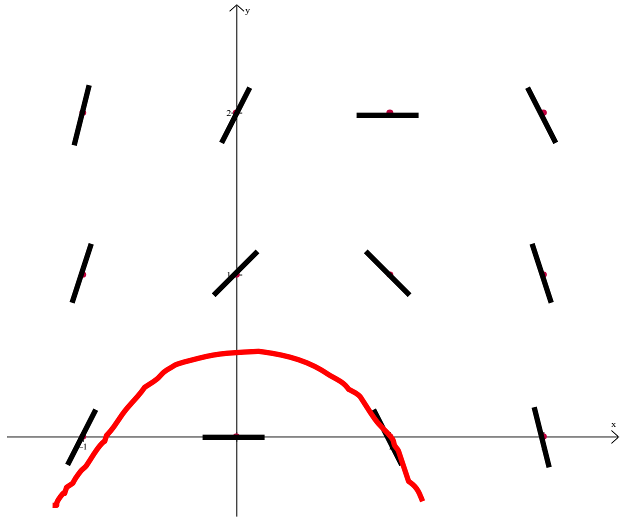


7. Sketch a slope field for the differential equation on the axes provided. Then use it to sketch three solution curves.

$y'=1+y$

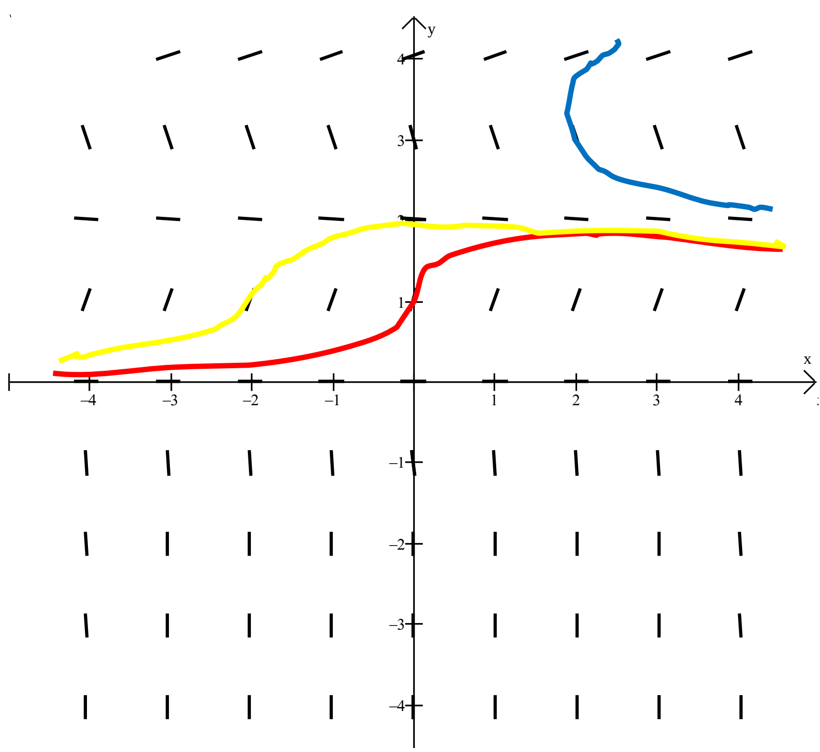


8. Sketch the slope field of the differential equation. Then use it to sketch a solution curve that passes through the given point.
$y' = y - 2x$; (1,0)



9. A slope field for the differential equation $y' = y(y-2)(y-4)$ is shown. Sketch the graphs of the solutions that satisfy the given initial conditions.

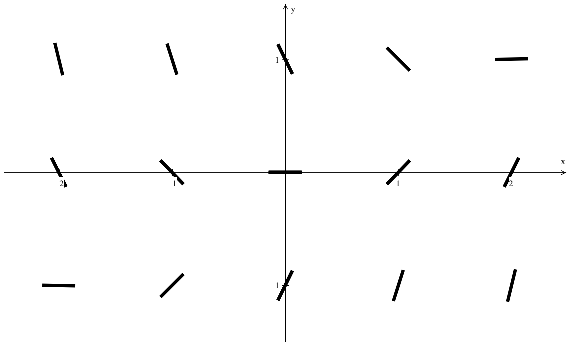
(a) $y(0)=1$ (b) $y(2)=3$ (c) $y(-2)=1$



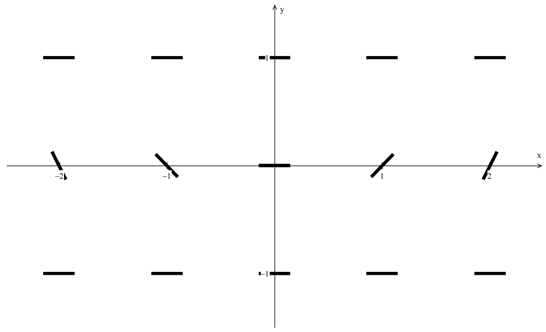
2.5 Homework Set B

Draw the slope fields for the intervals $x \in [-2,2]$ and $y \in [-1,1]$

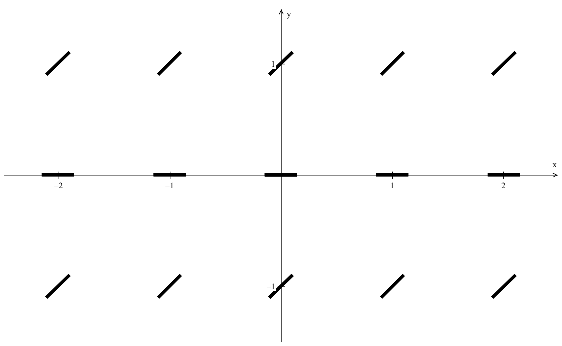
1. $\dfrac{dy}{dx} = x - 2y$



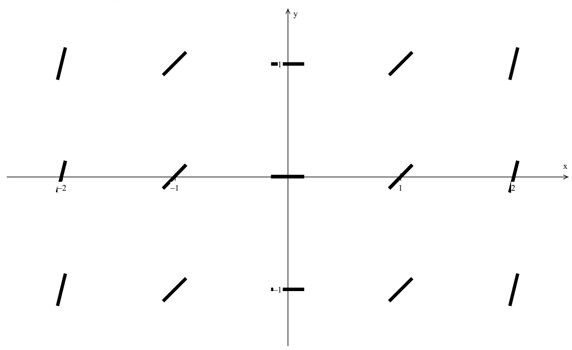
2. $\dfrac{dy}{dx} = x\cos\left(\dfrac{\pi}{2}y\right)$



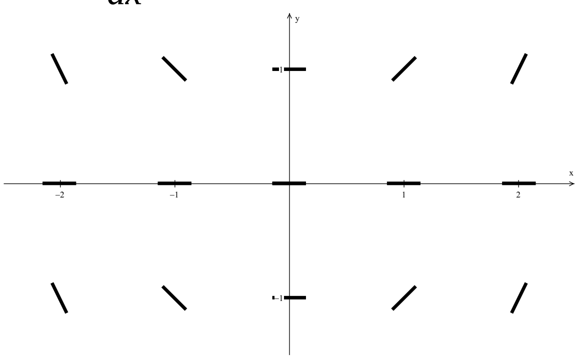
3. $\dfrac{dy}{dx} = y^2$



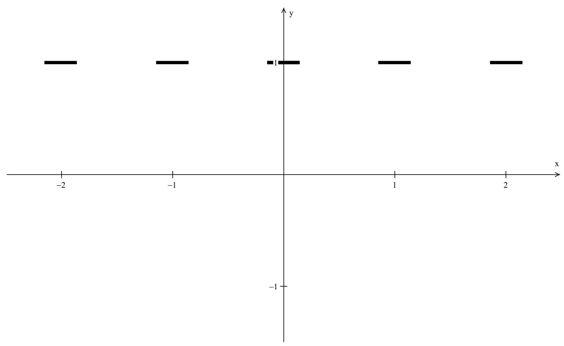
4. $\dfrac{dy}{dx} = x^2$



5. $\dfrac{dy}{dx} = xy^2$

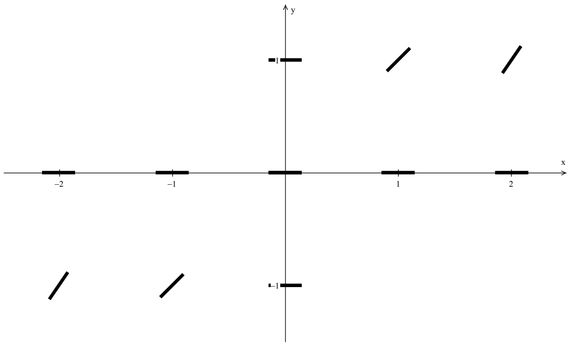


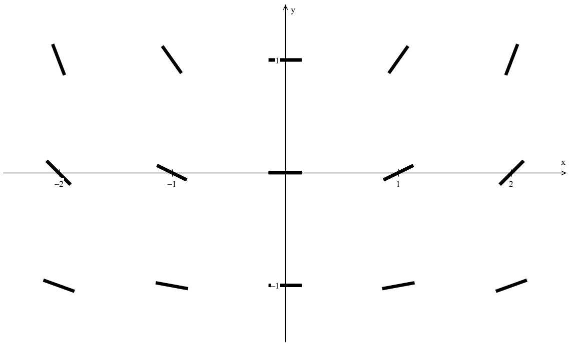
6. $\dfrac{dy}{dx} = x\ln y$



7. $\dfrac{dy}{dx} = \sqrt{xy}$

8. $\dfrac{dy}{dx} = \dfrac{xe^{y}}{2}$



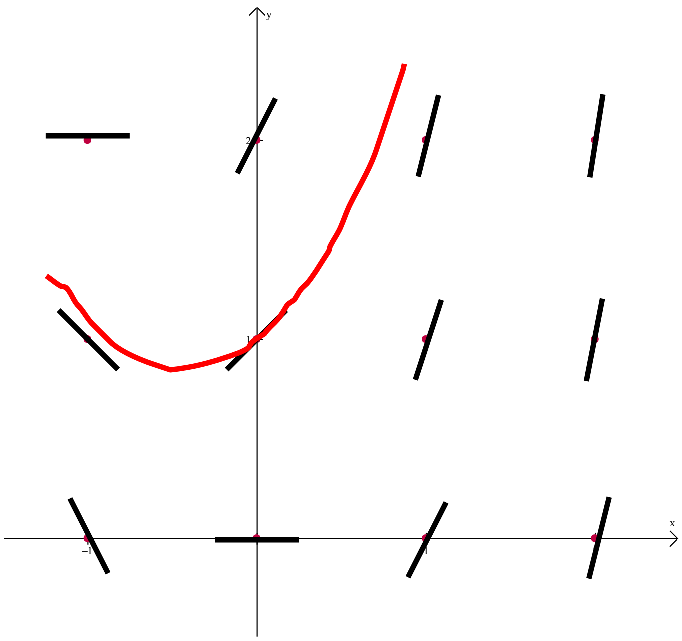


9. Given the differential equation, $\dfrac{dy}{dx} = 2x + y$

a. On the axis system provided, sketch the slope field for the $\dfrac{dy}{dx}$ at all points plotted on the graph.

b. If the solution curve passes through the point (0, 1), sketch the solution curve on the same set of axes as your slope field.

c. Find the equation for the solution curve of $\dfrac{dy}{dx} = 2xy$, given that $y(0) = 1$



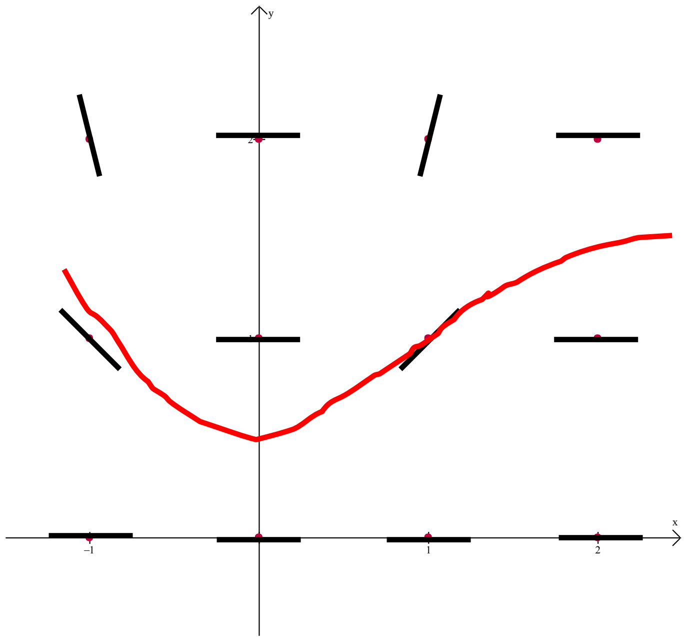
c. $y = e^{x^2}$

10. Given the differential equation, $\frac{dy}{dx} = y^2 \sin\left(\frac{\pi}{2}x\right)$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.

b. There is some value of *c* such that the horizontal line *y* = *c* satisfies the differential equation. Find the value of c.

c. Find the equation for the solution curve, given that $y(1) = 1$



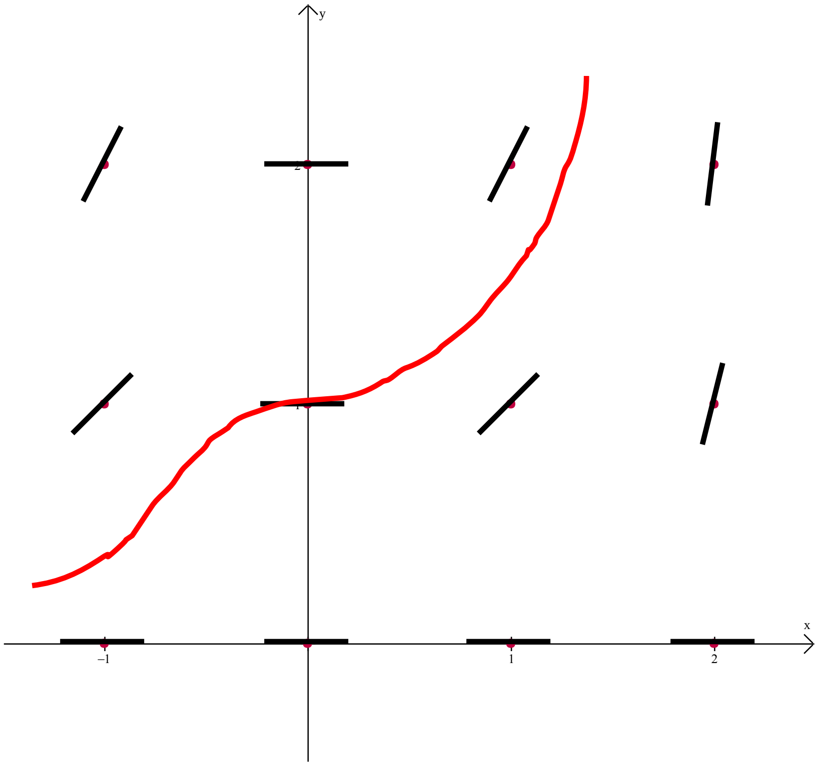
c. $y = \dfrac{1}{\frac{2}{\pi}\cos\left(\frac{\pi}{2}x\right) + 1}$

11. Given the differential equation, $\frac{dy}{dx} = x^2 y$

a. On the axis system provided, sketch the slope field for the $\frac{dy}{dx}$ at all points plotted on the graph.

b. If the solution curve passes through the point (0, 1), sketch the solution curve on the same set of axes as your slope field.

c. Find the equation for the solution curve, given that $y(0) = 1$



c. $y = e^{x^3/3}$

Chapter 2 Practice Test
Integral Test

Directions: Round at 3 decimal places.
Show all work.

$$\int u^n \, du = \begin{cases} \\ \end{cases}$$

$$\int e^u \, du =$$

$$\int a^u \, du =$$

$$\int \sin u \, du =$$

$$\int \cos u \, du =$$

$$\int \sec^2 u \, du =$$

$$\int \csc^2 u \, du =$$

$$\int \sec u \tan u \, du =$$

$$\int \csc u \cot u \, du =$$

$$\int \sin^2 u \, du =$$

$$\int \cos^2 u \, du =$$

$$\int \frac{du}{\sqrt{a^2 - u^2}} =$$

$$\int \frac{du}{a^2 + u^2} =$$

$$\int \frac{du}{u\sqrt{u^2 - a^2}} =$$

1. $\int \left(x^3 + 3x - \frac{2\pi}{x} + \sqrt[3]{x^4} + \frac{5}{\sqrt{x^7}} \right) dx$

2. $\int \frac{\cos\left(\sin^{-1} x\right) dx}{\sqrt{1 - x^2}}$

3. $\int e^{\cos 5x} \sin 5x \, dx$

4. $\int \csc^2 x \cot^{4/3} x \, dx$

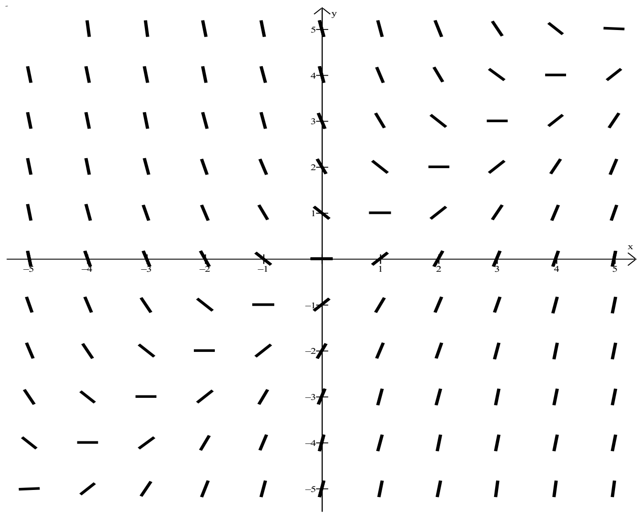
5. $\displaystyle\int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)^3}$

6. $\displaystyle\int \frac{\ln^2 x^2}{x}\,dx$

7. $a(t) = -4\sin 2t$ describes the acceleration of a particle. Find $v(t)$ and $x(t)$ if $v(0) = 2$ and $x(0) = -3$.

8. $\int x\sqrt{2x^2+8}\,dx$

9. $f''(x) = (2+3x)^3$, $f'(0) = 4$ and $f(0) = 3$. Find *f(x)*.



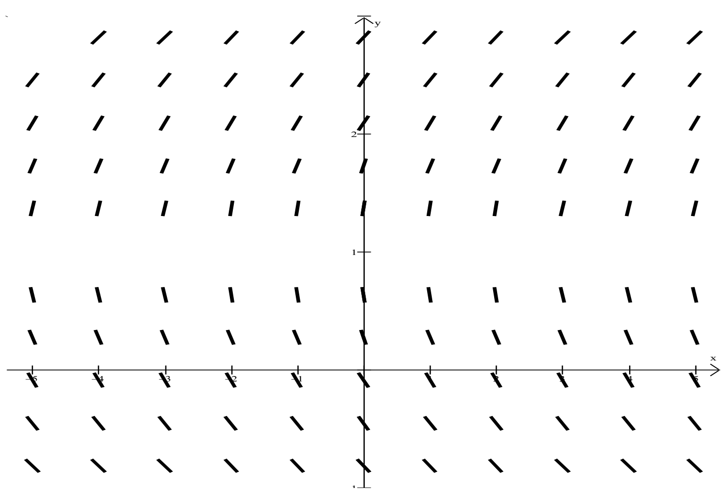
10) Shown above is the slope field for which of the following differential equations?

(A) $\frac{dy}{dx} = x - y$ (B) $\frac{dy}{dx} = y - x$ (C) $\frac{dy}{dx} = 1 + x$ (D) $\frac{dy}{dx} = x^2 - y$ (E) $\frac{dy}{dx} = y^2 - x$

---

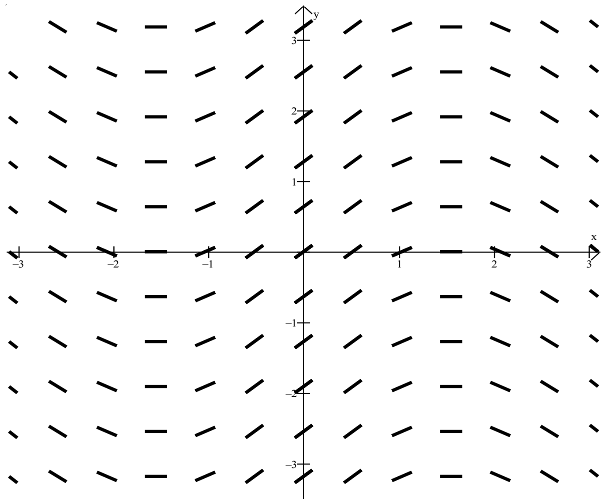
11) The slope field for a differential equation $\frac{dy}{dx} = f(x, y)$ is shown in the figure. Which of the following statements must be true?

I. All solution curves have the same slope for a given x value.
II. As y approaches 1, the solution curve has a vertical tangent.
III. The solution curve is undefined at $y = 1$.



(A) I. (B) II. (C) I. and II. (D) II. and III. (E) None

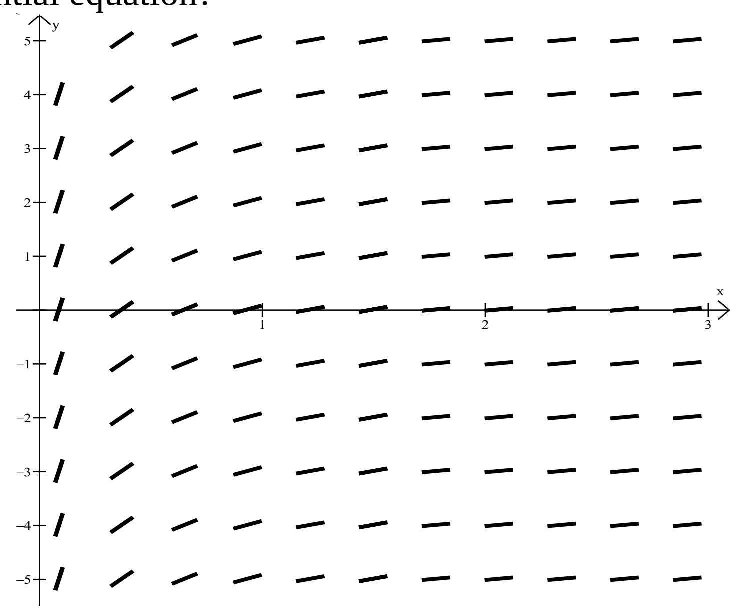
12) The slope field below corresponds to which of the following differential equations?



(A) $\dfrac{dy}{dx} = \cos x$ (B) $\dfrac{dy}{dx} = \sin x$ (C) $\dfrac{dy}{dx} = -\csc^2 x$ (D) $\dfrac{dy}{dx} = \ln x^2$ (E) $\dfrac{dy}{dx} = e^{x^2}$

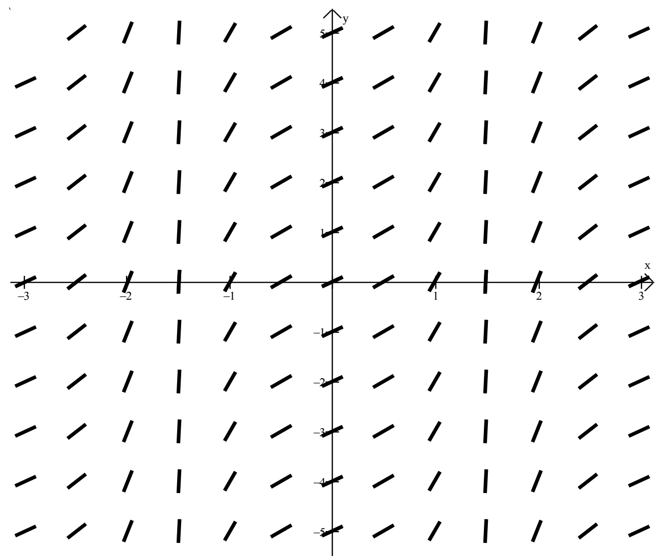
---

13) The slope field below represents solutions to a differential equation, $\dfrac{dy}{dx} = f(x, y)$. Which of the following could be a specific solution to that differential equation?



(A) $y = \sin x$ (B) $y = e^{-x}$ (C) $y = \ln x$ (D) $y = \dfrac{x^2}{2}$ (E) $y = -\sin x$

14) The slope field below represents solutions to a differential equation, $\frac{dy}{dx} = f(x, y)$. Which of the following could be a specific solution to that differential equation?



(A) $y = 2 + \tan x$ (B) $y = 2 + \cot x$ (C) $y = x^3 + 2$ (D) $y = \frac{1}{x+2}$ (E) $y = 2 + \ln|x|$

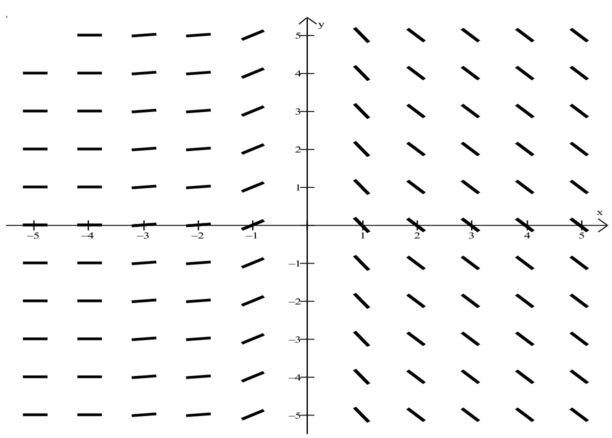
---

15) The slope field for a differential equation $\frac{dy}{dx} = f(x, y)$ is shown in the figure. Which of the following statements are true?

I. The curve is symmetrical about the y axis

II. At $x = 2$, the solution curve has a tangent with a slope of –1.

III. The solution curve is undefined at $y = 0$.



(A) I. (B) II. (C) II. and III. (D) I. and III. (E) I., II., and III.

Integral Test

1. $\int \left(x^3 + 3x - \frac{2\pi}{x} + \sqrt[3]{x^4} + \frac{5}{\sqrt{x^7}} \right) dx$

$= \frac{1}{4}x^4 + \frac{3}{2}x^2 + \frac{3}{7}x^{7/3} - 2x^{-5/2} + C$

2. $\int \frac{\cos\left(\sin^{-1} x\right) dx}{\sqrt{1-x^2}}$

$= x + C$

3. $\int e^{\cos 5x} \sin 5x \, dx$

$= -\frac{1}{5}e^{\cos 5x} + C$

4. $\int \csc^2 x \cot^{4/3} x \, dx$

$= -\frac{3}{7}\cot^{7/3} x + C$

5. $\int \frac{dx}{\sqrt{x}\left(1+\sqrt{x}\right)^3}$

$= -\left(1+\sqrt{x}\right)^{-2} + C$

6. $\int \frac{\ln^2 x^2}{x} dx$

$= \frac{4}{3}\ln^3 x + C$

7. $a(t) = -4\sin 2t$ describes the acceleration of a particle. Find $v(t)$ and $x(t)$ if $v(0) = 2$ and $x(0) = -3$.

$v(t) = 2\cos 2t$

$x(t) = \sin(2t) - 3$

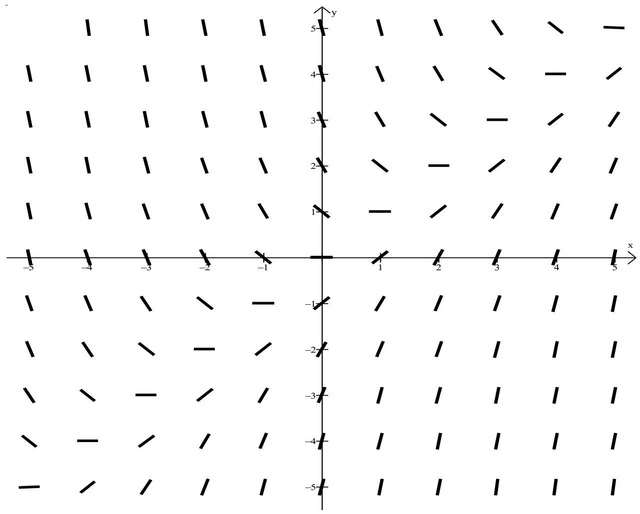
8. $\int x\sqrt{2x^2+8}\,dx$

$$=\frac{1}{6}\left(2x^2+8\right)^{3/2}$$

9. $f''(x)=(2+3x)^3$, $f'(0)=4$ and $f(0)=3$. Find *f(x)*.

$$f'(x)=\frac{1}{12}(2+3x)^4+\frac{8}{3}$$

$$f'(x)=\frac{1}{180}(2+3x)^5+\frac{8}{3}x+\frac{127}{45}$$



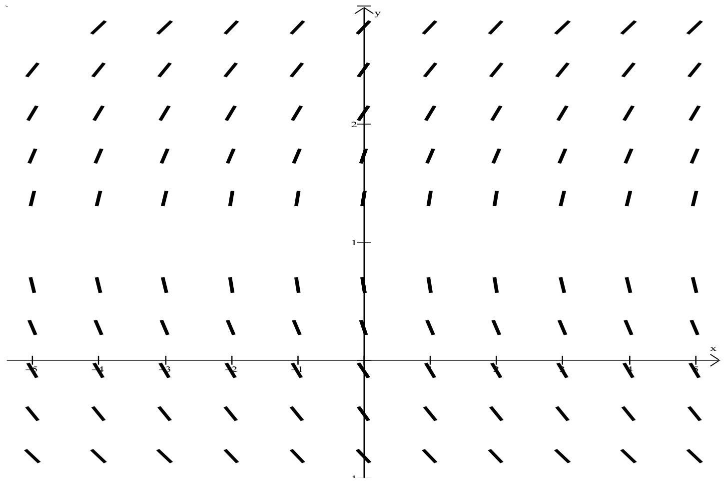
10. Shown above is the slope field for which of the following differential equations?

(A) $\frac{dy}{dx}=x-y$ (B) $\frac{dy}{dx}=y-x$ (C) $\frac{dy}{dx}=1+x$ (D) $\frac{dy}{dx}=x^2-y$ (E) $\frac{dy}{dx}=y^2-x$

---

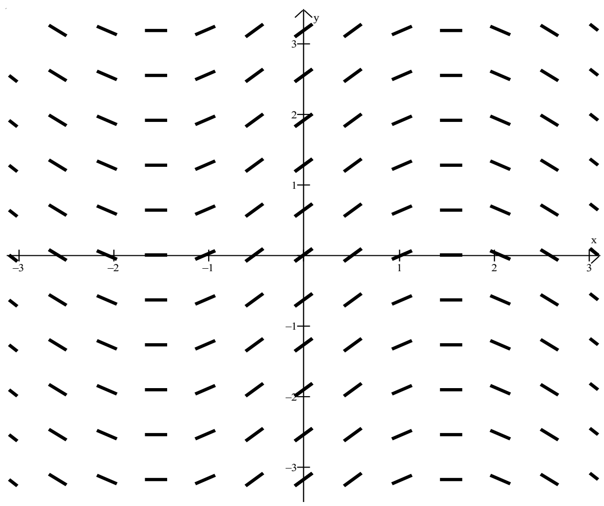
11. The slope field for a differential equation $\frac{dy}{dx}=f(x,y)$ is shown in the figure. Which of the following statements must be true?

I. All solution curves have the same slope for a given *x* value.
II. As *y* approaches 1, the solution curve has a vertical tangent.
III. The solution curve is undefined at *y* = 1.



(A) I. (B) II. (C) I. and II. (D) II. and III. (E) None

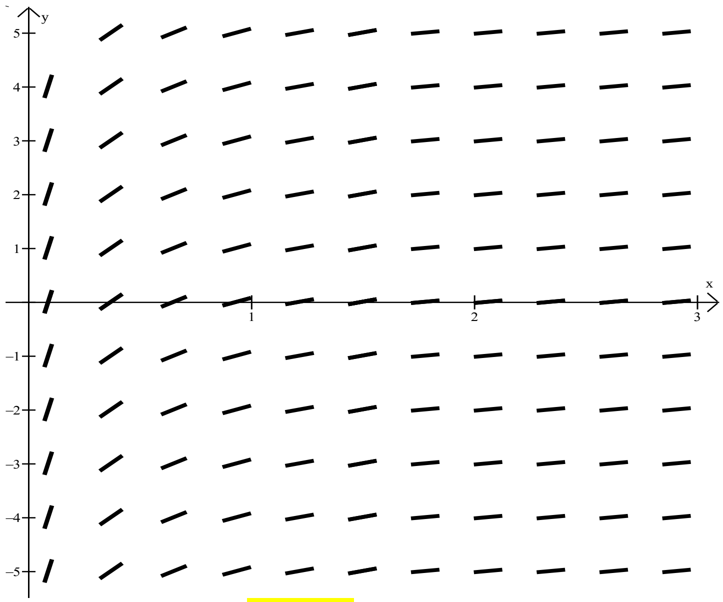
12. The slope field below corresponds to which of the following differential equations?



(A) $\frac{dy}{dx} = \cos x$ (B) $\frac{dy}{dx} = \sin x$ (C) $\frac{dy}{dx} = -\csc^2 x$ (D) $\frac{dy}{dx} = \ln x^2$ (E) $\frac{dy}{dx} = e^{x^2}$

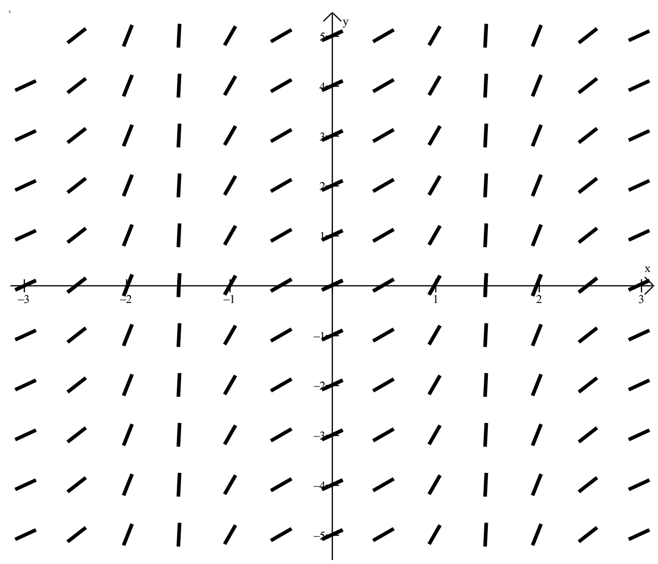
---

13. The slope field below represents solutions to a differential equation, $\frac{dy}{dx} = f\left(x, y\right)$. Which of the following could be a specific solution to that differential equation?



14. $y = \sin x$ (B) $y = e^{-x}$ (C) $y = \ln x$ (D) $y = \frac{x^2}{2}$ (E) $y = -\sin x$

15. The slope field below represents solutions to a differential equation, $\frac{dy}{dx} = f(x, y)$. Which of the following could be a specific solution to that differential equation?



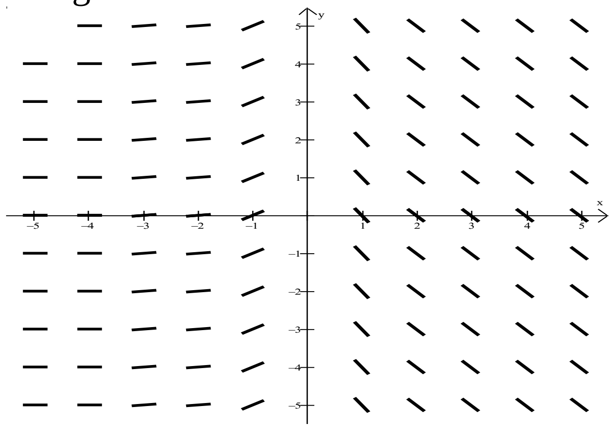
(A) $y = 2 + \tan x$ (B) $y = 2 + \cot x$ (C) $y = x^3 + 2$ (D) $y = \frac{1}{x+2}$ (E) $y = 2 + \ln|x|$

16. The slope field for a differential equation $\frac{dy}{dx} = f(x, y)$ is shown in the figure. Which of the following statements are true?

I. The curve is symmetrical about the y axis

II . At $x = 2$, the solution curve has a tangent with a slope of –1.

III. The solution curve is undefined at $y = 0$.



(A) I. (B) II. (C) II. and III. (D) I. and III. (E) I., II., and III.

Chapter 3 Overview: Derivatives Applications

Most of last year's derivative work (and the first chapter) concentrated on tangent lines and extremes—the geometric applications of the derivatives. The broader context for derivatives is that of "change." Slope is, indeed, a measure of change, but only one kind of change. If the axes of the graph are labeled as time and distance, slope become velocity (distance/time). There are two main contexts for derivatives in this chapter: motion and graphing.

In this first portion of this chapter we will concentrate on the applications of the derivative to motion. Much of this was covered last year. The underlying idea is that the independent variable provides context to the problem. The mechanics of differentiating in terms of x and differentiating in terms of time t are the same but the interpretations of the answers are very different. The Chain Rule takes on an added dimension here with Implicit Differentiation and Related Rates problems.

In the latter half of the chapter, we will consider the graphical applications of the derivative. Much of this is also a review of material covered last year, but with significant pieces added to emphasize the Calculus. Key topics include:

- Finding extremes
- The Mean Value Theorem
- The First Derivative Test
- The Second Derivative Test
- Optimization
- Graphing from the derivatives
- Making inferences regarding the original graph from the graph of its derivative

Several multiple-choice questions and at least one full free response question (often parts of others), have to do with this general topic.

3.1: Extrema and the Mean Value Theorem

One of the most valuable aspects of Calculus is that it allows us to find extreme values of functions. The ability to find maximum or minimum values of functions has wide-ranging applications. Every industry has uses for finding extremes, from optimizing profit and loss, to maximizing output of a chemical reaction, to minimizing surface areas of packages. This one tool of Calculus eventually revolutionized the way the entire world approached every aspect of industry. It allowed people to solve formerly unsolvable problems.

OBJECTIVES
Find critical values and extreme values for functions.
Understand the connection between slopes of secant lines and tangent lines
Apply the Mean Value Theorem to demonstrate that extremes exist within an interval.

It will be helpful to keep in mind a few things from last year for this chapter (and all other chapters following).

REMINDER: *Vocabulary*:

1. *Critical Value*--The *x*-coordinate of the extreme
2. *Maximum Value*--The *y*-coordinate of the high point.
3. *Minimum Value*--The *y*-coordinate of the low point.
4. *Relative Extremes*--the highest or lowest points in any section of the curve.
5. *Absolute Extremes*--the highest or lowest points of the whole curve.
6. *Interval of Increasing*--the interval of *x*-values for which the curve is rising from left to right.
7. *Interval of Decreasing*--the interval of *x*-values for which the curve is dropping from left to right.

Critical Values of a function occur when

i. $\frac{dy}{dx}=0$

ii. $\frac{dy}{dx}$ does not exist

iii. At the endpoints of its domain.

It is also helpful to remember that a **critical value** is referring specifically to the value of the *x*, while the **extreme value** refers to the value of the *y*.

Ex 1 Find the critical values of $y = x^3 - 9x$ on $x \in [-1, 6]$

i. $\frac{dy}{dx} = 3x^2 - 9 = 0$

$x^2 = 3$

$x = \pm\sqrt{3}$

but $-\sqrt{3}$ is not in the given domain, so $x = \sqrt{3}$.

ii. $\frac{dy}{dx}$ always exists

iii. $x = -1, 6$

Therefore the critical values are $x = -1, -\sqrt{3}, \text{and } 6$

Ex 2 Find the extremes points for $y = -\sqrt{x^4 - 6x^2 + 8}$

$$\frac{dy}{dx} = -\frac{1}{2}\left(x^4 - 6x^2 + 8\right)^{-1/2}\left(4x^3 - 12x\right)$$

$$= \frac{6x - 2x^3}{\left(x^4 - 6x^2 + 8\right)^{1/2}}$$

i) $$\frac{dy}{dx} = \frac{6x - 2x^3}{\left(x^4 - 6x^2 + 8\right)^{1/2}} = 0$$

$6x - 2x^3 = 0$

$6x - 2x^3 = 0$

$2x\left(3 - x^2\right) = 0$

$x = 0, \ \pm\sqrt{3}$ but $x = \pm\sqrt{3}$ are not in the domain so

$\left(0, \ -\sqrt{8}\right)$ or $\left(0, \ -2\sqrt{2}\right)$

ii) $\dfrac{dy}{dx} = \dfrac{6x - 2x^3}{\left(x^4 - 6x^2 + 8\right)^{1/2}}$ *dne*

$$\left(x^4 - 6x^2 + 8\right)^{1/2} = 0$$

$$x = \pm 2,\ \pm\sqrt{2}$$

$$y = 0$$

iii) There is no arbitrary domain.

So the extreme points are $\left(0,\ -2\sqrt{2}\right)$, $\left(\pm 2,\ 0\right)$, and $\left(\pm\sqrt{2},\ 0\right)$.

Ex 3 Find the extreme values of $y = 2xe^x$

i. $\dfrac{dy}{dx} = 2xe^x + e^x\left(2\right)$

$$2xe^x + e^x\left(2\right) = 0$$

$$2e^x\left(x+1\right) = 0$$

$e^x = 0$ $\qquad\qquad\qquad\qquad$ $\left(x+1\right) = 0$

x = *no solution* or $\qquad x = -1$

$\qquad\qquad\qquad\qquad\qquad\qquad y = -.736$

ii. $\dfrac{dy}{dx}$ always exists

iii. There is no arbitrary domain.

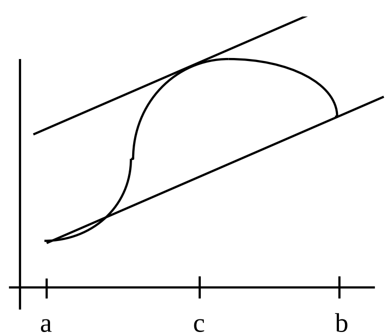
So $y = -.736$ is the extreme value.

The Mean Value and Rolle's Theorems

The Mean Value Theorem is an interesting piece of the history of Calculus that was used to prove a lot of what we take for granted. The Mean Value Theorem was used to prove that a derivative being positive or negative told you that the function was increasing or decreasing, respectively. Of course, this led directly to the first derivative test and the intervals of concavity.

> **Mean Value Theorem**
>
> If f is a function that satisfies these two hypotheses
> 1. f is continuous on the closed interval $\left[a,b\right]$
> 2. f is differentiable on the closed interval $\left(a,b\right)$
>
> Then there is a number c in the interval $\left(a,b\right)$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Again, translating from math to English, this just says that, if you have a smooth, continuous curve, the slope of the line connecting the endpoints has to equal the slope of a tangent somewhere in that interval. Alternatively, it says that the secant line through the endpoints has the same slope as a tangent line.



MEAN VALUE THEOREM

Rolle's Theorem is a specific a case of the Mean Value theorem, though Joseph-Louis Lagrange used it to prove the Mean Value Theorem. Therefore, Rolle's Theorem was used to prove all of the rules we have used to interpret derivatives for the last couple of years. It was a very useful theorem, but it is now something of a historical curiosity.

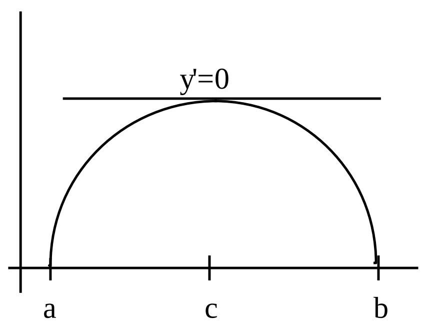
Rolle's Theorem

If f is a function that satisfies these three hypotheses

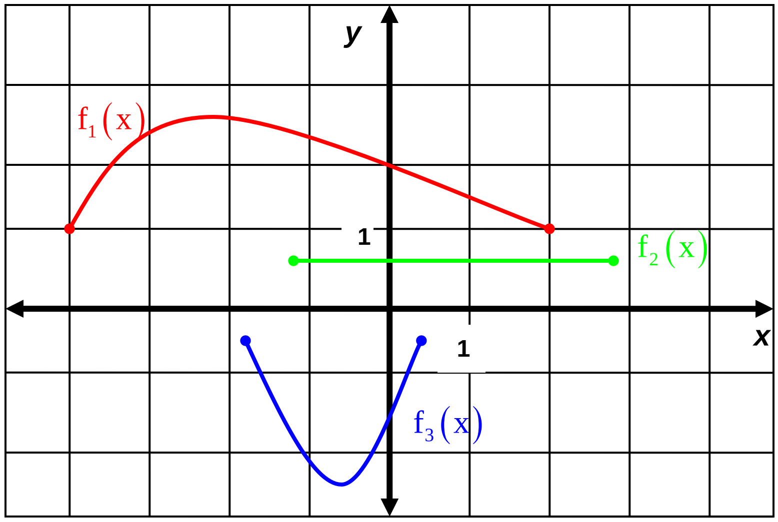
1. f is continuous on the closed interval $\left[a,b\right]$
2. f is differentiable on the open interval $\left(a,b\right)$
3. $f(a) = f(b)$

Then there is a number c in the interval $\left(a,b\right)$ such that $f'\left(c\right)=0$.

Written in this typically mathematical way, it is a bit confusing, but it basically says that if you have a continuous, smooth curve with the initial point and the ending point at the same height, there is some point in the curve that has a derivative of zero. If you look at this from a graphical perspective, it should be pretty obvious.



ROLLE'S THEOREM



Examples of Functions that satisfy Rolle's Theorem

Ex 4 Show that the function $f(x)=x^2-4x+1,\ [0,4]$ satisfies all the conditions of the Mean Value Theorem and find c.

Polynomials are continuous throughout their domain, so the first condition is satisfied.
Polynomials are also differentiable throughout their domain, so the second condition is satisfied.

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$ According to the Mean Value Theorem

$$f'(c)=\frac{f(4)-f(0)}{4-0}=\frac{\left(4^2-4(4)+1\right)-\left(0^2-4(0)+1\right)}{4}=0$$

$$f'(x)=2x-4$$
$$f'(c)=2c-4$$

Since, by applying the Mean Value Theorem, we found that $f'(c)=0$, so

$$2c-4=0$$
$$c=2$$

Ex 5 Suppose that $f(0) = -3$, and $f'(x) \le 5$ for all values of *x*. What is the largest possible value for $f(2)$?

We know that *f* is differentiable for all values of *x* and must therefore also be continuous. So we just make up an interval to look at using the mean value theorem. The interval will be [0, 2] because we are looking at $f(0)$ and $f(2)$.

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$f'(c) = \frac{f(2) - (-3)}{2}$$

$$2f'(c) = f(2) + 3$$

$$2f'(c) - 3 = f(2)$$

Since we know the maximum value for $f'(c)$ is 5, plug in 5 for $f'(c)$, and we get

$$2(5) - 3 = f(2)$$

$$f(2) = 7$$

Therefore the maximum possible value for $f(2)$ is 7.

3.1 Homework

Find the critical values and extreme values for each function.

1. $y = 2x^3 + 9x^2 - 168x$

2. $y = 3x^4 + 2x^3 + 12x^2 + 12x - 42$

3. $y = \dfrac{x^2 + 1}{x^3 - 4x}$

4. $y = \dfrac{x - 5}{x^2 + 9}$

5. $y = \sqrt{9x^3 - 4x^2 - 27x + 12}$

6. $y = \sqrt{\dfrac{3x}{9 - x^2}}$

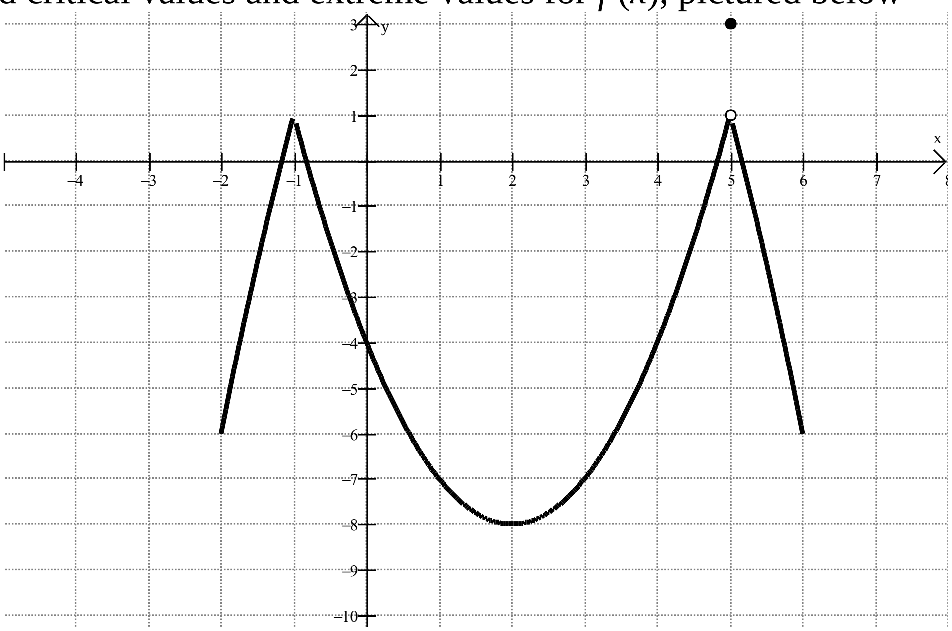
7. $y = \left(x^2\right)\sqrt[3]{9 - x^2}$

8. $y = \left(x - x^2\right)e^x$

9. $y = \frac{\sqrt{3}}{2}x + \cos x$ on $x \in [0,\ 2\pi]$

10. $y = \cot^{-1}\left(\frac{1}{x}\right) - \tan^{-1} x$

11. Find critical values and extreme values for $f(x)$, pictured below



Verify that the following functions fit all the conditions of Rolle's Theorem, and then find all values of *c* that satisfy the conclusion of Rolle's Theorem.

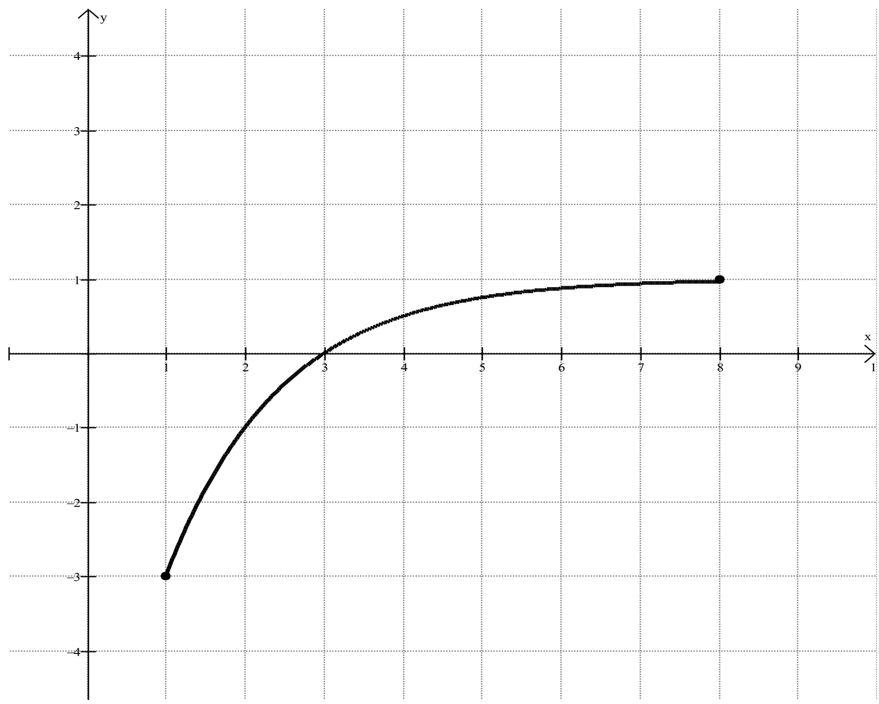
12. $f(x) = x^3 - 3x^2 + 2x + 5,\ [0,2]$

13. $f(x) = \sin(2\pi x),\ [-1,1]$

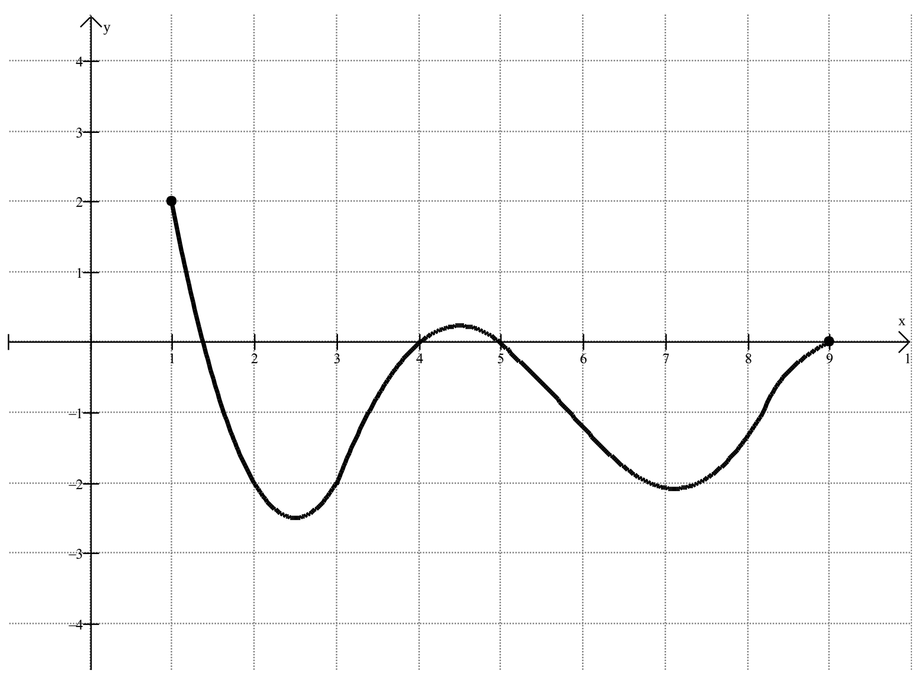
14. $g(t) = t\sqrt{t+6},\ [-6,0]$

Given the graphs of the functions below, estimate all values of c that satisfy the conclusion of the Mean Value Theorem for the interval [1,7].

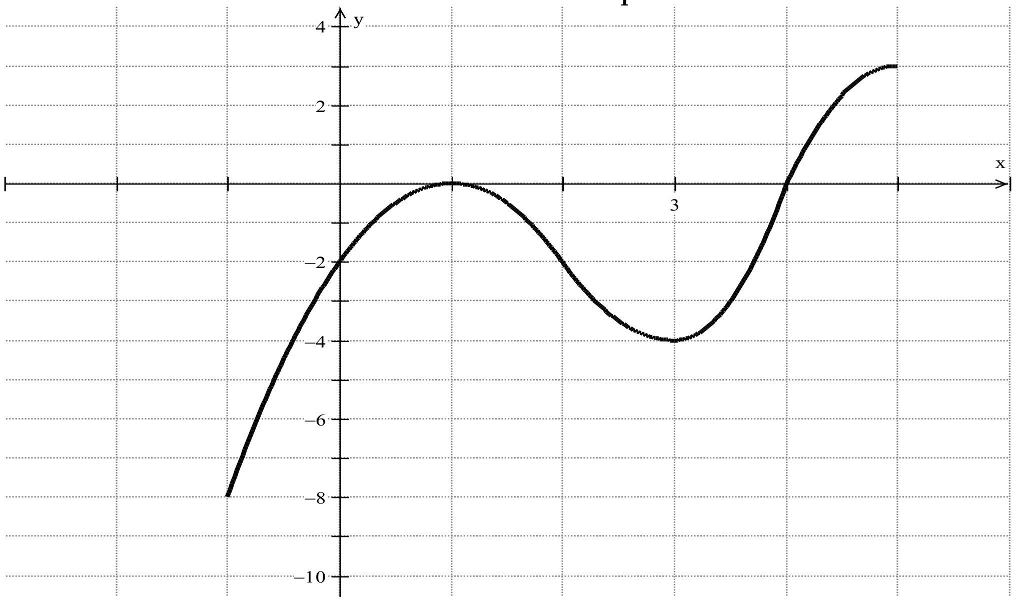
15.



16.



17. Given that $f(x)$ is a twice differentiable function with $f(2) = 6$. The **derivative** of $f(x)$, $f'(x)$, is pictured below on the closed interval $-1 \le x \le 5$. The graph of $f'(x)$ has horizontal tangent lines at $x = 1$ and at $x = 3$. Use this information to answer the questions below.



a) Find the x-coordinate of each point of inflection on f. Explain your reasoning.

b) Find where the function f attains its absolute maximum value and its absolute minimum value on the closed interval $-1 \le x \le 5$. Show the work that leads to this conclusion.

c) Let g be defined as the function $g(x) = x \cdot f(x)$. Find the equation of the graph of the tangent line to g at $x = 2$.

Answers: 3.1 Homework

1. $y = 2x^3 + 9x^2 - 168x$

   C.V. at x = 4 and –7
   E.V. at y = –400 and 931

2. $y = 3x^4 + 2x^3 + 12x^2 + 12x - 42$

   C.V. at x = –0.5
   E.V. at y = –45.063

3. $y = \dfrac{x^2 + 1}{x^3 - 4x}$

   C.V. at x = –0.729 and 0.729
   E.V. at x = 0.606 and –0.606

4. $y = \dfrac{x - 5}{x^2 + 9}$

   C.V. at x = –0.831 and 10.831
   E.V. at x = 0.046 and –0.602

5. $y = \sqrt{9x^3 - 4x^2 - 27x + 12}$

   C.V. at x = 4 and –7
   E.V. at y = 0 and 5.151

6. $y = \sqrt{\dfrac{3x}{9 - x^2}}$

   C.V. at x = –0.5
   E.V. at y = 0

7. $y = \left(x^2\right)\sqrt[3]{9 - x^2}$

   C.V. at x = 0 and ± 2.598
   E.V. at y = 0 and 8.845

8. $y = \left(x - x^2\right)e^x$

   C.V. at x = –1.618 and .618
   E.V. at y = –0.840 and 0.438

9. $y = \dfrac{\sqrt{3}}{2}x + \cos x$ on $x \in \left[0,\ 2\pi\right]$

   C.V. at x = 0 and 2π and 1.047 and 2.094
   E.V. at $y =$ 1 and 6.441 and 1.407 and 1.314

10. $y = \cot^{-1}\left(\dfrac{1}{x}\right) - \tan^{-1} x$

    None

11. Find critical values and extreme values for $f(x)$, pictured below

    C.V. at x = –2 and –1 and 2 and 5 and 6
    E.V. at y = –6 and 1 and –8 and 3

12. $f(x) = x^3 - 3x^2 + 2x + 5,\ x \in [0, 2]$

    Function is continuous and differentiable on the interval.
    $f(0) = f(2)$
    $c = 0.423$

13. $f(x)=\sin(2\pi x),\ x\in[-1,1]$

Function is continuous and differentiable on the interval.

$f(-1)=f(1)$

$c=\pm\frac{3}{4}$ and $\pm\frac{1}{4}$

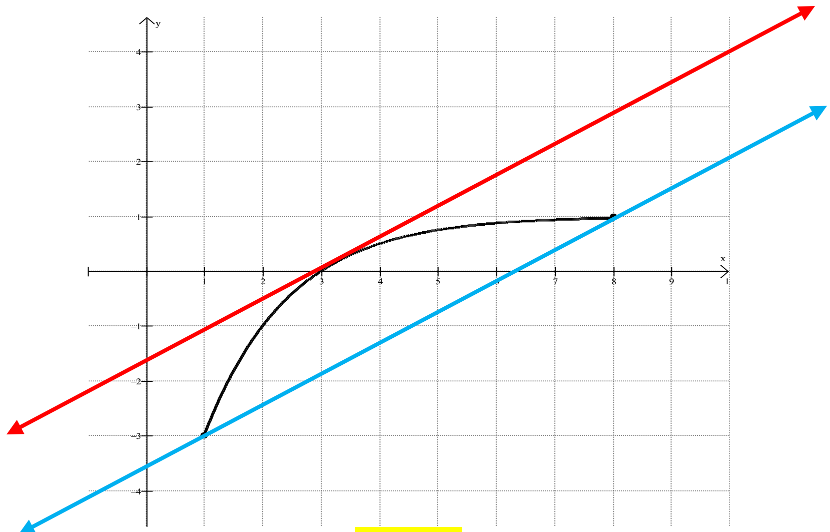
14. $g(t)=t\sqrt{t+6},\ t\in[-6,0]$

Function is continuous and differentiable on the interval.

$f(-6)=f(0)$

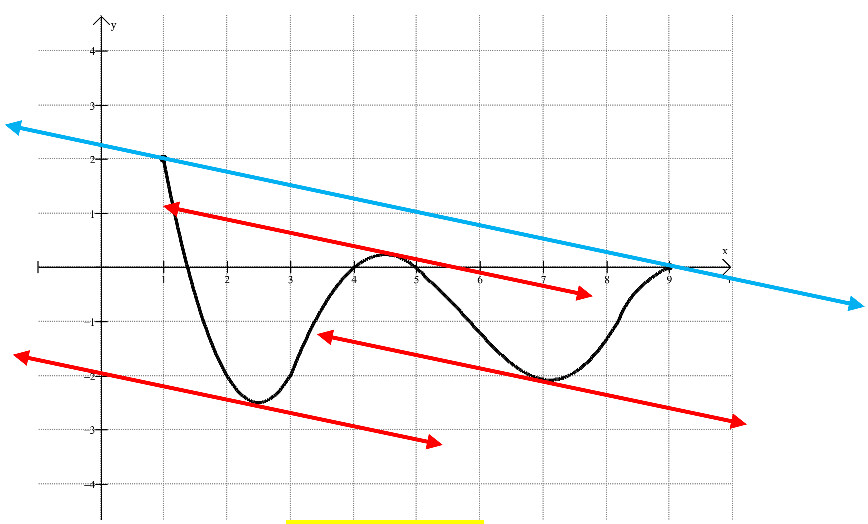
$c=-4$

15.



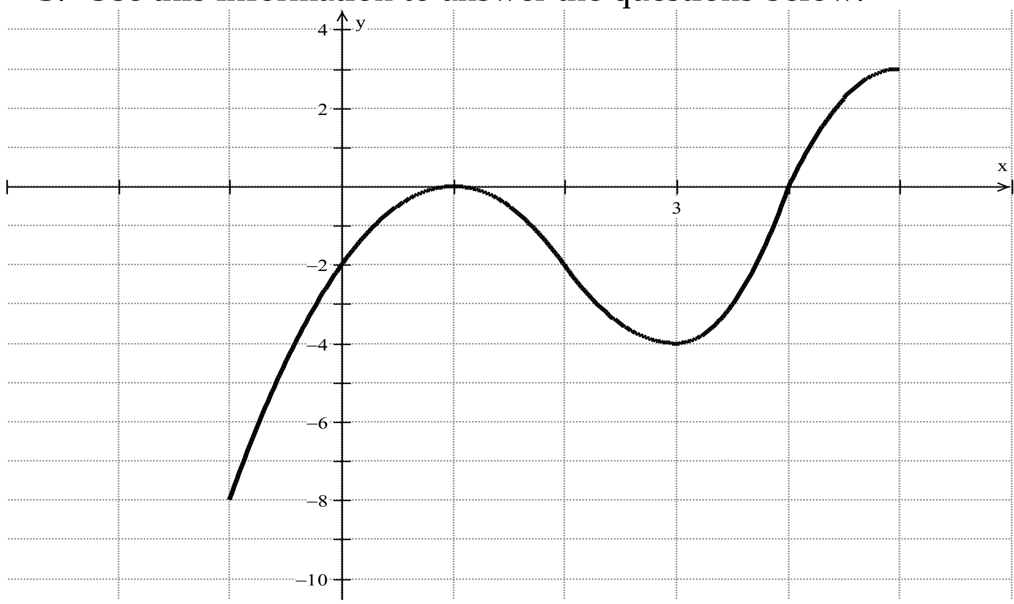
$c\approx 3.4$

16.



$c\approx 2.4, 4.6, 7$

17. Given that $f(x)$ is a twice differentiable function with $f(2) = 6$. The **derivative** of $f(x)$, $f'(x)$, is pictured below on the closed interval $-1 \le x \le 5$. The graph of $f'(x)$ has horizontal tangent lines at $x = 1$ and at $x = 3$. Use this information to answer the questions below.



a) Find the x-coordinate of each point of inflection on f. Explain your reasoning.

$x = 1$ and $x = 3$ because these are places where f' has a slope of 0. This means that $f'' = 0$ at these points. In addition, f'' changes from positive to negative or negative to positive at these values. Therefore, f has concavity changes at these x values.

b) Find where the function f attains its absolute maximum value and its absolute minimum value on the closed interval $-1 \le x \le 5$. Show the work that leads to this conclusion.

f has critical values at $x = -1$, 4, and 5, (endpoints of the interval or 0 of f'). Maxima occur at $x = -1$ and 5, with the absolute maximum occurring at $x = -1$. This is because f decreases from -1 (because f' is negative) and most of the area under f' is negative, meaning the value of $f(5) < f(-1)$. The absolute minimum occurs at $x = 4$ because it is the only minimum on the function (f switches from decreasing to increasing at this point).

c) Let g be defined as the function $g(x) = x \cdot f(x)$. Find the equation of the graph of the tangent line to g at $x = 2$.

$y - 12 = -2(x - 2)$

3.2: Rate of Change

As you read, one of the original concepts for the derivative comes from the slope of a line. Since we also know that slopes represent rates in algebra, it is just a small leap from there to recognizing that derivatives can represent the rate of change of any function.

Any time a dependent variable changes in comparison to an independent variable, you have a rate of change of the dependent variable with respect to the independent variable. The most common time you may have seen this is with the rate of change of position with respect to time. Hopefully, you recognize that change in position over change in time is velocity.

OBJECTIVES
Recognize and evaluate derivatives as rates of change. Interpret derivatives as rates of change. Utilize the language of rates of change with respect to derivatives.

Ex 1 For the function $S = 2\pi r^2 + \pi r L$, find the rate of change of S with respect to r, assuming L is constant.

The first thing we should notice is that when we are asked for the change of S with respect to r, that we are really being asked for $\frac{dS}{dr}$, so we simply take the derivative.

$$\frac{d}{dr}\left[S = 2\pi r^2 + \pi r L\right]$$
$$\frac{dS}{dr} = 4\pi r + \pi L$$

The rate of change of S with respect to r is simply $\frac{dS}{dr}$.

Ex. 2 A restaurant determines that its revenue from hamburger sales is given by $R = x\left(\frac{60,000 - x}{20,000}\right)$, where x is the number of hamburgers. Find the increase in revenue (marginal revenue) for monthly sales at 20,000 hamburgers.

Since we want to find an increase in revenue, we must be looking at a derivative, so we will take the derivative of the function. It would help to simplify the function first.

$$R = x\left(\frac{60,000 - x}{20,000}\right)$$
$$= 3x - \frac{x^2}{20,000}$$

$$\frac{dR}{dx} = 3 - \frac{x}{10,000}$$
$$\left.\frac{dR}{dx}\right|_{x=20,000} = 3 - \frac{20,000}{10,000} = \$1/\text{hamburger}$$

Ex 3 The energy in joules of a photon varies with the wavelength of the light, λ according to the function $E = \frac{1.9864\times10^{-25}}{\lambda}$. Find how fast the energy is changing for a photon of light from Superman's X-ray vision (light with $\lambda = 1.0\times10^{-9}\ m$). What does this number mean?

$$\frac{d}{d\lambda}\left[E = \frac{1.9864\times10^{-25}}{\lambda}\right]$$
$$\frac{dE}{d\lambda} = -1.9864\times10^{-25}\lambda^{-2}$$

But we want $\left.\frac{dE}{d\lambda}\right|_{\lambda=1.0\times10^{-9}}$, so we plug in $\lambda = 1.0\times10^{-9}$ m and find that

$$\left.\frac{dE}{d\lambda}\right|_{\lambda=1.0\times10^{-9}} = -1.9864\times10^{-7}\ \text{joules/meter}$$

This means that at the particular wavelength, $\lambda = 1.0\times10^{-9}$ m, energy goes down by 1.9864×10^{-7} joules for every 1 meter increase in the wavelength.

Ex 4 For the equation in example 2, find the rate of change of the wavelength with respect to the energy when $E = 1.9878\times10^{-16}$.

Notice, we are being asked for $\frac{d\lambda}{dE}$ this time, so we have one of two approaches. We could solve the equation for λ, and then take the derivative.

$$E = \frac{1.9864\times10^{-25}}{\lambda}$$
$$E\lambda = 1.9864\times10^{-25}$$
$$\lambda = \frac{1.9864\times10^{-25}}{E}$$
$$\frac{d}{dE}\left[\lambda = \frac{1.9864\times10^{-25}}{E}\right]$$
$$\frac{d\lambda}{dE} = -1.9864\times10^{-25}E^{-2}$$
$$\left.\frac{d\lambda}{dE}\right|_{E=1.9878\times10^{-16}} = -5.0271\times10^{6} \text{ meters/Joule}$$

Alternatively, we could find the value of $\frac{dE}{d\lambda}$ for this value of E and then take the reciprocal to find $\frac{d\lambda}{dE}$.

$\frac{dE}{d\lambda} = -1.9864\times10^{-25}\lambda^{-2}$, and when E=1.9878×10^{-16}, λ=9.99296×10^{-10}

And at this value for λ, $\frac{dE}{d\lambda} = -1.9892\times10^{-7}$ Joules/meter.

The reciprocal of this is $\frac{d\lambda}{dE} = -5.0271\times10^{6}$ meters/Joule

Clearly, in this case, both ways are relatively simple, but there may be times when it is much easier to find the reciprocal of the derivative if it is too difficult to isolate one of the variables.

Ex 5 The height of a person diving from a high dive is given by the equation $y = -4.9t^2 + 1.2t + 10$ where *y* is in meters and *t* is in seconds. Use this fact to answer each of the following.

a) Find the rate of change of height with respect to time at $t = 0.5$

b) Interpret the meaning of the value from a)

c) Find when the value of $\frac{dy}{dt} = -2.5 \,{}^{\text{meters}}\!/\!{}_{\text{second}}$. Use this to find the height of the diver at this time. Interpret the meaning of all of this.

a) The rate of change is the derivative, so

$$\frac{dy}{dt} = -9.8t + 1.2$$

$$\left.\frac{dy}{dt}\right|_{t=0.5} = -9.8(0.5) + 1.2 = -3.7 \,{}^{\text{meters}}\!/\!{}_{\text{second}}$$

b) $-3.7 \,{}^{\text{meters}}\!/\!{}_{\text{second}}$ means that the person is traveling downward at $3.7 \,{}^{\text{meters}}\!/\!{}_{\text{second}}$ at 0.5 seconds. You may notice that this is a velocity (because it is a rate of change of height versus time).

c) We just need to set the derivative equal to the value given and solve for *t*.

$$\frac{dy}{dt} = -9.8t + 1.2 = -2.5$$

$$-9.8t = -3.7$$

$$t = 0.378 \text{ seconds}$$

We need the height, so we go back to our initial equation we were given.

$$y(0.378) = -4.9(0.378)^2 + 1.2(0.378) + 35$$

$$y(0.378) = 9.755 \text{ meters}$$

So at 0.378 seconds, the diver is 9.755 meters high and is traveling downwards at $2.5 \,{}^{\text{meters}}\!/\!{}_{\text{second}}$

What you may have noticed from the previous example, an interesting application of rates of change comes from basic physics. For a displacement function, the rate of change is clearly the velocity, and the rate of change of the velocity is acceleration. Of course this means that the derivative of displacement is velocity and the derivative of velocity is acceleration.

Derivatives are used extensively in Physics to describe particle motion (or any other linear motion). If the variables represent time and distance, the derivative will be a rate or velocity of the particle. Because of their meaning, the letters *t* and either *s* or *x* are used to represent time and distance, respectively.

Rectilinear Motion--Defn: Movement that occurs in a straight line.
Parameter--Defn: a dummy variable that determines *x*- and *y*-coordinates independent of one another.
Parametric Motion--Defn: Movement that occurs in a plane.
Velocity--Defn: directed speed.
Means: How fast something it is going and whether it is moving right or left.

Average Velocity--Defn: distance traveled/time or $\dfrac{x_2 - x_1}{t_2 - t_1}$
Means: The average rate, as we used it in Algebra.
Instantaneous Velocity--Defn: Velocity at a particular time t.
Means: $\dfrac{ds}{dt}$, $\dfrac{dx}{dt}$,or $\dfrac{dy}{dt}$, or the rate at any given instant.

Acceleration--Defn: the rate of change of the velocity, or $\dfrac{dv}{dt}$.

OBJECTIVE

Given a distance function of an object in rectilinear or parametric motion, find the velocity and acceleration functions or vectors.
Use the velocity and acceleration functions to describe the motion of an object.

We will consider rectilinear motion first and then parametric motion. In either case, though, there are three things implied in the definitions above:

1. If we have a distance equation, its derivative will be the velocity equation.

2. The derivative of velocity is acceleration.

3. If the velocity is 0, the particle is stopped--usually paused in order to switch directions.

Rectilinear Motion

For rectilinear motion, it is helpful to remember some things from Precalculus.

$$h = -\frac{1}{2}at^2 + v_0 t + h_0$$

where a = the gravitational constant
(a = 32 ft/sec^2 or 9.8 m/sec^2 on Earth),
v_0 is the initial velocity and h_0 is the initial height

EX 6 A gun is fired up in the air from a 1600 foot tall building at 240 ft/second. How fast is the bullet going when it hits the ground?

$h(t) = -16t^2 + 240t + 1600 = 0$

$t^2 - 15t - 100 = 0$

$(t-20)(t+5) = 0$

$t = 20$ seconds is when it hits the ground.

$v(t) = -32t + 240$

$v(20) = -32(20) + 240 = -400\frac{\text{ft}}{\text{sec}}$ is the velocity.

The velocity is negative because the bullet is coming down. The speed at which it is going is $400\frac{\text{ft}}{\text{sec}}$.

EX 7 What is the acceleration due to gravity of any falling object?

We know from before that any object launched from height h_0 feet with initial velocity v_0 ft/sec follows the equation

$$h = -16t^2 + v_0 t + h_0$$

$$v = h' = -32t + v_0$$
$$a = v' = -32\frac{ft}{\sec^2}$$

This number is known as the Gravitational Constant.

EX 8 A particle's distance $x(t)$ from the origin at time $t \geq 0$ is described by $x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$. How far from the origin is it when it stops to switch directions?

$$x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$$
$$v(t) = 4t^3 - 6t^2 - 22t + 12 = 0$$
$$= 2t^3 - 3t^2 - 11t + 6 = 0$$
$$= (2t-1)(t+2)(t-3) = 0$$
$$t = \frac{1}{2}, -2, \text{ and } 3$$

But the problem specifies that $t \geq 0$, so $t = \frac{1}{2}$ and 3.

$$x\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^3 - 11\left(\frac{1}{2}\right)^2 + 12\left(\frac{1}{2}\right) + 1 = 4.0625$$
$$x(3) = (3)^4 - 2(3)^3 - 11(3)^2 + 12(3) + 1 = -35$$

Therefore, this particle stops 4.0625 units to the right of the origin (at 1/2 seconds) and 35 units left of the origin (at 3 seconds).

EX 9 A particle's distance $x(t)$ from the origin at time $t \geq 0$ is described by $x(t) = t^4 - 2t^3 - 11t^2 + 12t + 1$. What is the acceleration when the particle stops to switch directions?

Since this is the same problem as EX 3 in 2-7, we already know

$$\begin{aligned}
x(t) &= t^4 - 2t^3 - 11t^2 + 12t + 1 \\
v(t) &= 4t^3 - 6t^2 - 22t + 12 = 0 \\
&= 2t^3 - 3t^2 - 11t + 6 = 0 \\
&= (2t-1)(t+2)(t-3) = 0 \\
t &= \frac{1}{2}, -2, \text{ and } 3
\end{aligned}$$

and $t = \frac{1}{2}$ and 3 because the time $t \geq 0$. All we need to do is substitute these times into the acceleration equation.

$$\begin{aligned}
v(t) &= 4t^3 - 6t^2 - 22t + 12 \\
a(t) &= 12t^2 - 12t - 22
\end{aligned}$$

$$\begin{aligned}
a(3) &= 12(3)^2 - 12(3) - 22 = 50 \\
a\left(\frac{1}{2}\right) &= 12\left(\frac{1}{2}\right)^2 - 12\left(\frac{1}{2}\right) - 22 = -25
\end{aligned}$$

Note that the interpretation of a negative acceleration is **not**, like velocity, the direction it is moving, but which way the acceleration is affecting the particle. The acceleration and velocity's directions together determine if the particle is slowing down or speeding up.

Ex 10 A model rocket's height is given by the equation $x(t)=100\tan\left(\frac{\pi}{14}t\right)-4.9t^2+1$ on $t\in[0,5]$ where t is in seconds, and x is in meters. Use this to find each of the following:

a) $v(2.7)$

b) $a(2.7)$

c) Whether the rocket is speeding up or slowing down at t = 2.7 seconds.

a) $x'(t)=v(t)=\frac{50\pi}{7}\sec^2\left(\frac{\pi}{14}t\right)-9.8t$

$$v(2.7)=\frac{50\pi}{7}\sec^2\left(\frac{2.7\pi}{14}\right)-9.8(2.7)$$

$$v(2.7)=6.751 \text{ meters/second}$$

b) $x'(t)=v(t)=\frac{50\pi}{7}\sec^2\left(\frac{\pi}{14}t\right)-9.8t$

$$v'(t)=a(t)=\frac{100\pi}{7}\sec^2\left(\frac{\pi}{14}t\right)\tan\left(\frac{\pi}{14}t\right)-9.8$$

$$a(2.7)=\frac{100\pi}{7}\sec^2\left(\frac{2.7\pi}{14}\right)\tan\left(\frac{2.7\pi}{14}\right)-9.8$$

$$a(2.7)=10.326 \text{ meters/second}^2$$

c) Since both v and a are positive at t = 2.7, the rocket is speeding up.

The question of speeding up or slowing down is very common on the AP test. Since these are questions of *speed* rather than *velocity*, they require a little bit of extra thought.

Something is speeding up if it's speed is getting bigger (obviously), but since speed is non-directional, a large negative velocity is a large speed. If a negative velocity gets more negative (that is, the absolute value gets larger) then it is speeding up. If a positive value gets more positive, it is also speeding up.

Therefore, a particle speeds up when its velocity and acceleration have the same sign, and it slows down if they have opposite signs.

<u>Speeding Up/Slowing Down</u>

1. If the velocity and acceleration have the same sign (both positive or both negative), the particle is **speeding up.**

2. If the velocity and acceleration have the opposite signs (one positive and the other negative), the particle is **slowing down.**

EX 11 Describe the motion of a particle whose distance from the origin $x(t)$ is given by $x(t)=2t^3-3t^2-12t+1$.

$x(t)=2t^3-3t^2-12t+1$

$v(t)=6t^2-6t-12$

Since the particle stops at v = 0,

$$v(t)=6t^2-6t-12=0$$
$$v(t)=t^2-t-2=0$$
$$v(t)=(t+1)(t-2)=0$$
$$t=-1 \text{ or } 2$$

$x(-1)=8$ and $x(2)=-19$

The sign pattern shows:

v	+	0	–	0	+
t		-1		2	

which means the particle was moving right. One second before we started timing, it stopped 8 units to the right of the origin and began moving left. At two seconds, it stopped again 19 units left of the origin and began moving right.

(For a review of sign patterns, check out appendix A, and we will look at more on interpreting them later in this chapter)

3.2 Homework Set A

1. The height of a particle is given by the function $h(t) = t^3 - 4.5t^2 - 7t$.
   a. When does the particle reach a velocity of 7 m/s?

   b. Find the particle's velocity at 5 seconds and 10 seconds.

2. Newton's Law of Gravitation states that the magnitude of the force (*F*) exerted by a mass (*M*) on another mass (*m*) is given by $F = \dfrac{GMm}{r^2}$.
   a. Find the rate of change of the force with respect to the radius assuming that the masses are constant. Explain what it means.
   b. Find the rate of change of the force with respect to mass (*M*) if all other values are held constant and explain its meaning.
   c. If you know that a force changes at a rate of 0.002 N/m at $r = 2.0 \times 10^8$ m, find how fast the force is changing when $r = 1.0 \times 10^8$ m.

3. An object in free fall with no forces acting on it other than gravity is governed by the equation $y(t) = y_0 + v_0 t + \frac{1}{2}gt^2$.

a. Find the equations for the object's velocity and acceleration.
b. Find when the object's height (y) is increasing
c. Find when its velocity is increasing and when it is decreasing.

In problems 4 -7, the motion of a particle is described by the following distance equations. For each, find
a) when the particle is stopped,
b) which direction it is moving at t = 3 seconds,
c) <u>where</u> it is at t = 3, and
d) $a(3)$
e) Whether the particle is speeding up or slowing down at t = 3 seconds.

4. $x(t) = 2t^3 - 21t^2 + 60t + 4$

5. $x(t)=t^3-6t^2+12t+5$

6. $x(t)=9t^4-4t^3-240t^2+576t-48$

7. $x(t)=12t^5-15t^4-220t^3+270t^2+1080t$

Find maximum height of a projectile launched vertically from the given height and with the given initial velocity.

8. $h_0=25$ feet, $v_0=64\dfrac{\text{ft}}{\text{sec}}$

9. $h_0=10$ meters, $v_0=294\dfrac{\text{m}}{\text{sec}}$

10. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars, $s = 11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec on each planet?

11. Find $x(t)$ when v = 0.8. $x(t) = t^2 - 5t + 4$

Find $v(t)$ and $a(t)$ for problems 12 and13. Then find whether the particle is speeding up or slowing down at t = 1.

12. $x(t) = t^3 - 6t^2 - 63t + 4$

13. $x(t) = 6t^5 - 15t^4 - 8t^3 + 24t^2 + 12$

For problems 14 to 16, find $x(t)$ and $v(t)$ when $a(t)=0$.

14. $x(t)=2t^3-21t^2+60t+4$

15. $x(t)=t^3-6t^2+12t+5$

16. $x(t)=9t^4-4t^3-240t^2+576t-48$

3.2 Homework Set B

1. The distance of a particle from the origin is given by the function $x(t)=\sqrt{9t-t^3}$.
   a. When does the particle reach a velocity of 5 m/s to the left? 2 m/s to the right?
   b. Find the particle's velocity at 3 seconds.

2. A circular puddle is increasing in size as water drips into it (hint: remember the area of a circle)
   a. Find the rate of change of the area with respect to the radius when r = 25 cm, r = 15 cm, and r = 1 m.
   b. Find the rate of change of the radius with respect to the area when A = 4 meters^2 and when A = 7 meters^2.
   c. Assume that the puddle is actually a cylinder with a depth of 0.5 cm. Find the rate of change of the volume with respect to the radius when r = 25 cm, r = 15 cm, and r = 1 m. Is there any relationship between these answers and the answers in part a? Explain.

3. Coulomb's Law states that the magnitude of the force (F) exerted by one charge (Q) on another charge (q) is given by $F = \frac{k_C Qq}{r^2}$, where $k_C = 8.99 \times 10^9 \ N\ m^2 / C^2$.

a. Find the rate of change of the force with respect to the radius assuming that the charges are constant. Explain what it means.

b. Find the rate of change of the force with respect to charge (Q) if all other values are held constant and explain its meaning.

c. If you know that a force changes at a rate of 0.6 N/m at $r = 2.0 \times 10^{-2}\ m$, find how fast the force is changing when $r = 1.0 \times 10^{-2}\ m$.

4. An object flying in the x direction with no forces acting on it is governed by the equation $x(t) = x_0 + v_0 t$.

a. Find the equations for the object's velocity and acceleration.

b. Find when the object's distance (x) is increasing; find when its velocity is increasing and when it is decreasing, assuming v_0 is positive.

c. Explain why the answers you got in part b make sense.

5. Find the acceleration and velocity equations for a particle whose position is given by $x(t) = \tan^{-1}(t+2)$. Then find the position, velocity, and acceleration of the particle when $t = 3$.

6. Find $a(\pi)$ when $y(t) = \sec(5t)$

7. Find when the velocity of a particle is increasing if the particle's position is given by $x(t) = -t^4 + 9t^2 + 10$.

8. Find when the particle whose position is given by the function in problem 7 has a decreasing acceleration.

9. Find the maximum distance from the origin for the particle whose position is given by $y(t)=t^3-29t^2-t+29$. Then find the acceleration of the particle at that point.

10. If the velocity of a particle moving along the x axis is given by $v(t)=\sec(2t)\tan(2t)+t^2$, find the acceleration of the particle when $t=\pi$. Then find the position equation of the particle if $x(\pi)=\dfrac{\pi^3}{3}+\dfrac{1}{2}$

11. If the position of a particle is given by $y(t)=t^3+6t+\cos t$ on $t\in[0,2\pi]$, find the position of the particle when the acceleration is 3π.

Answers: 3.2 Homework Set A

1. The height of a particle is given by the function $h(t)=t^3-4.5t^2-7t$.
   a. When does the particle reach a velocity of 7 m/s?
   b. Find the particle's velocity at 5 seconds and 10 seconds.

   a. $t = -1.130, 4.130$ b. $v(5) = 23$, $v(10) = 203$

2. Newton's Law of Gravitation states that the magnitude of the force (F) exerted by a mass (M) on another mass (m) is given by $F=\dfrac{GMm}{r^2}$.
   a. Find the rate of change of the force with respect to the radius assuming that the masses are constant. Explain what it means.
   b. Find the rate of change of the force with respect to mass (M) if all other values are held constant and explain its meaning.
   c. If you know that a force changes at a rate of 0.002 N/m^3 at $r=2.0\times10^8$ m, find how fast the force is changing when $r=1.0\times10^8$ m.

a. $\dfrac{dF}{dr}=-2GMmr^{-3}$; this is how fast the force changes as the radius changes.

b. $\dfrac{dF}{dM}=\dfrac{Gm}{r^2}$; this is how fast the force changes as the mass (M) changes.

c. 0.016 N/m^3

3. An object in free fall with no forces acting on it other than gravity is governed by the equation $y(t)=y_0+v_0t+\dfrac{1}{2}gt^2$.
   a. Find the equations for the object's velocity and acceleration.
   b. Find when the object's height (y) is increasing, assuming g is negative and v_0 is positive.
   c. Find when its velocity is increasing and when it is decreasing, assuming g is negative.

   a. $v(t)=v_0+gt$, $a(t)=g$

   b. y is increasing when $t<-\dfrac{v_0}{g}$.

   c. v is always decreasing.

4. $x(t)=2t^3-21t^2+60t+4$

a) $t = 2, 5$
b) moving left
c) 49 units right of origin
d) $a(3)=-6$
e) speeding up

5. $x(t)=t^3-6t^2+12t+5$

a) $t = 2$
b) moving right
c) 14 units right of origin
d) $a(3)=2$
e) speeding up

6. $x(t)=9t^4-4t^3-240t^2+576t-48$

a) $t = -4, 3, 4/3$
b) stopped
c) 141 units right of origin
d) $a(3)=420$
e) stopped

7. $x(t)=12t^5-15t^4-220t^3+270t^2+1080t$

a) $t = -3, -1, 2, 3$
b) stopped
c) 1461 units right of origin
d) $a(3)=1440$
e) stopped

8. $h_0 = 25$ feet, $v_0 = 64\dfrac{\text{ft}}{\text{sec}}$

89 feet

9. $h_0 = 10$ meters, $v_0 = 294\dfrac{\text{m}}{\text{sec}}$

3520 meters

10. The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s=1.86t^2$ on Mars, $s=11.44t^2$ on Jupiter. How long would it take a rock falling from rest to reach a velocity of 27.8 m/sec on each planet?

7.473 seconds on Mars; 1.215 seconds on Jupiter

11. Find $x(t)$ when $v = 0.8$. $x(t)=t^2-5t+4$

$x(2.9)=-2.09$

12. $x(t)=t^3-6t^2-63t+4$

$v(t)=3t^2-12t-63$

$a(t)=6t-12$

Speeding up

13. $x(t)=6t^5-15t^4-8t^3+24t^2+12$

$v(t)=30t^4-60t^3-24t^2+48t$

$a(t)=120t^3-180t^2-48t+48$

Speeding up

14. $x(t)=2t^3-21t^2+60t+4$

$v(3.5)=-13.5$

$x(3.5)=42.5$

15. $x(t)=t^3-6t^2+12t+5$

$v(2)=0$

$x(2)=13$

16. $x(t)=9t^4-4t^3-240t^2+576t-48$

$v(-2)=1200$ $\qquad$ $v\left(\frac{22}{9}\right)=-143.210$

$x(-2)=1984$ $\qquad$ $x\left(\frac{22}{9}\right)=188.840$

3.2 Homework Set B

1. The distance of a particle from the origin is given by the function $x(t)=\sqrt{9t-t^3}$.
   a. When does the particle reach a velocity of 5 m/s to the left? 2 m/s to the right?
   b. Find the particle's velocity at 3 seconds.
      a. $t = -3.255,\ 2.857;\quad t = 0.489$
      b. $v(3)$ = DNE
2. A circular puddle is increasing in size as water drips into it (hint: remember the area of a circle)
   a. Find the rate of change of the area with respect to the radius when r = 25 cm, r = 15 cm, and r = 1 m.
   b. Find the rate of change of the radius with respect to the area when A = 4 meters^2 and when A = 7 meters^2.
   c. Assume that the puddle is actually a cylinder with a depth of 0.5 cm. Find the rate of change of the volume with respect to the radius when r = 25 cm, r = 15 cm, and r = 1 m. Is there any relationship between these answers and the answers in part a? Explain.

   a. $\left.\frac{dA}{dr}\right|_{r=25}=50\pi\text{ cm},\ \left.\frac{dA}{dr}\right|_{r=15}=30\pi\text{ cm},\ \left.\frac{dA}{dr}\right|_{r=100}=200\pi\text{ cm}$

   b. $\left.\frac{dr}{dA}\right|_{A=4}=\frac{\sqrt{\pi}}{4},\ \left.\frac{dr}{dA}\right|_{A=7}=\frac{1}{2}\sqrt{\frac{\pi}{7}}$

   c. $\left.\frac{dV}{dr}\right|_{r=25}=25\pi\text{ cm}^2,\ \left.\frac{dV}{dr}\right|_{r=15}=15\pi\text{ cm}^2,\ \left.\frac{dV}{dr}\right|_{r=100}=100\pi\text{ cm}$

   They are directly related, since depth is a constant value.

3. Coulomb's Law states that the magnitude of the force (F) exerted by one charge (Q) on another charge (q) is given by $F=\frac{k_C Qq}{r^2}$, where $k_C=8.99\times10^9\ N\ m^2/C^2$.

a. Find the rate of change of the force with respect to the radius assuming that the charges are constant. Explain what it means.

b. Find the rate of change of the force with respect to charge (Q) if all other values are held constant and explain its meaning.

c. If you know that a force changes at a rate of 0.6 N/m at $r=2.0\times10^{-2}\ m$, find how fast the force is changing when $r=1.0\times10^{-2}\ m$.

a. $\frac{dF}{dr}=-2k_C Qqr^{-3}$; it is how fast force changes as radius changes

b. $\frac{dF}{dQ}=\frac{k_C q}{r^2}$; it is how fast force changes as charge (Q) changes

c. 4.8 N/m

4. An object flying in the x direction with no forces acting on it is governed by the equation $x(t)=x_0+v_0t$.

a. Find the equations for the object's velocity and acceleration.

b. Find when the object's distance (x) is increasing; find when its velocity is increasing and when it is decreasing, assuming v_0 is positive.

c. Explain why the answers you got in part b make sense.

a. $v(t)=v_0,\ a=0$

b. Distance is always increasing, because v_0 is always positive. Velocity is constant because acceleration is 0.

c. This makes sense, because if no force is acting, then there is no acceleration, and it should move at a constant velocity.

5. Find the acceleration and velocity equations for a particle whose position is given by $x(t)=\tan^{-1}(t+2)$. Then find the position, velocity, and acceleration of the particle when $t=3$.

$v(t)=\frac{1}{t^2+4t+5}$ $\quad a(t)=-(t^2+4t+5)^{-2}(2t+4)$

$x(3)=\tan^{-1}5$ $\quad v(t)=\frac{1}{26}$ $\quad a(t)=-\frac{5}{338}$

6. Find $a(\pi)$ when $y(t)=\sec(5t)$ $a(\pi)=-25$

7. Find when the velocity of a particle is increasing if the particle's position is given by $x(t)=-t^4+9t^2+10$.
$t\in(-1.225,1.225)$

8. Find when the particle whose position is given by the function in problem 3 has a decreasing acceleration.
$t<0$

9. Find the maximum distance from the origin for the particle whose position is given by $y(t)=t^3-29t^2-t+29$. Then find the acceleration of the particle at that point.
29.009 units from origin.
$a(-0.017) = -58.103$

10. If the velocity of a particle moving along the x axis is given by $v(t)=\sec(2t)\tan(2t)+t^2$, find the acceleration of the particle when $t=\pi$.
Then find the position equation of the particle if $x(\pi)=\dfrac{\pi^3}{3}+\dfrac{1}{2}$
$a(\pi)=2+2\pi$
$x(t)=\dfrac{1}{2}\sec(2t)+\dfrac{t^3}{3}$

11. If the position of a particle is given by $y(t)=t^3+6t+\cos t$ on $t\in[0,2\pi]$, find the position of the particle when the acceleration is 3π.
$y\left(\dfrac{\pi}{2}\right)=\dfrac{\pi^3}{8}+3\pi$

3.3: Implicit Differentiation, Part 2

One of the more useful aspects of the chain rule that we reviewed earlier is that we can take derivatives of more complicated equations that would be difficult to take the derivative of otherwise. One of the key elements to remember is that we already know the derivative of *y* with respect to *x* – that is, $\frac{dy}{dx}$. This can be a powerful tool as it allows us to take the derivative of relations as well as functions while bypassing a lot of tedious algebra.

OBJECTIVES
Take derivatives of relations implicitly. Use implicit differentiation to find higher order derivatives.

Ex 1 Find the derivative of $\ln(y)=3x^2+5x+7$

Notice there are two different ways of doing this problem. First, we could simply solve for y and then take the derivative.

$$\ln(y)=3x^2+5x+7$$

$$y=e^{3x^2+5x+7} \qquad \text{Now take the derivative}$$

$$\frac{dy}{dx}=\left(e^{3x^2+5x+7}\right)(6x+5)$$

Implicit Differentiation allows us the luxury of taking the derivative without the first algebra step because of the chain rule

$$\frac{d}{dx}\left[\ln(y)=3x^2+5x+7\right]$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right)=6x+5$$

Notice when we took the derivative, we had to use the chain rule;

$$\frac{d}{dx}[y]=\frac{dy}{dx}$$

$$\frac{dy}{dx}=y(6x+5)$$

You might not immediately recognize that the two answers are the same, but since $y = e^{3x^2+5x+7}$, a simple substitution can show you that they are actually the same.

In terms of functions, this may not be very interesting or important, because it is often simple to isolate y. But consider a non-function, like this circle.

Ex 2 Find if $x^2 + y^2 = 25$

$$\frac{d}{dx}\left[x^2 + y^2 = 25\right] \to \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$
$$2x + 2y\frac{dy}{dx} = 0$$

We can now isolate $\frac{dy}{dx}$

$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

But even with this function, we could have solved for *y* and then found $\frac{dy}{dx}$.

$$x^2 + y^2 = 25$$
$$y^2 = 25 - x^2$$
$$y = \pm\sqrt{25 - x^2}$$
$$\frac{dy}{dx} = \frac{-x}{\pm\sqrt{25 - x^2}}$$

Notice that this is the exact same answer as we found with implicit differentiation. You could substitute *y* for $\pm\sqrt{25 - x^2}$ in the denominator and come up with the same derivative, $\frac{dy}{dx} = -\frac{x}{y}$.

The other thing that you may notice is that this is the differential equation we solved last chapter – and the solution to the differential was a circle.

Ex 3 Find $\frac{dy}{dx}$ for the hyperbola $x^2 - 3xy + 3y^2 = 2$

It would be very difficult to solve for *y* here, so implicit differentiation is really our only option.

$$\frac{d}{dx}\left[x^2 - 3xy + 3y^2 = 2\right]$$
$$2x - 3x\frac{dy}{dx} - 3y + 6y\frac{dy}{dx} = 0$$
$$2x - 3y = 3x\frac{dy}{dx} - 6y\frac{dy}{dx}$$
$$2x - 3y = (3x - 6y)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x - 3y}{3x - 6y}$$

Of course if we want to find a second derivative, we can use implicit differentiation a second time.

Ex 4 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the hyperbola $x^2 - 3y^2 + 4x - 12y - 2 = 0$

$$\frac{d}{dx}\left[x^2 - 3y^2 + 4x - 12y - 2 = 0\right]$$
$$2x - 6y\frac{dy}{dx} + 4 - 12\frac{dy}{dx} = 0$$
$$2x + 4 = 6y\frac{dy}{dx} + 12\frac{dy}{dx}$$
$$2x + 4 = (6y + 12)\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x + 4}{6y + 12}$$
$$\frac{dy}{dx} = \frac{x + 2}{3y + 6}$$

Now we just take the derivative again to find $\frac{d^2y}{dx^2}$.

$$\frac{d}{dx}\left[\frac{dy}{dx}=\frac{x+2}{3y+6}\right]$$

$$\frac{d^2y}{dx^2}=\frac{(3y+6)-(x+2)3\frac{dy}{dx}}{(3y+6)^2}$$

Since we already know $\frac{dy}{dx}$, we can substitute

$$\frac{d^2y}{dx^2}=\frac{(3y+6)-(x+2)(3)\left(\frac{x+2}{3y+6}\right)}{(3y+6)^2}$$

$$=\frac{(3y+6)^2-3(x+2)^2}{(3y+6)^3}$$

Ex 5 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for $\sin(y)=2\cos(3x)$

$$\frac{d}{dx}\left[\sin(y)=2\cos(3x)\right]$$

$$\cos(y)\frac{dy}{dx}=-6\sin(3x)$$

$$\frac{dy}{dx}=\frac{-6\sin(3x)}{\cos(y)}$$

$$\frac{d^2y}{dx^2}=\frac{-18\cos(y)\cos(3x)-6\sin(3x)\sin(y)\left(\frac{dy}{dx}\right)}{\cos^2(y)}$$

$$\frac{d^2y}{dx^2}=\frac{-18\cos^2(y)\cos(3x)+36\sin^2(3x)\sin(y)}{\cos^3(y)}$$

So $\frac{dy}{dx}=\frac{-6\sin(3x)}{\cos(y)}$ and $\frac{d^2y}{dx^2}=\frac{-18\cos^2(y)\cos(3x)+36\sin^2(3x)\sin(y)}{\cos^3(y)}$

Be Careful! There is a lot of algebraic simplification that happens in these problems, and it is easy to make mistakes. Take your time with the simplifications so that you don't make careless mistakes.

Another issue that arises is the need to use both the Product Rule and the Quotient Rule. Make sure you look for these when you are working through a problem.

Ex 6 Find $\frac{dy}{dx}$ for $e^{x^2}+xy^2-16=\frac{\tan y}{3x}$

$$\frac{d}{dx}\left[e^{x^2}+xy^2-16=\frac{\tan y}{3x}\right]$$

$$2xe^{x^2}+2xy\frac{dy}{dx}+y^2=\frac{3x\sec^2 y\frac{dy}{dx}-3\tan x}{9x^2}$$

$$2xe^{x^2}+2xy\frac{dy}{dx}+y^2=\frac{x\sec^2 y\frac{dy}{dx}-\tan x}{3x^2}$$

$$6x^3e^{x^2}+6x^3y\frac{dy}{dx}+3x^2y^2=x\sec^2 y\frac{dy}{dx}-\tan x$$

$$6x^3e^{x^2}+3x^2y^2+\tan x=x\sec^2 y\frac{dy}{dx}-6x^3y\frac{dy}{dx}$$

$$6x^3e^{x^2}+3x^2y^2+\tan x=\frac{dy}{dx}\left(x\sec^2 y-6x^3y\right)$$

$$\frac{dy}{dx}=\frac{6x^3e^{x^2}+3x^2y^2+\tan x}{x\sec^2 y-6x^3y}$$

3.3 Homework Set A

Find $\frac{dy}{dx}$ for each of these equations, first by implicit differentiation, then by solving for *y* and differentiating. Show that $\frac{dy}{dx}$ is the same in both cases.

1. $xy + 2x + 3x^2 = 4$

2. $\frac{1}{x} + \frac{1}{y} = 1$

3. $\sqrt{x} + \sqrt{y} = 4$

Find $\frac{dy}{dx}$ for each of these equations by implicit differentiation.

4. $x^2 + y^2 = 1$

5. $x^3 + 10x^2 y + 7y^2 = 60$

6. $x^2 y^2 + x\sin(y) = 4$

7. $4\cos(x)\sin(y) = 1$

8. $e^{x^2 y} = x + y$

9. $\tan(x - y) = \frac{y}{1+x^2}$

Find the equation of the line tangent to each of the following relations at the given point.

10. $x^2 - y^2 - 6y - 3 = 0$ at $\left(\sqrt{3}, 0\right)$

11. $9x^2 + 4y^2 + 36x - 8y + 4 = 0$ at $\left(0, -2\right)$

12. $12x^2 - 4y^2 + 72x + 16y + 44 = 0$ at $\left(-1, -3\right)$

13. Find the equation of the lines tangent and normal to $y - \frac{4}{\pi^2}x^2 = 2e^{y\sin x} + y^3 - 3$ through the point $\left(\frac{\pi}{2}, 0\right)$.

14. Find the equation of the line tangent to $x^3 + \frac{y}{x} + y^2 = 7$, through the point (1,2).

15. Find the equation of the line tangent to $x^2 + 3xy + y^2 = 8$, through the point (1,2).

16. Find $\frac{d^2y}{dx^2}$ if $xy + y^2 = 1$

17. Find $\frac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$

3.3 Homework Set B

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y^4+3xy=11$

2. Use implicit differentiation to find $\frac{dy}{dx}$ for $4xy^2-\sin(3x)=5\tan(y)+7$

3. Find $\frac{dy}{dx}$ for $5x^2+x^3y-\ln y=16$

4. Find $\frac{dy}{dx}$ for $\cos y+\sin x=\frac{x^2}{y^2}+16$

5. Find $\frac{dy}{dx}$ if $y^4+\frac{4}{y^2}+\ln(xy)=-12$

6. Find $\frac{dy}{dx}$ for $xy^2-y^4=3x+15$

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^2+y^2=16$

8. Find $\frac{dy}{dx}$ if $3x^3-e^{xy^2}-17=\cos y+x$

Answers: 3.3 Homework Set A

1. $xy+2x+3x^2=4$

Imp: $\dfrac{dy}{dx}=\dfrac{-6x-y-2}{x}$; Exp: $\dfrac{dy}{dx}=\dfrac{-4-3x^2}{x^2}$

2. $\dfrac{1}{x}+\dfrac{1}{y}=1$

Imp: $\dfrac{dy}{dx}=\dfrac{-y^2}{x^2}$; Exp: $\dfrac{dy}{dx}=\dfrac{-1}{(1-x)^2}$

3. $\sqrt{x}+\sqrt{y}=4$

Imp: $\dfrac{dy}{dx}=\dfrac{-y^{1/2}}{x^{1/2}}$; Exp: $\dfrac{dy}{dx}=\dfrac{-4+\sqrt{x}}{\sqrt{x}}$

4. $x^2+y^2=1$

$\dfrac{dy}{dx}=\dfrac{-x}{y}$

5. $x^3+10x^2y+7y^2=60$

$\dfrac{dy}{dx}=\dfrac{-3x^2-20xy}{10x^2+14}$

6. $x^2y^2+x\sin(y)=4$

$\dfrac{dy}{dx}=\dfrac{-\sin y-2xy^2}{2x^2y+x\cos y}$

7. $4\cos(x)\sin(y)=1$

$\dfrac{dy}{dx}=\tan(x)\tan(y)$

8. $e^{x^2y}=x+y$

$\dfrac{dy}{dx}=\dfrac{1-2xye^{x^2y}}{x^2e^{x^2y}-1}$

9. $\tan(x-y)=\dfrac{y}{1+x^2}$

$\dfrac{dy}{dx}=\dfrac{\sec(x-y)(1+x^2)^2+2xy}{1+x^2+\sec(x-y)(1+x^2)^2}$

2

10. $x^2-y^2-6y-3=0$ at $(\sqrt{3},0)$

$y=\dfrac{\sqrt{3}}{3}\left(x-\sqrt{3}\right)$

11. $9x^2+4y^2+36x-8y+4=0$ at $(0,-2)$

$y+2=\dfrac{2}{3}x$

12. $12x^2 - 4y^2 + 72x + 16y + 44 = 0$ at $(-1, -3)$

$y + 3 = \frac{-6}{5}(x+1)$

13. Find the equation of the lines tangent and normal to $y - \frac{4}{\pi^2}x^2 = 2e^{y\sin x} + y^3 - 3$ through the point $\left(\frac{\pi}{2}, 0\right)$.

Tangent: $y - 0 = -\frac{4}{\pi}\left(x - \frac{\pi}{2}\right)$ Normal: $y - 0 = \frac{\pi}{4}\left(x - \frac{\pi}{2}\right)$

14. Find the equation of the line tangent to $x^3 + \frac{y}{x} + y^2 = 7$, through the point (1,2).

$y - 2 = -\frac{10}{17}(x-1)$

15. Find the equation of the line tangent to $x^2 + 3xy + y^2 = 8$, through the point (1,2).

$y - 2 = -\frac{8}{7}(x-1)$

16. Find $\frac{d^2y}{dx^2}$ if $xy + y^2 = 1$

$\frac{d^2y}{dx^2} = \frac{2y(x+y)}{(x+2y)^3}$

17. Find $\frac{d^2y}{dx^2}$ if $4x^2 + 9y^2 = 36$

$\frac{d^2y}{dx^2} = \frac{-36y^2 - 16x^2}{81y^2}$

3.3 Homework Set B

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $y^4 + 3xy = 11$

$\frac{dy}{dx} = \frac{-3y}{4y^3 + 3x}$

$\frac{d^2y}{dx^2} = \frac{-36y^4 + 56xy}{(4y^3 + 3x)^3}$

2. Use implicit differentiation to find $\frac{dy}{dx}$ for $4xy^2 - \sin(3x) = 5\tan(y) + 7$

$\frac{dy}{dx} = \frac{3\cos(3x)}{8xy - 3\cos(3x)}$

3. Find $\frac{dy}{dx}$ for $5x^2+x^3y-\ln y=16$

$$\frac{dy}{dx}=\frac{-3x^2y^2-10xy}{x^3y+1}$$

4. Find $\frac{dy}{dx}$ for $\cos y+\sin x=\frac{x^2}{y^2}+16$

$$\frac{dy}{dx}=\frac{y^3\cos x-2xy}{y^3\sin y-2x^2}$$

5. Find $\frac{dy}{dx}$ if $y^4+\frac{4}{y^2}+\ln(xy)=-12$

$$\frac{dy}{dx}=\frac{-xy^3}{4xy^6-8x+xy^2}$$

6. Find $\frac{dy}{dx}$ for $xy^2-y^4=3x+15$

$$\frac{dy}{dx}=\frac{3-y^2}{2xy-4xy^3}$$

7. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ if $x^2+y^2=16$

$$\frac{dy}{dx}=-\frac{x}{y} \qquad \frac{d^2y}{dx^2}=\frac{-y^2+x^2}{y^3}$$

8. Find $\frac{dy}{dx}$ if $3x^3-e^{xy^2}-17=\cos y+x$

$$\frac{dy}{dx}=\frac{1-9x^2+e^{xy^2}y^2}{\sin y-2e^{xy^2}xy}$$

3.4: Related Rates

We have been looking at derivatives as rates of change for this chapter, and one of the principle ways that this comes up in Calculus is in the topic “related rates”. Hopefully, you recall this section from your Precalculus class – the topic in chapter 4 where everyone just skipped this problem on the test. Well, it is a very important topic for Calculus, so you don’t get to skip this topic anymore.

OBJECTIVE
Solve related rates problems.

As we saw in the previous section, parametric equations related an x and y to some other variable, t. This is very common in advanced mathematics; having some independent variable on which several other variables depend. In the case of related rates, the variable is time, and we seldom actually see that variable in the problem. This is very similar to both implicit differentiation in the previous section and motion in the section before that. With related rates, each of the variables given to us are actually in terms of another variable, t, (usually time). All of the variables are actually functions of time, but we never see t anywhere in the equations we use.

These problems are “rate” problems. This means that time is a factor, even though we never see it as a variable. When we see a rate with time in the denominator, it should be clear that we have a rate of change with respect to t (that is $\frac{d}{dt}$).

Suppose, for example, I am inflating a spherical balloon. It is obvious as you watch the balloon growing in size that the volume, radius, and surface area of the balloon are all increasing as you inflate the balloon. The problem arises in that I do not have an equation in terms of t for each of those variables. I do have an equation relating volume and radius, and another one relating surface area and radius.

$$V = \frac{4}{3}\pi r^3 \qquad\qquad S = 4\pi r^2$$

Since I know that both of the equations have time as a component (even though I do not see t as a variable). I could take the derivative of each **with respect to time.**

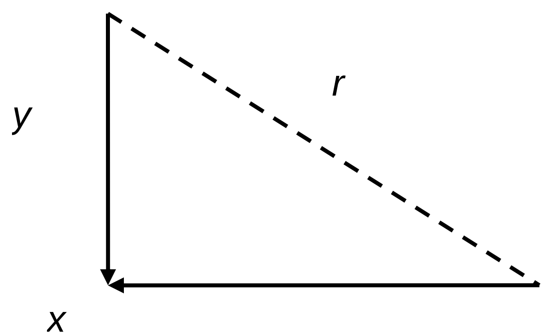
$$\frac{d}{dt}\left[V=\frac{4}{3}\pi r^3\right] \qquad\qquad \frac{d}{dt}\left[S=4\pi r^2\right]$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt} \qquad\qquad \frac{dS}{dt}=8\pi r\frac{dr}{dt}$$

Notice that we had to use implicit differentiation and The Chain Rule for all of the variables. Since each was defined in terms of t, when we took the derivative, The Chain Rule gave us the implicit fraction $\frac{d\text{ whatever}}{dt}$.

For example, the derivative of V with respect to t must be $\frac{dV}{dt}$, and the derivative of r^3 with respect to t must be $3r^2\frac{dr}{dt}$.

Ex 1 Two cars approach an intersection, one traveling south at 20 mph and the other traveling west at 30 mph. How fast is the direct distance between them decreasing when the westbound car is 0.6 miles and the southbound car is 0.8 miles from the intersection?



As we can see in the picture, the distance between the two cars are related by the Pythagorean Theorem.

$$x^2+y^2=r^2$$

We know several pieces of information. The southbound car is moving at 20 mph; i.e. $\frac{dy}{dt} = -20$. By similar logic we can deduce each of the following:

$$\frac{dy}{dt} = -20 \qquad \frac{dx}{dt} = -30$$
$$y = 0.8 \qquad x = 0.6$$

And, by the Pythagorean Theorem, $r = 1.0$

It is very important to notice that even though we solved for r using the Pythagorean Theorem, r is still a variable (the distance between the two cars is actually changing).

Now we take the derivative of the Pythagorean Theorem and get

$$\frac{d}{dt}\left[x^2 + y^2 = r^2\right]$$
$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2r\frac{dr}{dt}$$

This is essentially an equation in six variables. But we know five of those six variables, so just substitute and solve.

$$2(0.8)(-30) + 2(0.6)(-20) = 2(1.0)\frac{dr}{dt}$$

$$\frac{dr}{dt} = -36 \text{ miles/hour}$$

It should make sense that $\frac{dr}{dt}$ is negative since the two cars are approaching one another. We know the units based on the fraction, $\frac{dr}{dt}$. Since r was in miles and t was in hours, our final units must be miles/hour.

Ex 2 Sand is dumped onto a pile at 30 ft^3/min. The pile forms a cone with a height equal to the base diameter. How fast is the base area changing when the pile is 10 feet high?

The units on the 30 tell us that it is the change in volume, or $\frac{dV}{dt}$. We know that the volume of a cone is $V = \frac{1}{3}\pi r^2 h$. But this equation has too many variables for us to differentiate it as it stands, and we are looking for $\frac{dA}{dt}$, or how fast the base area is changing.

We should start by organizing what exactly we know, and what we want. It is also often helpful (but not always necessary) to draw a picture to help us identify what we know.

<u>What we know:</u>

$\frac{dV}{dt} = 30$

$h = d$
(the problem said height was the same as diameter

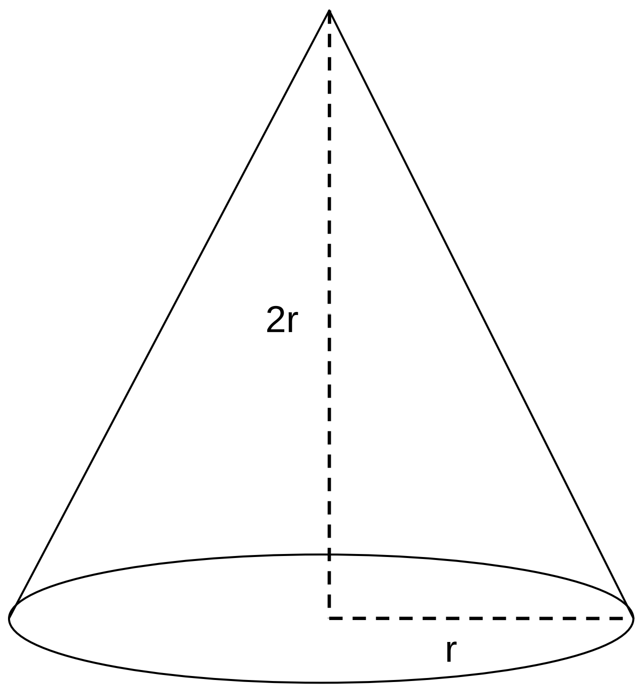
$V = \frac{1}{3}\pi r^2 h$

$A_{\text{base}} = \pi r^2$
(since the base of a cone is a circle)

<u>What I need to find:</u>

$\frac{dA}{dt}$

when h = 10 feet



$$V = \frac{1}{3}\pi r^2 (2r) \text{ and } A = \pi r^2$$

At this point, you could differentiate both equations and substitute to solve for $\frac{dA}{dt}$, or you could substitute A into the volume equation and differentiate to find $\frac{dA}{dt}$. We will do the latter.

$$V = \frac{2}{3}\pi\left(\sqrt{\frac{A}{\pi}}\right)^3$$

$$V = \frac{2}{3\sqrt{\pi}} A^{\frac{3}{2}}$$

$$\frac{dV}{dt} = \frac{1}{\sqrt{\pi}} A^{\frac{1}{2}} \frac{dA}{dt}$$

Since we know that $\frac{dV}{dt} = 30 \ {ft^3}/{\text{min}}.$ and $A = 25\pi \ ft^2$ (because r = 5 feet),

$$\frac{dA}{dt} = 6 \ {ft.^2}/{\text{min}}.$$

Obviously, remembering formulas from algebra and geometry classes is really important, and we will expect that you have certain formulas committed to memory. Here is a short list that you should know.

Common Formulas for Related Rates Problems

Pythagorean Theorem:

$x^2 + y^2 = r^2$

Area Formulas:

Circle $A = \pi r^2$

Rectangle $A = lw$

Trapezoid $A = \frac{1}{2}h\left(b_1 + b_2\right)$

Volume Formulas:

Sphere $V = \frac{4}{3}\pi r^3$

Right Prism $V = Bh$

Cylinder $V = \pi r^2 h$

Cone $V = \frac{1}{3}\pi r^2 h$

Right Pyramid $V = \frac{1}{3}Bh$

Surface Area Formulas:

Sphere $S = 4\pi r^2$

Cylinder $S = 2\pi r^2 + 2\pi rl$

Cone $S = \pi r^2 + \pi rl$

Right Prism $S = 2B + Ph$

Ex 3 If two resistors are connected in parallel, then the total resistance is given by the formula $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$, where all values for R are in Ohms (Ω). If R_1 and R_2 are increasing at rates of $0.3\ \Omega/\text{sec}$ and $0.2\ \Omega/\text{sec}$, respectively, find how fast R is changing when $R_1 = 80\ \Omega$ and $R_2 = 100\ \Omega$.

$$\frac{d}{dt}\left[\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}\right]$$

$$-R^{-2}\left(\frac{dR}{dt}\right)=-R_1^{-2}\left(\frac{dR_1}{dt}\right)-R_2^{-2}\left(\frac{dR_2}{dt}\right)$$

$$-\left(\frac{400}{9}\right)^{-2}\left(\frac{dR}{dt}\right)=-(80)^{-2}(0.3)-(100)^{-2}(0.2)$$

$$\frac{dR}{dt}=\frac{107}{810}\text{ or }0.132\ \Omega/\text{sec}$$

3.4 Homework Set A

1. If V is the volume of a cube with edge length, s, and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{ds}{dt}$.

2. A particle is moving along the curve $y = \sqrt{1+x^3}$. As it reaches the point (2,3), the *y*-coordinate is increasing at a rate of 4 cm/sec. How fast is the *x*-coordinate changing at that moment?

3. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph flies directly over a radar station. Find the rate at which the horizontal distance is increasing when it is 2 miles from the station. Find the rate at which the distance between the station and the plane is increasing when the plane is 2 miles from the station.

4. If a snowball melts so that its surface area is decreasing at a rate of 1 cm^2/min, find the rate at which the diameter is decreasing when it has a diameter of 10 cm.

5. A street light is mounted at the top of a 15 foot tall pole. A 6 foot tall man walks away from the pole at a speed of 5 ft/sec in a straight line. How fast is the tip of his shadow moving when he is 40 feet from the pole?

6. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

7. The altitude of a triangle is increasing at a rate of 1 cm/min, while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?

3.4 Homework Set B

1. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min. If the tank is 8 m tall and has a diameter of 4 m. Find the rate at which the height is decreasing when the water level is at 3 m. Then find the rate of change of the radius at that same instant.

2. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min at the same time that water is being pumped in at an unknown constant rate. If the tank is 8 m tall and has a diameter of 4 m, and when the water level is 6 m the height of the water is increasing at 20 cm/min, find the rate at which the water is being added.

3. A man starts walking north at 5 feet/sec from a point P. 3 minutes later a woman starts walking east at a rate of 4 feet/sec from a point 500 feet east of point P. At what rate are the two moving apart at 12 minutes after the woman starts walking.

4. A particle moves along the curve $x^2 + xy + y = 17$. When $y = 2$, $\frac{dx}{dt} = 10$, find all values of $\frac{dy}{dt}$.

5. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm^2/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm^2?

6. The total resistance of a certain circuit is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$. If R_1 is increasing at a rate of $0.4 \,\Omega/\text{sec}$ and R_2 is decreasing at a rate of $0.2 \,\Omega/\text{sec}$, how fast is R changing when $R_1 = 20\ \Omega$ and $R_2 = 50\ \Omega$?

Answers: 3.4 Homework Set A

1. If V is the volume of a cube with edge length, s, and the cube expands as time passes, find $\frac{dV}{dt}$ in terms of $\frac{ds}{dt}$.

   $\frac{dV}{dt} = 3s^2 \frac{ds}{dt}$

2. A particle is moving along the curve $y = \sqrt{1+x^3}$. As it reaches the point (2,3), the *y*-coordinate is increasing at a rate of 4 cm/sec. How fast is the *x*-coordinate changing at that moment? 2 cm/sec

3. A plane flying horizontally at an altitude of 1 mile and a speed of 500 mph flies directly over a radar station. Find the rate at which the horizontal distance is increasing when it is 2 miles from the station. Find the rate at which the distance between the station and the plane is increasing when the plane is 2 miles from the station.

   $250\sqrt{3}$ mph

4. If a snowball melts so that its surface area is decreasing at a rate of 1 cm^2/min, find the rate at which the diameter is decreasing when it has a diameter of 10 cm.

   $\frac{-1}{20\pi}$ cm/min

5. A street light is mounted at the top of a 15 foot tall pole. A 6 foot tall man walks away from the pole at a speed of 5 ft/sec in a straight line. How fast is the tip of his shadow moving when he is 40 feet from the pole?

   $\frac{25}{3}$ ft/min

6. Two cars start moving away from the same point. One travels south at 60 mph, and the other travels west at 25 mph. At what rate is the distance between the cars increasing two hours later?

   65 mph

7. The altitude of a triangle is increasing at a rate of 1 cm/min, while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle increasing when the altitude is 10 cm and the area is 100 cm^2?

   –1.6 cm/min

3.4 Homework Set B

1. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min. If the tank is 8 m tall and has a diameter of 4 m. Find the rate at which the height is decreasing when the water level is at 3 m. Then find the rate of change of the radius at that same instant.

   $\frac{dh}{dt} = \frac{80{,}000}{27\pi}$ cm/min $\quad\quad$ $\frac{dr}{dt} = \frac{20{,}000}{27\pi}$ cm/min

2. Water is leaking out of an inverted conical tank at a rate of 5000 cm^3/min at the same time that water is being pumped in at an unknown constant rate. If the tank is 8 m tall and has a diameter of 4 m, and when the water level is 6 m the height of the water is increasing at 20 cm/min, find the rate at which the water is being added.

   $\frac{dV}{dt} = 540\pi + 5{,}000$ cm/min

3. A man starts walking north at 5 feet/sec from a point P. 3 minutes later a woman starts walking east at a rate of 4 feet/sec from a point 500 feet east of point P. At what rate are the two moving apart at 12 minutes after the woman starts walking?

   4.641 feet/sec

4. A particle moves along the curve $x^2 + xy + y = 17$. When $y = 2$, $\frac{dx}{dt} = 10$, find all values of $\frac{dy}{dt}$.

   $\frac{dy}{dt} = -20, -22.5$

5. The altitude of a triangle is increasing at a rate of 2 cm/sec at the same time that the area of the triangle is increasing at a rate of 5 cm^2/sec. At what rate is the base increasing when the altitude is 12 cm and the area is 144 cm^2?

   $\frac{db}{dt} = -\frac{19}{6}$ cm/sec

6. The total resistance of a certain circuit is given by $\frac{1}{R}=\frac{1}{R_1}+\frac{1}{R_2}$. If R_1 is increasing at a rate of $0.4\,\Omega/\text{sec}$ and R_2 is decreasing at a rate of $0.2\,\Omega/\text{sec}$, how fast is R changing when $R_1 = 20\ \Omega$ and $R_2 = 50\ \Omega$?

$$\frac{dR}{dt}=0.188\ \Omega/\text{sec}$$

3.5 The First and Second Derivative Tests

OBJECTIVE

Use the 1st and 2nd derivative tests to find maxima and minima.

While the first derivative is what allows us to algebraically find extremes, and BOTH derivative tests allow us to interpret critical values as maxima or minima. Since the sign pattern of the first derivative tells us when that function is increasing or decreasing, we can figure out if a critical value is associated with a maximum or minimum depending on the sign change of the derivative.

The 1st Derivative Test

As the sign pattern of the 1st derivative is viewed left to right, the critical value represents a
1) relative maximum if the sign changes from + to –
2) relative minimum if the sign changes from – to +
or 3) neither a max. or min. if the sign does not change.

Ex 1 Apply the first derivative test to the function $y = 5x^4 - 10x^2$

$$\frac{dy}{dx} = 20x^3 - 20x = 0$$
$$20x\left(x^2 - 1\right) = 0$$
$$x = -1, 0, 1$$

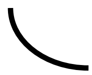
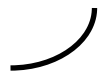
$$\begin{array}{c|ccccccc} y' & - & 0 & + & 0 & - & 0 & + \\ x & & -1 & & 0 & & 1 & \end{array}$$

So there are critical values at x values of –1, 0, and 1. When we look at the sign pattern, we can see we have a minimum at –1, a maximum at 0, and another minimum at 1.

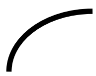
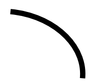
The second derivative can tell us about the concavity of a curve, in essence which way the curve is "bending":

If $f''(x) > 0$, then $f(x)$ is concave up. If $f''(x) < 0$, then $f(x)$ is concave down.

"Concave up" means the curve is turning in an upwards direction:

Looking either like this:  Or this: 

"Concave down" means the curve is turning in an downwards direction:

Looking either like this:  Or this: 

(Note that we would need the first derivative to tell if a section of a curve was increasing or decreasing, the second derivative only tells us which way the curve "bends")

If you have a critical value on a section of curve that is concave up, the critical value has to give a minimum value for the function (because of the shape of the curve). Similarly, if you have a critical value on a section of a curve that is concave down, the critical value must give a maximum value for the function (again, because of the shape of the curve). This can be generalized into what is called "The Second Derivative Test" – a test to find maximum or minimum values using the second derivative (essentially, using a function's concavity).

The 2nd Derivative Test

For a function *f*,

1) If $f'(c) = 0$ and $f''(c) > 0$, then *f* has a relative minimum at *c*.
2) If $f'(c) = 0$ and $f''(c) < 0$, then *f* has a relative maximum at *c*.

Ex 2 Use the 2nd Derivative Test to determine if the critical values of $g(t) = 27t - t^3$ are at a maximum or minimum value

$g'(t) = 27 - 3t^2 = 0$
$t = \pm 3$

So –3 and 3 are critical values.

$$g''(t) = -6t$$
$$g''(-3) = -6(-3) = 18$$
$$g''(3) = -6(3) = -18$$

Therefore, g has a minimum value at t = –3 and has a maximum value at t = 3. (Note that the numerical value "18" is irrelevant.)

Ex 3 Find the maximum values for the function $g(x) = \sqrt{x^3 - 9x}$

Domain: $x^3 - 9x \geq 0$

$$\begin{array}{c|ccccccc} y & - & 0 & + & 0 & - & 0 & + \\ \hline x & & -3 & & 0 & & 3 & \end{array}$$

$$x^3 - 9x \geq 0 \rightarrow x \in [-3,0] \cup [3,\infty)$$

$g'(x) = \dfrac{3x^2 - 9}{2\sqrt{x^3 - 9x}}$
$g'(x) = 0$ when $3x^2 - 9 = 0$, so $x = \pm\sqrt{3}$
$g'(x)$ does not exist when $x^3 - 9x = 0$, so $x = \pm 3, 0$

$$\begin{array}{c|ccccccccc} y' & & dne & + & 0 & - & dne & & dne & + \\ \hline x & & -3 & & -\sqrt{3} & & 0 & \sqrt{3} & 3 & \end{array}$$

Since $x = \sqrt{3}$ is not in the domain of the function, our critical values are at $x = \pm 3, 0, -\sqrt{3}$. From our sign pattern, we can conclude that at $x = -\sqrt{3}$, we have a maximum value, and by substituting this value into the function, we find that the maximum value is at $y = 3.224$.

3.5 Homework

Find the absolute maximum and minimum values of *f* on the given intervals.

1. $f(x) = \frac{x}{x^2+1},\ [0,2]$

2. $f(t) = \sqrt[3]{t}(8-t),\ [0,8]$

3. $f(z) = ze^{-z},\ [0,2]$

4. $f(x) = \frac{\ln(x)}{x},\ [1,3]$

5. $f(x) = e^{-x} - e^{-2x},\ [0,1]$

For each of the following functions, apply the 1^{st} Derivative Test, then verify the results with the 2^{nd} Derivative Test.

6. $f(x) = x^2 \ln x$

7. $h(f) = f^3 - 12f + 21$

8. $f(x) = xe^{-x^2}$

Answers: 3.6 Homework

1. $f(x)=\dfrac{x}{x^2+1},\ [0,2]$

   $f(0)=0 \therefore abs.\min.,\ f(1)=\frac{1}{2} \therefore abs.\max.$

2. $f(t)=\sqrt[3]{t}(8-t),\ [0,8]$

   $f(0)=0 \therefore abs.\min.,\ f(2)=7.560 \leftarrow \therefore abs.\max.,\ f(8)=0 \therefore abs.\min.$

3. $f(z)=ze^{-z},\ [0,2]$

   $f(0)=0 \therefore abs.\min.,\ f(1)=\frac{1}{e} \therefore abs.\max.$

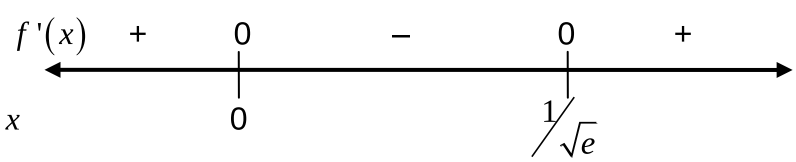
4. $f(x)=\dfrac{\ln(x)}{x},\ [1,3]$

   $f(1)=0 \therefore abs.\min.,\ f(e)=\frac{1}{e} \therefore abs.\max.$

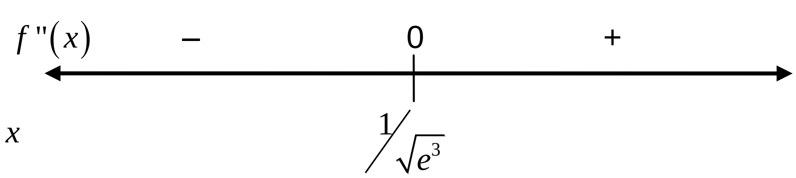
5. $f(x)=e^{-x}-e^{-2x},\ [0,1]$

   $f(0)=0 \therefore abs.\min.,\ f(\ln 2)=.25 \therefore abs.\max.$

6. $f(x)=x^2\ln x$



1st D_x Test:
Min: $x=\frac{1}{\sqrt{e}}$
Max: $x=0$



2nd D_x Test:
Min: $x=\frac{1}{\sqrt{e}}$
Max: $x=0$

7. $h(f)=f^3-12f+21$

$h'(f)$	+	0	–	0	+
x		$-\sqrt{12}$		$\sqrt{12}$	

1st D_x Test:
Min: $x=\sqrt{12}$
Max: $x=-\sqrt{12}$

$h''(f)$	–	0	+
x		0	

2nd D_x Test:
Min: $x=\sqrt{12}$
Max: $x=-\sqrt{12}$

8. $f(x)=xe^{-x^2}$

$f'(x)$	–	0	+	0	–
x		$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{1}{2}}$	

1st D_x Test:
Min: $x=-\sqrt{\frac{1}{2}}$
Max: $x=\sqrt{\frac{1}{2}}$

$f''(x)$	–	0	+	0	–	0	+
x		$-\sqrt{\frac{3}{2}}$		0		$\sqrt{\frac{3}{2}}$	

2nd D_x Test:
Min: $x=-\sqrt{\frac{1}{2}}$
Max: $x=\sqrt{\frac{1}{2}}$

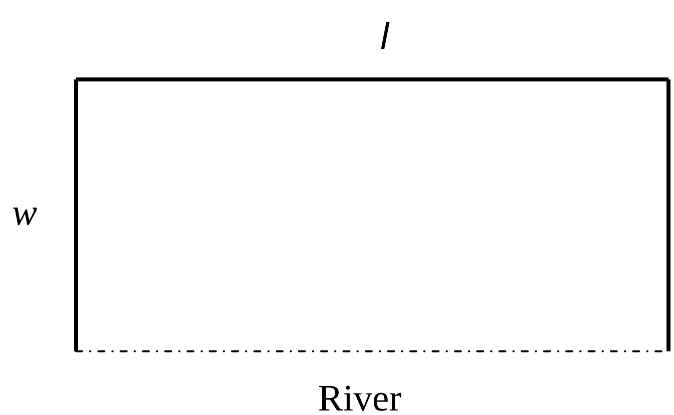
3.6: Optimization

In the last section, we looked at extrema and the derivative tests. Optimization is a practical application of finding maxima and minima for functions. As I mentioned before, this revolutionized thinking and is a critical component of all industries. You might remember this topic from chapter 2 of book 2 last year; the word problems many of you avoided on the test last year. This year, they form a much more fundamental part of what we need to be able to do, so we can no longer simply skip these problems on tests.

OBJECTIVES
Solve optimization problems.

Every optimization problem looks a bit different, but they all follow a similar progression. You must first identify your variables and any formula you need. Use algebra to eliminate variables, and **take the derivative of the function you are trying to optimize**. This is the most common mistake in optimization problems; taking the derivative of the wrong function.

Ex 1 The owner of the Rancho Grande has 3000 yards of fencing material with which to enclose a rectangular piece of grazing land along the straight portion of a river. If fencing is not required along the river, what are the dimensions of the largest area he can enclose? What is the area?



$A = lw$ The problem with maximizing this area formula lies in the fact that we have two independent variables (l and w). We need the fact about perimeter to complete the problem.

$$P = l + 2w$$
$$3000 = l + 2w$$
$$3000 - 2w = l$$
$$A = (3000 - 2w)w$$
$$A = 3000w - 2w^2$$

Now, since we have an equation with one independent variable, we can take the derivative easily.

$$\frac{dA}{dw} = 3000 - 4w$$
$$\frac{dA}{dw} = 3000 - 4w = 0$$
$$w = 750$$
$$l = 1500$$

So we would want a width of 750 yards and a length of 1500 yards. This would give us an area of 1,125,000 yards^2.

Ex 2 A cylindrical cola can has a volume $32\pi \text{ in}^3$. What is the minimum surface area?

$$V = \pi r^2 h$$
$$32\pi = \pi r^2 h$$
$$\frac{32}{r^2} = h$$

$$S = 2\pi r^2 + 2\pi r h$$
$$S = 2\pi r^2 + \pi r\left(\frac{64}{r^2}\right)$$
$$S = 2\pi r^2 + 64\pi r^{-1}$$
$$\frac{dS}{dr} = 4\pi r - 64\pi r^{-2} = 0$$
$$4\pi r\left(1 - \frac{16\pi}{r^3}\right) = 0$$
$$r = 0 \text{ or } 3.691$$

There is an implied domain here. You cannot have a radius of 0 inches, so 3.691 inches is the radius for the minimum area. The sign pattern verifies this:

$$\begin{array}{c|cccc} {}^{dA}\!/_{dr} & 0 & - & 0 & + \\ \hline r & 0 & & 3.691 & \end{array}$$

So the minimum surface area would be

$$S = 2\pi(3.691)^2 + 2\pi(3.691)\left(\frac{32}{(3.691)^2}\right) = 140.058 \text{ in}^2$$

Ex 3 Find the point on the curve $y = \dfrac{e^{-x^2}}{2}$ that is closest to the origin.

We want to minimize the distance to the origin, so we will be using the Pythagorean theorem to find the distance.

$$D = \sqrt{x^2 + y^2}$$
$$y = \frac{e^{-x^2}}{2}$$
$$D = \sqrt{x^2 + \left(\frac{e^{-x^2}}{2}\right)^2}$$
$$\frac{dD}{dx} = \frac{1}{2}\left(x^2 + \frac{e^{-2x^2}}{4}\right)^{-1/2}\left(2x - xe^{-2x^2}\right) = 0$$
$$x = 0,\ \pm .841$$

So the minimum distance from the origin is at the point (.377, .434).

Given the diverse nature of optimization problems, it is helpful to remember all the formulas from geometry.

3.6 Homework

Solve these problems algebraically.

1. Find two positive numbers whose product is 110 and whose sum is a minimum.

2. Find a positive number such that the sum of the number and its reciprocal is a minimum.

3. A farmer with 750 feet of fencing material wants to enclose a rectangular area and divide it into four smaller rectangular pens with sides parallel to one side of the rectangle. What is the largest possible total area?

4. If 1200 cm^2 of material is available to make a box with an open top and a square base, find the maximum volume the box can contain.

5. Find the point on the line $y = 4x + 7$ that is closest to the origin.

6. Find the points on the curve $y = \frac{1}{x^2 + 1}$ that are closest to the origin.

7. Find the area of the largest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

8. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, while the other is bent into an equilateral triangle. Find where the wire should be cut to maximize the area enclosed, then find where the wire should be cut to minimize the area enclosed.

9. You need to enclose 500 cm^3 of fluid in a cylinder. If the material you are using costs 0.001 dollars per cm^3, find the size of the cylinder that minimizes the cost. If the product that you are containing is a sports drink, do you think that the size that minimizes cost is the most efficient size? Explain.

10. The height of a man jumping off of a high dive is given by the function $h(t) = -4.9t^2 + 2t + 10$ on the domain $0 \le t \le 1.64715$. Find the absolute maximum and minimum heights modeled by this function.

11. Given that the area of a triangle can be calculated with the formula $A = \frac{1}{2}ab\sin\theta$, what value of θ will maximize the area of a triangle (given that a and b are constants)?

12. You operate a tour service that offers the following rates for the tours: \$200 per person if the minimum number of people book the tour (50 people is the minimum) and for each person past 50, up to a maximum of 80 people, the cost per person is decreased by \$2.

    It costs you \$6000 to operate the tour plus \$32 per person.

    a) Write a function that represents cost, $C(x)$.
    b) Write a function that represents revenue, $R(x)$.
    c) Given that profit can be represented as $P(x) = R(x) – C(x)$, write a function that represents profit and state the domain for the function.
    d) Find the number of people that maximizes your profit. What is the maximum profit?

Answers: 3.6 Homework

1. Find two positive numbers whose product is 110 and whose sum is a minimum.
   $x = y = \sqrt{110}$

2. Find a positive number such that the sum of the number and its reciprocal is a minimum.
   $x = 1$

3. A farmer with 750 feet of fencing material wants to enclose a rectangular area and divide it into four smaller rectangular pens with sides parallel to one side of the rectangle. What is the largest possible total area?
   $18000\,ft^2$

4. If 1200 cm^2 of material is available to make a box with an open top and a square base, find the maximum volume the box can contain.
   $5700cm^3$

5. Find the point on the line $y = 4x + 7$ that is closest to the origin.
   $x = \frac{-28}{17},\ y = \frac{7}{17}$

6. Find the points on the curve $y = \frac{1}{x^2 + 1}$ that are closest to the origin.
   $(\pm.510, .794)$

7. Find the area of the largest rectangle that can be inscribed in the ellipse
   $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
   Max. Area = ab

8. A piece of wire 10 m long is cut into two pieces. One piece is bent into a square, while the other is bent into an equilateral triangle. Find where the wire should be cut to maximize the area enclosed, then find where the wire should be cut to minimize the area enclosed.
   Maximum is to use all the wire for the square, min is x = 2.795cm for the square.

9. You need to enclose 500 cm^3 of fluid in a cylinder. If the material you are using costs 0.001 dollars per cm^3, find the size of the cylinder that minimizes the cost. If the product that you are containing is a sports drink, do you think that the size that minimizes cost is the most efficient size? Explain.
r = 4.301 cm, h = 8.603 cm. This seems to be an inefficient size for the purpose of holding it in one's hand: it is 8.603 cm tall (almost 3 and a half inches) and the same distance across – not a comfortable fit for most people's hands.

10. The height of a man jumping off of a high dive is given by the function $h(t) = -4.9t^2 + 2t + 10$ on the domain $0 \le t \le 1.64715$. Find the absolute maximum and minimum heights modeled by this function.
Absolute Maximum is at y = 10.204 meters
Absolute Minimum is at y = 0 meters

11. Given that the area of a triangle can be calculated with the formula $A = \frac{1}{2}ab\sin\theta$, what value of θ will maximize the area of a triangle (given that a and b are constants)?
Implied domain of $\theta \in (0, \pi)$
$\theta = \frac{\pi}{2}$

12. You operate a tour service that offers the following rates for the tours: $200 per person if the minimum number of people book the tour (50 people is the minimum) and for each person past 50, up to a maximum of 80 people, the cost per person is decreased by $2.
It costs you $6000 to operate the tour plus $32 per person.
a) Write a function that represents cost, $C(x)$.
b) Write a function that represents revenue, $R(x)$.
c) Given that profit can be represented as $P(x) = R(x) - C(x)$, write a function that represents profit and state the domain for the function.
d) Find the number of people that maximizes your profit. What is the maximum profit?
a) $C(x) = 6000 + 32x$
b) $R(x) = (200 - 2(x - 50))x$
c) $P(x) = (200 - 2(x - 50))x - 6000 - 32x$ for $x \in [50, 80]$
d) 80 people, for a maximum profit of $2,640

3.7: Graphing with Derivatives

OBJECTIVE

- Sketch the graph of a function using information from its first and/or second derivatives.
- Sketch the graph of a first and/or second derivative from the graph of a function.

All last year, we concerned ourselves with sketching graphs based on traits of a function. We tended to look at the one key aspect of the derivative – that is finding extremes – as it applied to a function. Toward the end of the year, we looked at the first and second derivatives as traits of the function. They gave a much wider range of information than specific details.

Remember:

Critical values representing extremes of a function occur when

i. $f'(x)=0$

ii. $f'(x)$ does not exist

or iii. at endpoints of an arbitrary domain.

If $f'(x)>0$, then $f(x)$ is increasing.

If $f'(x)<0$, then $f(x)$ is decreasing.

Critical values representing a Point of Inflection (POI of a function occur when

i. $f''(x)=0$

or ii. $f''(x)$ does not exist

If $f''(x)>0$, then $f(x)$ is concave up.

If $f''(x)<0$, then $f(x)$ is concave down.

Ex 1 Find the sign patterns of *y*, *y'*, and *y''* and sketch $y = xe^{2x}$

Zeros: $xe^{2x} = 0 \rightarrow x = 0$

$x = 0 \rightarrow y = 0$

$$\begin{array}{c|ccc} y & - & 0 & + \\ \hline x & & 0 & \end{array}$$

Extremes: $\dfrac{dy}{dx} = xe^{2x}(2) + e^{2x}(1)$

$$xe^{2x}(2) + e^{2x}(1) = 0$$

$$e^{2x}(2x+1) = 0$$

$$x = -\frac{1}{2}$$

$$y = -.184$$

$$\begin{array}{c|ccc} y' & - & 0 & + \\ \hline x & & -\frac{1}{2} & \end{array}$$

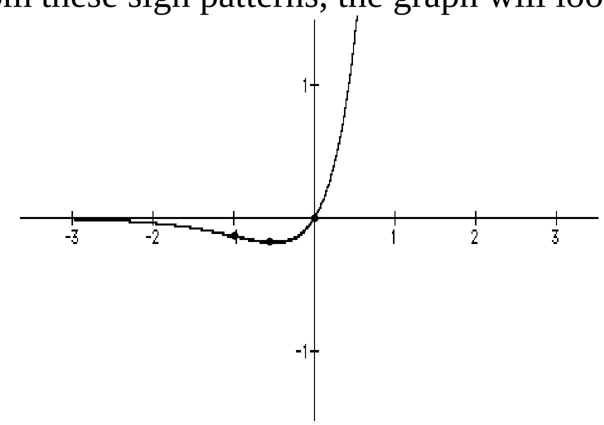
POI $\dfrac{d^2y}{dx^2} = e^{2x}(2) + (2x+1)\left(e^{2x}(2)\right)$

$$= e^{2x}(4x+4) = 0$$

$x = \text{-}1 \rightarrow y = \text{-}.135$

$$\begin{array}{c|ccc} y'' & - & 0 & + \\ \hline x & & -1 & \end{array}$$

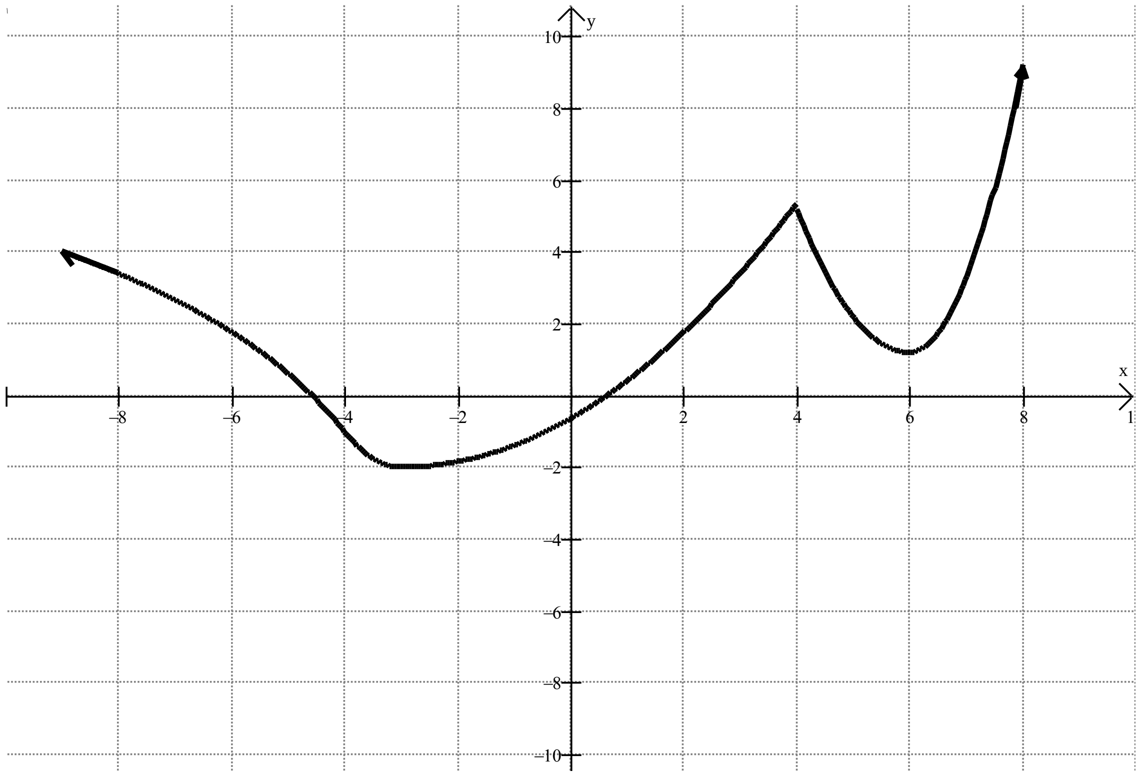
Putting together the points, increasing/decreasing and concavity that can be determined from these sign patterns, the graph will look something like this:



$y = xe^{2x}$

Ex 2 Sketch the function described as follows: decreasing from $(-\infty,-3)\cup(4,6)$, increasing from $(-3,4)\cup(6,\infty)$, concave down from $(-\infty,-4)$, concave up from $(-4,4)\cup(4,\infty)$.

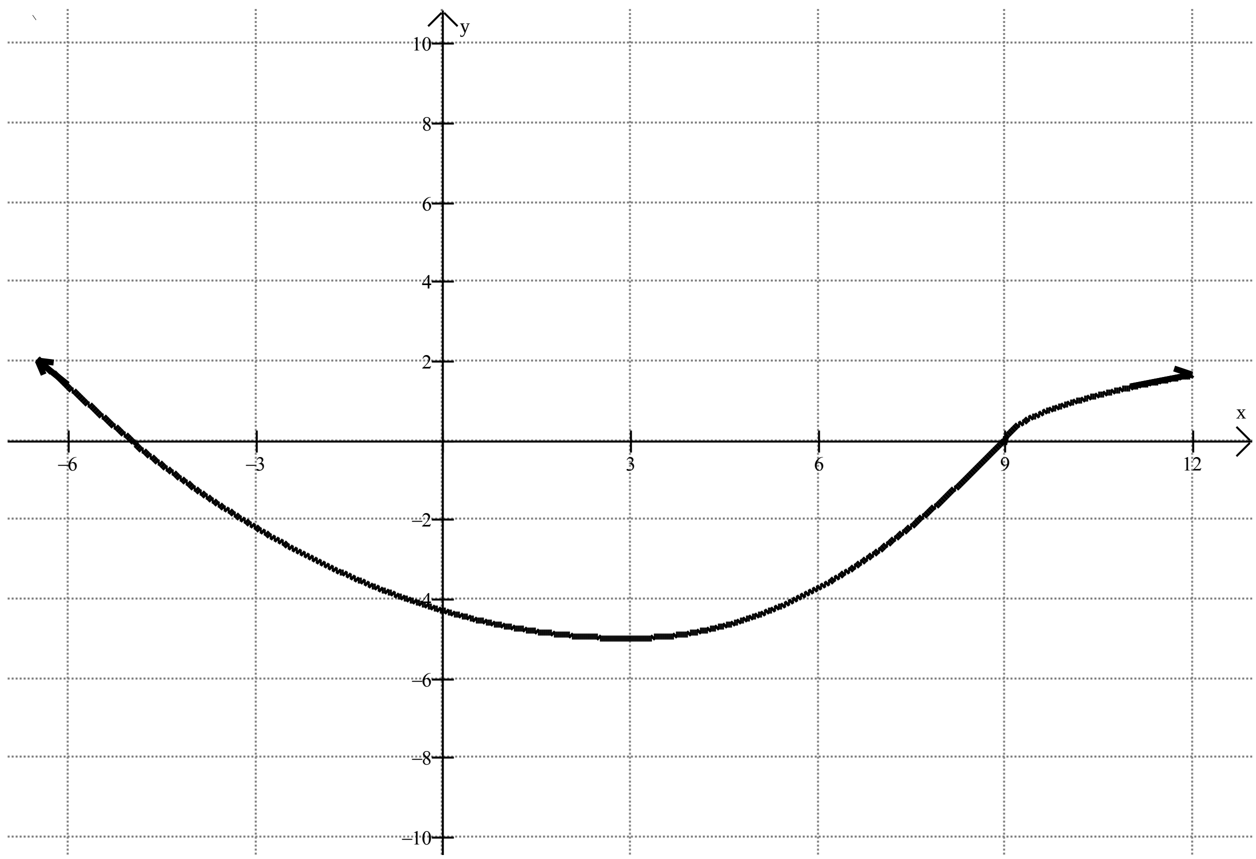
Note that this is only one possible answer. Since no *y*-values are given, the points could be at any height.



The trait and sign pattern information could be given in table form rather than in a description.

Ex 3 Sketch the graph of the function whose traits are given below.

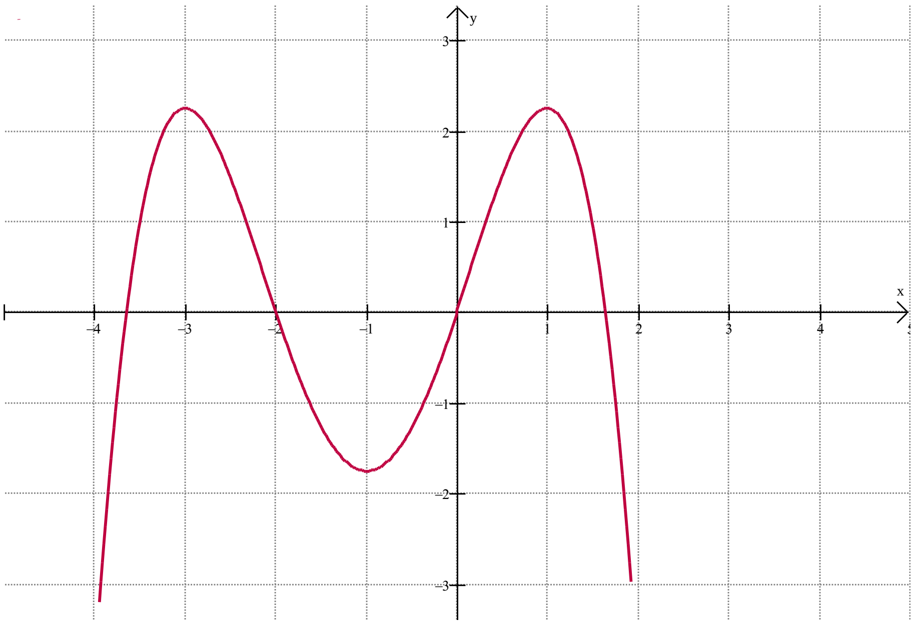
x	$f(x)$	$f'(x)$	$f''(x)$
$x < -5$	Positive	Negative	Positive
$x = -5$	0	Negative	Positive
$-5 < x < 3$	Negative	Negative	Positive
$x = 3$	-5	0	Positive
$3 < x < 9$	Negative	Positive	Positive
$x = 9$	0	Positive	0
$9 < x$	Positive	Positive	Negative



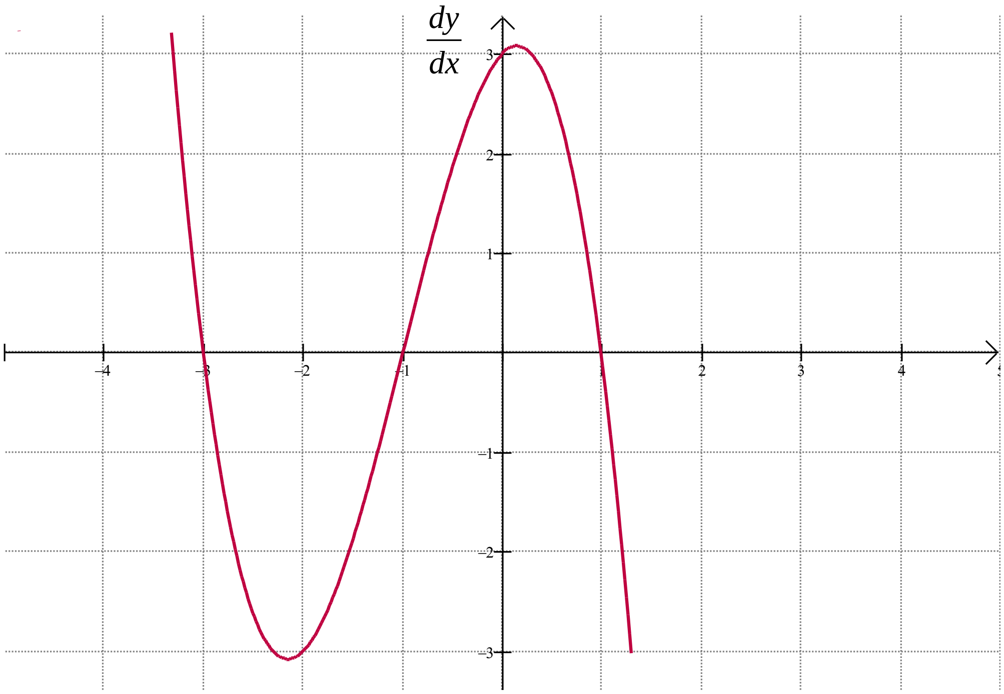
Notice that the point of inflection just happens to occur at a zero of the function. It is very possible that the traits can occur at the same point; you can conceive of functions that have maximums that are also points of inflection, minimums that are also zeroes, etc.

The derivatives also have graphs and we can discern what they look like from the information gleaned from the original graph.

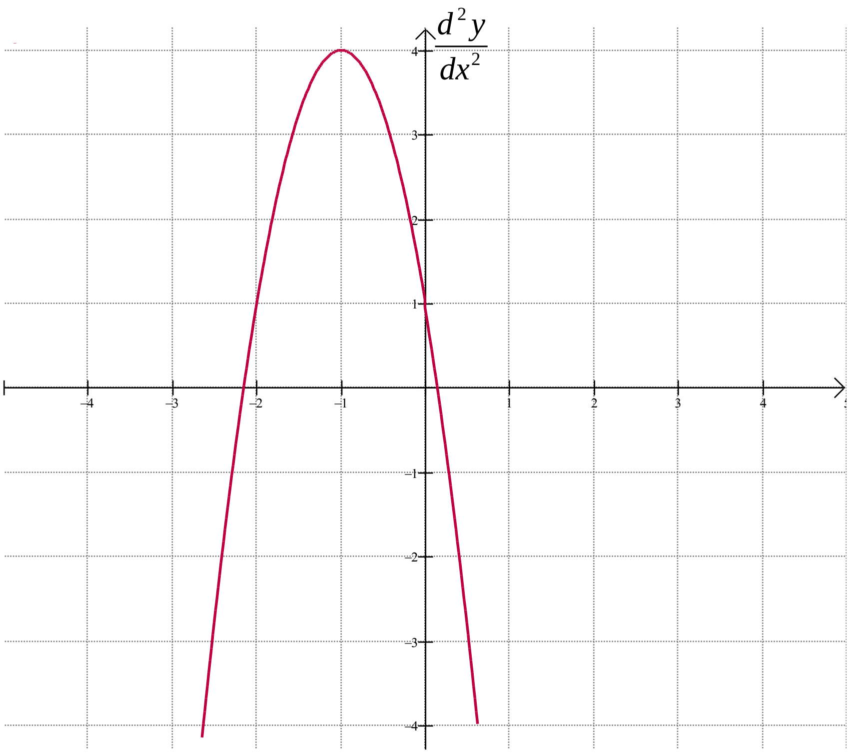
Ex 4 Sketch a possible graph of the first and second derivative of the function shown below.



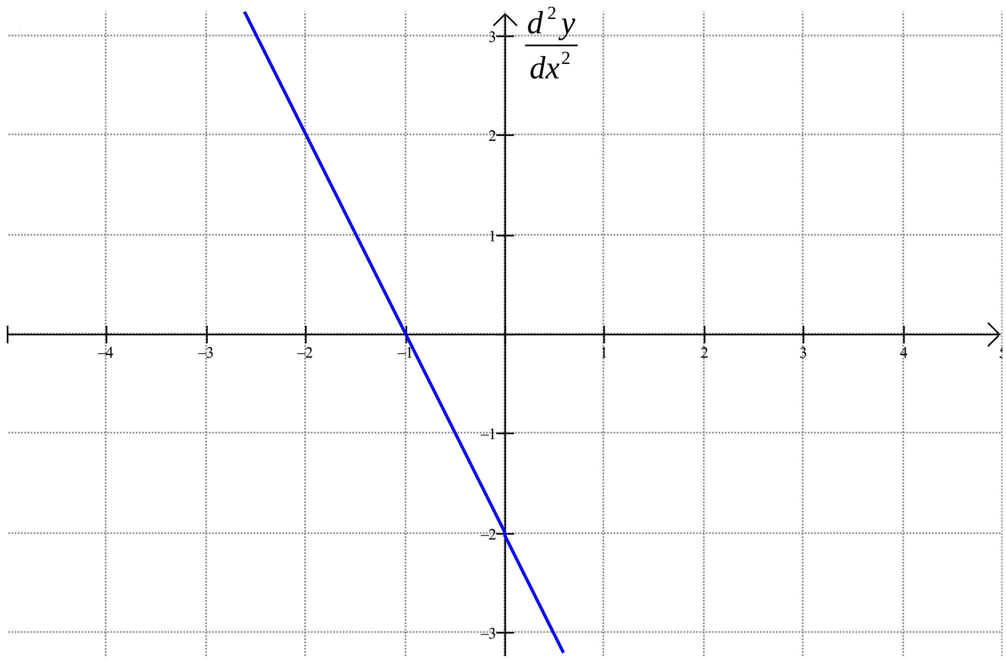
Notice we have critical values at $x = –3, –1$, and 1. These should be zeroes on the graph of the derivative. We can also see where the graph is increasing, and where it is decreasing; these regions correspond to where the graph of the derivative should be positive and negative, respectively.



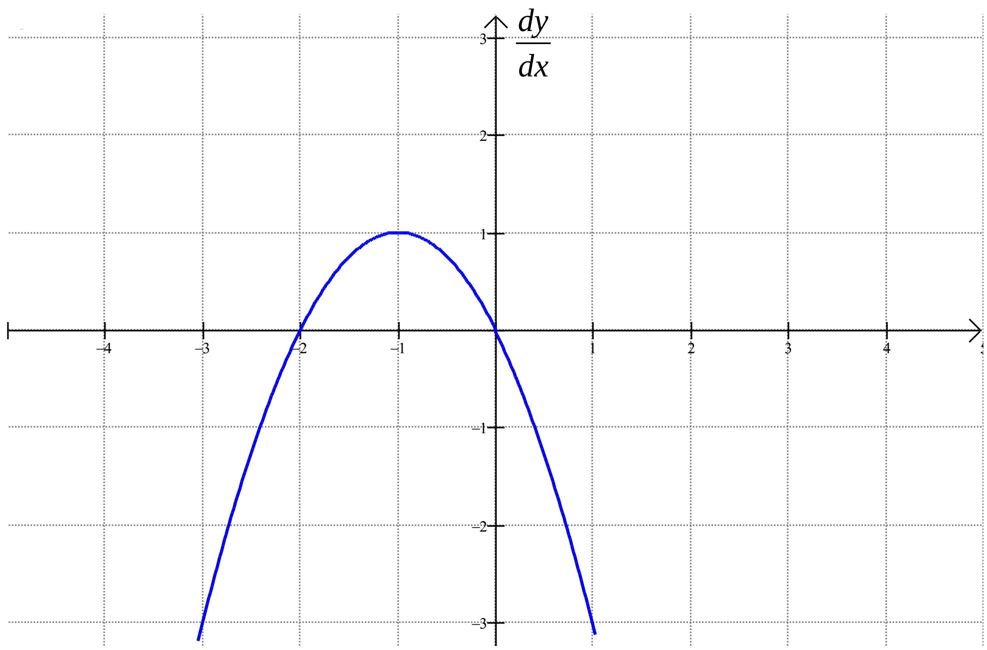
Notice that the points of inflection on the graph of y are maximums or minimums on the graph of the derivative. Those points will also be zeroes on the second derivative.



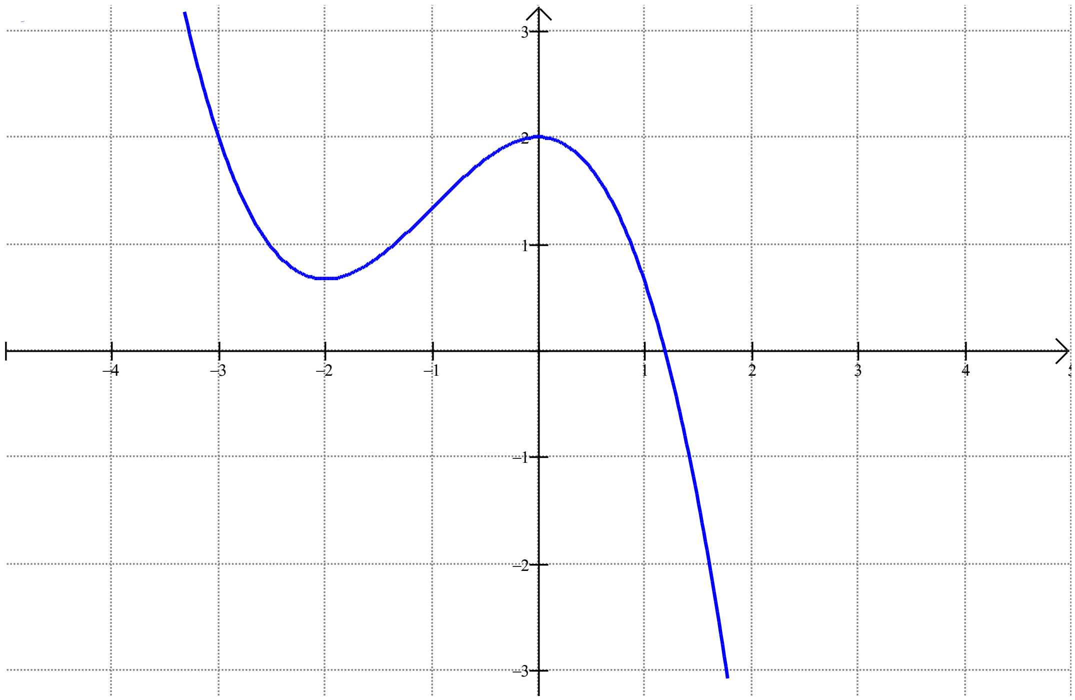
Ex 5 Sketch the possible graphs of a function and its first derivative given the graph of the second derivative below.



Looking at this graph, we can see a zero at x = –1, and the graph is positive for x < –1, and negative for x > –1. This should correspond to the graph of the first derivative increasing, then decreasing, with a maximum at x = –1.



Notice that this looks like a parabola. Since its derivative was a line this should make sense. However, we don't actually know the height of the maximum, nor do we actually know where the zeroes of the derivative are, or even if there are any. This is a sketch that is one of many possible functions that could have had the previous graph as its derivative. If you remember the "+*C*" from integration, we could have an infinite number of graphs that would match – each with a different *C* value.



Again, the zeroes of the previous graph are extremes of this graph, and the zero on the initial graph is a point of inflection on this graph.

3.8 Homework

Sketch these graphs using the sign patterns of the derivatives.

1. $y = 3x^4 - 15x^2 + 7$

2. $y = x^3 + 5x^2 + 3x - 4$

3. $y = \dfrac{x+1}{x^2 - 2x - 3}$

4. $y = \dfrac{x^2 - 1}{x^2 + x - 6}$

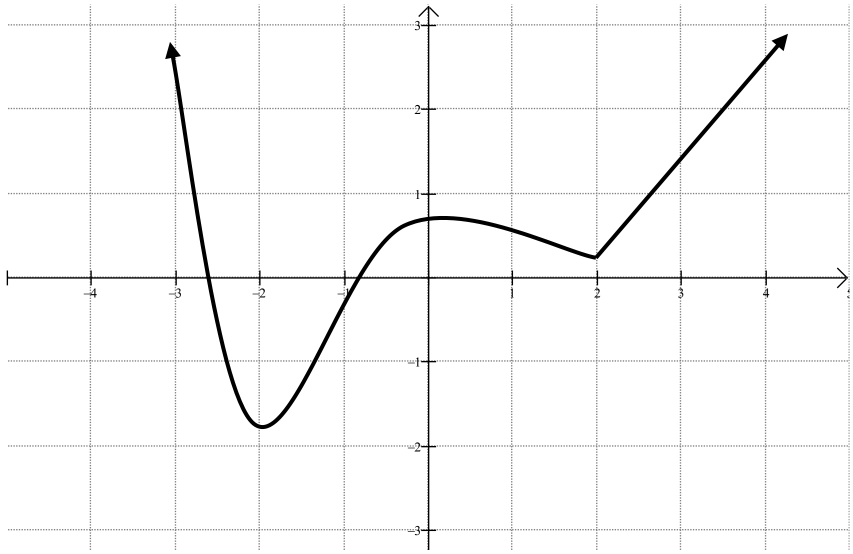
5. $y = 5x^{2/3} - x^{5/3}$

6. $y = \left(x^2\right)\sqrt{4 - x}$

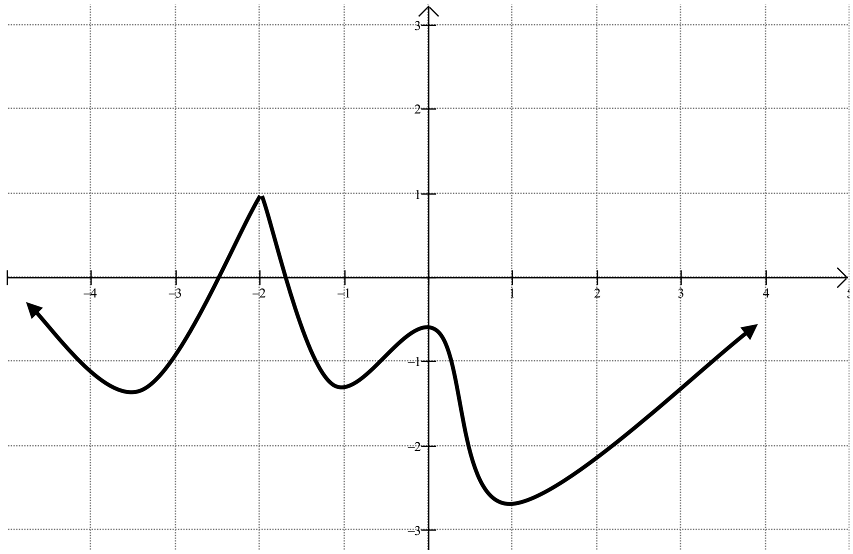
7. $y = x^2 e^{-x}$

For problems 8 through 10, sketch the graph of the first and second derivatives for the function shown in the graphs.

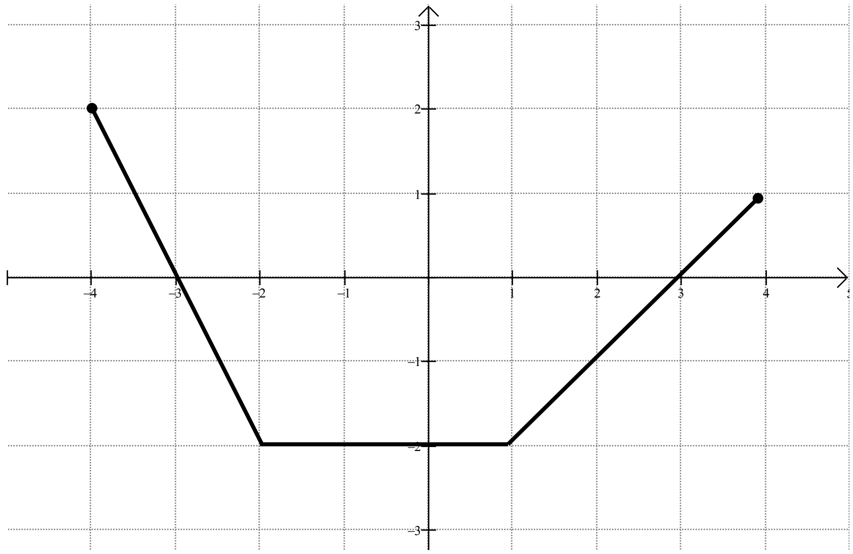
8.



9.



10.



Sketch the possible graph of a function that satisfies the conditions indicated below

11. Increasing from $(-\infty,5)\cup(7,10)$, decreasing from $(5,7)\cup(10,\infty)$

12. Increasing from $(-\infty,-3)\cup(5,\infty)$, decreasing from $(-3,5)$, concave up from $(-\infty,-2)\cup(2,\infty)$, concave down from $(-2,2)$

13. Decreasing from $(-\infty,-5)\cup(5,\infty)$, increasing from $(-5,5)$, concave down from $(-\infty,-7)\cup(-3,3)\cup(7,\infty)$, concave up from $(-7,-3)\cup(3,7)$

14. Increasing and concave up from (2,4), decreasing and concave down from (4,7), increasing and concave up from (7,10), with a domain of [2,10).

For problems 15 through 20 sketch the possible graph of a function that has the traits shown.

15.

x	$f(x)$	$f'(x)$	$f''(x)$
$x < 2$	Positive	Negative	Positive
$x = 2$	0	Negative	Positive
$2 < x < 3$	Negative	Negative	Positive
$x = 3$	–5	0	Positive
$3 < x < 5$	Negative	Positive	Positive
$x = 5$	0	Positive	0
$5 < x < 7$	Positive	Positive	Negative
$x = 7$	9	0	Negative
$7 < x < 9$	Positive	Negative	Negative
$x = 9$	1	Negative	0
$9 < x < 10$	Positive	Negative	Positive
$x = 10$	0	Negative	Positive
$10 < x$	Negative	Negative	Positive

16.

x	$f(x)$	$f'(x)$	$f''(x)$
$x < -1$	Positive	Positive	Positive
$x = -1$	3	DNE	DNE
$-1 < x < 4$	Positive	Negative	Positive
$x = 4$	1	0	Positive
$4 < x < 9$	Positive	Positive	Positive
$x = 9$	3	DNE	DNE
$9 < x$	Positive	Negative	Positive

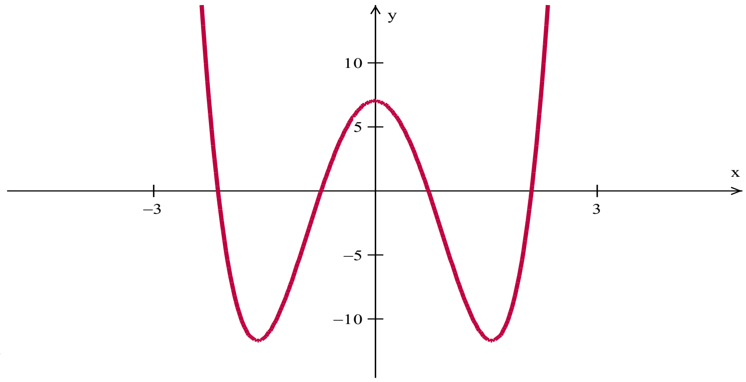
17. Absolute minimum at 1, absolute maximum at 3, local minima at 4 and 7, local maximum at 6

18. Absolute minimum at 2, absolute maximum at 7, local minima at 4 and 6, local maxima at 3 and 5

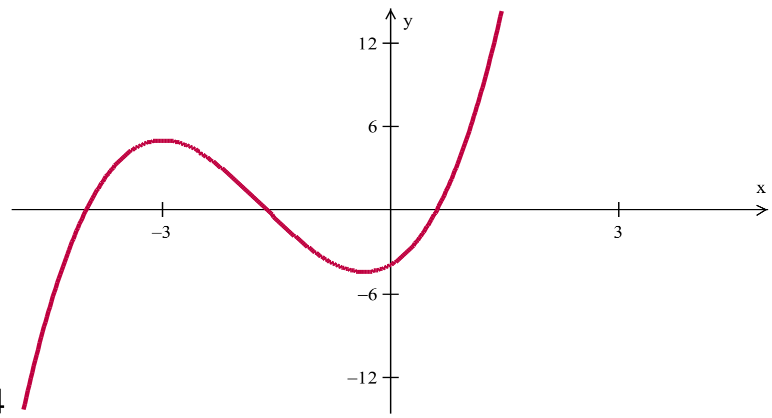
19. Absolute maximum at 1, absolute minimum at 7, no local maxima

20. Absolute minimum at 1, relative minimum at 7, absolute maximum at 4, no local maxima.

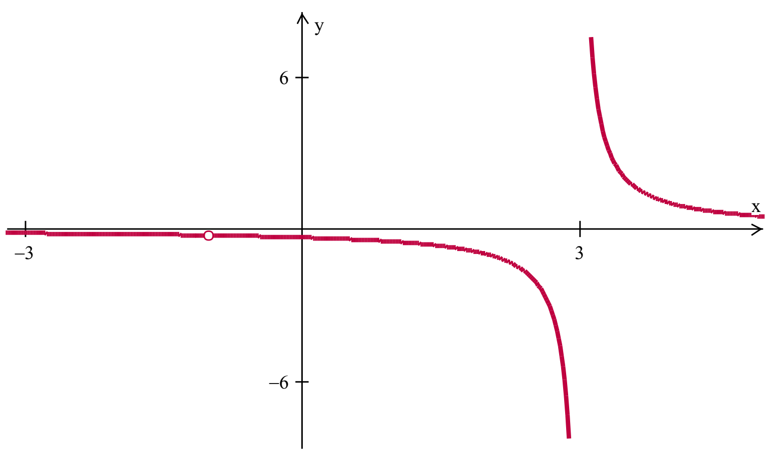
Answers: 3.7 Homework



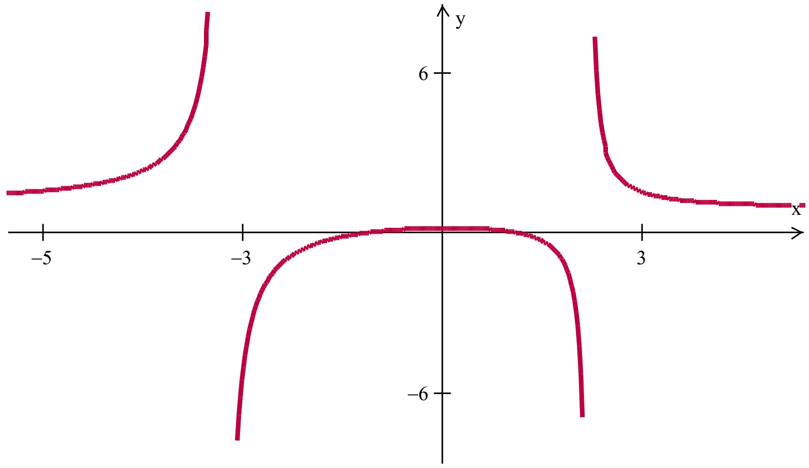
1. $y = 3x^4 - 15x^2 + 7$



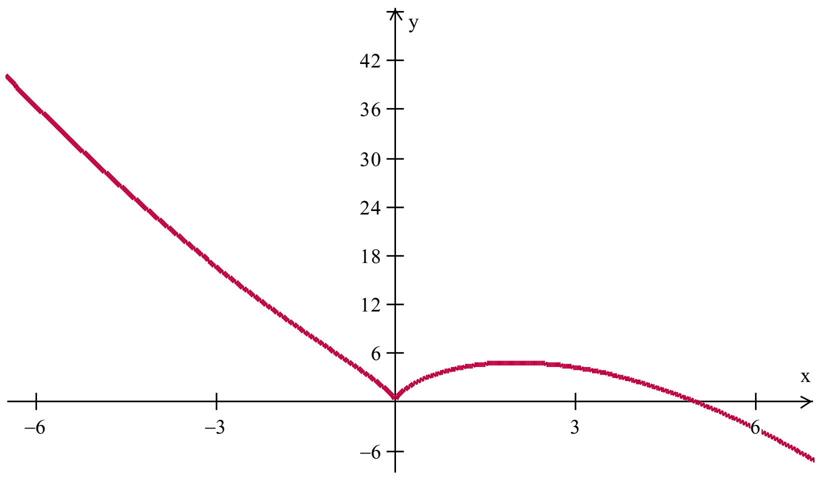
2. $y = x^3 + 5x^2 + 3x - 4$



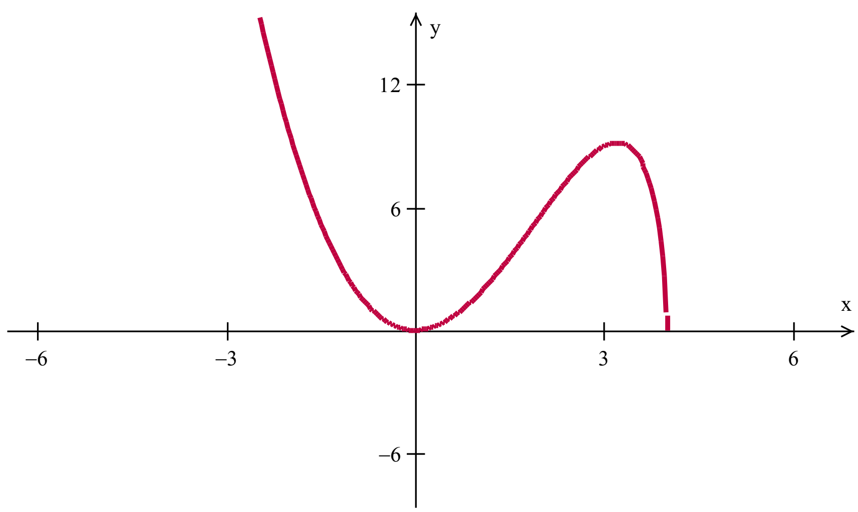
3. $y = \dfrac{x+1}{x^2 - 2x - 3}$



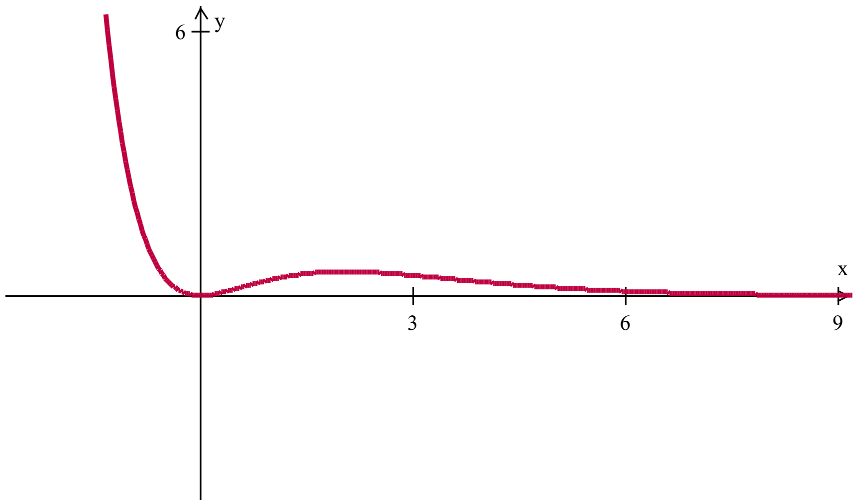
4. $y = \dfrac{x^2 - 1}{x^2 + x - 6}$



5. $y = 5x^{2/3} - x^{5/3}$

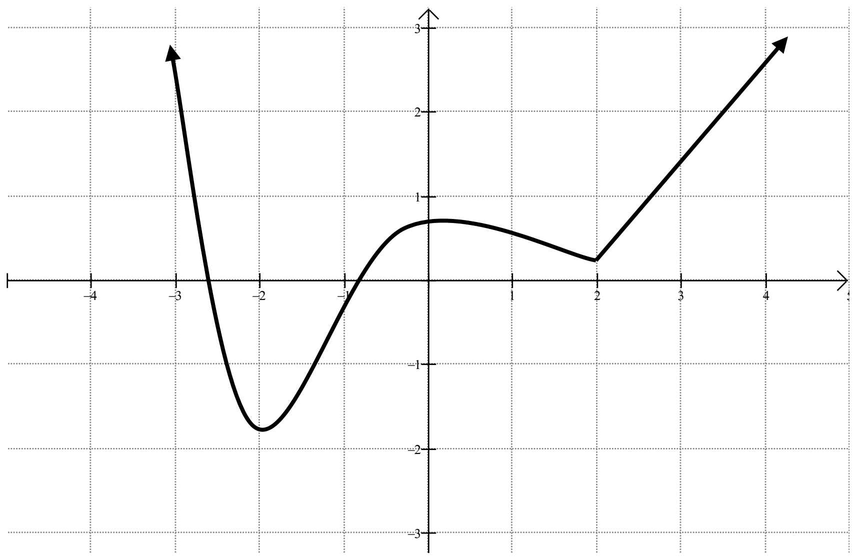


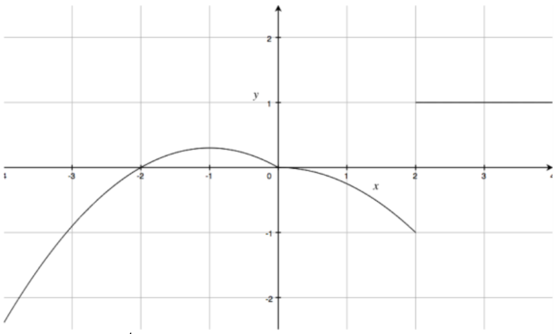
6. $y = \left(x^2\right)\sqrt{4 - x}$



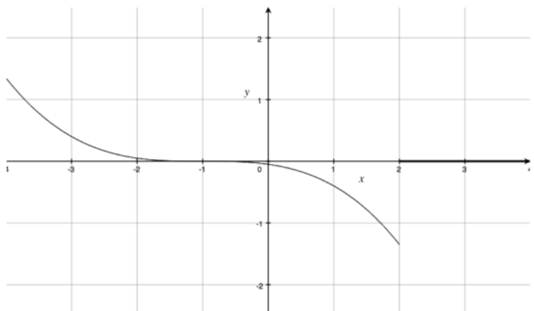
7. $y = x^2 e^{-x}$

8.



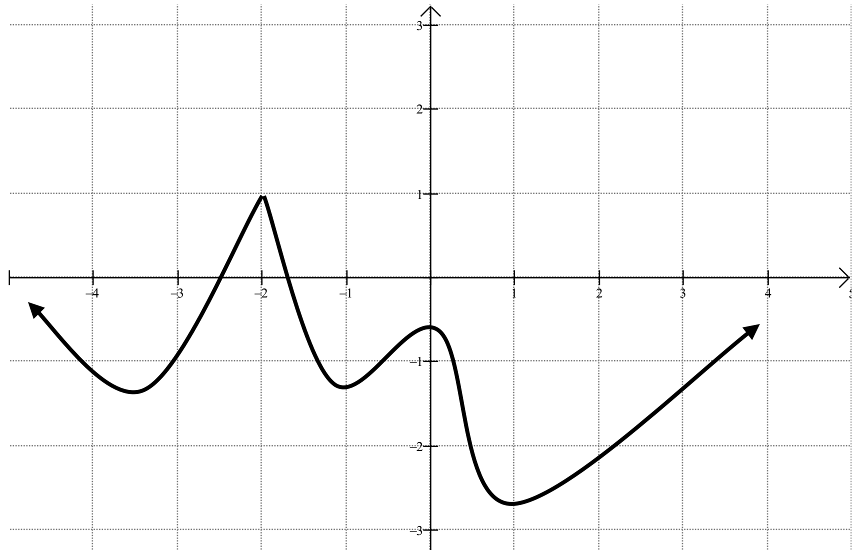


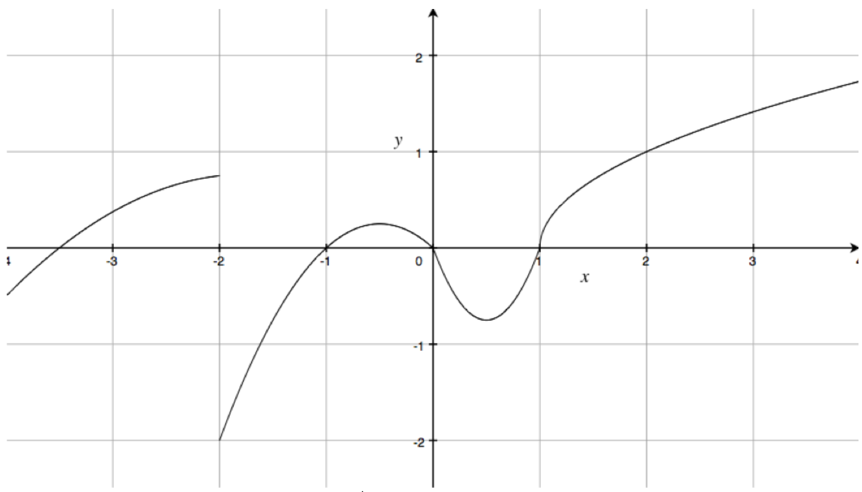
1^{st} Derivative



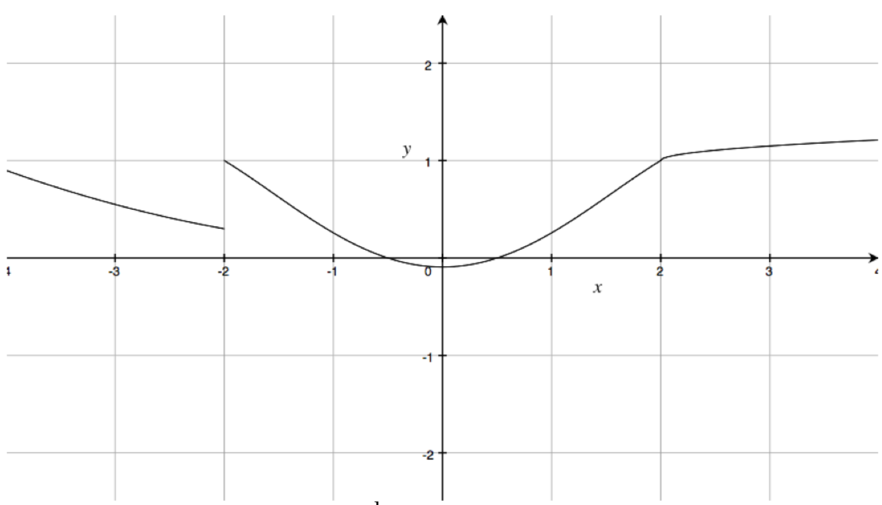
2^{nd} Derivative

9.



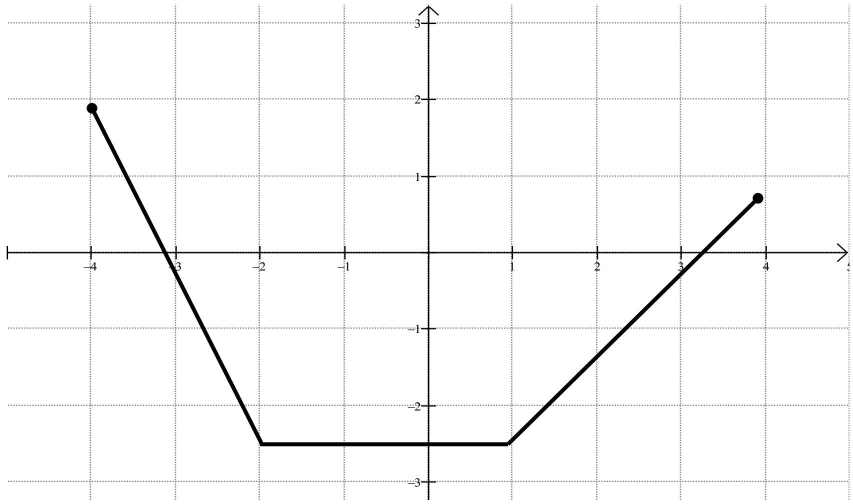


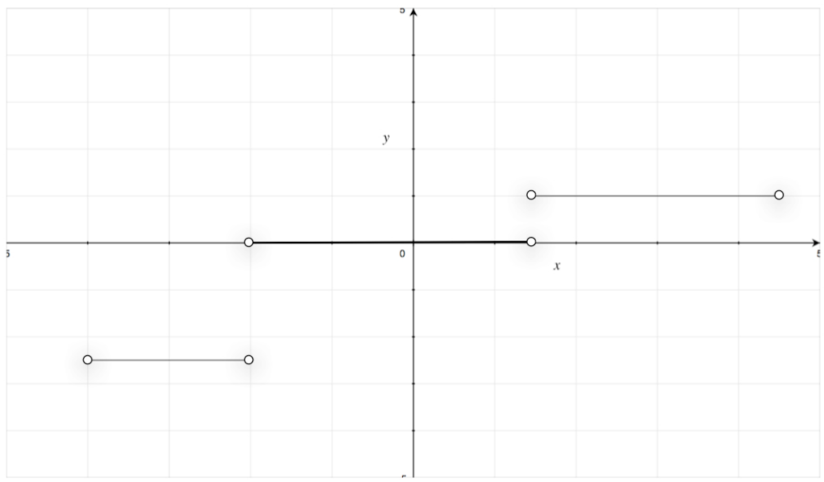
1st Derivative



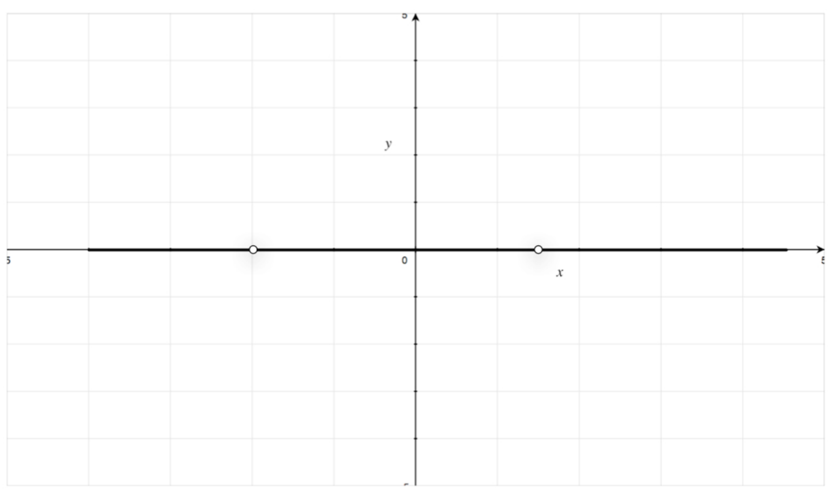
2nd Derivative

10.



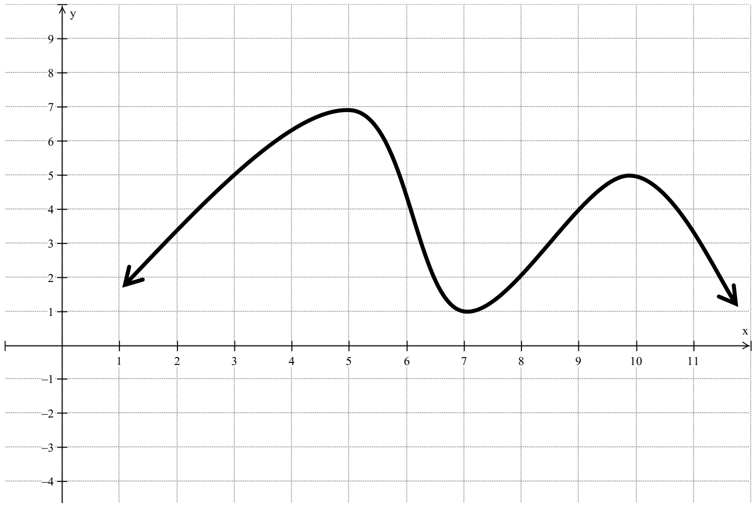


1st Derivative

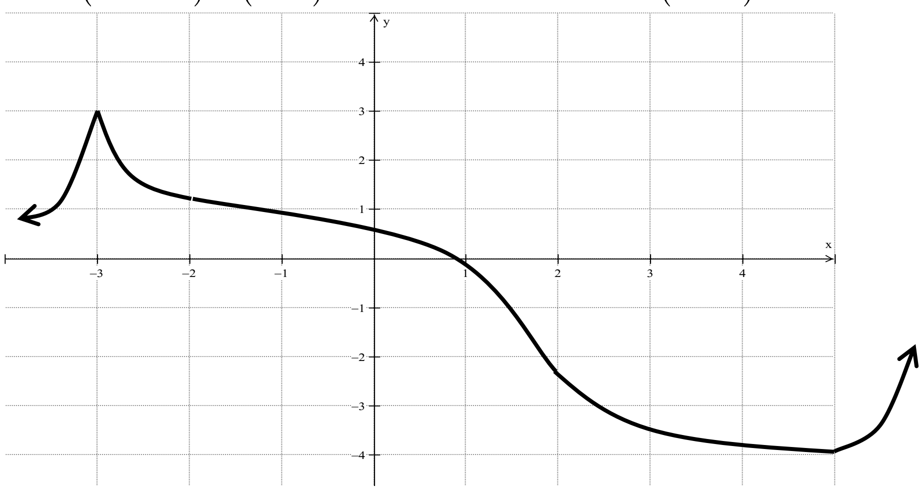


2nd Derivative

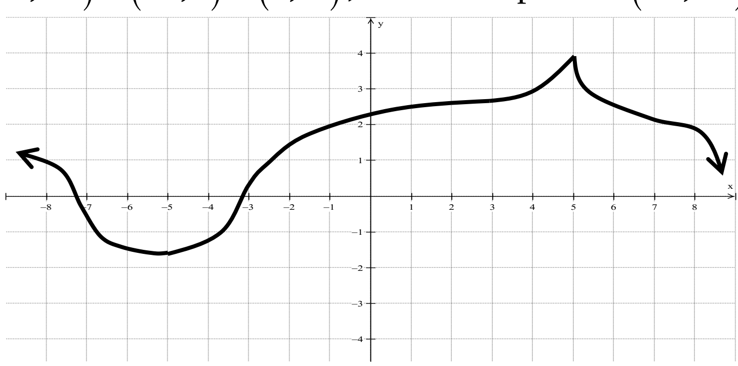
11. Increasing from $(-\infty,5)\cup(7,10)$, decreasing from $(5,7)\cup(10,\infty)$



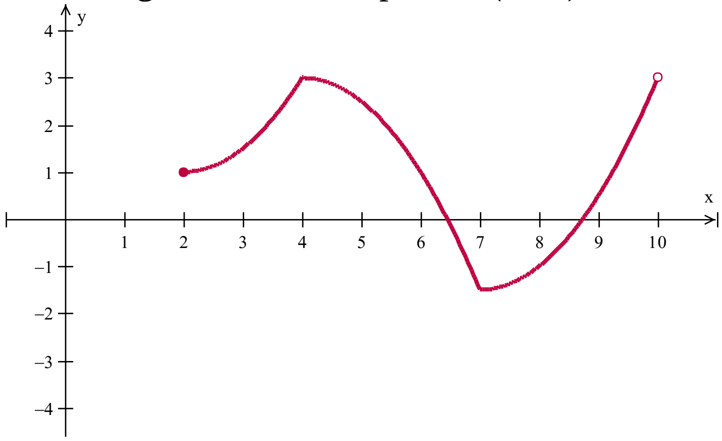
12. Increasing from $(-\infty,-3)\cup(5,\infty)$, decreasing from $(-3,5)$, concave up from $(-\infty,-2)\cup(2,\infty)$, concave down from $(-2,2)$



13. Decreasing from $(-\infty,-5)\cup(5,\infty)$, increasing from $(-5,5)$, concave down from $(-\infty,-7)\cup(-3,3)\cup(7,\infty)$, concave up from $(-7,-3)\cup(3,7)$

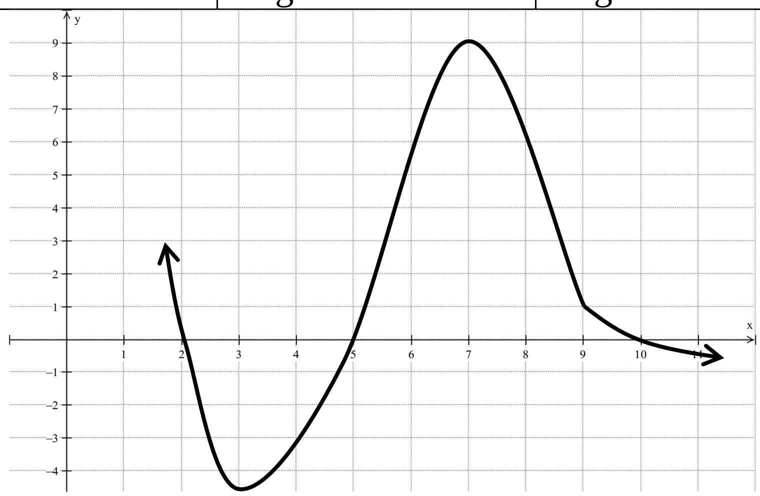


14. Increasing and concave up from (2,4), decreasing and concave down from (4,7), increasing and concave up from (7,10), with a domain of [2,10).



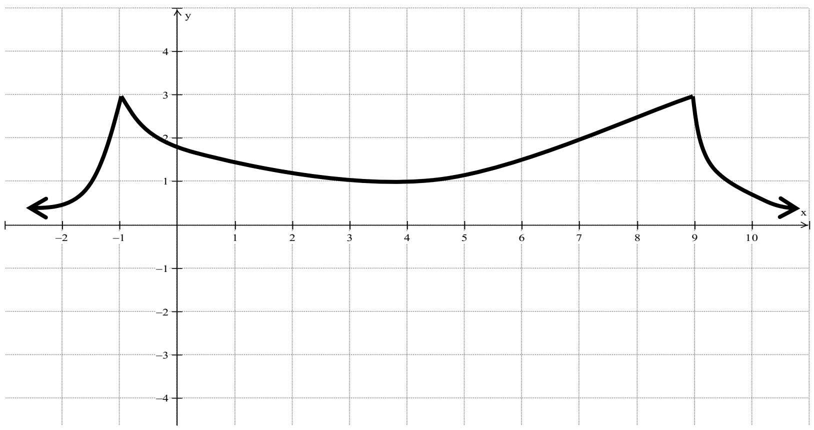
15.

x	$f(x)$	$f'(x)$	$f''(x)$
$x < 2$	Positive	Negative	Positive
$x = 2$	0	Negative	Positive
$2 < x < 3$	Negative	Negative	Positive
$x = 3$	–5	0	Positive
$3 < x < 5$	Negative	Positive	Positive
$x = 5$	0	Positive	0
$5 < x < 7$	Positive	Positive	Negative
$x = 7$	9	0	Negative
$7 < x < 9$	Positive	Negative	Negative
$x = 9$	1	Negative	0
$9 < x < 10$	Positive	Negative	Positive
$x = 10$	0	Negative	Positive
$10 < x$	Negative	Negative	Positive

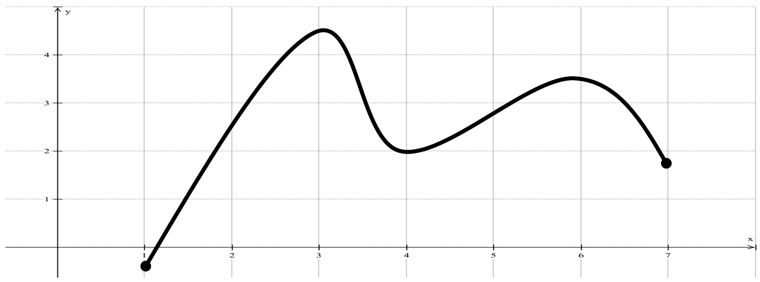


16.

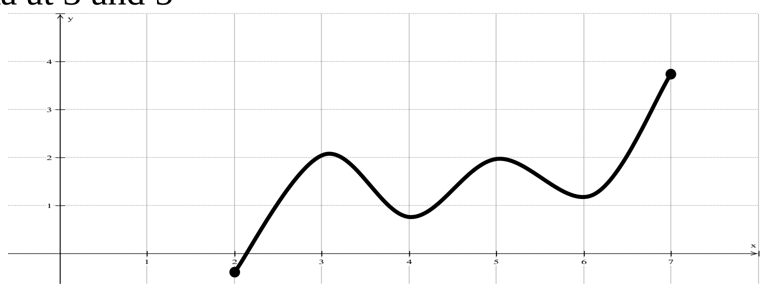
x	$f(x)$	$f'(x)$	$f''(x)$
$x < -1$	Positive	Positive	Positive
$x = -1$	3	DNE	DNE
$-1 < x < 4$	Positive	Negative	Positive
$x = 4$	1	0	Positive
$4 < x < 9$	Positive	Positive	Positive
$x = 9$	3	DNE	DNE
$9 < x$	Positive	Negative	Positive



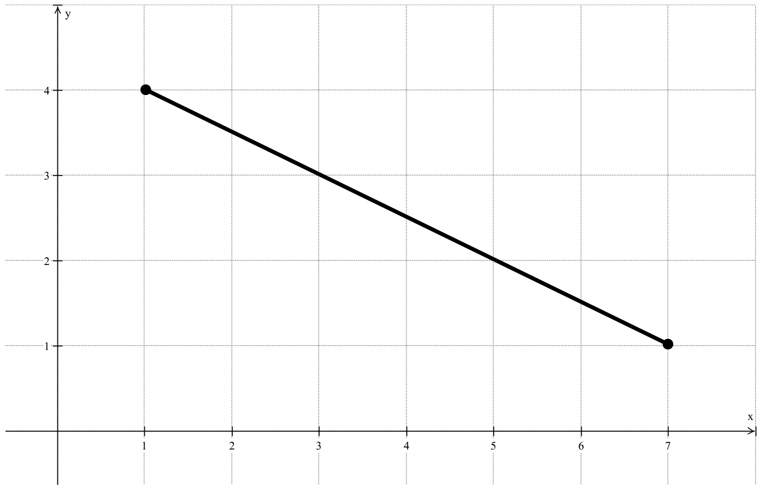
17. Absolute minimum at 1, absolute maximum at 3, local minima at 4 and 7, local maximum at 6



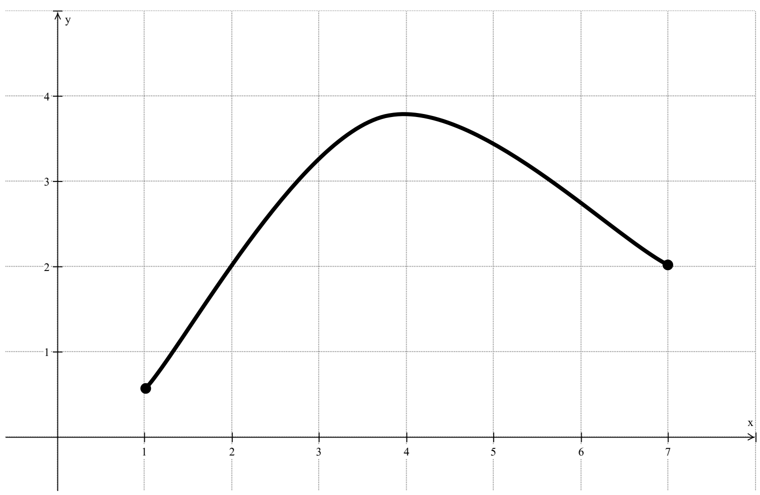
18. Absolute minimum at 2, absolute maximum at 7, local minima at 4 and 6, local maxima at 3 and 5



19. Absolute maximum at 1, absolute minimum at 7, no local maxima



20. Absolute minimum at 1, relative minimum at 7, absolute maximum at 4, no local maxima.



3.8: Graphical Analysis with Derivatives

In the last section, we looked at graphing functions and derivatives, but now we will reverse that process.

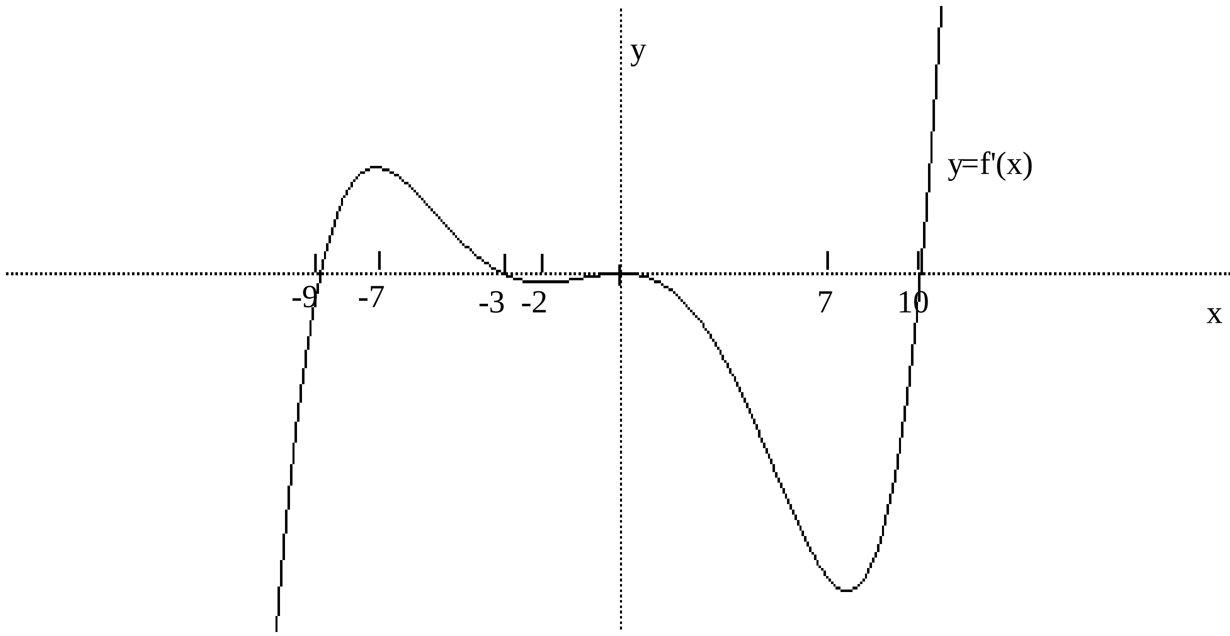
OBJECTIVES
Interpret information in the graph of a derivative in terms of the graph of the "original" function.

As we noted in the last example of the last section, there is a layering and parallelism between the function, its derivative and its second derivative. The zeros and signs of one tell us about increasing, decreasing, and extremes or the concavity and POIs of another. That interconnectedness can be summarized thus:

	$f(x)$			$f'(x)$			$f''(x)$		
	+	**0**	**-**	**+**	**0**	**-**	**+**	**0**	**-**
$f(x)$	*y* value positive	*y* value zero	*y* value negative	Interval of increasing	max. or min.	Interval of decreasing	Concave up	POI	Concave down
$f'(x)$				y' value positive	y' value zero	y' value negative	Interval of increasing	max. or min.	Interval of decreasing
$f''(x)$							y'' value positive	y'' value zero	y'' value negative

The relationship presented here is upward. The zeros of a function refer to the extremes of the one above it and to the POIs of the one two levels up. If we had a function defined as $\int_0^x F(t)dt$ it would be a level above *F(x)* and it would relate to *F'(x)* in the same way that *F(x)* related to *F''(x)*. We will explore this idea further in a later chapter.

Ex 1 If the curve below is $y = f'(x)$, what are (a) the critical values for the maximums and minimums, (b) intervals of increasing and decreasing, (c) the POI, and (d) the intervals of concavity of $y = f(x)$?



What we have here is a two dimensional representation of the sign pattern of the first derivative.

$$\begin{array}{c|ccccccccc} dy/dx & - & 0 & + & 0 & - & 0 & - & 0 & + \\ \hline x & & -9 & & -3 & & 0 & & 10 & \end{array}$$

(a) –9 and 10 must be the critical values of the minimums, because they are zeros of the derivative and the sign of the derivative changes from – to + (the 1st Derivative Test). –3 is the critical value of the maximum. 0 is a POI because the sign does not change on either side of it.

(b) *f*(*x*) is increasing on $x \in (-9, -3)$ and $x \in (10, \infty)$, and it is decreasing on $x \in (-\infty, -9) \cup (-3, 10)$.

The intervals of increasing and decreasing on *f ’* are the intervals of concavity on *f*. So signs of the slopes of *f ’* make up the second derivative sign pattern:

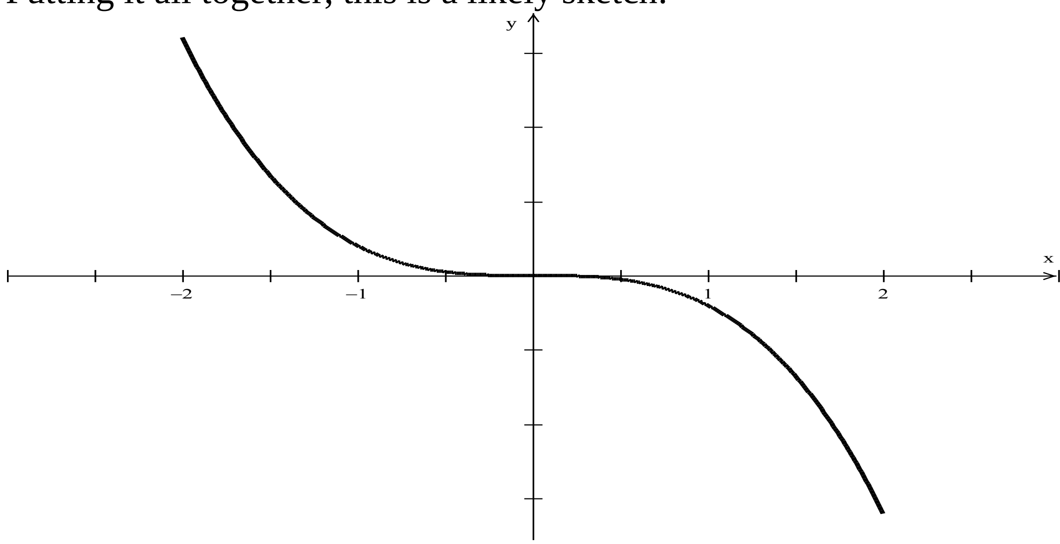
$$\begin{array}{c|ccccccccc} {d^2y}/{dx^2} & + & 0 & - & 0 & + & 0 & - & 0 & + \\ x & & -7 & & -2 & & 0 & & 7 & \end{array}$$

(c) -7, -2, 0 and 7 are Points of Inflection, because that is where the derivative of the derivative equals 0 and the second derivative signs change.

(d) $f(x)$ is concave up on $x \in (-\infty, -7), (-2, 0) \cup (7, \infty)$, and it is concave down on $x \in (-7, -3) \cup (0, 7)$.

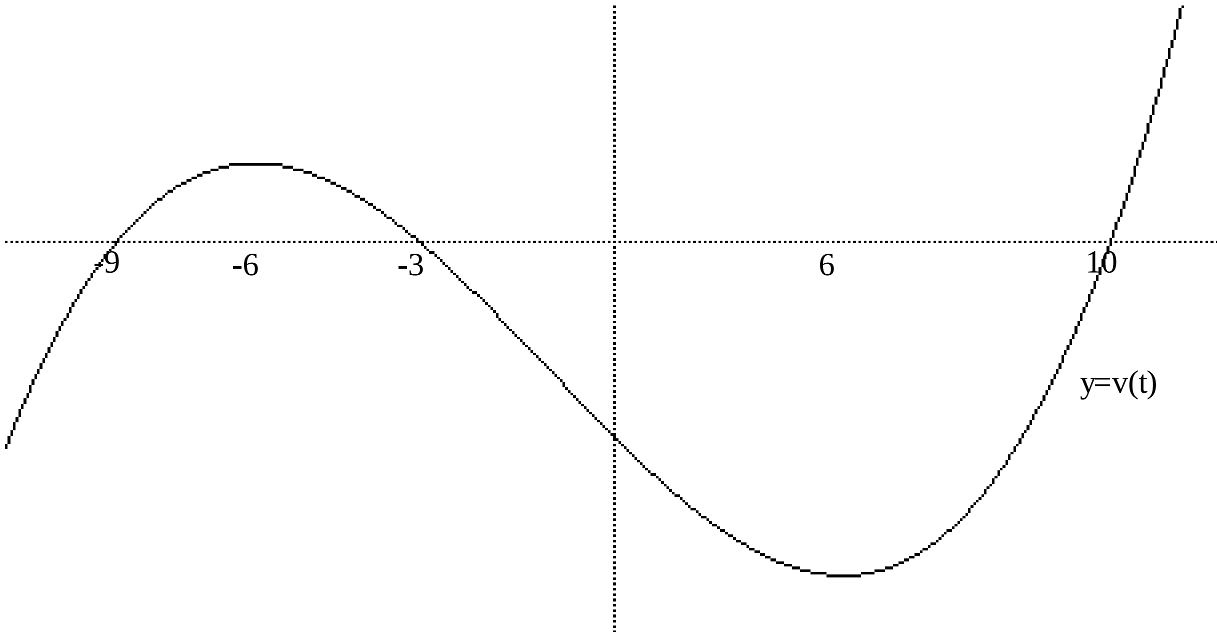
Ex 2 Given the same graph of $y = f'(x)$ in Ex 1, if $f(0) = 0$, sketch a likely curve for $f(x)$ on $x \in [-2, 2]$.

We know that on $x \in [-2, 2]$ $f'(x)$ is negative so $f(x)$ in decreasing on that interval. Since $f(x)$ is decreasing and $f(0)$ is the zero, the curve must be above the x-axis on $x \in [-2, 0)$ and below the x-axis on $x \in (0, 2]$.
On $x \in (-2, 0)$, the slope of f' (which is f'') is positive , so $f(x)$ is concave up. Similarly, on $x \in (0, 2)$, f'' is negative since f' is decreasing, so $f(x)$ is concave down. So (0, 0) is not only a zero, it is also a Point of Inflection. Putting it all together, this is a likely sketch:



Note that there are not markings for scale on the y-axis. This is because we cannot know, from the given information, the y-values of the endpoints of the given domain.

Ex 3 If the following graph is the velocity of a particle in rectilinear motion, what can be deduced about the acceleration and the distance?



The particle is accelerating until t = –6, it decelerates from t = –6 to t = 6, and then it accelerates again. The distance is a relative maximum at t = –3 and a relative minimum at t = –9 and t = 10, but we do not know what the distances from the origin are.

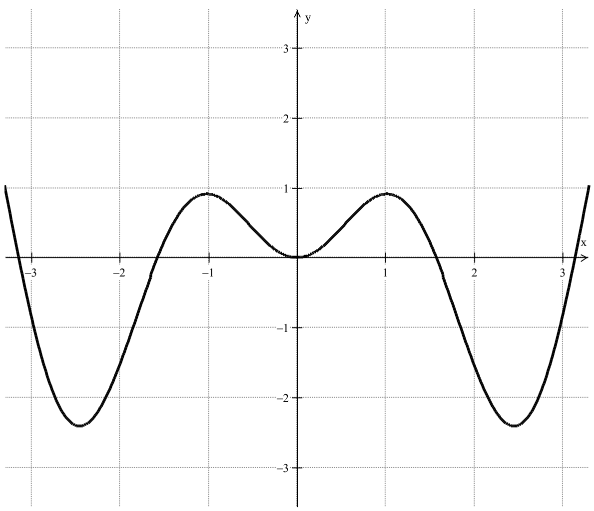
3.8 Homework

Each curve in problems 1 through 6 is $y = f\,'(x)$; show the sign patterns of the first and second derivatives. Then find, on the interval $x \in [-3, 3]$,

(a) the critical values for the maximums and minimums for *f*,
(b) the *x* values for the points of inflection for *f*,
(c) intervals of increasing and decreasing for *f*,
(d) the intervals of concavity of $y = f(x)$, and
(e) sketch a possible curve for $f(x)$ on $x \in [-3, 3]$ with *y*-intercept (0, 0).

Note that you may need to make decimal approximations for some of the critical values and/or interval endpoints.

1.



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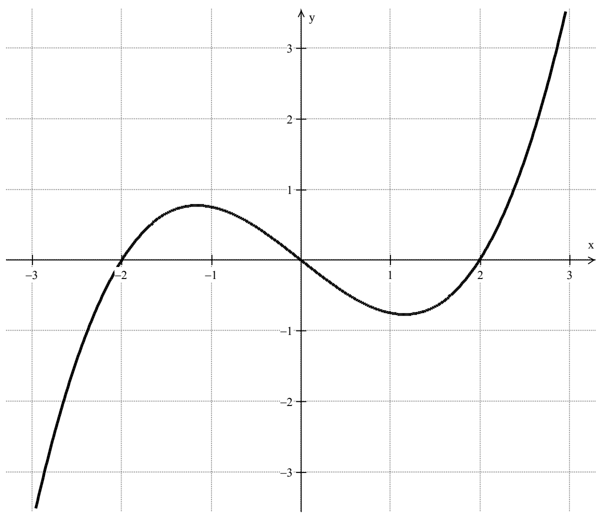

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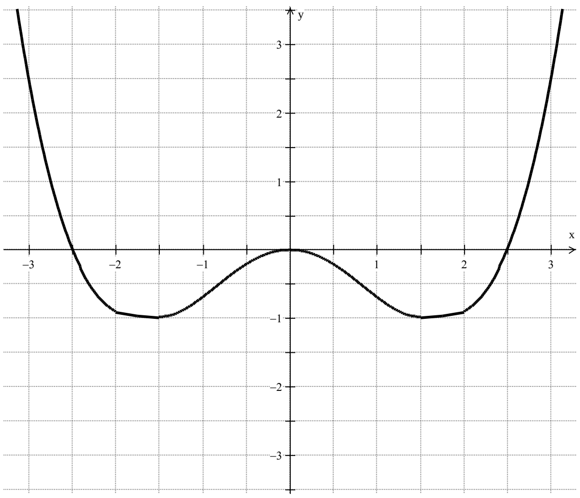

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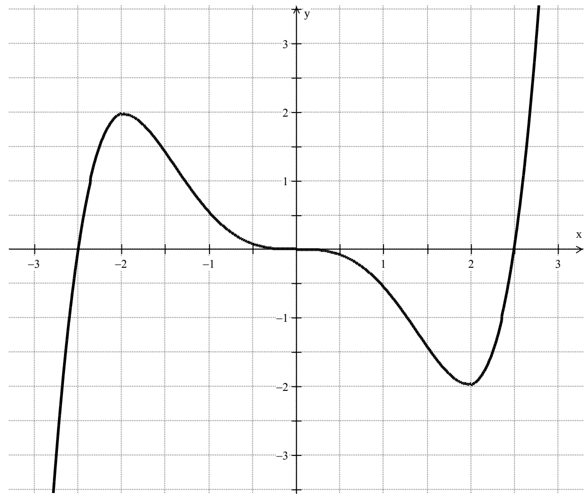

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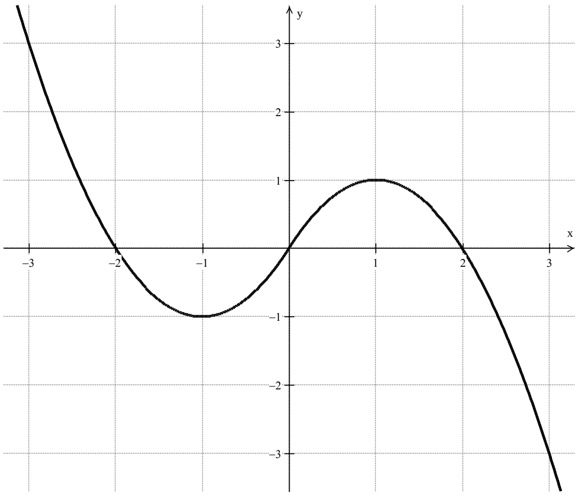

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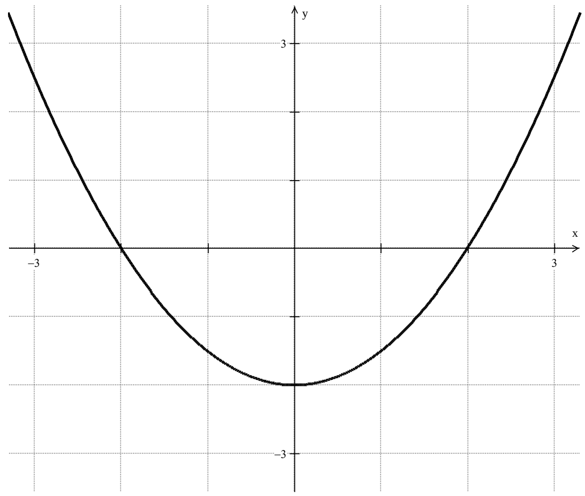
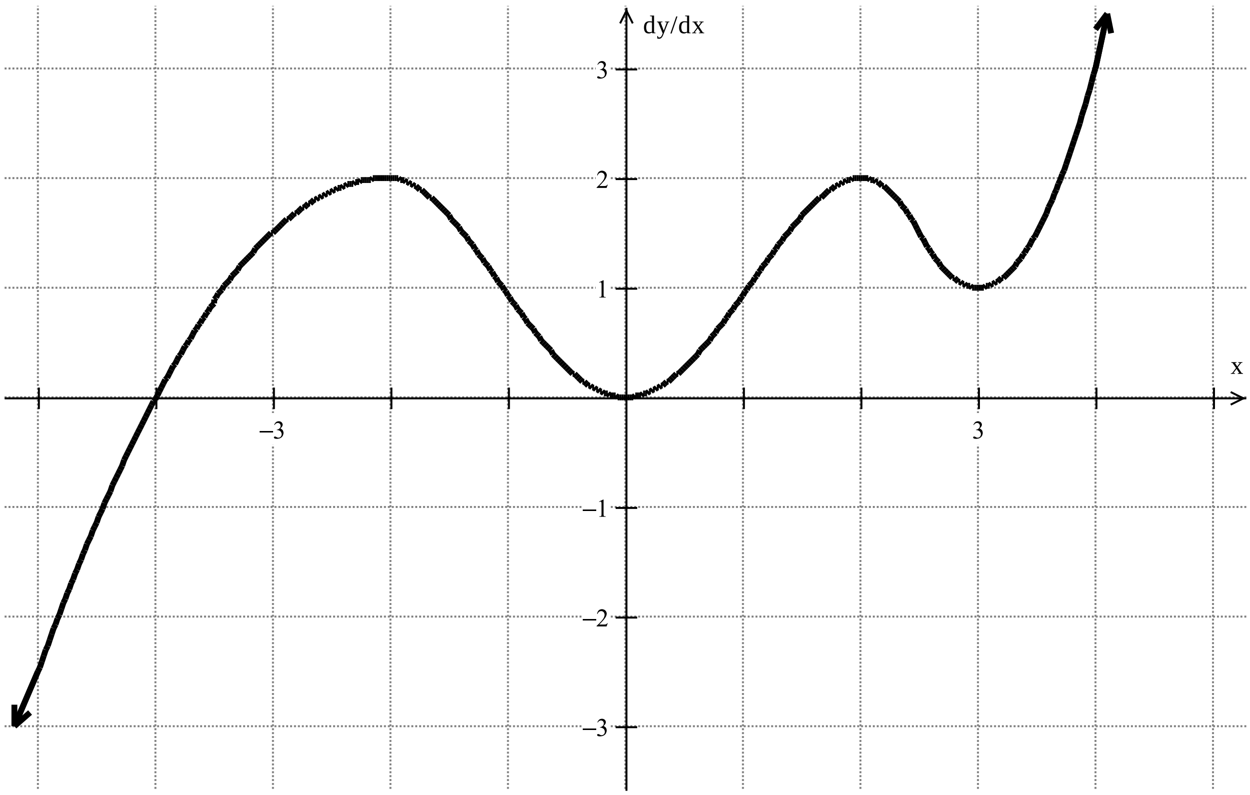
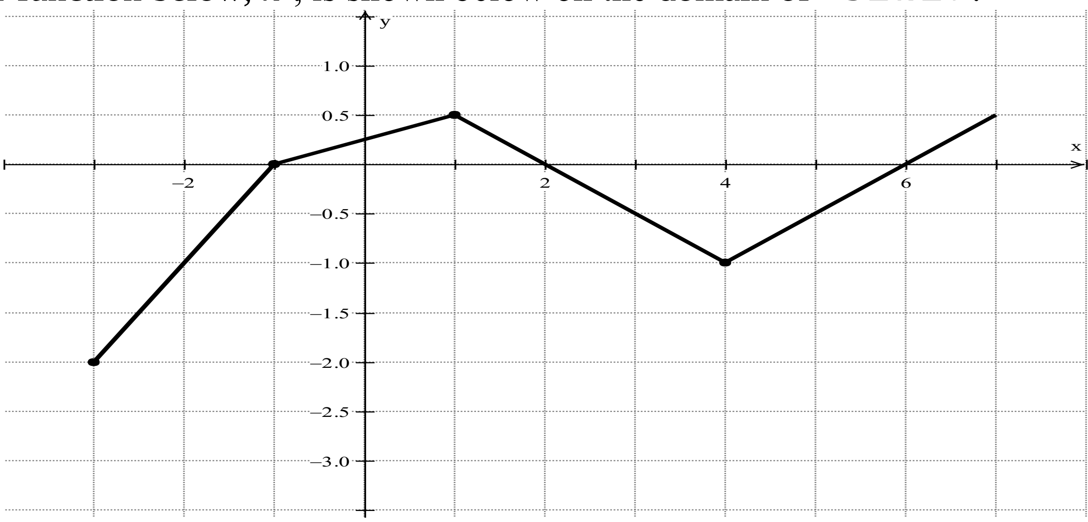


Sketch the possible graph of a function whose derivative is shown below.

7.



8. Let h be a continuous function with $h(2) = 5$. The graph of the piecewise linear function below, h', is shown below on the domain of $-3 \le x \le 7$.



a) Find the x-coordinates of all points of inflection on the graph of $y = h(x)$ for the interval $-3 < x < 7$. Justify your answer.

b) Find the absolute maximum value of h on the interval $-3 \le x \le 7$. Justify your answer.

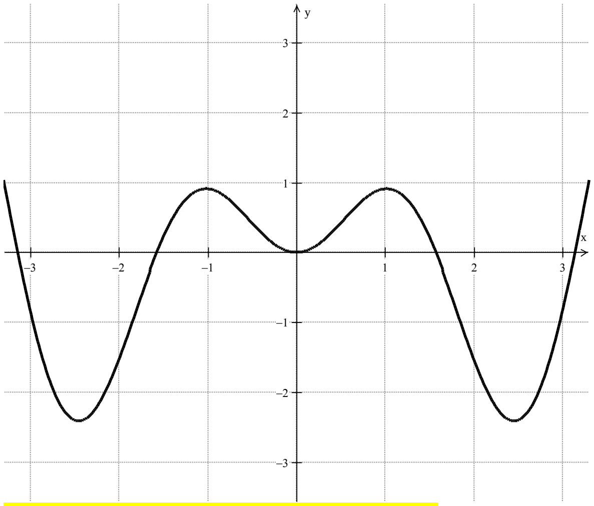
c) Find the average rate of change of h on the interval $-3 \le x \le 7$.

d) Find the average rate of change of h' on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on this interval guarantee a value of c, for $-3 < c < 7$, such that $h''(c)$ is equal to this average rate of change? Explain why or why not.

AP Handout: BC2003#4, AB2000#3, AB1996 # 1

Answers: 3.8 Homework

1.



(a) CV: Max at $x = -3.2,\ 1.6$

Min at $x = -1.6,\ 3.2$

(b) $x = -2.4, -1, 0, 1, 2.4$

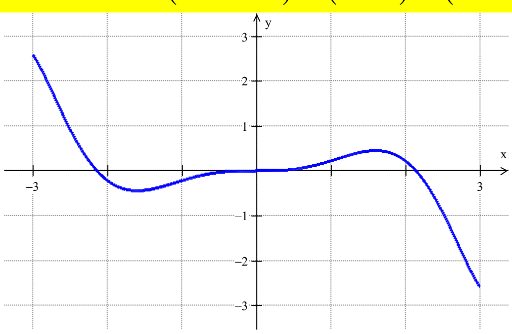
(c) Inc: $x \in (-1.6, 0) \cup (0, 1.6)$

Dec: $x \in (-3, -1.6) \cup (1.6, 3)$

(d) Up: $x \in (-2.4, -1) \cup (0, 1) \cup (2.4, 3)$

Down: $x \in (-3, -2.4) \cup (-1, 0) \cup (1, 2.4)$

(e)



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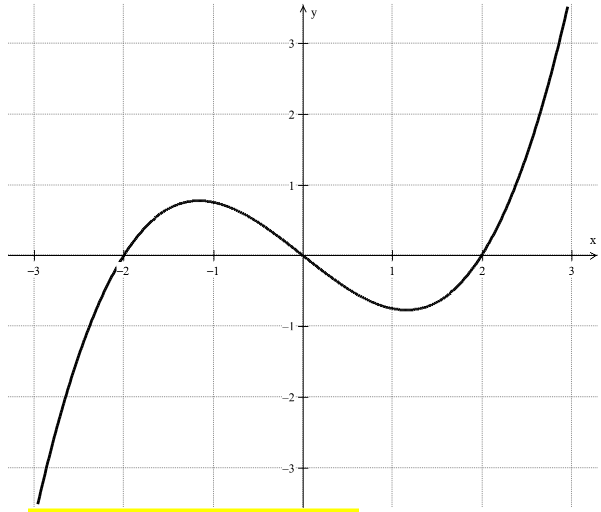


(a) CV: Max at $x = 0$

Min at $x = -2,\ 2$

(b) $x = -1.2, 1.2$

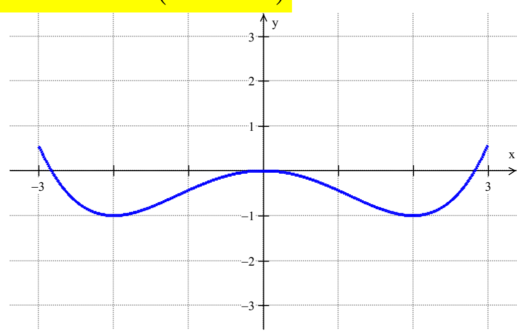
(c) Inc: $x \in (-2, 0) \cup (2, 3)$

Dec: $x \in (-3, -2) \cup (0, 2)$

(d) Up: $x \in (-3, -1.2) \cup (1.2, 3)$

Down: $x \in (-1.2, 1.2)$

(e)



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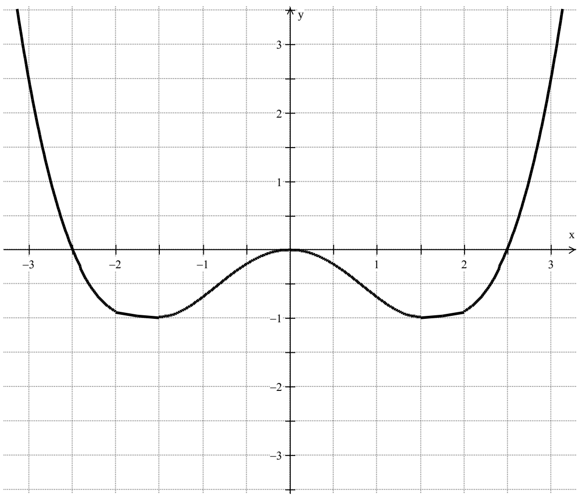


(a) CV: Max at $x = -2.5$

Min at $x = 2.5$

(b) $x = -1.5, 0, 1.5$

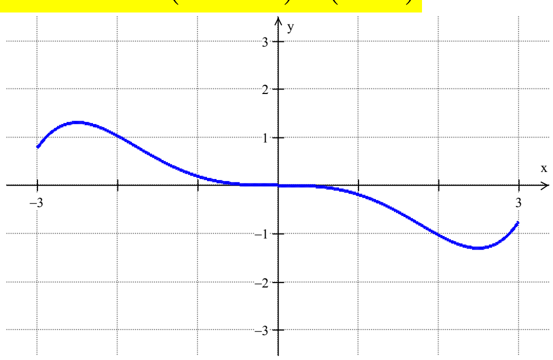
(c) Inc: $x \in (-3,-2.5) \cup (2.5,3)$

Dec: $x \in (-2.5,0) \cup (0,2.5)$

(d) Up: $x \in (-1.5,0) \cup (1.5,3)$

Down: $x \in (-3,-1.5) \cup (0,1.5)$

(e)



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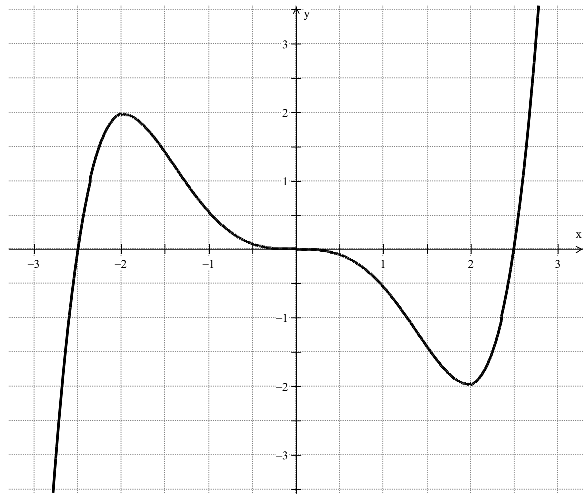


(a) CV: Max at $x = 0$

Min at $x = -2.5, 2.5$

(b) $x = -2, 2$

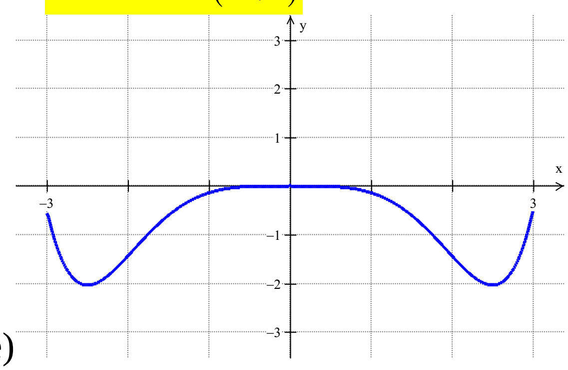
(c) Inc: $x \in (-2.5,0) \cup (2.5,3)$

Dec: $x \in (-3,-2.5) \cup (0,2.5)$

(d) Up: $x \in (-3,-2) \cup (2,3)$

Down: $x \in (-2,2)$

(e)



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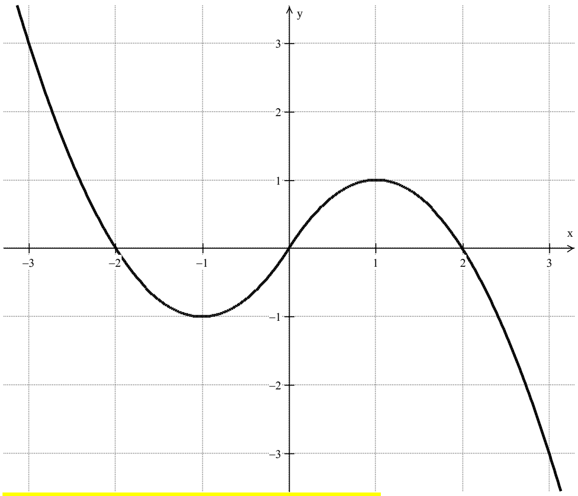


(a) CV: Max at $x = -2, 2$

Min at $x = 0$

(b) $x = -1, 1$

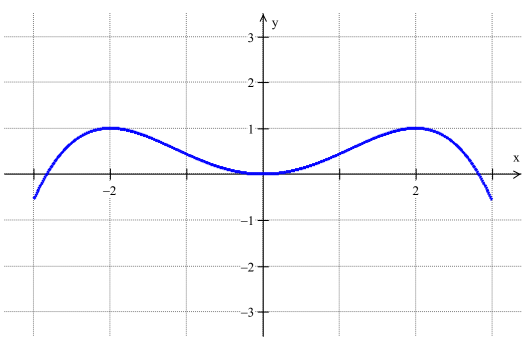
(c) Inc: $x \in (-3,-2) \cup (0,2)$

Dec: $x \in (-2,0) \cup (2,3)$

(d) Up: $x \in (-1,1)$

Down: $x \in (-3,-1) \cup (1,3)$

(e)



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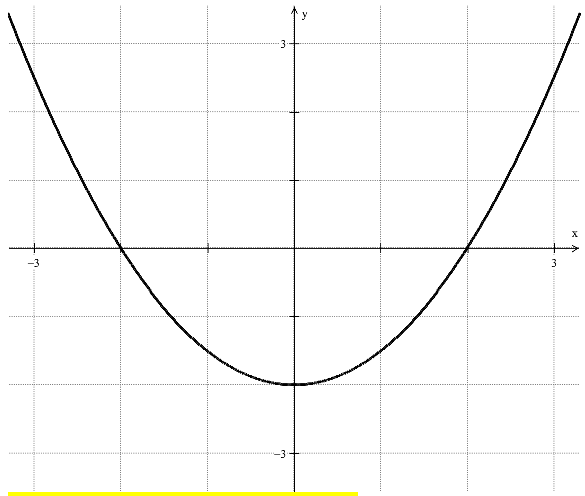


(a) CV: Max at $x = -2$

Min at $x = 2$

(b) $x = 0$

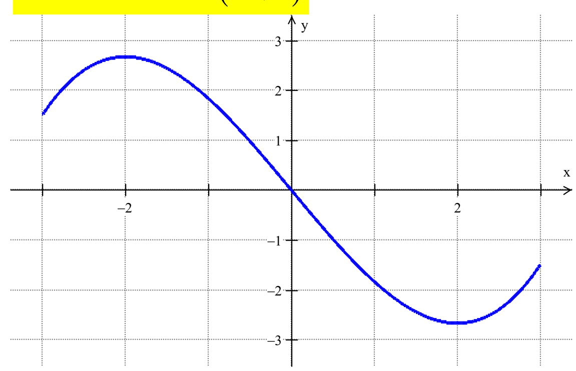
(c) Inc: $x \in (-3,-2) \cup (2,3)$

Dec: $x \in (-2,2)$

(d) Up: $x \in (0,3)$

Down: $x \in (-3,0)$

(e)



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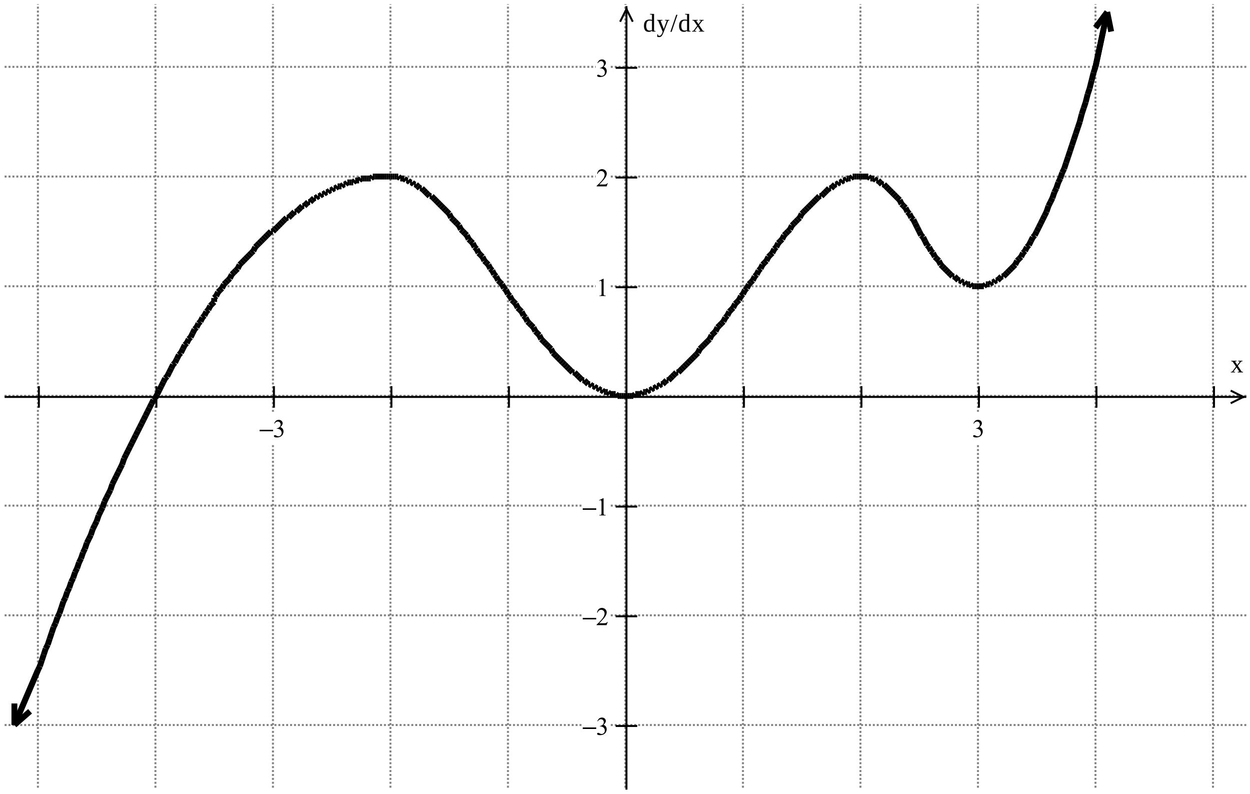
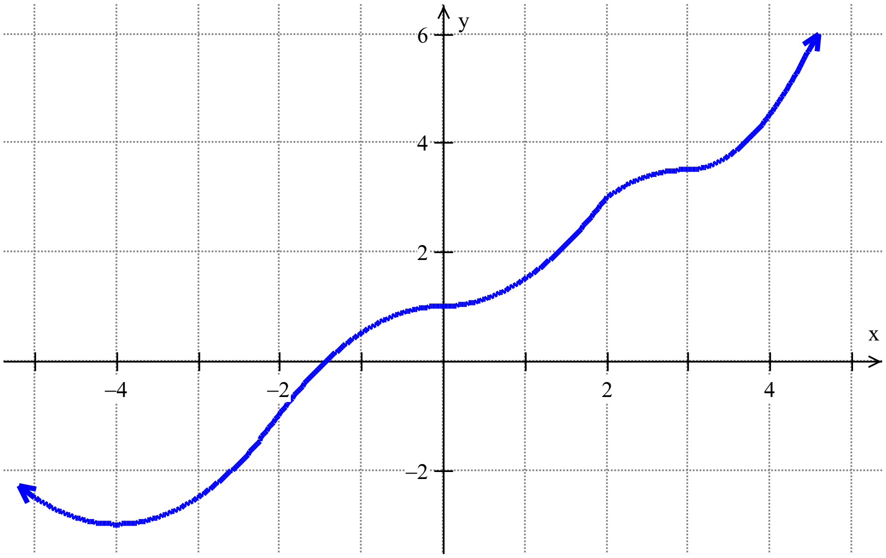




9. Let h be a continuous function with $h(2) = 5$. The graph of the piecewise linear function below, h', is shown below on the domain of $-3 \le x \le 7$.

a) Find the x-coordinates of all points of inflection on the graph of $y = h(x)$ for the interval $-3 < x < 7$. Justify your answer.
POIs occur where *h''* = 0 or does not exist, and switches signs. This occurs at x = 1 and 4; *h''* does not exist because *h'* is not differentiable for those x. The sign of *h''* switches because the slopes of *h'* switch from positive to negative or negative to positive.

b) Find the absolute maximum value of h on the interval $-3 \le x \le 7$. Justify your answer.
Maxima occur at x = –3, 2, and 7. $h(-3) = 5 + \int_{2}^{-3} h(x)\,dx = 3.5$, $h(2) = 5$, and $h(7) = 5 + \int_{2}^{7} h(x)\,dx = 3.25$. Therefore, the absolute maximum is at x = 2.

c) Find the average rate of change of h on the interval $-3 \le x \le 7$.
Average rate of change of h would be the slope of h on that interval. This would be $\frac{h(7) - h(-3)}{7 - (-3)} = -\frac{1}{40}$

d) Find the average rate of change of h' on the interval $-3 \le x \le 7$. Does the Mean Value Theorem applied on this interval guarantee a value of c, for $-3 < c < 7$, such that h''(c) is equal to this average rate of change? Explain why or why not.
$\frac{0.5 - (-2)}{7 - (-3)} = \frac{1}{4}$; the Mean Value Theorem does not guarantee this value for c because h' is not differentiable throughout this interval.

AP Handout: BC2003#4, AB2000#3, AB1996 # 1

Chapter 3 Test

1) If the position of a particle is given by $x(t)=\ln(t^2+9)$, find $v(t)$, $a(t)$, and find when the particle is stopped. When the particle is stopped, what is the position and acceleration?

2) For the function $V=\frac{1}{3}\pi r^2 h+2\pi r^3$

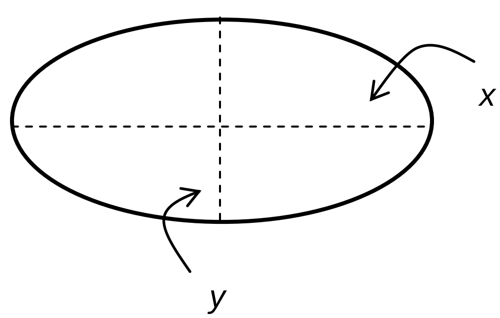
a) find the rate of change of *V* with respect to *r*, assuming *h* is constant.

b) find the rate of change of *V* with respect to *h*, assuming *r* is constant.

c) find the rate of change of *V* with respect to *t*, assuming *r* and *h* are both variables.

3) Two cars are leaving an intersection; one headed north, the other headed east. The northbound car is traveling at 35 miles per hour, while the eastbound car is traveling at 45 miles per hour. Find the rate at which the ***direct distance*** is increasing when the eastbound car is 0.6 miles from the intersection and the northbound car is 0.8 miles from the intersection.

4) You spill some milk on a tablecloth, and you notice that the stain is elliptical and that the major axis (x) is always twice the minor axis (y). Given that the area of an ellipse is $A = \frac{\pi}{4}xy$, find how fast the area of the stain is increasing when x = 6 mm, y = 3 mm, and the minor axis is increasing at 0.2 mm per second.



5) Find $\frac{dy}{dx}$ for $e^{y^2}+5xy=\tan\left(y+1\right)+\ln\left(x+1\right)$

6) Given the function $x^2-y^2=16$

a) Use implicit differentiation to show that $\frac{dy}{dx}=\frac{x}{y}$

b) Use implicit differentiation on $\frac{dy}{dx}$ to show that $\frac{d^2y}{dx^2}=\frac{y^2-x^2}{y^3}$

7) Given the position function $x(t) = 2\tan^{-1}(t)$

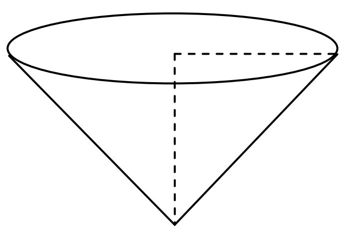
a) Find the velocity and the acceleration functions for this position function.

b) Find the position, velocity, and acceleration at $t = -1$

c) In which direction is the particle moving at that time?

d) Is the particle speeding up or slowing down at that time?

8) You are filling a conical glass with a total height of 5 inches and a radius of 5 inches at the top. If you pour water into the glass at a rate of 3 cubic inches per second, how fast is the height increasing when the radius of the liquid in the glass is 3 inches.



9) Your friend has an ice cream cone that is melting. He notices that the ice cream is dripping from the bottom of the cone at a rate of $0.4 \text{ cm}^3/\text{sec}$. He also notices that the level of the ice cream in the cone is at a height of 4 cm, and that the radius of the cone at that point is equal to the height. He poses the following solution to how fast the height of the ice cream is changing:

$$V_{cone} = \frac{1}{3}\pi r^2 h \qquad r = h$$

$$V_{cone} = \frac{1}{3}\pi h^3 \qquad h = 4$$

$$\text{so } V_{cone} = \frac{1}{3}\pi(4)^3$$

$$V_{cone} = \frac{64\pi}{3}$$

$$\frac{dV}{dt} = 0$$

He says, "Obviously, my ice cream cone isn't really melting."

Your other friend says, "No, no, no – you have that all wrong. This is what you should do:"

$$V_{cone} = \frac{1}{3}\pi r^2 h \qquad r = h$$

$$V_{cone} = \frac{1}{3}\pi h^3$$

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \qquad h = 4, \frac{dV}{dt} = 0.4$$

$$0.4 = 16\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0.4}{16\pi} = 0.00796 \text{ cm/sec}$$

"So the level of ice cream is rising at 0.00796 cm/sec."

Is either of your friends' solution right? Correct the mistakes in their work, if there are any, and explain what went wrong.

Find the critical values and extremes for each of the following functions in problems 10 and 11:

10) $y=(x+1)e^{-x}$

Critical Values:
Extreme Values:

11) $y=2x-2\tan x$ on $x\in\left[-\frac{\pi}{4},\ \frac{\pi}{4}\right]$

Critical Values:
Extreme Values:

12) Given the table of values for $f(x)$ and its first and second derivatives, find the points at which the curve is at a maximum or minimum. Explain why this is the case.

x	-4	-2	0	6	9	12
$f(x)$	18	-27	30	42	67	-11
$f'(x)$	19	0	-22	0	0	83
$f''(x)$	18	-28	23	19	-12	23

13) For each of the following, list the point and describe what is happening at each point on $f(x)$:

a) $x=2$, $f(2)=-5$, $f'(2)=-13$, $f''(2)=27$

b) $x=3$, $f(3)=9$, $f'(3)=0$, $f''(3)=-29$

c) $x=-1$, $f(-1)=0$, $f'(-1)=0$, $f''(-1)=18$

14) Below is a chart showing the volume of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	2	4	6	8	10	12
$V(t)$ (in m^3)	12	14	8	12	19	18	18

a) Find the approximate value of $V'(2)$. Show your work and use units.

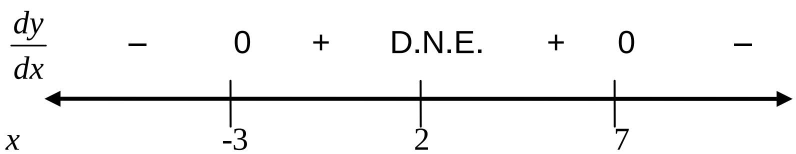
b) Find the approximate value of $V'(4)$. Show your work and use units.

c) Find the approximate value of $V'(10)$. Show your work and use units.

d) Given that $V(t)$ is continuous and differentiable on the interval $0< t <12$. Must there be a value of c in that interval such that $V'(c)$ equals $\frac{V(12)-V(0)}{12-0}$. Explain why or why not.

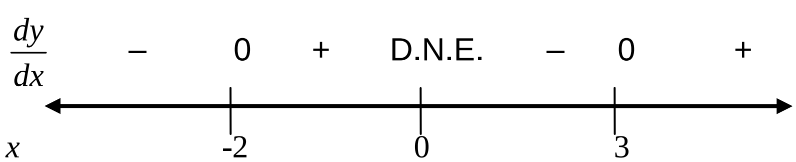
15) Given the sign patterns below, find the critical values and determine which ones are associated with maximums and which are associated with minimums. Explain why. Assume *y* is a continuous function.

a)



Maximums: Minimums:

b)



Maximums: Minimums:

16. Sketch a graph with the following traits:

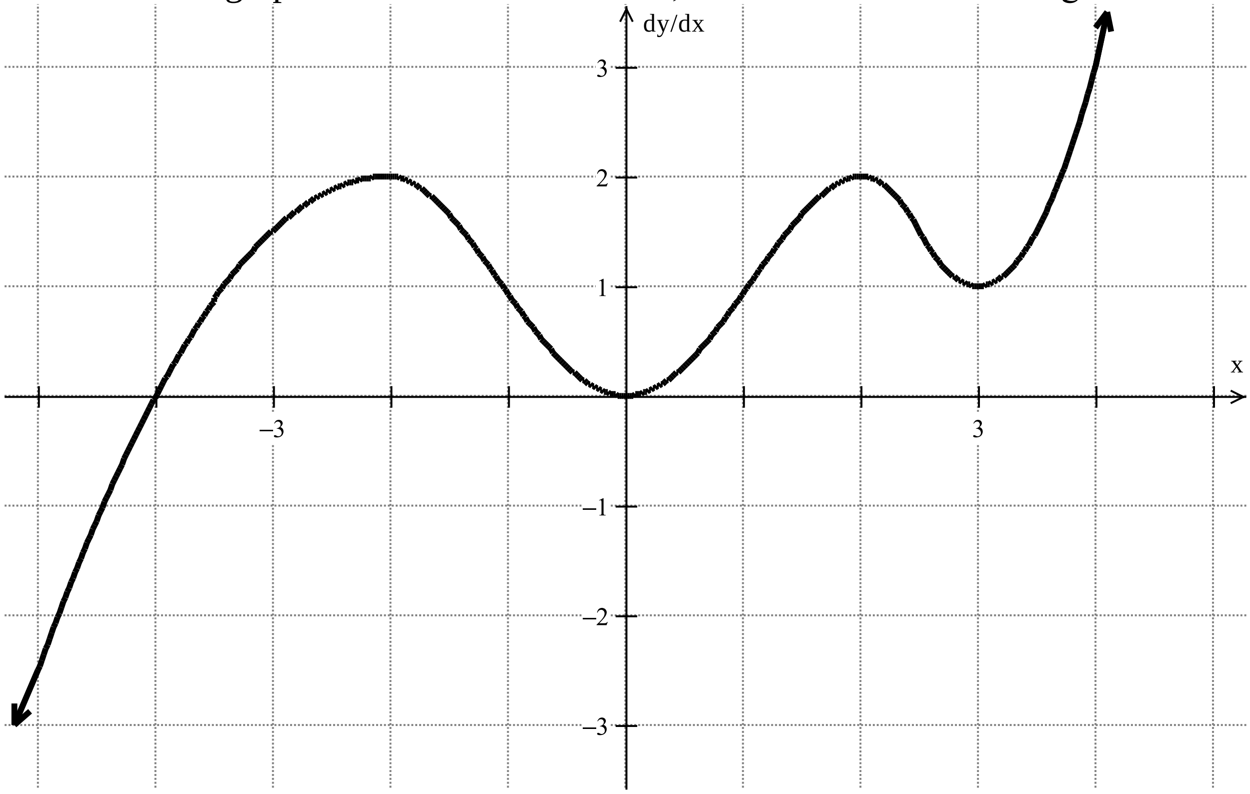
x	$f(x)$	$f'(x)$	$f''(x)$
$x < 2$	Negative	Positive	Negative
$x = 2$	0	Positive	Negative
$2 < x < 3$	Positive	Positive	Negative
$x = 3$	5	0	Negative
$3 < x < 5$	Positive	Negative	Negative
$x = 5$	0	Negative	0
$5 < x < 7$	Negative	Negative	Positive
$x = 7$	–9	0	Positive
$7 < x < 9$	Negative	Positive	Positive
$x = 9$	–1	Positive	0
$9 < x < 10$	Negative	Positive	Negative
$x = 10$	0	Positive	Negative
$10 < x$	Positive	Positive	Negative

17. It costs you c dollars each to manufacture and distribute video games. If you sell the games at x dollars each, you will sell n games, where $n = \frac{x - c}{a} + b(100 - x)$, where a and b are positive constants, and $a < b$.

a) Write an equation that represents profit for selling n video games.

b) What price, x, will maximize profit?

18. Given the graph of *h*' illustrated below, find each of the following for *h*.



a) Find all critical values for *h*.

b) Find which critical values are associated with maxima or minima. Justify your answer.

c) Find all points of inflection for *h*. Explain why they are points of inflection.

19. In a certain community, an epidemic spreads in such a way that the percentage P of the population that is infected after t months is modeled by

$$P(t) = \frac{kt^2}{\left(C + t^2\right)^2},$$

where C and k are constants. Find t, in terms of C, such that P is least.

Answers: Chapter 3 Test

1) If the position of a particle is given by $x(t)=\ln(t^2+9)$, find $v(t)$, $a(t)$, and find when the particle is stopped. When the particle is stopped, what is the position and acceleration?

$v(t)=\dfrac{2t}{t^2+9}$ Particle stops at $t = 0$; $x(0)=\ln(9)$, $a(0)=\dfrac{2}{9}$

$a(t)=\dfrac{18-2t^2}{(t^2+9)^2}$

2) For the function $V=\dfrac{1}{3}\pi r^2h+2\pi r^3$

a) find the rate of change of *V* with respect to *r*, assuming *h* is constant.

$\dfrac{dV}{dr}=\dfrac{2}{3}\pi rh+6\pi r^2$

b) find the rate of change of *V* with respect to *h*, assuming *r* is constant.

$\dfrac{dV}{dh}=\dfrac{1}{3}\pi r^2$

c) find the rate of change of *V* with respect to *t*, assuming *r* and *h* are both variables.

$\dfrac{dV}{dt}=\dfrac{2}{3}\pi rh\dfrac{dr}{dt}+\dfrac{1}{3}\pi r^2\dfrac{dh}{dt}+6\pi r^2\dfrac{dr}{dt}$

3) Two cars are leaving an intersection; one headed north, the other headed east. The northbound car is traveling at 35 miles per hour, while the eastbound car is traveling at 45 miles per hour. Find the rate at which the ***direct distance*** is increasing when the eastbound car is 0.6 miles from the intersection and the northbound car is 0.8 miles from the intersection.

55 miles per hour

4) You spill some milk on a tablecloth, and you notice that the stain is elliptical and that the major axis (*x*) is always twice the minor axis (*y*). Given that the area of an ellipse is $A=\dfrac{\pi}{4}xy$, find how fast the area of the stain is increasing when $x = 6$ mm, $y = 3$ mm, and the minor axis is increasing at 0.2 mm per second.

$\dfrac{dA}{dt}=0.6\pi\ \text{mm}^2/\text{sec}$

5) Find $\frac{dy}{dx}$ for $e^{y^2}+5xy=\tan(y+1)+\ln(x+1)$

$$\frac{dy}{dx}=\frac{1-5xy-5y}{(x+1)\left(2ye^{y^2}+5x-\sec^2(y+1)\right)}$$

6) Given the function $x^2-y^2=16$

a) Use implicit differentiation to show that $\frac{dy}{dx}=\frac{x}{y}$

$$\frac{d}{dx}\left[x^2-y^2=16\right]$$

$$2x-2y\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{x}{y}$$

b) Use implicit differentiation on $\frac{dy}{dx}$ to show that $\frac{d^2y}{dx^2}=\frac{y^2-x^2}{y^3}$

$$\frac{d}{dx}\left[\frac{dy}{dx}=\frac{x}{y}\right]$$

$$\frac{d^2y}{dx^2}=\frac{y-x\frac{dy}{dx}}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{y-x\frac{x}{y}}{y^2}$$

$$\frac{d^2y}{dx^2}=\frac{y^2-x^2}{y^3}$$

7) Given the position function $x(t)=2\tan^{-1}(t)$

a) Find the velocity and the acceleration functions for this position function.

$v(t)=\frac{2}{1+t^2}$ $\qquad$ $a(t)=-4t\left(1+t^2\right)^{-2}$

b) Find the position, velocity, and acceleration at $t=-1$

$x(-1)=-\frac{\pi}{2}$ $\qquad$ $v(-1)=1$ $\qquad$ $a(-1)=1$

c) In which direction is the particle moving at that time?
The particle is moving right because the velocity is positive.

d) Is the particle speeding up or slowing down at that time?
It is speeding up because velocity and acceleration are both positive

8) You are filling a conical glass with a total height of 5 inches and a radius of 5 inches at the top. If you pour water into the glass at a rate of 3 cubic inches per second, how fast is the height increasing when the radius of the liquid in the glass is 3 inches.

$$\frac{dh}{dt} = \sqrt{\frac{1}{3\pi}} \quad \text{cm/sec}$$

9) Your friend has an ice cream cone that is melting. He notices that the ice cream is dripping from the bottom of the cone at a rate of $0.4 \text{ cm}^3/\text{sec}$. He also notices that the level of the ice cream in the cone is at a height of 4 cm, and that the radius of the cone at that point is equal to the height. He poses the following solution to how fast the height of the ice cream is changing:

$$V_{cone} = \frac{1}{3}\pi r^2 h \qquad r = h$$

$$V_{cone} = \frac{1}{3}\pi h^3 \qquad h = 4$$

$$\text{so } V_{cone} = \frac{1}{3}\pi(4)^3$$

$$V_{cone} = \frac{64\pi}{3}$$

$$\frac{dV}{dt} = 0$$

He says, “Obviously, my ice cream cone isn’t really melting.”

Your other friend says, "No, no, no – you have that all wrong. This is what you should do:"

$$V_{cone} = \frac{1}{3}\pi r^2 h \qquad r = h$$

$$V_{cone} = \frac{1}{3}\pi h^3$$

$$\frac{dV}{dt} = \pi h^2 \frac{dh}{dt} \qquad h = 4, \frac{dV}{dt} = 0.4$$

$$0.4 = 16\pi \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0.4}{16\pi} = 0.00796 \text{ cm/sec}$$

"So the level of ice cream is rising at 0.00796 cm/sec."

Is either of your friends' solution right? Correct the mistakes in their work, if there are any, and explain what went wrong.

Both the solutions are wrong. The first friend assumed that *h* was constant (which it was not) leading to a derivative = 0. The second friend missed the fact that the ice cream dripping out was a decreasing volume, making $\frac{dV}{dt}$ negative.

Find the critical values and extremes for each of the following functions in problems 10 and 11:

10) $y = (x+1)e^{-x}$

Critical Values: $x = 0$

Extreme Values: $y = 1$

11) $y = 2x - 2\tan x$ on $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

Critical Values: $x = -\frac{\pi}{4},\ 0,\ \frac{\pi}{4}$

Extreme Values: $x = -\frac{\pi}{2} + 2,\ 0,\ \frac{\pi}{2} - 2$

12) Given the table of values for $f(x)$ and its first and second derivatives, find the points at which the curve is at a maximum or minimum. Explain why this is the case.

x	-4	-2	0	6	9	12
$f(x)$	18	-27	30	42	67	-11
$f'(x)$	19	0	-22	0	0	83
$f''(x)$	18	-28	23	19	-12	23

(–2, –27) is a maximum because the first derivative = 0 and the curve is concave down.
(6, 42) is a minimum because the first derivative = 0 and the curve is concave up.
(9, 67) is a maximum because the first derivative = 0 and the curve is concave down.

13) For each of the following, list the point and describe what is happening at each point on $f(x)$:

a) $x=2$, $f(2)=-5$, $f'(2)=-13$, $f''(2)=27$

(2, –5)decreasing, concave up

b) $x=3$, $f(3)=9$, $f'(3)=0$, $f''(3)=-29$

(3, 9) Maximum

c) $x=-1$, $f(-1)=0$, $f'(-1)=0$, $f''(-1)=18$

(–1,0) Minimum

14) Below is a chart showing the volume of water flowing through a pipeline according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	2	4	6	8	10	12
$V(t)$ (in m^3)	12	14	8	12	19	18	18

a) Find the approximate value of $V'(2)$. Show your work and use units.

$V'(2)\approx -1$ m^3/minute

b) Find the approximate value of $V'(4)$. Show your work and use units.

$V'(4)\approx -\frac{1}{2}$ m^3/minute

c) Find the approximate value of $V'(10)$. Show your work and use units.

$V'(10)\approx -\frac{1}{4}$ m^3/minute

d) Given that $V(t)$ is continuous and differentiable on the interval $0< t <12$. Must there be a value of c in that interval such that $V'(c)$ equals $\frac{V(12)-V(0)}{12-0}$. Explain why or why not.

Yes, because of the Mean Value Theorem

15) Given the sign patterns below, find the critical values and determine which ones are associated with maximums and which are associated with minimums. Explain why. Assume y is a continuous function.

a)

$\frac{dy}{dx}$	–	0	+	D.N.E.	+	0	–
x		-3		2		7	

Maximums:

$x = 7$

$\frac{dy}{dx} = 0$ and switches positive to negative.

Minimums:

$x = -3$

$\frac{dy}{dx} = 0$ and switches negative to posative.

b)

$\frac{dy}{dx}$	–	0	+	D.N.E.	–	0	+
x		-2		0		3	

Maximums:

$x = 0$

$\frac{dy}{dx} = \text{DNE}$ and switches positive to negative.

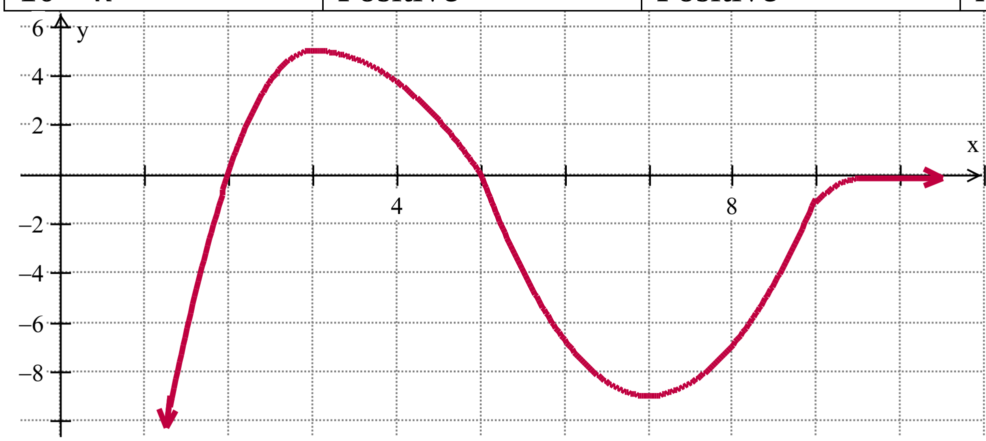
Minimums:

$x = -2$ and 3

$\frac{dy}{dx} = 0$ and switches negative to posative.

16. Sketch a graph with the following traits:

x	$f(x)$	$f'(x)$	$f''(x)$
$x < 2$	Negative	Positive	Negative
$x = 2$	0	Positive	Negative
$2< x <3$	Positive	Positive	Negative
$x = 3$	5	0	Negative
$3< x <5$	Positive	Negative	Negative
$x = 5$	0	Negative	0
$5< x <7$	Negative	Negative	Positive
$x = 7$	–9	0	Positive
$7< x <9$	Negative	Positive	Positive
$x = 9$	–1	Positive	0
$9< x <10$	Negative	Positive	Negative
$x = 10$	0	Positive	Negative
$10 < x$	Positive	Positive	Negative



17. It costs you c dollars each to manufacture and distribute video games. If you sell the games at x dollars each, you will sell n games, where $n=\dfrac{x-c}{a}+b(100-x)$, where a and b are positive constants, and $a < b$.

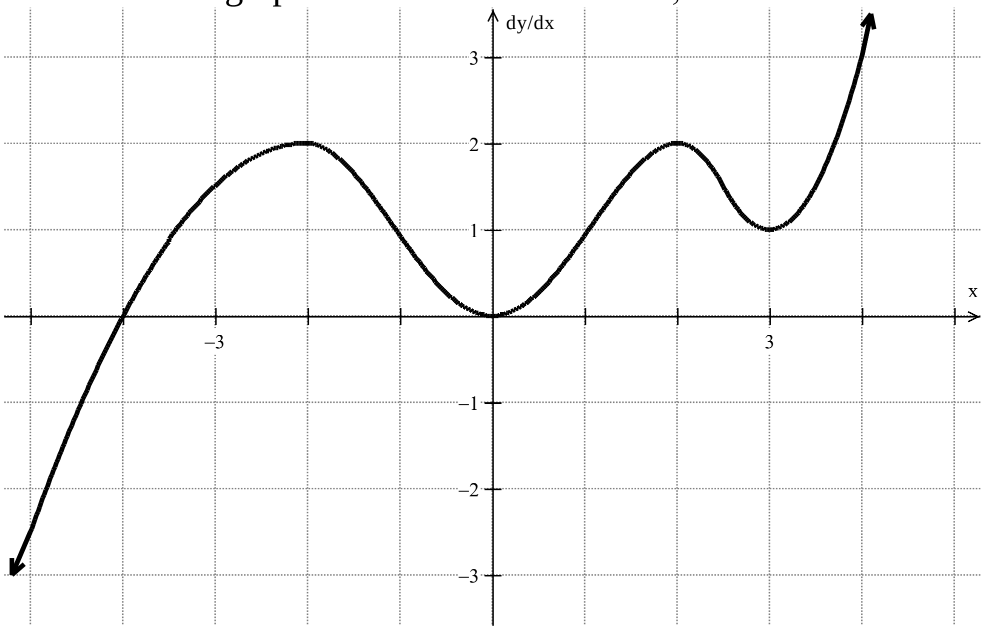
a) Write an equation that represents profit for selling n video games.

$$P(x)=\frac{x(x-c)}{a}+bx(100-x)-cx$$

b) What price, x, will maximize profit?

$$x=\frac{ac-100ab+c}{2-2ab}$$

18. Given the graph of h' illustrated below, find each of the following for h.



a) Find all critical values for h.
$x = -4$

b) Find which critical values are associated with maxima or minima. Justify your answer.
It is associated with a minimum; f' switches from negative to positive, which means f switches from decreasing to increasing.

c) Find all points of inflection for h. Explain why they are points of inflection.
$x = -2, 0, 2, 3$, because f' goes increasing to decreasing (for $x = -2, 2$) or decreasing to increasing (for $x = 0, 3$) which means that f'' switches signs; therefore f switches concavity at those values of x.

19. In a certain community, an epidemic spreads in such a way that the percentage P of the population that is infected after t months is modeled by

$$P(t) = \frac{kt^2}{\left(t^2 - C\right)^2},\ t \geq 0$$

where C and k are positive constant. Find t, in terms of C and k, such that P is least.

P is at a maximum when $t = \sqrt{C}$

3.__: Optional Topic: Logarithmic Differentiation

With implicit differentiation and the chain rule, we learned some powerful tools for differentiating functions and relations. The product and quotient rules also allowed us to take derivatives of certain functions that would otherwise be impossible to differentiate. Sometimes, however, with very complex functions, it becomes easier to manipulate an equation so that it is easier to take the derivative. This is where logarithmic differentiation comes in.

OBJECTIVES

- Determine when it is appropriate to use logarithmic differentiation.
- Use logarithmic differentiation to take the derivatives of complicated functions.

Before we begin, it would be helpful to look at a few rules that we should remember from algebra and precalculus concerning logarithms.

$$a^x a^y = a^{x+y} \qquad \log_a x + \log_a y = \log_a (xy)$$
$$\frac{a^x}{a^y} = a^{x-y} \qquad \log_a x - \log_a y = \log_a \frac{x}{y}$$
$$\left(a^x\right)^y = a^{xy} \qquad \log_a x^n = n \log_a x$$

Since logarithms are exponents expressed in a different form, all of the above rules are derived from the rules for exponents, and you can see the corresponding exponential rule. Because of our algebraic rules, we can do whatever we want to both sides of an equation. In algebra, we usually used this to solve for a variable. In Calculus, we can use this principle to make many derivative problems significantly easier.

Ex 1 Find the derivative of $y=\left(x^2+7x-3\right)\left(\sin(x)\right)$

What we would traditionally use to take the derivative of this function is the product rule.

$$\frac{d}{dx}\left[y=\left(x^2+7x-3\right)\left(\sin(x)\right)\right]$$
$$\frac{dy}{dx}=\left(x^2+7x-3\right)\left(\cos(x)\right)+\left(2x+7\right)\left(\sin(x)\right)$$

Obviously, this is a straightforward problem that can be easily done using the product rule. If, however, I took the natural log of both sides of the equation, I can achieve the same results, and never use the product rule. (Remember, we will almost exclusively use the natural log because it works so well within the framework of Calculus)

$$\ln(y)=\ln\left[\left(x^2+7x-3\right)\left(\sin(x)\right)\right]$$
$$\ln(y)=\ln\left(x^2+7x-3\right)+\ln\left(\sin(x)\right)$$ This is simplifying using log rules.
$$\frac{d}{dx}\left[\ln(y)=\ln\left(x^2+7x-3\right)+\ln\left(\sin(x)\right)\right]$$
$$\frac{1}{y}\left(\frac{dy}{dx}\right)=\frac{2x+7}{x^2+7x-3}+\frac{\cos(x)}{\sin(x)}$$
$$\frac{dy}{dx}=\left(\frac{2x+7}{x^2+7x-3}+\frac{\cos(x)}{\sin(x)}\right)(y)$$ Now just substitute *y* back in and simplify.
$$\frac{dy}{dx}=\left(\frac{2x+7}{x^2+7x-3}+\frac{\cos(x)}{\sin(x)}\right)\left(x^2+7x-3\right)\left(\sin(x)\right)$$
$$\frac{dy}{dx}=\left(2x+7\right)\left(\sin(x)\right)+\left(\cos(x)\right)\left(x^2+7x-3\right)$$

Clearly, we got the same answer that we got from the product rule, but with significantly more effort. Logarithmic differentiation is a tool we can use, but we have to use it judiciously, we don't want to make problems more difficult than they have to be.

Interestingly enough, logarithmic differentiation can be used to easily derive the product and the quotient rules.

Ex 2 If $y=(u)(v)$, and u and v are both functions of x, find $\frac{dy}{dx}$.

$\ln y=\ln\left[(u)(v)\right]$ Take the log of both sides.

$\ln y=\ln(u)+\ln(v)$ Apply the log rules.

$\frac{d}{dx}\left[\ln y=\ln(u)+\ln(v)\right]$ Take the derivative.

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}$$

$\frac{dy}{dx}=\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}\right)y$ Solve for $\frac{dy}{dx}$, substitute for y, and simplify.

$$\frac{dy}{dx}=\left(\frac{1}{u}\frac{du}{dx}+\frac{1}{v}\frac{dv}{dx}\right)\left((u)(v)\right)$$

$$\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$$

Notice that we have just proved the product rule. This is significantly easier than some of the first proofs of the product rule (you can look them up if you are interested, they interesting in that they involve the limit definition of the derivative). The proof for the quotient rule is very similar, and you will be doing it yourself in the homework.

Where logarithmic differentiation is really useful is in functions that are excessively painful to work with (or impossible to take the derivative of any other way) because of multiple operations.

Ex 3 Find $\frac{dy}{dx}$ for $y=\frac{\left(x^2+5\right)\sin\left(3x^3\right)}{\tan(5x+2)}$

We could take the derivative by applying the chain rule, quotient rule, and product rule, but that would be a time-consuming and tedious process. It's much easier to take the log of both sides, simplify and then take the derivative.

$$\ln y = \ln\left[\frac{\left(x^2+5\right)\sin\left(3x^3\right)}{\tan\left(5x+2\right)}\right]$$

$$\ln y = \ln\left(x^2+5\right)+\ln\left(\sin\left(3x^3\right)\right)-\ln\left(\tan\left(5x+2\right)\right)$$

$$\frac{d}{dx}\left[\ln y = \ln\left(x^2+5\right)+\ln\left(\sin\left(3x^3\right)\right)-\ln\left(\tan\left(5x+2\right)\right)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+5}+\frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)}-\frac{5\sec^2\left(5x+2\right)}{\tan\left(5x+2\right)}$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2+5}+\frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)}-\frac{5\sec^2\left(5x+2\right)}{\tan\left(5x+2\right)}\right)y$$

$$\frac{dy}{dx} = \left(\frac{2x}{x^2+5}+\frac{9x^2\cos\left(3x^3\right)}{\sin\left(3x^3\right)}-\frac{5\sec^2\left(5x+2\right)}{\tan\left(5x+2\right)}\right)\left(\frac{\left(x^2+5\right)\sin\left(3x^3\right)}{\tan\left(5x+2\right)}\right)$$

Now that may seem long and messy, but try it any other way, and you might end up taking a lot more time, with a lot more algebra and a lot more potential spots to make mistakes.

Ex 4 Find $f'(\pi)$ for $f(z) = z^{\cos z}$

$$\ln\left[f(z)\right] = \ln\left[z^{\cos z}\right]$$

$$\frac{d}{dx}\left(\ln\left[f(z)\right] = (\cos z)(\ln z)\right)$$

$$\frac{f'(z)}{f(z)} = \frac{\cos z}{z}-(\ln z)(\sin z)$$

$$f'(z) = \left(\frac{\cos z}{z}-(\ln z)(\sin z)\right)f(z)$$

$$f'(z) = \left(\frac{\cos z}{z}-(\ln z)(\sin z)\right)\left(z^{\cos z}\right)$$

$$f'(\pi) = \left(\frac{\cos \pi}{\pi}-(\ln \pi)(\sin \pi)\right)\left(\pi^{\cos \pi}\right) = -\frac{1}{\pi^2}$$

You could have also done this problem using the change of base property that you learned in Precalculus and you would get the same answer in about the same number of steps, but you would have to remember how to change the base of an exponential function using an old algebra rule.

Ex 4B Find $f'(\pi)$ for $f(z)=z^{\cos z}$

$$f(z)=z^{\cos z}$$

$$f(z)=e^{\cos z \cdot \ln z} \qquad \text{Since } a^x=e^{x\ln a}$$

$$f'(z)=e^{\cos z \cdot \ln z}\left(\cos z \cdot \frac{1}{z}-\sin z \cdot \ln z\right)$$

$$f'(z)=e^{\cos z \cdot \ln z}\left(\frac{\cos z}{z}-(\ln z)(\sin z)\right)$$

$$f'(z)=z^{\cos z}\left(\frac{\cos z}{z}-(\ln z)(\sin z)\right)$$

$$f'(\pi)=\left(\frac{\cos \pi}{\pi}-(\ln \pi)(\sin \pi)\right)\left(\pi^{\cos \pi}\right)=-\frac{1}{\pi^2}$$

Again, there are often more than one way to do a specific problem, and part of what we do as mathematicians is decide on the simplest **correct** method to solving a problem.

The issue many people have when learning more difficult mathematical concepts is that they try to oversimplify a problem and end up getting it wrong as a result.

As Albert Einstein once said, “Make things as simple as possible but not simpler.”

3.__ Homework Set A

Find the derivatives of the following functions. Use logarithmic differentiation **when appropriate.**

1. $y = (2x+1)^4 (x^3 - 3)^5$

2. $z = (y^3 - 3)e^{(2y+1)}$

3. $y = \dfrac{\sin^2 x \ \tan^4 x}{(x^2+5)^2}$

4. $g(t) = t \ \ln(t)$

5. $y = \ln^x(x)$

6. $p(v) = v^{e^v}$

7. Use logarithmic differentiation to prove the quotient rule.

8. For the function, $f(x) = x^{\ln x}$, find an equation for a tangent line at $x = e$, and use that to approximate the value for $f(2.7)$. Find the percent difference between this and the actual value of $f(2.7)$.

10. Explain why you think we use natural logs rather than other bases for logs in Calculus. (Hint: think back to the derivative rules)

3.__ Homework Set B

1. Use logarithmic differentiation to find $\frac{dq}{dt}$ if $q = \frac{e^{t^4-15}\sin^5 3t}{(\ln t)^{10}}$

2. Use logarithmic differentiation to find $\frac{dy}{dx}$ when $y = e^{150x-19}\ln(\sin x)^{100}\sqrt{x^2-1}$

3. Use logarithmic differentiation to find $\frac{dq}{dt}$ if $q = \frac{e^{t^4-15}\csc^5 3t}{\ln t^{10}}$

4. Find $\frac{dy}{dx}$ for the function $y = \left(e^{17x^4}\right)\left(\sin^7 x\right)\left(5x - 17\right)^{12}\left(\cot 5x\right)$

Answers: 3.__ Homework Set A

1. $y=(2x+1)^4(x^3-3)^5$

$$\frac{dy}{dx}=(2x+1)^4(x^3-3)^5\left(\frac{8}{2x+1}+\frac{15x^2}{x^3-3}\right)$$

2. $z=(y^3-3)e^{(2y+1)}$

$$\frac{dz}{dy}=(y^3-3)e^{(2y+1)}\left[\frac{3y^2}{y^3-3}+2\right]$$

3. $y=\dfrac{\sin^2 x\ \tan^4 x}{(x^2+5)^2}$

$$\frac{dy}{dx}=\frac{\sin^2 x\ \tan^4 x}{(x^2+5)^2}\left(2\cot x+\frac{4}{\sin x\cos x}+\frac{4x}{x^2+5}\right)$$

4. $g(t)=t\ \ln(t)$

$$g'(t)=1+\ln(t)$$

5. $y=\ \ln^x(x)$

$$\frac{dy}{dx}=\ \ln^x(x)\left[\frac{1}{\ln x}+\ln(\ln x)\right]$$

6. $p(v)=v^{e^v}$

$$\frac{dp}{dv}=v^{e^v}\left[\frac{e^v}{v}+e^v\ln v\right]$$

7. Use logarithmic differentiation to prove the quotient rule.

$$y=\frac{u}{v}$$

$$\ln(y)=\ln(u)-\ln(v)$$

$$\frac{d}{dx}\left[\ln(y)=\ln(u)-\ln(v)\right]$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}-\frac{1}{v}\frac{dv}{dx}$$

$$\left[\frac{1}{y}\frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}-\frac{1}{v}\frac{dv}{dx}\right]y$$

$$\frac{dy}{dx}=\frac{y}{u}\frac{du}{dx}-\frac{y}{v}\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{u/v}{u}\frac{du}{dx}-\frac{u/v}{v}\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{1}{v}\frac{du}{dx}-\frac{u}{v^2}\frac{dv}{dx}$$

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

8. For the function, $f(x) = x^{\ln x}$, find an equation for a tangent line at $x = e$, and use that to approximate the value for $f(2.7)$. Find the percent difference between this and the actual value of $f(2.7)$.

$y - e = 1(x - e)$ and $f(2.7) \approx 2.7$

$f(2.7) = 2.681963256...$ which is a 0.673% difference.

9. Explain why you think we use natural logs rather than other bases for logs in Calculus. (Hint: think back to the derivative rules)

We use natural logs because the derivative rule is significantly easier for problems involving the natural log, rather than the common (or other) logs.

3.__ Homework Set B

1. Use logarithmic differentiation to find $\frac{dq}{dt}$ if $q = \frac{e^{t^4-15}\sin^5 3t}{(\ln t)^{10}}$

$$\frac{dq}{dt} = \left(\frac{e^{t^4-15}\sin^5 3t}{(\ln t)^{10}}\right)\left(4t^3 + 15\cot(3t) - \frac{10}{t\ln t}\right)$$

2. Use logarithmic differentiation to find $\frac{dy}{dx}$ when

$$\frac{dy}{dx} = \left(e^{150x-19}\ln(\sin x)^{100}\sqrt{x^2-1}\right)\left(-19 + \frac{\cot x}{\ln(\sin x)} + \frac{x}{x^2-1}\right)$$

3. Use logarithmic differentiation to find $\frac{dq}{dt}$ if $q = \frac{e^{t^4-15}\csc^5 3t}{\ln t^{10}}$

$$\frac{dq}{dt} = \left(\frac{e^{t^4-15}\csc^5 3t}{\ln t^{10}}\right)\left(4t^3 - 15\cot(3t) - \frac{1}{t\ln t}\right)$$

4. Find $\frac{dy}{dx}$ for the function $y = \left(e^{17x^4}\right)\left(\sin^7 x\right)(5x-17)^{12}(\cot 5x)$

$$\frac{dy}{dx} = \left[\left(e^{17x^4}\right)\left(\sin^7 x\right)(5x-17)^{12}(\cot 5x)\right]\left(68x^3 + 7\cot x + \frac{60}{5x-17} - \frac{5\csc^2 5x}{\cot 5x}\right)$$

Chapter 4 Overview: Definite Integrals

In the Introduction to this book, we pointed out that there are four tools or operations in Calculus. This chapter presents the fourth—the Definite Integral. Where the indefinite integral was defined as the inverse operation to the derivative (i.e., the Anti-Derivative), the definite integral links this algebraic process to the geometric concept of the area under the curve.

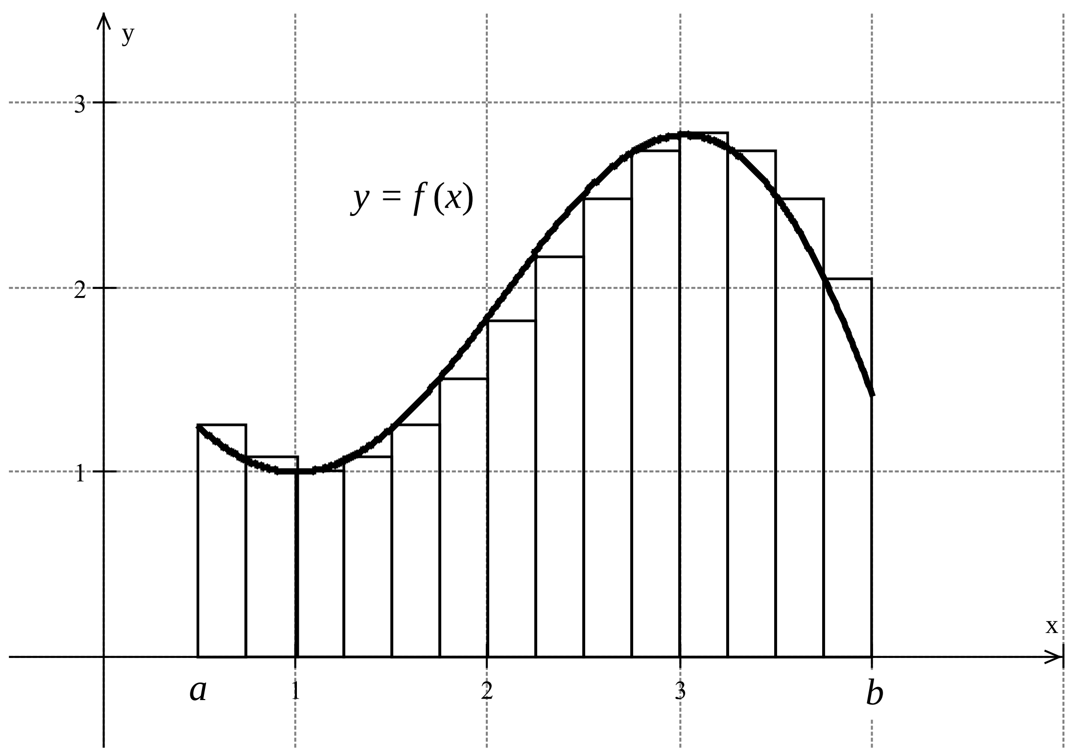
We will study the Fundamental Theorem of Calculus, which establishes the link between the algebra and the geometry, with an emphasis on the mechanics of how to find the definite integral. We will consider the differences implied between the context of the definite integral as an operation and as an area accumulator. We will learn some approximation techniques for definite integrals and see how they provide theoretical foundation for the integral. We will revisit graphical analysis in terms of the definite integral and view another typical AP context for it. Finally, we will consider what happens when trying to integrate at or near an asymptote.

The integral function is actually an accumulator. The symbol $\int$ comes from the 17^{th} century S and stands for sum. In fact, it is a sum of an infinite number of terms which are infinitely small. One of the key applications of integration (which we will see in a later chapter as well as this one) involves finding the area under a curve by using this "addition" process.

4.1 The Fundamental Theorem of Calculus

As noted in the overview, Anti-derivatives are known as Indefinite Integrals and this is because the answer is a function, not a definite number. But there is a time when the integral represents a number. That is when the integral is used in an Analytic Geometry context of area. Though it is not necessary to know the theory behind this to be able to do it, the theory is a major subject of Integral Calculus, so we will explore it briefly.

We know, from Geometry, how to find the exact area of various polygons, but we never considered figures where one side is not made of a line segment. Here we want to consider a figure where one side is the curve $y = f(x)$ and the other sides are the x-axis and the lines $x = a$ and $x = b$.



As we can see above, the area can be approximated by rectangles whose height is the y-value of the equation and whose width we will call Δx. The more rectangles we make, the better the approximation. The area of each rectangle would be $f(x)\cdot\Delta x$ and the total area of n rectangles would be $A = \sum_{i=1}^{n} f(x_i)\cdot\Delta x$. If we could make an infinite number of rectangles (which would be infinitely thin), we would have the exact area. The rectangles can be drawn several ways--with the left side at the height of the curve (as drawn above), with the right side at the curve, with the rectangle straddling the curve, or even with rectangles of different widths. But once they become infinitely thin, it will not matter how they were drawn--they will have virtually no width (represented by *dx* instead of the Δx) and a height equal to the *y*-value of the curve.

We can make an infinite number of rectangles mathematically by taking the Limit as *n* approaches infinity, or

$$\underset{n\to\infty}{Lim}\sum_{i=1}^{n} f\left(x_i\right)\cdot \Delta x\,.$$

This limit is rewritten as the Definite Integral:

$$\int_a^b f\left(x\right)\,dx$$

b is the "upper bound" and *a* is the "lower bound," and would not mean much if it were not for the following rule:

The Fundamental Theorem of Calculus

If $f\left(x\right)$ is a continuous function on $\left[a,b\right]$, then

1) $\frac{d}{dx}\int_c^x f\left(t\right)\,dt = f\left(x\right)$ or $\frac{d}{dx}\int_c^u f\left(t\right)\,dt = f\left(u\right)D_u$

2) If $F'\left(x\right)= f\left(x\right)$, then $\int_a^b f\left(x\right)\,dx = F\left(b\right) - F\left(a\right)$.

The first part of the Fundamental Theorem of Calculus simply says what we already know--that an integral is an anti-derivative. The second part of the Fundamental Theorem says the answer to a definite integral is the difference between the anti-derivative at the upper bound and the anti-derivative at the lower bound.

This idea of the integral meaning the area may not make sense initially, mainly because we are used to Geometry, where area is always measured in square units. But that is only because the length and width are always measured in the same kind of units, so multiplying length and width must yield square units. We are expanding our vision beyond that narrow view of things here. Consider a graph where the x-axis is time in seconds and the y-axis is velocity in feet per second. The area under the curve would be measured as seconds multiplied by feet/sec--that is, feet. So the area under the curve equals the distance traveled in feet. In other words, the integral of velocity is distance.

Objectives
Evaluate Definite Integrals Find the average value of a continuous function over a given interval Differentiate integral expressions with the variable in the boundary

Let us first consider Part 2 of the Fundamental Theorem, since it has a very practical application. This part of the Fundamental Theorem gives us a method for evaluating definite integrals.

Ex 1 Evaluate $\int_2^8 (4x+3)dx$

$\int_2^8 (4x+3)dx = 2x^2+3x\Big\|_2^8$	The antiderivative of $4x + 3$ is $2x^2 + 3x$. We use this notation when we apply the Fundamental Theorem
$=\left[(128+24)-(8+6)\right]$	Plug the upper limit of integration into the antiderivative and subtract it from the lower limit when plugged into the antiderivative
$= 138$	Check with Math 9.

Ex 2 Evaluate $\int_1^4 \frac{1}{\sqrt{x}}dx$

$$\int_1^4 \frac{1}{\sqrt{x}}dx = 2\sqrt{x}\Big|_1^4$$
$$=\left[(4)-(2)\right]$$
$$=2$$

Ex 3 Evaluate $\int_0^{\pi/2} \sin x dx$

$$\int_0^{\pi/2} \sin x dx = -\cos x \Big|_0^{\pi/2}$$
$$= -[0-1]$$
$$= 1$$

Ex 4 Evaluate $\int_1^2 \frac{4+u^2}{u^3} du$

$$\int_1^2 \frac{4+u^2}{u^3} du = \int_1^2 \left(4u^{-3} + u^{-1}\right) du$$
$$= -2u^{-2} + \ln|u| \Big|_1^2$$
$$= \left[\left(-2 \cdot \frac{1}{2^2} + \ln 2\right) - \left(-2 + \ln 1\right)\right]$$
$$= \frac{3}{2} + \ln 2$$

Ex 5 Evaluate $\int_{-5}^5 \frac{1}{x^3} dx$

This is a trick! We use the Fundamental Theorem on this integral because the curve is not continuous on $x \in [a,b]$. When $x = 0$, $\frac{1}{x^3}$ does not exist, so the Fundamental Theorem of Calculus does not apply.

One simple application of the definite integral is the Average Value Theorem. We all recall how to find the average of a finite set of numbers, namely, the total of the numbers divided by how many numbers there are. But what does it mean to take the average value of a continuous function? Let's say you drive from home to school – what was your average velocity (velocity is continuous)? What was the average temperature today (temperature is continuous)? What was your average height for the first 15 years of your life (height is continuous)? Since the integral is the sum of infinite number of function values, the formula below answers those questions.

Average Value Formula:

The average value of a function *f* on a closed interval $\left[a,b\right]$ is defined by

$$f_{avg} = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

If we look at this formula in the context of the Fundamental Theorem of Calculus, it can start to make a little more sense.

$\int_a^b f\left(x\right)\,dx = F\left(b\right) - F\left(a\right)$ The Fundamental Theorem of Calculus

$$\frac{1}{b-a}\int_a^b f\left(x\right)\,dx = \frac{1}{b-a}\left[F\left(b\right)-F\left(a\right)\right]$$
$$= \frac{F\left(b\right)-F\left(a\right)}{b-a}$$

Notice that this is just the average slope for $F\left(x\right)$ on $x \in \left[a,b\right]$. The average slope of $F\left(x\right)$ would be the average value of $F'\left(x\right)$. But since the definition in the Fundamental Theorem of Calculus says that $F'\left(x\right) = f\left(x\right)$, this is actually just the average value of $f\left(x\right)$.

Ex 6 Find the average value of $f\left(x\right) = x^2 + 1$ on $\left[0,5\right]$.

$$f_{avg} = \frac{1}{b-a}\int_a^b f\left(x\right)dx$$

$f_{avg} = \frac{1}{5-0}\int_0^5 \left(x^2+1\right)dx$ Substitute in the function and interval

$f_{avg} = {}^{28}\!/_{3}$ Use Math 9 (or integrate analytically) to calculate answer

Ex 7 Find the average value of $h(\theta) = \sec\theta\tan\theta$ on $\left[0, \frac{\pi}{4}\right]$.

$$\begin{aligned} h_{avg} &= \frac{1}{b-a}\int_a^b h(\theta)\,d\theta \\ &= \frac{1}{\frac{\pi}{4}-0}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta \\ &= .524 \end{aligned}$$

Now let's take a look at the First Part of the Fundamental Theorem of Calculus:

$$\boxed{\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \text{ or } \frac{d}{dx}\int_a^u f(t)\,dt = f(u)\cdot\frac{du}{dx}}$$

This part of the Fundamental Theorem of Calculus says that derivatives and integrals are inverses operations of each other. Notice how we have a function that is defined as the integral of function *f* – with the variable in the upper limit of integration – and when we take the derivative of that function, we are only left with function *f*. That is, when we take the derivative of an integral, they cancel out … like squaring and square rooting.

Confusion usually pops up with the seemingly "extra" variable *t* showing up. It is a dummy variable, somewhat like the parameter in parametric mode. It would disappear in the process of integrating. In terms of the AP Test, the symbology is what is important.

Ex 8 Let $g(x)=\int_1^x \cos(t)dt$. Show that if we take the derivative of *g* we will wind up with $\cos x$.

From Part 1 of the Fundamental Theorem of Calculus, we know that

$$g(x)=\int_1^x \cos(t)\,dt \;=\sin(t)\Big|_1^x$$
$$= \sin x-\sin 1$$

We have established that for our function defined *as an integral*, $g(x)=\sin x-\sin 1$. Let us now take the derivative of *g*.

$$g(x)=\sin x-\sin 1\Rightarrow g'(x)=\cos x$$

One thing that you might notice is that when we use the Fundamental Theorem of Calculus and plug in the boundaries, one of the function values becomes constant (sin 1 in this case), and the derivative of a constant is 0. This is why it doesn't matter what constant is the lower bound in this formula. It will always disappear with the derivative.

You should also note that the variable in this equation was in the upper limit of integration and that it was simply the variable *x*. If it was something other than just *x*, then we would have to use the chain rule in the derivative.

Ex 9 Find the derivatives of (a) $f(x)=\int_1^x \ln t\,dt$

(b) $g(x)=\int_1^u \frac{1}{x+x^2}dx$

(c) $h(x)=\int_1^{x^2} \cos t\,dt$

(d) $H(x)=\int_1^{\sqrt{x}} 2t^2\,dt$

(a) $f(x)=\int_1^x \ln t\,dt \Rightarrow f'(x)=\ln x$

(b) $g(u)=\int_3^u \frac{1}{x+x^2}dx\Rightarrow g'(u)=\frac{1}{u+u^2}$

(c) $h(x)=\int_1^{x^2}\cos t\,dt \Rightarrow h'(x)=\cos x^2\cdot 2x$

(d) $H(x)=\int_0^{\sqrt{x}} 2t^2 dt \Rightarrow H'(x)=2\left(x^{1/2}\right)^2 \frac{1}{2}x^{-1/2}=x^{1/2}$

Notice that in each case where the upper bound was not simply a variable, we had to apply the Chain Rule. We never bothered to actually integrate, because we know that the derivative and the integral cancel each other out. We rely on that fact and shortcut a lot of work (and sometimes, the function we are given cannot actually be integrated, so we have to use the rule).

4.1 Homework Set A

Use Part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why it does not exist.

1. $\int_{-1}^{3} x^5 dx$

2. $\int_{2}^{7} (5x-1) dx$

3. $\int_{-5}^{5} \frac{2}{x^3} dx$

4. $\int_{-3}^{-1} \left(\frac{x^7 - 4x^3 - e}{x} \right) dx$

5. $\int_{1}^{2} \frac{3}{t^4} dt$

6. $\int_{\pi/4}^{3\pi/4} \csc y \cot y \, dy$

7. $\int_{0}^{\pi/4} \sec^2 y \, dy$

8. $\int_{1}^{9} \frac{3}{2z} dz$

9. If $F(x)=\int_1^x f(t)dt$, where $f(t)=\int_1^{t^2}\frac{\sqrt{1+u^4}}{u}du$, find $F''(2)$.

Find the Average Value of each of the following functions.

10. $F(x)=(x-3)^2$ on $x\in[3,\ 7]$

11. $H(x)=\sqrt{x}$ on $x\in[0,\ 3]$

12. $F(x)=\sec^2 x$ on $x\in\left[0,\ \frac{\pi}{4}\right]$

13. $F(x)=\frac{1}{x}$ on $x\in[1,\ 3]$

14. If a cookie taken out of a 450˚F oven cools in a 60˚F room, then according to Newton's Law of Cooling, the temperature of the cookie t minutes after it has been taken out of the oven is given by

$$T(t) = 60 + 390e^{-.205t}.$$

What is the average value for the temperature of the cookie during its first 10 minutes out of the oven?

15. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the given equation

$$L(t) = 10 - 3\cos\left(\frac{\pi t}{182}\right),$$

where t the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

16. During one summer in the Sunset, the temperature is modeled by the function $T(t) = 50 + 15\sin\frac{\pi}{12}t$, where T is measured in F° and t is measured in

hours after 7 am. What is the average temperature in the Sunset during the six-hour Chemistry class that runs from 9 am to 3 pm?

Use Part I of the Fundamental Theorem of Calculus to find the derivative of the function.

17. $g(y)=\int_2^y t^2 \sin t\,dt$

18. $g(x)=\int_0^x \sqrt{1+2t}\,dt$

19. $F(x)=\int_x^2 \cos(t^2)\,dt$

20. $h(x)=\int_2^{1/x} \arctan t\,dt$

21. $y=\int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt$

4.1 Homework Set B

1. $\int_{0}^{e^2-1} \frac{1}{x+1}\,dx$

2. $\int_{\pi}^{\frac{5\pi}{4}} \sin y\,dy$

3. Find the average value of $f(t)=t^2-\sqrt{t}+5$ on $t\in[1,4]$

4. $\int_{3}^{5}\left(x^2+5x+6\right)dx$

5. $\int_{3}^{e^2+2} \frac{1}{x-2}\,dx$, show your work, you may use the calculator to verify your answer.

6. $\int_{\pi}^{\frac{3\pi}{4}} \cos y\,dy$, show your work, you may use the calculator to verify your answer.

7. $\int_{5}^{e^3+4} \frac{1}{x-4}\,dx$, show your work, you may use the calculator to verify your answer.

8. Find the average value of $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [4,9]$

9. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin y\,dy$, show your work, you may use the calculator to verify your answer.

10. $\int_{1}^{2} \left(\frac{x^2 - 4x + 7}{x} \right) dx$

11. $\frac{d}{dx}\left[\int_{e}^{x^2} \ln(t^2 + 1)\,dt \right]$

12. If $h(x) = \int_{\pi}^{\sqrt{x}} e^{5t}\,dt$, find $h'(x)$

13. $\dfrac{d}{dx}\int_{10}^{x^2} t\ln(t)\,dt$

14. $h(m) = \int_{5}^{\cos m} t^2 \cos^{-1}(t)\,dt$, find $h'(m)$

15. $h(y) = \int_{5}^{\ln y} \dfrac{e^t}{t^4}\,dt$, find $h'(y)$

16. $\dfrac{d}{dx}\int_{e^x}^{5} \left(t^3 + t + 1\right)dt$

Answers: 4.1 Homework Set A

Use Part II of the Fundamental Theorem of Calculus to evaluate the integral, or explain why it does not exist.

1. $\int_{-1}^{3} x^5 dx$

$= \dfrac{364}{3}$

2. $\int_{2}^{7} (5x-1) dx$

$= 107.5$

3. $\int_{-5}^{5} \dfrac{2}{x^3} dx$

= D.N.E.

4. $\displaystyle\int_{-3}^{-1} \left(\frac{x^7 - 4x^3 - e}{x} \right) dx$

$= 280.605$

5. $\displaystyle\int_{1}^{2} \frac{3}{t^4} dt$

$= \dfrac{7}{8}$

6. $\int_{\pi/4}^{3\pi/4} \csc y \cot y \, dy$

$= 0$

7. $\int_{0}^{\pi/4} \sec^2 y \, dy$

$= 1$

8. $\displaystyle\int_{1}^{9} \frac{3}{2z} dz$

$= 3\ln 3$

9. If $F(x) = \int_{1}^{x} f(t)dt$, where $f(t) = \displaystyle\int_{1}^{t^2} \frac{\sqrt{1+u^4}}{u} du$, find $F''(2)$.

$F''(2) = \dfrac{16}{3}$

10. $F(x) = (x-3)^2$ on $x \in [3, 7]$

Average value of $F(x) = \dfrac{2\sqrt{3}}{3}$

11. $H(x) = \sqrt{x}$ on $x \in [0, 3]$

Average value of $H(x) = \dfrac{4}{\pi}$

12. $F(x) = \sec^2 x$ on $x \in \left[0, \frac{\pi}{4}\right]$

Average value of $F(x) = \frac{1}{2}\ln 3$

13. $F(x) = \frac{1}{x}$ on $x \in [1, 3]$

Average value of $F(x) = \sqrt{257}$

14. If a cookie taken out of a 450˚F oven cools in a 60˚F room, then according to Newton's Law of Cooling, the temperature of the cookie t minutes after it has been taken out of the oven is given by

$$T(t) = 60 + 390e^{-.205t}.$$

What is the average value for the temperature of the cookie during its first 10 minutes out of the oven?

225.753°F

15. We know as the seasons change so do the length of the days. Suppose the length of the day varies sinusoidally with time by the given equation

$$L(t) = 10 - 3\cos\left(\frac{\pi t}{182}\right),$$

where t the number of days after the winter solstice (December 22, 2007). What was the average day length from January 1, 2008 to March 31, 2008?

8.201 hrs

16. During one summer in the Sunset, the temperature is modeled by the function $T(t) = 50 + 15\sin\frac{\pi}{12}t$, where T is measured in F° and t is measured in hours after 7 am. What is the average temperature in the Sunset during the six-hour Chemistry class that runs from 9 am to 3 pm?

63.045°F

17. $g(y) = \int_2^y t^2 \sin t\,dt$

$g'(y) = y^2 \sin y$

18. $g(x) = \int_0^x \sqrt{1+2t}\,dt$

$g'(x) = \sqrt{1+2x}$

19. $F(x) = \int_x^2 \cos(t^2)\,dt$

20. $h(x) = \int_2^{1/x} \arctan t\,dt$

$F'(x) = -\cos(x^2)$

$h'(x) = -\frac{1}{x^2}\arctan \frac{1}{x}$

21. $y = \int_3^{\sqrt{x}} \frac{\cos t}{t}\,dt$

$\frac{dy}{dx} = \frac{\cos\sqrt{x}}{2x}$

4.1 Homework Set B

1. $\int_0^{e^2-1} \frac{1}{x+1}\,dx$

$=2$

2. $\int_{\pi}^{\frac{5\pi}{4}} \sin y\,dy$

$= \frac{1}{\sqrt{2}} - 1$

3. Find the average value of $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [1,4]$

Average value of $f(t) = \frac{94}{3}$

4. $\int_3^5 (x^2 + 5x + 6)\,dx$

$= \frac{254}{3}$

5. $\int_3^{e^2+2} \frac{1}{x-2}\,dx$, show your work, you may use the calculator to verify your answer.

$=2$

6. $\int_{\pi}^{\frac{3\pi}{4}} \cos y\,dy$, show your work, you may use the calculator to verify your answer.

$= \frac{1}{\sqrt{2}}$

7. $\int_5^{e^3+4} \frac{1}{x-4}\,dx$, show your work, you may use the calculator to verify your answer.

$=3$

8. Find the average value of $f(t) = t^2 - \sqrt{t} + 5$ on $t \in [4,9]$

Average value of $f(t) = 234$

9. $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin y\, dy$, show your work, you may use the calculator to verify your answer.

$= \sqrt{2}$

10. $\int_1^2 \left(\frac{x^2 - 4x + 7}{x} \right) dx$

$= -2.5$

11. $\frac{d}{dx}\left[\int_e^{x^2} \ln(t^2+1)\, dt \right]$

$= 2x\ln(x^4+1)$

12. If $h(x) = \int_{\pi}^{\sqrt{x}} e^{5t}\, dt$, find $h'(x)$

$h'(x) = \frac{e^{5\sqrt{x}}}{2\sqrt{x}}$

13. $\frac{d}{dx}\int_{10}^{x^2} t\ln(t)\, dt$

$= 2x^3\ln(x^2)$

14. $h(m) = \int_5^{\cos m} t^2 \cos^{-1}(t)\, dt$, find $h'(m)$

$h'(m) = -m\sin m\cos^2 m$

15. $h(y) = \int_5^{\ln y} \frac{e^t}{t^4}\, dt$, find $h'(y)$

$h'(y) = \frac{y}{y\ln^4 y}$

16. $\frac{d}{dx}\int_{e^x}^{5} (t^3 + t + 1)\, dt$

$= -e^x\left(e^{3x} + e^x + 1\right)$

4.2 Definite Integrals, Riemann Sums, and the Trapezoidal Rule

In Chapter 2, we focused on finding antiderivatives of analytic functions (that is functions that we could actually integrate). But what if we have an analytical function but we don't know it's antiderivative (think $\int_{-2}^{3} e^{x^2}\,dx$)? What can we do then? Or what if we have a set of values for a function, but no actual expression representing the function (like a table of experimentally collected data)? We can use the fact that the integral and the area under a curve are closely related to find approximations.

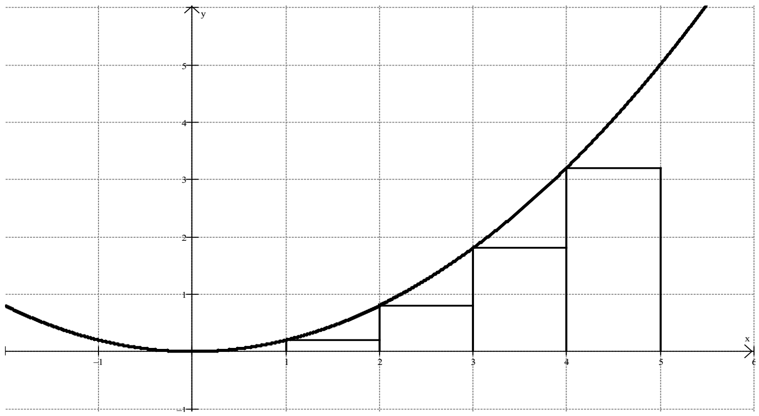
Objectives:
Find approximations of integrals using different rectangles. Use proper notation when dealing with integral approximation.

What if you forgot how to integrate $\frac{1}{5}x^2$? We can use a graphical approach to approximate this integral: $\int_1^5 \frac{1}{5}x^2 dx$. Mathematicians came up with this technique of dividing the area in question into rectangles, and then finding the area of each rectangle.

Ex 1 Use a left end Riemann sum with four equal subintervals to approximate $\int_1^5 \frac{1}{5}x^2 dx$.

Let's first figure out the width of our rectangles. If we need four rectangles in between $x = 1$ and $x = 5$, then our rectangles will be 1 unit wide.

The height will be determined by how high the function $\frac{1}{5}x^2$ is for each rectangle.



$$\int_1^5 \frac{1}{5}x^2dx \approx 1\cdot\frac{1}{5}\cdot 1^2 + 1\cdot\frac{1}{5}\cdot 2^2 + 1\cdot\frac{1}{5}\cdot 3^2 + 1\cdot\frac{1}{5}\cdot 4^2 = 6$$

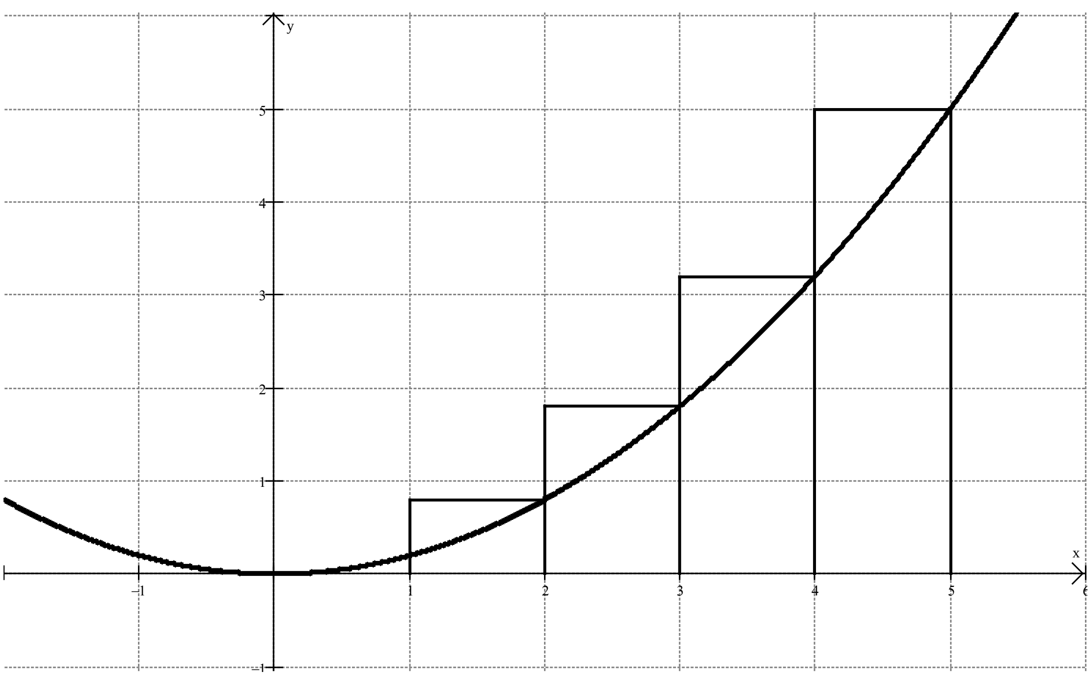
Notice the use of $\approx$ and =

Notice that this is clearly an underestimate because the rectangles are below the curve (and the area they cover is less than the area under the curve).

If we used more rectangles, we could get a more accurate estimation.

Ex 2 Use a right end Riemann sum with four rectangles to approximate $\int_1^5 \frac{1}{5}x^2dx$.

Again, our rectangles would be 1 unit wide.



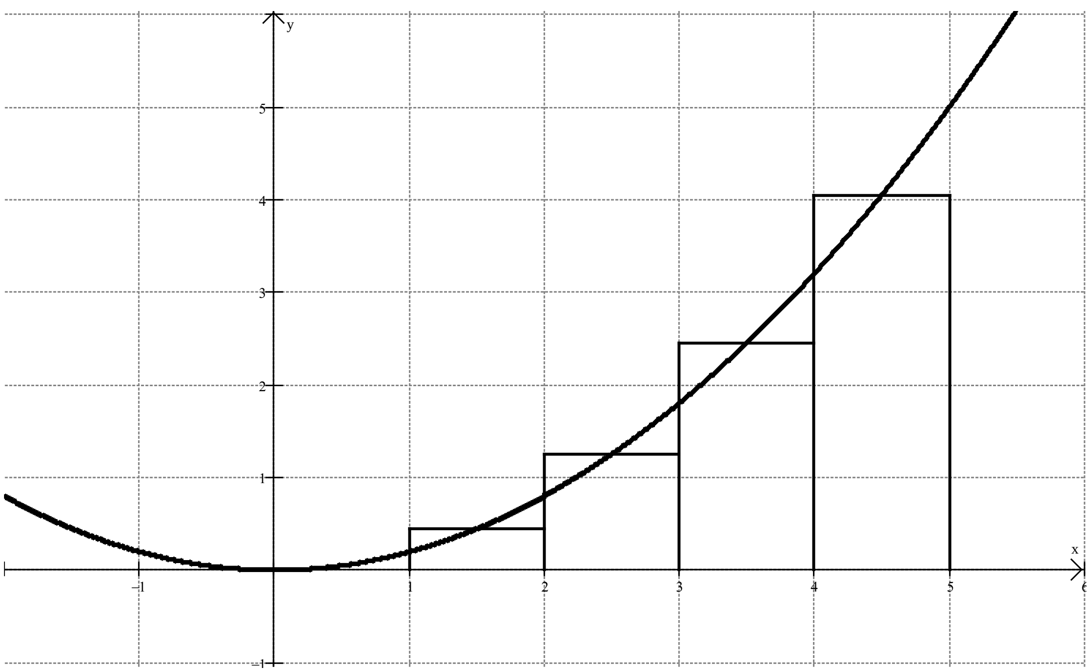
$$\int_1^5 \frac{1}{5}x^2dx \approx 1\cdot\frac{1}{5}\cdot 2^2 + 1\cdot\frac{1}{5}\cdot 3^2 + 1\cdot\frac{1}{5}\cdot 4^2 + 1\cdot\frac{1}{5}\cdot 5^2 = 10.8$$

****Notice the use of $\approx$ and =**

These two approximations are very different, and both aren't too close to the true value of the integral – you remembered how to find $\int_1^5 \frac{1}{5}x^2dx$ using the FUNDAMENTAL THEOREM OF CALCULUS … $\int_1^5 \frac{1}{5}x^2dx = 124/15 = 8.2\overline{6}$

Ex 3 Use a midpoint Riemann sum with four rectangles to approximate $\int_1^5 \frac{1}{5}x^2dx$.

Yet again, our rectangles would be 1 unit wide.



$$\int_1^5 \frac{1}{5}x^2dx \approx 1\cdot\frac{1}{5}\cdot(1.5)^2 + 1\cdot\frac{1}{5}\cdot(2.5)^2 + 1\cdot\frac{1}{5}\cdot(3.5)^2 + 1\cdot\frac{1}{5}\cdot(4.5)^2 = 8.2$$

This is a much better approximation.

If we used more rectangles, we would get better and better approximations because our Δx would become closer and closer to dx.

Steps to Approximating an Integral with Rectangles:

1. Determine the width of your rectangle.
2. Determine the height of your rectangle – whether that is using the left endpoint, right endpoint, midpoint, or any other height requested.
3. Calculate the areas of each rectangle.
4. Add the areas together for your approximation.
5. State answer using proper notation.

Of course, since we know the Fundamental Theorem of Calculus, approximations are of little value, unless we are trying to integrate a function that we don't know how to integrate, or we have a problem where we don't know the function at all.

Ex 4 Use the midpoint rule and the given data to approximate the value of $\int_0^{2.6} f(x)dx$.

x	$f(x)$	x	$f(x)$
0	7.5	1.6	7.7
0.4	7.3	2.1	7.9
0.8	7.2	2.6	8.0
1.2	7.3		

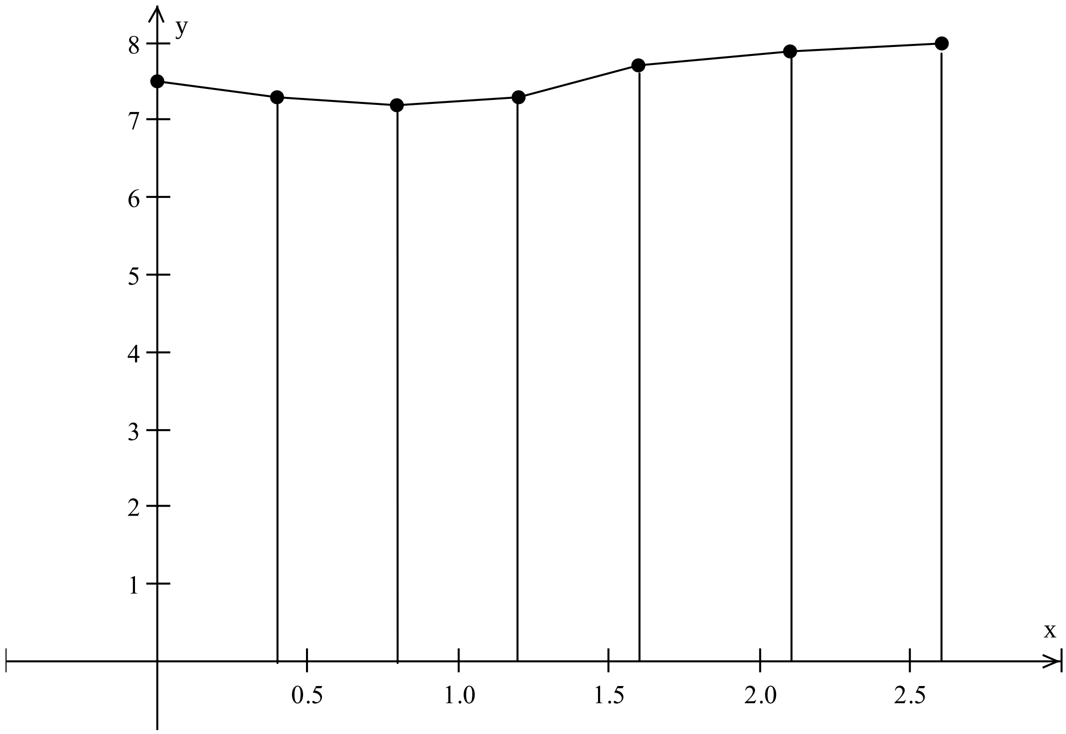
$$\int_0^{2.6} f(x)dx \approx 0.8(7.3)+0.8(7.3)+1.0(7.9)=19.58$$

A geometric alternative to rectangles would be to use trapezoids.

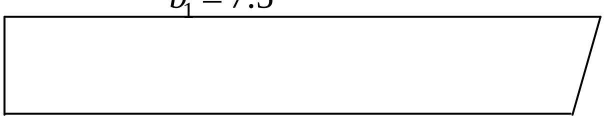
Ex 5 Use 6 trapezoids to approximate $\int_0^{2.6} f(x)dx$.
Use the table of data from Ex 4.

Keep in mind the area formula for a trapezoid is $A_{\text{Trap}} = \frac{1}{2}(b_1 + b_2)\cdot h$, where b_1 and b_2 are the parallel sides of the trapezoid.

Each trapezoid must have a height equal to the change in *x* and the bases must be the *y*-values (see the illustration below).



These look much more like trapezoids if you turn the page sideways – then it becomes obvious that the height is Δx, and that the *y*-values are the bases:



$$\begin{aligned}\int_0^{2.6} f(x)dx &\approx \frac{1}{2}(0.4)(7.5+7.3)+\frac{1}{2}(0.4)(7.3+7.2)+\frac{1}{2}(0.4)(7.2+7.3)\\ &+\frac{1}{2}(0.4)(7.3+7.7)+\frac{1}{2}(0.5)(7.7+7.9)+\frac{1}{2}(0.5)(7.9+8.0)\\ &=19.635\end{aligned}$$

This problem could not be done with the Fundamental Theorem of Calculus because we did not have a defined function to integrate.

Notice what happened when we did our last calculation – each trapezoid has a $\frac{1}{2}$ in its area formula and if the intervals had been identical (as they often are) the Δx would be common as well; the outer endpoints showed up once while the inner endpoints showed up twice.

This can be summed up in the Trapezoidal Rule formula – feel free to memorize this formula, or just derive it when you come across these problems.

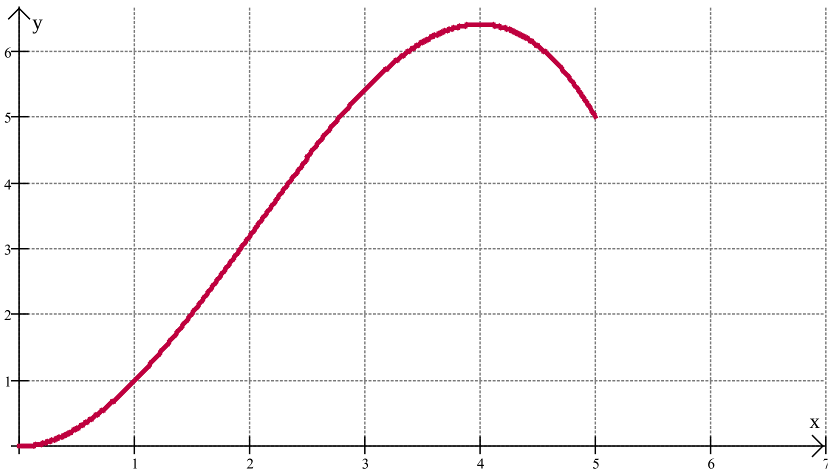
The Trapezoidal Rule*

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\ldots+2f(x_{n-1})+f(x_n)\right]$$

*The Trapezoidal Rule requires equal sub-intervals

On the AP test it is becoming more common that they ask problems with unequal sub-intervals; this formula is not useful in that case – it is far more useful to understand the geometry and be able to find the area of the individual trapezoids.

Ex 6 Estimate the area under the graph in the figure by first using Midpoint Rule and then by the Trapezoidal Rule with n = 5.



$$A_{\text{Mid}} \approx 1\cdot(.3)+1\cdot(2)+1\cdot(4.4)+1\cdot(6.2)+1\cdot(6)=18.9$$

$$A_{\text{Trap}} \approx \frac{1}{2}(0+1)\cdot 1+\frac{1}{2}(1+3.2)\cdot 1+\frac{1}{2}(3.2+5.4)\cdot 1+\frac{1}{2}(5.4+6.4)\cdot 1+\frac{1}{2}(6.4+5)\cdot 1=18.5$$

Ex 7 The following table gives values of a continuous function. Approximate the average value of the function using the midpoint rule with 3 equal subintervals. Repeat the process using the Trapezoidal Rule.

x	10	20	30	40	50	60	70
$f(x)$	3.649	4.718	6.482	9.389	14.182	22.086	35.115

Midpoint Rule:

$$f_{avg} = \frac{1}{70-10}\int_{10}^{70} f(x)dx \approx \frac{1}{70-10}\left[20(4.718)+20(9.389)+20(22.086)\right] = 12.064$$

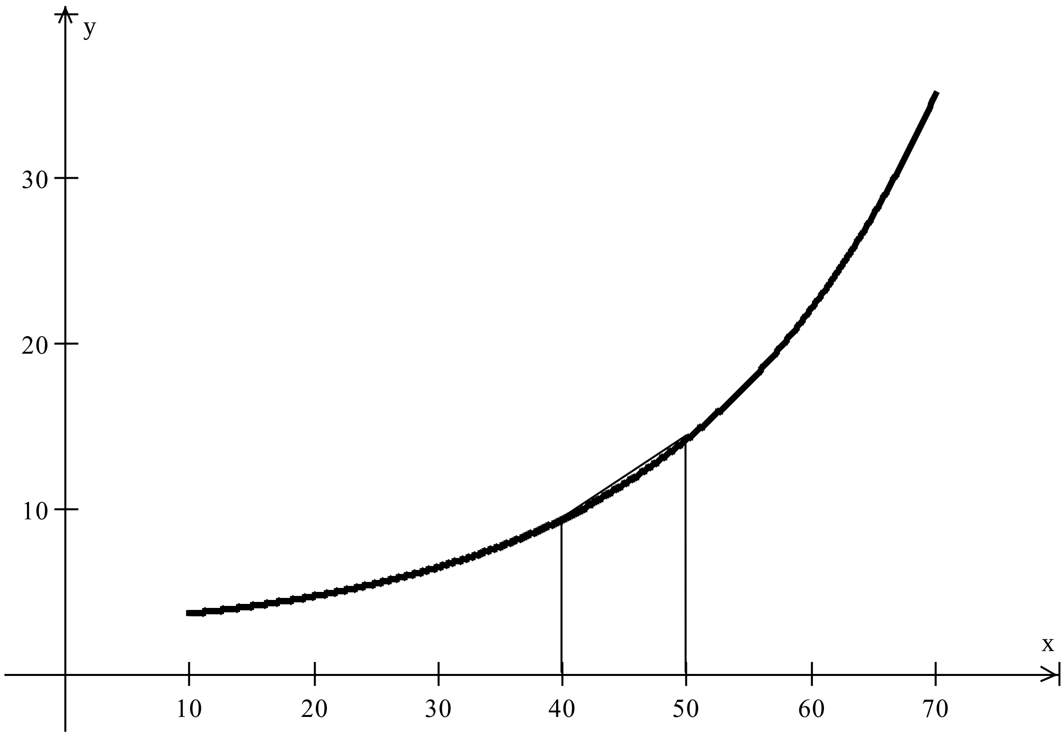
Trapezoidal Rule:

$$\begin{aligned}f_{avg} &= \frac{1}{70-10}\int_{10}^{70} f(x)dx \\ &\approx \frac{1}{70-10}\left[\frac{70-10}{12}\begin{pmatrix}3.649+2(4.718)+2(6.482)+ \\ 2(9.389)+2(14.182)+2(22.086)+35.115\end{pmatrix}\right] \\ &= 12.707\end{aligned}$$

Ex 8 Let us assume that the function that determined the values in the chart above is $f(x) = 2+e^{0.05x}$. Calculate the average value of the function and compare it to your approximations. (Can you figure out ahead of time whether our approximation will be an overestimate or an underestimate?)

$$f_{avg} = \frac{1}{70-10}\int_{10}^{70}\left(2+e^{0.05x}\right)dx = 12.489$$

If we want to figure out if our approximations are overestimates or underestimates, we have to look at the graph of the function.



Since this is concave up throughout, the area of the trapezoid would be larger than the area under the curve (see the trapezoid above for an example, it is just slightly above the curve).

In looking at whether an approximation is an over- or under- estimate, we always look at the concavity and compare whether our rectangles (or trapezoids) are above the curve or below it. If they are above the curve, we have overestimates, below the curve, we have underestimates.

As noted in the Trapezoidal Rule footnote, the Trapezoidal Rule in the stated form only works for equal subintervals. This is because the Trapezoidal Rule as stated is a factored form of setting up the problem using the trapezoidal area formula from Geometry:

$$\boxed{\text{Trapezoid Area: } A = \frac{1}{2}h\left(b_1 + b_2\right)}$$

Ex 8 Given the table of values below, find an approximation for

a) $\int_0^6 v(t)\,dt$ using 3 midpoint rectangles.

b) $\int_4^8 v(t)\,dt$ using 4 trapezoids.

c) $\int_0^4 v(t)\,dt$ using 4 left Riemann rectangles.

Make sure you indicate the units for each.

t (in sec.)	0	1	2	3	4	5	6	7	8
v(*t*) (in m/sec.)	8	12	17	10	3	8	6	12	10

a) $\int_0^6 v(t)\,dt \approx 2(12)+2(10)+2(8)=60$ meters

b) $\int_4^8 v(t)\,dt \approx \frac{1}{2}(3+8)+\frac{1}{2}(8+6)+\frac{1}{2}(6+12)+\frac{1}{2}(12+10)=32.5$ meters

c) $\int_0^4 v(t)\,dt \approx 1(8)+1(12)+1(17)+1(10)=47$ meters

The units were in meters for all of them by virtue of the multiplication: the widths were always in seconds, and the heights were in meters per second. When we multiply these quantities, the seconds cancel.

Ex 9 The function g is continuous on the closed interval [2,14] and has values shown on the table below. Using the subintervals [2,5], [5,10], and [10,14], what is the approximation of $\int_{2}^{14} g(x)\,dx$ found by using a right Riemann sum?

x	2	5	10	14
$g(x)$	1.2	2.8	3.4	3.0

(A) 29.6 (B) 31.2 (C) 34.3 (D) 37.4 (E) 39.0

4.2 Homework Set A

1. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the given integral with the specified value of n.

   $$\int_0^2 \sqrt[4]{1+x^2}\,dx, n=8$$

2. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the given integral with the specified value of n.

   $$\int_1^2 \frac{\ln x}{1+x}\,dx, n=10$$

3. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

t(s)	v(mi/h)	t(s)	v(mi/h)
0	0	60	56
10	38	70	53
20	52	80	50
30	58	90	47
40	55	100	45
50	51		

4. The following table gives values of a continuous function. Estimate the average value of the function on $x \in [0, 8]$ using (a) Right-Hand Riemann rectangles, (b) Left-Hand Riemann rectangles, and (c) Midpoint Riemann rectangles.

X	0	1	2	3	4	5	6	7	8
F(x)	10	15	17	12	3	-5	8	-2	10

5. The temperature in degrees Celsius (°C), of the water in a lake is a differentiable function W of t. The table below shows the water temperature recorded every 3 days for a 15 day period. Use this table to answer the questions below.

t (days)	$W(t)$ (°C)
0	20
3	31
6	28
9	24
12	22
15	21

a) Approximate the average temperature for the pond over the 15 days in the table above. Use a trapezoidal approximation with subintervals of 3 days.

b) A biologist proposes the following function C, given by $C(t) = 20 + 10te^{-t/3}$, as a model for the temperature of the lake after t days. Find the average value of the temperature of the lake over the same 15 days as part a. Does the value seem reasonable compared to your answer in a?

4.2 Homework Set B

1. Use the trapezoidal rule with $n = 4$ to approximate the value of $\int_0^4 e^{x^2}\,dx$. Why do you think that approximations would be necessary for evaluation of this integral?

2. Given the table of values below for velocities of a glider, find an approximation for the displacement of the glider using a left Riemann sum, a right Riemann sum, and a trapezoidal sum (make sure you include units in your answer). Which do you think is the most accurate? Explain.

t (minutes)	1	5	11	13	19
$V(t)$ (m/min)	450	550	400	350	500

3. The following table is for a continuous function, $f(x)$. Use the information in the table to find each of the following:

x	0	1	2	3	4	5	6	7	8	9
$f(x)$	2.7	3.1	4.2	2.1	1.0	0.1	-0.3	-1.5	-0.4	0.2

a. Approximate $\int_0^5 f(x)dx$ using a right approximation

b. Approximate $\int_0^5 f(x)dx$ using a left approximation

c. Approximate $\int_0^6 f(x)dx$ using a midpoint approximation

d. Approximate $\int_0^6 f(x)dx$ using a trapezoidal approximation

e. Approximate $\int_0^9 f(x)dx$ using a right approximation

f. Approximate $\int_0^9 f(x)dx$ using a left approximation

g. Approximate $\int_1^9 f(x)dx$ using a midpoint approximation

h. Approximate $\int_1^9 f(x)dx$ using a trapezoidal approximation

4. Given the table of values below, find each of the following:

t in seconds	$R(t)$ in meters/second
0	12
5	15
10	17
15	13
20	16
25	18
30	21

a) Find a left Riemann approximation for $\int_{10}^{30} R(t)\,dt$

b) Find a left Riemann approximation for $\int_{0}^{15} R(t)\,dt$

c) Find a right Riemann approximation for $\int_{10}^{30} R(t)\,dt$

d) Find a right Riemann approximation for $\int_{0}^{15} R(t)\,dt$

e) Find a midpoint Riemann approximation for $\int_{0}^{30} R(t)\,dt$

f) Find a midpoint Riemann approximation for $\int_{10}^{30} R(t)\,dt$

g) Find a trapezoidal approximation for $\int_{0}^{30} R(t)\,dt$

Answers: 4.2 Homework Set A

1. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the given integral with the specified value of *n*.

   $\int_0^2 \sqrt[4]{1+x^2}\,dx, n=8$

   a. 2.414 b. 2.411

2. Use (a) the Trapezoidal Rule and (b) the Midpoint Rule to approximate the given integral with the specified value of *n*.

   $\int_1^2 \frac{\ln x}{1+x}\,dx, n=10$

   a. 0.147 b. 0.147

3. The velocity of a car was read from its speedometer at 10-second intervals and recorded in the table. Use the Midpoint Rule to estimate the distance traveled by the car.

   4940 miles

4. The following table gives values of a continuous function. Estimate the average value of the function on $x \in [0, 8]$ using (a) Right-Hand Riemann rectangles, (b) Left-Hand Riemann rectangles, and (c) Midpoint Riemann rectangles.

   a. 7.25 b. 7.25 c. 5

5. The temperature in degrees Celsius (°C), of the water in a lake is a differentiable function *W* of *t*. The table below shows the water temperature recorded every 3 days for a 15 day period. Use this table to answer the questions below.

   a) Approximate the average temperature for the pond over the 15 days in the table above. Use a trapezoidal approximation with subintervals of 3 days. 25.1 °C

   b) A biologist proposes the following function *C*, given by $C(t) = 20 + 10te^{-t/3}$, as a model for the temperature of the lake after *t* days. Find the average value of the temperature of the lake over the same 15 days as part a. Does the value seem reasonable compared to your answer in a?

   25.757 °C – it seems like a fairly reasonable answer, but it is off by about 2.6%, so it may not be the best model.

4.2 Homework Set B

1. Use the trapezoidal rule with $n = 4$ to approximate the value of $\int_0^4 e^{x^2}\,dx$. Why do you think that approximations would be necessary for evaluation of this integral?

   4,451,216.161

   We would need to approximate because we have no way of integrating this function.

2. Given the table of values below for velocities of a glider, find an approximation for the displacement of the glider using a left Riemann sum, a right Riemann sum, and a trapezoidal sum (make sure you include units in your answer). Which do you think is the most accurate? Explain.

   Left sum = 8,000 meters
   Right sum = 8,300 meters
   Trapezoidal sum = 8,150 meters

3. The following table is for a continuous function, $f(x)$. Use the information in the table to find each of the following:

   i. Approximate $\int_0^5 f(x)dx$ using a right approximation $\int_0^5 f(x)dx \approx 10.5$

   j. Approximate $\int_0^5 f(x)dx$ using a left approximation $\int_0^5 f(x)dx \approx 13.1$

   k. Approximate $\int_0^6 f(x)dx$ using a midpoint approximation $\int_0^6 f(x)dx \approx 10.6$

   l. Approximate $\int_0^6 f(x)dx$ using a trapezoidal approximation $\int_0^6 f(x)dx \approx 11.7$

   m. Approximate $\int_0^9 f(x)dx$ using a right approximation $\int_0^9 f(x)dx \approx 8.5$

   n. Approximate $\int_0^9 f(x)dx$ using a left approximation $\int_0^9 f(x)dx \approx 11$

   o. Approximate $\int_1^9 f(x)dx$ using a midpoint approximation $\int_1^9 f(x)dx \approx 9$

   p. Approximate $\int_1^9 f(x)dx$ using a trapezoidal approximation $\int_1^9 f(x)dx \approx 6.85$

4. Given the table of values below, find each of the following:

a) Find a left Riemann approximation for $\int_{10}^{30} R(t)\,dt$ $\int_{10}^{30} R(t)\,dt \approx 320$ meters

b) Find a left Riemann approximation for $\int_{0}^{15} R(t)\,dt$ $\int_{0}^{15} R(t)\,dt \approx 220$ meters

c) Find a right Riemann approximation for $\int_{10}^{30} R(t)\,dt$ $\int_{10}^{30} R(t)\,dt \approx 290$ meters

d) Find a right Riemann approximation for $\int_{0}^{15} R(t)\,dt$ $\int_{0}^{15} R(t)\,dt \approx 225$ meters

e) Find a midpoint Riemann approximation for $\int_{0}^{30} R(t)\,dt$ $\int_{0}^{30} R(t)\,dt \approx 460$ meters

f) Find a midpoint Riemann approximation for $\int_{10}^{30} R(t)\,dt$ $\int_{10}^{30} R(t)\,dt \approx 310$ meters

g) Find a trapezoidal approximation for $\int_{0}^{30} R(t)\,dt$ $\int_{0}^{30} R(t)\,dt \approx 477.5$ meters

4.3 Definite Integrals and the Substitution Rule

Let's revisit *u* – substitution with definite integrals and pick up a couple of more properties for the definite integral.

Properties of Definite Integrals

1. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

2. $\int_a^a f(x)dx = 0$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$, where $a < c < b$

The first property comes from the second part of the Fundamental Theorem of Calculus. Since the theorem is defined as a difference, this means that

$$\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$$

Since $\int_a^b f(x)\ dx = F(b) - F(a)$, if we reverse the boundaries, the subtraction is reversed as well. When subtraction is reversed, we get the opposite result (a negative instead of a positive or vice versa). When we put the negative in front of the integral, we cancel the switching of the boundaries.

The second property is also based on the fact that the Fundamental Theorem of Calculus is defined as a difference. If we subtract the same value from itself, $F(a) - F(a)$, we always get 0. Therefore, we never bother to integrate a problem where the boundaries are identical.

The third property just says that you can break up one integral into multiple integrals by using a value, *c*, that is in between *a* and *b*.

Objectives

Evaluate definite integrals using the Fundamental Theorem of Calculus.
Evaluate definite integrals applying the Substitution Rule, when appropriate.
Use proper notation when evaluating these integrals.

Ex 1 Evaluate $\int_0^2 \left(t^2\sqrt{t^3+1}\right)dt$.

$$\int_0^2 \left(t^2\sqrt{t^3+1}\right)dt = \frac{1}{3}\int_0^2 \left(3t^2\sqrt{t^3+1}\right)dt$$

Let $u = t^3+1$ $du = 3t^2 dt$	$u(0)=1$ $u(2)=9$

$$= \frac{1}{3}\int_1^9 \left(\sqrt{u}\right)du$$
$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_1^9$$
$$= \frac{2}{9}\left[9^{3/2} - 1^{3/2}\right]$$
$$= \frac{52}{9}$$

Ex 2 Evaluate $\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)dx$.

$$\int_0^{\sqrt{\pi}} \left(x\cos\left(x^2\right)\right)dx \quad = \frac{1}{2}\int_{x=0}^{x=\sqrt{\pi}} \cos u \, du$$

Let $u = x^2$ $du = 2x dx$	$u(0)=0$ $u\left(\sqrt{\pi}\right)=\pi$

$$= \frac{1}{2}\sin u\Big|_0^{\pi}$$
$$=0$$

Ex 3 $\int_1^2 \frac{e^{1/x}}{x^2}dx$

$$\int_1^2 \frac{e^{1/x}}{x^2}dx \quad = -\int_1^{1/2} e^u du$$

$$\boxed{\begin{aligned} \text{Let } u &= 1/x \\ du &= -1/x^2\, dx \end{aligned}} \quad \boxed{\begin{aligned} u(1) &= 1 \\ u(2) &= 1/2 \end{aligned}}$$

$$= -e^u \Big|_1^{1/2}$$

$$= -e^{1/2} + e$$

Ex 4 $\quad \int_2^0 x\sqrt{x^2 + a^2}\,dx \qquad a > 0$

$$\int_2^0 x\sqrt{x^2+a^2}\,dx \quad = \quad -\int_0^2 x\sqrt{x^2+a^2}\,dx$$

$$\boxed{\begin{aligned} \text{Let } u &= x^2 + a^2 \\ du &= 2xdx \end{aligned}}$$

$$= -\int_0^2 x\sqrt{x^2+a^2}\,dx$$

$$= -\frac{1}{2}\int_{a^2}^{4+a^2} \sqrt{u}\,\hat{\text{E}}du$$

$$=$$

$$= -\frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_{a^2}^{4+a^2}$$

$$= -\frac{1}{3}\left[\left(4+a^2\right)^{3/2} - a^3\right]$$

Ex 5 Find the derivatives of the following functions.

(a) $F(x) = \int_x^{10} \tan\theta d\theta$

$$F(x) = \int_x^{10} \tan\theta d\theta = -\int_{10}^{x} \tan\theta d\theta \Rightarrow F'(x) = -\tan x$$

(b) $y = \int_{e^x}^{0} \sin^3 t dt$

$$y = \int_{e^x}^{0} \sin^3 t dt \Rightarrow y' = -\sin^3 e^x \cdot e^x$$

(c) $F(x) = \int_{10}^{10} \tan\theta d\theta$

$$F(x) = \int_{10}^{10} \tan\theta d\theta = 0$$

4.3 Homework Set A

Evaluate the definite integral, if it exists.

1. $\int_0^1 x^2(1+2x^3)^5 dx$

2. $\int_0^\pi \sec^2(t/4)dt$

3. $\int_1^2 \frac{e^{1/x}}{x^2} dx$

4. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta} d\theta$

5. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

6. $\int_{-1}^{2} \frac{dx}{2x+5}$

7. $\int_0^\pi \frac{\sin x}{2-\cos x} dx$

8. $\int_2^4 \frac{dx}{x\ln(x)}$

9. $\displaystyle\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\, dx$

10. $\displaystyle\int_{\pi/6}^{\pi/2} \cos^5 x \sin x\, dx$

11. $\displaystyle\int_{\pi}^{2\pi} \cos\left(\frac{1}{2}\theta\right) d\theta$

12. $\displaystyle\int_0^2 f(x)dx$ where $f(x)=\begin{cases} x^4, & if\ 0\le x<1 \\ x^5, & if\ 1\le x\le 2 \end{cases}$

Find the Average Value of each of the following functions.

13. $f(x)=\cos x \sin^4 x$ on $x\in[0,\ \pi]$

14. $g(x)=xe^{-x^2}$ on $x\in[1,\ 5]$

15. $G(x) = \dfrac{x}{\left(1+x^2\right)^3}$ on $x \in [0, 2]$

16. $h(x) = \dfrac{x}{\left(1+x^2\right)^2}$ on $x \in [0, 4]$

4.3 Homework Set B

1. $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$

2. $\displaystyle\int_0^{\pi} \frac{\cos x}{1+\sin x}\,dx$

3. $\displaystyle\int_3^5 \frac{x^2-16}{\left(x^2+5x+6\right)}\,dx$

4. $\displaystyle\int_0^{\pi} \frac{\cos x}{2+\sin x}\,dx$

5. $\displaystyle\int_0^{\frac{\pi}{16}} \sec^2(4x)\,dx$

6. $\displaystyle\int_0^{\pi} \frac{\sin y}{2+\cos y}\,dy$

7. $\displaystyle\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y}\,dy$

8. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} m\sec\left(m^2\right)\tan\left(m^2\right)dm$

9. $\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\,dx$

10. $\int_{0}^{\frac{\pi}{4}} \sec^2 x \tan^3 x\,dx$

11. $\int_{\sqrt{3}}^{\sqrt{4}} y e^{y^2-3}\,dy$

12. If $a(t) = \cos(2t)$, find $v(t)$ and $x(t)$ if $v(\pi) = 6$ and $x\left(\frac{\pi}{2}\right) = 0$.

13. If $a(t) = e^{3t} + \cos(2t)$, find $v(t)$ when $v(0) = 3$. Then find $x(t)$ when $x(0) = 0$.

14. If $f'(\theta) = \tan^6(2\theta)\sec^2(2\theta)$, find $f(\theta)$ when $f\left(\frac{3\pi}{8}\right) = 2$.

15. If $v(t) = \sec(3t)\tan(3t)$, find $a(t)$ and $x(t)$ when $x\left(\frac{\pi}{12}\right) = 3$

16. If $f'(x) = \frac{\cos\sqrt{x}}{\sqrt{x}}$, find $f(x)$ when $f\left(\frac{\pi^2}{4}\right) = 2$

17. $\displaystyle\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec^2 2x\,dx$

18. Find the exact value of $\displaystyle\int_{0}^{e^2-1} \frac{1}{x+1}\,dx$

19. $\displaystyle\int_{\pi}^{\frac{5\pi}{4}} \sin y\,dy$

20. $\displaystyle\int_0^e \frac{5}{\sqrt[4]{m^2+3m+1}}\,dm$

21. $\displaystyle\int_2^{e^3+1} \frac{\left(\ln(x-1)\right)^4}{x-1}\,dx$

22. $\displaystyle\int_0^{\pi} \cos^6\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$

23. $\displaystyle\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y}dy$

24. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} m\sec\left(m^2\right)\tan\left(m^2\right)dm$

25. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\,dx$

26. Find the average value of $f(x) = \csc 2x \cot 2x$ on $x \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

Answers: 4.3 Homework Set A

1. $\int_0^1 x^2(1+2x^3)^5\,dx$

$=\frac{182}{9}$

2. $\int_0^{\pi} \sec^2(t/4)\,dt$

$=4$

3. $\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

$=e-e^{1/2}$

4. $\int_0^{\pi/3} \frac{\sin\theta}{\cos^2\theta}\,d\theta$

$=1$

5. $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

$=2$

6. $\int_{-1}^{2} \frac{dx}{2x+5}$

$=\frac{1}{2}\ln 3$

7. $\int_0^{\pi} \frac{\sin x}{2-\cos x}\,dx$

$=\ln 3$

8. $\int_2^4 \frac{dx}{x\ \ln x}$

$=\ln\left(\frac{\ln 4}{\ln 2}\right)$

9. $\int_0^{\ln 2} \frac{e^x}{1+e^{2x}}\,dx$

$=\tan^{-1}2-\frac{\pi}{4}$

10. $\int_{\pi/6}^{\pi/2} \cos^5 x\ \sin x\,dx$

$=\frac{9}{128}$

11. $\int_{\pi}^{2\pi} \cos\frac{1}{2}\theta\,d\theta$

$=-2$

12. $\int_0^2 f(x)\,dx$ where $f(x)=\begin{cases} x^4, & \text{if } 0\le x<1 \\ x^5, & \text{if } 1\le x\le 2 \end{cases}$

$=\frac{107}{10}$

13. $f(x)=\cos x\sin^4 x$ on $x\in[0,\pi]$

14. $g(x)=xe^{-x^2}$ on $x\in[1,5]$

Avg. Value = 0

Avg. Value = $-\frac{1}{8}\left(e^{-25}-e^{-1}\right)$

15. $G(x)=\dfrac{x}{\left(1+x^2\right)^3}$ on $x\in[0,2]$

Avg. Value $=-\dfrac{3}{25}$

16. $h(x)=\dfrac{x}{\left(1+x^2\right)^2}$ on $x\in[0,4]$

Avg. Value = $\dfrac{2}{17}$

4.3 Homework Set B

1. $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)\,dx$

$=\dfrac{1}{2}$

2. $\displaystyle\int_0^{\pi} \frac{\cos x}{1+\sin x}dx$

$=0$

3. $\displaystyle\int_3^5 \frac{x^2-16}{\left(x^2+5x+6\right)}dx$

$=\ln\left(\dfrac{28}{15}\right)$

4. $\displaystyle\int_0^{\pi} \frac{\cos x}{2+\sin x}dx$

$=0$

5. $\displaystyle\int_0^{\frac{\pi}{16}} \sec^2(4x)\,dx$

$=\dfrac{1}{4}$

6. $\displaystyle\int_0^{\pi} \frac{\sin y}{2+\cos y}dy$

$=\ln\left(\dfrac{1}{3}\right)$

7. $\displaystyle\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2\left(\ln y\right)}{y}dy$

$=1$

8. $\displaystyle\int_0^{\sqrt{\frac{\pi}{4}}} m\sec\left(m^2\right)\tan\left(m^2\right)dm$

$=\dfrac{\sqrt{2}}{2}$

9. $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\,dx$

$=\dfrac{1}{10}$

10. $\displaystyle\int_0^{\frac{\pi}{4}} \sec^2 x\tan^3 x\,dx$

$=\dfrac{1}{4}$

11. $\int_{\sqrt{3}}^{\sqrt{4}} ye^{y^2-3}dy$

$=\frac{1}{2}(e-1)$

12. If $a(t)=\cos(2t)$, find $v(t)$ and $x(t)$ if $v(\pi)=6$ and $x\left(\frac{\pi}{2}\right)=0$.

$v(t)=\frac{1}{2}\cos(2t)+6$

$x(t)=-\frac{1}{4}\sin(2t)+6t-3\pi$

13. If $a(t)=e^{3t}+\cos(2t)$, find $v(t)$ when $v(0)=3$. Then find $x(t)$ when $x(0)=0$.

$v(t)=\frac{1}{3}e^{3t}+\frac{1}{2}\cos(2t)+\frac{13}{6}$

$x(t)=\frac{1}{9}e^{3t}-\frac{1}{4}\sin(2t)+\frac{13}{6}t-\frac{1}{9}$

14. If $f'(\theta)=\tan^6(2\theta)\sec^2(2\theta)$, find $f(\theta)$ when $f\left(\frac{3\pi}{8}\right)=2$.

$f(\theta)=\frac{1}{14}\tan^7(2\theta)+\frac{29}{14}$

15. If $v(t)=\sec(3t)\tan(3t)$, find $a(t)$ and $x(t)$ when $x\left(\frac{\pi}{12}\right)=3$

$x(t)=\frac{1}{3}\sec(3t)+3-\frac{\sqrt{2}}{3}$

$a(t)=3\sec^2(3t)\left(\tan(3t)+\sec(3t)\right)$

16. If $f'(x)=\frac{\cos\sqrt{x}}{\sqrt{x}}$, find $f(x)$ when $f\left(\frac{\pi^2}{4}\right)=2$

$f(x)=\frac{1}{4}\cos^2\sqrt{x}+2$

17. $\int_{\frac{\pi}{8}}^{\frac{\pi}{6}} \sec^2 2x\,dx$

$= \frac{1}{2}\left(\sqrt{3}-1\right)$

18. Find the exact value of $\int_0^{e^2-1} \frac{1}{x+1}dx$

$\int_0^{e^2-1} \frac{1}{x+1}dx = 2$

19. $\int_{\pi}^{\frac{5\pi}{4}} \sin y\,dy$

$= \frac{1}{\sqrt{2}}+1$

20. $\int_0^e \frac{5}{\sqrt[4]{m^2+3m+1}}dm$

$= 8.854$

21. $\int_2^{e^3+1} \frac{\left(\ln(x-1)\right)^4}{x-1}dx$

$= \frac{243}{5}$

22. $\int_0^{\pi} \cos^6\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$

$= \frac{4}{7}$

23. $\int_{e^{\frac{\pi}{4}}}^{e^{\frac{\pi}{2}}} \frac{\csc^2(\ln y)}{y}dy$

$=1$

24. $\int_0^{\sqrt{\frac{\pi}{4}}} m\sec\left(m^2\right)\tan\left(m^2\right)dm$

$= \frac{1}{2}\left(\sqrt{2}-1\right)$

25. $\int_{\frac{\pi}{2}}^{\pi} \cos^9(x)\sin(x)\,dx$

$= \frac{1}{10}$

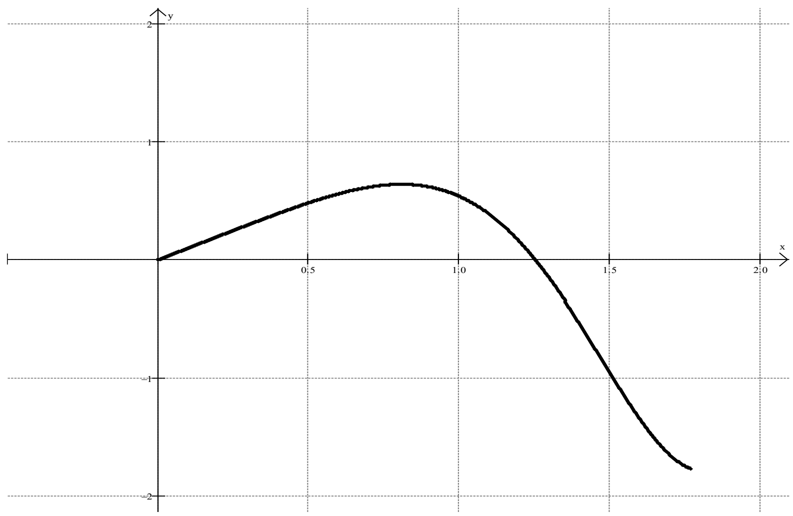
26. Find the average value of $f(x) = \csc 2x \cot 2x$ on $x \in \left[\frac{\pi}{8}, \frac{\pi}{4}\right]$

Average Value $= \frac{4}{\pi}\left(-1+\sqrt{2}\right)$

4.4 Context: Area, Displacement, and Net Change

Since we originally defined the definite integrals in terms of "area under a curve", we need to consider what this idea of "area" really means in relation to the definite integral.

Let's say that we have a function, $y=\left(x\cos\left(x^2\right)\right)$ on $x\in\left[0,\sqrt{\pi}\right]$. The graph looks like this:



In the last section, we saw that $\int_0^{\sqrt{\pi}}\left(x\cos\left(x^2\right)\right)dx=0$. But we can see there is area under the curve, so how can the integral equal the area and equal 0? Remember that the integral was created from rectangles with width *dx* and height *f* (*x*). So the area below the *x*-axis would be negative, because the *f* (*x*)-values are negative.

Ex 1 What is the area under $y=\left(x\cos\left(x^2\right)\right)$ on $x\in\left[0,\sqrt{\pi}\right]$?

We already know that $\int_0^{\sqrt{\pi}}\left(x\cos\left(x^2\right)\right)dx=0$, so this integral cannot represent the area. As with example 1b, we really are looking for the positive number that represents the area (total distance), not the difference between the positive and negative "areas" (displacement). The commonly accepted context for area is a positive value. So,

$$
\begin{aligned}
\text{Area} &= \int_0^{\sqrt{\pi}} \left| x\cos\left(x^2\right) \right| dx \\
&= \int_0^{1.244} x\cos\left(x^2\right) dx - \int_{1.244}^{\sqrt{\pi}} x\cos\left(x^2\right) dx \\
&= 1
\end{aligned}
$$

We could have used our calculator to find this answer:

```
fnInt(abs(Xcos(X
²)),X,0,√(π))
        1.00000159
█
```

Parentheses are very important with Math 9 entries in the calculator. Be careful!

Note that if you have a newer version of the graphing calculator, the operating system symbolically represents the definite integral the same way we do in the book, making the process of entering a problem in the calculator much easier.

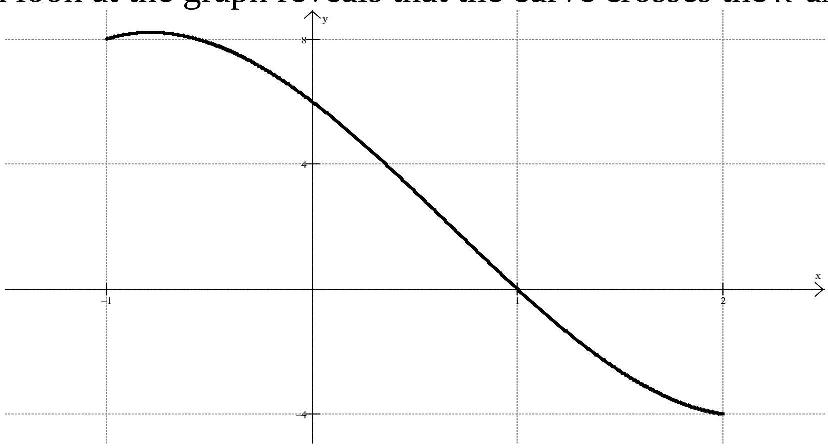
When we use the phrase "area under the curve", we really mean the area between the curve and the *x*-axis. CONTEXT IS EVERYTHING. The area under the curve is only equal to the definite integral when the curve is completely above the *x*-axis. When the curve goes below the x-axis, the definite integral is negative, but the area, by definition, is positive.

Objectives:

- Relate definite integrals to area under a curve.
- Understand the difference between displacement and total distance.
- Extend that idea to understanding the difference between the two concepts in other contexts.

Ex 2 Find the area under $y = x^3 - 2x^2 - 5x + 6$ on $x \in [-1, 2]$.

A quick look at the graph reveals that the curve crosses the *x*-axis at *x*=1.



If we integrate y on $x \in [-1, 2]$, we will get the **difference** between the areas, not the sum. To get the total area, we need to set up two integrals:

$$\begin{aligned}
\text{Area} &= \int_{-1}^{1}\left(x^3 - 2x^2 - 5x + 6\right)dx + \left(-\int_{1}^{2}\left(x^3 - 2x^2 - 5x + 6\right)dx\right) \\
&= \frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\Bigg]_{-1}^{1} - \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{5x^2}{2} + 6x\right]_{1}^{2} \\
&= \frac{32}{3} + \frac{29}{12} \\
&= \frac{157}{12}
\end{aligned}$$

F could be a function that describes anything – volume, weight, time, height, temperature. F' represents its rate of change. The left side of that equation accumulates the rate of change of *F* from *a* to *b* and the right side of the equation says that accumulation is difference in *F* from *a* to *b*.

Imagine if you were leaving your house to go to school, and that school is 6 miles away. You leave your house and halfway to school you realize you have forgotten your calculus homework (gasp!). You head back home, pick up your assignment, and then head to school.

There are two different questions that can be asked here. How far are you from where you started? And how far have you actually traveled? You are six miles from where you started but you have traveled 12 miles. These are the two different ideas behind displacement and total distance.

Vocabulary:

Displacement – How far apart the starting position and ending position are. (It can be positive or negative.)
Total Distance – how far you travel in total. (This can only be positive.)

$$\text{Displacement} = \int_a^b v(t)dt$$
$$\text{Total Distance} = \int_a^b |v(t)|dt$$

Ex 3 A particle moves along a line so that its velocity at any time t is $v(t)=t^2-t-6$ (measured in meters per second).
(a) Find the displacement of the particle during the time period $1 \le t \le 4$.
(b) Find the distance traveled during the time period $1 \le t \le 4$.

$$(a) \quad \int_a^b v dt = \int_1^4 \left(t^2 - t - 6\right) dt$$
$$= \frac{t^3}{3} - \frac{t^2}{2} - 6t \Big|_1^4$$
$$= -4.5$$

(b) $$\int_a^b |v|\,dt = \int_1^4 \left(t^2 - t - 6\right) dt$$
$$= -\int_1^3 \left(t^2 - t - 6\right) dt + \int_3^4 \left(t^2 - t - 6\right) dt$$
$$= -\frac{t^3}{3} + \frac{t^2}{2} + 6t\Big|_1^3 + \frac{t^3}{3} - \frac{t^2}{2} - 6t\Big|_3^4$$
$$= \frac{61}{6}$$

Note that we used the properties of integrals to split the integral into two integrals that represent the separate positive and negative distanced and then made the negative one into a positive value by putting a – in front. We split the integral at $t = 3$ because that would be where $v(t) = 0$.

Ex 4 (a) If $r'(t)$ represents the growth of a puppy in pounds per month, what does $\int_2^{10} r'(t)\,dt$ represent?

(b) If water is leaking from a water heater at a rate of $w(t)$ gallons per minute, what does $\int_0^{360} w(t)\,dt$ represent?

(c) A bacteria population starts with 200 specimens and increases at a rate $b'(t)$ specimens per month. What does $\int_0^{12} b'(t)\,dt$ represent? What does $200 + \int_0^{12} b'(t)\,dt$ represent?

(a) $\int_2^{10} r'(t)\,dt$ would represent the number of pounds gained or lost by this puppy between two and ten months.

(b) $\int_0^{360} w(t)\,dt$ would represent the total number of gallons of water that leaked from the water heater in the first six hours.

(c) $\int_0^{12} b'(t)\,dt$ would represent the increase in bacteria specimens in the first year. $200 + \int_0^{12} b'(t)\,dt$ would represent the total number of bacteria specimens at the end of the first year.

4.4 Homework Set A

Find the area under the curve of the given equation on the given interval.

1. $y = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$ on $x \in \left[0,\ \pi^2\right]$

2. $y = x\sqrt{2x^2 - 18}$ on $x \in \left[-2,\ 1\right]$

3. $y = 3\sin x\ \sqrt{1 - \cos x}$ on $x \in \left[-\pi, 0\right]$

4. $y = x^2 e^{x^3}$ on $x \in \left[0,\ 1.5\right]$

5. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$ for $0 \le t \le 3$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

6. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$ for $1 \le t \le 6$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

7. If $w'(t)$ is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)dt$ represent?

8. If oil leaks from a tank at a rate of $r(t)$ gallons per minute at time t, what does $\int_0^{120} r(t)dt$ represent?

9. The marginal revenue function $R'(x)$ is the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)dx$ represent?

10. If x is measured in meters and $f(x)$ is measured in Newtons, what are the units for $\int_{0}^{100} f(x)dx$?

4.4 Homework Set B

1. Find the area under the curve $f(x) = e^{-x^2} - x$ on $x \in [-1, 2]$ (do not use absolute values in your setup, break it into multiple integrals, then evaluate on your calculator using absolute values).

2. Find the area under the curve $f(x) = e^{-x^2} - 2x$ on $x \in [-1, 2]$ (do not use absolute values in your setup, break it into multiple integrals, then evaluate on your calculator using absolute values).

3. Find the area under the curve $f(x) = \frac{x}{x^2+1} + \cos(x)$ on $x \in [0, \pi]$

4. Find the area under the curve $g(x) = -1 - x\sin x$ on $x \in [0, 2\pi]$

Answers: 4.4 Homework Set A

1. $y = \dfrac{\sin\sqrt{x}}{\sqrt{x}}$ on $x \in \left[0,\ \pi^2\right]$

   $= -2\cos\sqrt{\pi} + 2$

2. $y = x\sqrt{2x^2 - 18}$ on $x \in \left[-2,\ 1\right]$

   $= \dfrac{2}{17}$

3. $y = 3\sin\ x\ \sqrt{1 - \cos x}$ on $x \in \left[-\pi, 0\right]$

   $= -2^{5/2}$

4. $y = x^2 e^{x^3}$ on $x \in \left[0,\ 1.5\right]$

   $= \dfrac{1}{3}\left(e^{3.375} - 1\right)$

5. The velocity function (in meters per second) for a particle moving along a line is $v(t) = 3t - 5$ for $0 \le t \le 3$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

   (a) $-\dfrac{3}{2}m$ and (b) $\dfrac{41}{6}m$

6. The velocity function (in meters per second) for a particle moving along a line is $v(t) = t^2 - 2t - 8$ for $1 \le t \le 6$. Find (a) the displacement and (b) the distance traveled by the particle during the given time interval.

   (a) $-\dfrac{10}{3}m$ and (b) $\dfrac{98}{3}m$

7. If *w'*(*t*) is the rate of growth of a child in pounds per year, what does $\int_5^{10} w'(t)dt$ represent?

   $\int_5^{10} w'(t)dt$ represents the change of the child's weight in pounds from ages 5 to 10 years

8. If oil leaks from a tank at a rate of *r*(*t*) gallons per minute at time *t*, what does $\int_0^{120} r(t)dt$ represent?

   $\int_0^{120} r(t)dt$ represents how much oil, in gallons, has leaked from the tank in the first 120 minutes.

9. The marginal revenue function $R'(x)$ is the derivative of the revenue function $R(x)$, where x is the number of units sold. What does $\int_{1000}^{5000} R'(x)dx$ represent?

$\int_{1000}^{5000} R'(x)dx$ represent the amount of revenue generated from the sale of the four thousand washing machines starting at 1000.

10. If x is measured in meters and $f(x)$ is measured in Newtons, what are the units for $\int_{0}^{100} f(x)dx$?

Newton meters, or joules

4.4 Homework Set B

1. Find the area under the curve $f(x) = e^{-x^2} - x$ on $x \in [-1,2]$ (do not use absolute values in your setup, break it into multiple integrals, then evaluate on your calculator using absolute values).

$$A = \int_{-1}^{0.65292} \left(e^{-x^2} - x\right)dx - \int_{0.65292}^{2} \left(e^{-x^2} - x\right)dx$$

A = 3.080

2. Find the area under the curve $f(x) = e^{-x^2} - 2x$ on $x \in [-1,2]$ (do not use absolute values in your setup, break it into multiple integrals, then evaluate on your calculator using absolute values).

$$A = \int_{-1}^{0.419365} \left(e^{-x^2} - 2x\right)dx - \int_{0.419365}^{2} \left(e^{-x^2} - 2x\right)dx$$

A = 5.305

3. Find the area under the curve $f(x) = \dfrac{x}{x^2+1} + \cos(x)$ on $x \in [0,\pi]$

A = 2.235

4. Find the area under the curve $g(x) = -1 - x\sin x$ on $x \in [0,2\pi]$

A = 14.330

4.5 Graphical Analysis with Integration

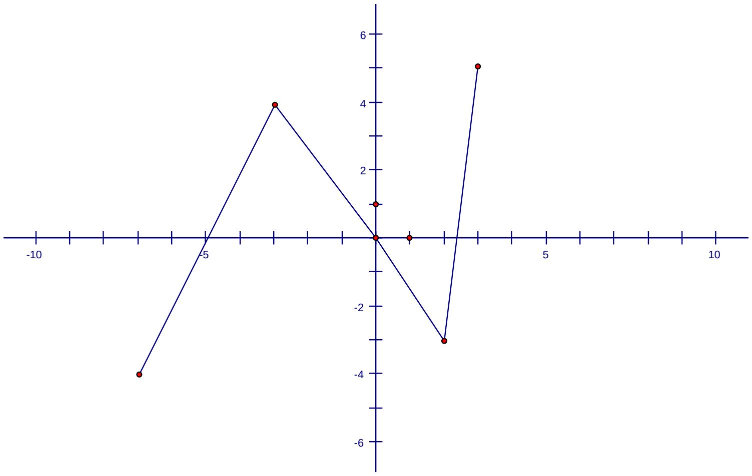
An important part of calculus is being able to read information from graphs. We have looked at graphs and sign patterns of first derivatives and answered questions concerning the traits of the "original" function. This section will answer the same questions, even with similar graphs – but this time the "original" function will be defined as an integral in the form of the FUNDAMENTAL THEOREM OF CALCULUS. **This is a major AP topic.**

Objectives
Use the graph of a function to answer questions concerning the area under a curve.

We have learned that the graph of a function and its derivative are closely related. It should be no surprise that we can apply this same logic to a graph and its integral.

$G(x) = \int_0^x F(t)\,dt$	The y-values of $G(x)$	increasing EXTREME negative	concave up POI concave down
$F(x)$	Area under $F(x)$	positive ZERO Negative	increasing EXTREME negative
$F'(x)$			positive ZERO Negative

Ex 1 The graph of f is shown below. f consists of three line segments.



Let $g(x) = \int_{-3}^{x} f(t)\,dt$.

(a) Find $g(3)$.

(b) Find $g(-6)$.

(c) Find $g(1)$, $g'(1)$, and $g''(1)$.

(d) On what intervals is g decreasing?

(e) Where is g concave up?

(f) Make a sketch of the original function.

(g) How do the answers in (a) – (e) change if $m(x) = 4 + \int_{-3}^{x} f(t)\,dt$?

(a) Find $g(3)$.

$$g(3) = \int_{-3}^{3} f(t)\,dt$$

We must split this integral into four parts – the two triangles above the x – axis and the two triangles below the x- axis.

$$g(3) = \int_{-3}^{3} f(t)\,dt = \int_{-3}^{0} f(t)\,dt - \int_{0}^{2} f(t)\,dt - \int_{2}^{19/8} f(t)\,dt + \int_{19/8}^{3} f(t)\,dt$$

Where did $x = 19/8$ come from? That is the x – intercept of that line segment.

$$g(3) = 6 - 3 - \frac{9}{16} + \frac{25}{16} = 4$$

Use the area formula for a triangle to calculate the different areas. Area below the x – axis must be subtracted.

(b) Find $g(-6)$.

$g(-6) = \int_{-3}^{-6} f(t)dt = -\int_{-6}^{-3} f(t)dt$ Use the properties of integrals.

$= -\left(\frac{1}{2}\cdot 2\cdot 4 - \frac{1}{2}\cdot 1\cdot 2\right)$ Area of triangles – triangle above the x – axis is added to the integral, triangle below the x – axis is subtracted from the integral.

$= -3$

(c) Find $g(1)$, $g'(1)$, and $g''(1)$.

$g(1) = \int_{-3}^{1} f(t)dt = \int_{-3}^{0} f(t)dt - \int_{0}^{1} f(t)dt$ Again, we must split the integral into two parts – one for each separate triangle.

$g(1) = 6 - \frac{3}{4} = \frac{21}{4}$ How did we conclude that $3/4$ as the area of that second triangle? Find the equation of the line making up the hypotenuse of the triangle, plugged in $x = 1$, and use that number as the height of the triangle. Subtract this number because that triangle was below the x – axis.

Now in order to find $g'(1)$, we must find $g'(x)$. But since g was defined as an integral, the FUNDAMENTAL THEOREM OF CALCULUS kicks in when we take the derivative of g.

$g'(x) = \frac{d}{dx}\left[\int_{-3}^{x} f(t)dt\right] = f(x)$ Derivatives and integrals are inverse operations – the FUNDAMENTAL THEOREM OF CALCULUS

$g'(1) = f(1) = -\frac{3}{2}$ You can read the value from the graph.

In order to find $g''(1)$, we must find $g''(x)$. But this follows from what we just did. Since $g'(x) = f(x)$, we can see that $g''(x) = f'(x)$.

$g''(1) = f'(1) = -\frac{3}{2}$

The derivative is the slope of *f* – and the slope of that line segment is $-\frac{3}{2}$.

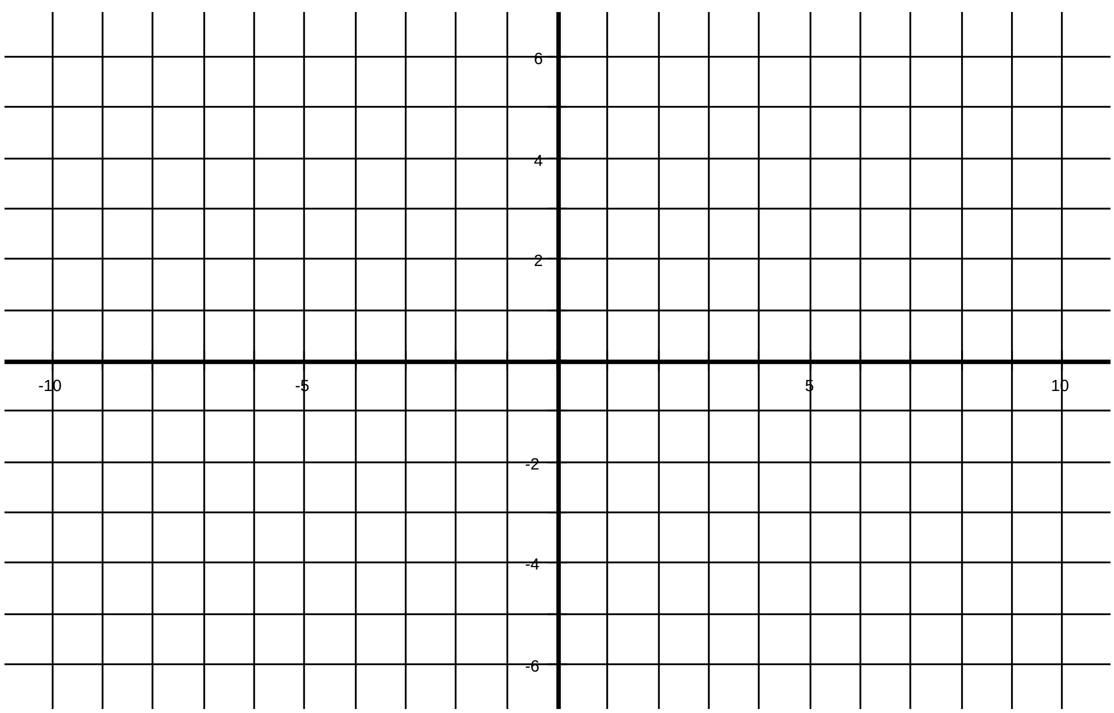
(d) On what intervals is *g* decreasing?

g is decreasing when $g' < 0$ -- but as stated above $g'(x) = f(x)$. And we can see graphically that *f* is negative from $x \in (-7,-5) \cup \left(0, \frac{19}{8}\right)$.

(e) Where is *g* concave up?

g is concave up when f' is increasing. So *g* is concave up from $x \in (-7,-3) \cup (2,3)$.

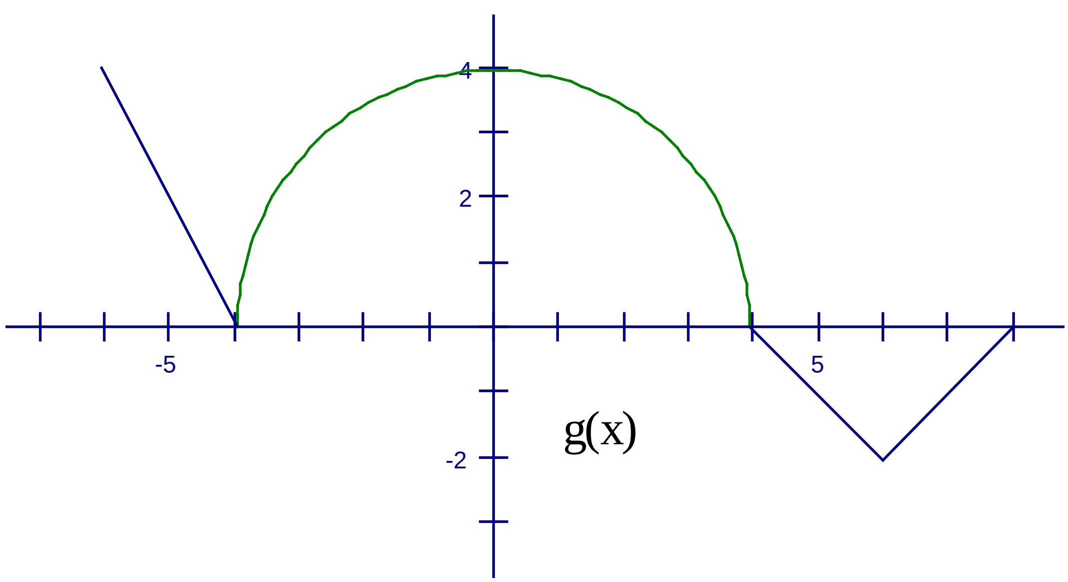
(f) Make a sketch of the original function.



(g) How do the answers in (a) – (e) change if $m(x) = 4 + \int_{-3}^{x} f(t)\,dt$?

The answers in (a) and the first part of (b) would both need a 4 added to them. The graph in part (e) would be shifted up 4 units. And all other answers would remain the same.

Ex 2 Let $h(x) = \int_{0}^{x} g(t)\,dt$. The graph of g is shown below.



(a) Find $h(4)$.

(b) Find $h(0)$, $h'(0)$, and $h''(0)$.

(c) What is the instantaneous rate of change of *h(x)* at $x = 1$?

(d) Find the absolute minimum value of *h* on $[-6,9]$.

(e) Find the x – coordinate of all points of inflection of *h* on $(-6,9)$.

(f) Let $F(x) = \int_x^0 g(t)dt$. When is *F* increasing?

(a) $h(4) = \int_0^4 g(t)dt$ would equal the area from 0 to 4. This is one quarter of the circle with radius 4. So, $h(4) = \int_0^4 g(t)dt = 4\pi$.

(b) $h(0) = \int_0^0 g(t)dt = 0$

$h'(0) = \dfrac{d}{dx}\int_0^x g(t)dt = g(0) = 4$

$h''(0) = g'(0) = 0$

(c) The instantaneous rate of change of $h(x)$ at $x = 1$ would be $g(1)$. To find $g(1)$, we would need the equation of the circle. Since I t has its center at the origin and radius = 4, the equation is $x^2 + y^2 = 16$ or $y = \sqrt{16 - x^2}$.

$$g(1) = \sqrt{16 - 1^2} = \sqrt{15}$$

(d) Since $g(x) = h'(x)$, the critical values of h would be the zeros and endpoints of g, namely, $x = -6, -4, 4, 9$. The graph of $g(x)$ yields a sign pattern:

g	0	+	0	+	0	–	0
x	–6		–4		4		9

From this, we can see that x = -4 and 2 cannot be minimums. x = 2 is a maximum because the sign of $g(x) = h'(x)$ switches from + to – and $x = -4$ is not an extreme because the sign does not change. Now all we need are $h(-6)$ and $h(-9)$. Whichever is smaller is the absolute minimum.

$h(-6) = \int_0^{-6} g(t)dt = -\int_{-6}^{0} g(t)dt = -(4+4\pi)$

$h(9) = \int_0^{9} g(t)dt = 4\pi - 4$

$h(-6)$ is the smaller value so the absolute minimum is $-(4+4\pi)$

(e) $g'(x) = h''(x)$ so the points of inflection on h would be where the slope of g changes sign, namely at $x = -4$, 0 and 6.

(f) $F(x) = \int_x^0 g(t)dt \to F'(x) = -g(x)$. So F is increasing when – is negative, namely on $x \in (4,8)$.

4.5 Homework

Handout of AP Questions

1. BC 2002B #4
2. BC 1999 #5
3. BC 2003 #4
4. AB 2000 #3
5. AB 1996 #1

4.6 Accumulation of Rates

Beginning in 2002, AP shifted emphasis on understanding of the accumulation aspect of the Fundamental Theorem to a new kind of rate problem. The following Amusement Park problem caught many off guard. Almost every year since then, the test has included this kind of problem, usually #2.

Objective
Analyze the interplay between rates and accumulation in context.

Ex 1 The rate at which people enter a park is given by the function $E(t)=\dfrac{15600}{t^2-24t+160}$, and the rate at which they are leaving is given by $L(t)=\dfrac{9890}{t^2-38t+370}-76$. Both $E(t)$ and $L(t)$ are measured in people per hour where *t* is the number of hours past midnight. The functions are valid for when the park is open, $8\le t\le 24$. At *t* = 8 there are no people in the park.

a) How many people have entered the park at 4 pm (*t* = 16)? Round your answer to the nearest whole number.

b) The price of admission is $36 until 4 pm (*t* = 16). After that, the price drops to $20. How much money is collected from admissions that day? Round your answer to the nearest whole number.

c) Let $H(t)=\int_8^t E(x)-L(x)\,dx$ for $8\le t\le 24$. The value of $H(16)$ to the nearest whole number is 5023. Find the value of $H'(16)$ and explain the meaning of $H(16)$ and $H'(16)$ in the context of the amusement park.

d) At what time *t*, for $8\le t\le 24$, does the model predict the number of people in the park is at a maximum.

a) Since $E(t)$ is a rate in people per hour, the number of people who have entered the park will be an integral from 8 to 16.

$$\text{Total entered} = \int_8^{16} E(t)\,dt = 6126 \text{ people}$$

b) Since there is an entry fee per person, and we saw in the first part that the integral gave us the number of people the total revenue that the park gets from admissions just means multiplying the number of people by the admission charge.

$$\text{Total entered at \$36/person} = \int_8^{16} E(t)\,dt = 6126 \text{ people}$$

$$\text{Total entered at \$20/person} = \int_{16}^{24} E(t)\,dt = 1808 \text{ people}$$

Total revenue = \$36(6126) + \$20(1808) = \$256,696

c) Since $H(t) = \int_8^t E(x) - L(x)\,dx$, we can use the Fundamental Theorem of Calculus to determine the derivative.

$$H'(t) = \frac{d}{dt}\int_8^t E(x) - L(x)\,dx$$

$$H'(t) = E(t) - L(t)$$

$$H'(16) = E(16) - L(16)$$

So we just plug 16 into the two original functions and subtract.

$$H'(16) = 14$$

But we still need to interpret the meaning of the numbers.

$H(16) = 5023$ people – since it was defined as an integral, we knew that integrating the rates gave us people. This is how many people are in the park at that moment.

$H'(16) = 14$ people per hour – since the original equations we were provided were rates, and $H'(t) = E(t) - L(t)$, so this is how many people are entering the park at t = 16. Since the value is positive, this is the rate at which the number of people in the park is increasing.

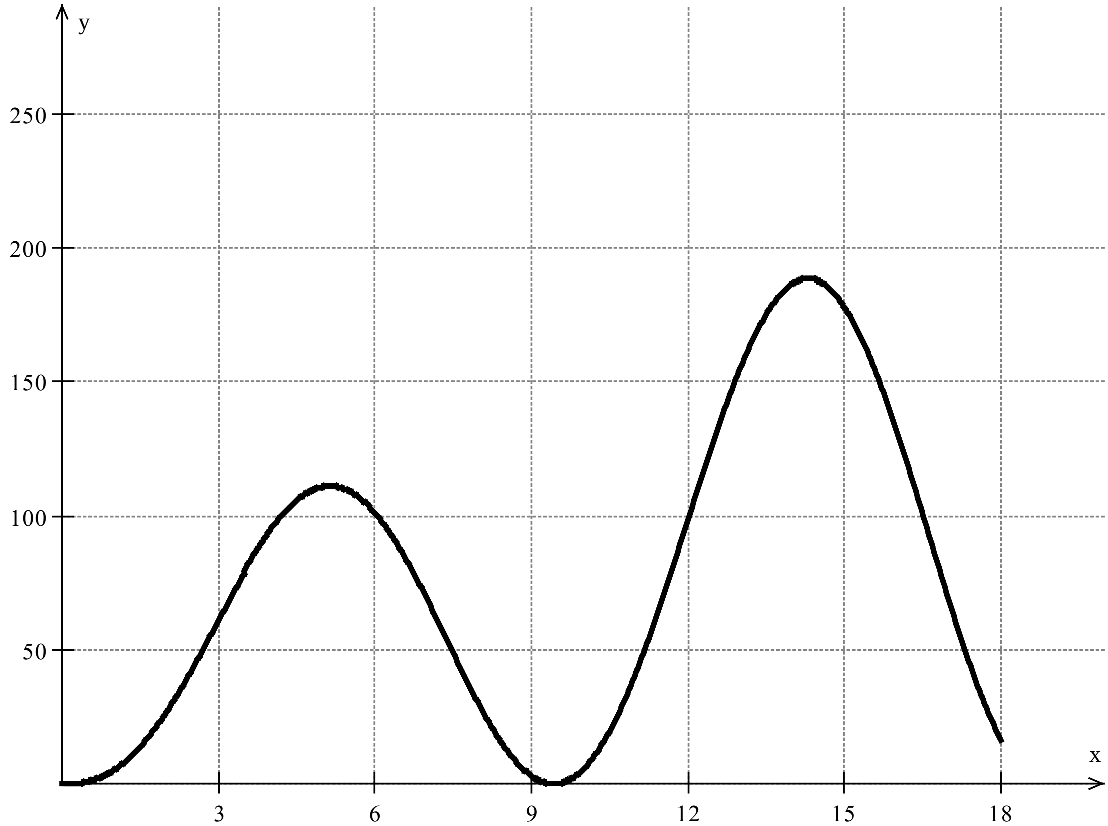
d) To find the maximum number of people, we have to set the derivative, $H'(t)$, equal to zero. We should also check the endpoints, which are also critical values (we could just do a sign pattern instead to verify that the zero is the maximum).

$$H'(t) = E(t) - L(t) = 0$$

Using a graphing calculator, we find the zero is at t = 16.046
Therefore we have critical values at t = 0, 16.046, and 24.

When t = 0, H = 0 (this was given at the beginning of the problem)
When t = 16.046, H = 5023
When t = 24, H = 1453

Ex 2 At an intersection in San Francisco, cars turn left at the rate $L(t) = 50\sqrt{t}\sin^2\left(\frac{t}{3}\right)$ cars per hour for the time interval $0 \le t \le 18$. The graph of L is shown below.



a) To the nearest whole number, find the total number of cars turning left on the time interval given above.

b) Traffic engineers will consider turn restrictions if $L(t)$ equals or exceeds 125 cars per hour. Find the time interval where $L(t) \ge 125$, and find the average value of $L(t)$ for this time interval. Indicate units of measure.

c) San Francisco will install a traffic light if there is a two hour time interval in which the product of the number of cars turning left and the number of cars travelling through the intersection exceeds 160,000. In every two hour interval, 480 cars travel straight through the intersection. Does this intersection need a traffic light? Explain your reasoning.

4.6 Homework

1. A certain industrial chemical reaction produces synthetic oil at a rate of $S(t)=\dfrac{15t}{1+3t}$. At the same time, the oil is removed from the reaction vessel by a skimmer that has a rate of $R(t)=2+5\sin\left(\dfrac{4\pi}{25}t\right)$. Both functions have units of gallons per hour, and the reaction runs from t = 0 to t = 6. At time of t = 0, the reaction vessel contains 2500 gallons of oil.

   a) How much oil will the skimmer remove from the reaction vessel in this six hour period? Indicate units of measure.

   b) Write an expression for $P(t)$, the total number of gallons of oil in the reaction vessel at time t.

   c) Find the rate at which the total amount of oil is changing at t = 4.

   d) For the interval indicated above, at what time t is the amount of oil in the reaction vessel at a minimum? What is the minimum value? Justify your answers.

2. The number of parts per million (ppm), $C(t)$, of chlorine in a pool changes at the rate of $P'(t) = 1 - 3e^{-0.2\sqrt{t}}$ ounces per day, where t is measured in days. There are 50 ppm of chlorine in the pool at time $t = 0$. Chlorine should be added to the pool if the level drops below 40 ppm.

a) Is the amount of chorine increasing or decreasing at $t = 9$? Why or why not?

b) For what value of t is the amount of chlorine at a minimum? Justify your answer.

c) When the value of chlorine is at a minimum, does chlorine need to be added? Justify your answer.

3. The basement of a house is flooded, and water keeps pouring in at a rate of $w(t) = 95\sqrt{t}\sin^2\left(t/6\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t) = 275\sin^2\left(t/3\right)$. When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

a) Is the amount of water increasing at $t = 15$? Why or why not?

b) To the nearest whole number, how many gallons are in the basement at the time $t = 18$?

c) At what time t, for $0 \le t \le 18$, is the amount of water in the basement at an absolute minimum? Show the work that leads to this conclusion.

d) For $t > 18$, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of k.

4. A tank at a sewage processing plant contains 125 gallons of raw sewage at time $t = 0$. During the time interval $0 \le t \le 12$ hours, sewage is pumped into the tank at the rate $E(t) = 2 + \dfrac{10}{1+\ln(t+1)}$. During the same time interval, sewage is pumped out at a rate of $L(t) = 12\sin\left(\dfrac{t^2}{47}\right)$.

a) How many gallons of sewage are pumped into the tank during the time interval $0 \le t \le 12$ hours?

b) Is the level of sewage rising or falling at $t = 6$? Explain your reasoning.

c) How many gallons of sewage are in the tank at $t = 12$ hours?

d) At what time t, for $0 \le t \le 12$, is the volume of the sewage at an absolute maximum? Show the analysis that leads to your answer. If the sewage level ever exceeds 150 gallons, the tank overflows. Is there a time at which the tank overflows? Explain.

Answers: 4.6 Homework

1. A certain industrial chemical reaction produces synthetic oil at a rate of $S(t)=\frac{15t}{1+3t}$. At the same time, the oil is removed from the reaction vessel by a skimmer that has a rate of $R(t)=2+5\sin\left(\frac{4\pi}{25}t\right)$. Both functions have units of gallons per hour, and the reaction runs from t = 0 to t = 6. At time of t = 0, the reaction vessel contains 2500 gallons of oil.

a) How much oil will the skimmer remove from the reaction vessel in this six hour period? Indicate units of measure.

31.816 Gallons

b) Write an expression for $P(t)$, the total number of gallons of oil in the reaction vessel at time t.

$$P(t)=2500+\int_0^t\left(\frac{15x}{1+3x}\right)-\left(2+5\sin\left(\frac{4\pi}{25}x\right)\right)dx$$

c) Find the rate at which the total amount of oil is changing at t = 4.

– 0.371 Gallons per Hour

d) For the interval indicated above, at what time t is the amount of oil in the reaction vessel at a minimum? What is the minimum value? Justify your answers.

There are Critical Values at t = 0, 5.118, and 6. The extremes associated with these critical values are 2500, 2492.369, and 2493.277 gallons, respectively. Therefore the minimum value is 2492.369 occurring at t = 5.118

2. The number of parts per million (ppm), $P(t)$, of chlorine in a pool changes at the rate of $P'(t)=1-3e^{-0.2\sqrt{t}}$ ounces per day, where t is measured in days. There are 50 ppm of chlorine in the pool at time t = 0. Chlorine should be added to the pool if the level drops below 40 ppm.

a) Is the amount of chorine increasing or decreasing at t = 9? Why or why not?

$P'(9)=-0.646$; since this rate is negative, the amount of chlorine is decreasing at t = 9.

b) For what value of t is the amount of chlorine at a minimum? Justify your answer.

The value of $P'(t)$ is 0 when t = 30.174 days. Since P' is negative before this time and positive after, P is decreasing and then increasing, indicating a minimum value on P.

c) When the value of chlorine is at a minimum, does chlorine need to be added? Justify your answer.

When the chlorine is at its minimum, $P(30.173724) = 50 + \int_0^{30.173724} P'(t)\,dt$.

Therefore, the minimum value is 35.104 ppm of chlorine. Therefore, chlorine should be added.

3. The basement of a house is flooded, and water keeps pouring in at a rate of $w(t) = 95\sqrt{t}\sin^2\left(t/6\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t) = 275\sin^2\left(t/3\right)$. When the pump is started, at time t = 0, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

a) Is the amount of water increasing at t = 15? Why or why not?

Rate = $w(t) - r(t)$. At t = 15, the rate is –121.090 gallons per hour. Since this is negative, the amount of water is decreasing.

b) To the nearest whole number, how many gallons are in the basement at the time t = 18?

$1200 + \int_0^{18} w(t) - r(t)\,dt$ = 1309.788 gallons.

c) At what time t, for $0 \le t \le 18$, is the amount of water in the basement at an absolute minimum? Show the work that leads to this conclusion.

Critical values occur at t = 0, 6.495, 12.975, and 18, and respective extreme values are 1200, 525.242, 1697.441, and 1309.788 gallons, the absolute minimum occurs at t = 6.495 hours.

d) For t > 18, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of k.

$$1309.788 = \int_{18}^{k} r(t)\,dt$$

4. A tank at a sewage processing plant contains 125 gallons of raw sewage at time $t = 0$. During the time interval $0 \le t \le 12$ hours, sewage is pumped into the tank at the rate $E(t) = 2 + \dfrac{10}{1 + \ln(t+1)}$. During the same time interval, sewage is pumped out at a rate of $L(t) = 12\sin\left(\dfrac{t^2}{47}\right)$.

a) How many gallons of sewage are pumped into the tank during the time interval $0 \le t \le 12$ hours?

$$\int_{0}^{12} E(t)\,dt = 70.571 \text{ gallons}$$

b) Is the level of sewage rising or falling at $t = 6$? Explain your reasoning.

Rate = $E(t) - L(t)$. The rate at $t = 6$ is –2.924 gallons per hour. Since this value is negative, the level of the sewage is falling.

c) How many gallons of sewage are in the tank at $t = 12$ hours?

$$125 + \int_{0}^{12} E(t) - L(t)\,dt = 122.026 \text{ gallons}$$

d) At what time t, for $0 \le t \le 12$, is the volume of the sewage at an absolute maximum? Show the analysis that leads to your answer. If the sewage level ever exceeds 150 gallons, the tank overflows. Is there a time at which the tank overflows? Explain.

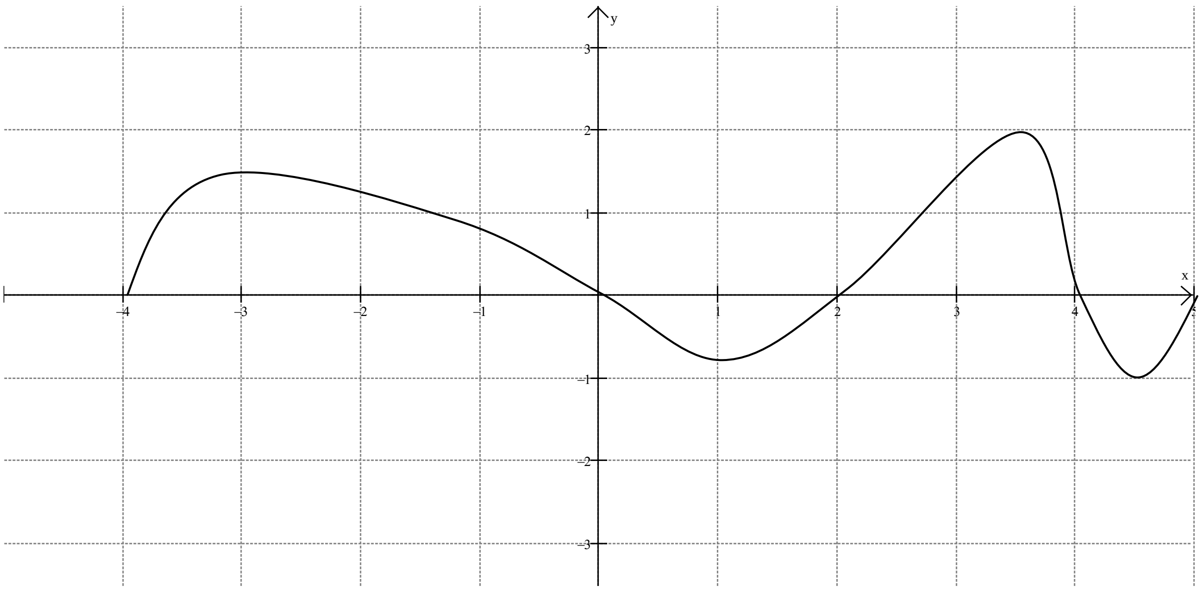
Critical values occur at $t = 0$, 4.790, 11.318, and 12. The respective extreme values are 125, 149.408, 120.738, 122.026 gallons. Therefore, the absolute maximum occurs at $t = 4.790$ hours. Since the absolute maximum value is less than 150, the tank never overflows.

Chapter 4 Test

A.M.D.G.

Instructions: Round all answers at three decimal places. You must show the set-up for all integrations, but you may perform them using Math 9 on your calculator.

1) Given the illustration of $y = f(x)$ and information below, find the values for each of the following:



$$\int_{-4}^{0} f(x)dx = 4,\ \int_{0}^{2} f(x)dx = -1,\ \int_{2}^{4} f(x)dx = 2,\ \int_{4}^{5} f(x)dx = \frac{-1}{2}$$

a) $\int_{-4}^{5} f(x)dx =$ ____________

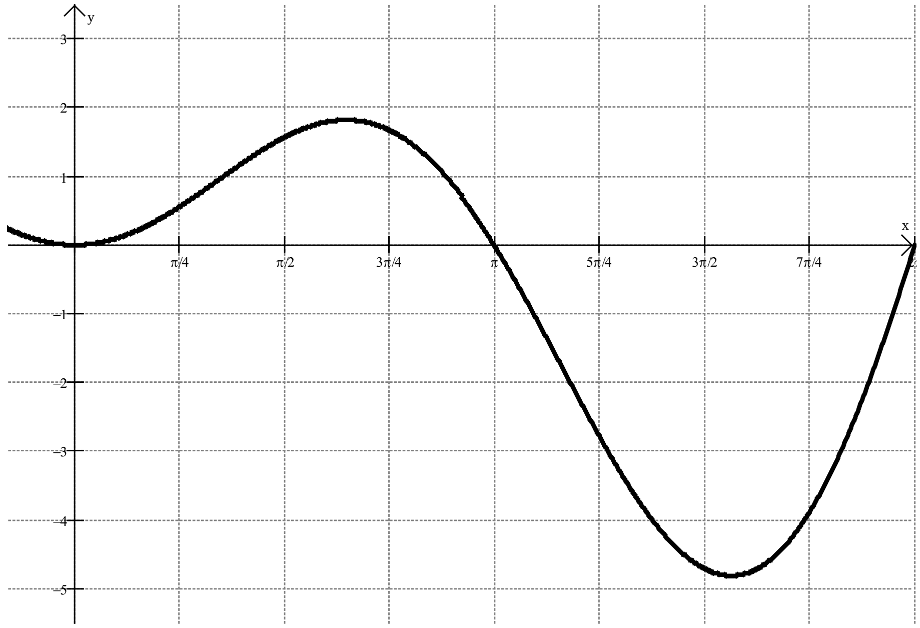
b) Area under $f(x)$ from $x = -4$ to $x = 5$ ____________

c) $\int_{-4}^{5} |f(x)|dx =$ ____________

d) Area under $f(x)$ from $x = -4$ to $x = 4$ ____________

e) $\int_{-4}^{4} f(x)dx =$ ____________

2) Given the function $y = x\sin(x)$ on $x \in [0, 2\pi]$, pictured below.

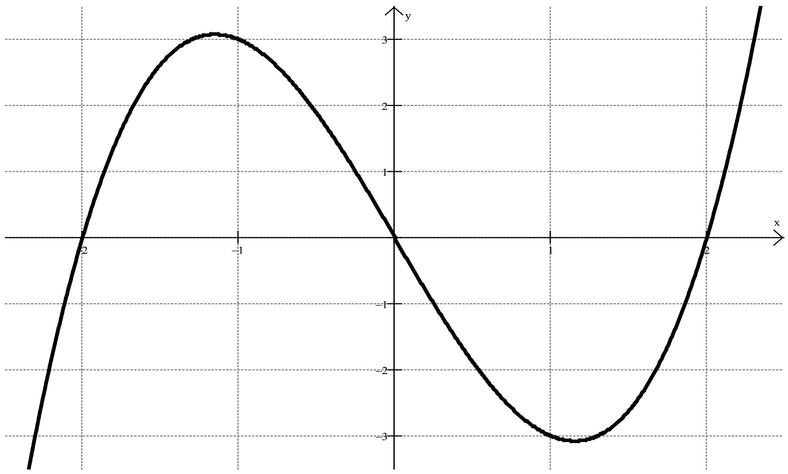


a) Find the value of the integral, $\int_0^{2\pi} x\sin(x)\,dx$.

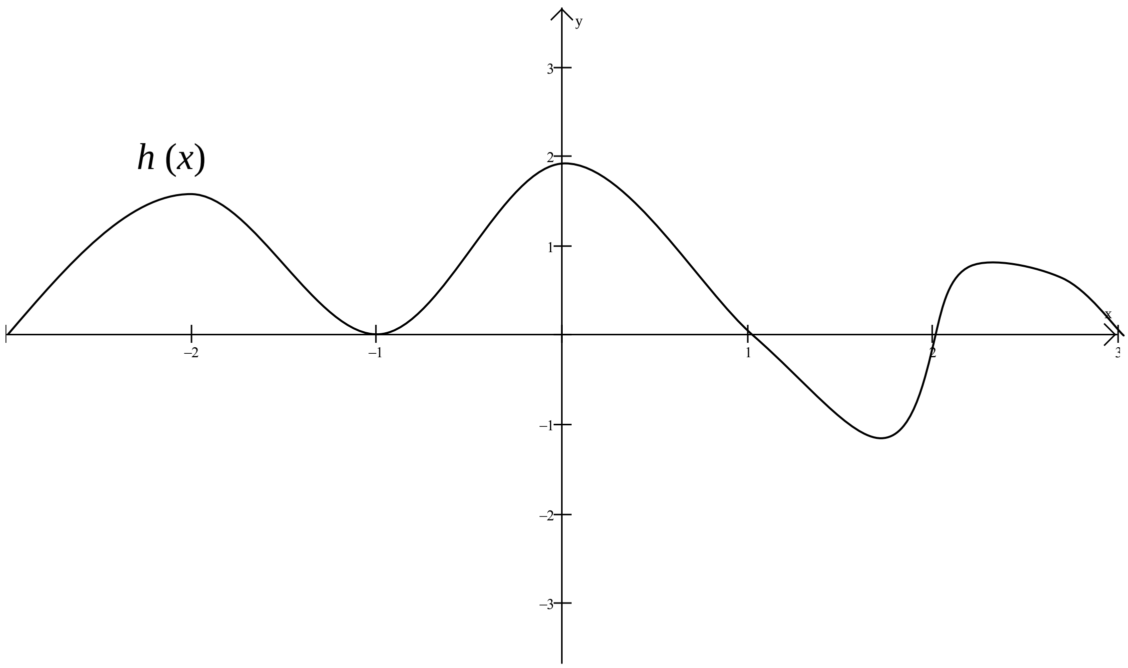
b) Find the area bounded by the curve and the x-axis on this interval. Write the integral expression for the area; you may use Math 9 to evaluate the area.

c) Find the average value of $y = x\sin(x)$ on $x \in [0, 2\pi]$

3) Find the area between the curve $f(x) = x^3 - 4x$ and the *x*-axis on $x \in [-2, 2]$. You must show the setup to receive credit for this problem (do not use absolute values in your setup, break it into multiple integrals). Show the integration as well. You may use Math 9 to calculate the value. Then find the average value of $f(x) = x^3 - 4x$ on $x \in [-2, 2]$.



4) Write an integral expression representing the area between the curve and the x-axis for the function, $h(x)$, on the interval $x \in [-3,3]$. Do not use absolute values in your setup.



Area =____

5) Find the values of the integrals below. Show all work.

a) $\int_0^1 x^3 \left(7+\frac{1}{2}x^4\right)^9 dx$

b) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 y \tan^8 y dy$

6) Integrate the following functions. Show all work.

a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 x dx$

b) $\int_{-1}^{0} \frac{1}{1+y^2} dy$

7) Below is a chart showing the rate of sewage flow through a pipe according to time in minutes. Use this information to answer each of the questions below.

t (in minutes)	0	6	12	16	20	30	40
v(*t*) (in m^3/min)	16	23	32	18	16	8	6

a) Find an approximation for $\int_0^{40} v(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_0^{16} v(t)\,dt$ using right Riemann rectangles. Make sure you express your answer in correct units.

8) Below is a chart showing the rate of sewage flow through a pipe according to time in minutes. Use this information to answer each of the questions below.

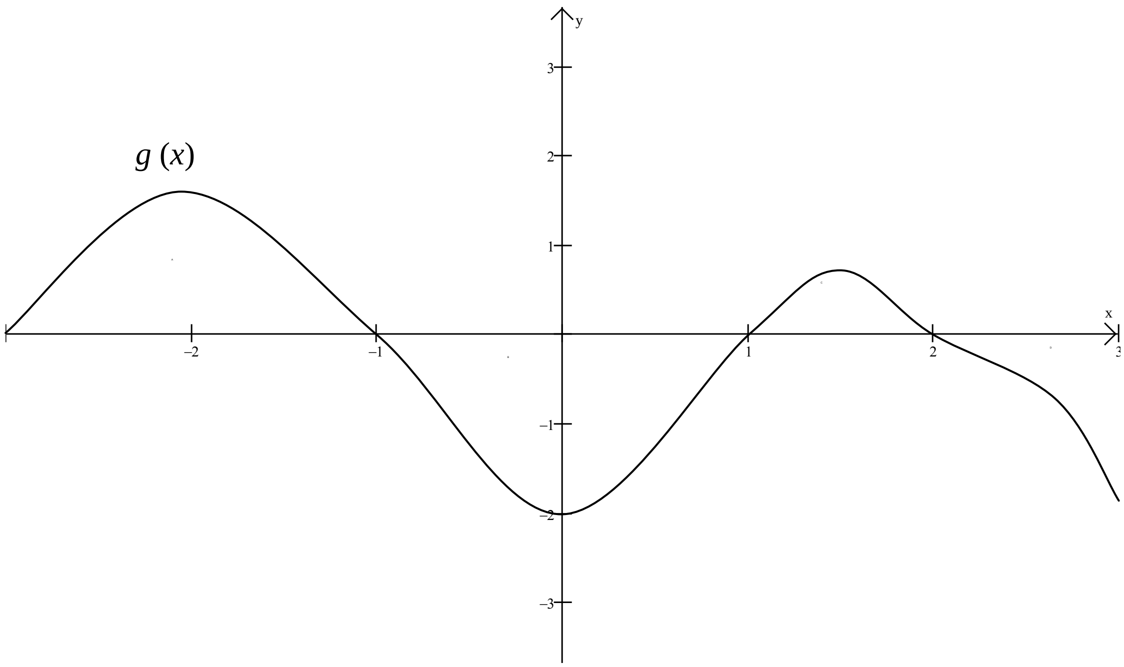
t (in minutes)	0	6	12	16	20	30	40
$v(t)$ (in m^3/min)	16	23	32	18	16	8	6

a) Find an approximation for $\int_{30}^{40} v(t)\,dt$ using left Riemann rectangles. Make sure you express your answer in correct units.

b) Find an approximation for $\int_{0}^{12} v(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

9) For the function, $g(x)$, below, the areas of each of the regions are given below:

A=12
B=14
C=1
D=2



Find each of the following:

a) $\int_{-3}^{3} g(x)dx =$

c) $\int_{-1}^{3} g(x)dx =$

b) $\int_{-3}^{1} g(x)dx =$

d) $\int_{-1}^{2} g(x)dx =$

10) The basement of a house is flooded, and water keeps pouring in at a rate of $w(t) = 95\sqrt{t}\sin^2\left(t/6\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t) = 275\sin^2\left(t/3\right)$. When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

a) Is the amount of water increasing at $t = 15$? Why or why not?

b) To the nearest whole number, how many gallons are in the basement at the time $t = 18$?

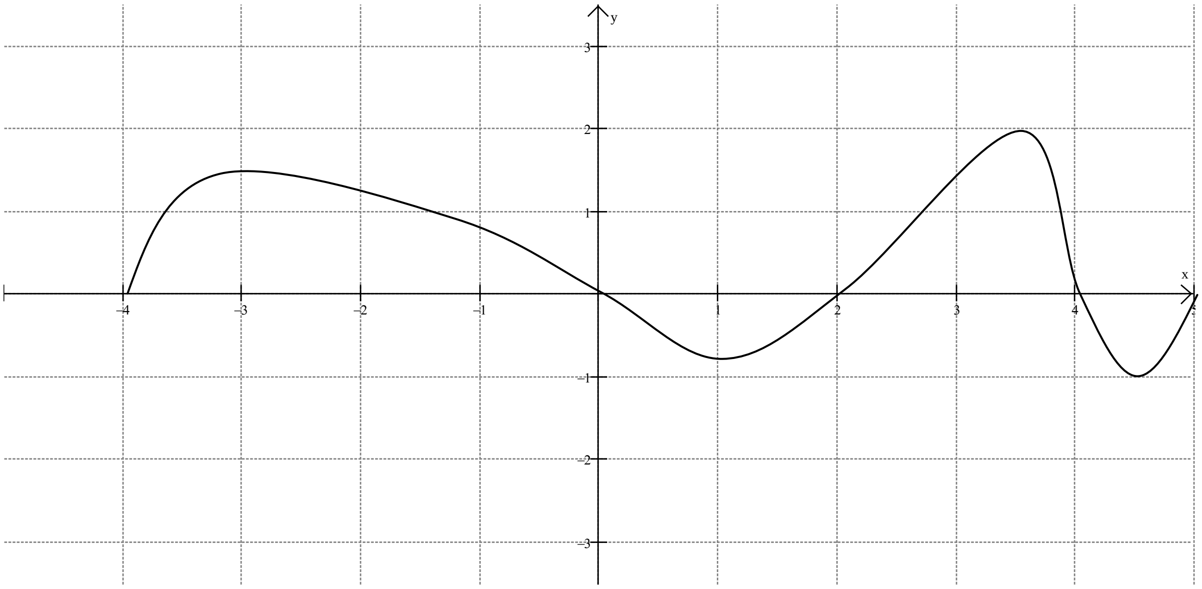
c) At what time t, for $0 \le t \le 18$, is the amount of water in the basement at an absolute minimum? Show the work that leads to this conclusion.

d) For $t > 18$, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of k.

Answers: Chapter 4 Test

A.M.D.G.

1) Given the illustration of $y = f(x)$ and information below, find the values for each of the following:



$$\int_{-4}^{0} f(x)dx = 4,\ \int_{0}^{2} f(x)dx = -1,\ \int_{2}^{4} f(x)dx = 2,\ \int_{4}^{5} f(x)dx = \frac{-1}{2}$$

a) $\int_{-4}^{5} f(x)dx =$ 4.5

b) Area under $f(x)$ from $x = -4$ to $x = 5$ 7.5

c) $\int_{-4}^{5} |f(x)|dx =$ 7.5

d) Area under $f(x)$ from $x = -4$ to $x = 4$ 7

e) $\int_{-4}^{4} f(x)dx =$ 5

2) Given the function $y = x\sin(x)$ on $x \in [0, 2\pi]$, pictured below.

a) Find the value of the integral, $\int_{0}^{2\pi} x\sin(x)dx$.

$$\int_{0}^{2\pi} x\sin(x)dx = -6.283$$

b) Find the area bounded by the curve and the x-axis on this interval. Write the integral expression for the area; you may use Math 9 to evaluate the area.

Area = 12.566

c) Find the average value of $y = x\sin(x)$ on $x \in [0, 2\pi]$

$$\frac{1}{2\pi}\int_0^{2\pi} x\sin(x)\,dx = -1$$

3) Find the area between the curve $f(x) = x^3 - 4x$ and the *x*-axis on $x \in [-2, 2]$. You must show the setup to receive credit for this problem (do not use absolute values in your setup, break it into multiple integrals). Show the integration as well. You may use Math 9 to calculate the value. Then find the average value of $f(x) = x^3 - 4x$ on $x \in [-2, 2]$.

$$Area = \int_{-2}^{0}(x^3 - 4x)dx - \int_0^2 (x^3 - 4x)dx = 8$$

$$\text{Average value } = \frac{1}{2-(-2)}\int_{-2}^{0}(x^3 - 4x)dx - \int_0^2 (x^3 - 4x)dx = 0$$

4) Write an integral expression representing the area between the curve and the *x*-axis for the function, *h*(*x*), on the interval $x \in [-3, 3]$. Do not use absolute values in your setup.

$$\text{Area } = \int_{-3}^{1} h(x)dx - \int_1^2 h(x)dx + \int_2^3 h(x)dx$$

5) Find the values of the integrals below. Show all work.

a) $\displaystyle\int_0^1 x^3\left(7 + \frac{1}{2}x^4\right)^9 dx$

=14,032,994.9

b) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sec^2 y \tan^8 y\,dy$

$= \dfrac{2}{9}$

6) Integrate the following functions. Show all work.

a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \csc^2 x dx$

$= 1$

b) $\int_{-1}^{0} \frac{1}{1+y^2} dy$

$= \frac{\pi}{4}$

7) Below is a chart showing the rate of sewage flow through a pipe according to time in minutes. Use this information to answer each of the questions below.

a) Find an approximation for $\int_0^{40} v(t)\,dt$ using midpoint rectangles. Make sure you express your answer in correct units.

$\int_0^{40} v(t)\,dt \approx 580 \text{ meters}^3$

b) Find an approximation for $\int_0^{16} v(t)\,dt$ using right Riemann rectangles. Make sure you express your answer in correct units.

$\int_0^{16} v(t)\,dt \approx 402 \text{ meters}^3$

8) Below is a chart showing the rate of sewage flow through a pipe according to time in minutes. Use this information to answer each of the questions below.

a) Find an approximation for $\int_{30}^{40} v(t)\,dt$ using left Riemann rectangles. Make sure you express your answer in correct units.

$\int_{30}^{40} v(t)\,dt \approx 80 \text{ meters}^3$

b) Find an approximation for $\int_0^{12} v(t)\,dt$ using trapezoids. Make sure you express your answer in correct units.

$\int_0^{12} v(t)\,dt \approx 282 \text{ meters}^3$

9) For the function, $g(x)$, below, the areas of each of the regions are given below:

A=12
B=14
C=1
D=2

Find each of the following:

a) $\int_{-3}^{3} g(x)\,dx = -3$

c) $\int_{-1}^{3} g(x)\,dx = -15$

b) $\int_{-3}^{1} g(x)\,dx = -2$

d) $\int_{-1}^{2} g(x)\,dx = -13$

10) The basement of a house is flooded, and water keeps pouring in at a rate of $w(t) = 95\sqrt{t}\sin^2\left(\frac{t}{6}\right)$ gallons per hour. At the same time, water is being pumped out at a rate of $r(t) = 275\sin^2\left(\frac{t}{3}\right)$. When the pump is started, at time $t = 0$, there is 1200 gallons of water in the basement. Water continues to pour in and be pumped out for the interval $0 \le t \le 18$.

a) Is the amount of water increasing at $t = 15$? Why or why not?

Rate = $w(t) - r(t)$. At $t = 15$, the rate is –121.090 gallons per hour. Since this is negative, the amount of water is decreasing.

b) To the nearest whole number, how many gallons are in the basement at the time $t = 18$?

$1200 + \int_0^{18} w(t) - r(t)\,dt$ = 1309.788 gallons.

c) At what time t, for $0 \le t \le 18$, is the amount of water in the basement at an absolute minimum? Show the work that leads to this conclusion.

Critical values occur at $t = 0, 6.495, 12.975$, and 18, and respective extreme values are 1200, 525.242, 1697.441, and 1309.788 gallons, the absolute minimum occurs at $t = 6.495$ hours.

d) For $t > 18$, the water stops pouring into the basement, but the pump continues to remove water until all of the water is pumped out of the basement. Let k be the time at which the tank becomes empty. Write, but do not solve, an equation involving an integral expression that can be used to find a value of k.

$$1309.788 = \int_{18}^{k} r(t)\,dt$$

Chapter 5 Overview: Limits, Continuity and Differentiability

Derivatives and Integrals are the core practical aspects of Calculus. They were the first things investigated by Archimedes and developed by Liebnitz and Newton. The process involved examining smaller and smaller pieces to get a sense of a progression toward a goal. This process was not formalized algebraically, though, at the time. The theoretical underpinnings of these operations were developed and formalized later by Bolzano, Weierstrauss and others. These core concepts in this area are Limits, Continuity and Differentiability. Derivatives and Integrals are defined in terms of limits. Continuity and Differentiability are important because almost every theorem in Calculus begins with the assumption that the function is continuous and differentiable.

The Limit of a function is the function value (*y*-value) expected by the trend (or sequence) of *y*-values yielded by a sequence of *x*-values that approach the *x*-value being investigated. In other words, the Limit is what the *y*-value should be for a given *x*-value, even if the actual *y*-value does not exist. The limit was created/defined as an operation that would deal with *y*-values that were of an indeterminate form.

Indeterminate Form of a Number--Defn: "A number for which further analysis is necessary to determine its value."

Means: the number equals $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, 1^∞,or other strange things.

The formal definition is rather unwieldy and we will not deal with it in this course other than to show the Formal Definition and translate it:

Formal Definition of a Limit

$\lim\limits_{x\to a} f(x) = \mathrm{L}$ if and only if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$.

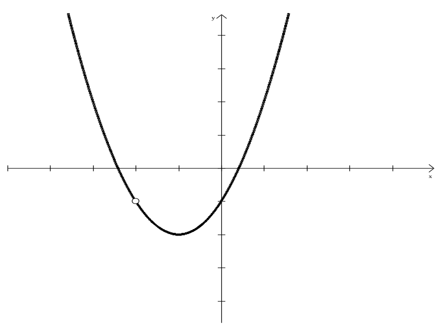
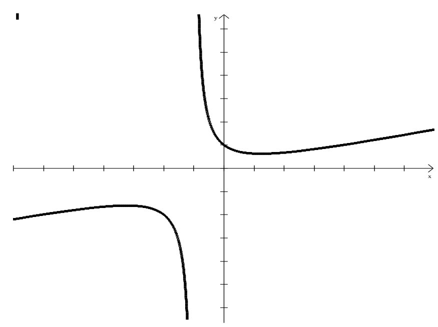
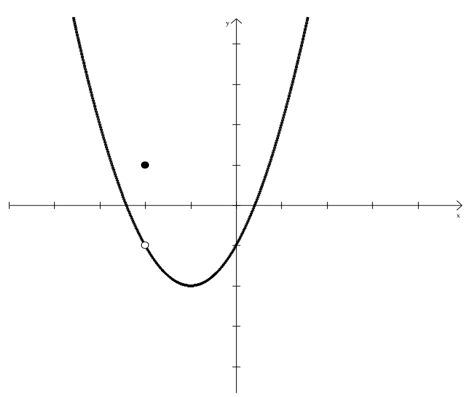
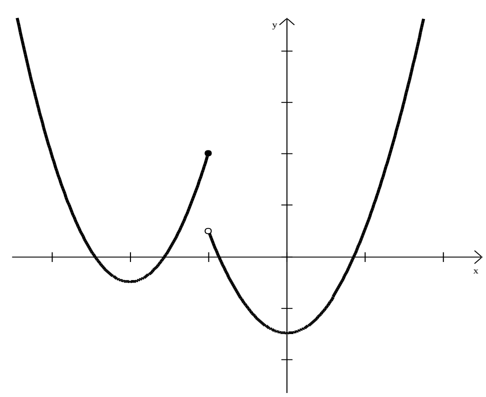
Formal Definition of a Limit means:

When x almost equals a, the limit almost equals y.

There are four kinds of Limits:

- Two-sided Limits (most often just referred to as Limits)
- One-sided Limits
- Infinite Limits
- Limits at Infinity

Continuity basically means a function's graph has no breaks in it. The formal definition involved limits. Since all the families of functions investigated in PreCalculus are continuous in their domain, it is easier to look at when a curve is discontinuous rather than continuous. There are four kinds of discontinuity:

	Removable Discontinuity ($\underset{x\to a}{Lim} f(x)$ does exist)	***Essential Discontinuity*** ($\underset{x\to a}{Lim} f(x)$ does not exist)
$f(a)$ does not exist	 Point of Exclusion (POE)	 Vertical asymptotes
$f(a)$ exists	 Point of Displacement (POD)	 Jump Discontinuity

5.1: Limits, L'Hopital's Rule, and The Limit Definitions of a Derivative

As mentioned in the introduction to this chapter and last year, the limit was created/defined as an operation that would deal with *y*-values that were of an indeterminate form.

$\lim_{x\to a} f(x)$ is read "the limit, as *x* approaches *a*, of *f* of *x*." What the definition means is, if *x* is almost equal to *a* (the difference is smaller than some small number δ), $f(x)$ is almost equal to L (the difference is smaller than some small number ε). In fact, they are so close, we could round off and consider them equal. In practice, usually $\lim_{x\to a} f(x) = f(a)$. In other words, the limit is the *y* for a given *x* = *a*--as long as $y \neq$ 0/0. If *y* = 0/0, we are allowed to factor and cancel the terms that gave the zeros. No matter how small the factors get, they cancel to 1 as long as they do not quite equal 0/0.

Ex 1 Find $\lim_{x\to 5}(x+2)$ and $\lim_{x\to -4}(x^2+3)$

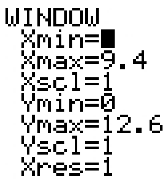
$\lim_{x\to 5}(x+2) = 5+2 =$ 7 $\qquad$ $\lim_{x\to -4}(x^2+3) = (-4)^2+3 =$ 19

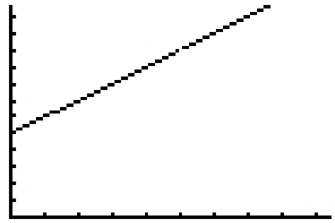
Ex 2 Find $\lim_{x\to 5}\frac{x^2-25}{x-5}$

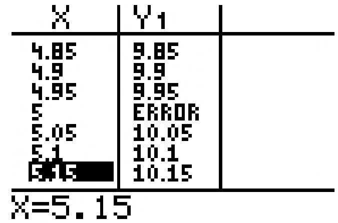
If *x* = 5 here, $\frac{x^2-25}{x-5}$, which is $\frac{(x-5)(x+5)}{x-5}$, would $=\frac{0}{0}$. But with a Limit, *x* is only *almost* equal to 5, and, therefore, $\frac{x-5}{x-5}=1$. So

$$\begin{aligned}\lim_{x\to 5}\frac{x^2-25}{x-5} &= \lim_{x\to 5}\frac{(x+5)(x-5)}{x-5}\\ &= \lim_{x\to 5}(x+5)\\ &= 5+5\\ &= 10\end{aligned}$$

Notice that throughout this process, we kept the limit notation in the problem until we actually evaluated the limit (that is, plugged in *a*). Not writing this notation actually makes the problem wrong – it is like getting rid of an operation. The limit is essentially what allows you to do the cancelling and/or plugging in. You must use proper notation when writing these up.







As we can see from both the graph (in the given window) and the table, while no *y*-value exists for *x* = 5, the *y*-values of the points on either side of *x* = 5 show *y* should be 10.

Basically, the limit allows us to factor and cancel before we substitute the number "a" for x.

OBJECTIVE

Evaluate Limits algebraically.
Evaluate Limits using L’Hopital’s Rule.
Recognize and evaluate Limits which are derivatives.
Use the *nDeriv* function on the calculator to find numerical derivatives.

Ex 3 Find $\lim\limits_{x\to 2}\dfrac{x^2+4x-12}{x^2+3x-10}$

Since if x = 2, $\dfrac{x^2+4x-12}{x^2+3x-10}=\dfrac{0}{0}$, we must be able to factor and cancel this fraction. And, in fact, we know one of the factors in each must be $(x-2)$, otherwise the fraction would not yield zeros. So,

$$\begin{aligned}\lim_{x\to 2}\frac{x^2+4x-12}{x^2+3x-10}&=\lim_{x\to 2}\frac{(x-2)(x+6)}{(x-2)(x+5)}\\&=\lim_{x\to 2}\frac{x+6}{x+5}\\&=\frac{8}{7}\end{aligned}$$

Ex 4 Find $\lim_{x\to -3} \frac{2x^3+x^2-13x+6}{x^2+x-6}$

$$\begin{aligned}\lim_{x\to -3} \frac{2x^3+x^2-13x+6}{x^2+x-6} &= \lim_{x\to -3} \frac{(x+3)(2x^2-5x+2)}{(x+3)(x-2)} \\ &= \lim_{x\to -3} \frac{(2x^2-5x+2)}{(x-2)} \\ &= \frac{35}{-5} \\ &= -7\end{aligned}$$

Ex 5 Find $\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2}$

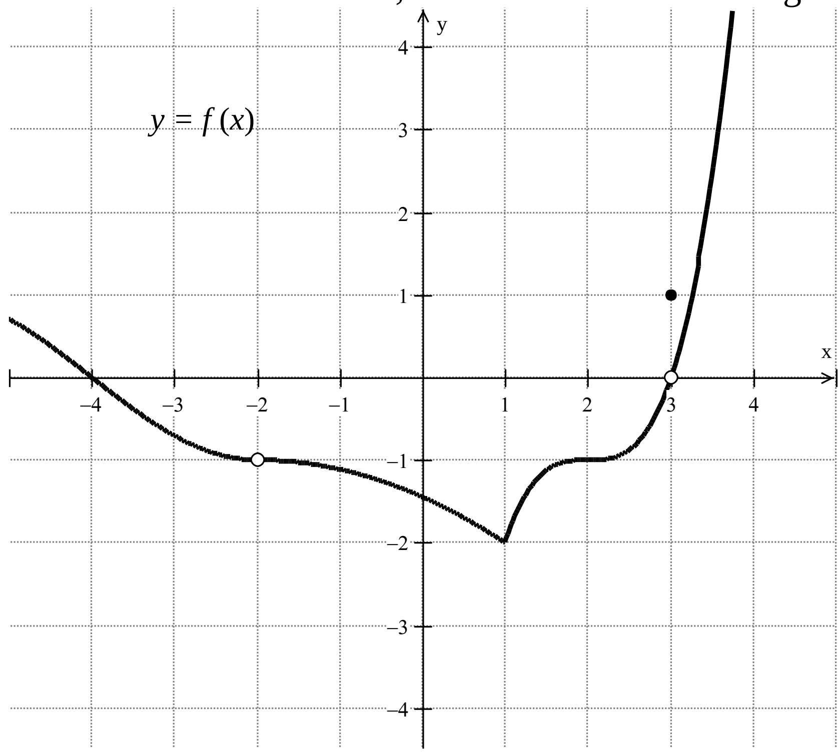
Unlike the previous examples, this fraction does not factor. Yet it must simplify, somehow, to eliminate the Indeterminate Number. If we multiply by conjugates to eliminate the radicals from the numerator:

$$\begin{aligned}\lim_{x\to 2} \frac{\sqrt{4-x}-\sqrt{2}}{x-2} &= \lim_{x\to 2} \frac{(\sqrt{4-x}-\sqrt{2})(\sqrt{4-x}+\sqrt{2})}{(x-2)(\sqrt{4-x}+\sqrt{2})} \\ &= \lim_{x\to 2} \frac{4-x-2}{(x-2)(\sqrt{4-x}+\sqrt{2})} \\ &= \lim_{x\to 2} \frac{2-x}{(x-2)(\sqrt{4-x}+\sqrt{2})} \\ &= \lim_{x\to 2} \frac{-1}{\sqrt{4-x}+\sqrt{2}} \\ &= \frac{-1}{\sqrt{2}+\sqrt{2}} = \frac{-1}{2\sqrt{2}} \text{ or } \frac{-\sqrt{2}}{4}\end{aligned}$$

Again, notation is very important. Essentially, the limit is the operation you are performing. Until you have actually evaluated the expression by plugging in the *a*, you must still write the notation $\lim_{x\to a}$. Just like you would not drop a square root from an equation until you actually do the "square rooting", you don't get rid of your limit notation until you've actually taken your limit. You will get marked wrong in this class, on the AP test, and/or in college if you don't do this.

Fundamentally, a limit is just telling us where a *y*-value **should be** for a particular function. It does not necessarily tell us that the *y*-value does or does not exist, it just tells us where it is supposed to be on a given curve.

Ex 6 For the function illustrated below, find each of the following limits:



a) $\lim\limits_{x\to -2} f(x)$ b) $\lim\limits_{x\to 1} f(x)$ c) $\lim\limits_{x\to 3} f(x)$

Note that the *y*-values are on the function $f(x)$, and it doesn't matter that we don't know what $f(x)$ is as an equation – it is the graph.

a) $\lim\limits_{x\to -2} f(x) = -1$ When $x \to -2$, the *y*-value approaches –1. It doesn't matter that there is a hole there.

b) $\lim\limits_{x\to 1} f(x) = -2$ When $x \to 1$, the *y*-value approaches –2. There is an actual point on the curve there, but that doesn't matter.

c) $\lim\limits_{x\to 3} f(x) = 0$ Even though there is an actual *y*-value (*y* = 1 when *x* = 3), the curve heads to the hole (at *y* = 0)

One of the more powerful tools in Calculus for dealing with Indeterminate Forms and limits is called L'Hôpital's Rule.

L'Hôpital's Rule

$$\text{If } \frac{f(a)}{g(a)} = \frac{0}{0} \text{ or } \frac{\infty}{\infty}, \text{ then } \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

This rule allows us to evaluate limits of functions that do not factor, such as those that involve transcendental functions.

It is very important to note that this is **<u>not</u>** the Quotient Rule. Since the original problem is a limit and not a derivative, we are not using the quotient rule. L'Hôpital's Rule allows us a powerful tool to find the limits of quotients.

We are using derivatives in the process of L'Hôpital's Rule, but we are not taking the derivative of a quotient, so we don't use the Quotient Rule!

Note also that some books spell it "L'Hospital's Rule" – both are acceptable spellings of the French name.

Ex 5 (again) Find $\lim_{x\to 2}\frac{\sqrt{4-x}-\sqrt{2}}{x-2}$ using L'Hôpital's Rule.

Since, at $x = 0$, $\lim_{x\to 2}\frac{\sqrt{4-x}-\sqrt{2}}{x-2} = \frac{\sqrt{4-2}-\sqrt{2}}{2-2} = \frac{0}{0}$, the condition for L'Hôpital's Rule is satisfied.

$$\lim_{x\to 2}\frac{\sqrt{4-x}-\sqrt{2}}{x-2} \overset{\text{L'H}}{=} \lim_{x\to 2}\frac{D_x\left(\sqrt{4-x}-\sqrt{2}\right)}{D_x(x-2)}$$
$$= \lim_{x\to 2}\frac{\frac{1}{\sqrt{4-x}}(-1)}{1}$$
$$= \frac{-1}{2\sqrt{2}}$$

Recall, to do this algebraically, we would have to rationalize the numerator by multiplying the numerator and denominator by the conjugate – a rather tedious algebraic process.

Obviously, this is a much faster process than the algebraic one.

Usually, when we apply L'Hôpital's Rule, we write a "L'H" over the equal sign to indicate what we are doing.

Ex 7 Demonstrate that $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$.

Last year, we used this limit to prove the derivative of sin x, but we never proved that the limit actually equals 1. This limit is relatively difficult to evaluate without L'Hôpital's Rule (and requires a complicated theorem called the Squeeze Theorem), but it is very easy with L'Hôpital's Rule.

At x = 0, $\frac{\sin x}{x}=\frac{0}{0}$, L'Hôpital's Rule applies.

$$\lim_{x\to 0}\frac{\sin\ x}{x}\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{\cos\ x}{1}=1$$

Obviously, this is not a proof, because of the circular reasoning. We used a derivative whose proof involved the limit that we were evaluating. But we are just interested in the process of applying L'Hôpital's Rule here.

Ex 8 Evaluate $\lim\limits_{x\to 0}(1+2x)^{\csc x}$.

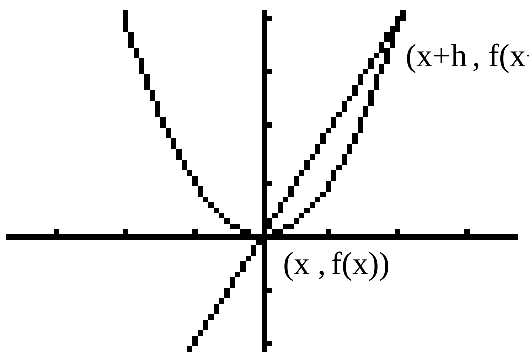
This limit yields a different indeterminate form from before, namely, 1^{∞}. Because the variable is in the base and the exponent, we might apply the logarithm rules before applying L'Hôpital's Rule.

$$\begin{aligned}
y&=\lim_{x\to 0}(1+2x)^{\csc x}\\
\ln y&=\lim_{x\to 0}\csc x\ \ln(1+2x)\\
&=\lim_{x\to 0}\frac{\ln(1+2x)}{\sin x}\\
&\overset{\text{L'H}}{=}\lim_{x\to 0}\frac{\left(\frac{2}{1+2x}\right)}{\cos x}\\
\ln y&=2\\
y&=e^2
\end{aligned}$$

Therefore, $\lim\limits_{x\to 0}(1+2x)^{\csc x}=e^2$

The Limit Definitions of a Derivative

As we recall from last year, the derivative was initially created as a function that would yield the slope of the tangent line. The slope formula for any line though two points is $m = \frac{y_2 - y_1}{x_2 - x_1}$. When considering a tangent line, though, that equation has too many variables. We can simplify this some by realizing $y = f(x)$. If we consider h to represent the horizontal distance between the points and realize that $y = f(x)$, then the two points that form the secant line would be $(x, f(x))$ and $(x+h, f(x+h))$.



Then the slope formula becomes $m = \frac{f(x+h) - f(x)}{x+h-x}$ or $m = \frac{f(x+h) - f(x)}{h}$.
When h = 0, the points would merge and we would have the tangent line. h = 0 gives $m = \frac{0}{0}$, therefore, we can use the $\lim\limits_{h \to 0}$ and $m_{\text{tangent line}} = \lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ will represent the slope of the tangent line.

$$\boxed{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ is the derivative.}}$$

Since we already know the derivative rules from the last chapter, we will not be using this very often. What we might see instead is the Numerical Derivative, which yields the slope of the tangent line at a specific point.

There are two versions of the Numerical Derivative formula:

The Numerical Derivative

$$f\ '(a) = \lim_{x\to a}\frac{f(x)\ -\ f(a)}{x\ -\ a}\ \text{ or }\ f\ '(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Often, there are questions on the AP test that look like limit questions, but which are really questions about recognition of these formulas.

You might notice that the second one is simply the same as the general form for the limit definition of the derivative where $x = a$, but the first one looks a little different.

The first Numerical Derivative is just the slope formula through the points $(a, f(a))$ and $(x, f(x))$.

Ex 9   Evaluate $\lim\limits_{h\to 0}\dfrac{(2+h)^3-(2)^3}{h}$.

We could FOIL the numerator out (using Pascal's Triangle for ease) and solve this limit algebraically:

$$\begin{aligned}\lim_{h\to 0}\frac{(2+h)^3-(2)^3}{h} &= \lim_{h\to 0}\frac{8+12h+3h^2+h^3-8}{h}\\ &= \lim_{h\to 0}\frac{12h+3h^2+h^3}{h}\\ &= \lim_{h\to 0}\left(12+2h+h^2\right)\\ &= 12\end{aligned}$$

Or we could apply L'Hôpital's Rule:

$$\begin{aligned}\lim_{h\to 0}\frac{(2+h)^3-(2)^3}{h} &\overset{\text{L'H}}{=} \lim_{h\to 0}\frac{3(2+h)^2}{1}\\ &= 3(2)^2\\ &= 12\end{aligned}$$

But the quickest way is to recognize that $\lim\limits_{h\to 0}\dfrac{(2+h)^3-(2)^3}{h}$ is really $f\,'(2)$, where $f(x)=x^3$. Of course, $f\,'(x)=3x^2$ and $f\,'(2)=3(2)^2=12$.

Ex 10 Evaluate $\lim\limits_{h\to 0}\dfrac{\ln(e+h)-1}{h}$

$$\begin{aligned}
&\lim_{h\to 0}\frac{\ln(e+h)-1}{h}\\
&=\left.\frac{d}{dx}\right|_{x=e}\left(\ln\ x\right)\\
&=\left.\frac{1}{x}\right|_{x=e}\\
&=\frac{1}{e}
\end{aligned}$$

Ex 11 Evaluate $\lim\limits_{x\to 2\pi}\dfrac{\cos x-1}{x-2\pi}$

$$\begin{aligned}
\lim_{x\to 2\pi}\frac{\cos x-1}{x-2\pi}&=\left.\frac{d}{dx}\right|_{x=2\pi}\left(\cos x\right)\\
&=-\sin\left(2\pi\right)\\
&=0
\end{aligned}$$

We have to recognize that this is in the form of

$$f\ '\left(a\right)=\lim_{x\to a}\frac{f(x)\ -\ f(a)}{x\ -\ a},$$

where $a=2\pi$

$$f\left(x\right)=\cos x$$

$$f\left(a\right)=\cos 2\pi=1$$

Again, if we failed to recognize the definition of the derivative, we could always use L'Hôpital's Rule if this was a multiple choice question on an AP test – this is completely inappropriate from a mathematician's point of view, but it will get you the right answer in a multiple choice test where they cannot see your work.

5.1 Homework Set A

Evaluate the following Limits.

1. $\lim\limits_{x\to -3} \dfrac{x^2+7x+12}{x^2-9}$

2. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

3. $\lim\limits_{x\to 0} \dfrac{x^2}{\tan x}$

4. $\lim\limits_{x\to 2} \dfrac{4x-12}{x^2-3x-10}$

5. $\lim\limits_{x\to 0} x(\cot\ x)$

6. $\lim\limits_{x\to 1} \dfrac{x^2-x}{e^x-1}$

7. $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^2+3x-4}$

8. $\lim\limits_{x\to 4} \dfrac{\ln\left(\frac{x}{4}\right)}{\sqrt{x}-2}$

9. $\lim\limits_{x\to 1} \dfrac{\ln x}{\tan x}$

10. $\lim\limits_{x\to -3} \dfrac{x^2+4x-1}{x^3+3x}$

11. $\lim\limits_{x\to \sqrt{2}} \dfrac{x^4+4x^2-12}{x^3+x^2-2x-2}$

12. $\lim\limits_{x\to 9} \dfrac{3-\sqrt{x}}{x^2-81}$

13. $\lim\limits_{x\to \pi} \dfrac{\cos^2\ x-1}{2\cos^2\ x-5\cos\ x-7}$

14. $\lim\limits_{x\to 1} \dfrac{x^2-1}{\ln x}$

15. $\lim\limits_{x\to -1} \dfrac{1-x^2}{e^{x+1}-1}$

16. $\lim\limits_{x\to\pi} \csc x\left(1+\sec x\right)$

17. $\lim\limits_{x\to -1} \dfrac{x+1}{e^{x+1}}$

18. $\lim\limits_{x\to 5} \dfrac{1-\sqrt{x-5}}{x-6}$

19. $\lim\limits_{x\to 0} \left(1+\dfrac{x}{2}\right)^{\cot x}$

20. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1-\sin^2 x}{3\sin^2 x-\sin x-2}$

21. $\lim\limits_{x\to 5} \csc\left(x-5\right)\ln\left(x-4\right)$

22. $\lim\limits_{x\to\pi} \dfrac{1+\cos x}{e^x-1}$

23. $\operatorname*{Lim}\limits_{h\to 0} \dfrac{\left(3+h\right)^3-27}{h}$

24. $\operatorname*{Lim}\limits_{x\to\pi} \dfrac{\sin x}{x-\pi}$

25. $\operatorname*{Lim}\limits_{h\to 0} \dfrac{e^{3+h}-e^3}{h}$

26. $\operatorname*{Lim}\limits_{x\to\frac{\pi}{2}} \dfrac{1+\sin x}{x+\frac{\pi}{2}}$

27. $\operatorname*{Lim}\limits_{h\to 0} \dfrac{\cos\left(\pi+h\right)+1}{h}$

28. $\operatorname*{Lim}\limits_{x\to 2} \dfrac{\ln x-\ln 2}{x-2}$

29. $\operatorname*{Lim}\limits_{x\to 4} \dfrac{\sin x-\sin 4}{x-3}$

30. $\operatorname*{Lim}\limits_{x\to a} \dfrac{x^3-a^3}{x-a}$

31. $\lim\limits_{x \to a} \dfrac{x^3 - a^3}{x^2 - a^2}$

32. $\lim\limits_{h \to 0} \dfrac{2(3+h)^4 - 162}{h}$

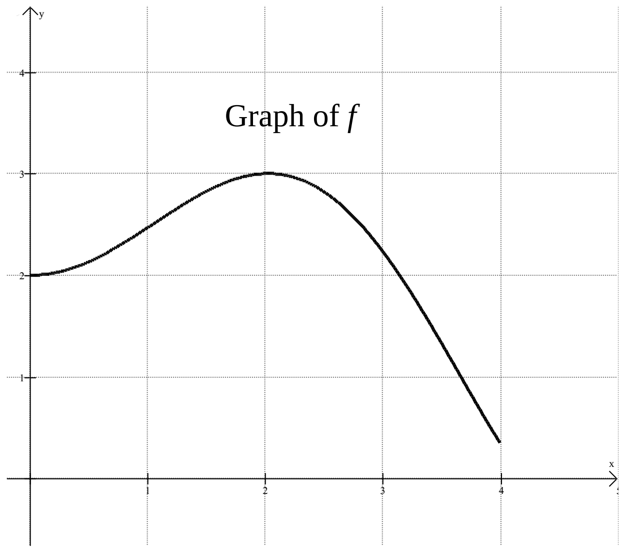
Solve the following multiple choice problems

33. Given the graph of $f(x)$ below, tell which of the following are **TRUE**.

I. $\lim\limits_{x \to 2} \dfrac{f(x) - 3}{x - 2}$ does not exist.

II. $f(2) = 3$

III. $\lim\limits_{x \to 2} f(x) = 3$



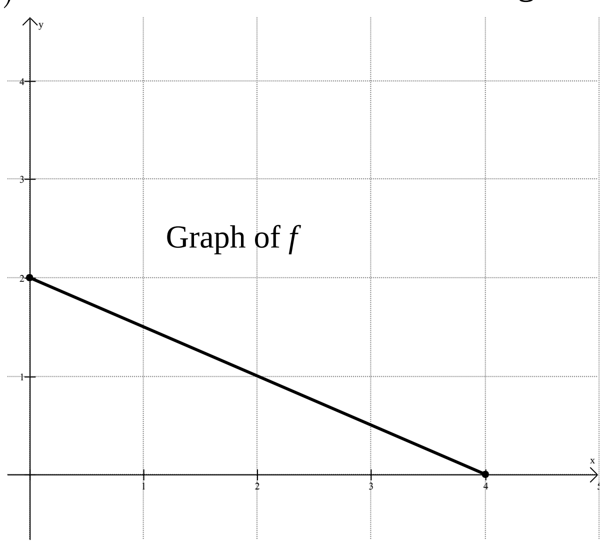
A. I only B. I and II C. II and III D. I, II, and III E. III only

34. Given the graph of $f(x)$ below, tell which of the following are **TRUE**.

I. $\lim\limits_{h \to 0} \dfrac{f(2+h) - 1}{h} = -\dfrac{1}{2}$

II. $f(2) = 1$

III. $\lim\limits_{x \to 2} f(x) = 1$



A. I only B. I and II C. II and III D. I, II, and III E. III only

35. If $a \neq 0$, then $\lim_{x \to a} \frac{x^4 - a^4}{x - a}$ is

A. a B. $3a^2$ C. $4a^3$ D. 0 E. nonexistent

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36. If the average rate of change of a function f over the interval from $x = 2$ to $x = 2 + h$ is given by $4e^h - 4\sin h$, then $f'(2) =$

A. 0 B. 1 C. 2 D. 3 E. 4

5.1 Homework Set B

1. $\lim_{t\to 2}\frac{\ln(t-1)}{\tan(t-1)}$

2. $\lim_{x\to e}\frac{(\ln x^2)-2}{x-e}$

3. $\lim_{x\to \pi}\frac{\tan x}{1+\cos x}$

4. $\lim_{x\to 0}\frac{\ln(1-15x)}{\tan x}$

5. $\lim_{x\to 0}\frac{x^2-9}{x^2+16}$

6. $\lim_{x\to 0}\frac{x^{12}-5x^6+6}{e^x}$

7. $\lim_{x\to 2}\frac{\sin x-\sin 2}{x-2}$

8. $\lim_{x\to 3}\frac{x^2-9}{x+4}$

9. $\lim_{x\to 0}\frac{e^x-\cos x}{\tan x}$

10. $\lim_{x\to e}\frac{1-\ln x}{x-e}$

11. $\lim_{t\to 3}\frac{t^3-9t}{t^2-5t+6}$

12. $\lim_{x\to \pi}\frac{\tan^2 x+1}{\cos x}$

13. $\lim_{x\to 4}\frac{\sqrt{x+5}-3}{x-4}$

14. $\lim_{x\to e}(2+\ln x)$

15. $\lim_{t\to 3}\dfrac{e^{t-3}-1}{3-t}$

22. $\lim_{t\to 2}\dfrac{\tan^{-1}(t-1)}{\ln\left(\dfrac{2e}{t}\right)}$

16. $\lim_{y\to \pi}\dfrac{y+\pi\cos y}{\sin y}$

23. $\lim_{x\to e}\dfrac{\left(\ln x^4\right)-4}{x-e}$

17. $\lim_{x\to e}\dfrac{1-\ln x}{\sin\left(\dfrac{\pi x}{e}\right)}$

24. $\lim_{x\to 2}\dfrac{\sin x-\sin 2}{x-2}$

18. $\lim_{x\to 9}\dfrac{x^3-9x^2-5x+45}{x^2-11x+18}$

25. $\lim_{x\to \pi}\dfrac{\tan x}{x-\pi}$

19. $\lim_{x\to \frac{\pi}{2}}\dfrac{\cos x}{x-\dfrac{\pi}{2}\sin x}$

26. $\lim_{h\to 0}\dfrac{e^{5+h}-e^5}{h}$

20. $\lim_{x\to 4}\sqrt{\dfrac{x^2-16}{x+4}}$

27. $\lim_{x\to \sqrt{e}}\dfrac{\left(\ln x^2\right)-1}{x^2-e}$

21. $\lim_{x\to 0}\csc x\ln(1+2x)$

28. $\lim_{h \to 0} \frac{2(x+h)^2 - 7(x+h) + 6 - (2x^2 - 7x + 6)}{h}$

29. $\lim_{h \to 0} \frac{(2+h)^5 - 32}{h}$

30. $\lim_{x \to e} \frac{(\ln x) - 1}{x - e}$

31. $\lim_{h \to 0} \frac{(4+h)^3 - 64}{h}$

Answers 5.1 Homework Set A

1. $\lim\limits_{x\to -3} \dfrac{x^2+7x+12}{x^2-9} = -\dfrac{1}{6}$

2. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2} = \dfrac{1}{2}$

3. $\lim\limits_{x\to 0} \dfrac{x^2}{\tan x} = 0$

4. $\lim\limits_{x\to 2} \dfrac{4x-12}{x^2-3x-10} = \dfrac{1}{3}$

5. $\lim\limits_{x\to 0} x(\cot x) = 1$

6. $\lim\limits_{x\to 1} \dfrac{x^2-x}{e^x-1} = 0$

7. $\lim\limits_{x\to 1} \dfrac{x^3-1}{x^2+3x-4} = \dfrac{3}{5}$

8. $\lim\limits_{x\to 4} \dfrac{\ln\left(\dfrac{x}{4}\right)}{\sqrt{x}-2} = 1$

9. $\lim\limits_{x\to 1} \dfrac{\ln x}{\tan x} = 0$

10. $\lim\limits_{x\to -3} \dfrac{x^2+4x-1}{x^3+3x} = \dfrac{2}{9}$

11. $\lim\limits_{x\to\sqrt{2}} \dfrac{x^4+4x^2-12}{x^3+x^2-2x-2} = \dfrac{8\sqrt{2}}{3+\sqrt{2}}$

12. $\lim\limits_{x\to 9} \dfrac{3-\sqrt{x}}{x^2-81} = -\dfrac{1}{108}$

13. $\lim\limits_{x\to\pi} \dfrac{\cos^2 x-1}{2\cos^2 x-5\cos x-7} = \dfrac{2}{9}$

14. $\lim\limits_{x\to 1} \dfrac{x^2-1}{\ln x} = 2$

15. $\lim\limits_{x\to -1} \dfrac{1-x^2}{e^{x+1}-1} = 2$

16. $\lim\limits_{x\to\pi} \csc x(1+\sec x) = 0$

17. $\lim\limits_{x\to -1} \dfrac{x+1}{e^{x+1}} = 1$

18. $\lim\limits_{x\to 5} \dfrac{1-\sqrt{x-5}}{x-6} = -1$

19. $\lim\limits_{x\to 0} \left(1+\dfrac{x}{2}\right)^{\cot x} = e$

20. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1-\sin^2 x}{3\sin^2 x-\sin x-2} = 2$

21. $\lim\limits_{x\to 5} \csc(x-5)\ln(x-4) = 1$

22. $\lim\limits_{x\to\pi} \dfrac{1+\cos x}{e^x-1} = 0$

23. $\mathrm{Lim}_{h\to 0} \dfrac{(3+h)^3-27}{h} = 27$

24. $\mathrm{Lim}_{x\to\pi} \dfrac{\sin x}{x-\pi} = -1$

25. $\mathrm{Lim}_{h\to 0} \dfrac{e^{3+h}-e^3}{h} = e^3$

26. $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{1+\sin x}{x+\dfrac{\pi}{2}} = \dfrac{2}{\pi}$

27. $\lim\limits_{h\to 0} \dfrac{\cos(\pi+h)+1}{h} = 0$

28. $\lim\limits_{x\to 2} \dfrac{\ln x-\ln 2}{x-2} = \dfrac{1}{2}$

29. $\lim\limits_{x\to 4} \dfrac{\sin x-\sin 4}{x-3} = \cos 4$

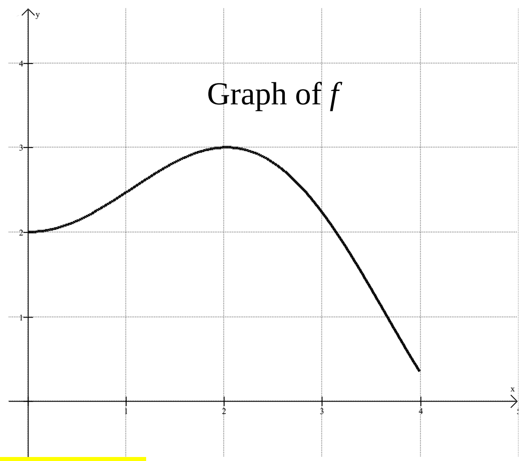
30. $\lim\limits_{x\to a} \dfrac{x^3-a^3}{x-a} = 3a^2$

31. $\lim\limits_{x\to a} \dfrac{x^3-a^3}{x^2-a^2} = \dfrac{3a}{2}$

32. $\lim\limits_{h\to 0} \dfrac{2(3+h)^4-162}{h} = 21$

33. Given the graph of $f(x)$ below, tell which of the following are **TRUE**.

I. $\lim\limits_{x\to 2} \dfrac{f(x)-3}{x-2}$ does not exist.



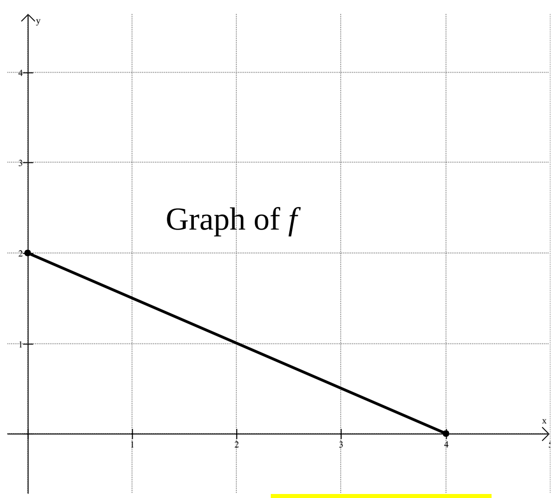
II. $f(2)=3$

III. $\lim\limits_{x\to 2} f(x)=3$

A. I only B. I and II C. II and III D. I, II, and III E. III only

34. Given the graph of $f(x)$ below, tell which of the following are **TRUE**.

I. $\lim\limits_{h\to 0} \dfrac{f(2+h)-1}{h} = -\dfrac{1}{2}$



II. $f(2)=1$

III. $\lim\limits_{x\to 2} f(x)=1$

A. I only B. I and II C. II and III D. I, II, and III E. III only

35. If $a \neq 0$, then $\lim_{x \to a} \frac{x^4 - a^4}{x - a}$ is

A. a B. $3a^2$ C. $4a^3$ D. 0 E. nonexistent

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36. If the average rate of change of a function f over the interval from $x = 2$ to $x = 2 + h$ is given by $4e^h - 4\sin h$, then $f'(2) =$

A. 0 B. 1 C. 2 D. 3 E. 4

5.1 Homework Set B

1. $\lim_{t \to 2} \frac{\ln(t-1)}{\tan(t-1)} = 0$

2. $\lim_{x \to e} \frac{(\ln x^2) - 2}{x - e} = \frac{2}{e}$

3. $\lim_{x \to \pi} \frac{\tan x}{1 + \cos x} =$ D.N.E.

4. $\lim_{x \to 0} \frac{\ln(1 - 15x)}{\tan x} = -15$

5. $\lim_{x \to 0} \frac{x^2 - 9}{x^2 + 16} = -\frac{9}{16}$

6. $\lim_{x \to 0} \frac{x^{12} - 5x^6 + 6}{e^x} = 6$

7. $\lim_{x \to 2} \frac{\sin x - \sin 2}{x - 2} = \cos 2$

8. $\lim_{x \to 3} \frac{x^2 - 9}{x + 4} = 0$

9. $\lim_{x \to 0} \frac{e^x - \cos x}{\tan x} = 1$

10. $\lim_{x \to e} \frac{1 - \ln x}{x - e} = -\frac{1}{e}$

11. $\lim_{t \to 3} \frac{t^3 - 9t}{t^2 - 5t + 6} = 18$

12. $\lim_{x \to \pi} \frac{\tan^2 x + 1}{\cos x} = -1$

13. $\lim_{x \to 4} \frac{\sqrt{x+5} - 3}{x - 4} = \frac{1}{6}$

14. $\lim_{x \to e} (2 + \ln x) = 3$

15. $\lim_{t\to 3}\frac{e^{t-3}-1}{3-t}$ $= -3$

22. $\lim_{t\to 2}\frac{\tan^{-1}(t-1)}{\ln\left(\frac{2e}{t}\right)}$ $= \frac{\pi}{4}$

16. $\lim_{y\to \pi}\frac{y+\pi\cos y}{\sin y}$ $= -1$

23. $\lim_{x\to e}\frac{(\ln x^4)-4}{x-e}$ $= \frac{4}{e}$

17. $\lim_{x\to e}\frac{1-\ln x}{\sin\left(\frac{\pi x}{e}\right)}$ $= \pi$

24. $\lim_{x\to 2}\frac{\sin x-\sin 2}{x-2}$ $= \cos 2$

18. $\lim_{x\to 9}\frac{x^3-9x^2-5x+45}{x^2-11x+18}$ $= \frac{76}{7}$

25. $\lim_{x\to \pi}\frac{\tan x}{x-\pi}$ $= 1$

19. $\lim_{x\to \frac{\pi}{2}}\frac{\cos x}{x-\frac{\pi}{2}\sin x}$ $= -1$

26. $\lim_{h\to 0}\frac{e^{5+h}-e^5}{h}$ $= e^5$

20. $\lim_{x\to 4}\sqrt{\frac{x^2-16}{x+4}}$ $= 0$

27. $\lim_{x\to \sqrt{e}}\frac{(\ln x^2)-1}{x^2-e}$ $= \frac{1}{e}$

21. $\lim_{x\to 0}\csc x\ln(1+2x)$ $= 2$

28. $\lim_{h\to 0}\frac{2(x+h)^2-7(x+h)+6-(2x^2-7x+6)}{h}$ $= 4x^2 - 7$

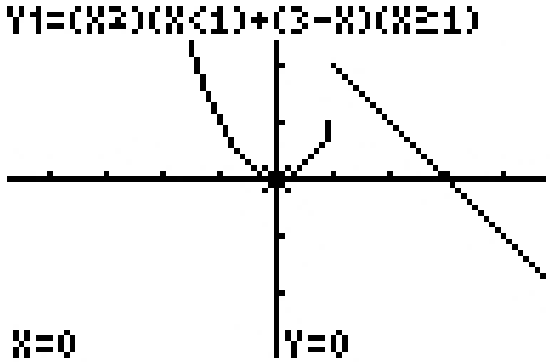
29. $\lim_{h\to 0}\frac{(2+h)^5-32}{h}$ $=80$

30. $\lim_{x\to e}\frac{(\ln x)-1}{x-e}$ $= \frac{1}{e}$

31. $\lim_{h\to 0}\frac{(4+h)^3-64}{h}$ $= 48$

5.2: Continuity, One-sided Limits, and Infinite Limits

In 5.1, we said that $\lim_{x \to a} f(x)$ is basically what the *y*-value should be when $x = a$, even if *a* is not in the domain. For all of the families of functions that we studied last year, this was enough. But what if that *y*-value might be two different numbers? In a graph of a piece-wise defined function like this,



It is not clear whether the *y*-value for $x = 1$ is 1 or 2. As we can see from the table of values, *x*-values less than 1 have *y*-values that approach 1 while *x*-values greater than 1 have *y*-values that approach 2:

X	Y1	
.85	.7225	
.9	.81	
.95	.9025	
1	2	
1.05	1.95	
1.1	1.9	
1.15	1.85	

X=1

Basically, we get a different *y*-value if we approach $x = 1$ from the left or the right.

The algebraic ways to describe theses differences are one-sided limits. The symbols we use are:

$$\lim_{x \to a^-} f(x) \quad \text{or} \quad \lim_{x \to a^+} f(x)$$

You read $\lim\limits_{x \to a^-} f(x)$ as, "the limit as x approaches a from the left."

You read $\lim\limits_{x \to a^+} f(x)$ as, "the limit as x approaches a from the right."

In this example, $\lim\limits_{x \to 1^-} f(x) = 1$ and $\lim\limits_{x \to 1^+} f(x) = 2$. The $\lim\limits_{x \to 1} f(x)$ does not exist, because the one-sided limits do not equal each other.

By "the $\lim\limits_{x \to 1} f(x)$ does not exist," we mean there is not <u>one</u> REAL number that the limit equals.

OBJECTIVE
Evaluate one-sided limits graphically, numerically, and algebraically. Evaluate two-sided limits in terms of one-sided limits. Prove continuity or discontinuity of a given function. Interpret Vertical Asymptotes in terms of one-sided limits.

Ex 1 Does $\lim\limits_{x \to -1} f(x)$ exist for $f(x) = \begin{cases} x^2 + 1, & \text{if } x \geq -1 \\ 3 - x, & \text{if } x < -1 \end{cases}$

For $\lim\limits_{x \to -1} f(x)$ to exist, $\lim\limits_{x \to -1^-} f(x)$ must equal $\lim\limits_{x \to -1^+} f(x)$. The domain states that any number less than $x = -1$ goes into $3 - x$. Therefore,

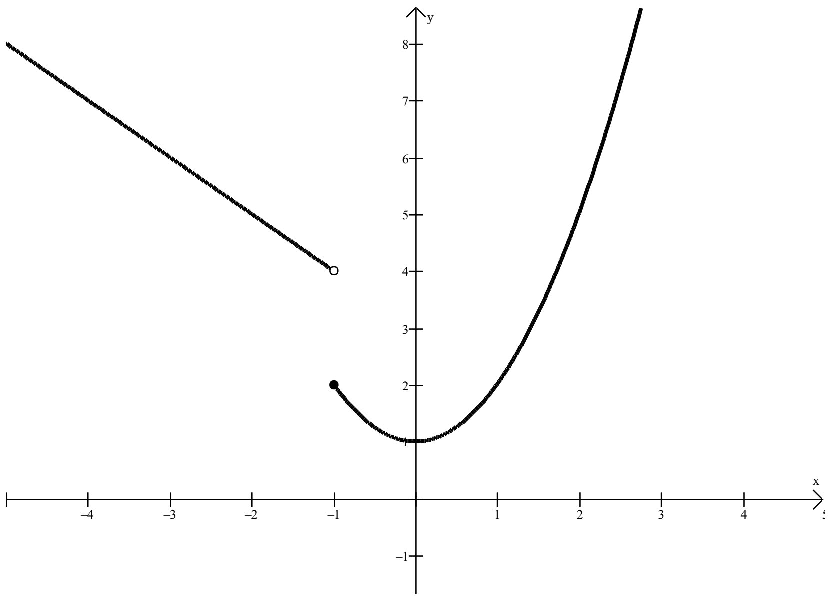
$$\begin{aligned} \lim_{x \to -1^-} f(x) &= \lim_{x \to -1^-} (3 - x) \\ &= 3 - (-1) \\ &= 4 \end{aligned}$$

Similarly, numbers greater than $x = -1$ go into $x^2 + 1$, and

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} (x^2 + 1) \\ &= (-1)^2 + 1 \\ &= 2 \end{aligned}$$

You can see that the two one-sided limits are not equal. Therefore, $\lim\limits_{x \to -1} f(x)$ does not exist.

We see in the graph that the two parts do not come together:

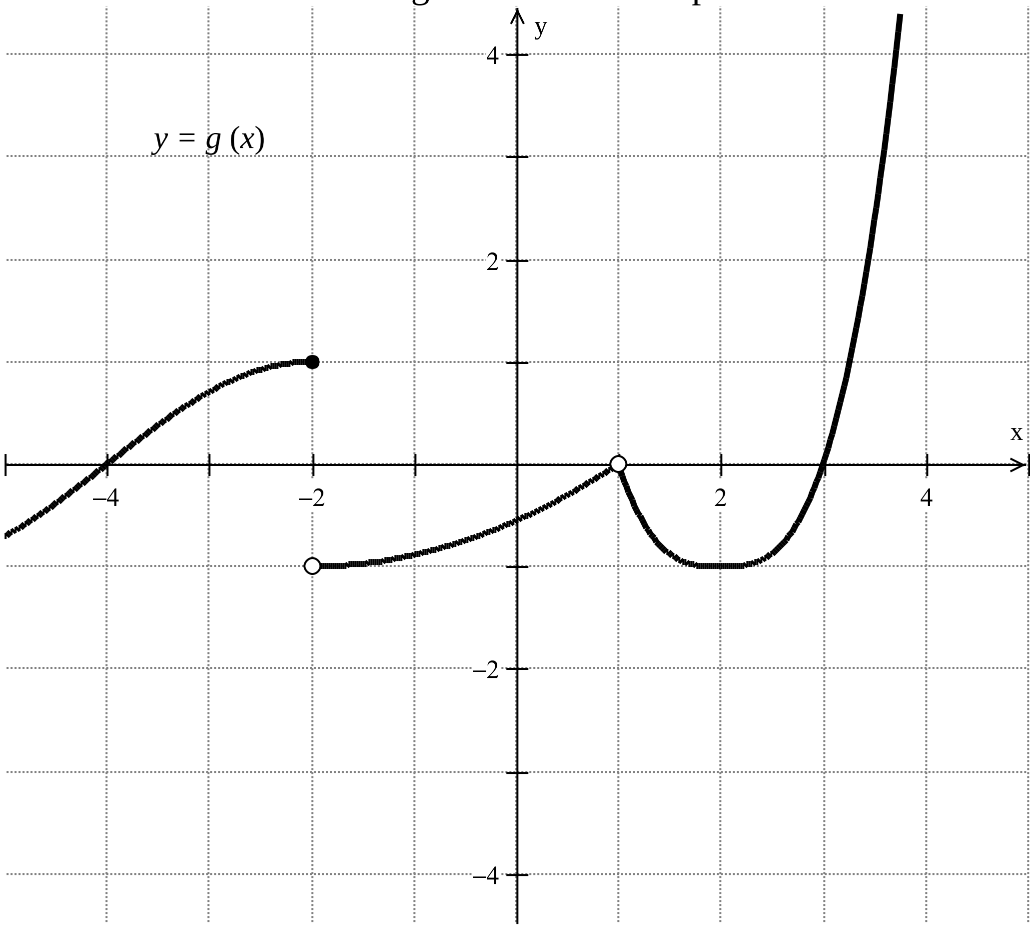


There are two main topics where one-sided limits most often come into play: Continuity and Infinite Limits.

A limit for a function only exists if the left and right limits are equal. Since this was not much of a concern for the functions in Precalculus, we pretty much ignored the fact, but in Calculus (both here and in college) teachers love to deal with this fact.

$$\boxed{\lim_{x\to a} f(x) \text{ exists if and only if } \lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)}$$

Ex 2   Evaluate each of the limits given the function pictured below.



a) $\lim\limits_{x\to -2^-} g(x)$	e) $\lim\limits_{x\to 1^-} g(x)$	i) $\lim\limits_{x\to 2^-} g(x)$
b) $\lim\limits_{x\to -2^+} g(x)$	f) $\lim\limits_{x\to 1^+} g(x)$	j) $\lim\limits_{x\to 2^+} g(x)$
c) $\lim\limits_{x\to -2} g(x)$	g) $\lim\limits_{x\to 1} g(x)$	k) $\lim\limits_{x\to 2} g(x)$
d) $g(-2)$	h) $g(1)$	l) $g(2)$

a) $\lim_{x\to -2^-} g(x) = 1$

b) $\lim_{x\to -2^+} g(x) = -1$

c) $\lim_{x\to -2} g(x) = \text{D.N.E.}$

d) $g(-2) = 1$

The left and right limits are different because the curve approaches different values from each side, therefore the $\lim_{x\to -2} g(x)$ does not exist. The actual value of the function ($g(-2)$) is 1 because that is where the function actually has a value.

e) $\lim_{x\to 1^-} g(x) = 0$

f) $\lim_{x\to 1^+} g(x) = 0$

g) $\lim_{x\to 1} g(x) = 0$

h) $g(1) = \text{D.N.E.}$

The left and right limits are the same because the curve approaches the same value from each side, therefore the $\lim_{x\to 1} g(x)$ is 0 (the y-value the curve approaches). The actual value of the function ($g(1)$) does not exist because that is where the function has no value (there is a hole in the graph).

i) $\lim_{x\to 2^-} g(x) = -1$

j) $\lim_{x\to 2^+} g(x) = -1$

k) $\lim_{x\to 2} g(x) = -1$

l) $g(2) = -1$

The left and right limits are the same because the curve approaches the same value from each side, therefore the $\lim_{x\to 2} g(x)$ is –1. The actual value of the function ($g(2)$) is –1 because that is where the function actually has a value.

CONTINUITY

One of the main topics early in Calculus is CONTINUITY. It really is a simple concept, which, like the Limit, is made complicated by its mathematical definition. Let us take a look at the formal definition of continuous:

Continuous**--**Defn: "A function $f(x)$ is continuous at $x = a$ if and only if:

i. $f(a)$ exists*,

ii. $\lim_{x\to a} f(x)$ exists*,

and iii. $\lim_{x\to a} f(x) = f(a)$."

**By "exists," we mean that it equals a real number.*

i) "$f(a)$ exists" means a must be in the domain.
ii) "$\lim_{x\to a} f(x)$ exists" means $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$.
iii) "$\lim_{x\to a} f(x) = f(a)$" should be self explanatory.

NB. All the families of functions which were explored in PreCalculus are continuous in their domain.

Ex 3 $g(x) = \begin{cases} x^2 - 2x + 1, & \text{if } x > -1 \\ 2, & \text{if } x = -1 \\ 3 - x, & \text{if } x < -1 \end{cases}$ Is $g(x)$ continuous at $x = -1$?

To answer this question, we must check each part of the definition.

i) Does $g(-1)$ exist? Yes, the middle line says that –1 is in the domain and it tells us that $y = 2$ if $x = -1$.

ii) Does the $\lim_{x\to -1} g(x)$ exist? We need to check the two one-sided limits.

$$\left.\begin{array}{l} \lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} 3 - x = 4 \\ \lim_{x\to -1^+} g(x) = \lim_{x\to -1^+} (x-1)^2 = 4 \end{array}\right\} \therefore \lim_{x\to -1} g(x) = 4$$

Since both one-sided limits equal 4, then $\lim_{x\to -1} g(x) = 4$

iii) Does $\lim_{x\to -1} g(x) = g(-1)$? No. $\lim_{x\to -1} g(x) = 4$, while $g(-1) = 2$

So $g(x)$ is not continuous at $x = -1$ because the limit does not equal the function.

Ex 4 $F(x)=\begin{cases} x^2-5, & \text{if } x>0 \\ x+2, & \text{if } x<0 \end{cases}$ Is $F(x)$ continuous at x = 0? Why not?

i) Does $F(0)$ exist? No, we cannot plug x = 0 into the function.

$F(x)$ is not continuous at x = 0 because 0 is not in the domain. Notice neither inequality includes an equal sign.

Ex 5 If $G(x)=\begin{cases} x+2, & \text{if } x>-1 \\ 3, & \text{if } x=-1 \\ x^2-5, & \text{if } x<-1 \end{cases}$, is $G(x)$ continuous at x = –1? Why not?

i) Does G(1) exist? Yes, the second line says that x = –1 is in the domain (and it tells us that y = 3, if x = 1).

ii) Does the $\lim_{x\to -1} G(x)$ exist? No. The two-sided limit only exists if the two one-sided limits are equal. But,

$$\left.\begin{aligned} \lim_{x\to -1^-} G(x) &= \lim_{x\to -1^-} x^2-5=-4 \\ \lim_{x\to -1^+} G(x) &= \lim_{x\to -1^+} x+2=1 \end{aligned}\right\} \therefore \lim_{x\to -1} g(x)=\text{D.N.E.}$$

The two one-sided limits are not equal. Therefore,

$G(x)$ is not continuous at x = 1, because $\lim_{x\to 1} G(x)$ Does Not Exist.

INFINITE LIMITS

The other situation where one-sided limits come into play is at vertical asymptotes. Here, the *y*-value goes to infinity (or negative infinity), which is why these limits are also called "Limits to Infinity".

Vocabulary

Infinite Limit— Defn: a limit where *y* approaches infinity

Ex 6 Evaluate a) $\lim\limits_{x\to 2^-} \dfrac{2}{x-2}$ and b) $\lim\limits_{x\to 2^+} \dfrac{1-x}{x-2}$

In both these cases, we are considering a vertical asymptote. The limits, being *y* values, would be either positive or negative infinity, depending on if the curve went up or down on that side of the asymptote. We can look at the limit algebraically (vs. graphically) though:

a) $\lim\limits_{x\to 2^-} \dfrac{2}{x-2} = \dfrac{2}{0^-} = -\infty$

Note that the 0^- is not "from the left" because the 2 is from the left, but rather that $x-2$ is negative for any *x* values less than 2.

b) $\lim\limits_{x\to 2^+} \dfrac{1-x}{x-2} = \dfrac{-1}{0^+} = -\infty$

Note in this case that the numerator's sign affects the outcome.

Notice that just because you approached from the left or right does not necessarily mean you approach negative or positive infinity – you determine this from the signs in the expression.

Also, we never really write anything like $\dfrac{2}{0^-}$. Mathematically speaking, that is grammatically bad. That describes how we think about the problem. We really just write the solution to the problem, and either write the $\dfrac{2}{0^-}$ as scratch-work off to the side or just do it in our heads.

Note: It is debatable whether the Infinite Limits exist or not. It depends on whether "exist" is defined as equal to a real number or not. Some books would

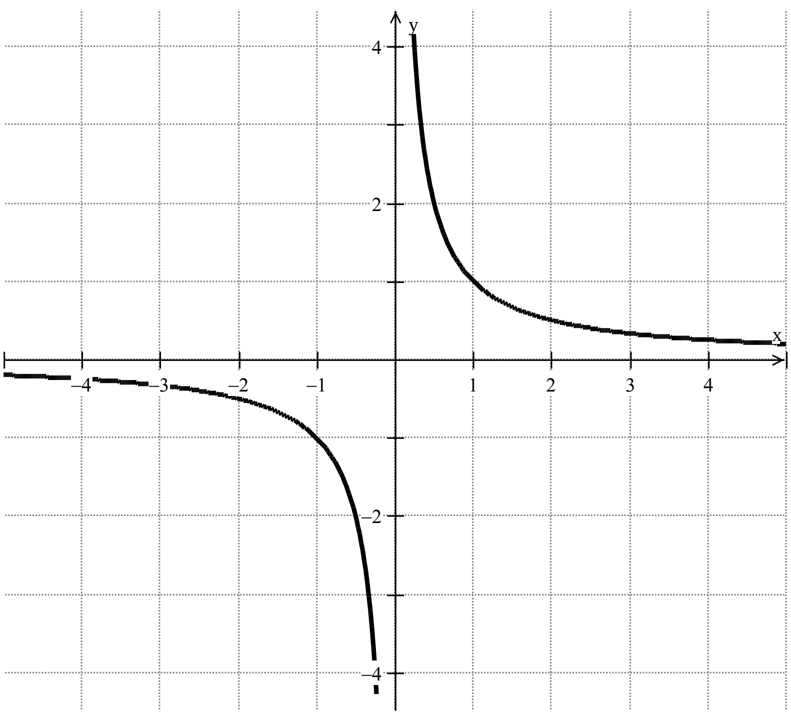
say that $\lim\limits_{x\to 2^-} \dfrac{2}{x-2}$ ***exists because there is one answer. It just happens to be a transfinite number. Other books would say*** $\lim\limits_{x\to 2^-} \dfrac{2}{x-2}$ ***does not exist (D.N.E.) because the answer is not a Real number.***

There are certain Infinite Limits that we just need to know.

$\lim\limits_{x\to 0^+} \dfrac{a}{x}=\infty,\text{ if } a>0$	$\lim\limits_{x\to 0^-} \dfrac{a}{x}=-\infty,\text{ if } a>0$	$\lim\limits_{x\to \infty} \dfrac{x}{a}=\infty$
$\lim\limits_{x\to 0^+} (\ln x)=-\infty$	$\lim\limits_{x\to \frac{\pi}{2}^-} (\tan x)=\infty$	$\lim\limits_{x\to \frac{\pi}{2}^+} (\tan x)=-\infty$

The first two rules ($\lim\limits_{x\to 0^+} \dfrac{a}{x}=\infty,\text{ if } a>0$ and $\lim\limits_{x\to 0^-} \dfrac{a}{x}=-\infty,\text{ if } a>0$) reverse signs if a is negative.

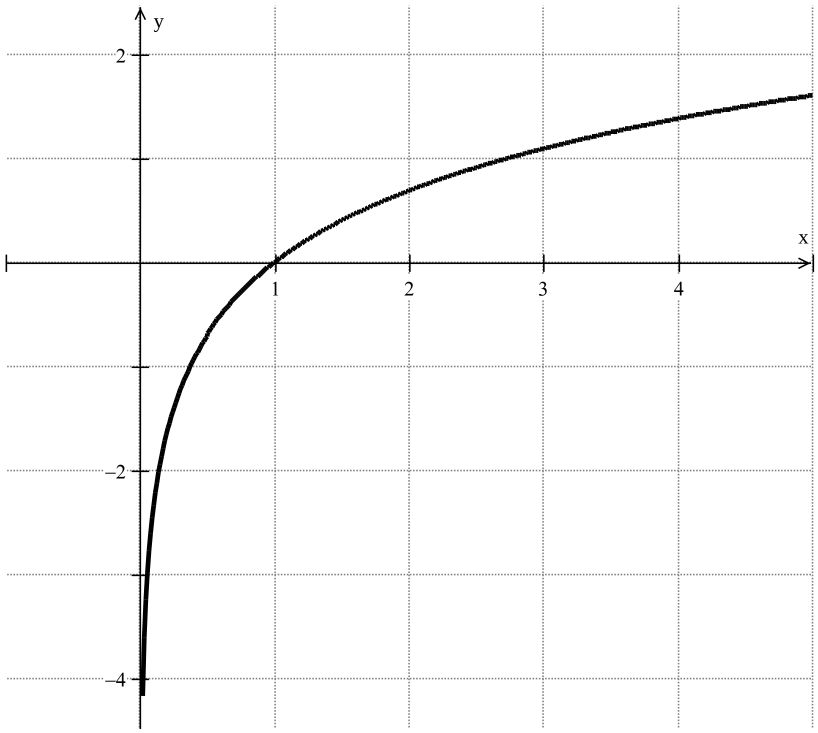
All these rules are based on the asymptotic behavior of the functions.



Here is the graph of $y=\dfrac{1}{x}$.

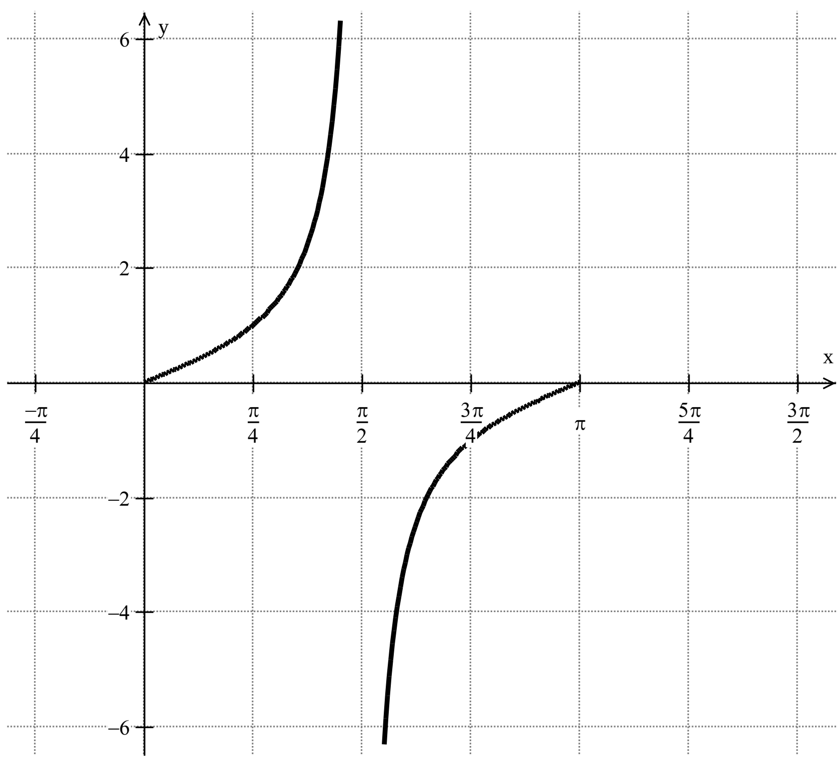
As we trace along the graph approaching 0 from the right side, the graph heads up (to ∞).

As we trace along the graph approaching 0 from the left side, the graph heads down (to $-\infty$).



Here is the graph of $y = \ln x$.

As we trace along the graph approaching 0 from the right side, the graph heads down (to $-\infty$).



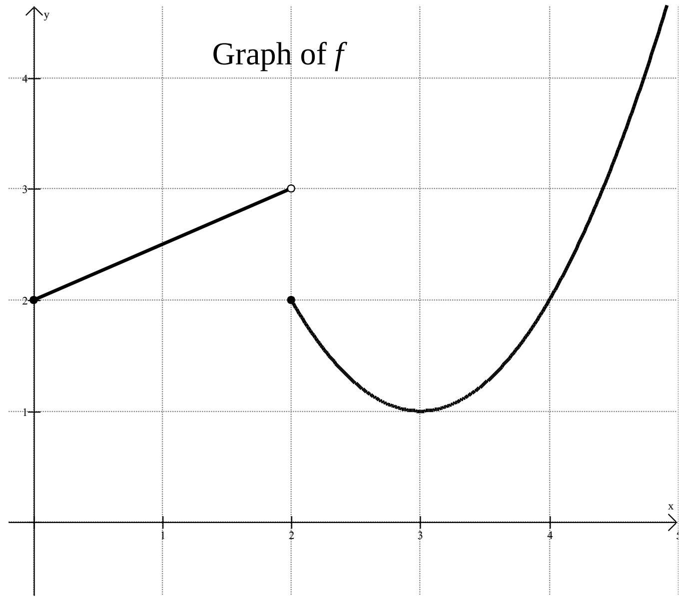
Here is the graph of $y = \tan x$ on $x \in [0, \pi]$.

As we trace along the graph approaching $\frac{\pi}{2}$ from the right side, the graph heads down (to $-\infty$).

As we trace along the graph approaching $\frac{\pi}{2}$ from the left side, the graph heads up (to ∞).

5.2 Homework

1. The graph of a function, *f*, is shown below. Which of the following statements are **TRUE**?



I. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

II. $f(2) = 2$

III. $\lim_{x \to 2} f(x)$ does not exist

A. I only B. II only C. I, II, and III D. I and III E. II and III

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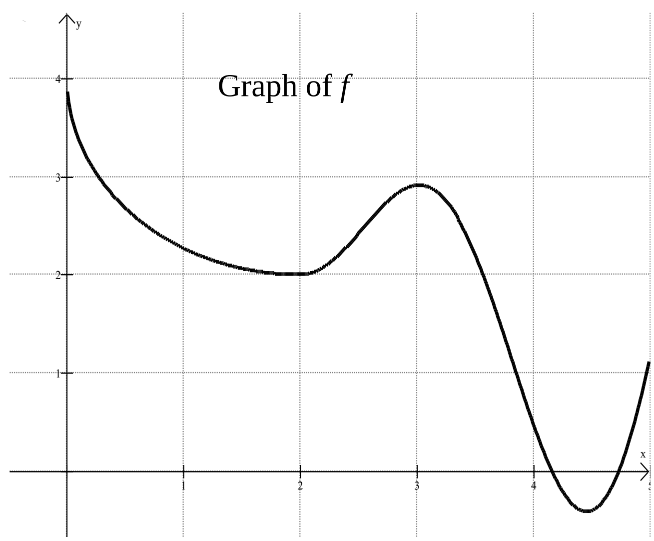
2. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



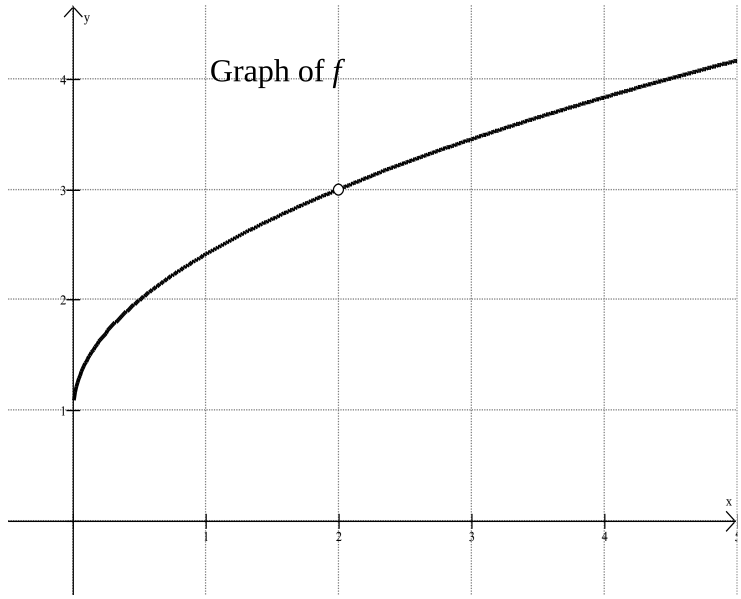
I. $f'(2) > \lim_{x \to 2} f(x)$

II. $f'(1) < \lim_{x \to 2} f(x)$

III. $\lim_{x \to 2} f(x)$ does not exist

A. I only B. II only C. I, II, and III D. I and III E. II and III

3. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. f is continuous at $x = 3$

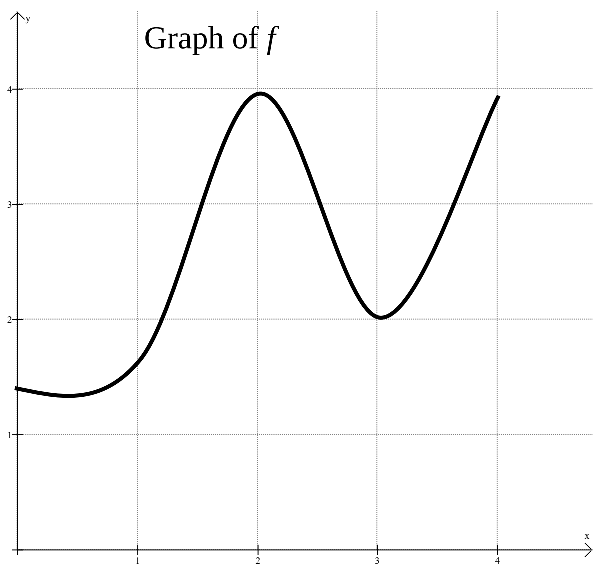
II. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x)$

III. $\lim\limits_{x\to 2} f(x)$ does not exist

A. I only B. II only C. I, II, and III D. I and II E. II and III

---

4. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



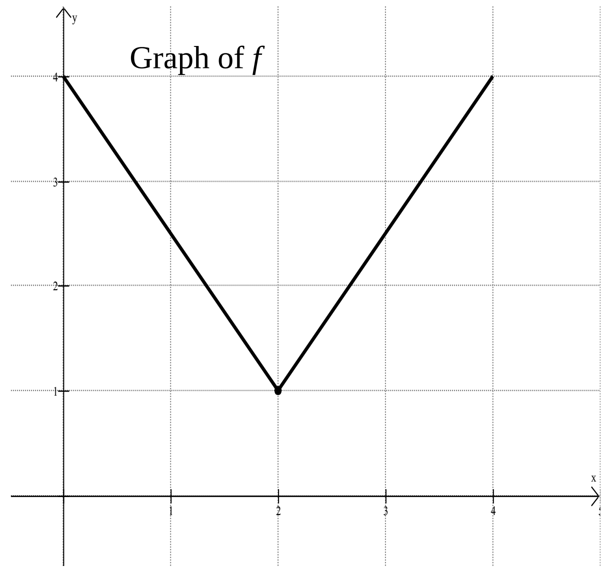
I. $\lim\limits_{x\to 2} f(x) > \lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$

II. $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 3} f(x)$

III. $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2} = \lim\limits_{x\to 3} \dfrac{f(x)-f(3)}{x-3}$

A. I only B. II only C. I, II, and III D. I and III E. II and III

5. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. $\lim\limits_{x\to 2} f(x) = f(2)$

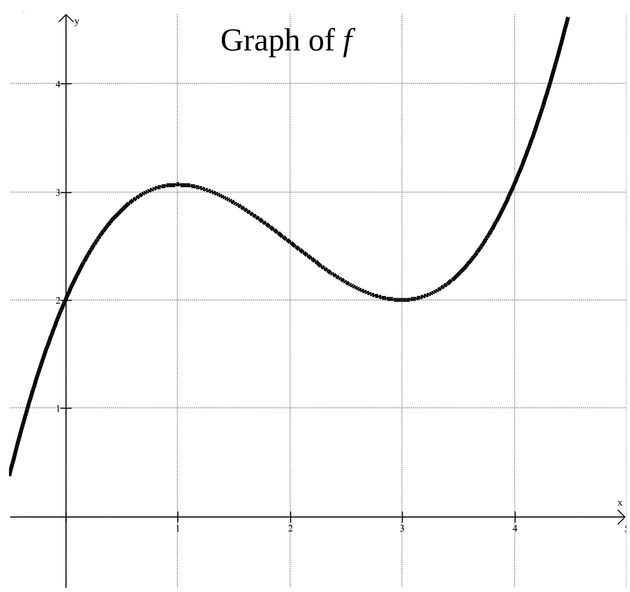
II. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x)$

III. $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ does not exist

A. I only B. II only C. I, II, and III D. I and III E. II and III

---

6. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. $\lim\limits_{x\to 1} f(x) = f(1)$

II. $\lim\limits_{x\to 1^+} f(x) \neq \lim\limits_{x\to 1^-} f(x)$

III. $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2} < \lim\limits_{x\to 4} \dfrac{f(x)-f(4)}{x-4}$

A. I only B. I and III C. I, II, and III D. II only E. II and III

For problems 7 through 16, determine if each of the following functions is continuous at $x = a$ and use the definition of continuity to prove it.

7. $f(x)=\begin{cases} x^2-1, & \text{if } x>-1 \\ 0, & \text{if } x=-1 \\ 4-x, & \text{if } x<-1 \end{cases}$; $a = -1$

8. $g(x)=\dfrac{x^2-4x-5}{x^2-1}$; $a = -1$

9. $h(x)=\begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x\neq 2 \\ 4, & \text{if } x=2 \end{cases}$; $a = 2$

10. $k(x)=\begin{cases} 2x+1, & \text{if } x\leq -1 \\ 3-x^2, & \text{if } x>-1 \end{cases}$; $a = 3$

11. $k(x)=\begin{cases} 2x+1, & \text{if } x\leq -1 \\ 3-x^2, & \text{if } x>-1 \end{cases}$; $a = -1$

12. $F(x)=\begin{cases} 2x+1, & \text{if } x\le -1 \\ -x^2, & \text{if } x>-1 \end{cases}; \ a=-1$

13. $H(x)=\begin{cases} 2x+1, & \text{if } x>-1 \\ 0, & \text{if } x=-1 \\ -x^2, & \text{if } x<-1 \end{cases}; \ a=-1$

14. $G(x)=\begin{cases} \cos x, & \text{if } x\le \pi \\ x^2-\pi x-1, & \text{if } x>\pi \end{cases}; \ a=\pi$

15. $f(x)=\begin{cases} \tan^{-1}(x-3) & \text{if } x\le 3 \\ 1-\cos(x-3) & \text{if } x>3 \end{cases}; \ a=3$

16. $f(x)=\begin{cases}\ln(1+x) & \text{if } x<0 \\ x^2+5x & \text{if } x>0\end{cases}$; $a=0$

Evaluate the following Limits.

17. $\lim\limits_{x\to 0^+}\dfrac{\ln x}{\sin x}$

18. $\lim\limits_{x\to 2^-}\dfrac{7x}{x^2-4}$

19. $\lim\limits_{x\to 0^-}\dfrac{\cot(x^2-5x)}{\ln x^2}$

20. $\lim\limits_{x\to 2^+}\dfrac{x-2}{\tan x}$

21. $\lim\limits_{x\to 5^-}\dfrac{\cos(\pi x)}{5-x}$

22. $\lim\limits_{x\to 2^+}\dfrac{\tan\left(\frac{\pi}{4}x\right)}{x-3}$

23. $\lim\limits_{x\to 0^+}\left((x-2)\ln x\right)$

24. $\lim\limits_{x\to -1^+}\dfrac{e^x}{\cot\left(-\frac{\pi}{2}x\right)}$

25. $\lim\limits_{x\to 4^-}\dfrac{7-x}{16-x^2}$

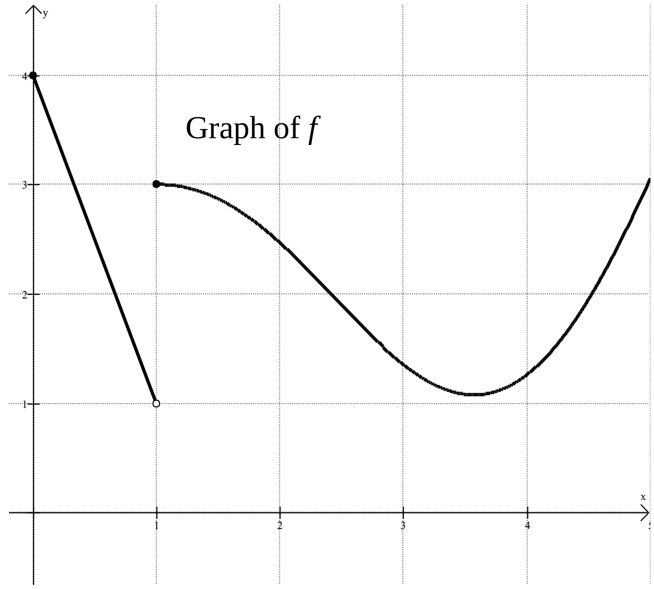
Solve the following multiple choice problems.

26. Given the graph of the function, $f(x)$, below, which of the following statements are TRUE?

I. $\lim_{x \to 1} f(x)$ does not exist.

II. $\lim_{x \to 1^+} f(x) = 1$

III. $\lim_{x \to 1^-} f(x) = 3$



A. I only B. I and II C. II and III D. I, II, and III E. III only

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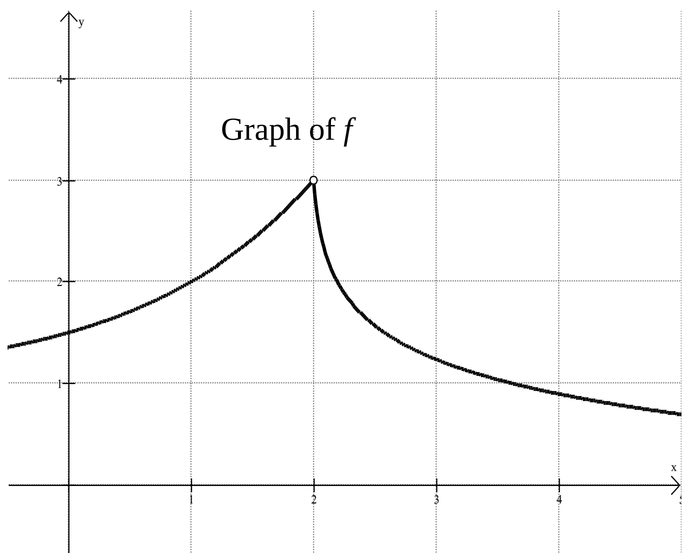
27. Given the graph of the function, $f(x)$, below, which of the following statements are TRUE?

I. $\lim_{x \to 2} f(x)$ does not exist.

II. $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^-} f(x)$

III. $\lim_{x \to 2} f(x) = 3$



A. I only B. I and II C. II and III D. I, II, and III E. III only

5.2 Homework Set B

Evaluate the following Limits.

1. $\lim_{x \to -4^+} \sqrt{\dfrac{x^2+16}{x+4}}$

2. $\lim_{x \to 1^+} \dfrac{\ln(x-1)}{\tan\left(\dfrac{\pi x}{2}\right)}$

3. $\lim_{x \to 0^-} \ln(1-x)$

4. $\lim_{t \to 5^-} \dfrac{\ln(5-t)}{t-3}$

5. $\lim_{v \to e^-} \dfrac{v^2+1}{v-e}$

6. $\lim_{x \to 0^+} \dfrac{e^x-3}{x}$

7. $\lim_{x \to 2^-} \dfrac{\ln(2-x)}{x+4}$

8. $\lim_{x \to 4^+} \dfrac{x+16}{\ln(x-4)}$

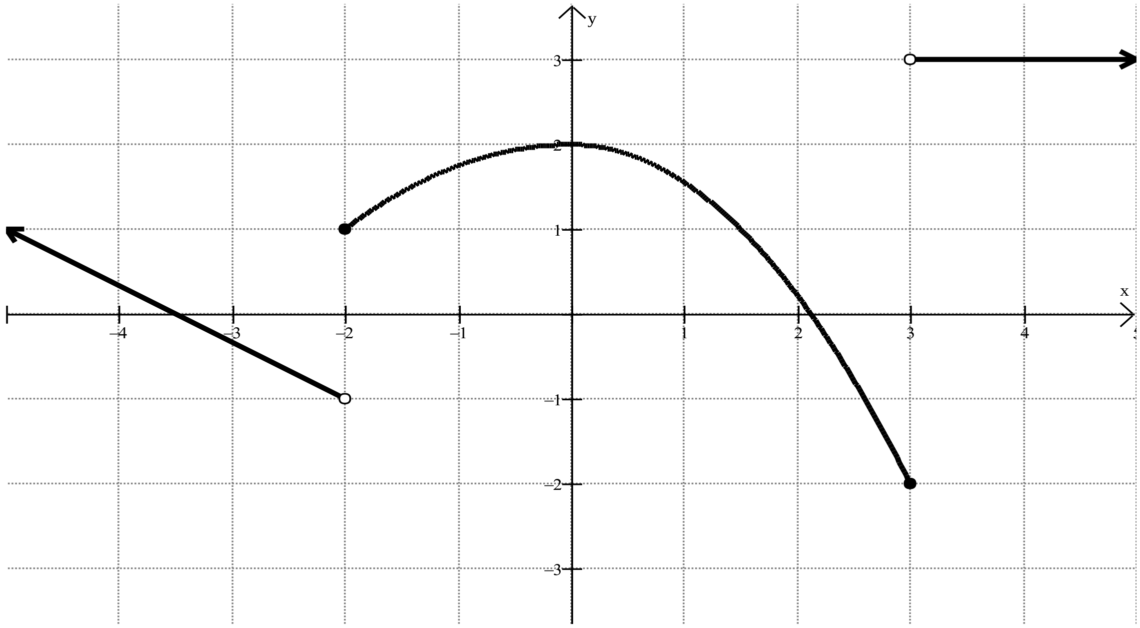
9. $\lim_{x \to 1^-} \ln(1-x)$

10. $\lim_{x \to 0^+} (5-\ln x)$

11. $\lim_{x \to 2^+} \ln\left|x^2-3\right|$

12. $\lim_{x \to 3^+} \dfrac{x^2-16}{9-x^2}$

13. Evaluate each of the following for the graph of $f(x)$, shown below.



a. $\lim_{x \to -2^-} f(x) =$ b. $\lim_{x \to -2^+} f(x) =$ c. $\lim_{x \to -2} f(x) =$

d. $f(-2) =$ e. $\lim_{x \to 0^+} f(x) =$ f. $\lim_{x \to 0^-} f(x) =$

g $\lim_{x \to 0} f(x) =$ h. $f(0) =$ i. $\lim_{x \to 3^+} f(x) =$

j. $\lim_{x \to 3^-} f(x) =$ k. $\lim_{x \to 3} f(x) =$ l. $f(3) =$

m. $\lim_{x \to 4^+} f(x) =$ n. $\lim_{x \to 4^-} f(x) =$

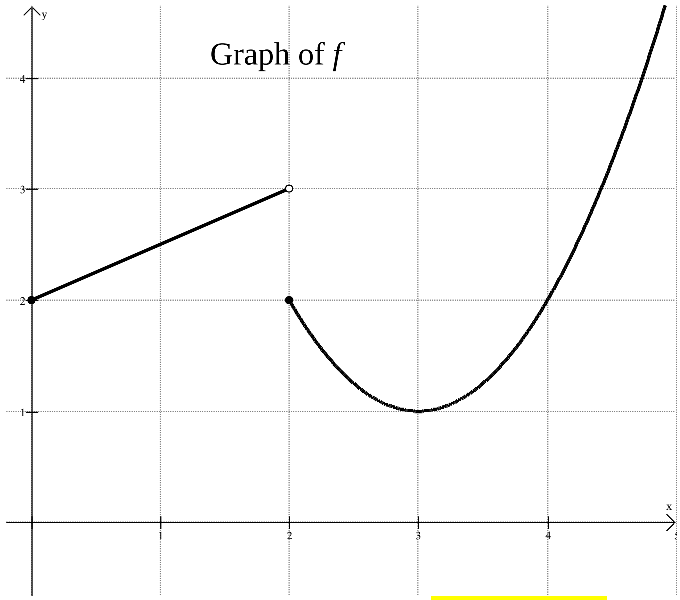
Answers: 5.2 Homework Set A

1. The graph of a function, *f*, is shown below. Which of the following statements are **TRUE**?

I. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x)$

II. $f(2) = 2$

III. $\lim\limits_{x\to 2} f(x)$ does not exist



A. I only B. II only C. I, II, and III D. I and III E. II and III

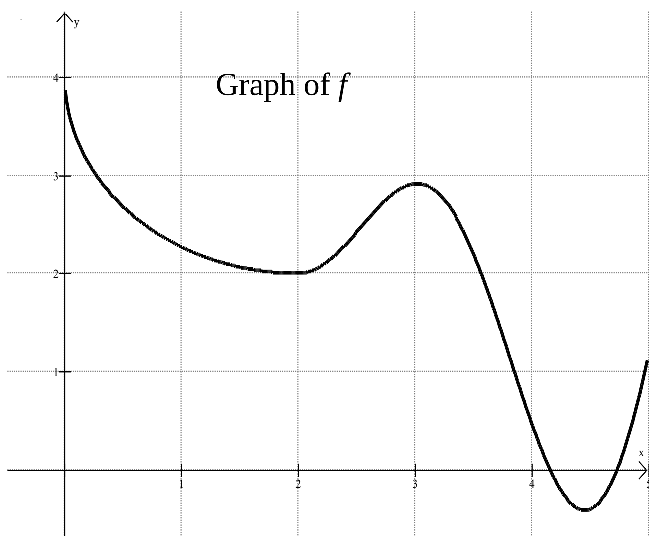
---

2. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?

I. $f'(2) > \lim\limits_{x\to 2} f(x)$

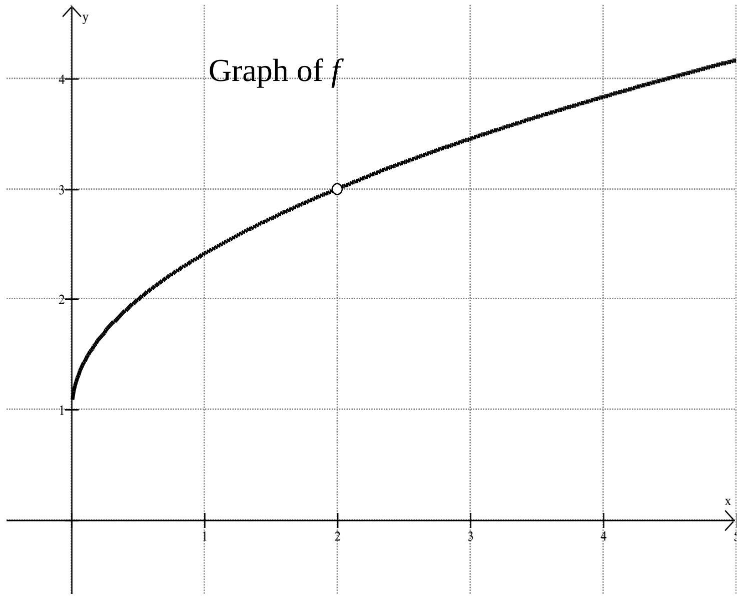
II. $f'(1) < \lim\limits_{x\to 2} f(x)$

III. $\lim\limits_{x\to 2} f(x)$ does not exist



A. I only B. II only C. I, II, and III D. I and III E. II and III

3. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. f is continuous at $x = 3$

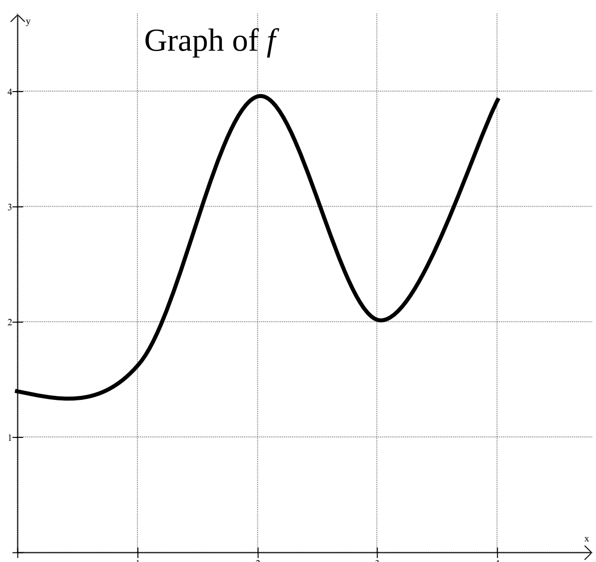
II. $\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^-} f(x)$

III. $\lim\limits_{x \to 2} f(x)$ does not exist

A. I only B. II only C. I, II, and III D. I and II E. II and III

---

4. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?



I. $\lim\limits_{x \to 2} f(x) > \lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2}$

II. $\lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 3} f(x)$

III. $\lim\limits_{x \to 2} \dfrac{f(x) - f(2)}{x - 2} = \lim\limits_{x \to 3} \dfrac{f(x) - f(3)}{x - 3}$

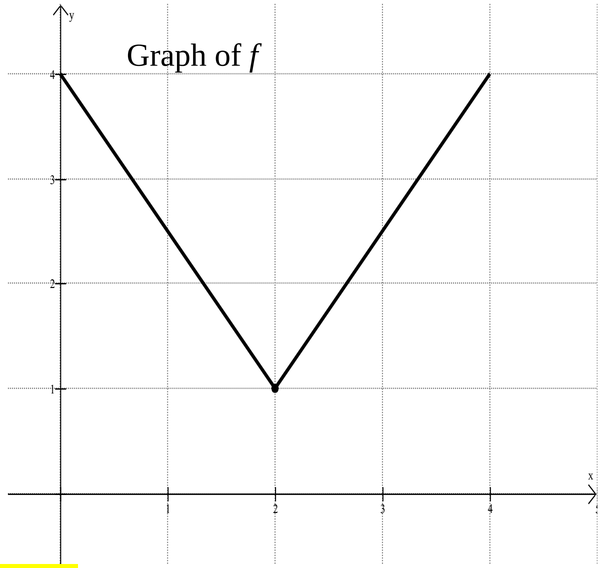
A. I only B. II only C. I, II, and III D. I and III E. II and III

5. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?

I. $\lim\limits_{x\to 2} f(x) = f(2)$

II. $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^-} f(x)$

III. $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2}$ does not exist



A. I only B. II only C. I, II, and III D. I and III E. II and III

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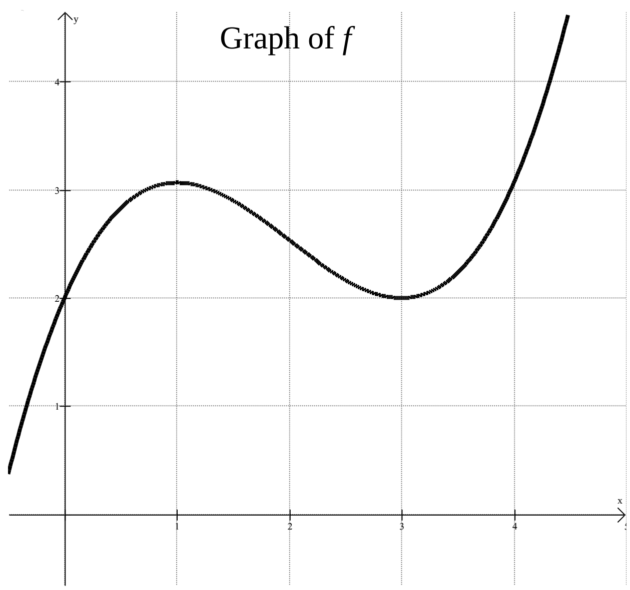
6. The graph of a function, f, is shown below. Which of the following statements are **TRUE**?

I. $\lim\limits_{x\to 1} f(x) = f(1)$

II. $\lim\limits_{x\to 1^+} f(x) \neq \lim\limits_{x\to 1^-} f(x)$

III. $\lim\limits_{x\to 2} \dfrac{f(x)-f(2)}{x-2} < \lim\limits_{x\to 4} \dfrac{f(x)-f(4)}{x-4}$



A. I only B. I and III C. I, II, and III D. II only E. II and III

7. $f(x)=\begin{cases} x^2-1, & \text{if } x>-1 \\ 0, & \text{if } x=-1 \\ 4-x, & \text{if } x<-1 \end{cases};$

$a=-1$

Not Continuous,
Limit does not exist

8. $g(x)=\dfrac{x^2-4x-5}{x^2-1};\ a=-1$

Not Continuous,
POE at $x=-1$

9. $h(x)=\begin{cases} \dfrac{x^2-4}{x-2}, & \text{if } x\neq 2 \\ 4, & \text{if } x=2 \end{cases};\ a=2$

Continuous

10. $k(x)=\begin{cases} 2x+1, & \text{if } x\leq -1 \\ 3-x^2, & \text{if } x>-1 \end{cases};\ a=3$

Continuous

11. $k(x)=\begin{cases} 2x+1, & \text{if } x\leq -1 \\ 3-x^2, & \text{if } x>-1 \end{cases};\ a=-1$

Not Continuous,
Limit does not exist

12. $F(x)=\begin{cases} 2x+1, & \text{if } x\leq -1 \\ -x^2, & \text{if } x>-1 \end{cases};\ a=-1$

Continuous

13. $H(x)=\begin{cases} 2x+1, & \text{if } x>-1 \\ 0, & \text{if } x=-1 \\ -x^2, & \text{if } x<-1 \end{cases};\ a=-1$

Not Continuous
$\lim\limits_{x\to -1} H(x)\neq H(-1)$

14. $G(x)=\begin{cases} \cos x, & \text{if } x\leq \pi \\ x^2-\pi x-1, & \text{if } x>\pi \end{cases};\ a=\pi$

Continuous

15. $f(x)=\begin{cases} \tan^{-1}(x-3) & \text{if } x\leq 3 \\ 1-\cos(x-3) & \text{if } x>3 \end{cases};\ a=3$

Continuous

16. $f(x)=\begin{cases} \ln(1+x) & \text{if } x<0 \\ x^2+5x & \text{if } x>0 \end{cases};\ a=0$

Continuous

17. $\lim\limits_{x\to 0^+}\dfrac{\ln x}{\sin x}=-\infty$

18. $\lim\limits_{x\to 2^-}\dfrac{7x}{x^2-4}=-\infty$

19. $\lim\limits_{x\to 0^-}\dfrac{\cot(x^2-5x)}{\ln x^2}=-\infty$

20. $\lim\limits_{x\to 2^+}\dfrac{x-2}{\tan x}=0$

21. $\lim\limits_{x\to 5^-}\dfrac{\cos(\pi x)}{5-x}=-\infty$

22. $\lim\limits_{x\to 2^+}\dfrac{\tan\left(\frac{\pi}{4}x\right)}{x-3}=\infty$

23. $\lim\limits_{x\to 0^+}\left((x-2)\ln x\right)=\infty$

24. $\lim\limits_{x\to -1^+}\dfrac{e^x}{\cot\left(-\frac{\pi}{2}x\right)}=\infty$

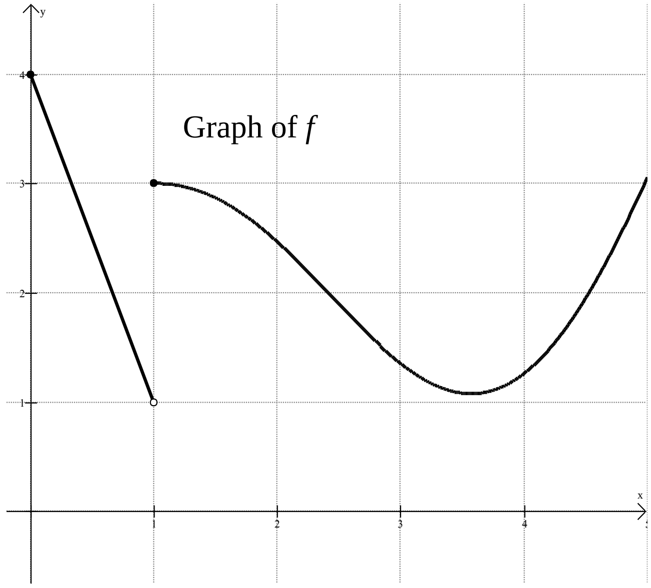
25. $\lim\limits_{x\to 4^-}\dfrac{7-x}{16-x^2}=\infty$

26. Given the graph of the function, $f(x)$, below, which of the following statements are TRUE?

I. $\lim\limits_{x\to 1} f(x)$ does not exist.

II. $\lim\limits_{x\to 1^+} f(x)=1$

III. $\lim\limits_{x\to 1^-} f(x)=3$



A. I only B. I and II C. II and III D. I, II, and III E. III only

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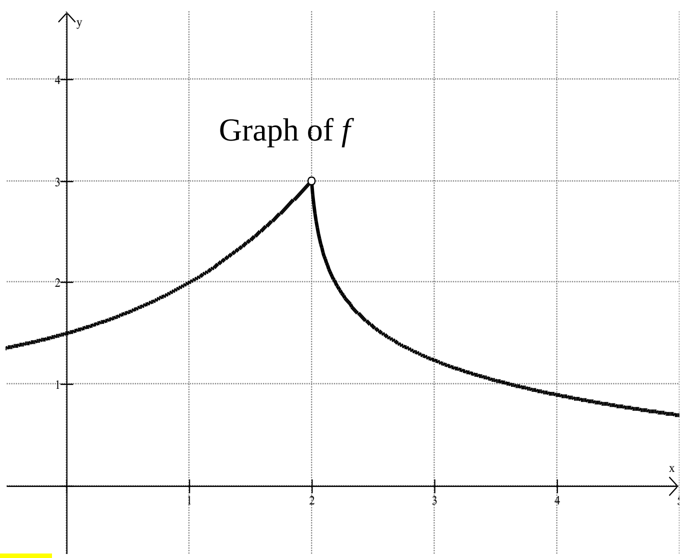
27. Given the graph of the function, $f(x)$, below, which of the following statements are TRUE?

I. $\lim\limits_{x\to 2} f(x)$ does not exist.

II. $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 2^-} f(x)$

III. $\lim\limits_{x\to 2} f(x)=3$



A. I only B. I and II C. II and III D. I, II, and III E. III only

5.2 Homework Set B

1. $\lim\limits_{x\to -4^+} \sqrt{\dfrac{x^2+16}{x+4}} = \infty$

2. $\lim\limits_{x\to 1^+} \dfrac{\ln(x-1)}{\tan\left(\dfrac{\pi x}{2}\right)} = 0$

3. $\lim\limits_{x\to 0^-} \ln(1-x) = 0$

4. $\lim\limits_{t\to 5^-} \dfrac{\ln(5-t)}{t-3} = -\infty$

5. $\lim\limits_{v\to e^-} \dfrac{v^2+1}{v-e} = -\infty$

6. $\lim\limits_{x\to 0^+} \dfrac{e^x-3}{x} = -\infty$

7. $\lim\limits_{x\to 2^-} \dfrac{\ln(2-x)}{x+4} = -\infty$

8. $\lim\limits_{x\to 4^+} \dfrac{x+16}{\ln(x-4)} = 0$

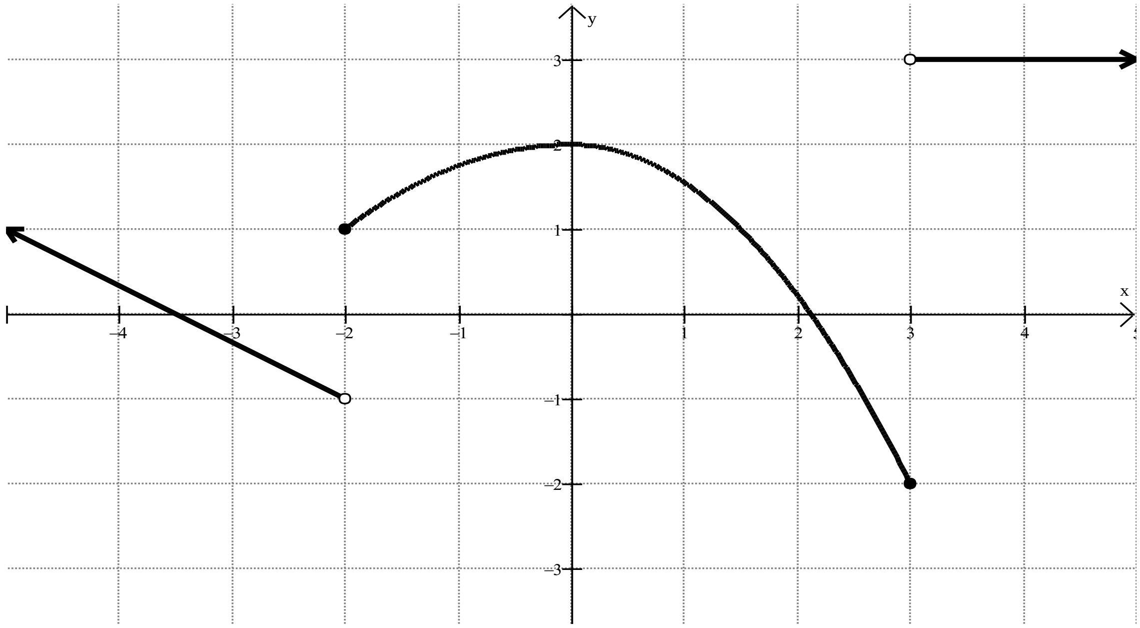
9. $\lim\limits_{x\to 1^-} \ln(1-x) = -\infty$

10. $\lim\limits_{x\to 0^+} (5-\ln x) = \infty$

11. $\lim\limits_{x\to 2^+} \ln\left|x^2-3\right| = -\infty$

12. $\lim\limits_{x\to 3^+} \dfrac{x^2-16}{9-x^2} = \infty$

13. Evaluate each of the following for the graph of $f(x)$, shown below.



a. $\lim_{x \to -2^-} f(x) = -1$

b. $\lim_{x \to -2^+} f(x) = 1$

c. $\lim_{x \to -2} f(x) =$ D.N.E.

d. $f(-2) = 1$

e. $\lim_{x \to 0^+} f(x) = 2$

f. $\lim_{x \to 0^-} f(x) = 2$

g $\lim_{x \to 0} f(x) = 2$

h. $f(0) = 2$

i. $\lim_{x \to 3^+} f(x) = 3$

j. $\lim_{x \to 3^-} f(x) = -2$

k. $\lim_{x \to 3} f(x) =$ D.N.E.

l. $f(3) = -2$

m. $\lim_{x \to 4^+} f(x) = 3$

n. $\lim_{x \to 4^-} f(x) = 3$

5.3: Limits at Infinity and End Behavior

Vocabulary

Limit at Infinity—Defn: the *y*-value when *x* approaches infinity or negative infinity

End Behavior—Defn: The graphical interpretation of a limit at infinity

OBJECTIVE
Evaluate Limits at infinity. Interpret Limits at infinity in terms of end behavior of the graph.

Evaluating limits at infinity for algebraic functions relies on one fact:

$$\boxed{\lim_{x\to\pm\infty}\frac{1}{x}=0}$$

You may notice that this is effectively the reciprocal of one of the rules for infinite limits, and therefore that 0 and $\pm\infty$ are reciprocals of one another.

This fact is what led to our general rule about horizontal asymptotes.

Ex 1 Evaluate $\lim\limits_{x\to-\infty}\dfrac{1-3x^2}{4x^2+3}$.

We already know, from last year, that this function, $y=\dfrac{1-3x^2}{4x^2+3}$, has a horizontal asymptote at $y=-\dfrac{3}{4}$. We could use factoring and limits to prove this:

$$\lim_{x\to-\infty}\frac{1-3x^2}{4x^2+3}=\lim_{x\to-\infty}\frac{x^2\left(\frac{1}{x^2}-3\right)}{x^2\left(4+\frac{3}{x^2}\right)}$$

$$=-\frac{3}{4}$$

Honestly, we never really need to do this algebraic process because of the end behavior rule from last year. These kinds of limits are generally more intuitive than analytical – while you could do this analytically (as on the previous page), it is much simpler to understand what is going on and use what we know from last year.

Ex 2 Find $\lim_{x\to\infty} e^x$ and $\lim_{x\to-\infty} e^x$ and interpret these limits in terms of the end behavior of $y=e^x$.

$\lim_{x\to\infty} e^x = e^{\infty} = \infty$. This means that the right end ($+\infty$) goes up.

$\lim_{x\to-\infty} e^x = e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0$. This means that, on the left ($-\infty$), $y=e^x$ has a horizontal asymptote at $y = 0$.

****NB. The syntax we used here—treating ∞ as if it were a real number and "plugging" it in—is incorrect. Though it works as a practical approach, writing this on the AP test will lose you points for process. Similarly, college Calculus teachers will not allow this notation on a test.**

There are certain Limits at Infinity (which come from our study of end behavior) that we just need to know.

$$\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2} \quad \text{or} \quad \lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$$

$$\lim_{x\to\infty} e^x = \infty \quad \text{and} \quad \lim_{x\to-\infty} e^x = 0$$

Similar to the last section, we could look at the graphs from last year to see this information.

Ex 3 Evaluate $\lim\limits_{x \to \infty} \tan^{-1}\left(x - x^3\right)$

$$\lim_{x \to \infty} \tan^{-1}\left(x - x^3\right) = \tan^{-1}\left(\lim_{x \to \infty}\left(x - x^3\right)\right)$$
$$= \tan^{-1}\left(-\infty\right)$$
$$= -\frac{\pi}{2}$$

While we could go through all the algebraic machinations above, it is easier to just look at the functions as if we are "plugging in" infinity, and evaluating each expression as we would in algebra.

In the above case, we would have to recognize that the polynomial is governed by the highest term ($-x^3$), and when we plug in infinity, we get $-\infty$. Then we see that we actually have $\tan^{-1}\left(-\infty\right)$, which is $-\frac{\pi}{2}$.

Ex 4 Evaluate $\lim\limits_{x \to \infty} \frac{x^5 + 2}{e^x}$

When we "plug in" infinity, we get the indeterminate form, $\frac{\infty}{\infty}$, which means we should be using L'Hôpital's Rule.

$$\lim_{x \to \infty} \frac{x^5 + 2}{e^x} \stackrel{\text{L'H}}{=} \lim_{x \to \infty} \frac{5x^4}{e^x}$$

But we still get $\frac{\infty}{\infty}$, so we should use L'Hôpital's Rule again.

$$\stackrel{\text{L'H}}{=} \lim_{x \to \infty} \frac{20x^3}{e^x}$$

This is still $\frac{\infty}{\infty}$, so we still need L'Hôpital's Rule. But what you may have already noticed is that the numerator's degree keeps decreasing, while the denominator stays the same (because it is an exponential function). This means that eventually, if we take enough derivatives, the numerator will become a constant, while the denominator stays an exponential function. This means we will eventually get a constant over infinity, which gives us 0. Therefore, we know that $\lim\limits_{x \to \infty} \frac{x^5 + 2}{e^x} = 0$.

Limits at Infinity often involve ratios and L'Hôpital's Rule can be applied. But some math teachers consider this to be "using a cannon to kill a fly." It is easier to just know the relative growth rates of functions—i.e., which families of functions grow faster than which. The End Behavior studied in PreCalculus gives a good sense of order.

The Hierarchy of Functions

1. Logs grow the slowest.
2. Polynomials, Rationals and Radicals grow faster than logs and the degree of their End Behavior determines which algebraic function grows fastest. For example, $y = x^{1/2}$ grows more slowly than $y = x^2$.
3. The trig inverses fall in between the algebraic functions at the value of their respective horizontal asymptotes.
4. Exponential functions grow faster than the others. (In BC Calculus, we will see the factorial function, $y = n\,!$, grows the fastest.)

Using the Hierarchy of Functions:

Basically, the fastest growing function dominates the problem and that is the function that determines the limit. With fractions, it's easiest to extend the rules from the end behavior model of last year:

$$\frac{\text{fast function}}{\text{slow function}} = \pm\infty \qquad \frac{\text{slow function}}{\text{fast function}} = 0 \qquad \frac{\text{same function}}{\text{same function}} = a$$

Where a is whatever the functions cancel to. By comparison to a fast function, the slow function basically acts like a constant. So if you look at it like that, we have the same rules from the previous two sections,

$$\frac{\text{fast}}{\text{slow}} = \frac{\pm\infty}{a} = \pm\infty \qquad x \in [-3,1] \cup [2,\infty)$$

The last rule is essentially the same as the End Behavior Model from last year.

Ex 5 Evaluate (a) $\lim_{x\to\infty} \frac{x-x^3}{e^x-95}$ and (b) $\lim_{x\to\infty} \frac{\sqrt{x^3-x}}{x^2-\ln x}$

(a) Repeated iterations of L'Hôpital's Rule will give the same result, but, because exponential grow faster than polynomials, $\lim_{x\to\infty} \frac{x-x^3}{e^x-95} = 0$

(b) Since $\sqrt{x^3-x}$ essentially has a degree of $\frac{3}{2}$ (since we can ignore the x just like in our End Behavior Model), the denominator has a higher degree and $\lim_{x\to\infty} \frac{\sqrt{x^3-x}}{x^2-\ln x} = 0$.

Ex 6 Use the concept of the Hierarchy of Functions to find the end behavior of:

(a) $y = 2xe^x$ (b) $y = x\ln\left(\frac{1}{x}\right)$ (c) $y = x^3e^{x^2}$

(a) For $y = 2xe^x$, the exponential $y = e^x$ determines the end behavior. So the right end goes up and the left end has a horizontal asymptote at $y = 0$.

(b) We still must consider the domain, first. There is no end behavior on left, because the domain is $y \in (0, \infty)$. For $y = x \ \ln\left(\frac{1}{x}\right)$, $y = x$ dominates, so, on the right, the curve goes up.

(c) For $y = x^3e^{x^2}$, the exponential determines the end behavior, but the polynomial might have an influence. The right end goes up, and the left end goes down. This is because e^{x^2} goes up on both ends, but when we "plug in" negative infinity to the x^3, we are multiplying by a negative, so the function goes down.

Ex 7 Evaluate a) $\lim\limits_{x\to-\infty}\dfrac{x-x^3}{e^x-95}$ and b) $\lim\limits_{x\to0^+}x^2\ln x$

a) At first glance, this appears to be a "hierarchy of functions" issue, but as we "plug in" negative infinity, only the x^3 matters in the numerator, and e^x goes to zero as x approaches negative infinity. This gives us $\dfrac{\infty}{-95}$ which gives us a result of $\lim\limits_{x\to-\infty}\dfrac{x-x^3}{e^x-95}=-\infty$

b) When we look at this one, we find that we have zero times negative infinity. **This is not necessarily zero!** It is an indeterminate form of a number. We can either convert it to a fraction to look at it with L'Hôpital's Rule or we can go a more intuitive approach with the Hierarchy of Functions.

i) L'Hôpital's Rule:

$$\lim_{x\to0^+}x^2\ln x$$

$$=\lim_{x\to0^+}\frac{\ln x}{x^{-2}}$$ (this gives negative infinity over infinity)

$$\overset{L'H}{=}\lim_{x\to0^+}\frac{\frac{1}{x}}{-2x^{-3}}$$

$$=\lim_{x\to0^+}\frac{-1}{2}x^2=0$$

ii) Hierarchy of Functions:
Since x^2 is a "faster" function than $\ln x$, the x^2 is what we look at as dominating the entire expression. It essentially gets to 0 "faster" than the natural logarithm reaches negative infinity. Therefore, $\lim\limits_{x\to0^+}x^2\ln x=0$

Ex 8 Evaluate $\lim_{x\to\infty}\frac{e^x+x}{2-5e^x}$

Again, we could evaluate this by using L'Hôpital's Rule (because it yields $\frac{\infty}{-\infty}$) or we could use the Hierarchy. With the Heirarchy of Functions,

$\lim_{x\to\infty}\frac{e^x+\cancel{x}}{\cancel{2}-5e^x}$ We ignore the x and the 2 because they are insignificant compared with e^x.

$\lim_{x\to\infty}\frac{e^x+x}{2-5e^x}=-\frac{1}{5}$ Then we simply cancel the e^x leaving us with the final value of $-\frac{1}{5}$

It would have taken two applications of L'Hôpital's Rule to get the analytical result we could reach fairly easily with an intuitive understanding of how these functions actually behave.

5.3 Homework Set A

Evaluate the Limit for each of the following.

1. $\lim\limits_{x\to\infty} \dfrac{1-15x+12x^2}{16x^2-1}$

2. $\lim\limits_{x\to -\infty} xe^x$

3. $\lim\limits_{x\to -\infty} \left(2+e^x\right)\tan^{-1}\left(x^2+1\right)$

4. $\lim\limits_{x\to\infty} \ln\left(\tan^{-1}\left(\dfrac{x}{e^x}+1\right)\right)$

5. $\lim\limits_{x\to\infty} \dfrac{\cos x}{18-4x+x^2}$

6. $\lim\limits_{x\to\infty} \dfrac{2^x+1}{x^{48}-48x}$

7, $\lim\limits_{x\to -\infty} \dfrac{\sin x}{\ln(-x)}$

8. $\lim\limits_{x\to -\infty} \left(\tan^{-1}\left(\dfrac{x^2+5x+1}{x-2}\right)\right)$

9. $\lim\limits_{x\to -\infty} \left(\tan^{-1}\left(\dfrac{x^3+7x^2+10x}{x+5}\right)\right)$

10. $\lim\limits_{x\to -\infty} \dfrac{\tan^{-1}\left(\dfrac{x^3}{x^3-3x}\right)}{\left(e^x+1\right)}$

11. $\lim\limits_{x\to\infty} \dfrac{\cot^{-1}x}{e^{-x}+1}$

Use limits to determine the end behavior of the following functions.

12. $f(x) = (e^x + 1)(\tan^{-1} x^3)$

13. $f(x) = \dfrac{\cos x}{x^2}$

14. $f(x) = \dfrac{\ln x}{\sqrt{x-4}}$

15. $f(x) = \dfrac{\ln x^2}{\sqrt{4-x}}$

5.3 Homework Set B

Evaluate the Limit for each of the following.

1. $\lim\limits_{x\to -\infty} 2xe^x$

2. $\lim\limits_{x\to\infty} \dfrac{x^2-9}{x^2+16}$

3. $\lim\limits_{x\to -\infty} \tan^{-1}\left(1+e^x\right)$

4. $\lim\limits_{x\to\infty} \dfrac{x^{12}-5x^6+6}{e^x}$

5. $\lim\limits_{x\to -\infty} \dfrac{e^x}{x^{12}-5x^6+6}$

6. $\lim\limits_{x\to -\infty} \tan^{-1}\left(x^2-1\right)$

7. $\lim\limits_{x\to\infty} \tan^{-1}\left(\dfrac{1}{1-x}\right)$

8. $\lim\limits_{x\to -\infty} \sin\left(\tan^{-1}\left(5x+1\right)\right)$

9. $\lim\limits_{t\to\infty} \dfrac{t^2+4}{e^t}$

10. $\lim\limits_{y\to\infty} \dfrac{y+5}{y^3-12y}$

11. $\lim\limits_{x\to\infty} \dfrac{5x^2-15x+27}{x^2-16}$

12. $\lim\limits_{x\to -\infty} 5+e^{3x}$

13. $\lim\limits_{x\to\infty} \dfrac{5x^2-15x+27}{e^x\left(x^2-16\right)}$

14. $\lim\limits_{x\to\infty} \tan^{-1}\left(1-x^2\right)$

15. $\lim\limits_{x\to -\infty} \dfrac{x^2-16}{9-x^2}$

16. $\lim\limits_{y\to\infty} \dfrac{16-5y^4}{y^4-8y^2+12}$

17. $\lim\limits_{t\to\infty} \sin\left(\tan^{-1}\left(\dfrac{t^5}{5^t}+1\right)\right)$

18. $\lim\limits_{x\to -\infty} \dfrac{e^x\left(x^2-25\right)}{x^2-5x+4}$

19. $\lim\limits_{t\to -\infty} \left(e^{2t}-\tan^{-1}\left(t^2+t+1\right)\right)$

20. $\lim_{t\to\infty} \cos\left(\tan^{-1}\left(\frac{e^t}{t^e} + 1 \right) \right)$

21. $\lim_{y\to -\infty} \frac{\left(16 - 5y^4\right)e^{x^3}}{y^4 - 8y^2 + 12}$

22. $\lim_{x\to-\infty} \tan^{-1}\left(1 - x^3\right)$

23. $\lim_{t\to-\infty} \left(e^{2x} - \tan^{-1}\left(\frac{x^2 + x + 1}{5 + x^2} \right) \right)$

Answers: 5.3 Homework Set A

1. $\lim_{x\to\infty} \frac{1-15x+12x^2}{16x^2-1} = \frac{3}{4}$

2. $\lim_{x\to -\infty} xe^x = 0$

3. $\lim_{x\to -\infty} \left(2+e^x\right)\tan^{-1}\left(x^2+1\right) = \pi$

4. $\lim_{x\to\infty} \ln\left(\tan^{-1}\left(\frac{x}{e^x}+1\right)\right) = \ln\frac{\pi}{4}$

5. $\lim_{x\to\infty} \frac{\cos x}{18-4x+x^2} = 0$

6. $\lim_{x\to\infty} \frac{2^x+1}{x^{48}-48x} = \infty$

7. $\lim_{x\to -\infty} \frac{\sin x}{\ln(-x)} = 0$

8. $\lim_{x\to -\infty} \left(\tan^{-1}\left(\frac{x^2+5x+1}{x-2}\right)\right) = -\frac{\pi}{2}$

9. $\lim_{x\to -\infty} \left(\tan^{-1}\left(\frac{x^3+7x^2+10x}{x+5}\right)\right) = \frac{\pi}{2}$

10. $\lim_{x\to -\infty} \frac{\tan^{-1}\left(\frac{x^3}{x^3-3x}\right)}{\left(e^x+1\right)} = \frac{\pi}{4}$

11. $\lim_{x\to\infty} \frac{\cot^{-1}x}{e^{-x}+1} = 0$

12. $f(x) = \left(e^x+1\right)\left(\tan^{-1}x^3\right)$
Right: up
Left: $y = -\frac{\pi}{2}$

13. $f(x) = \frac{\cos x}{x^2}$
Right: $y = 0$
Left: $y = 0$

14. $f(x) = \frac{\ln x}{\sqrt{x-4}}$
Right: $y = 0$
Left: none

15. $f(x) = \frac{\ln x^2}{\sqrt{4-x}}$
Right: none
Left: $y = 0$

5.3 Homework Set B

1. $\lim_{x\to -\infty} 2xe^x$

$= 0$

2. $\lim_{x\to\infty} \frac{x^2-9}{x^2+16}$

$= 1$

3. $\lim_{x\to -\infty} \tan^{-1}\left(1+e^x\right)$

$= \frac{\pi}{4}$

4. $\lim_{x\to\infty} \frac{x^{12}-5x^6+6}{e^x}$

$= 0$

5. $\lim_{x\to -\infty} \frac{e^x}{x^{12}-5x^6+6}$

$= 0$

6. $\lim_{x\to -\infty} \tan^{-1}\left(x^2-1\right)$

$= \frac{\pi}{2}$

7. $\lim_{x\to\infty} \tan^{-1}\left(\frac{1}{1-x}\right)$

$= 0$

8. $\lim_{x\to -\infty} \sin\left(\tan^{-1}(5x+1)\right)$

$= -1$

9. $\lim_{t\to\infty} \frac{t^2+4}{e^t}$

$= 0$

10. $\lim_{y\to\infty} \frac{y+5}{y^3-12y}$

$= 0$

11. $\lim_{x\to\infty} \frac{5x^2-15x+27}{x^2-16}$

$=5$

12. $\lim_{x\to -\infty} 5+e^{3x}$

$=5$

13. $\lim_{x\to\infty} \frac{5x^2-15x+27}{e^x\left(x^2-16\right)}$

$= 0$

14. $\lim_{x\to\infty} \tan^{-1}\left(1-x^2\right)$

$= -\frac{\pi}{2}$

15. $\lim_{x\to -\infty} \frac{x^2-16}{9-x^2}$

$= -1$

16. $\lim_{y\to\infty} \frac{16-5y^4}{y^4-8y^2+12}$

$= -1$

17. $\lim_{t\to\infty} \sin\left(\tan^{-1}\left(\frac{t^5}{5^t}+1\right)\right)$

$= \frac{1}{\sqrt{2}}$

18. $\lim_{x\to -\infty} \frac{e^x\left(x^2-25\right)}{x^2-5x+4}$

$= 0$

19. $\lim_{t\to -\infty} \left(e^{2t}-\tan^{-1}\left(t^2+t+1\right)\right)$

$= -\frac{\pi}{2}$

20. $\lim_{t\to\infty} \cos\left(\tan^{-1}\left(\frac{e^t}{t^e}+1\right)\right)$

$= 0$

21. $\lim_{y\to -\infty} \frac{\left(16-5y^4\right)e^{x^3}}{y^4-8y^2+12}$

$= 0$

22. $\lim_{x\to -\infty} \tan^{-1}\left(1-x^3\right)$

$= \frac{\pi}{2}$

23. $\lim_{t\to -\infty} \left(e^{2x}-\tan^{-1}\left(\frac{x^2+x+1}{5+x^2}\right)\right)$

$= -\frac{\pi}{4}$

5.4: Differentiability and Smoothness

The second (after Continuity) important underlying concept to Calculus is differentiability. Almost all theorems in Calculus begin with "If a function is continuous and differentiable…" It is actually a very simple idea.

Differentiability—Defn: the derivative exists. $F(x)$ is differentiable at a if and only if $F'(a)$ exists and is differentiable on $[a, b]$ if and only $F(x)$ is differentiable at every point in the interval.

$F(x)$ is differentiable at $x = a$ if and only if

i. $F(x)$ is continuous at $x = a$,

ii. $\lim_{x \to a^-} F'(x)$ and $\lim_{x \to a^+} F'(x)$ both exist,

and iii. $\lim_{x \to a^-} F'(x) = \lim_{x \to a^+} F'(x)$

OBJECTIVE

Determine if a function is differentiable or not.

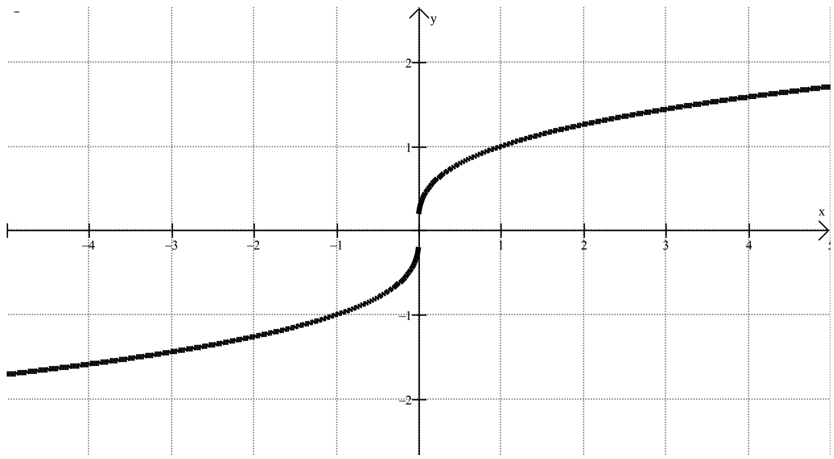
Demonstrate understanding of the connections and differences between differentiability and continuity.

There are two ways that a function could not be differentiable:

I. The tangent line could be vertical, causing the slope to be infinite.

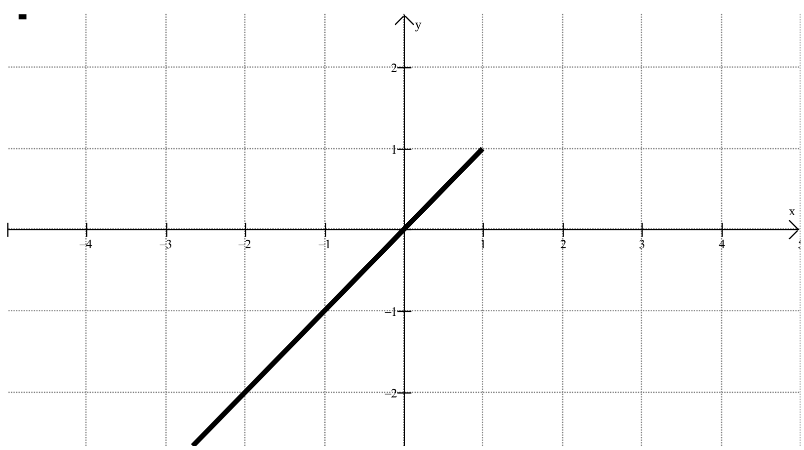
Ex 1 Is $y = \sqrt[3]{x}$ differentiable at $x = 0$?

If $y = \sqrt[3]{x}$, then $\frac{dy}{dx} = \frac{1}{3x^{2/3}}$. At $x = 0$, $\frac{dy}{dx} = \frac{1}{0} = \text{D.N.E.}$



II. Just as the Limit does not exist if the two one-sided limits are not equal, a derivative would not exist if the two one-sided derivatives are not equal. That is, the slopes of the tangent lines to the left of a point are not equal to the tangent slopes to the right. A curve like this is called non-smooth.

Ex 2 Is the function represented by this curve differentiable at $x = 1$?



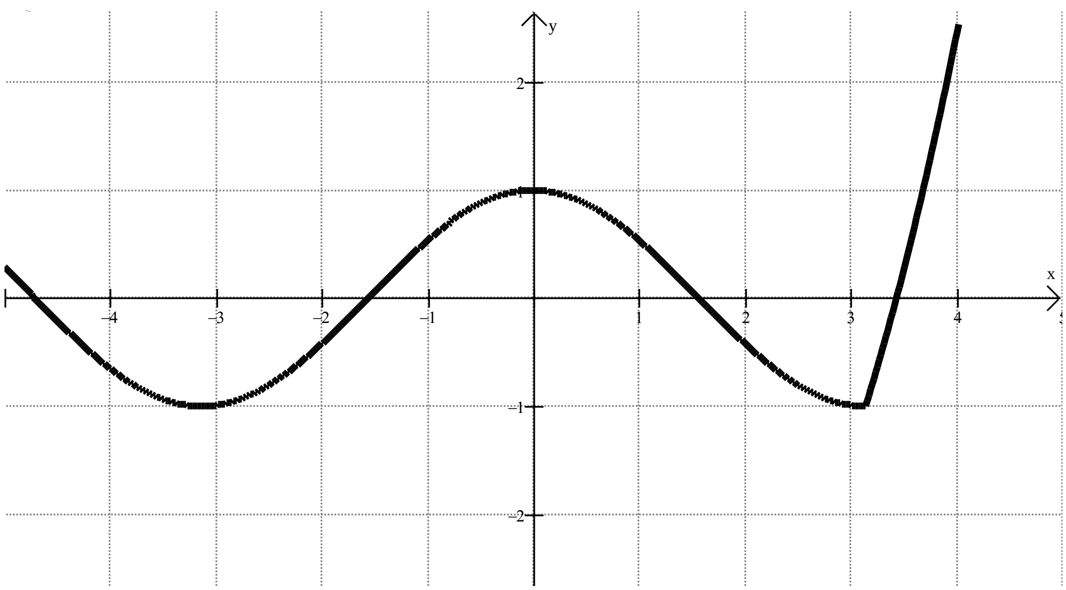
We can see that the slope to the left of $x = 1$ is 1, while the slope to the right of $x = 1$ is 0. So this function is not differentiable at $x = 1$. This is an example of a curve that is not smooth.

Ex 3 Is $G(x)=\begin{cases}\cos x, & \text{if } x\le\pi\\ x^2-\pi x-1, & \text{if } x>\pi\end{cases}$ differentiable at $x=\pi$?

$$G'(x)=\begin{cases}-\sin x, & \text{if } x\le\pi\\ 2x-\pi, & \text{if } x>\pi\end{cases}$$

$$\lim_{x\to\pi^-} G'(x)=\lim_{x\to\pi^-}\left(-\sin x\right)=0$$
$$\lim_{x\to\pi^+} G'(x)=\lim_{x\to\pi^+}\left(2x-\pi x\right)=2\pi-\pi=\pi$$

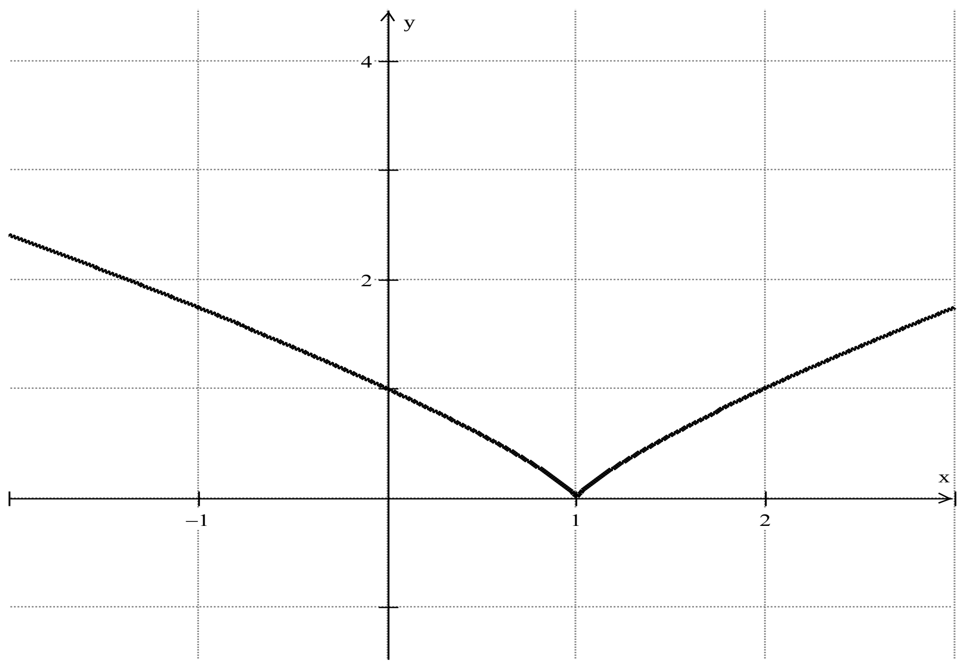
Clearly, these two one-sided derivatives are not equal. Therefore, $G(x)$ is not differentiable at $x=\pi$.



Ex 4 Is $h(x)=(x-1)^{4/5}$ differentiable at $x=1$?

$$h'(x)=\frac{4}{5}(x-1)^{-1/5}$$
$$h'(x)=\frac{4}{5\sqrt[5]{x-1}}$$
$$h'(1)=\frac{4}{5\sqrt[5]{1-1}}=\frac{4}{0}=\text{D.N.E.}$$

Therefore, *h*(*x*) is not differentiable at *x* = 1. The graph below illustrates how the slopes from the left and right aren't the same.



You can always identify if a function is differentiable visually (these are the same two rules as before):

I. If the curve has a vertical tangent, it is not differentiable at that point.

II. If the curve has a corner (also called a *cusp*), it is not differentiable at that point.

If the curve is not continuous, it cannot be differentiable (there is no point at which to find a slope).

All four of these examples show functions that are continuous, but not differentiable. Continuity does not ensure differentiability. But the converse is true.

If $f(x)$ is differentiable, it MUST BE continuous and If $f(x)$ is not continuous, it CANNOT BE differentiable

Ex 4 Given that $f(x)=\begin{cases} k\sqrt{x+1}, \text{ if } x\le 3 \\ 2+mx, \text{ if } x>3 \end{cases}$ is differentiable at $x = 3$, find m and k.

$$f'(x)=\begin{cases} k\dfrac{1}{2\sqrt{x+1}}, & \text{if } x\le 3 \\ m, & \text{if } x>3 \end{cases}$$

If $f(x)$ is differentiable at $x = 3$, then $k\dfrac{1}{2\sqrt{3+1}}=m \to \dfrac{1}{4}k=m$.

If $f(x)$ is differentiable at $x = 3$, it is also continuous.

Therefore, at $x = 3$, $k\sqrt{x+1}=2+mx$

$$k\sqrt{3+1}=2+m3$$
$$2k=3m+2$$

Since both $\frac{1}{4}k=m$ and $2k=3m+2$ must be true, we can use linear combination or substitution to solve for k and m.

$$2k=3\left(\frac{1}{4}k\right)+2$$
$$2k=\frac{3}{4}k+2$$
$$\frac{5}{4}k=2$$
$$k=\frac{8}{5}\to m=\frac{2}{5}$$

5.4 Homework

1. Determine if $f(x) = \sqrt[3]{x^2}$ is differentiable or not.

2. Determine if $f(x) = \begin{cases} \sin^{-1} x, \text{ if } -1 \le x < 1 \\ \ln x, \text{ if } x \ge 1 \end{cases}$ is differentiable or not at x = 1.

3. Determine if $f(x) = \begin{cases} x^2 - 2x + 1, \text{ if } x \le 1 \\ \ln x, \text{ if } x > 1 \end{cases}$ is differentiable or not at x = 1.

4. Determine if $f(x) = \begin{cases} x^2 + 2x - 5, \text{ if } x \le 1 \\ x^3 + x - 4, \text{ if } x > 1 \end{cases}$ is differentiable or not at x = 1.

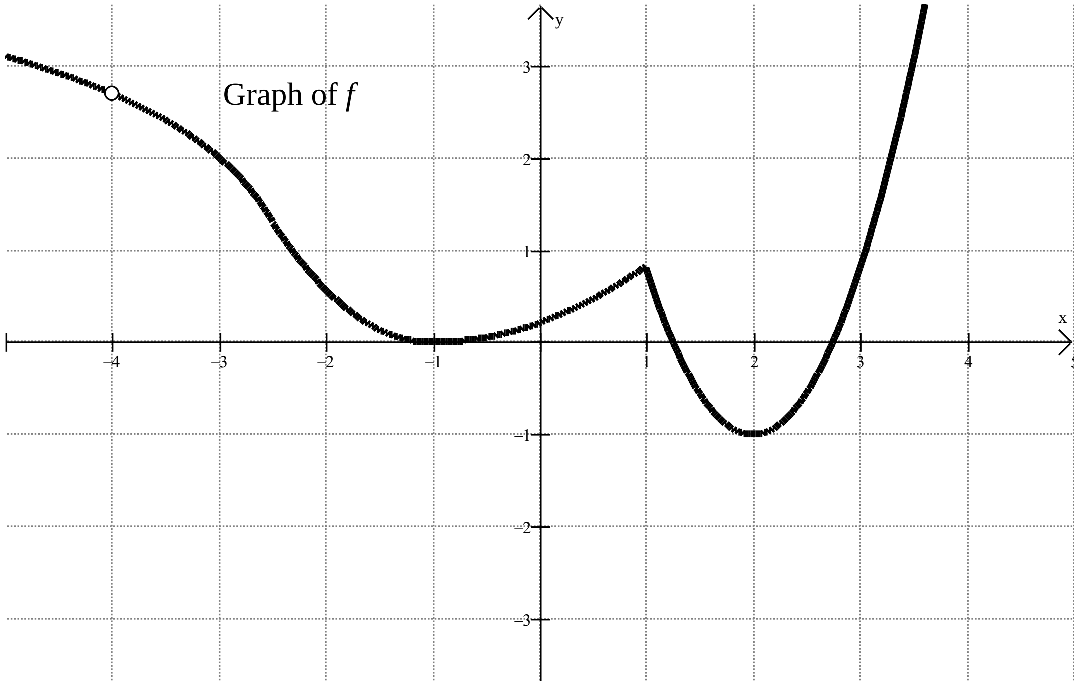
5. Given that $f\left(x\right)=\begin{cases} mx+2, & \text{if } x\le 1 \\ k\ln x, & \text{if } x>1 \end{cases}$ is differentiable at $x = 1$, find m and k.

6. Given that $f\left(x\right)=\begin{cases} mx-5, & \text{if } x\le -2 \\ kx^2+1, & \text{if } x>-2 \end{cases}$ is differentiable at $x = -2$, find m and k.

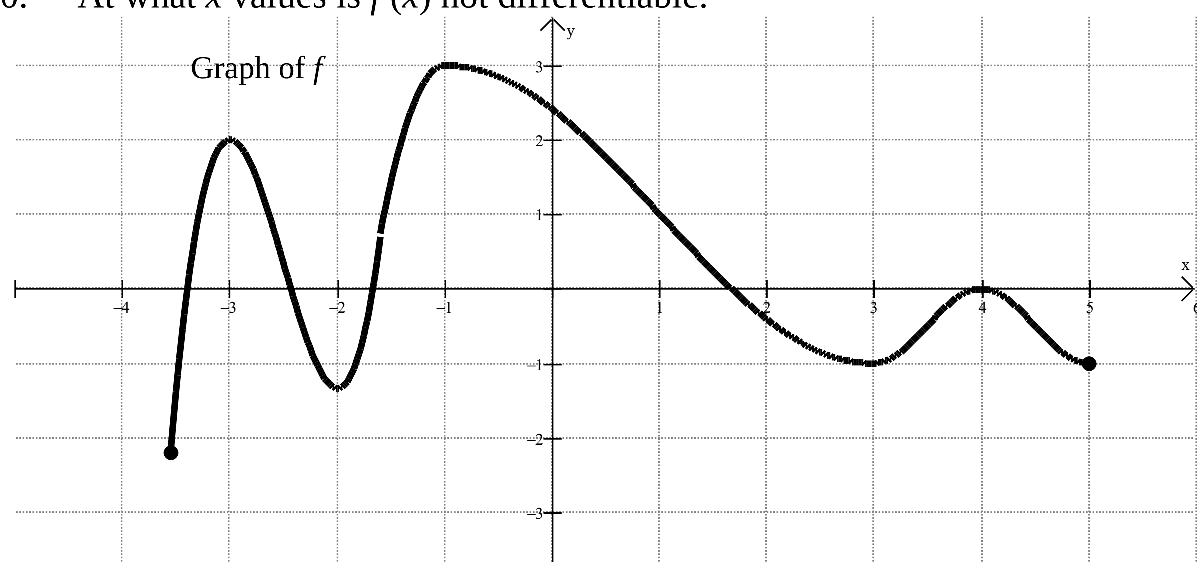
7. Given that $f\left(x\right)=\begin{cases} ke^{2x}, & \text{if } x\le 0 \\ 3-mx, & \text{if } x>0 \end{cases}$ is differentiable at $x = 0$, find m and k.

8. Given that $f\left(x\right)=\begin{cases} mx-2, & \text{if } x\le 2 \\ k\sqrt{x^2-3}, & \text{if } x>2 \end{cases}$ is differentiable at $x = 2$, find m and k.

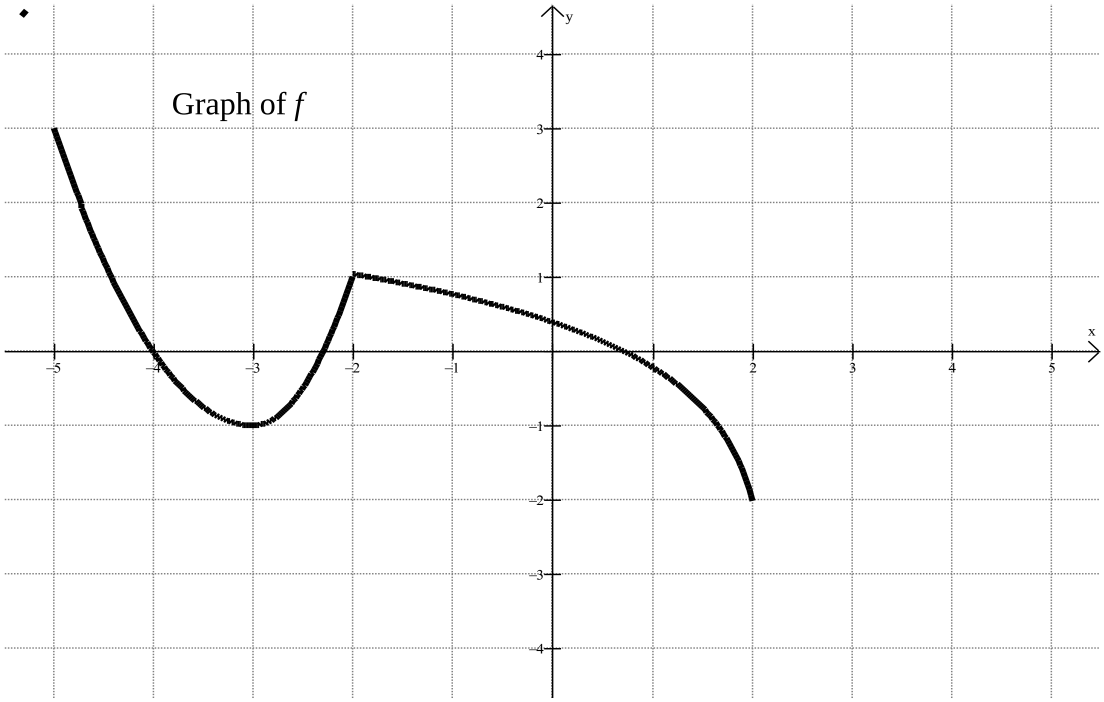
9. At what x values is $f(x)$ not differentiable.



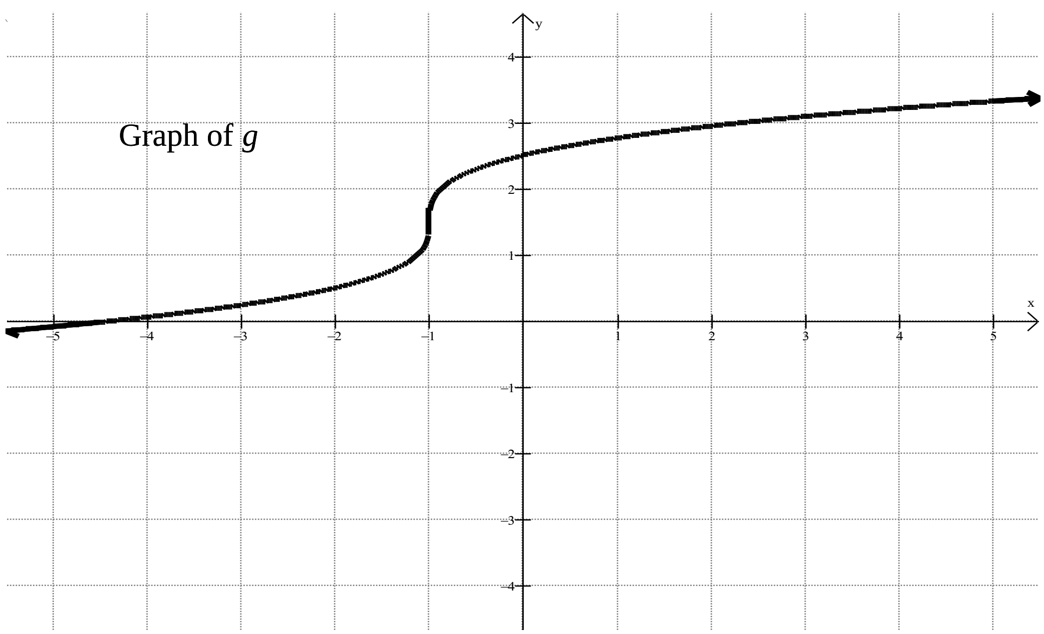
10. At what x values is $f(x)$ not differentiable.



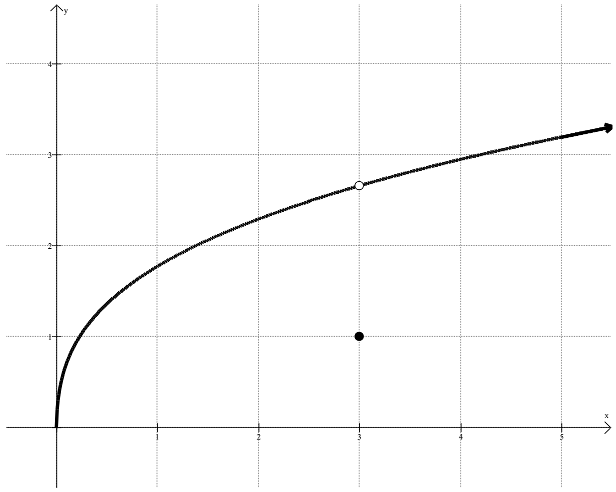
11. At what x values is $f(x)$ not differentiable.



12. At what x value is function g not differentiable.



13. Use the graph of f below to select the correct answer from the choices below.



A. $\lim_{x \to 3} f(x) = f(3)$

B. f is not continuous at $x = 3$

C. f is differentiable at $x = 3$

D. $f'(1) < f'(4)$

E. $\lim_{x \to 3} f(x)$ does not exist

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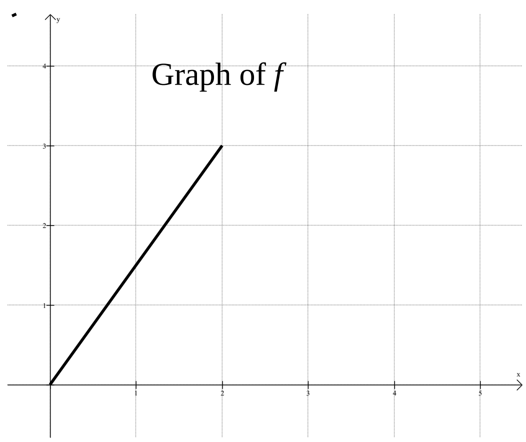
14. Use the graph of f below to select the correct answer from the choices below.



A. f has no extremes

B. f is continuous at $x = 2$

C. f is differentiable for $x \in (0, 4)$

D. f has a relative maximum at 2

E. f is concave up for $x \in (0, 4)$

Answers 5.4 Homework

1. Determine if $f(x)=\sqrt[3]{x^2}$ is differentiable or not.

   Not Differentiable at $x = 0$

2. Determine if $f(x)=\begin{cases}\sin^{-1}x, \text{ if } -1\le x<1\\ \ln x, \text{ if } x\ge 1\end{cases}$ is differentiable or not at $x = 1$.

   No, because it is not continuous at $x = 1$

3. Determine if $f(x)=\begin{cases}x^2-2x+1, \text{ if } x\le 1\\ \ln x, \text{ if } x>1\end{cases}$ is differentiable or not at $x = 1$.

   No, because $\lim\limits_{x\to 1^+} f'(x) \ne \lim\limits_{x\to 1^-} f'(x)$

4. Determine if $f(x)=\begin{cases}x^2+2x-5, \text{ if } x\le 1\\ x^3+x-4, \text{ if } x>1\end{cases}$ is differentiable or not at $x = 1$.

   Yes, it is differentiable

5. Given that $f(x)=\begin{cases}mx+2, \text{ if } x\le 1\\ k\ln x, \text{ if } x>1\end{cases}$ is differentiable at $x = 1$, find m and k.

   $m=-2,\ k=-2$

6. Given that $f(x)=\begin{cases}mx-5, \text{ if } x\le -2\\ kx^2+1, \text{ if } x>-2\end{cases}$ is differentiable at $x = -2$, find m and k.

   $m=-6,\ k=\dfrac{3}{2}$

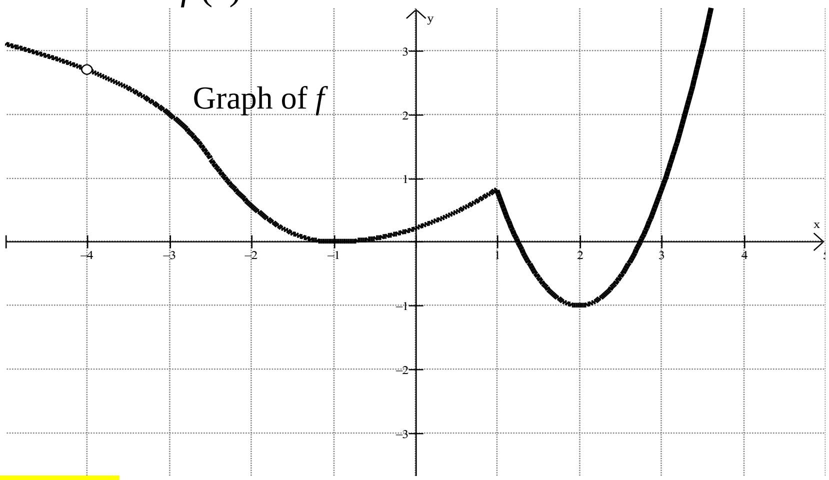
7. Given that $f(x)=\begin{cases}ke^{2x}, \text{ if } x\le 0\\ 3-mx, \text{ if } x>0\end{cases}$ is differentiable at $x = 0$, find m and k.

   $m=-6,\ k=3$

8. Given that $f(x)=\begin{cases}mx-2, \text{ if } x\le 2\\ k\sqrt{x^2-3}, \text{ if } x>2\end{cases}$ is differentiable at $x = 2$, find m and k.

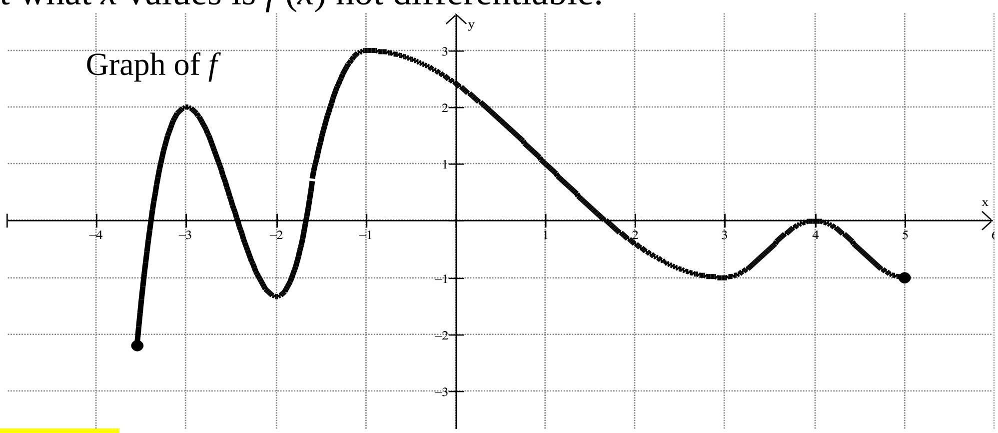
   $m=\dfrac{4}{3},\ k=\dfrac{2}{3}$

9. At what x values is $f(x)$ not differentiable.



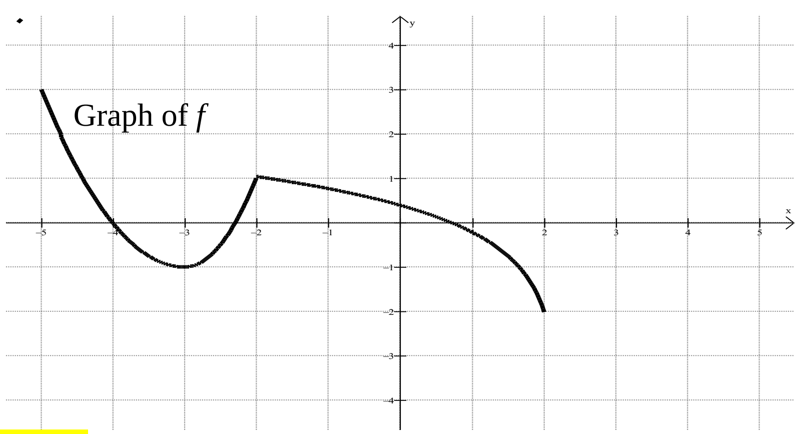
$x \in \{-5, -4, 1, 3.5\}$

10. At what x values is $f(x)$ not differentiable.



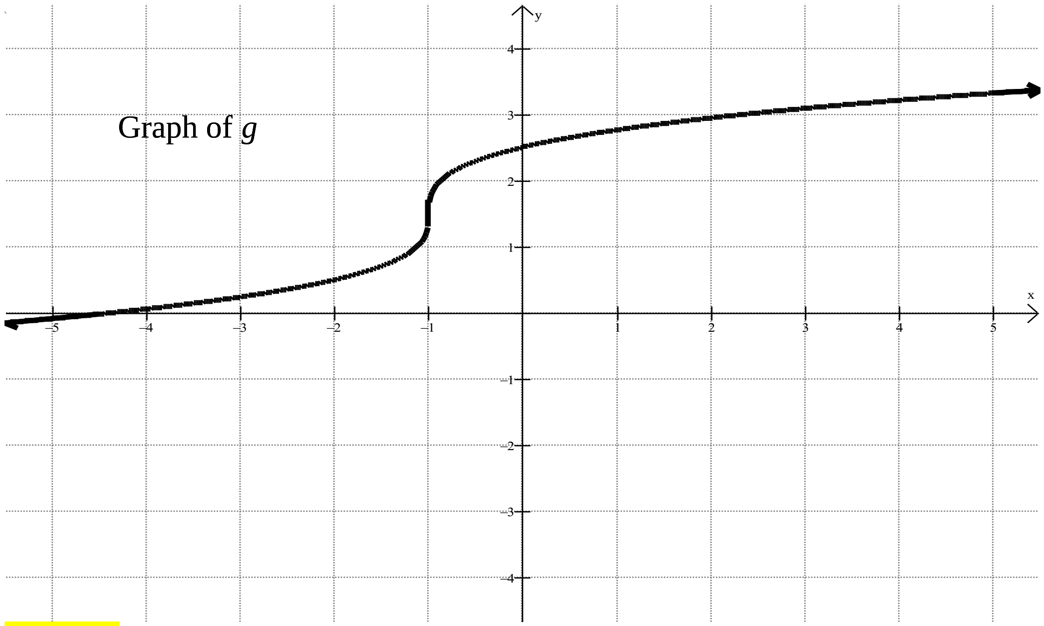
$x \in \{3.5, 5\}$

11. At what x values is $f(x)$ not differentiable.



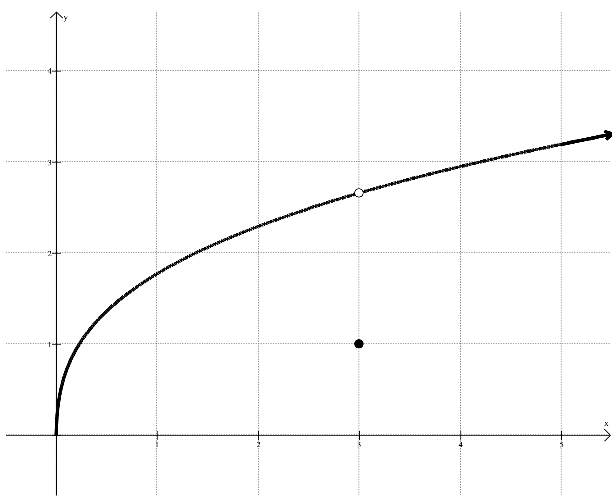
$x \in \{-5, -2, 2, 4\}$

12. At what x value is function g not differentiable.



$x \in \{-1\}$

13. Use the graph of f below to select the correct answer from the choices below.



F. $\lim_{x \to 3} f(x) = f(3)$

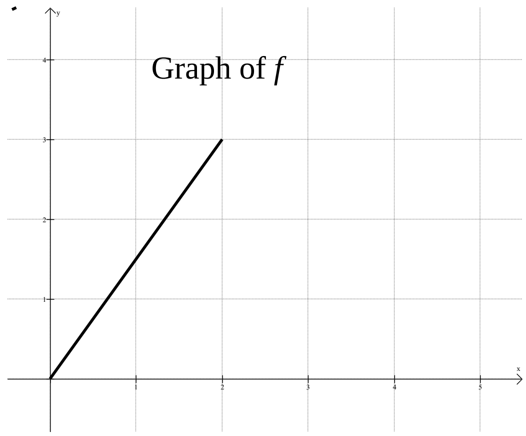
G. f is not continuous at $x = 3$

H. f is differentiable at $x = 3$

I. $f'(1) < f'(4)$

J. $\lim_{x \to 3} f(x)$ does not exist

14. Use the graph of f below to select the correct answer from the choices below.



A. f has no extremes

B. f is continuous at $x = 2$

C. f is differentiable for $x \in (0,4)$

D. f has a relative maximum at 2

E. f is concave up for $x \in (0,4)$

Chapter 5: Limit and Continuity Test

1. The function f is differentiable at $x = b$. Which of the following statements could be false?

(a) $\lim_{x\to b} f(x)$ exists (b) $\lim_{x\to b} f(x) = f(b)$ (c) $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} f(x)$

(d) $\lim_{x\to b^-} f'(x) = \lim_{x\to b^+} f'(x)$ (e) None of these

---

2. The function f is defined for all Reals such that $f(x) = \begin{cases} x^2 + kx & \text{for } x < 5 \\ 5\sin\frac{\pi}{2}x & \text{for } x \geq 5 \end{cases}$.

For which value of k will the function be continuous throughout its domain?

(a) –2 (b) –1 (c) $\frac{2}{3}$ (d) 1 (e) None of these

---

3. The graph of $y = \frac{1 - \cos x}{x}$ has

I. a point of exclusion at $x = 0$
II. a horizontal asymptote at $y = 0$
III. an infinite number of zeros

(a) I only (b) II only (c) II and III only

(d) I and III only (e) I, II, and III

---

4. If $f(x) = \begin{cases} x + 2 & \text{for } x \leq 3 \\ 4x - 7 & \text{for } x > 3 \end{cases}$, which of the following statements are true?

I. $\lim_{x\to 3} f(x)$ exists II. f is continuous at $x = 3$ III. f is differentiable at $x = 3$

(a) None (b) I only (c) II only

(d) I and II only (e) I, II, and III

5. Which of the following functions is differentiable at $x=0$?

(a) $f(x)=\sqrt{1+|x|}$ (b) $f(x)=|x|$ (c) $f(x)=\begin{cases} x^2\sin\dfrac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$

(d) $f(x)=\begin{cases} \dfrac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$ (e) $f(x)=\begin{cases} \cos x & \text{for } x<0 \\ \sin x & \text{for } x\geq 0 \end{cases}$

---

6. $\lim\limits_{h\to 0}\dfrac{\sin\left(\dfrac{\pi}{2}+h\right)-1}{h}=$

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{4}$ (c) 0 (d) $-\dfrac{\pi}{4}$ (e) DNE

---

7. $\lim\limits_{x\to 0}\dfrac{\int_0^x \sin t^2 dt}{x^3}$

(a) 0 (b) 1 (c) $\dfrac{1}{3}$ (d) 3 (e) DNE

---

8. $\lim\limits_{x\to\infty}\dfrac{4x^5+3x^4+2x^3+x^2+1}{3x^5-9x^4+4x^3+15}=$

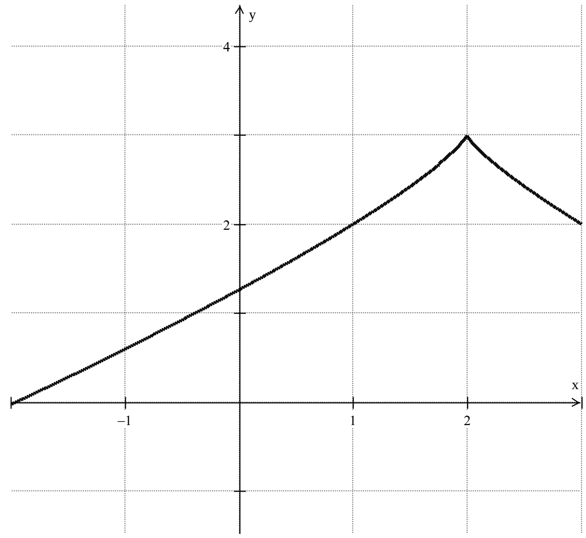
(a) 0 (b) $\dfrac{3}{4}$ (c) $\dfrac{4}{3}$ (d) 3 (e) DNE

9. The graph of a function f is given below. Which of the following statements are true?

I. $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$ D.N.E.

II. $\lim_{x\to 2} f(x)=4$

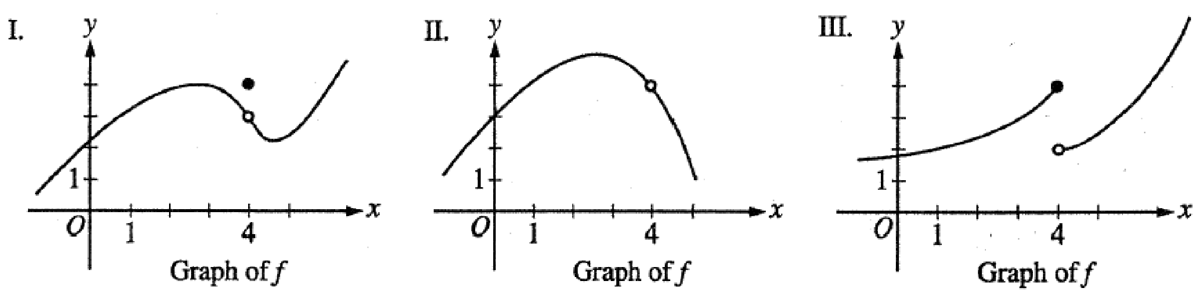
III. $\lim_{x\to 2} f(x)$ D.N.E.



(A)I only (B) II only (C) I and II (D) I, II, and III (E) II and III

---

10. For which of the following does $\lim_{x\to 4} f(x)$ exist?



(a) I only (b) II only (c) III only (d) I and II only (e) I and III only

---

For 11 through 14, find

a) Is $f(x)$ continuous at $x = a$? Why/Why not?

b) Is $f(x)$ differentiable at $x = a$? Why/Why not?

11. $f(x)=\begin{cases} \sin^{-1}\left[\pi(x-1)\right], & \text{if } x<1 \\ 0, & \text{if } x=1 \\ \ln x^2, & \text{if } x>1 \end{cases}$ $\quad a = 1$

12. $f(x)=\begin{cases} x^3-1, & \text{if } x\le 1 \\ 0, & \text{if } x=1 \\ 1+x^3, & \text{if } x>1 \end{cases}$ $\quad a = 1$

13. $f(x)=\begin{cases} e^{3-x}, & \text{if } x<3 \\ -1 & \text{if } x=3 \\ x^2-7x+10, & \text{if } x>3 \end{cases}$ $\quad a = 3$

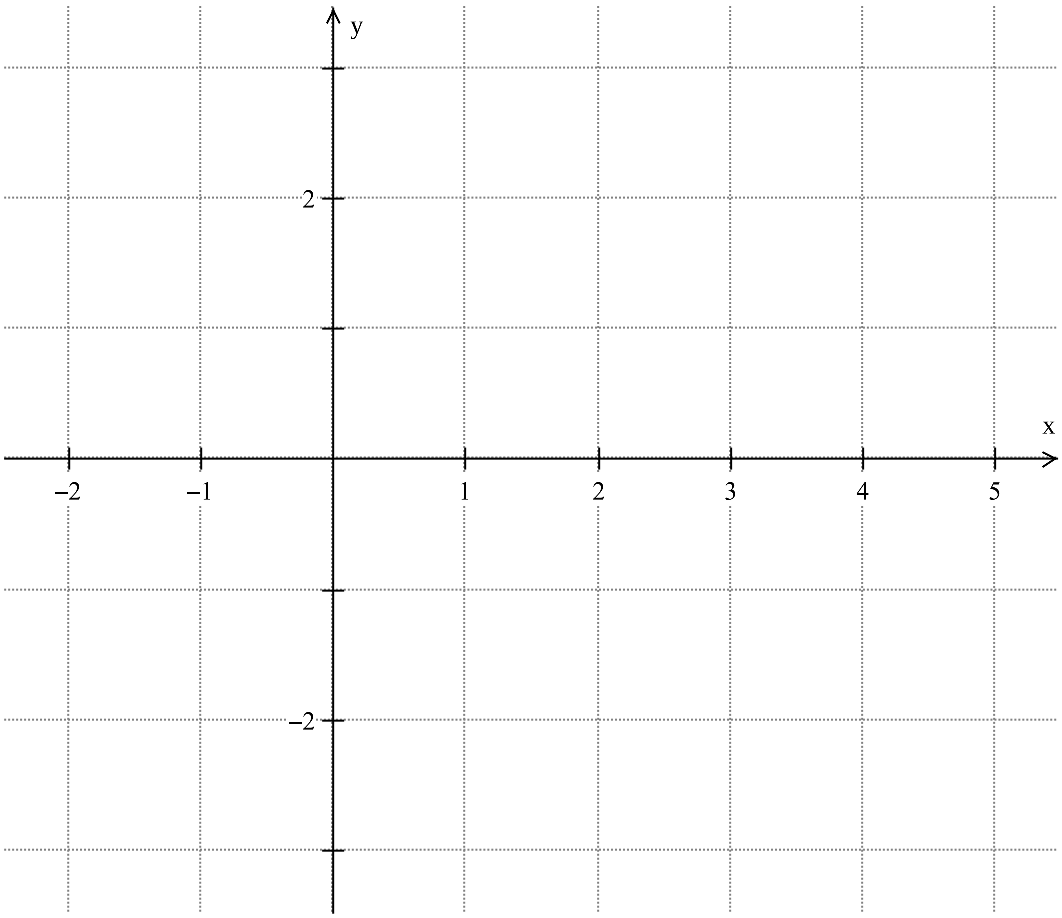
14. $f(x)=\begin{cases} \sin(x-1), & \text{if } x<1 \\ 0 \quad , & \text{if } x=1 \\ \ln x, & \text{if } x>1 \end{cases}$ $\quad a = 1$

15. The function *f(x)* is defined on $[-2,4]$ by $f(x)=\begin{cases} \sin(x-1), & \text{if } -2 \le x < 1 \\ 0\quad, & \text{if } x=1 \\ -\ln x, & \text{if } 1 < x < 4 \end{cases}$

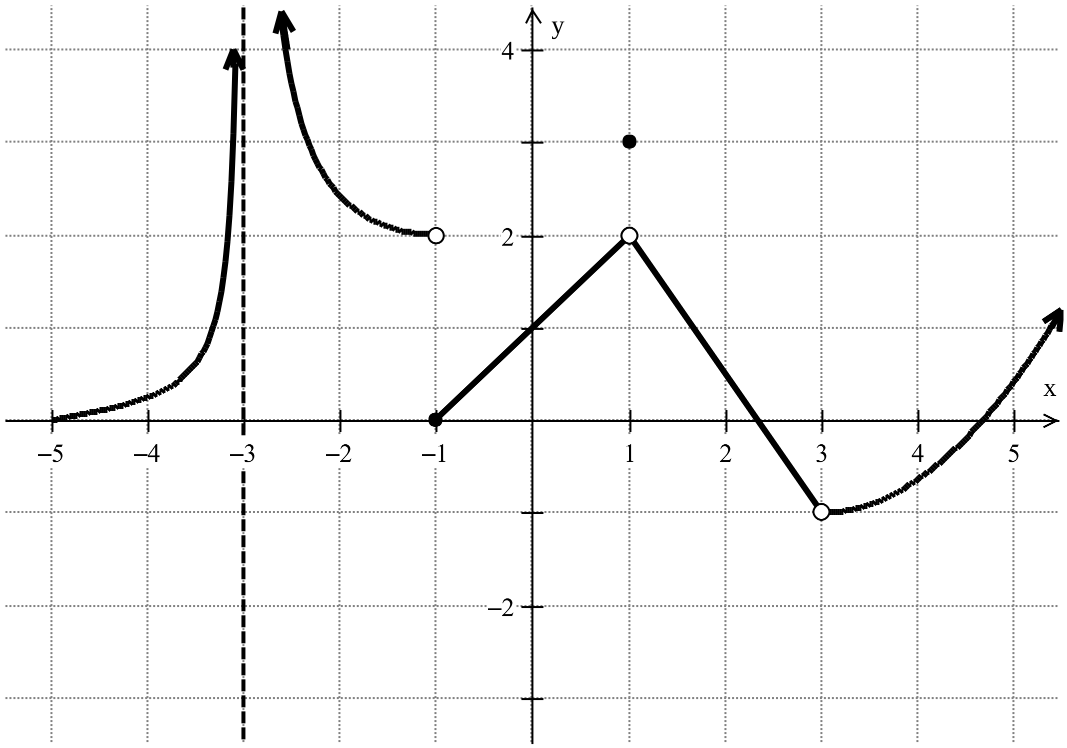
(a) Sketch the graph of *f(x)* on the axes provided.

(b) Determine where on $[-2,4]$ *f(x)* is not continuous. Explain.

(c) Determine where on $[-2,4]$ *f(x)* is not differentiable. Explain.



16. Given the graph of $f(x)$ below, find the values of the following:



a. $\lim_{x \to -3^-} f(x)$ b. $\lim_{x \to -3^+} f(x)$ c. $\lim_{x \to -3} f(x)$ d. $f(-3)$

e. $\lim_{x \to -1^-} f(x)$ f. $\lim_{x \to -1^+} f(x)$ g. $\lim_{x \to -1} f(x)$ h. $f(-1)$

i. $\lim_{x \to 1^-} f(x)$ j. $\lim_{x \to 1^+} f(x)$ k. $\lim_{x \to 1} f(x)$ l. $f(1)$

m. $\lim_{x \to 3^-} f(x)$ n. $\lim_{x \to 3^+} f(x)$ o. $\lim_{x \to 3} f(x)$ p. $f(3)$

Answers: Chapter 5: Limit and Continuity Test

1. The function f is differentiable at $x = b$. Which of the following statements could be false?

(a) $\lim_{x\to b} f(x)$ exists (b) $\lim_{x\to b} f(x) = f(b)$ (c) $\lim_{x\to b^-} f(x) = \lim_{x\to b^+} f(x)$

(d) $\lim_{x\to b^-} f'(x) = \lim_{x\to b^+} f'(x)$ (e) None of these

---

2. The function f is defined for all Reals such that $f(x) = \begin{cases} x^2 + kx & \text{for } x < 5 \\ 5\sin\frac{\pi}{2}x & \text{for } x \geq 5 \end{cases}$.

For which value of k will the function be continuous throughout its domain?

(a) -2 (b) -1 (c) $\frac{2}{3}$ (d) 1 (e) None of these

---

3. The graph of $y = \frac{1-\cos x}{x}$ has

I. a point of exclusion at $x = 0$
II. a horizontal asymptote at $y = 0$
III. an infinite number of zeros

(a) I only (b) II only (c) II and III only

(d) I and III only (e) I, II, and III

---

4. If $f(x) = \begin{cases} x+2 & \text{for } x \leq 3 \\ 4x-7 & \text{for } x > 3 \end{cases}$, which of the following statements are true?

I. $\lim_{x\to 3} f(x)$ exists II. f is continuous at $x = 3$ III. f is differentiable at $x = 3$

(a) None (b) I only (c) II only

(d) I and II only (e) I, II, and III

5. Which of the following functions is differentiable at $x = 0$?

(a) $f(x)=\sqrt{1+|x|}$ (b) $f(x)=|x|$ (c) $f(x)=\begin{cases} x^2\sin\frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$

(d) $f(x)=\begin{cases} \frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$ (e) $f(x)=\begin{cases} \cos x & \text{for } x<0 \\ \sin x & \text{for } x\geq 0 \end{cases}$

---

6. $\lim\limits_{h\to 0}\dfrac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}=$

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) 0 (d) $-\frac{\pi}{4}$ (e) DNE

---

7. $\lim\limits_{x\to 0}\dfrac{\int_0^x \sin t^2\,dt}{x^3}$

(a) 0 (b) 1 (c) $\frac{1}{3}$ (d) 3 (e) DNE

---

8. $\lim\limits_{x\to\infty}\dfrac{4x^5+3x^4+2x^3+x^2+1}{3x^5-9x^4+4x^3+15}=$

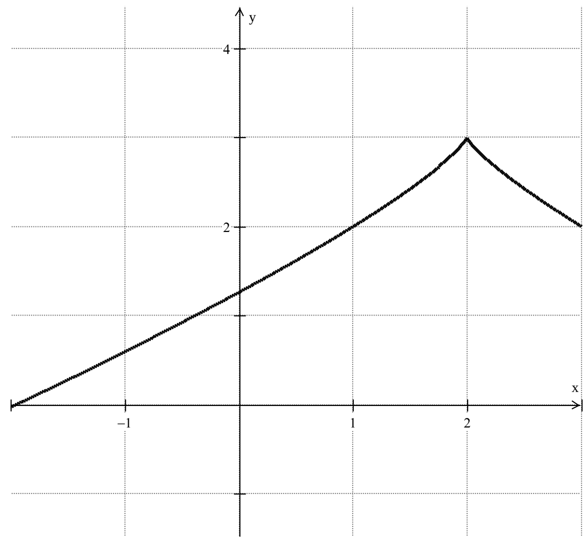
(a) 0 (b) $\frac{3}{4}$ (c) $\frac{4}{3}$ (d) 3 (e) DNE

9. The graph of a function f is given below. Which of the following statements are true?

I. $\lim_{x\to 2}\frac{f(x)-f(2)}{x-2}$ D.N.E.

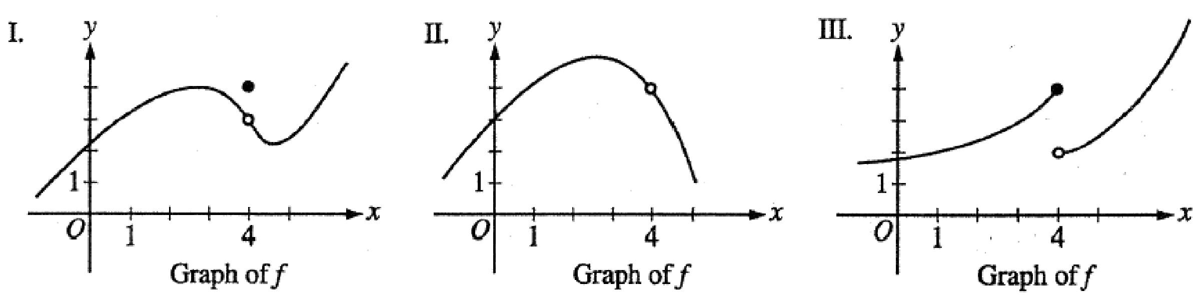
II. $\lim_{x\to 2} f(x)=4$

III. $\lim_{x\to 2} f(x)$ D.N.E.



(A)I only (B) II only (C) I and II (D) I, II, and III (E) II and III

10. For which of the following does $\lim_{x\to 4} f(x)$ exist?



(a) I only (b) II only (c) III only (d) I and II only (e) I and III only

For 11 through 14, find

a) Is $f(x)$ continuous at $x = a$? Why/Why not?

b) Is $f(x)$ differentiable at $x = a$? Why/Why not?

11. $f(x) = \begin{cases} \sin^{-1}\left[\pi(x-1)\right], & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ \ln x^2, & \text{if } x > 1 \end{cases}$ $a = 1$

a) continuous
b) not differentiable

12. $f(x) = \begin{cases} x^3 - 1, & \text{if } x \le 1 \\ 0, & \text{if } x = 1 \\ 1 + x^3, & \text{if } x > 1 \end{cases}$ $a = 1$

a) not continuous because the limit D.N.E.
b) not differentiable because it is not continuous

13. $f(x) = \begin{cases} e^{3-x}, & \text{if } x < 3 \\ -1 & \text{if } x = 3 \\ x^2 - 7x + 10, & \text{if } x > 3 \end{cases}$ $a = 3$

a) not continuous because the limit D.N.E.
b) not differentiable because it is not continuous

14. $f(x) = \begin{cases} \sin(x-1), & \text{if } x < 1 \\ 0, & \text{if } x = 1 \\ \ln x, & \text{if } x > 1 \end{cases}$ $a = 1$

a) continuous
b) differentiable

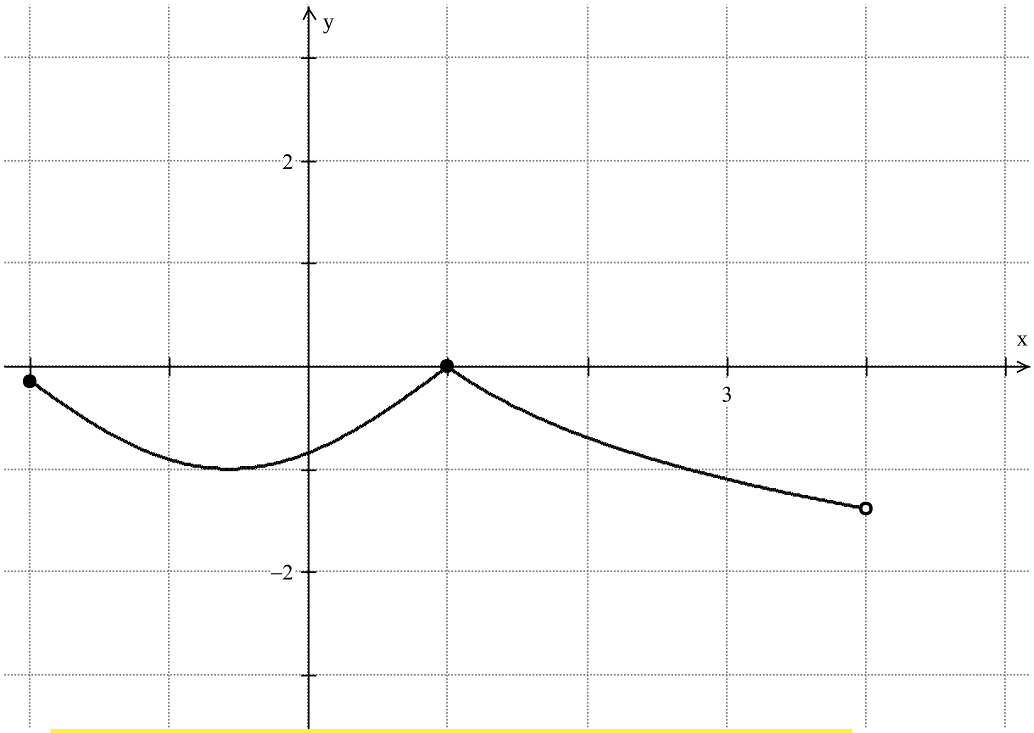
15. The function *f(x)* is defined on $\left[-2,4\right]$ by $f\left(x\right)=\begin{cases}\sin\left(x-1\right), & \text{if } -2\le x<1\\ 0\quad, & \text{if } x=1\\ -\ln x, & \text{if } 1<x<4\end{cases}$

(a) Sketch the graph of *f(x)* on the axes provided.

(b) Determine where on $\left[-2,4\right]$ *f(x)* is not continuous. Explain.

(c) Determine where on $\left[-2,4\right]$ *f(x)* is not differentiable. Explain.

a)



b) Not continuous at $x = 4$. There is no point at $x = 4$.

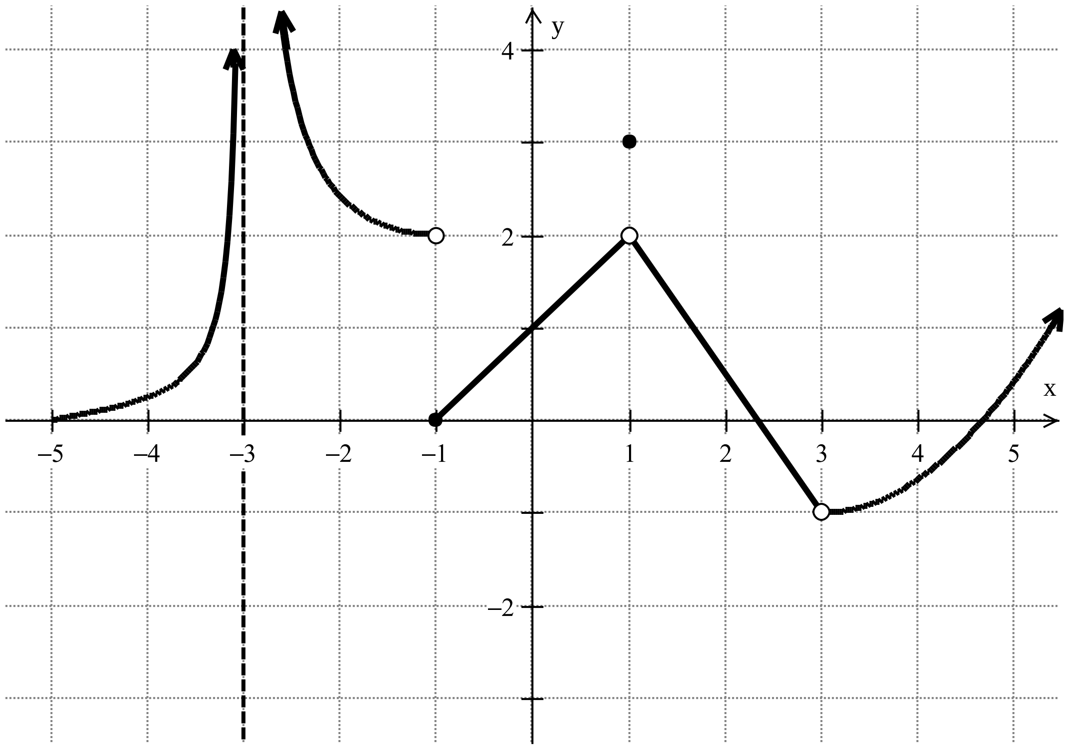
c) Not differentiable at $x = –2, 1, 4$

At $x = –2$, endpoint of curve

At $x = 1$, cusp

At $x = 4$, not continuous

16. Given the graph of $f(x)$ below, find the values of the following:



a. $\lim_{x\to -3^-} f(x) = \infty$

b. $\lim_{x\to -3^+} f(x) = \infty$

c. $\lim_{x\to -3} f(x) = \infty$

d. $f(-3)$ =D.N.E.

e. $\lim_{x\to -1^-} f(x) = 2$

f. $\lim_{x\to -1^+} f(x) = 0$

g. $\lim_{x\to -1} f(x)$ =D.N.E.

h. $f(-1) = 0$

i. $\lim_{x\to 1^-} f(x) = 2$

j. $\lim_{x\to 1^+} f(x) = 2$

k. $\lim_{x\to 1} f(x) = 2$

l. $f(1) = 3$

m. $\lim_{x\to 3^-} f(x) = -1$

n. $\lim_{x\to 3^+} f(x) = -1$

o. $\lim_{x\to 3} f(x) = -1$

p. $f(3)$ =D.N.E.

Chapter 6 Overview: Applications of Integrals

Calculus, like most mathematical fields, began with trying to solve everyday problems. The theory and operations were formalized later. As early as 270 BC, Archimedes was working on the problem of problem of finding the volume of a non-regular shapes. Beyond his bathtub incident that revealed the relationship between weight volume and displacement, He had actually begun to formalize the limiting process to explore the volume of a diagonal slice of a cylinder.

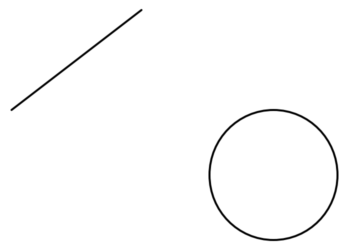
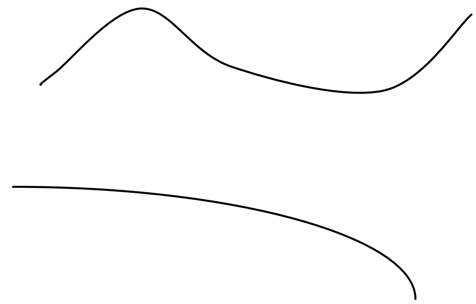
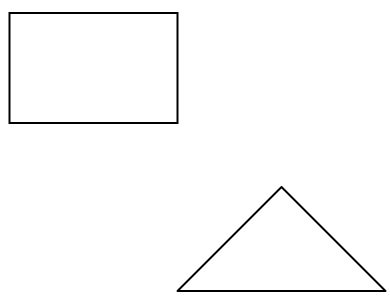
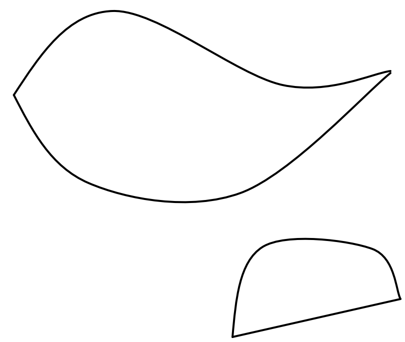
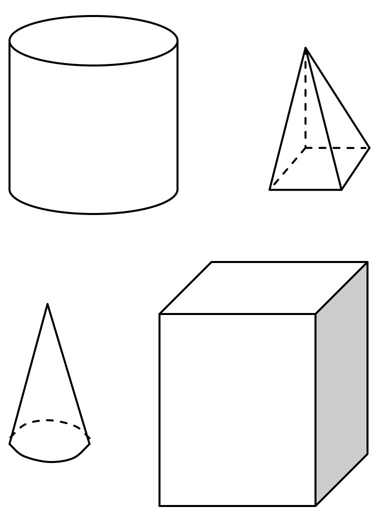
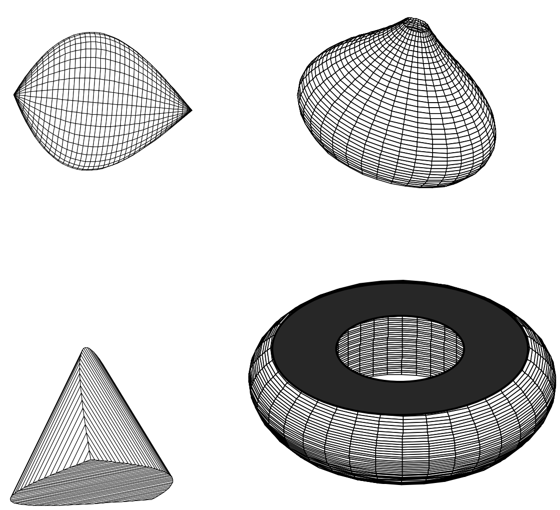
This is where Calculus can give us some very powerful tools. In geometry, we can find lengths of specific objects like line segments or arcs of circles, while in Calculus we can find the length of any arc that we can represent with an equation. The same is true with area and volume. Geometry is very limited, while Calculus is very open-ended in the problems it can solve. On the next page is an illustration of the difference.

In this chapter, we will investigate what have become the standard applications of the integral:

- Area
- Volumes of Rotation
- Volume by Cross-Section
- Arc Length

Just as AP, we will emphasize how the formulas relate to the geometry of the problems and the technological (graphing calculator) solutions rather than the algebraic solutions.

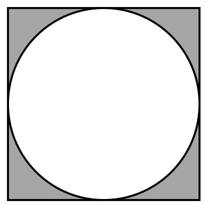
Below is an illustration of what we can accomplish with Calculus as opposed to geometry:

	Geometry	**Calculus**
Length		
Area		
Volume		

Calculus can also be used to generate surface areas for odd-shaped solids as well, but that is out of the scope of this class.

6.1 Area Between Two Curves

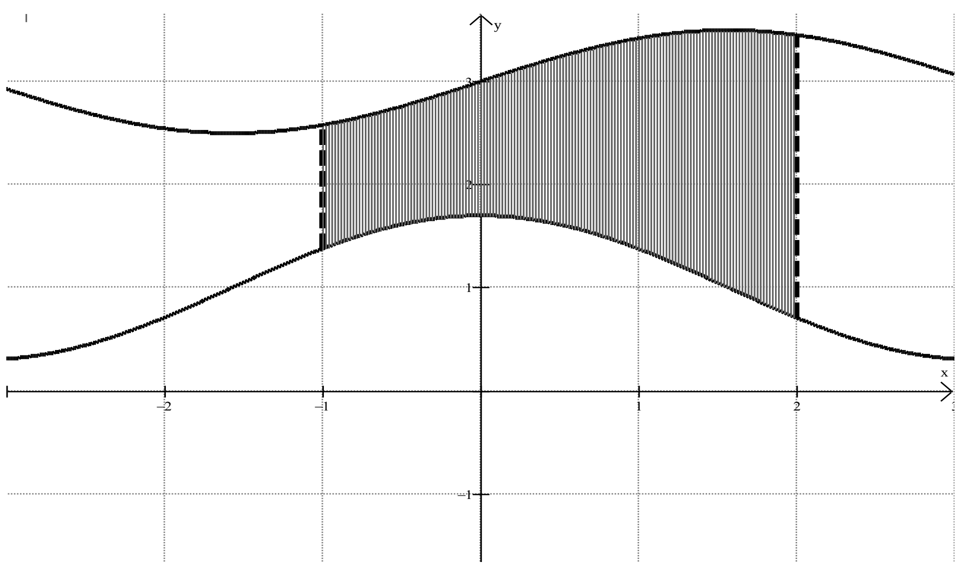
We have learned that the area under a curve is associated with an integral. But what about the area between two curves? It turns out that it is a simple proposition, very similar to some problems we encountered in geometry. In geometry, if we wanted to know the area of a shaded region that was composed of multiple figures, like the illustration below, we find the area of the larger and subtract the area of the smaller.



The area for this figure would be

$$A_{Total} = A_{square} - A_{circle} = s^2 - \pi\left(\frac{s}{2}\right)^2$$

Similarly, since the integral gets us a numeric value for the area between a curve and an axis, if we simply subtract the "smaller" curve from the "larger" one and integrate, we can find the area between two curves.



Objectives:
Find the area of the region between two curves.

Unlike before, when we had to be concerned about positives and negatives from a definite integral messing up our interpretation of area, the subtraction takes care of the negative values for us (if one curve is under the axis, the subtraction makes the negative value of the integral into the positive value of the area.

<u>Area Between Two Curves:</u>
The area of the region bounded by the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$ where f and g are continuous and $f \geq g$ for all x in $[a,b]$ is

$$A = \int_a^b \left[f(x) - g(x) \right] dx$$

You can also think of this expression as the '**top**' curve minus the '**bottom**' curve. If we associate integrals as 'area under a curve' we are finding the area under the top curve, subtracting the area under the bottom curve, and that leaves us with the area between the two curves.

The area of the region bounded by the curves $x = f(y)$, $x = g(y)$, and the lines $y = c$ and $y = d$ where f and g are continuous and $f \geq g$ for all y in $[c,d]$ is

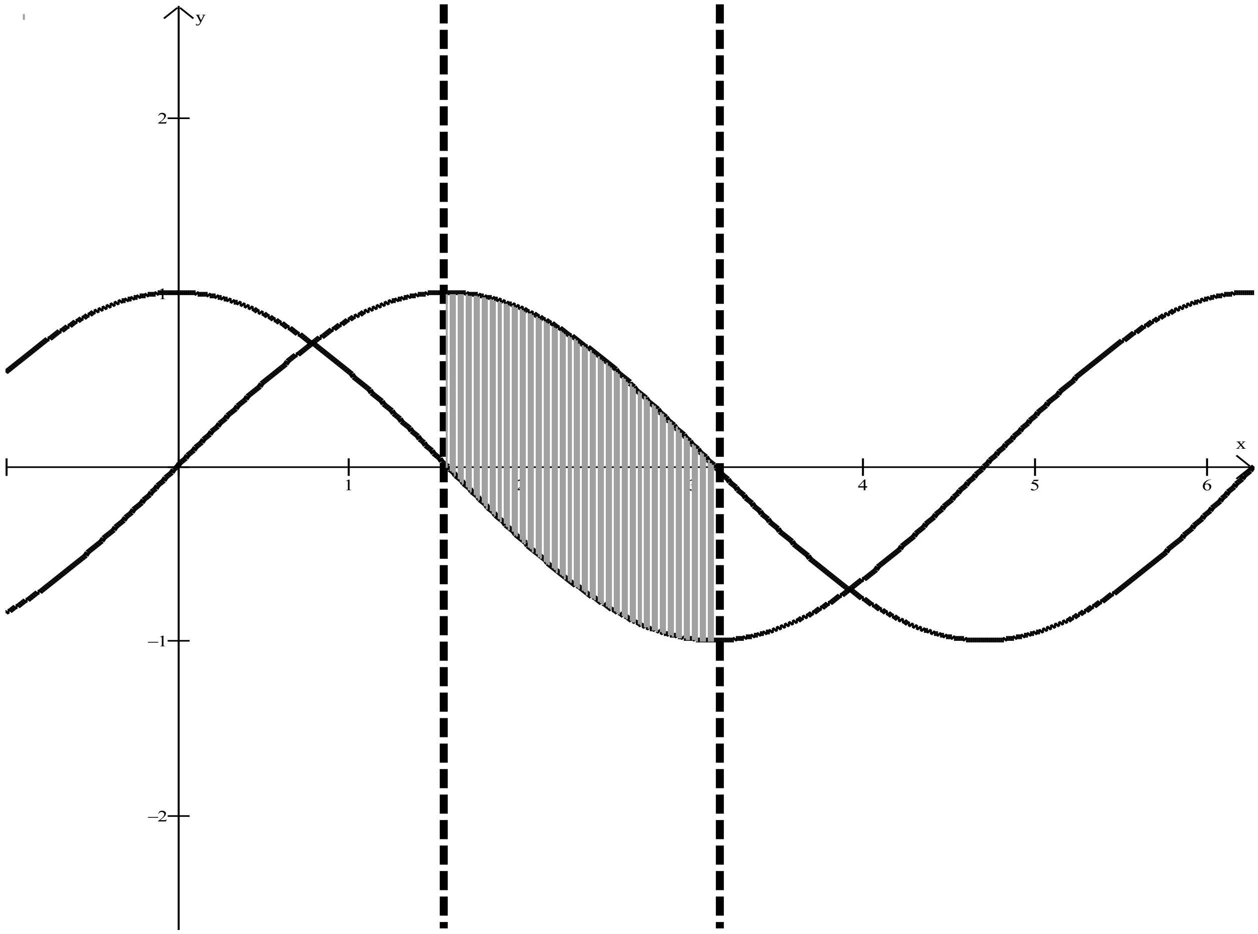
$$A = \int_c^d \left[f(y) - g(y) \right] dy$$

You can also think of this expression as the '**right**' curve minus the '**left**' curve.

<u>Steps to Finding the Area of a Region:</u>

1. Draw a picture of the region.
2. Sketch a Reimann rectangle – if the rectangle is vertical your integral will have a *dx* in it, if your rectangle is horizontal your integral will have a *dy* in it.
3. Determine an expression representing the length of the rectangle: (top – bottom) or (right – left).
4. Determine the endpoints the boundaries (points of intersection).
5. Set up an integral containing the limits of integration, which are the numbers found in Step 4, and the integrand, which is the expression found in Step 3.
6. Solve the integral.

Ex 1 Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = \frac{\pi}{2}$, and $x = \pi$.



$A = \int_a^b \left[f(x) - g(x) \right] dx$ Our pieces are perpendicular to the x – axis, so our integrand will contain dx

$A = \int_a^b \left[\sin(x) - \cos(x) \right] dx$ On our interval $\sin(x)$ is greater than $\cos(x)$

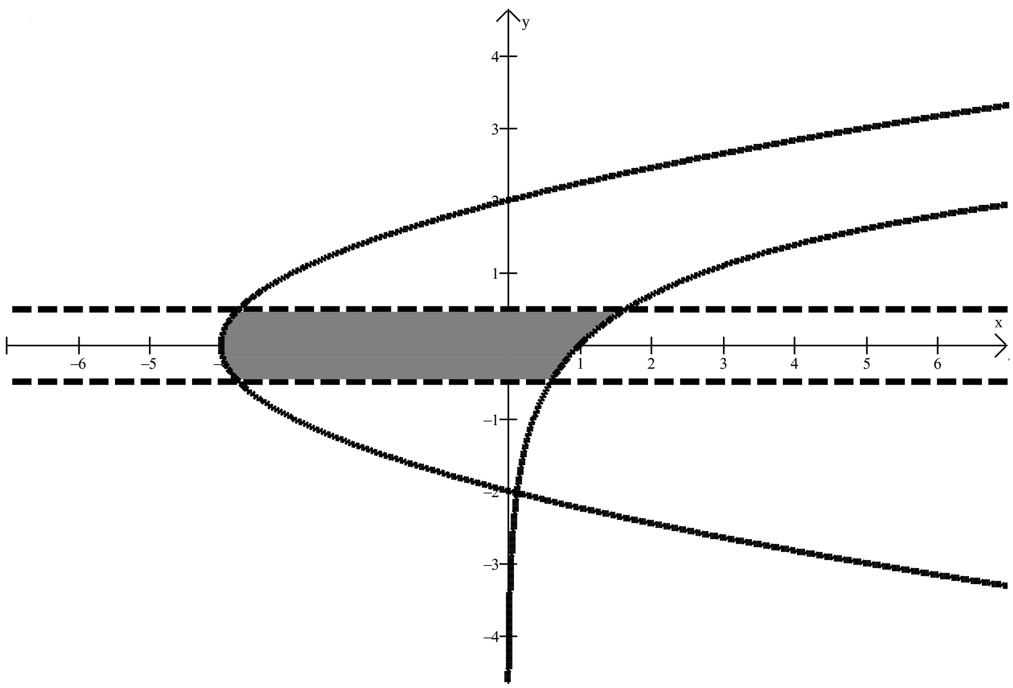
$A = \int_{\pi/2}^{\pi} \left[\sin(x) - \cos(x) \right] dx$ Our region extends from $x = \frac{\pi}{2}$ to $x = \pi$

$$A = \left[-\cos(x) - \sin(x) \right]_{\pi/2}^{\pi}$$

$$A = \left(-\cos(\pi) - \sin(\pi) \right) - \left(-\cos\left(\frac{\pi}{2}\right) - \sin\left(\frac{\pi}{2}\right) \right) = 2$$

What if the functions are not defined in terms of *x* but in terms of *y* instead?

Ex 2   Find the area of the region bounded by $x = e^y$, $x = y^2 - 4$, $y = \frac{1}{2}$, and $y = -\frac{1}{2}$.



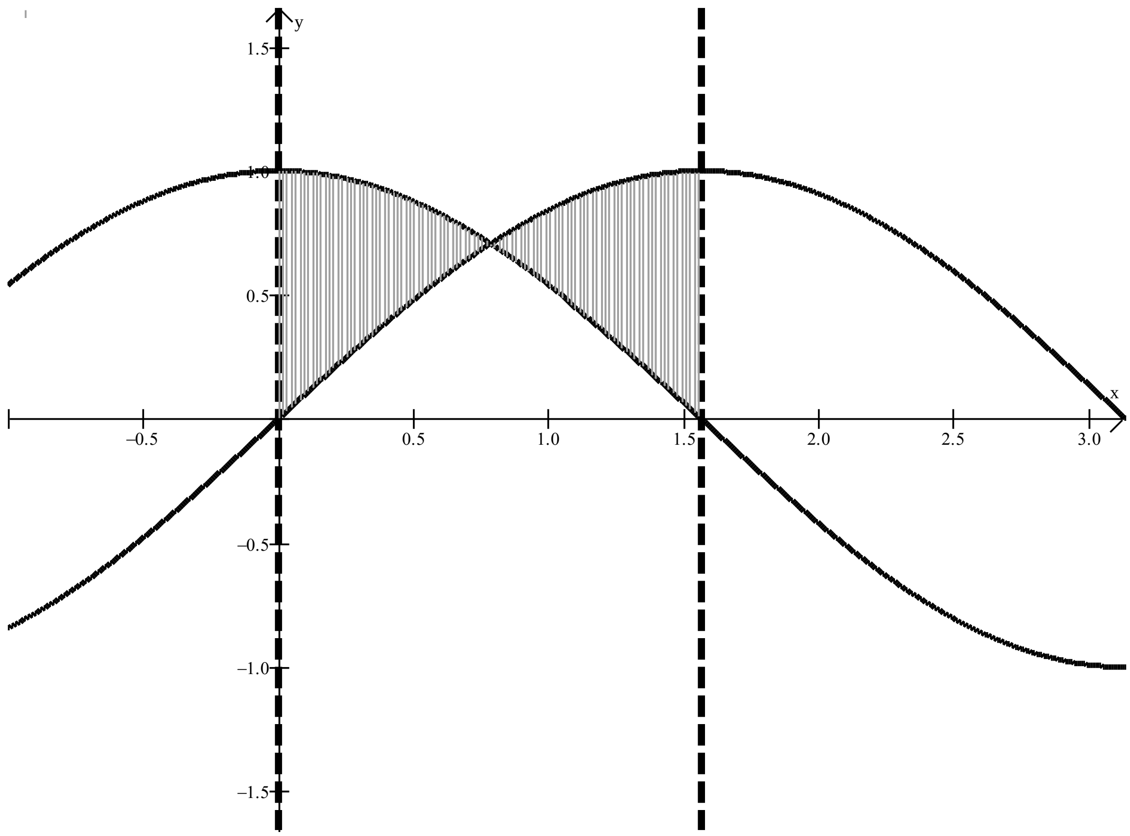
$A = \int_c^d \left[f(y) - g(y) \right] dy$ Our pieces are perpendicular to the y – axis, so our integrand will contain dy

$A = \int_c^d \left[e^y - \left(y^2 - 4 \right) \right] dy$ On our interval e^y is greater than $y^2 - 4$

$A = \int_{-1/2}^{1/2} \left[e^y - \left(y^2 - 4 \right) \right] dy$ Our region extends from $y = -\frac{1}{2}$ to $y = \frac{1}{2}$

$A = 4.959$

Ex 3 Find the area of the region bounded by $y = \sin x$, $y = \cos x$, $x = 0$, and $x = \frac{\pi}{2}$.



$$A = \int_0^{\pi/4} \left[\cos(x) - \sin(x)\right] dx + \int_{\pi/4}^{\pi/2} \left[\sin(x) - \cos(x)\right] dx = .828$$

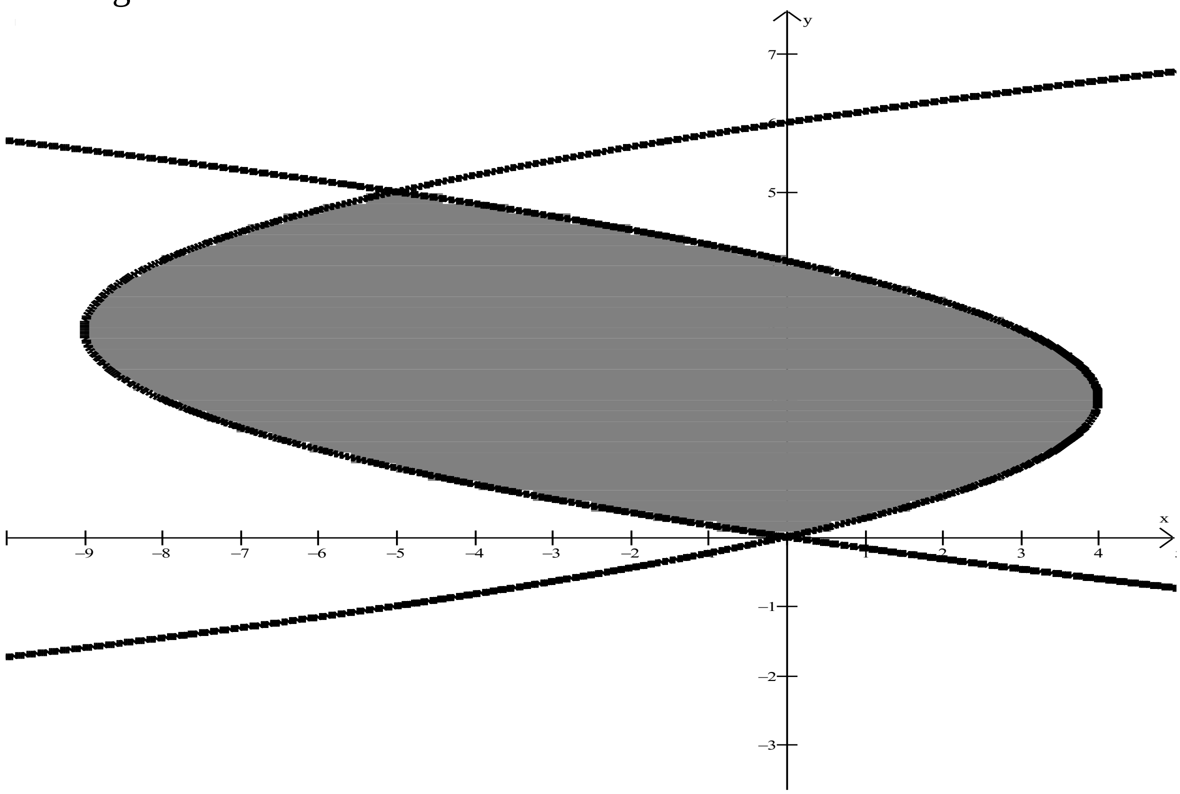
With this problem, we had to split it into two integrals, because the "top" and "bottom" curves switch partway through the region. We could have also done this with an absolute value:

$$A = \int_0^{\pi/2} \left|\cos(x) - \sin(x)\right| dx = .828$$

If we are asked this question on an AP test or in a college classroom, they usually want to see the setup. The absolute value is just an easy way to integrate using the calculator.

NOTE: Any integral can be solved going dy or dx. It is usually the case though, that one is "easier" than the other.

Ex 4   Sketch the graphs of $x = y^2 - 6y$ and $x = 4y - y^2$. If asked to find the area of the region bounded by the two curves which integral would you choose – an integral with a dy and an integral with a dx? Both will work but one is less time consuming.



We can see that if we went dx we would need three integrals (because the tops and bottoms are different on three different sections), but if we go dy we only need one integral. I like integrals as much as the next person, but I'll just do the one integral if that's OK right now.

$$A = \int_c^d \left[f(y) - g(y) \right] dy$$

$$A = \int_0^5 \left[\left(4y - y^2\right) - \left(y^2 - 6y\right) \right] dy$$

$$= \int_0^5 \left(10y - 2y^2\right) dy$$

$$= 5y^2 - \frac{2}{3} y^3 \Big]_0^5$$

$$A = 125/3$$

6.1 Homework

Draw and find the area of the region enclosed by the given curves.

1. $y = x + 1, y = 9 - x^2, x = -1, x = 2$

2. $y = \sin x, y = e^x, x = 0, x = \frac{\pi}{2}$

3. $y = x^2, y = 8 - x^3, x = -3, x = 3$

4. $y = \sin 2x, y = \cos x, x = 0, x = \dfrac{\pi}{2}$

5. $y = 5x - x^2, y = x$

6. $y = 1 + \sqrt{x}, y = \dfrac{3 + x}{3}$

7. $x = e^y, x = y^2 - 2, y = -1, y = 1$

8. $x = 1 - y^2, x = y^2 - 1$

9. $y = \sqrt{x+2}, y = \dfrac{1}{x^2}, x = 1, x = 2$

10. $x = 2y - y^2, x = y^2 - 4y$

Use your grapher to sketch the regions described below. Find the points of intersection and find the area of the region described region.

11. $y = \sqrt{x}, y = e^{-2x}, x = 1$

12. $y = \ln\left(x^2 + 1\right), y = \cos x$

13. $y = x^2, y = 2^x$

Answers: 6.1 Homework

1. $y=x+1, y=9-x^2, x=-1, x=2$

   A=19.5

2. $y=\sin x, y=e^x, x=0, x=\dfrac{\pi}{2}$

   A=2.810

3. $y=x^2, y=8-x^3, x=-3, x=3$

   A=60.252

4. $y=\sin 2x, y=\cos x, x=0, x=\dfrac{\pi}{2}$

   A=0.5

5. $y=5x-x^2, y=x$

   A=$\dfrac{32}{3}$

6. $y=1+\sqrt{x}, y=\dfrac{3+x}{3}$

   A=4

7. $x=e^y, x=y^2-2, y=-1, y=1$

   A=5.684

8. $x=1-y^2, x=y^2-1$

   A=$\dfrac{8}{3}$

9. $y=\sqrt{x+2}, y=\dfrac{1}{x^2}, x=1, x=2$

   A=1.369

10. $x=2y-y^2, x=y^2-4y$

    A=9

11. $y=\sqrt{x}, y=e^{-2x}, x=1$

    A=0.350

12. $y=\ln\left(x^2+1\right), y=\cos x$

    A=1.168

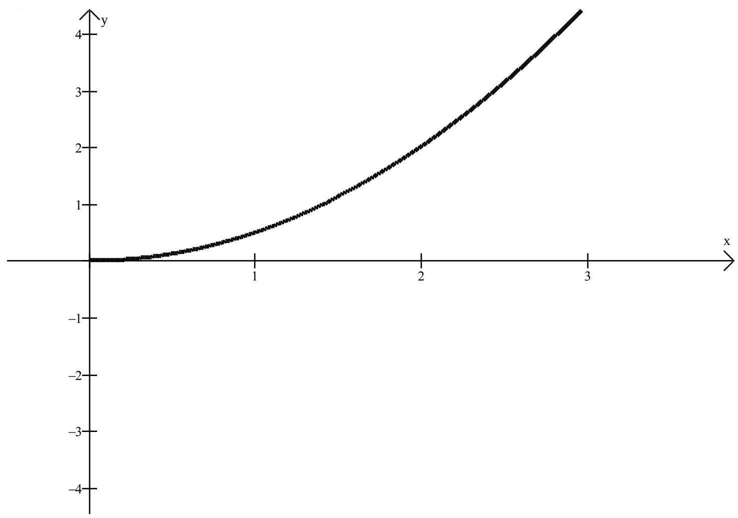
13. $y=x^2, y=2^x$

    A=2.106

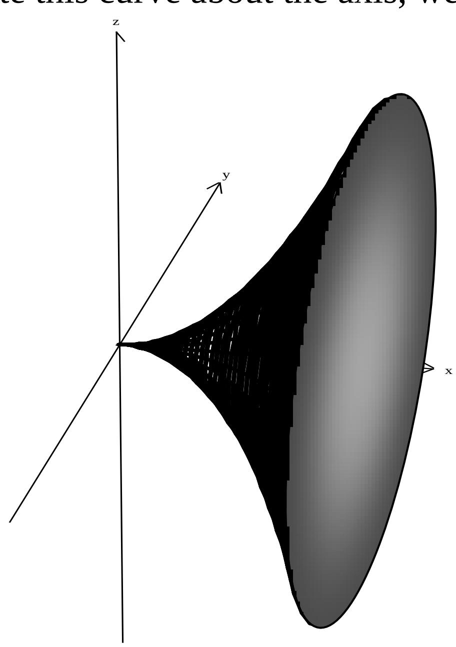
6.2 Volume by Rotation about an Axis

We know how to find the volume of many objects (remember those geometry formulas?) i.e. cubes, spheres, cones, cylinders … but what about non-regular shaped object? How do we find the volumes of these solids? Luckily we have Calculus, but the basis of the Calculus formula is actually geometry.

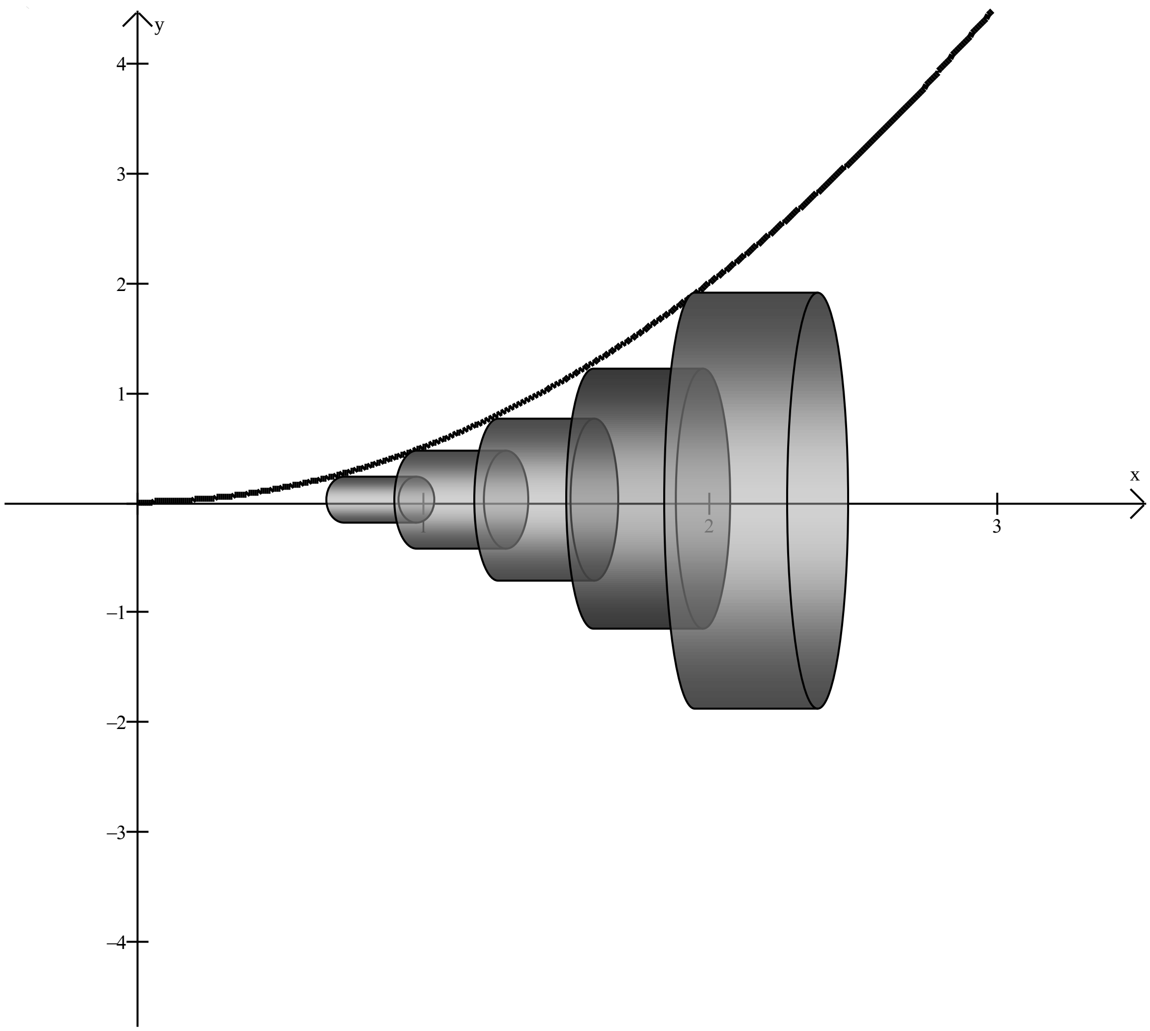
If we take a function (like the parabola below) and rotate it around the *x*-axis on an interval, we get a shape that does not look like anything we could solve with geometry.



When we rotate this curve about the axis, we get a shape like the one below:



This is sort of like a cone, but because the surface curves inward its volume would be less than a cone with the same size base. In fact, we cannot use a simple geometry formula to find the volume. But if we begin by thinking of the curve as having Reimann rectangles, and rotate those rectangles, the problem becomes a little more obvious:



Each of the rotated rectangles becomes a cylinder. The volume of a cylinder is

$$V = \pi r^2 h$$

Note that the radius (r) is the distance from the axis to the curve (which is $f(x)$) and the height is the change in x (Δx), giving us a volume of

$$V = \pi\left(f(x)\right)^2 \Delta x$$

To find the total volume, we would simply add up all of the individual volumes on the interval at which we are looking (let's say from $x = a$ to $x = b$).

$$V_{total} = \sum_{x=a}^{b} \pi \left(f\left(x\right) \right)^2 \Delta x$$

Of course, this would only give us an approximation of the volume. But if we made the rectangles very narrow, the height of our cylinders changes from Δx to dx, and to add up this infinite number of terms with these infinitely thin cylinders, the $\sum_{x=a}^{b}$ becomes $\int_a^b$ giving us the formula

$$V_{total} = \pi \int_a^b \left(f\left(x\right) \right)^2 dx$$

What you may realize is that the $f\left(x\right)$ is simply the radius of our cylinder, and we usually write the formula with an *r* instead. The reason for this will become more obvious when we rotate about an axis that is not an *x*- or *y*-axis.

And since integration works whether you have *dx* or *dy*, this same process works for curves that are rotated around the *y*-axis and are defined in terms of *y* as well.

<u>Volume by Disc Method (Part 1):</u> The volume of the solid generated when the function $f\left(x\right)$ from $x = a$ and $x = b$, where $f\left(x\right) \geq 0$, is rotated about the x – axis [or $g\left(y\right)$ from $y = c$ and $y = d$, where $g\left(y\right) \geq 0$, is rotated about the y – axis is given by

$$V = \pi \int_a^b \left[f\left(x\right) \right]^2 dx \text{ or } V = \pi \int_c^d \left[g\left(y\right) \right]^2 dy$$

or

$$V = \pi \int_a^b r^2 dx$$

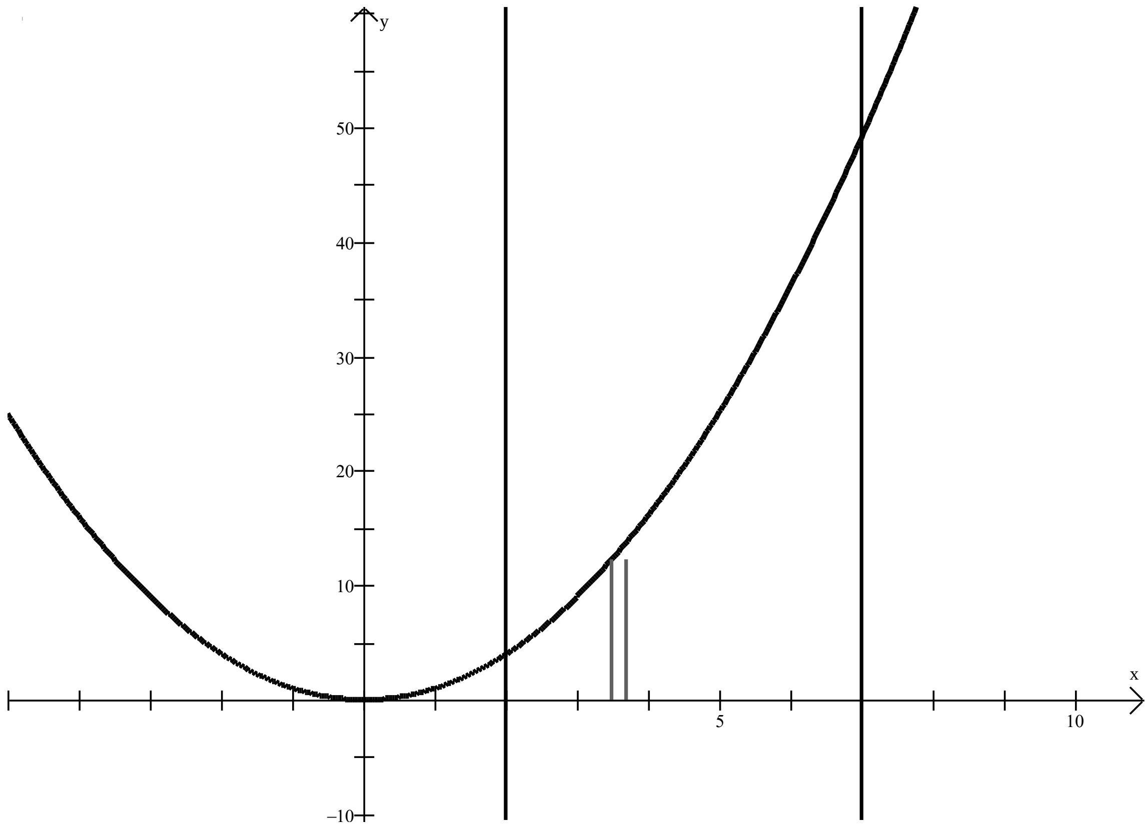
where *r* is the height (or length if they are horizontal) of your Riemann rectangle.

Steps to Finding the Volume of a Solid:

1. Draw a picture of the region to be rotated.
2. Draw the axis of rotation.
3. Sketch a Reimann rectangle – if the piece is vertical your integral will have a *dx* in it, if your piece is horizontal your integral will have a *dy* in it. Your rectangle should always be sketched perpendicular to the axis of rotation.
4. Determine an expression representing the radius of the rectangle (in these cases, the radius is the function value).
5. Determine the endpoints the region covers.
6. Set up an integral containing π outside the integrand, the limits of integration are the numbers found in Step 5, and the integrand is the expression found in Step 4.

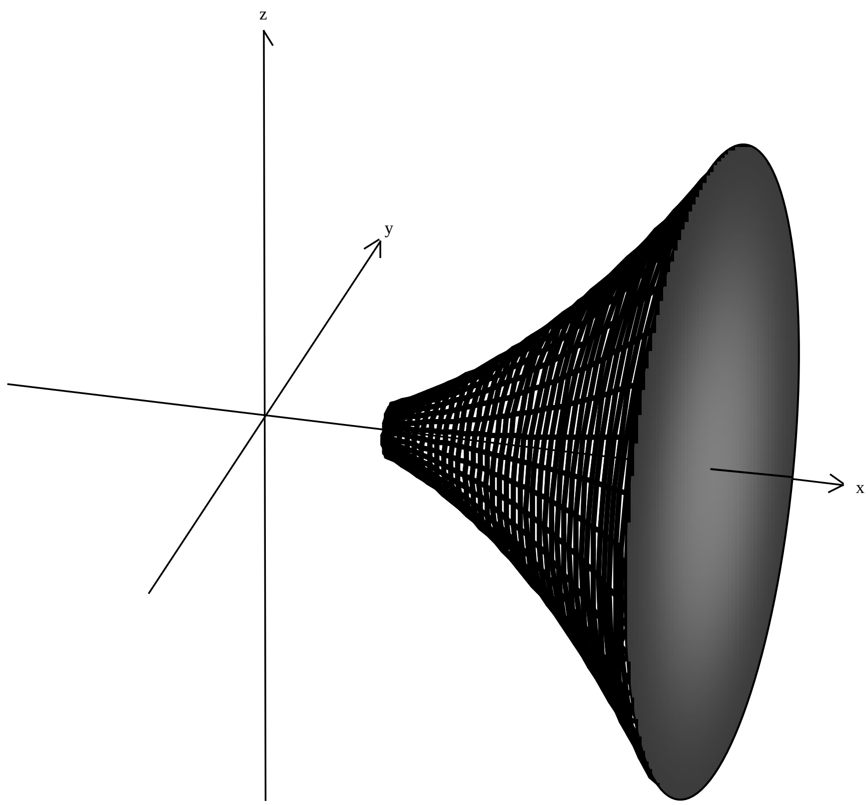
Objective: Find the volume of a solid rotated when a region is rotated about a given line.

Ex 1 Let R be the region bounded by the equations $y = x^2$, the x – axis, $x = 2$, and $x = 7$. Find the volume of the solid generated when R is rotated about the x – axis.



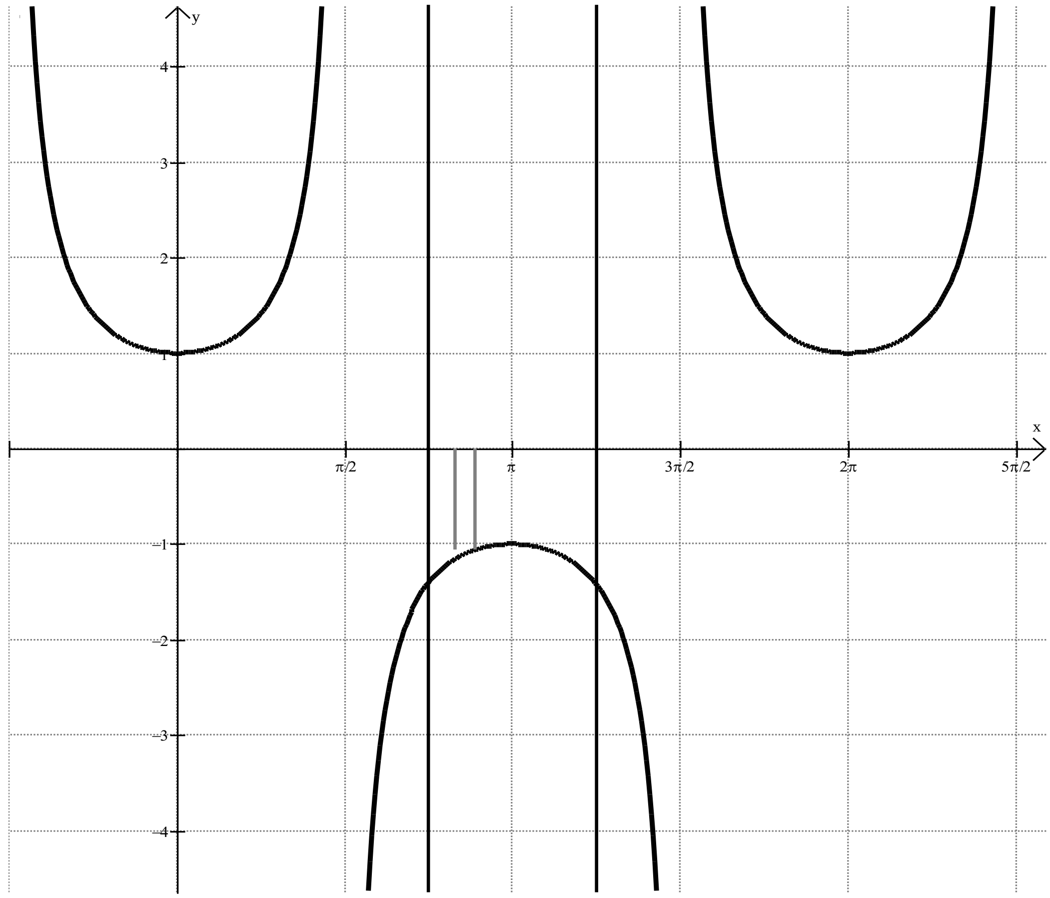
$V = \pi\int_a^b r^2 dx$	Our pieces are perpendicular to the x – axis, so our integrand will contain dx
$V = \pi\int_a^b \left[y\right]^2 dx$	We cannot integrate y with respect to x so we will substitute out for y
$V = \pi\int_a^b \left[x^2\right]^2 dx$	The expression for y is x^2
$V = \pi\int_2^7 \left[x^2\right]^2 dx$	Our region extends from $x = 2$ to $x = 7$
$V = 3355\pi$	

When the region, R, is rotated as in the example above, the solid generated would look like this:



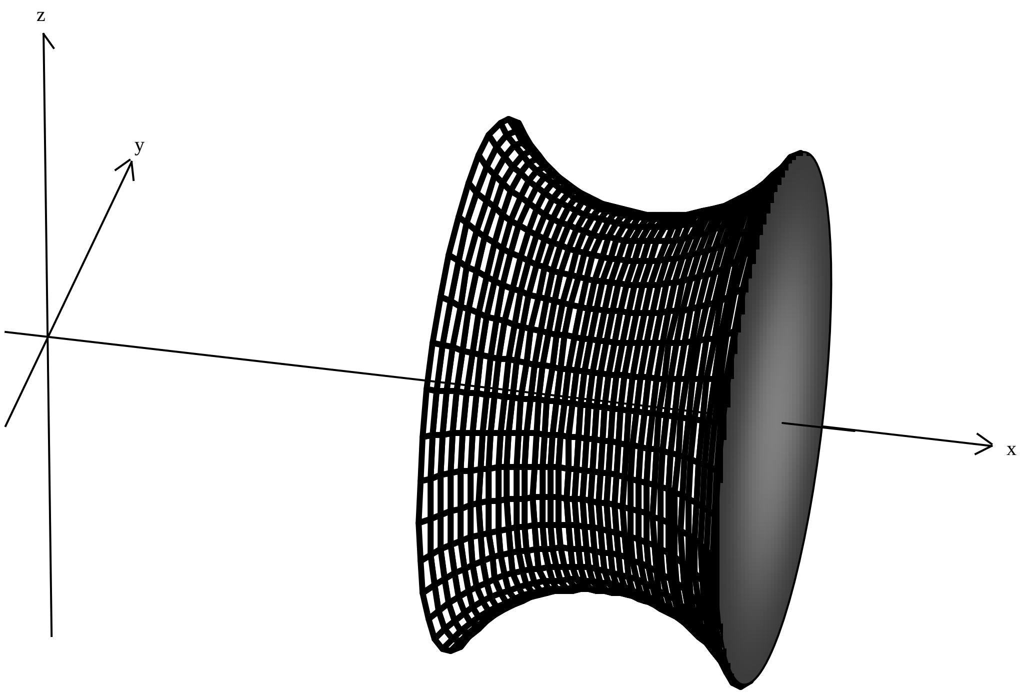
Note that knowing that this is what the rotated solid looks like has no bearing on the math. It makes an interesting shape, but being able to draw the image and being able to generate the volume by integration are two completely separate things. We are much more concerned with finding the volume.

Ex 2 Let R be the region bounded by the curves $y=\sec x$, the x – axis, $x=\dfrac{3\pi}{4}$, and $x=\dfrac{5\pi}{4}$. Find the volume of the solid generated when R is rotated about the x – axis.



$V=\pi\int_a^b r^2dx$	Our pieces are perpendicular to the x – axis, so our integrand will contain dx
$V=\pi\int_a^b y^2dx$	We cannot integrate y with respect to x so we will substitute out for y
$V=\pi\int_a^b\left[\sec x\right]^2 dx$	The expression for y is $\sec x$
$V=\pi\int_{3\pi/4}^{5\pi/4}\left[\sec x\right]^2 dx$	Our region extends from $x=\dfrac{3\pi}{4}$ to $x=\dfrac{5\pi}{4}$
$V=2\pi$	

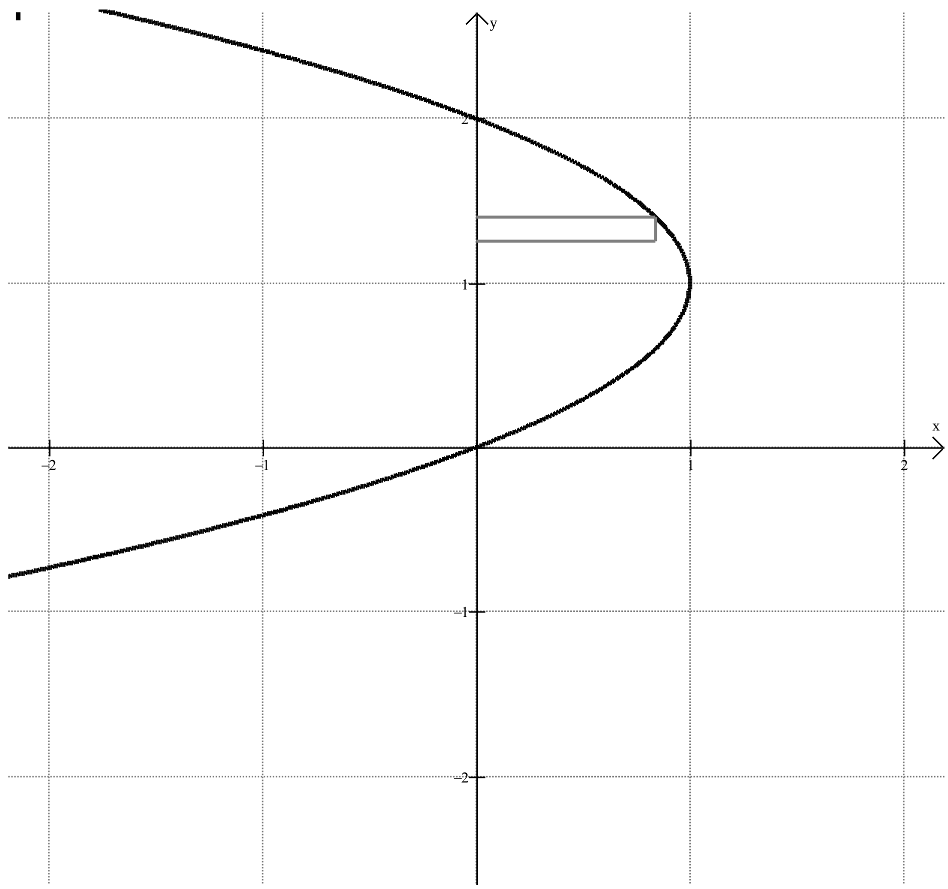
Again, when the region, *S*, is rotated as in the example above, the solid generated would look like this:



It's pretty cool-looking, but of no great consequence in terms of the math.

What if we had a function defined in terms of *y*?

Ex 3 Find the volume of the solid obtained by rotating about the y – axis the region bounded by $x = 2y - y^2$ and the y – axis.



$V = \pi \int_c^d r^2 dy$	Our pieces are perpendicular to the y – axis, so our integrand will contain dy
$V = \pi \int_c^d x^2 dy$	We cannot integrate x with respect to y so we will sub out for x
$V = \pi \int_c^d \left[2y - y^2\right]^2 dy$	The expression for x is $2y - y^2$
$V = \pi \int_0^2 \left[2y - y^2\right]^2 dy$	Our region extends from $y = 0$ to $y = 2$
$V = \dfrac{16\pi}{15}$	

Suppose we wanted to find the volume of a region bounded by two curves that was then rotated about an axis.

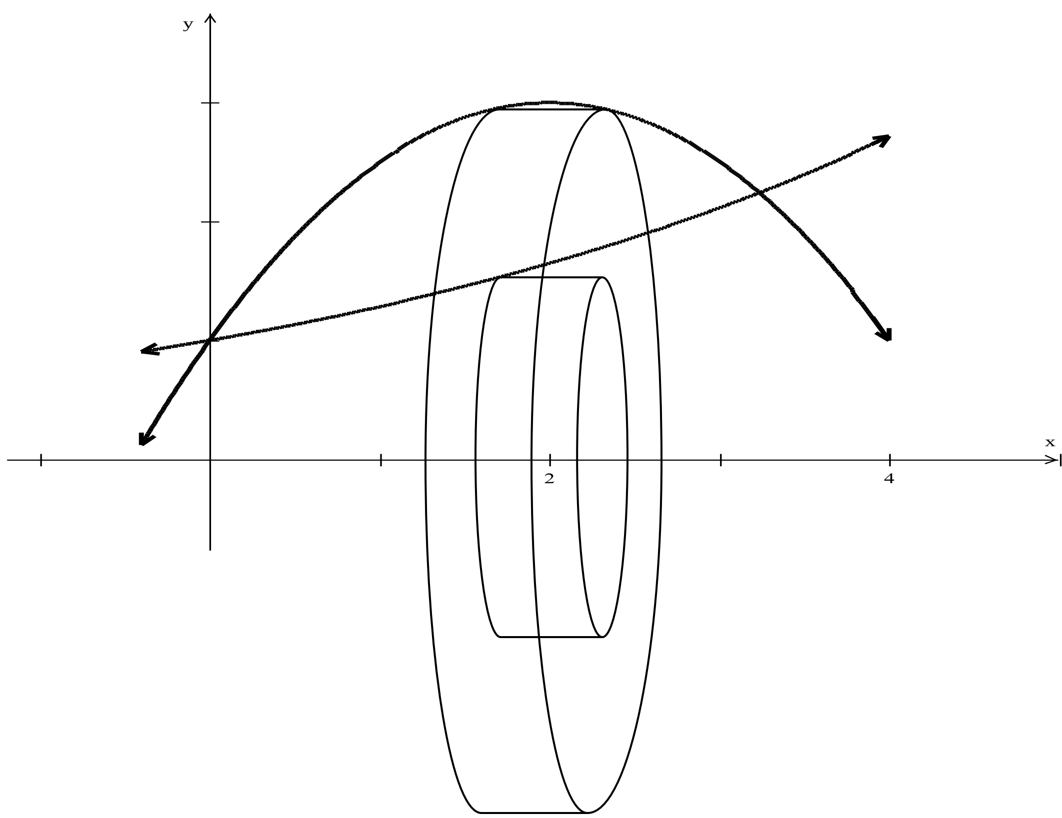
<u>Volume by Washer Method:</u> The volume of the solid generated when the region bounded by functions $f(x)$ and $g(x)$, from $x = a$ and $x = b$, where $f(x) \geq g(x)$ [or $f(y)$ and $g(y)$, from $y = c$ and $y = d$, where $f(y) \geq g(y)$], is rotated about the x – axis is given by

$$V = \pi\int_a^b \left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx \quad \text{or} \quad V = \pi\int_c^d \left(\left[f(y)\right]^2 - \left[g(y)\right]^2\right)dy$$

or

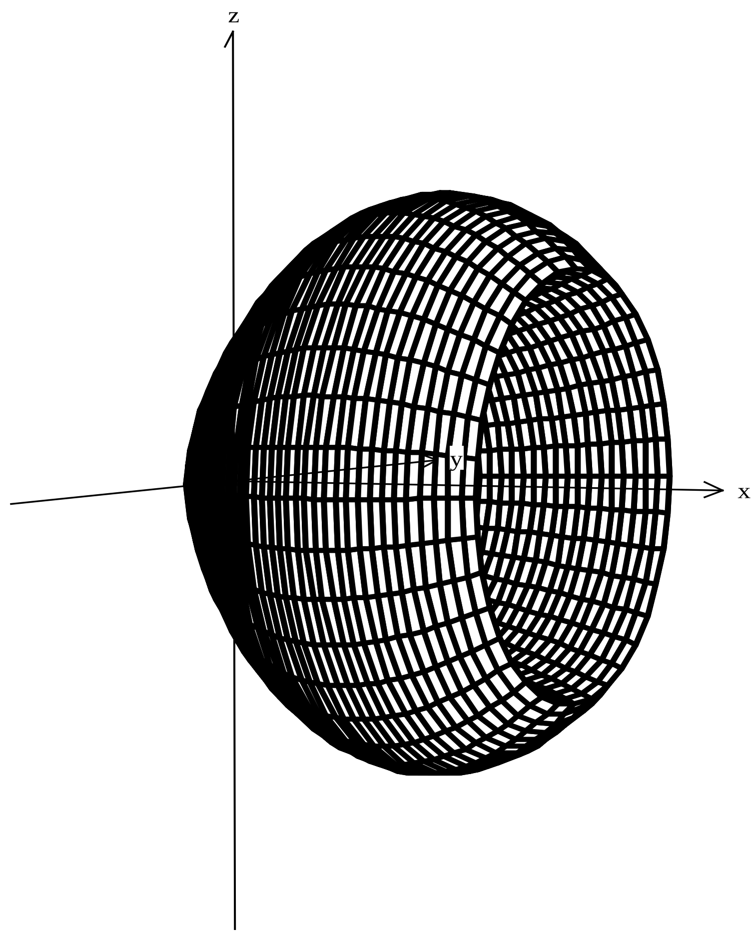
$$V = \pi\int_a^b \left(R^2 - r^2\right)dx \text{ or } V = \pi\int_c^d \left(R^2 - r^2\right)dy$$

Where R is the outer radius of your Riemann rectangle and r is the inner radius of your Riemann rectangle.



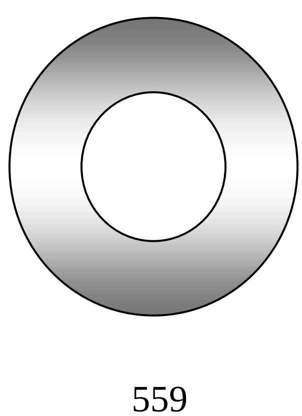
Again, think of taking a tiny strip of this region, a Riemann rectangle, and rotating it about the x – axis. Note that this gives us a cylinder (albeit a very narrow one) with another cylinder cut out of it. This is why the formula is what we stated above.

If we took all of those strips for the above example, the solid would look like this.



Notice that overall, this does not look like a "washer" per se, but if you cut a cross section, you would see washer shapes like the one illustrated below:

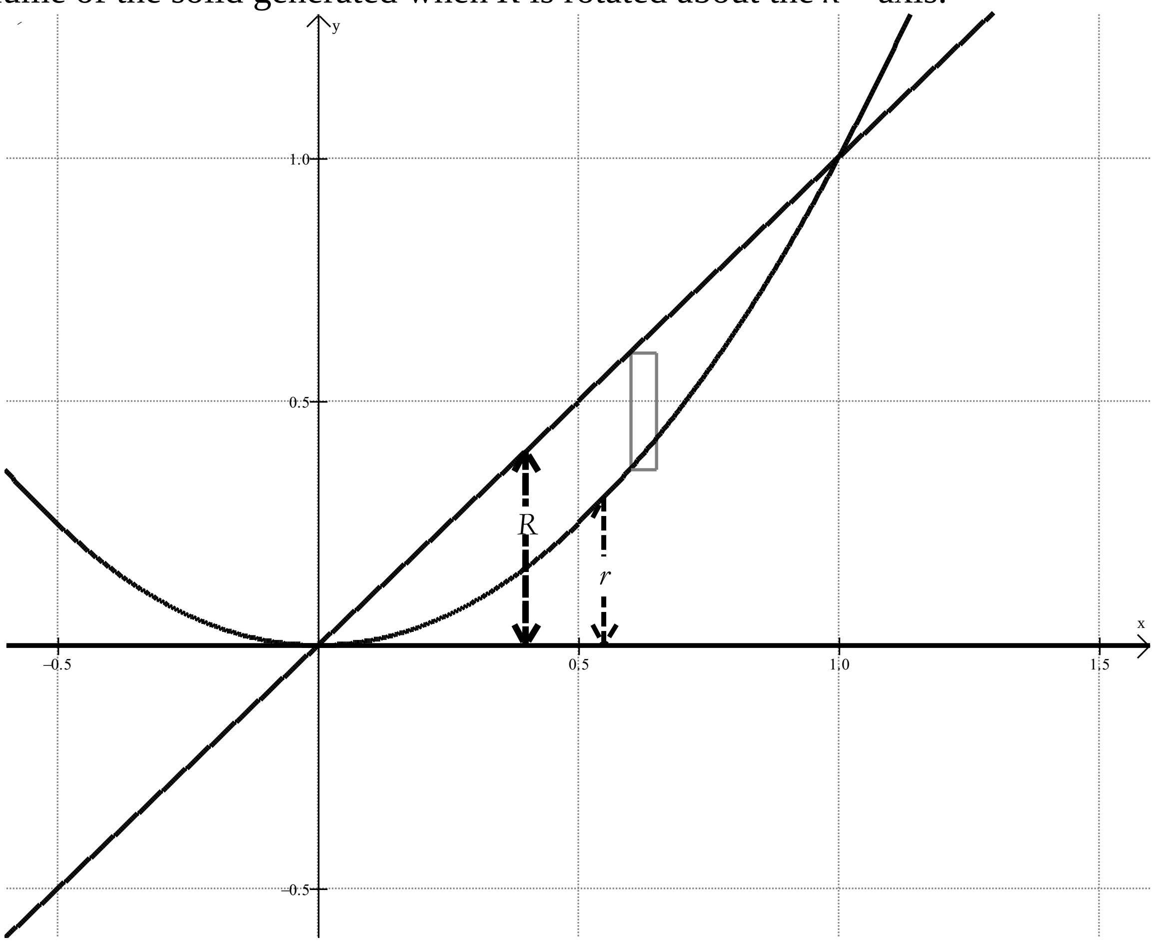
Cross section of the solid above:



Steps to Finding the Volume of a Solid With the Washer Method:

1. Draw a picture of the region to be rotated.
2. Draw the axis of rotation.
3. Sketch a Riemann rectangle – if the piece is vertical your integral will have a *dx* in it, if your piece is horizontal your integral will have a *dy* in it. Your rectangle should always be sketched perpendicular to the axis of rotation.
4. Determine an expression representing the radius of the rectangle – in the case of washers you will have an expression for the outer radius and an expression for the inner radius.
5. Determine the boundaries the region covers.
6. Set up an integral containing π outside the integrand, the limits of integration are the boundaries found in Step 5, and the integrand is the expression found in Step 4.

Ex 5 Let *R* be the region bounded by the equations $y = x^2$ and $y = x$. Find the volume of the solid generated when *R* is rotated about the *x* – axis.



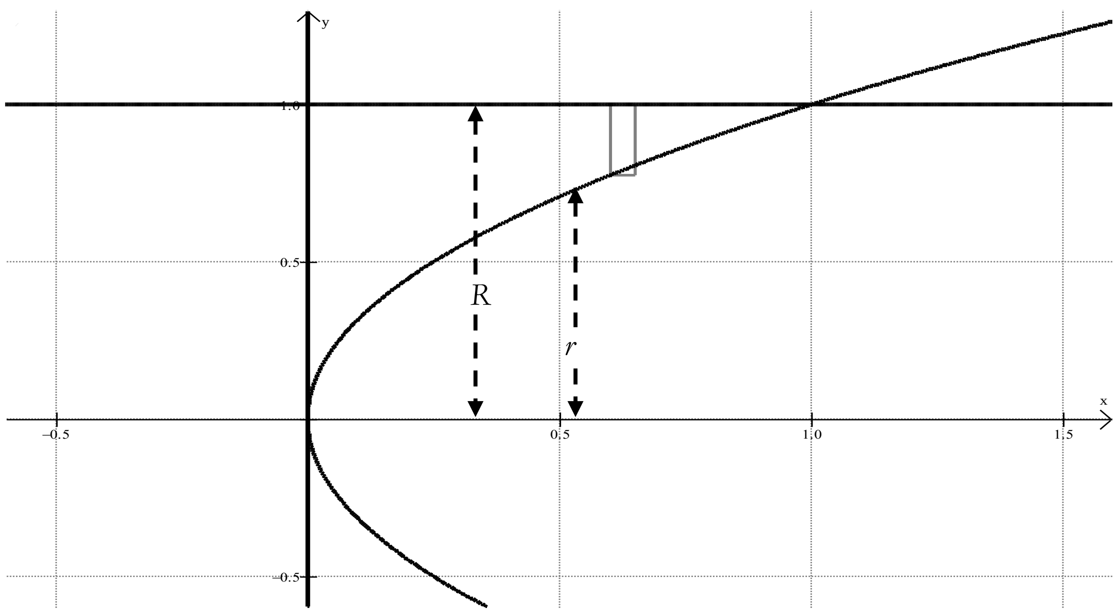
$V=\pi\int_a^b\left(R^2-r^2\right)dx$ Our pieces are perpendicular to the x – axis, so our integrand will contain dx

$V=\pi\int_a^b\left[(x)^2-\left(x^2\right)^2\right]dx$ The expression for R_1 is x and the expression for R_2 is x^2

$V=\pi\int_0^1\left[(x)^2-\left(x^2\right)^2\right]dx$ Our region extends from $x=0$ to $x=1$

$V=.419$

Ex 6 Let R be the region bounded by $x=y^2, y=1$, and $x=0$. Find the volume of the solid generated when R is rotated about the x – axis.



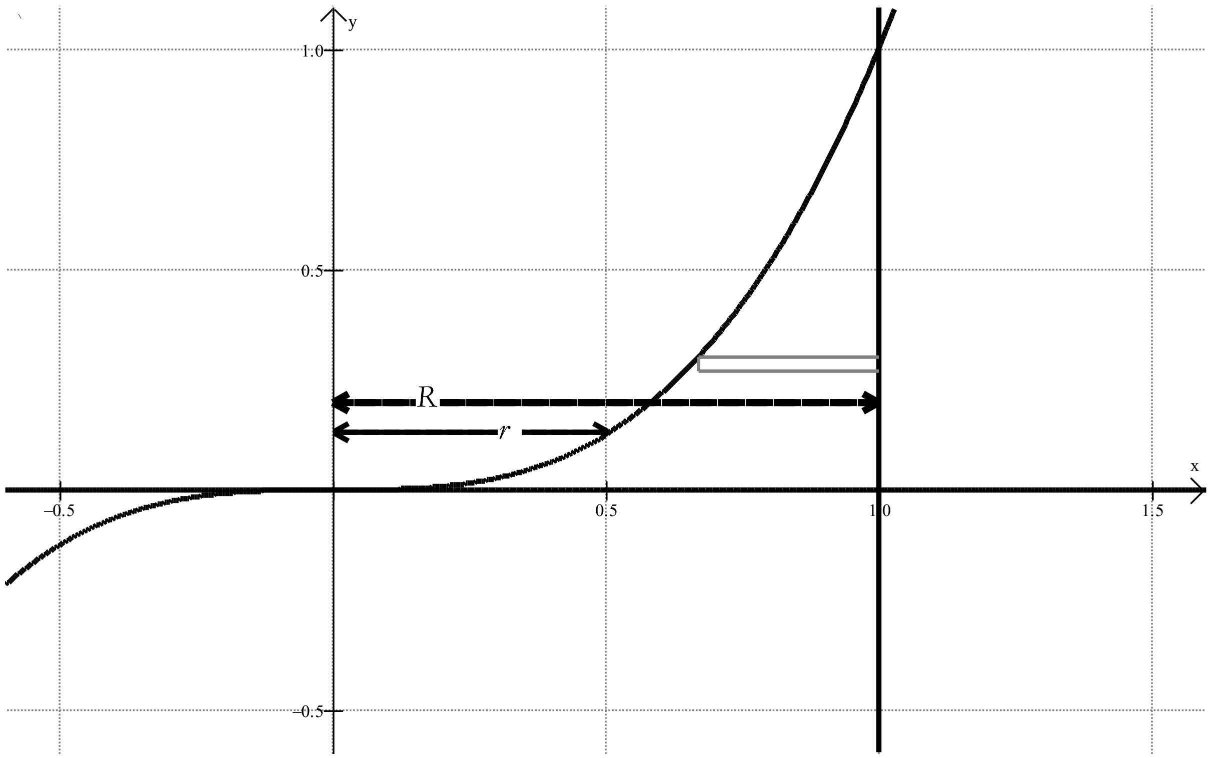
$V=\pi\int_a^b\left(R^2-r^2\right)dx$ Our pieces are perpendicular to the x – axis, so our integrand will contain dx

$V=\pi\int_a^b\left[(1)^2-\left(\sqrt{x}\right)^2\right]dx$ The expression for R_1 is 1and the expression for R_2 is $\sqrt{x}$

$V=\pi\int_0^1\left[(1)^2-\left(\sqrt{x}\right)^2\right]dx$ Our region extends from $x=0$ to $x=1$

$V=1.571$

Ex 7 Let R be the region bounded by $y = x^3, x = 1$, and $y = 0$. Find the volume of the solid generated when R is rotated about the y – axis.



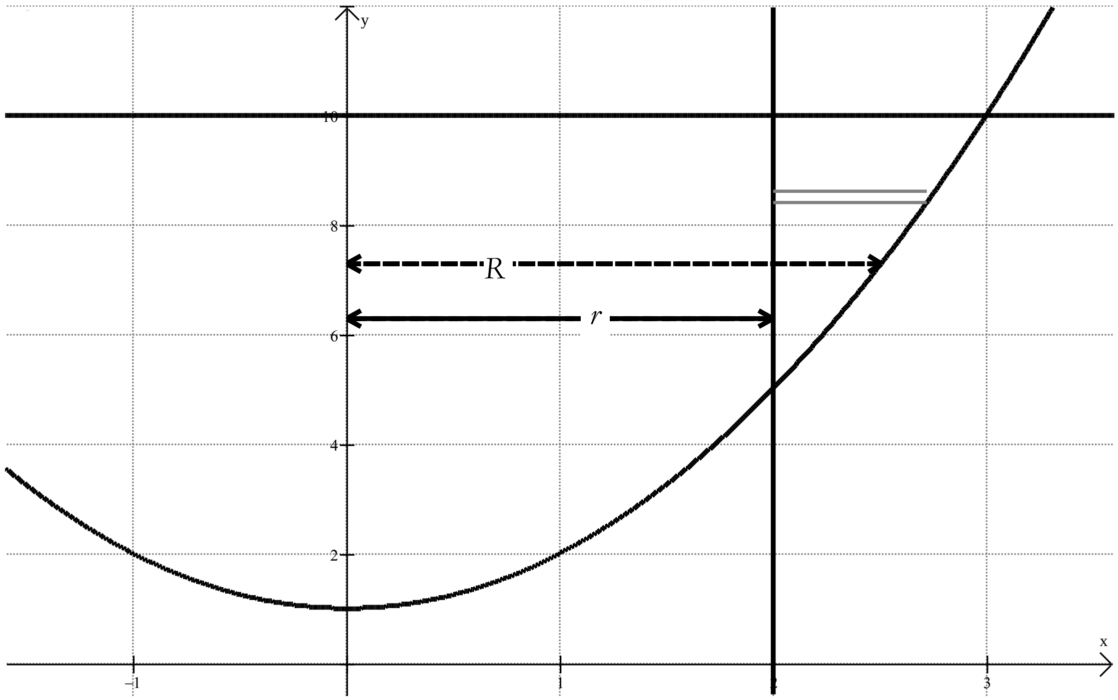
$V = \pi \int_c^d \left(R^2 - r^2\right) dy$ Our pieces are perpendicular to the y – axis, so our integrand will contain dy

$V = \pi \int_c^d \left[(1)^2 - \left(\sqrt[3]{y}\right)^2\right] dy$ The expression for R_1 is 1and the expression for R_2 is $\sqrt[3]{y}$

$V = \pi \int_0^1 \left[(1)^2 - \left(\sqrt[3]{y}\right)^2\right] dy$ Our region extends from $y = 0$ to $y = 1$

$V = 1.257$

Ex 8 Let R be the region bounded by $y=1+x^2, x=2$, and $y=10$. Find the volume of the solid generated when R is rotated about the y – axis.



$$V=\pi\int_c^d\left(R^2-r^2\right)dy$$
$$=\pi\int_5^{10}\left[\left(\sqrt{y-1}\right)^2-(2)^2\right]dy$$
$$=39.270$$

6.2 Homework

Find the volume of the solid formed by rotating the described region about the given line.

1. $y = x^2, x = 1, y = 0;$ about the $x-$axis.

2. $y = e^x, x = 0, x = 1, y = 0;$ about the x-axis.

3. $y = \dfrac{1}{x}, x = 1, x = 2, y = 0;$ about the x-axis.

4. $y = \sqrt{4 - x^2}, y = 0;$ about the x-axis.

5. $y^2 = x, x = 0, y = 1;$ about the y-axis.

6. $y = \sec x, y = 1, x = -\frac{\pi}{3}, x = \frac{\pi}{3};$ about the x-axis.

7. $y^2 = x, x = 2y;$ about the y-axis.

8. $y = x^{2/3}, x = 1, y = 0;$ about the y-axis.

9. $y=\sqrt{4-x^2}, y=2+x^2, x=-2, x=2;$ about the x-axis.

10. $y=x^2, x=y^2;$ about the x-axis.

11. $y=\sqrt{x}, y=e^{-2x}, x=1;$ about the x-axis.

12. $y=x^2, y=2^x;$ about the x-axis.

Answers: 6.2 Homework

1. $y=x^2, x=1, y=0;$ about the $x-$axis.

   $V=\frac{\pi}{5}$

2. $y=e^x, x=0, x=1, y=0;$ about the x-axis.

   V=10.036

3. $y=\frac{1}{x}, x=1, x=2, y=0;$ about the x-axis.

   $V=\frac{\pi}{2}$

4. $y=\sqrt{4-x^2}, y=0;$ about the x-axis.

   $V=\frac{32\pi}{3}$

5. $y^2=x, x=0, y=1;$ about the y-axis.

   $V=\frac{\pi}{5}$

6. $y=\sec x, y=1, x=-\frac{\pi}{3}, x=\frac{\pi}{3};$ about the x-axis.

   V=4.303

7. $y^2=x, x=2y;$ about the y-axis.

   $V=\frac{64\pi}{15}$

8. $y=x^{2/3}, x=1, y=0;$ about the y-axis.

   $V=\frac{3\pi}{4}$

9. $y=\sqrt{4-x^2}, y=2+x^2, x=-2, x=2;$ about the x-axis.

   $V=\frac{592\pi}{15}$

10. $y=x^2, x=y^2;$ about the x-axis.

    $V=\frac{3\pi}{10}$

11. $y=\sqrt{x}, y=e^{-2x}, x=1;$ about the x-axis.

    V=1.207

12. $y=x^2, y=2^x;$ about the x-axis.

    V=94.612

6.3 Volume by Rotation about a Line not an Axis

In the last section we learned how to find the volume of a non-regular solid. This section will be more of the same, but with a bit of a twist. Instead of rotating about an axis, let the region be revolved about a line not the origin.

<u>Volume by Disc Method (Form 2):</u> The volume of the solid generated when the function $f(x)$ from $x = a$ and $x = b$, where $f(x) \geq 0$, is rotated about the line $y = k$ [or $g(y)$ from $y = c$ and $y = d$, where $g(y) \geq 0$, is rotated about the line $x = h$ is given by

$$V = \pi \int_a^b \left[f(x) - k \right]^2 dx \text{ or } V = \pi \int_c^d \left[g(y) - h \right]^2 dy$$

or

$$V = \pi \int_a^b r^2 dx$$

where r is the Length of your Riemann rectangle.

<u>Volume by Washer Method (Part 2):</u> The volume of the solid generated when the region bounded by functions $f(x)$ and $g(x)$, from $x = a$ and $x = b$, where $f(x) \geq g(x)$ [or $f(y)$ and $g(y)$, from $y = c$ and $y = d$, where $f(y) \geq g(y)$], is rotated about the line $y = k$ is given by

$$V = \pi \int_a^b \left(\left[f(x) - k \right]^2 - \left[g(x) - k \right]^2 \right) dx$$

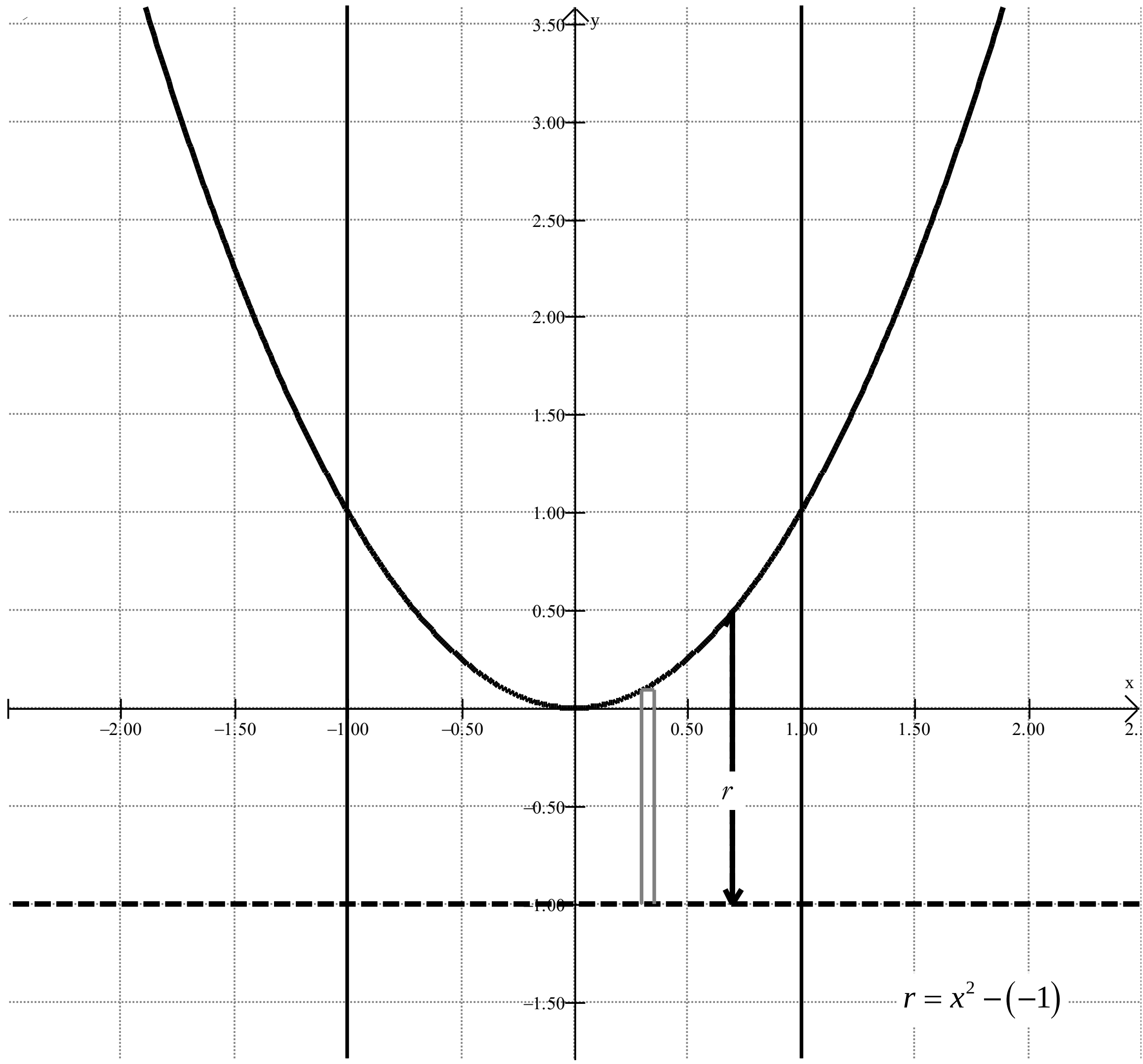
or

$$V = \pi \int_c^d \left(\left[f(y) - h \right]^2 - \left[g(y) - h \right]^2 \right) dy$$

$$V = \pi \int_c^d \left(R^2 - r^2 \right) dy$$

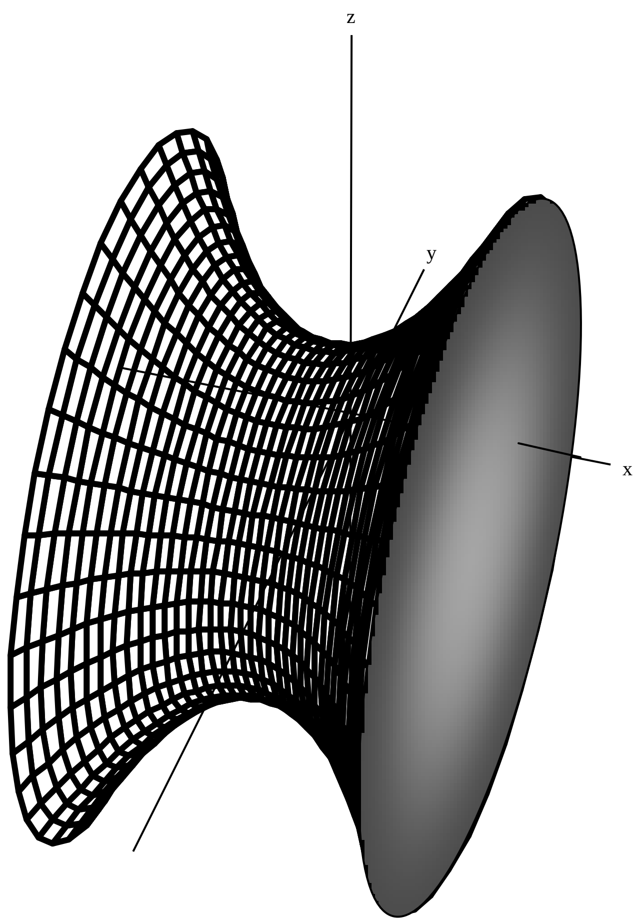
where R is the outer radius of the Riemann rectangle and r is the inner radius.

Ex 1 Let R be the region bounded by the equations $y = x^2$, $y = -1$, $x = -1$, and $x = 1$. Find the volume of the solid generated when R is rotated about the line $y = -1$.

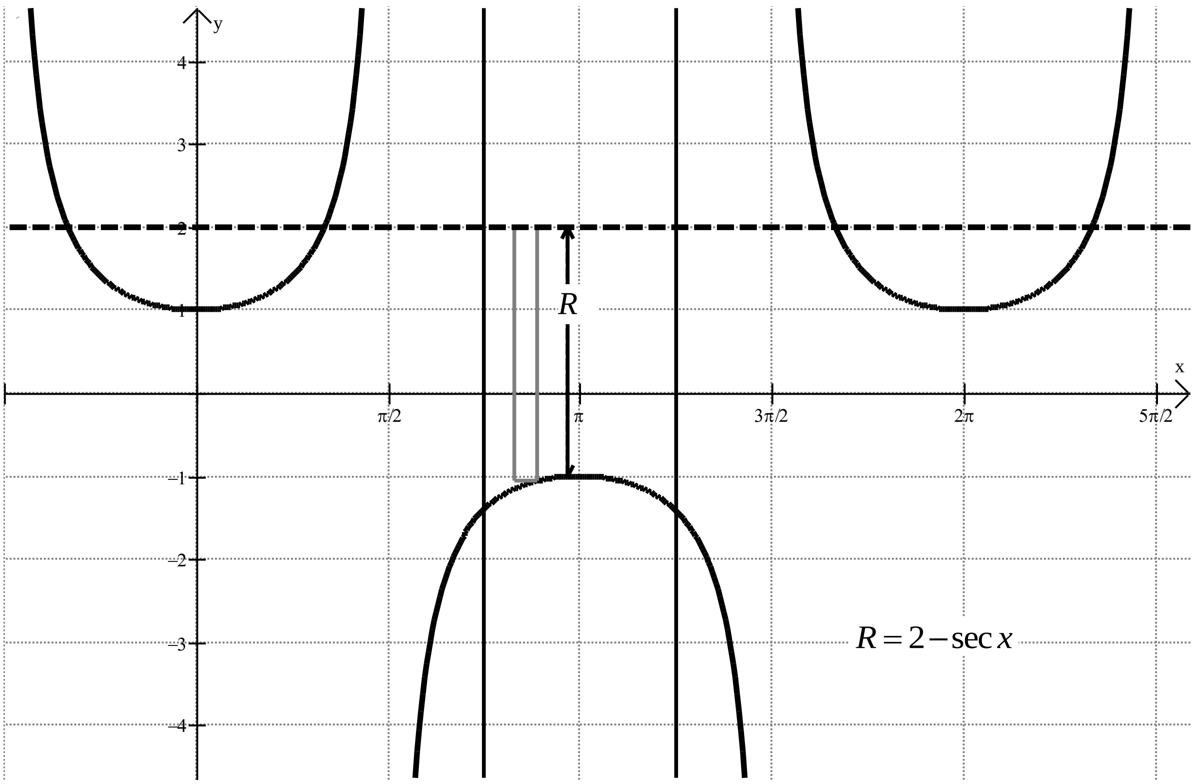


$$V = \pi \int_a^b r^2 dx$$
$$= \pi \int_2^7 \left[x^2 + 1\right]^2 dx$$
$$= 3583.333$$

As in the previous section, this makes an interesting shape, but is of very little use to us in actually solving for the volume. You could easily have found the value for the volume never knowing what the shape actually looked like.

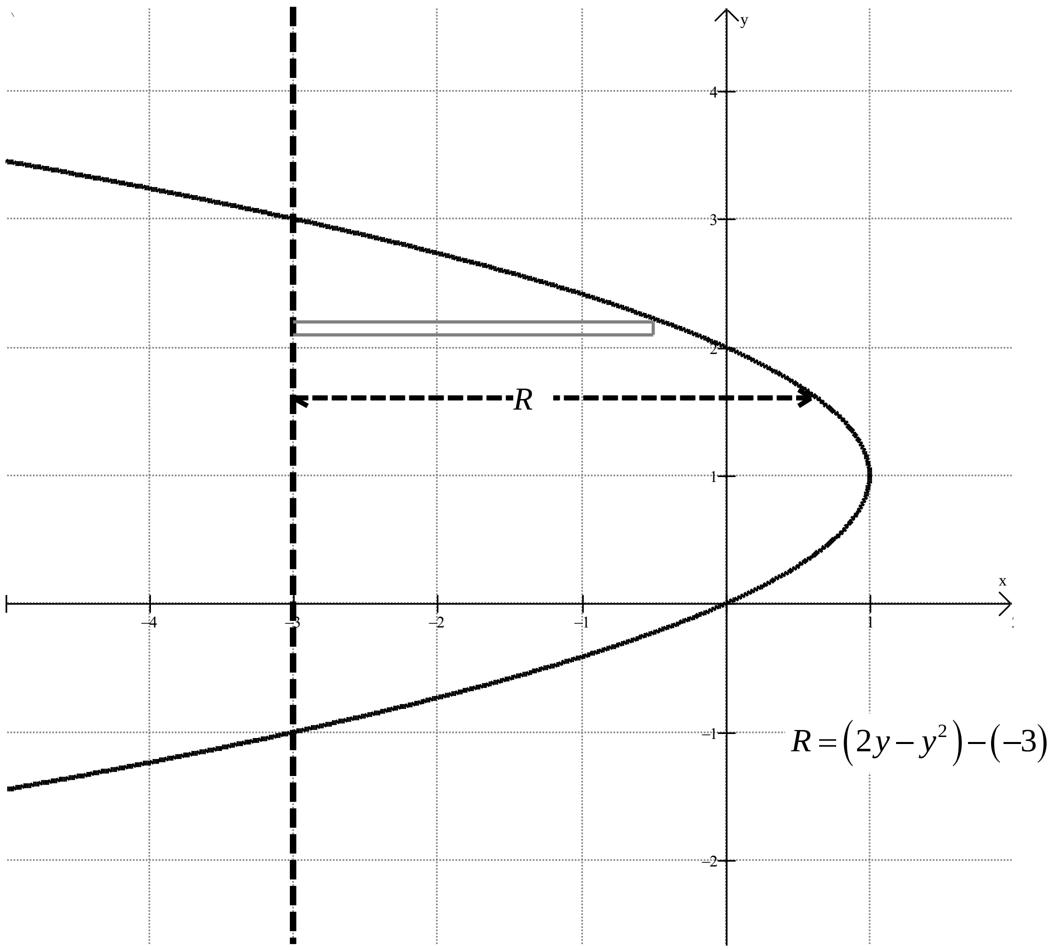

z
y
x

Ex 2 Let R be the region bounded by the curves $y = \sec x$, $y = 2$, $x = \frac{3\pi}{4}$, and $x = \frac{5\pi}{4}$. Find the volume of the solid generated when R is rotated about the line $y = 2$.



$$
\begin{aligned}
V &= \pi \int_a^b \left[2 - y\right]^2 dx \\
&= \pi \int_{3\pi/4}^{5\pi/4} \left[2 - \sec x\right]^2 dx \\
&= 48.174
\end{aligned}
$$

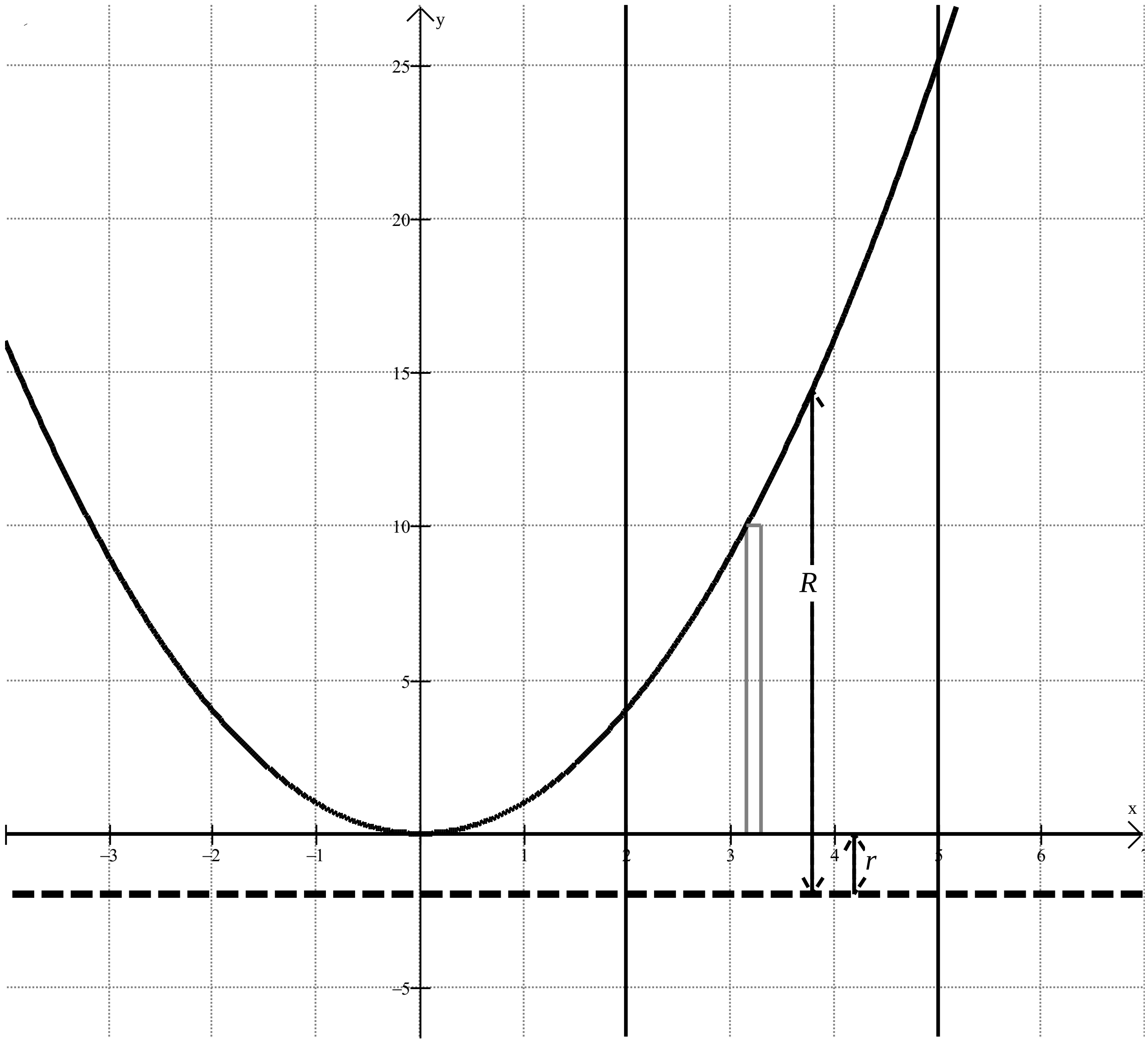
Ex 3 Find the volume of the solid obtained by rotating about the y – axis the region bounded by $x = 2y - y^2$ and the line $x = -3$ about the line $x = -3$.



$$V = \pi \int_c^d [3 + x]^2 \, dy$$
$$= \pi \int_{-1}^{3} [3 + 2y - y^2]^2 \, dy$$
$$= 961.746$$

What if you were asked to take the region bounded by $y = x$ and $y = x^2$ and rotate it about the x – axis? How is this different from the previous problems?

Ex 4 Let R be the region bounded by the equations $y=x^2$, the x – axis, $x=2$, and $x=5$. Find the volume of the solid generated when R is rotated about the line $y=-2$.



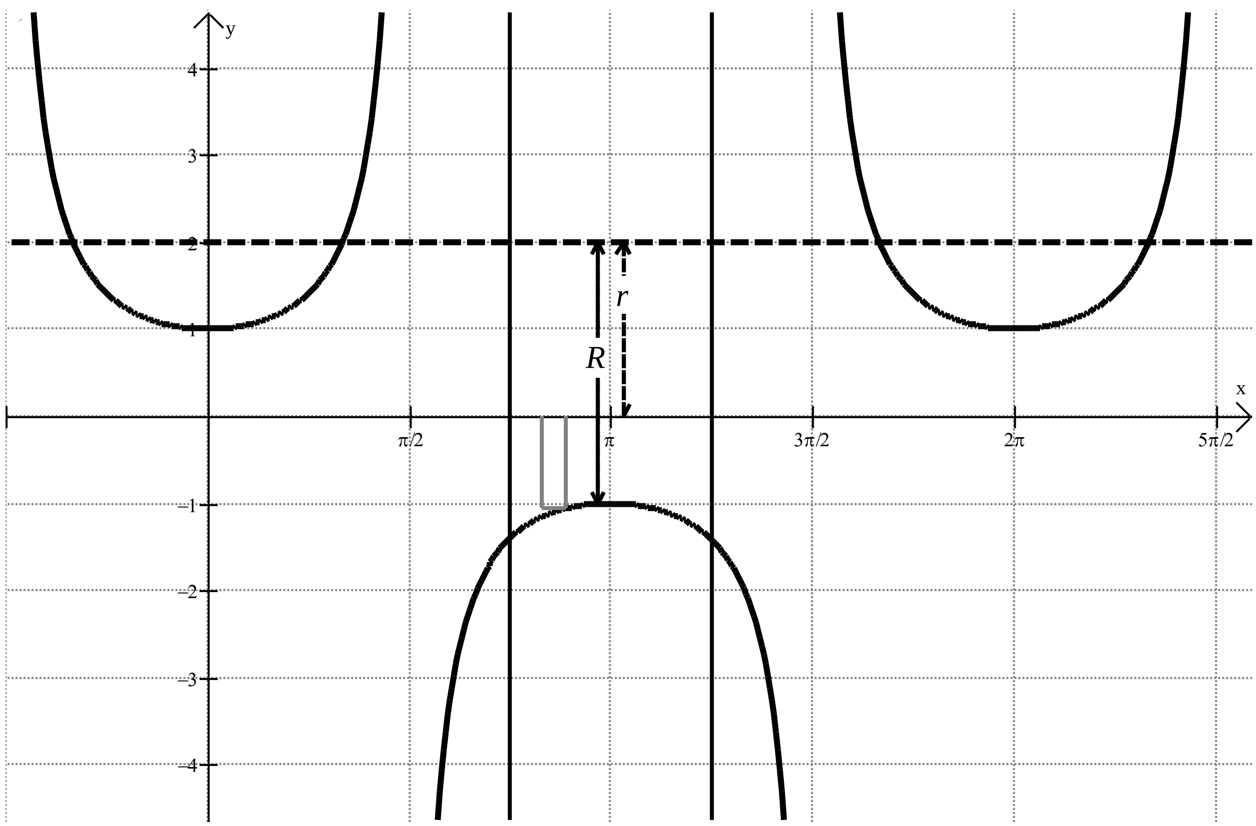
$$V=\pi\int_a^b\left(R^2-r^2\right)dx$$

$$=\pi\int_2^5\left[\left(x^2+2\right)^2-\left(0-(-2)\right)^2\right]dx$$

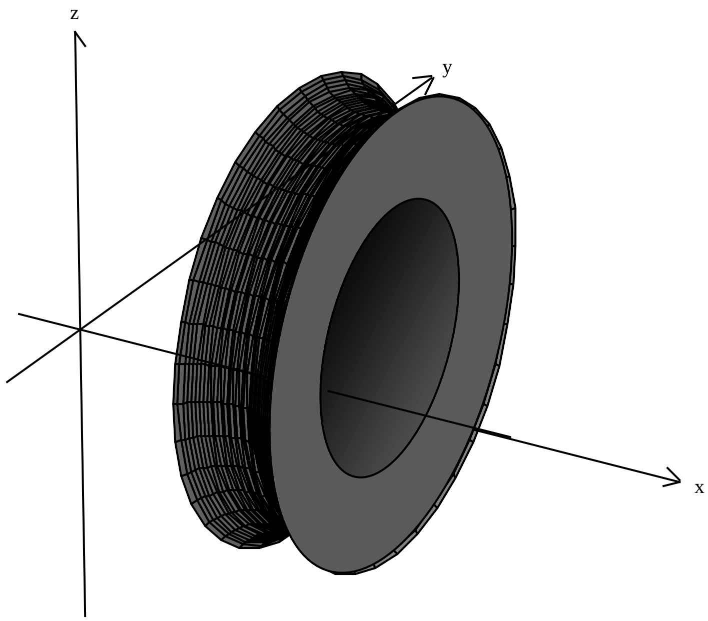
$$=2433.478$$

Ex 5 Let R be the region bounded by the curves $y = \sec x$, the x – axis, $x = \frac{3\pi}{4}$, and $x = \frac{5\pi}{4}$. Find the volume of the solid generated when R is rotated about the line $y = 2$.

$$V = \pi \int_a^b \left(R^2 - r^2 \right) dx$$
$$= \pi \int_{3\pi/4}^{5\pi/4} \left[\left(2 - \sec x\right)^2 - \left(2\right)^2 \right] dx$$
$$= 28.435$$



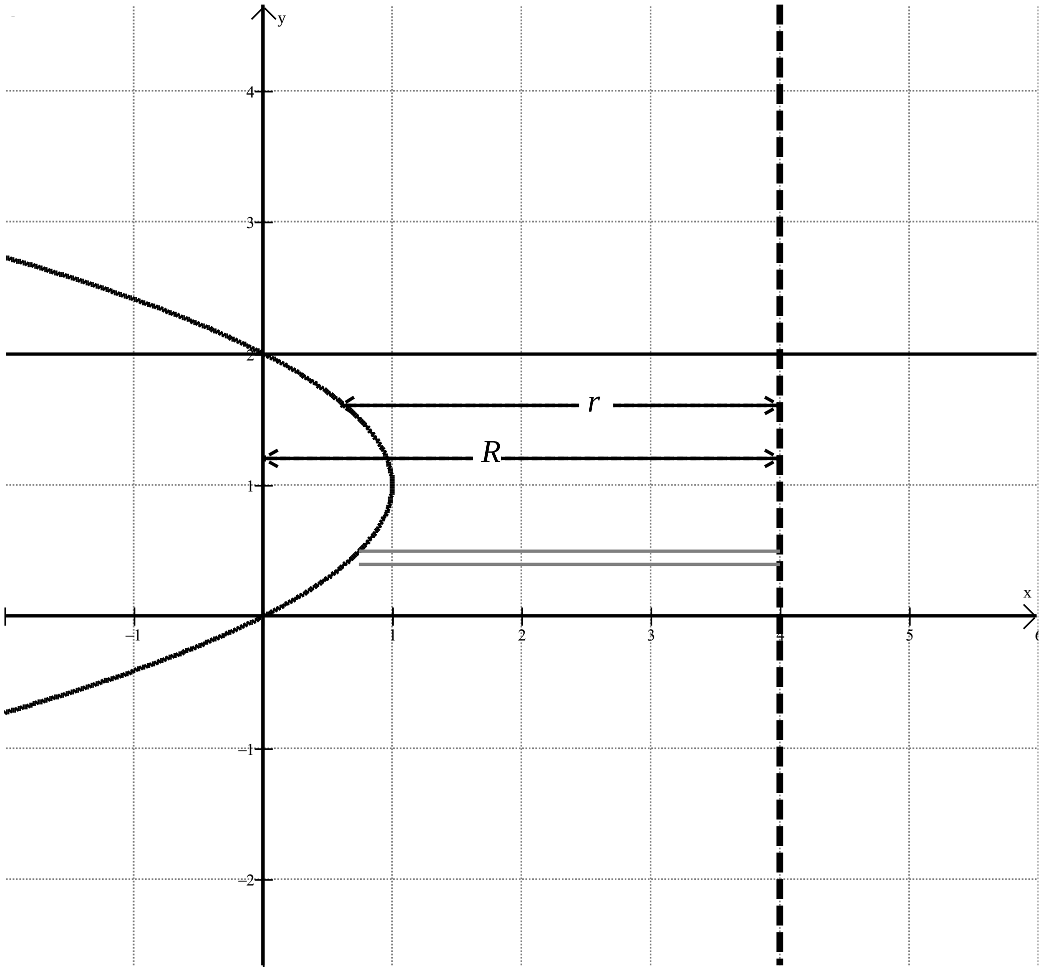
The rotated figure looks like this:



It would be difficult to sketch this by hand, and we definitely do not need it to solve the problem.

Ex 6 Find the volume of the solid obtained by rotating about the y – axis the region bounded by $x = 2y - y^2$ and the y – axis about line $x = 4$.

Given the information above and the sketch of $x = 2y - y^2$ below, sketch your boundaries, the axis of rotation, your Riemann rectangle, and your inner and outer radii.



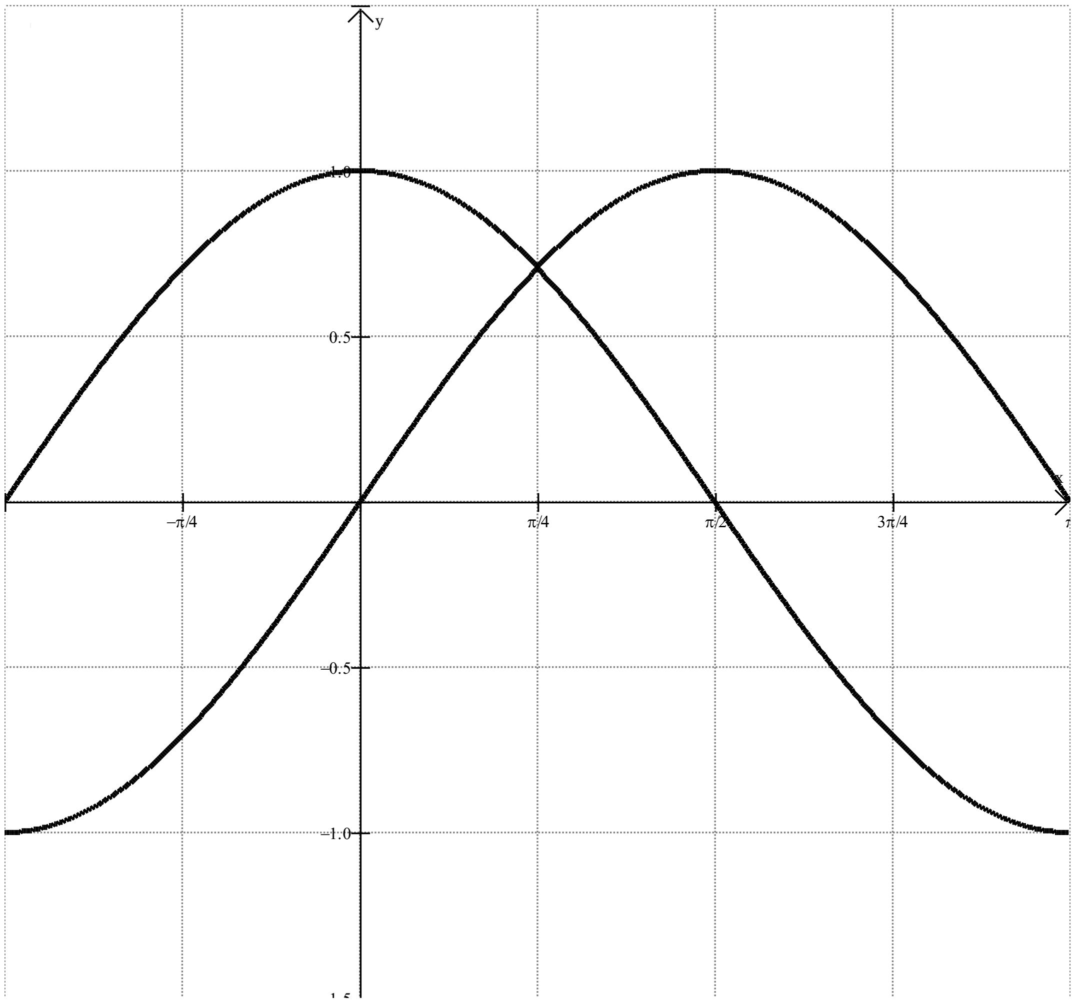
Set up the integration:

$$V = \pi \int_c^d \left(R^2 - r^2\right) dy$$

$$= \pi \int_0^2 \left[(4)^2 - \left(4 - 2y + y^2\right)^2 \right] dy$$

$$= 30.159$$

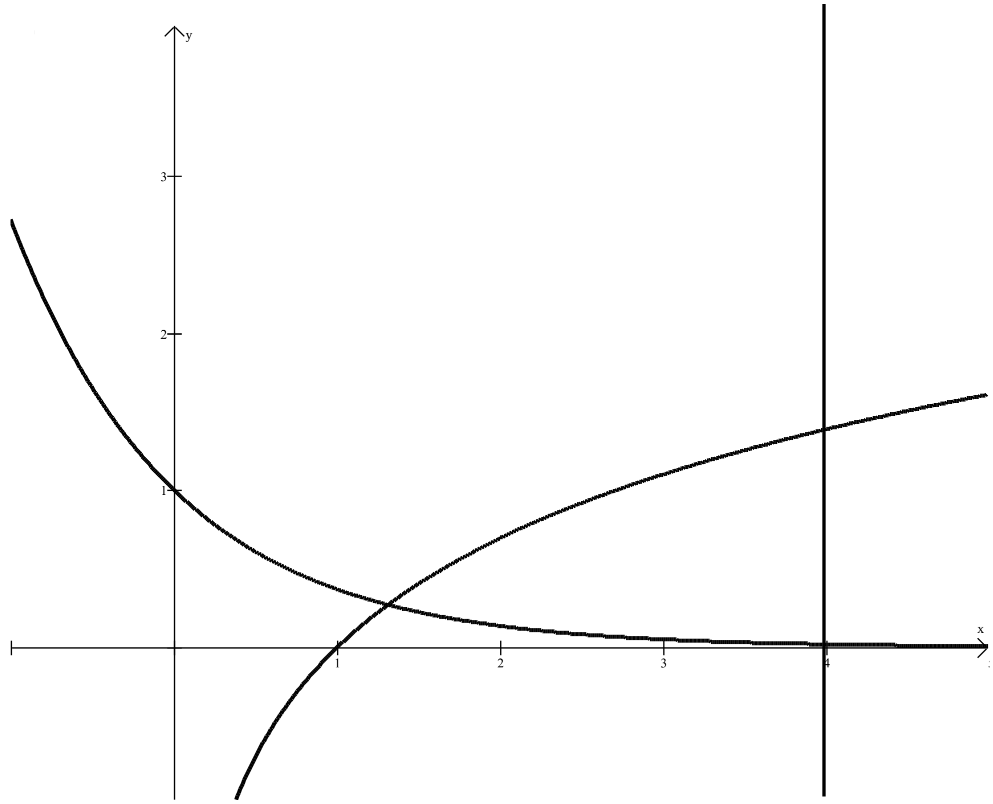
Ex 7 Let R be the region bounded by the curves $y=\sin x$, $y=\cos x$, $x=0$, and $x=\frac{\pi}{2}$. Find the volume of the solid obtained by rotating R about the x–axis.



$$
\begin{aligned}
V &= \pi\int_a^b \left(R^2 - r^2\right)dx + \pi\int_b^c \left(R^2 - r^2\right)dx \\
&= \pi\int_0^{\pi/4}\left(\left(\cos x\right)^2 - \left(\sin x\right)^2\right)dx + \pi\int_{\pi/4}^{\pi/2}\left(\left(\sin x\right)^2 - \left(\cos x\right)^2\right)dx \\
&= \pi
\end{aligned}
$$

6.3 Homework Set A

1. Given the curves $f(x)=\ln x$, $g(x)=e^{-x}$ and $x = 4$.

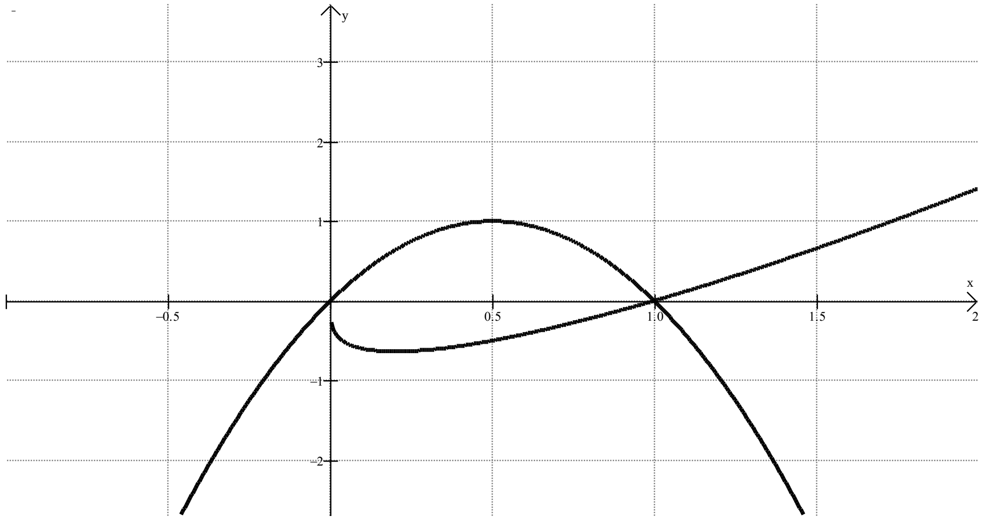


a. Find the area of the region bounded by three curves in the first quadrant.

b. Find the volume of the solid generated by rotating the region around the line $y = 2$.

c. Find the volume of the solid generated by rotating the region around the line $y = -1$.

2. Let f and g be the functions given by $f(x)=4x(1-x)$ and $g(x)=\sqrt[4]{2x}(x-1)$



a. Find the area of the region bounded by three curves in the first quadrant.

b. Find the volume of the solid generated by rotating the region around the line $y = 3$.

c. Find the volume of the solid generated by rotating the region around the line $y = -2$.

d. Set up an integral expression for the curve $g(x)=\sqrt[4]{2x}(x-1)$ and the curve $h(x)=x(1-x)$ which is not pictured above that is rotated around the line $y = k$, where $k > 2$, in which the volume equals 10. Do not solve this equation.

Find the volume of the solid formed by rotating the described region about the given line. Sketch the graph so that it is easier for you to apply the Disk/Washer Methods.

3. $y=\dfrac{1}{x}, y=0, x=1, x=3;$ about the line $y=-1$.

4. $y=2+\sin x, y=2, x=0, x=\pi;$ about the line $y=3$.

5. $y=\sqrt{x}, y=x^3;$ about the line $y=1$.

6. $y=\sqrt{x}, y=x^3;$ about the line $x=1$.

Use your grapher to sketch the regions described below. Find the points of intersection and find the volume of the solid formed by rotating the described region about the given line.

7. $y = \ln\left(x^2 + 1\right), y = \cos x;$ about the line $y = 1$.

8. $y = x^2, y = 4;$ about the line $y = 4$.

9. $y = 2 + \sin x, y = 2, x = 0, x = 2\pi;$ about the line $y = 2$.

10. $y^2 = x, x = 1;$ about the line $x = 1$.

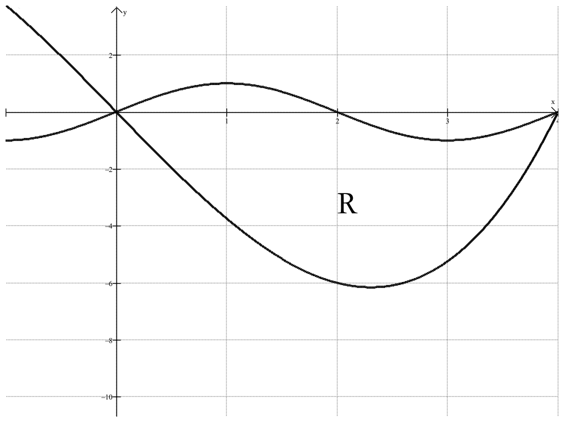
6.3 Homework Set B

1. Given the functions $f(x)=\dfrac{1}{1+x^2}$ and $g(x)=\dfrac{-1}{2}\left(x^2-6x+5\right)$, find the area bounded by the two curves in the first quadrant.

2. Find the volume of the solid generated when the function $y=\ln\left(9+x^3\right)$ is rotated around the line $y=4$ on the interval $x\in\left[-1,2\right]$

3. Find the volume of the solid generated when the area in the first quadrant bounded by the curves $x=\ln\left(y+1\right)$and $x=\left(y-2\right)^2$ is rotated around the *y*–axis

4. Let R be the region bounded by the graphs $y = \sin\left(\frac{\pi}{2}x\right)$ and $y = \frac{1}{4}\left(x^3 - 16x\right)$

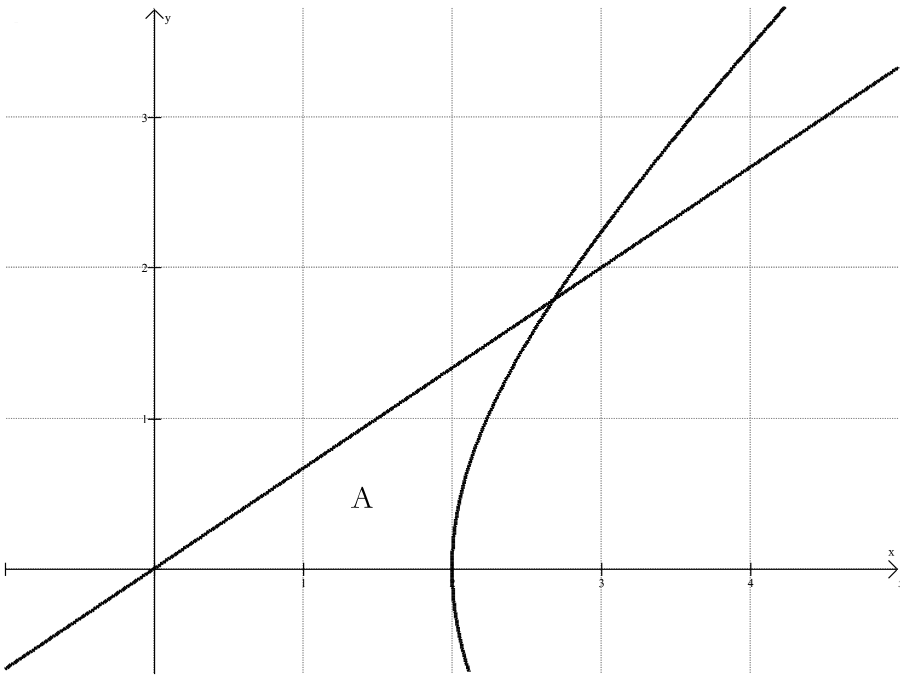


a. Find the area of the region R.

b. Find the volume of the solid when the region R is rotated around the line $y = -10$.

c. The line $y = -3$ cuts the region R into two regions. Write and evaluate an integral to find the area of the region below the line.

5. Given the functions below are $x = \dfrac{3y}{2}$ and $x = \sqrt{4 + y^2}$. Let A be the region bounded by the two curves and the x-axis.



a. Find the area of region A.

b. Find the value of $\dfrac{dx}{dy}$ for $x = \sqrt{4 + y^2}$ at the point where the two curves intersect.

c. Find the volume of the solid when region A is rotated around the y-axis.

Answers: 6.3 Homework Set A

1. Given the curves $f(x)=\ln x$, $g(x)=e^{-x}$ and $x = 4$.

   a. Find the area of the region bounded by three curves in the first quadrant.
   A=2.250

   b. Find the volume of the solid generated by rotating the region around the line $y = 2$.
   V=20.254

   c. Find the volume of the solid generated by rotating the region around the line $y = -1$.
   V=22.156

2. Let f and g be the functions given by $f(x)=4x(1-x)$ and $g(x)=\sqrt[4]{2x}(x-1)$

   a. Find the area of the region bounded by three curves in the first quadrant.
   A=1.089

   b. Find the volume of the solid generated by rotating the region around the line $y = 3$.
   V=19.538

   c. Find the volume of the solid generated by rotating the region around the line $y = -2$.
   V=14.689

   d. Set up an integral expression for the curve $g(x)=\sqrt[4]{2x}(x-1)$ and the curve $h(x)=x(1-x)$ which is not pictured above that is rotated around the line $y = k$, where $k > 2$, in which the volume equals 10. Do not solve this equation.

   $$10=\pi\int_0^1 \left(k-g(x)\right)^2-\left(k-h(x)\right)^2\,dx$$

3. $y=\frac{1}{x}, y=0, x=1, x=3;$ about the line $y=-1$.
   V=8.997

4. $y=2+\sin x, y=2, x=0, x=\pi;$ about the line $y=3$.
   V=7.632

5. $y=\sqrt{x}, y=x^3$; about the line $y=1$.

V= $\frac{5\pi}{14}$

6. $y=\sqrt{x}, y=x^3$; about the line $x=1$.

V=1.361

7. $y=\ln\left(x^2+1\right), y=\cos x$; about the line $y=1$.

V=3.447

8. $y=x^2, y=4$; about the line $y=4$.

V= $\frac{512\pi}{15}$

9. $y=2+\sin x, y=2, x=0, x=2\pi$; about the line $y=2$.

V=9.870

10. $y^2=x, x=1$; about the line $x=1$.

V= $\frac{16\pi}{15}$

6.3 Homework Set B

1. Given the functions $f(x)=\frac{1}{1+x^2}$ and $g(x)=\frac{-1}{2}\left(x^2-6x+5\right)$, find the area bounded by the two curves in the first quadrant.

A=4.798

2. Find the volume of the solid generated when the function $y=\ln\left(9+x^3\right)$ is rotated around the line $y=4$ on the interval $x\in[-1,2]$

V=27.286

3. Find the volume of the solid generated when the area in the first quadrant bounded by the curves $x=\ln(y+1)$ and $x=(y-2)^2$ is rotated around the y–axis

V=124.757

4. Let R be the region bounded by the graphs $y = \sin\left(\frac{\pi}{2}x\right)$ and $y = \frac{1}{4}\left(x^3 - 16x\right)$

a. Find the area of the region R.

R=16

b. Find the volume of the solid when the region R is rotated around the line $y = -10$.

V=766.489

c. The line $y = -3$ cuts the region R into two regions. Write and evaluate an integral to find the area of the region below the line.

$$V = \pi \int_{0.7796156}^{3.552799} \left(\frac{1}{4}\left(x^3 - 16x\right) + 10\right)^2 dx = 5.775$$

5. Given the functions below are $x = \frac{3y}{2}$ and $x = \sqrt{4 + y^2}$. Let A be the region bounded by the two curves and the x-axis.

a. Find the area of region A.

A=1.609

b. Find the value of $\frac{dx}{dy}$ for $x = \sqrt{4 + y^2}$ at the point where the two curves intersect.

$\frac{dx}{dy} = 3.578$

c. Find the volume of the solid when region A is rotated around the y-axis.

V=14.986

6.4 Volume by Cross Sections

In the last sections, we have learned how to find the volume of a solid created when a region was rotated about a line. In this section, we will find volumes of solids that are not made though revolutions, but cross sections.

<u>Volume by Cross Sections Formula:</u>
The volume of solid made of cross sections with area A is

$$V = \int_a^b A(x)\,dx \text{ or } V = \int_c^d A(y)\,dy$$

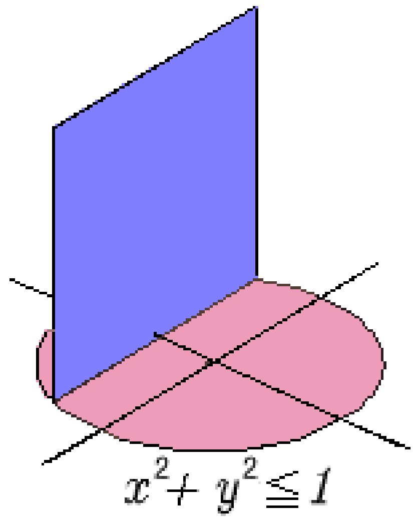
Objectives:

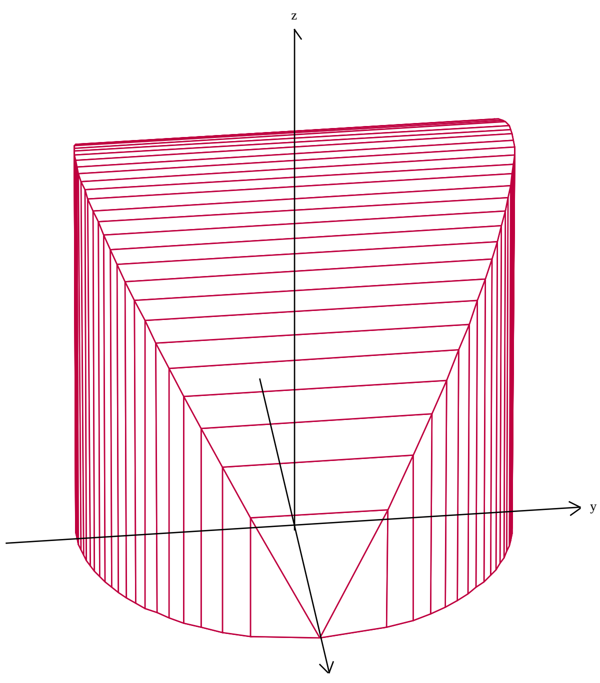
Find the volume of a solid with given cross sections.

<u>Steps to Finding the Volume of a Solid by Cross Sections:</u>

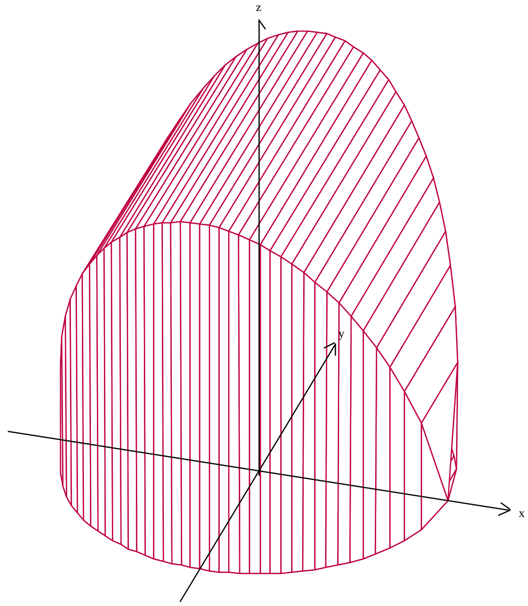
1. Draw a picture of the region of the base.
2. Sketch a sample cross section (the base of the sample cross section will lie on the base sketched in Step 1)– if the piece is vertical your integral will have a *dx* in it, if your piece is horizontal your integral will have a *dy* in it.
3. Determine an expression representing the area of the sample cross section (be careful when using semi-circles – the radius is **half** the distance of the base).
4. Determine the endpoints the region covers.
5. Set up an integral containing the limits of integration found in Step 5, and the integrand expression found in Step 4.

Ex 1 The base of solid S is $x^2 + y^2 = 1$. Cross sections perpendicular to the x – axis are squares. What is the volume of the solid?





View of the solid generated with the given cross-sections from the perspective of the x-axis coming out of the page.



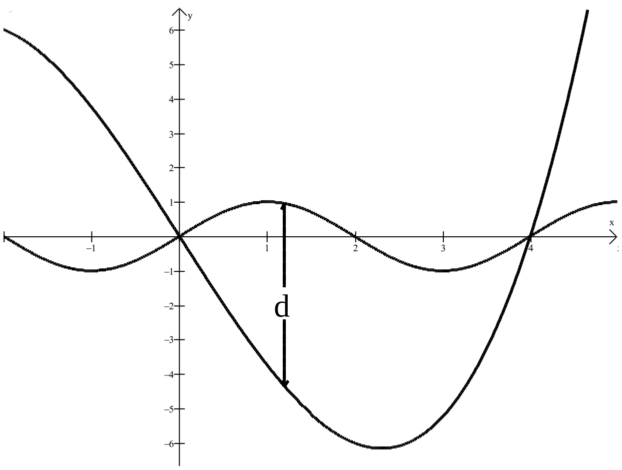
View of the solid generated with the given cross-sections from the perspective of the *x*-axis coming out of the page.

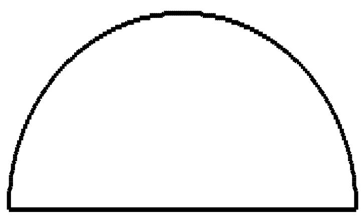
$V=\int_a^b A(x)\,dx$	Start here
$V=\int_a^b (\text{side})^2\,dx$	Area formula for our (square) cross section
$V=\int_{-1}^1 (\text{side})^2\,dx$	Endpoints of the region of our solid
$V=2\int_0^1 (\text{side})^2\,dx$	Simplify the integral
$V=2\int_0^1 (2y)^2\,dx$	Length of the side of our cross section
$V=2\int_0^1 4y^2dx$	We cannot integrate *y* with respect to *x* so we will sub out for *y*

$$V=2\int_0^1 4\left(1-x^2\right)dx$$
$$V=5.333$$

Ex 2 The base of solid is the region S which is bounded by the graphs $y = \sin\left(\frac{\pi}{2}x\right)$ and $y = \frac{1}{4}\left(x^3 - 16x\right)$. Cross sections perpendicular to the x – axis are semicircles. What is the volume of the solid?

It is easier for set-up to visualize the base region and the cross-section separately and not to try to imagine the 3D solid.





$V = \int_a^b A(x)\,dx$ Start here

$V = \int_a^b \frac{1}{2}\pi r^2 dx$ Area formula for our cross section

$V = \int_0^4 \frac{1}{2}\pi r^2 dx$ Endpoints of the region of our solid

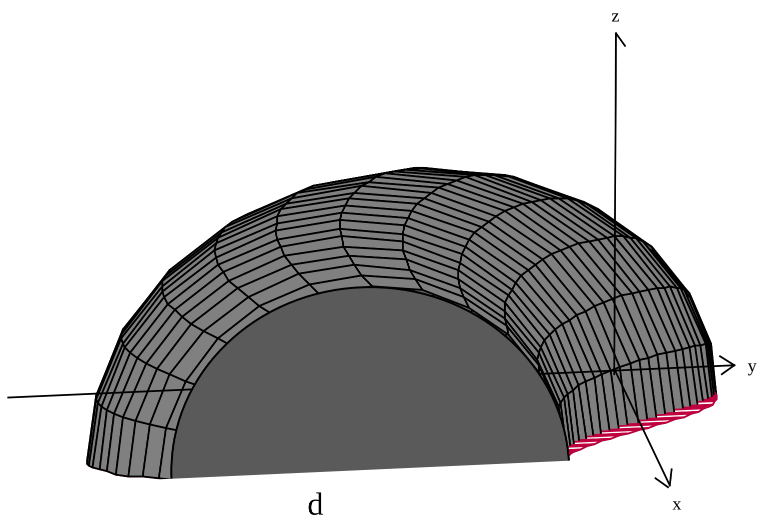
$V = \frac{1}{2}\pi \int_0^4 \left(\frac{d}{2}\right)^2 dx$

$V = \frac{1}{8}\pi \int_0^4 d^2 dx$ For our cross section $y = r$.

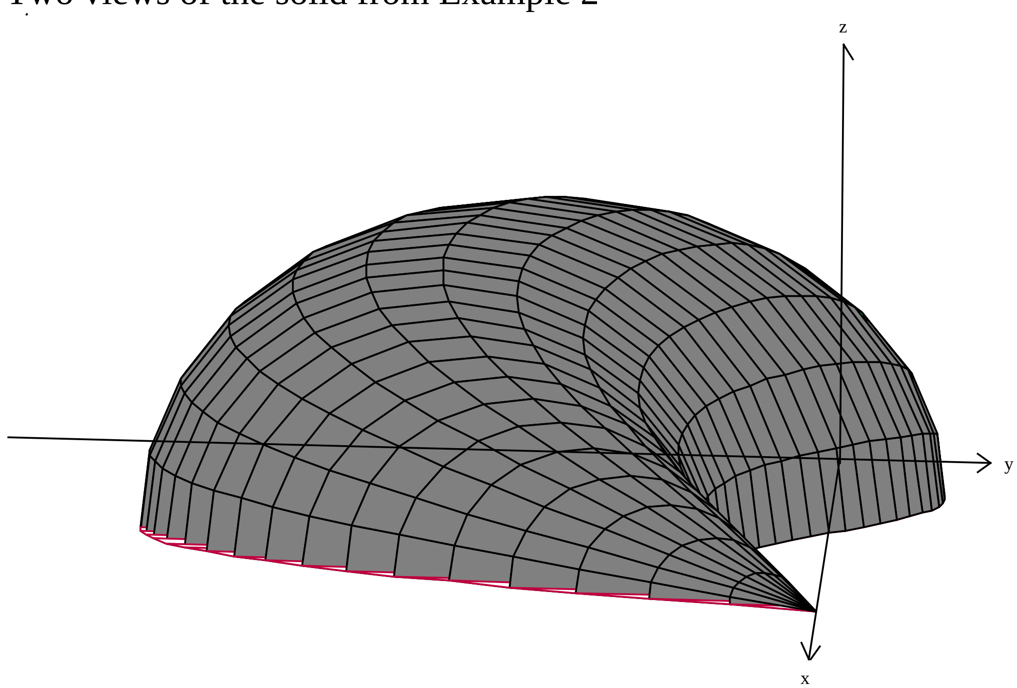
$V = \frac{\pi}{8}\int_0^4 \left(\sin\left(\frac{\pi}{2}x\right) - \frac{1}{4}\left(x^3 - 16x\right)\right)^2 dx$ We cannot integrate y with respect to x so we will substitute out for y.

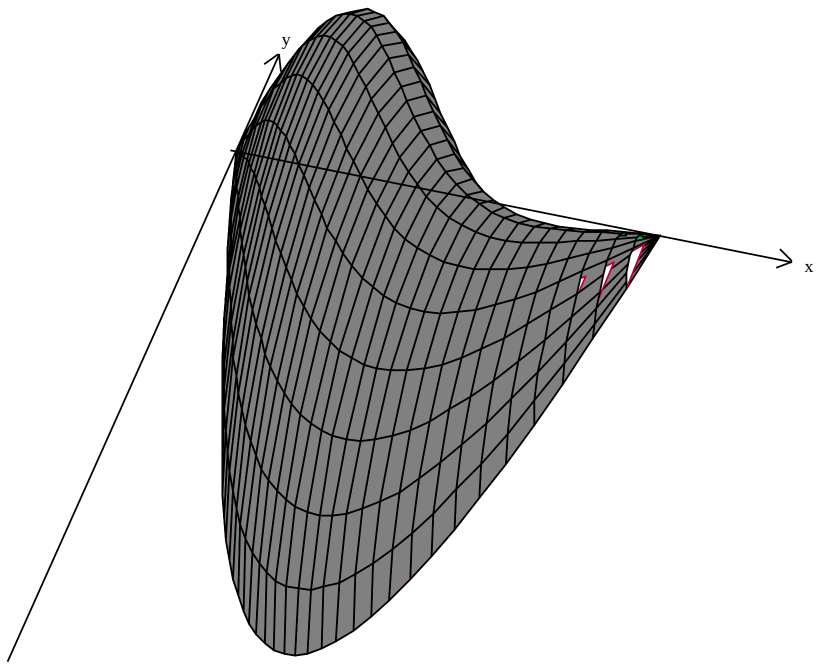
$\underline{V = 30.208}$

Here is an illustration of the solid, cut-away so that you can see the cross-section:

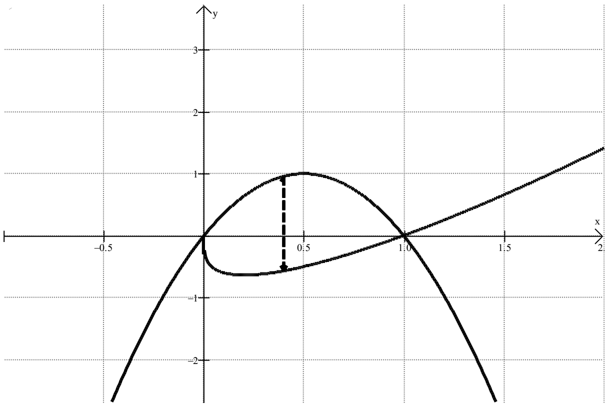


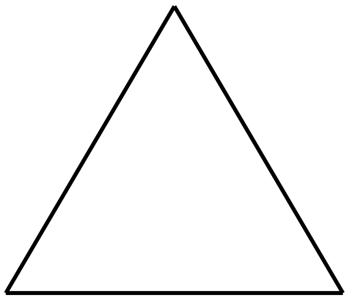
Two views of the solid from Example 2




y
x

Ex 3 The base of solid is the region S which is bounded by the graphs $f(x)=4x(1-x)$ and $g(x)=\sqrt[4]{2x}(x-1)$. Cross-sections perpendicular to the x – axis are equilateral triangles. What is the volume of the solid?





d

$$V=\int_a^b A(x)\,dx$$
$$=\int_0^1 \frac{1}{2}d\left(d\frac{\sqrt{3}}{2}\right)dx$$
$$=\int_0^1 \frac{\sqrt{3}}{4}d^2dx$$
$$=\frac{\sqrt{3}}{4}\int_0^1\left(4x(1-x)-\sqrt[4]{2x}(x-1)\right)^2dx$$
$$=.589$$

Ex 4 Let R be the region in the first quadrant bounded by $y = \cos x$, $y = e^x$, and $x = \frac{\pi}{2}$.

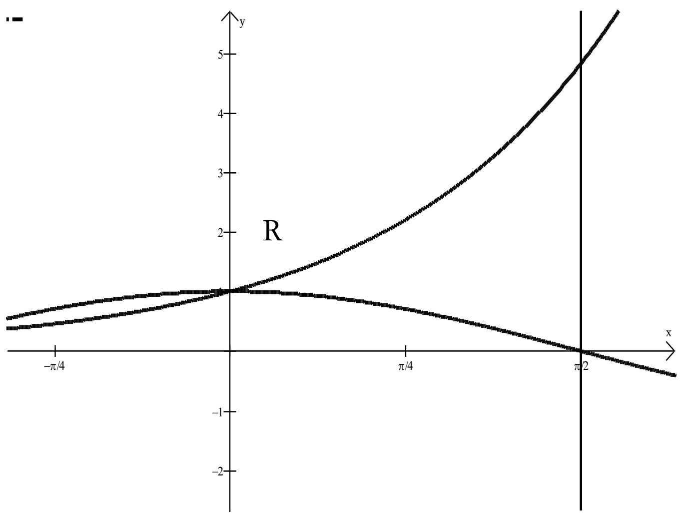
(a) Find the area of region R.

(b) Find the volume of the solid obtained when R is rotated about the line $y = -1$.

(c) The region R is the base of s solid. For this solid, each cross section perpendicular to the x – axis is a square. Find the volume of the solid.

(a) $A = \int_0^{\pi/2} \left(e^x - \cos x\right) dx = 2.810$

(b)



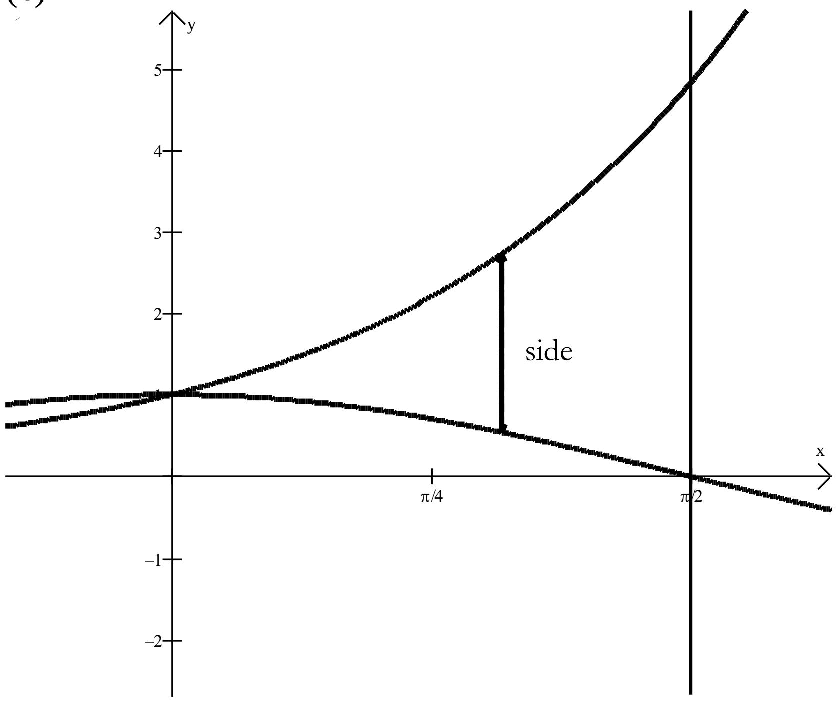
$$V = \pi \int_0^{\pi/2} \left(R^2 - r^2\right) dx$$

$$V = \pi \int_0^{\pi/2} \left[\left(1 + y_1\right)^2 - \left(1 + y_2\right)^2\right] dx$$

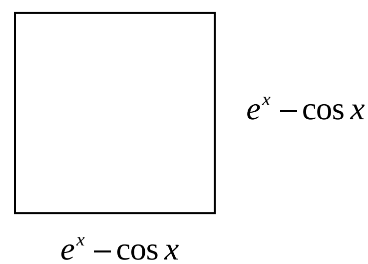
$$= \pi \int_0^{\pi/2} \left[\left(1 + e^x\right)^2 - \left(1 + \cos x\right)^2\right] dx$$

$$= 49.970$$

(c)



$$V = \int_0^{\pi/2} (\text{side})^2 \, dx$$
$$V = \int_0^{\pi/2} \left(e^x - \cos x\right)^2 dx$$
$$V = 8.045$$



<u>What you may have noticed is that the base of your cross-section is always either “top curve” – “bottom curve” or “right curve” – “left curve”</u>.

What this means for us is that all we really need to do is
1) Draw the cross-section and label the base with its length (top – bottom or right – left).
2) Figure out the area formula for that cross-section.
3) Integrate the area formula on the appropriate interval.

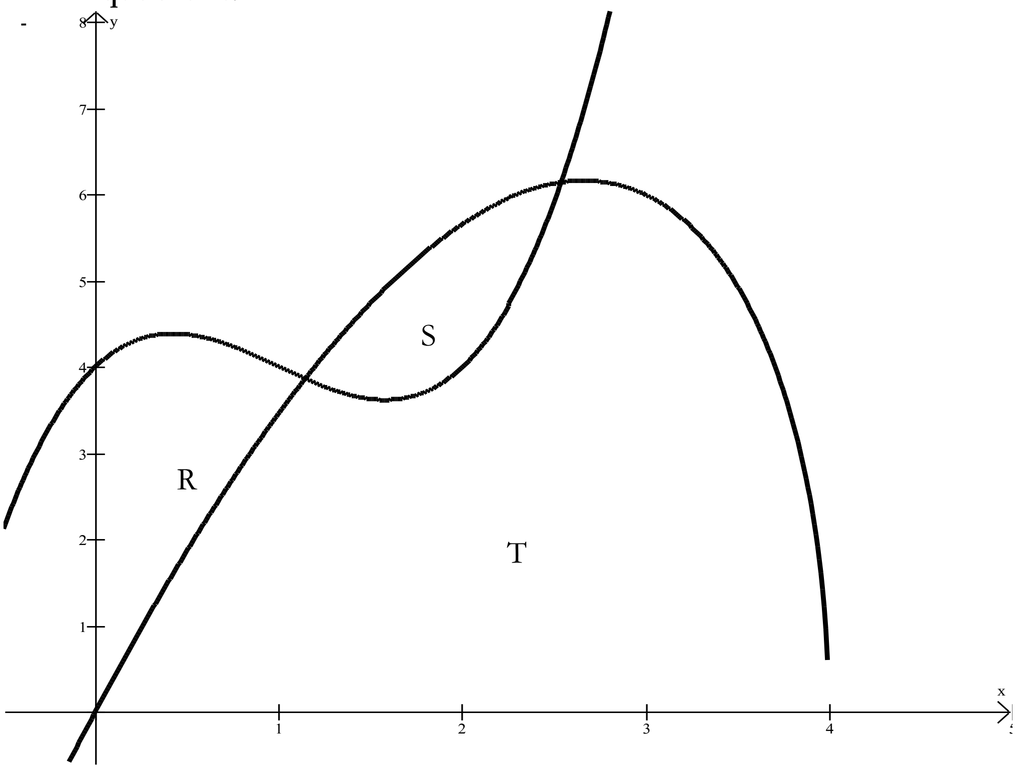
6.4 Homework

Use your grapher to sketch the regions described below. Find the points of intersection and find the volume of the solid that has the described region as its base and the given cross-sections.

1. $y=\sqrt{x}, y=e^{-2x}, x=1;$ the cross-sections are semi-circles.

2. $y=\ln\left(x^2+1\right)$ and $y=\cos x;$ the cross-sections are squares.

3. Given the curves $f(x) = x^3 - 3x^2 + 2x + 4$, $g(x) = 2x\sqrt{4-x}$ and in the first quadrant.



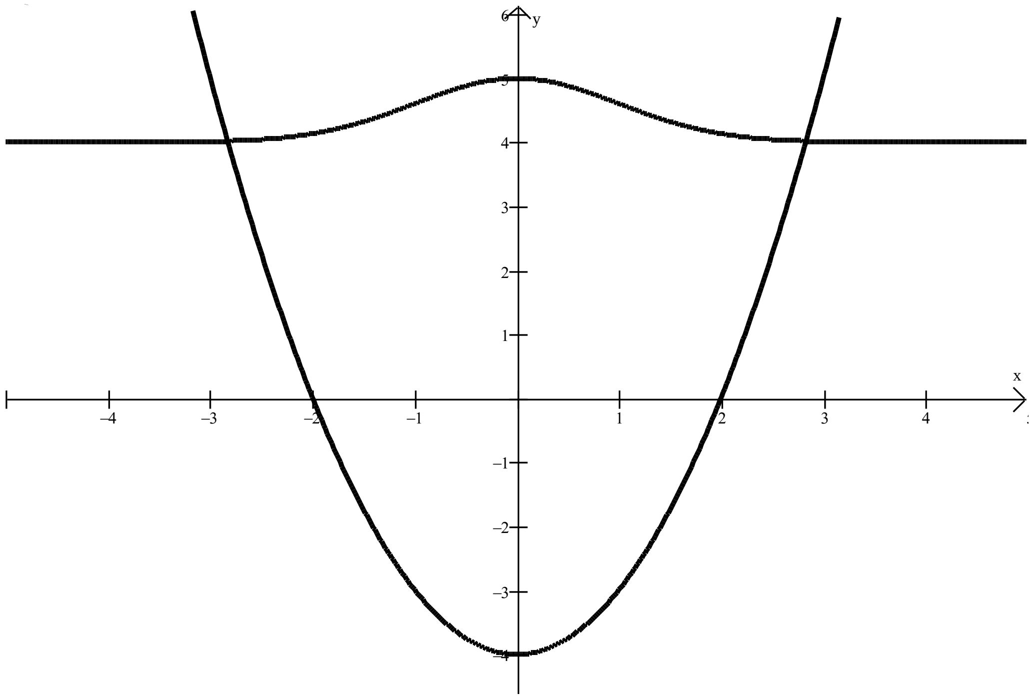
a. Find the area of the regions R, S, and T.

b. Find the volume of the solid generated by rotating the curve $g(x) = 2x\sqrt{4-x}$ around the line $y = 8$ on the interval $x \in [1,3]$.

c. Find the volume of the solid generated by rotating the region S around the line $y = -1$.

d. Find the volume of the solid generated if R forms the base of a solid whose cross sections are squares whose bases are perpendicular to the *x*-axis

4. Let f and g be the functions given by $f(x) = e^{\frac{-x^2}{2}} + 3$ and $g(x) = x^2 - 4$



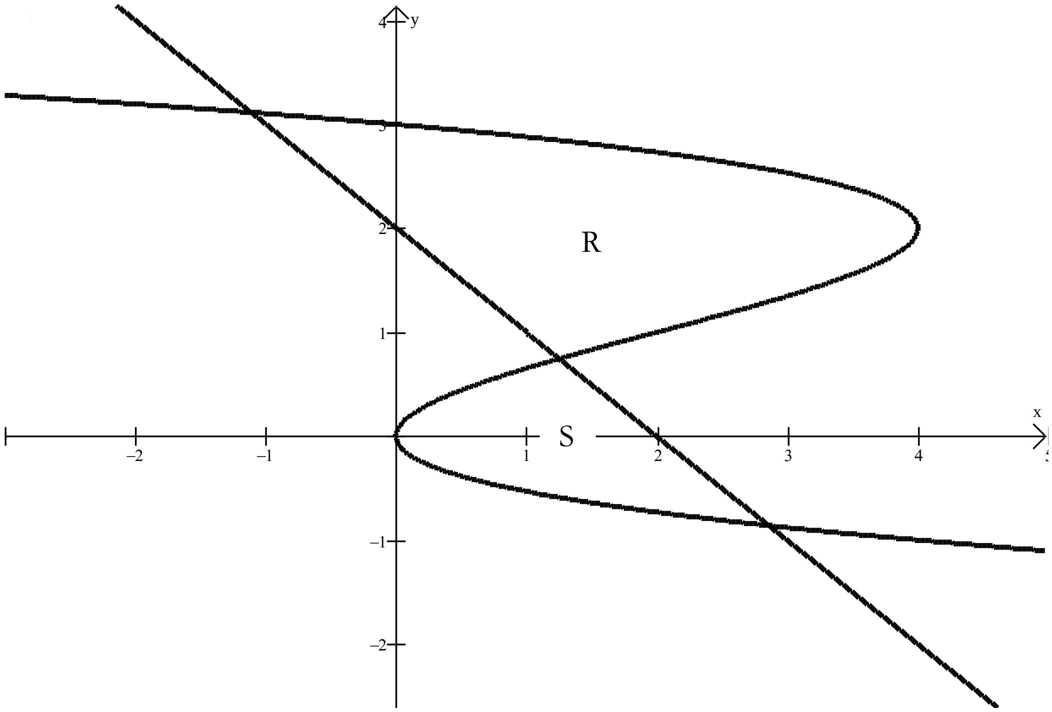
a. Find the area of the region bounded by two curves above.

b. Find the volume of the solid generated by rotating the region around the line $y = 4$.

c. Find the volume of the solid generated by rotating the area between $g(x)$ and the x-axis around the line $y = 0$.

d. Find the volume of the solid generated if the region above forms the base of a solid whose cross sections are semicircles with bases perpendicular to the x-axis.

5. Let R and S be the regions bounded by the graphs $x = y^2(3 - y)$ and $y = -x + 2$



a. Find the area of the regions R and S.

b. Find the volume of the solid when the region S is rotated around the line $x = 3$.

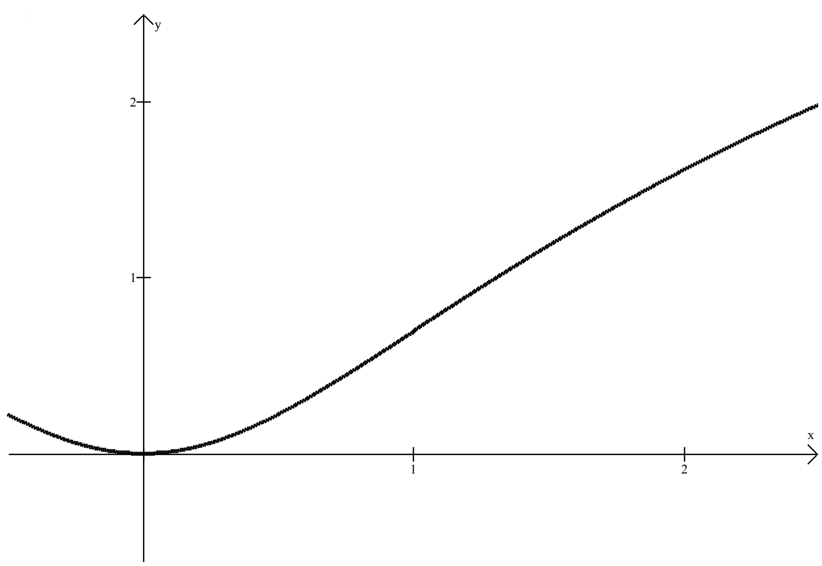
c. Find the volume of the solid when the region R is rotated around the line $x = -2$.

d. Find the volume of the solid if the region S forms the base of a solid whose cross sections are equilateral triangles with bases perpendicular to the y-axis. (Hint: the area of an equilateral triangle is $A = \frac{s^2\sqrt{3}}{4}$).

6. Let h be the function given by $h(x) = \ln(x^2 + 1)$ as graphed below. Find each of the following:

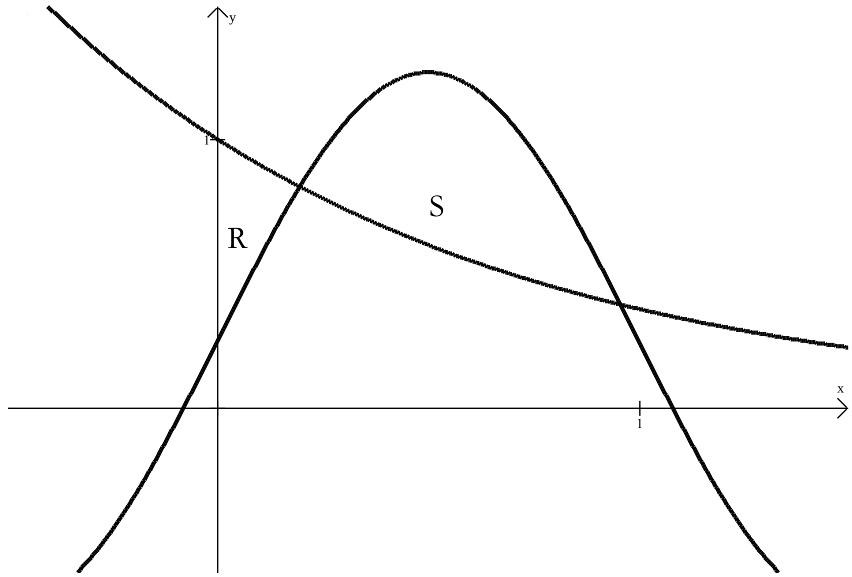
a. The volume of the solid generated when h is rotated around the line $y = 2$ on the interval $x \in [0, 2]$

b. The volume of the solid formed if the region between the curve and the lines $y = -1$, $x = 0$, and $x = 2$ forms the base of a solid whose cross sections are rectangles with heights twice their base and whose bases are perpendicular to the x-axis.



7. Let f and g be the functions given by $f(x)=\frac{1}{4}+\sin(\pi x)$ and $g(x)=e^{-x}$.

Let R be the region in the first quadrant enclosed by the y – axis and the graphs of f and g, and let S be the region in the first quadrant enclosed by the graphs of f and g, as shown in the figure below.



a. Find the area of R

b. Find the area of S

c. Find the volume of the solid generated when S is revolved around the horizontal line $y = -2$

d. The region R forms the base of a solid whose cross-sections are rectangles with bases perpendicular to the *x*-axis and heights equal to half the length of the base. Find the volume of the solid.

Answers: 6.4 Homework

1. $y=\sqrt{x}, y=e^{-2x}, x=1;$ the cross-sections are semi-circles.
   V=0.085
2. $y=\ln\left(x^2+1\right)$ and $y=\cos x;$ the cross-sections are squares.
   V=0.916
3. Given the curves $f(x)=x^3-3x^2+2x+4$, $g(x)=2x\sqrt{4-x}$ and in the first quadrant.
   a. Find the area of the regions R, S, and T.
      R=2.463 S=12.470 T=9.835
   b. Find the volume of the solid generated by rotating the curve $g(x)=2x\sqrt{4-x}$ around the line y = 8 on the interval $x\in[1,3]$.
      V=48.152
   c. Find the volume of the solid generated by rotating the region S around the line y = –1. V=55.727
   d. Find the volume of the solid generated if R forms the base of a solid whose cross sections are squares whose bases are perpendicular to the *x*-axis
      V=6.962

4. Let f and g be the functions given by $f(x)=e^{\frac{-x^2}{2}}+3$ and $g(x)=x^2-4$
   a. Find the area of the region bounded by two curves above.
      A=27.180
   b. Find the volume of the solid generated by rotating the region around the line y = 4.
      V=599.694
   c. Find the volume of the solid generated by rotating the area between $g(x)$ and the *x*-axis around the line y = 0.
      V=107.233
   d. Find the volume of the solid generated if the region above forms the base of a solid whose cross sections are semicircles with bases perpendicular to the *x*-axis.
      V=66.842

5. Let R and S be the regions bounded by the graphs $x = y^2(3-y)$ and $y = -x+2$

a. Find the area of the regions R and S.

R=2.193, S=6.185

b. Find the volume of the solid when the region S is rotated around the line $x = 3$.

V=55.345

c. Find the volume of the solid when the region R is rotated around the line $x = -2$.

V=45.081

d. Find the volume of the solid if the region S forms the base of a solid whose cross sections are equilateral triangles with bases perpendicular to the y-axis. (Hint: the area of an equilateral triangle is $A = \frac{s^2\sqrt{3}}{4}$).

V=8.607

6. Let h be the function given by $h(x) = \ln(x^2+1)$ as graphed below. Find each of the following:

a. The volume of the solid generated when h is rotated around the line $y = 2$ on the interval $x \in [0,2]$

V=4.881

b. The volume of the solid formed if the region between the curve and the lines $y = -1$, $x = 0$, and $x = 2$ forms the base of a solid whose cross sections are rectangles with heights twice their base and whose bases are perpendicular to the x-axis.

V=12.840

7. Let f and g be the functions given by $f(x) = \frac{1}{4} + \sin(\pi x)$ and $g(x) = e^{-x}$.

Let R be the region in the first quadrant enclosed by the y – axis and the graphs of f and g, and let S be the region in the first quadrant enclosed by the graphs of f and g, as shown in the figure below.

a. Find the area of R

R=0.071

b. Find the area of S

S=0.328

c. Find the volume of the solid generated when S is revolved around the horizontal line $y = -2$

V=5.837

d. The region R forms the base of a solid whose cross-sections are rectangles with bases perpendicular to the x-axis and heights equal to half the length of the base. Find the volume of the solid.

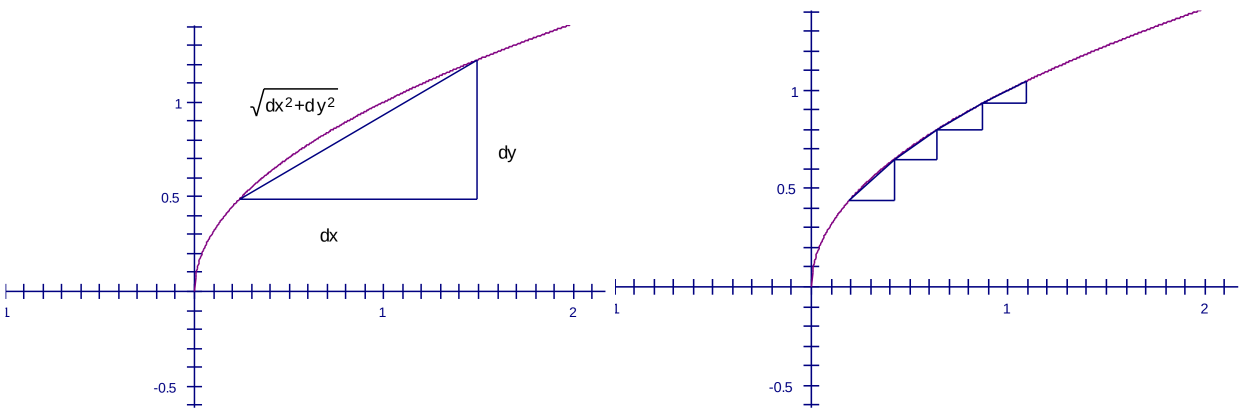
V=0.007

6.5 Arc Length

Back in your geometry days you learned how to find the distance between two points on a line. But what if we want to find the distance between to points that lie on a nonlinear curve?

Objectives:
Find the arc length of a function in Cartesian mode between two points.

Just as the area under a curve can be approximated by the sum of rectangles, arc length can be approximated by the sum of ever-smaller hypotenuses.



Here is our arc length formula:

Arc Length between Two Points:
Let $f(x)$ be a differentiable function such that $f'(x)$ is continuous, the length L of $f(x)$ over $[a,b]$ is given by

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \quad or \quad L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

As with the volume problems, we will use Math 9 in almost all cases to calculate arc length.

Ex 1 Find the length of the curve $y=1+e^x$ from $x=0$ to $x=3$.

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$ Start here

$$L=\int_0^3 \sqrt{1+\left(e^x\right)^2}\,dx$$ Substitute in for $\frac{dy}{dx}$

$$L=19.528$$ Math 9

Ex 2 Find the length of the curve $x=2y+y^3$ from $y=1$ to $y=4$.

$$L=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$
$$L=\int_1^4 \sqrt{1+\left(2+3y^2\right)^2}\,dy$$
$$L=69.083$$

Ex 3 Use right hand Riemann sums with n = 4 to estimate the arc length of $y=3+x\ln x$ from $x=1$ to $x=5$. How does this approximation compare with the true arc length (use Math 9)?

Let $f(x)=\sqrt{1+\left(1+\ln x\right)^2}$ Let $f(x)$ represent our integrand

$$L\approx\int_1^5 \sqrt{1+\left(1+\ln x\right)^2}\,dx$$
$$=1\cdot f(2)+1\cdot f(3)+1\cdot f(4)+1\cdot f(5)$$ Approximate using rectangles
$$=1.966+2.325+2.587+2.794$$
$$=9.672$$

Actual Value:

$$L=\int_1^5 \sqrt{1+\left(1+\ln x\right)^2}=9.026$$

<u>Note</u>: The integral is the arc length. Some people get confused as to how "rectangle areas" can get us length. Remember that the Riemann rectangles are a way of evaluating an integral by approximation. Just like the "area under the curve" could be a displacement if the curve was velocity, in this case, the "area under the curve $\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$ is the length of the arc along the curve *y*.

Ex 4 Find the length of the curve $f(x)=\int_2^{x^3} \sqrt{t+1}\,dt$ on $2\le t\le 7$.

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_a^b \sqrt{1+\left(3x^2\sqrt{x^3+1}\right)^2}\,dx \\ &= 4235.511 \end{aligned}$$

6.5 Homework

Find the arc length of the curve.

1. $y = 1 + 6x^{3/2}$ on $x \in [0,1]$

2. $y = \frac{1}{2}x^2 + \frac{1}{4}\ln x$ on $x \in [2,4]$

3. $x = \frac{1}{3}\sqrt{y}(y-3)$ on $y \in [1,9]$

4. $y = \ln(\sec x)$ on $x \in \left[0, \frac{\pi}{4}\right]$

5. $y = \ln x^{3/2}$ on $x \in \left[1, \sqrt{3}\right]$

6. $y = e^x$ on $x \in [0,1]$

7. Find the perimeter of each of the two regions bounded by $y = x^2$ and $y = 2^x$.

8. Find the length of the arc along $f(x) = \int_0^x \sqrt{\cos t}\, dt$ on $x \in \left[0, \frac{\pi}{2}\right]$.

9. Find the length of the arc along $f\left(x\right)=\int_{-2}^{x}\sqrt{3t^4-1}\,dt$ on $x\in\left[-2,\ 1\right]$.

Answers: 6.5 Homework

1. $y = 1 + 6x^{3/2}$ on $x \in [0,1]$

   L=6.103

2. $y = \frac{1}{2}x^2 + \frac{1}{4}\ln x$ on $x \in [2,4]$

   L=6.499

3. $x = \frac{1}{3}\sqrt{y}\,(y-3)$ on $y \in [1,9]$

   L=10.177

4. $y = \ln(\sec x)$ on $x \in \left[0, \frac{\pi}{4}\right]$

   L=0.881

5. $y = \ln x^{3/2}$ on $x \in \left[1, \sqrt{3}\right]$

   L=1.106

6. $y = e^x$ on $x \in [0,1]$

   L=2.003

7. Find the perimeter of each of the two regions bounded by $y = x^2$ and $y = 2^x$.

   L=34.553

8. Find the length of the arc along $f(x) = \int_0^x \sqrt{\cos t}\, dt$ on $x \in \left[0\ ,\ \frac{\pi}{2}\right]$.

   L=2

9. Find the length of the arc along $f(x) = \int_{-2}^x \sqrt{3t^4 - 1}\, dt$ on $x \in [-2,\ 1]$.

   L=1.337

Volume Test

1. The area of the region enclosed by $y = x^2 - 4$ and $y = x - 4$ is given by

(a) $\int_0^1 \left(x - x^2\right) dx$

(b) $\int_0^1 \left(x^2 - x\right) dx$

(c) $\int_0^2 \left(x - x^2\right) dx$

(d) $\int_0^2 \left(x^2 - x\right) dx$

(e) $\int_0^4 \left(x^2 - x\right) dx$

---

2. Which of the following integrals gives the length of the graph $y = \tan x$ between $x=a$ to $x=b$ if $0 < a < b < \dfrac{\pi}{2}$?

(a) $\int_a^b \sqrt{x^2 + \tan^2 x}\,dx$

(b) $\int_a^b \sqrt{x + \tan x}\,dx$

(c) $\int_a^b \sqrt{1 + \sec^2 x}\,dx$

(d) $\int_a^b \sqrt{1 + \tan^2 x}\,dx$

(e) $\int_a^b \sqrt{1 + \sec^4 x}\,dx$

3. Let R be the region in the first quadrant bounded by $y = e^{x/2}$, $y = 1$ and $x = \ln 3$. What is the volume of the solid generated when R is rotated about the *x*-axis?

(a) 2.80 (b) 2.83 (c) 2.86 (d) 2.89 (e) 2.92

---

4. The base of a solid is the region enclosed by $y = \cos x$ and the *x* axis for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. If each cross-section of the solid perpendicular to the *x*-axis is a square, the volume of the solid is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi^2}{2}$ (e) 2

---

5. A region is bounded by $y = \frac{1}{x}$, the x-axis, the line $x = m$, and the line $x = 2m$, where $m > 0$. A solid is formed by revolving the region about the *x*-axis. The volume of the solid

(a) is independent of *m*.

(b) increases as *m* increases.

(c) decreases as *m* increases.

(d) increases until $m = \frac{1}{2}$, then decreases.

(e) is none of the above

6. Let R be the region in the first quadrant bounded by $y = \sin^{-1} x$, the y-axis, and $y = \frac{\pi}{2}$. Which of the following integrals gives the volume of the solid generated when R is rotated about the y-axis?

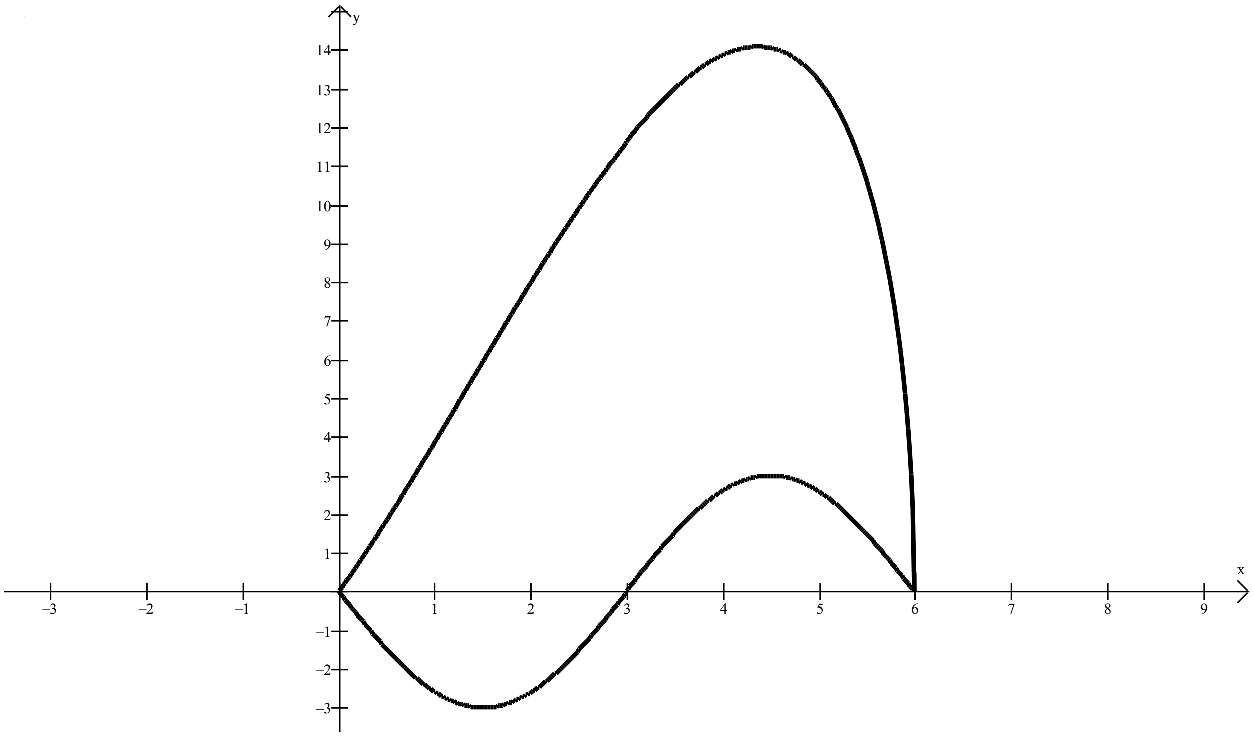
(a) $\pi \int_0^{\frac{\pi}{2}} y^2 dy$

(b) $\pi \int_0^{1} \left(\sin^{-1} x\right)^2 dx$

(c) $\pi \int_0^{\frac{\pi}{2}} \left(\sin^{-1} x\right)^2 dx$

(d) $\pi \int_0^{\frac{\pi}{2}} \left(\sin y\right)^2 dy$

(e) $\pi \int_0^{1} \left(\sin y\right)^2 dy$



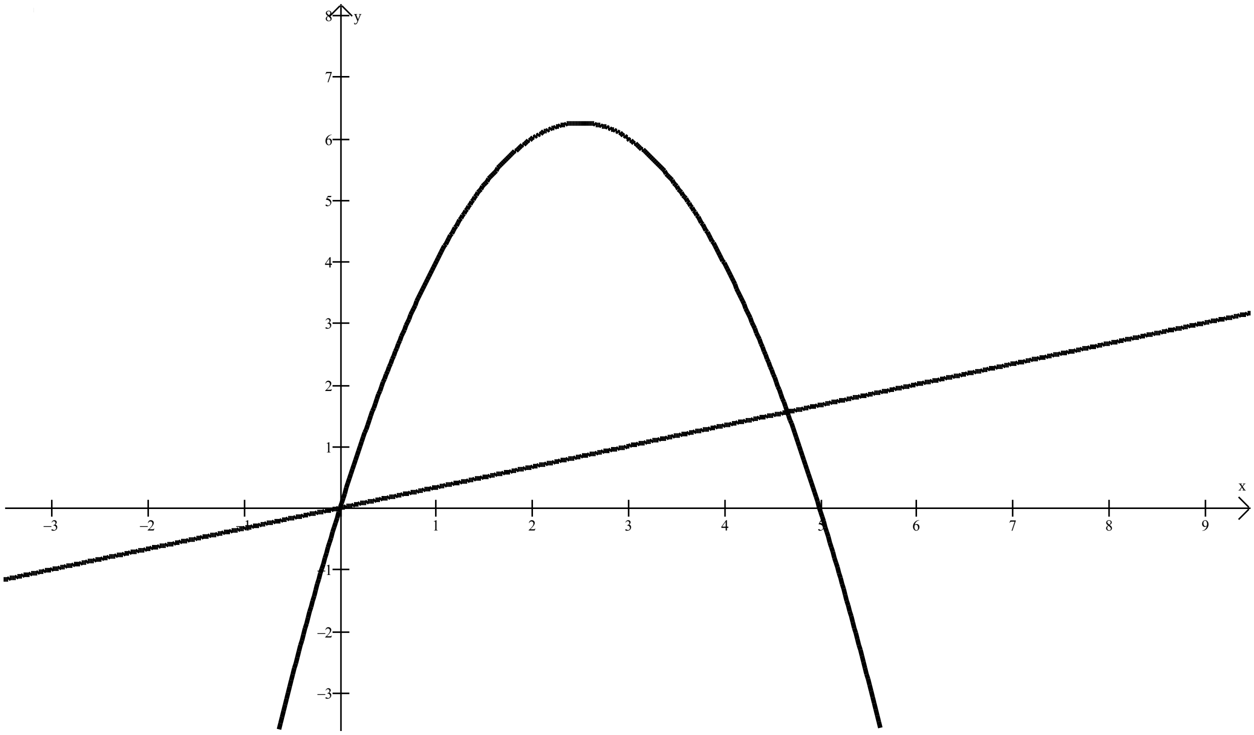
7. Let R be the region bounded by the graphs $f(x) = x\sqrt{12+4x-x^2}$ and $g(x) = -3\sin\left(\dfrac{\pi}{3}x\right)$ pictured above .

(a) Find the area of R.

(b) Find the volume of the figure if R is rotated around the line $y = -4$.

(c) The region, R, is the base of a solid whose cross-sections are squares perpendicular to the *x*-axis. Find the volume of this solid.

(d) Find the volume of the solid if **only** $f(x) = x\sqrt{12 + 4x - x^2}$ is rotated around the *x*-axis.



8. Let S be the region bounded by the curves $f(x)=\frac{1}{3}x$ and $g(x)=-x^2+5x$.

(a) Find the area between the two curves.

(b) Find the volume formed by rotating the region S around the line $y = 7$.

(c) Find the perimeter of the region S.

(d) Find the volume formed by rotating the region S around the x- axis.

Answers: Volume Test

1. The area of the region enclosed by $y = x^2 - 4$ and $y = x - 4$ is given by

(a) $\int_0^1 \left(x - x^2\right) dx$

(b) $\int_0^1 \left(x^2 - x\right) dx$

(c) $\int_0^2 \left(x - x^2\right) dx$

(d) $\int_0^2 \left(x^2 - x\right) dx$

(e) $\int_0^4 \left(x^2 - x\right) dx$

---

2. Which of the following integrals gives the length of the graph $y = \tan x$ between $x=a$ to $x=b$ if $0 < a < b < \frac{\pi}{2}$?

(a) $\int_a^b \sqrt{x^2 + \tan^2 x}\, dx$

(b) $\int_a^b \sqrt{x + \tan x}\, dx$

(c) $\int_a^b \sqrt{1 + \sec^2 x}\, dx$

(d) $\int_a^b \sqrt{1 + \tan^2 x}\, dx$

(e) $\int_a^b \sqrt{1 + \sec^4 x}\, dx$

3. Let R be the region in the first quadrant bounded by $y = e^{x/2}$, $y = 1$ and $x = \ln 3$. What is the volume of the solid generated when R is rotated about the x-axis?

(a) 2.80 (b) 2.83 (c) 2.86 (d) 2.89 (e) 2.92

---

4. The base of a solid is the region enclosed by $y = \cos x$ and the x axis for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. If each cross-section of the solid perpendicular to the x-axis is a square, the volume of the solid is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi^2}{2}$ (e) 2

---

5. A region is bounded by $y = \frac{1}{x}$, the x-axis, the line $x = m$, and the line $x = 2m$, where $m > 0$. A solid is formed by revolving the region about the x-axis. The volume of the solid

(a) is independent of *m*.

(b) increases as *m* increases.

(c) decreases as *m* increases.

(d) increases until $m = \frac{1}{2}$, then decreases.

(e) is none of the above

6. Let R be the region in the first quadrant bounded by $y = \sin^{-1} x$, the y-axis, and $y = \frac{\pi}{2}$. Which of the following integrals gives the volume of the solid generated when R is rotated about the y-axis?

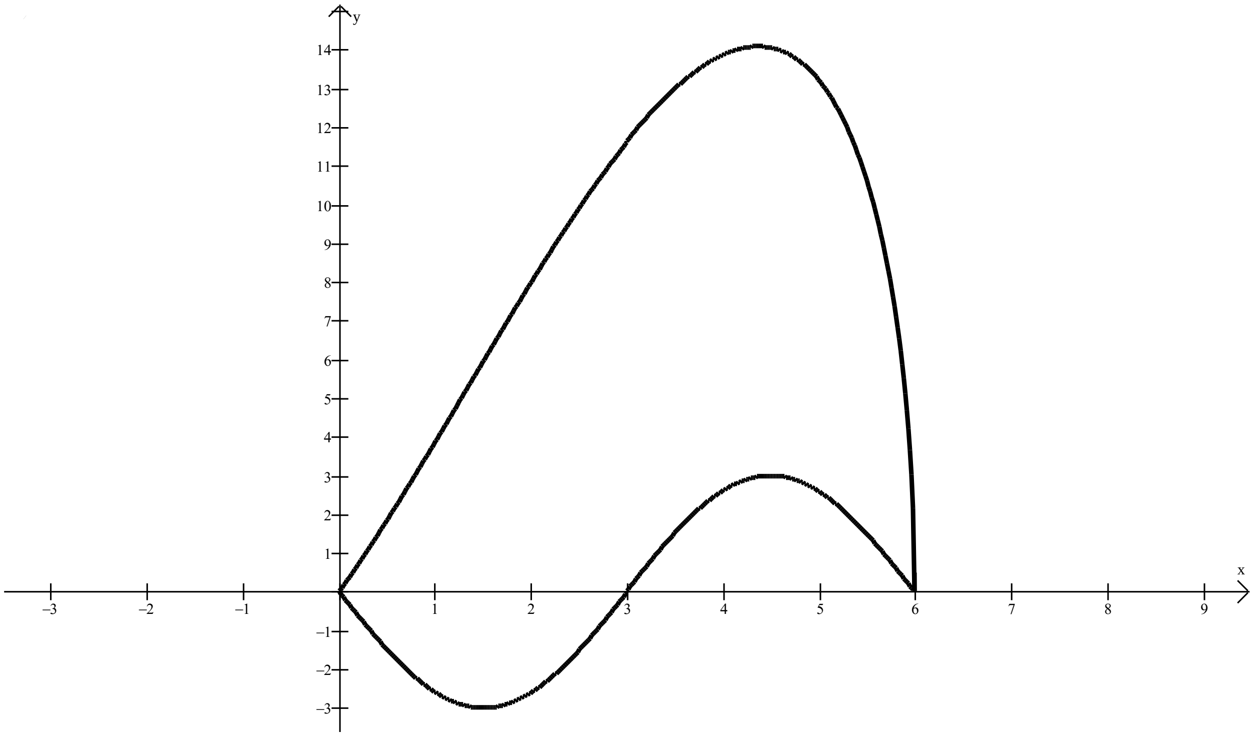
(a) $\pi \int_0^{\frac{\pi}{2}} y^2 dy$

(b) $\pi \int_0^1 \left(\sin^{-1} x\right)^2 dx$

(c) $\pi \int_0^{\frac{\pi}{2}} \left(\sin^{-1} x\right)^2 dx$

(d) $\pi \int_0^{\frac{\pi}{2}} \left(\sin y\right)^2 dy$

(e) $\pi \int_0^1 \left(\sin y\right)^2 dy$



7. Let R be the region bounded by the graphs $f(x) = x\sqrt{12+4x-x^2}$ and $g(x) = -3\sin\left(\dfrac{\pi}{3}x\right)$ pictured above .

(a) Find the area of R.

R=54.595

(b) Find the volume of the figure if R is rotated around the line $y = -4$.

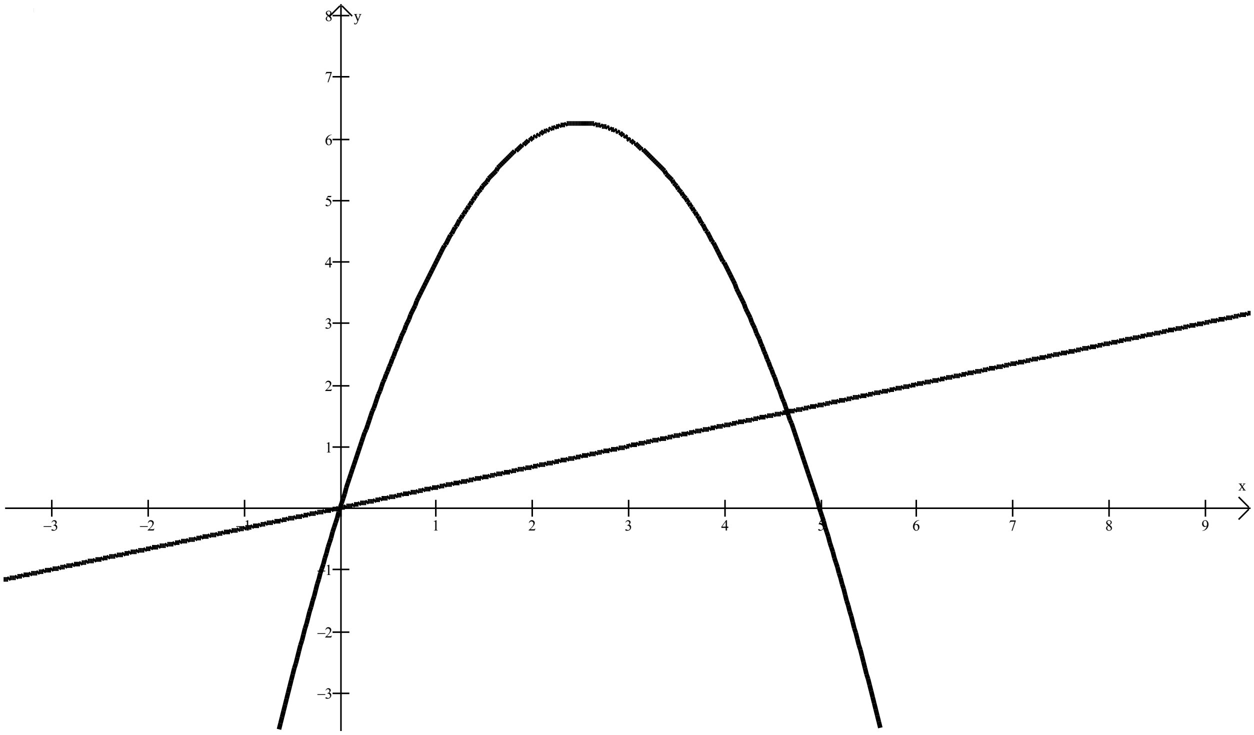
V=3179.787

(c) The region, R, is the base of a solid whose cross-sections are squares perpendicular to the x-axis. Find the volume of this solid.

V=549.698

(d) Find the volume of the solid if **only** $f(x) = x\sqrt{12+4x-x^2}$ is rotated around the x-axis.

V=1900.035



8. Let S be the region bounded by the curves $f(x)=\frac{1}{3}x$ and $g(x)=-x^2+5x$.

(a) Find the area between the two curves.

S=16.948

(b) Find the volume formed by rotating the region S around the line $y = 7$.

V=430.435

(c) Find the perimeter of the region S.

P=7.232

(d) Find the volume formed by rotating the region S around the x- axis.

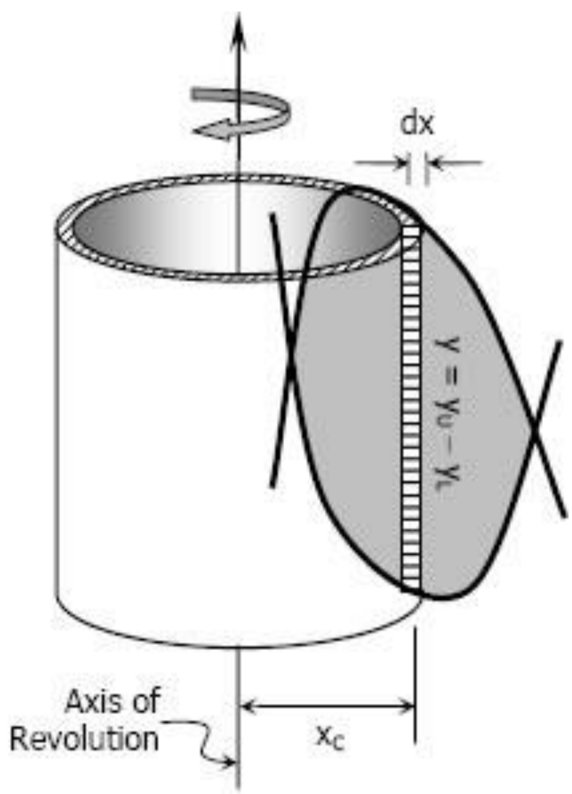
V=314.549

6.__ Bonus Section: Volume by the Shell Method

There is another method we can use to find the volume of a solid of rotation. With the Disc/Washer method we drew our sample pieces perpendicular to the axis of rotation. Now we will look into calculating volumes when our sample pieces are drawn parallel to the axis of rotation (**parallel** and **shell** rhyme – that's how you can remember they go together).

Objectives:
Find the volume of a solid rotated when a region is rotated about a given line.

Take the following function, let's call it $f(x)$, from a to b and rotate it about the y – axis.



$$V = 2\pi\int_{x_1}^{x_2} x_C(y_U - y_L)\,dx$$

How do we begin to find the volume of this solid?

Imagine taking a tiny strip of the function and rotating this little piece around the y – axis. What would that piece look like once it was rotated?
What is the surface area formula for a cylindrical shell? $2\pi rl$

Now would the surface area of just this one piece give you the volume of the entire solid?
How would we add up all of the surface areas of all these little pieces?

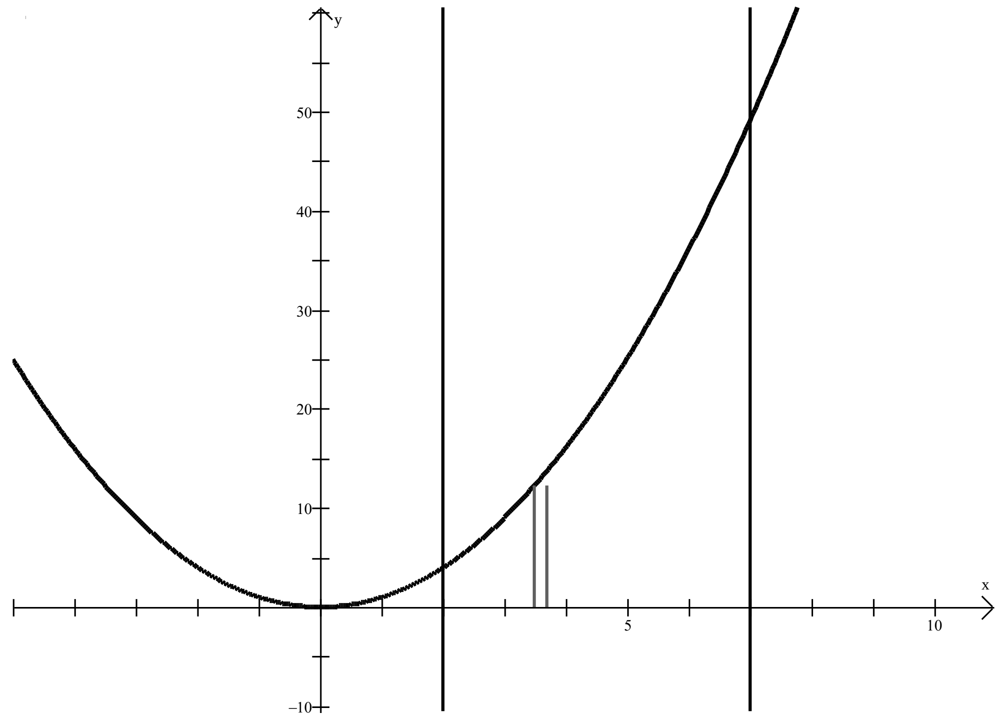
We arrive at our volume formula.

<u>Volume by Shell Method:</u> The volume of the solid generated when the region bounded by function $f(x)$, from $x = a$ and $x = b$, where $f(x) \geq 0$ [or $g(y)$, from $y = c$ and $y = d$, where $g(y) \geq 0$], is rotated about the y – axis is given by

$$V = 2\pi\int_a^b x \cdot f(x)dx \text{ or } V = 2\pi\int_c^d y \cdot g(y)dy$$

** Everything rhymes with the shell method – Shell/Parallel/ $2\pi rl$ so the disc method must be of the form Disc/Perpendicular/ πr^2.

Ex 1 Let R be the region bounded by the equations $y = x^2$, $y = 0$, $x = 1, x = 3$.
Find the volume of the solid generated when R is rotated about the y – axis.



$V = 2\pi\int_a^b x \cdot y\,dx$ Our pieces are parallel to the y – axis, so our integrand will contain dx

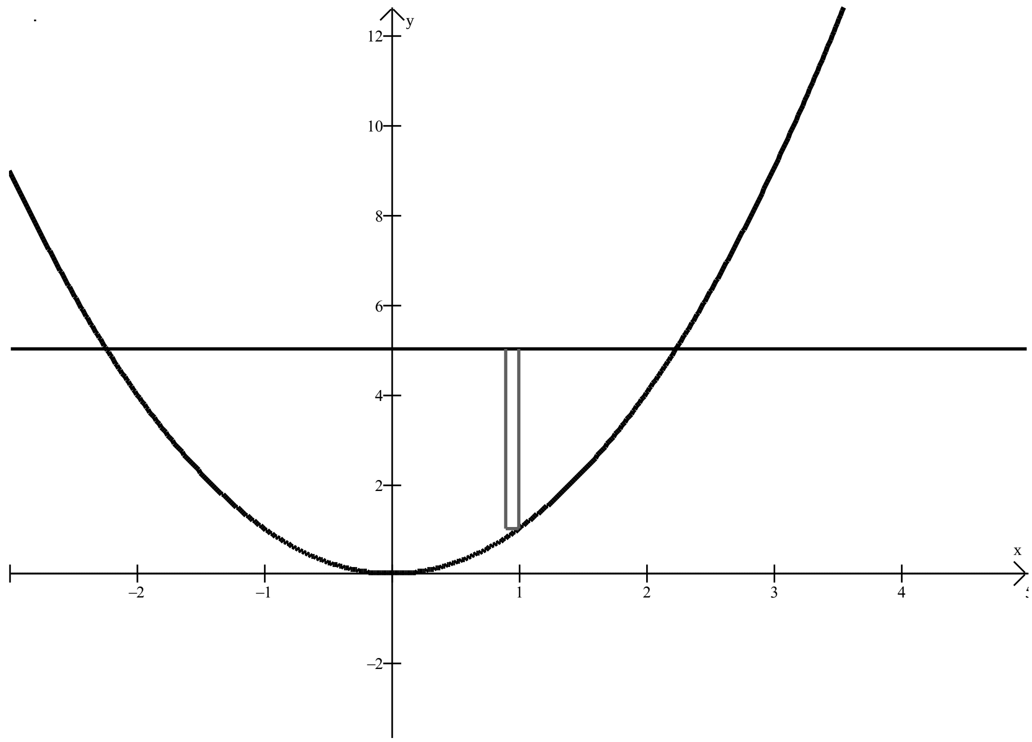
$V = 2\pi\int_a^b x \cdot y\,dx$ We cannot integrate y with respect to x so we will substitute out for y

$V = 2\pi\int_a^b x \cdot x^2 dx$ The expression for y is x^2

$V = 2\pi\int_1^3 x \cdot x^2 dx$ Our region extends from $x = 1$ to $x = 3$

$V = 40\pi$

Ex 2 Let R be the region bounded by $y = x^2, y = 5$, and $x = 0$. Find the volume of the solid generated when R is rotated about the y – axis.



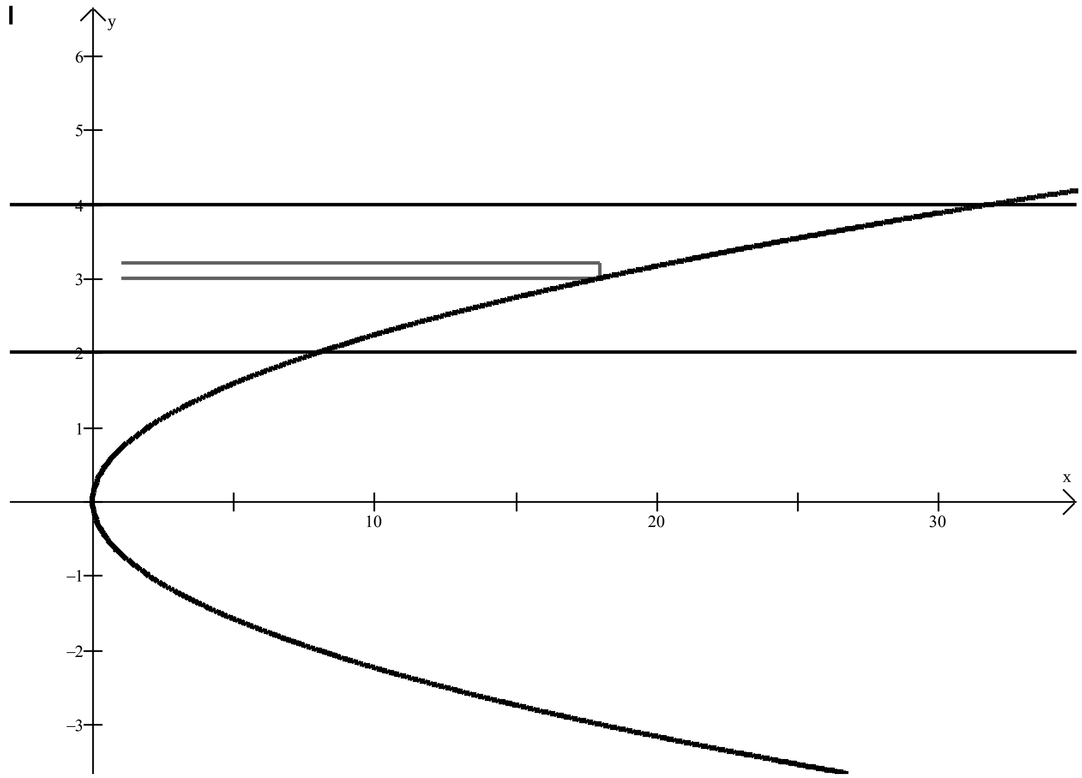
$V = 2\pi\int_a^b x\cdot(5-y)\,dx$ Our pieces are parallel to the y – axis, so our integrand will contain dx. We cannot integrate y with respect to x so we will substitute out for y.

$V = 2\pi\int_a^b x\cdot(5-x^2)\,dx$ The expression for y is x^2.

$V = 2\pi\int_0^{\sqrt{5}} x\cdot(5-x^2)\,dx$ Our region extends from $x=0$ to $x=\sqrt{5}$

$V = {}^{50\pi}\!/_{4}$

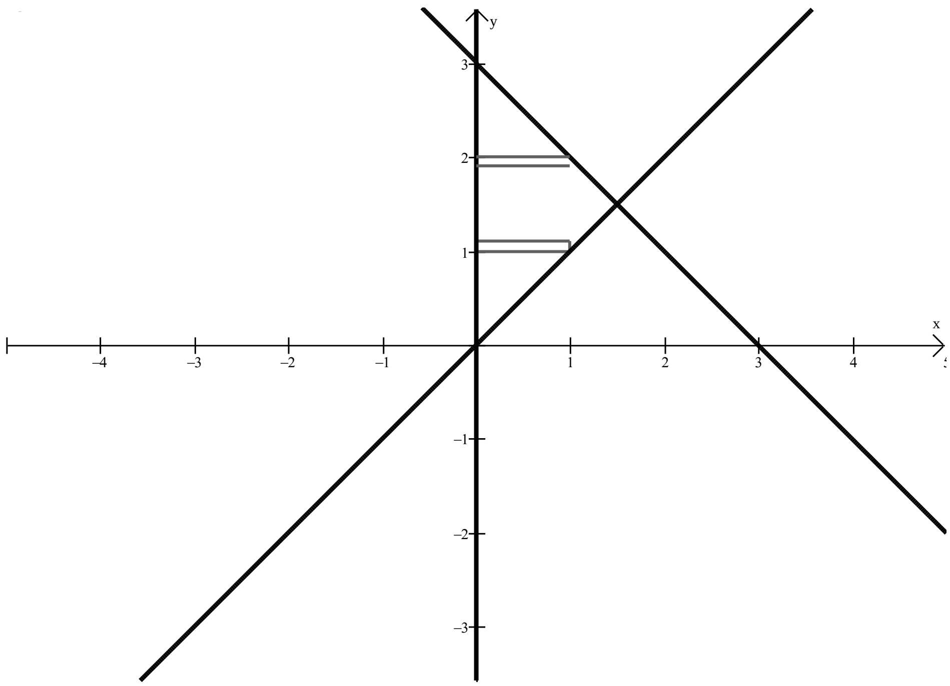
Ex 3 Let R be the region bounded by $x=2y^2, x=1,\ y=2$ and $y=4$. Find the volume of the solid generated when R is rotated about the x – axis.



$$V = 2\pi\int_c^d y \cdot x dy$$

$$V = 2\pi\int_2^4 y \cdot 2y^2 dy$$
$$V = 4\pi\int_2^4 y^3 dy$$
$$V = 240\pi$$

Ex 4 Let R be the region bounded by $y = x, x + y = 3$, and $x = 0$. Find the volume of the solid generated when R is rotated about the x – axis.



$$V = 2\pi \int_c^d y \cdot x dy$$

Our pieces are parallel to the x – axis, so our integrand will contain dy. But we will need two integrals as our region of rotation is bounded by different curves. The dividing line is $y = \frac{3}{2}$

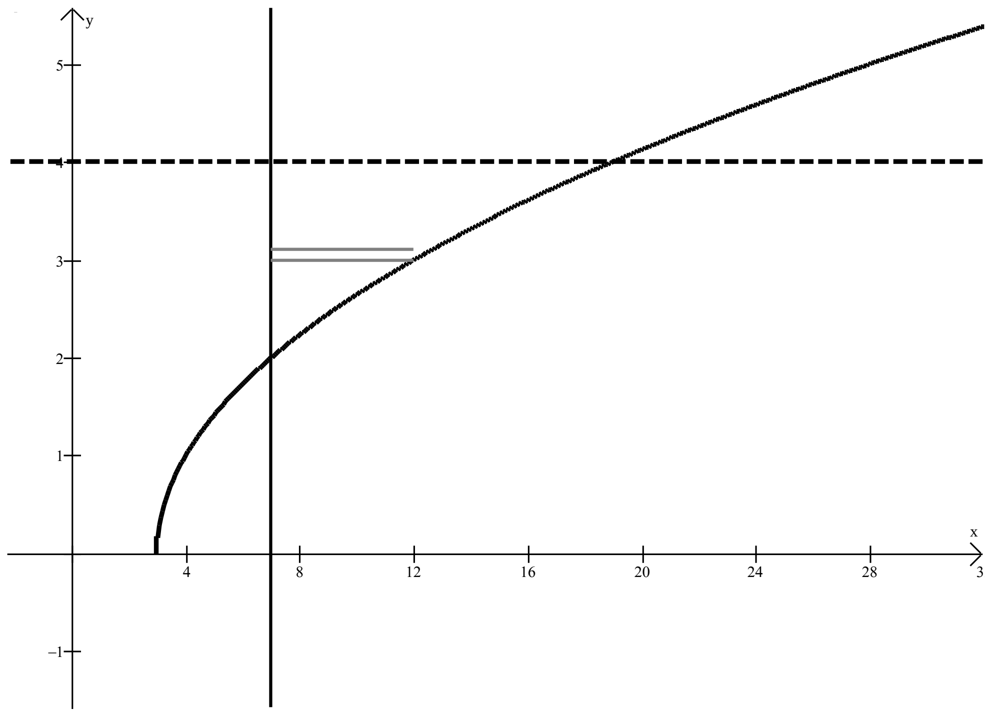
$$V = 2\pi \int_0^{3/2} y \cdot x_1 dy + 2\pi \int_{3/2}^{3} y \cdot x_2 dy$$

$$V = 2\pi \int_0^{3/2} y \cdot y dy + 2\pi \int_{3/2}^{3} y \cdot (-y + 3) dy$$

$$V = \frac{27\pi}{4}$$

The Shell Method can be applied to rotating regions about lines other than the axes, just as the Disk and Washer Methods were.

Ex 5 Let R be the region bounded by the equations $y=\sqrt{x-3}$, the x – axis, and $x=7$. Find the volume of the solid generated when R is rotated about the line $y=4$.



$V=2\pi\int_c^d y\cdot x\,dy$ Our pieces are parallel to the x – axis, so our integrand will contain dy

$V=2\pi\int_c^d (4-y)\cdot(7-x)\,dy$ We cannot integrate x with respect to y so we will substitute out for x

$V=2\pi\int_c^d (4-y)\cdot\left(7-\left(y^2+3\right)\right)dy$ The expression for x is y^2+3

$V=2\pi\int_0^2 (4-y)\cdot\left(7-\left(y^2+3\right)\right)dy$ Our region extends from $y=0$ to $y=2$

$V=108.909$

The Washer Method yields the same result.

$$V = \pi \int_a^b \left(R^2 - r^2 \right) dx$$

$$V = \pi \int_a^b \left((y_1)^2 - (4 - y_2)^2 \right) dx$$

$$V = \pi \int_3^7 \left[(4)^2 - \left(4 - \sqrt{x-3}\right)^2 \right] dx$$

$$V = 108.909$$

6.__ Homework Set A

Find the volume of the solid formed by rotating the described region about the given line.

1. $y = \sec x, y = 1, x = 0, x = \frac{\pi}{6}$; about the y-axis.

2. $y = x^3, x = 0, y = 8$; about the y-axis.

3. $y = \frac{1}{x^{2/3}}, x = 1, x = 8, y = 0$; about the y-axis.

4. $y = e^{-x^2}, y = 0, x = -2, x = 2$; about the x-axis.

5. $y = x(x-1)^2$ and the x-axis; about the y-axis.

6. $y = \sin x^2$ and the x-axis on $x \in \left[0, \sqrt{\pi}\right]$; about the y-axis.

Use your grapher to sketch the regions described below. Find the points of intersection and find the volume of the solid formed by rotating the described region about the given line.

7. $y = \sqrt{x}, y = e^{-2x}, x = 1$; about the line $x = 1$.

8. $y = \ln\left(x^2+1\right), y = \cos x$; about the line $x = 2$.

9. $y = x^2, y = 2^x$; about the line $x = -1$.

Answers: 6.4 Homework Set A

1. $y=\sec x, y=1, x=0, x=\frac{\pi}{6}$; about the y-axis.

   V=0.064

2. $y=x^3, x=0, y=8$; about the y-axis.

   V=60.319

3. $y=\frac{1}{x^{2/3}}, x=1, x=8, y=0$; about the y-axis.

   V=70.689

4. $y=e^{-x^2}, y=0, x=-2, x=2$; about the x-axis.

   V=1.969

5. $y=x(x-1)^2$ and the x-axis; about the y-axis.

   V=0.209

6. $y=\sin x^2$ and the x-axis on $x\in\left[0,\sqrt{\pi}\right]$; about the y-axis.

   V=2π

7. $y=\sqrt{x}, y=e^{-2x}, x=1$; about the line $x=1$.

   V=0.554

8. $y=\ln\left(x^2+1\right), y=\cos x$; about the line $x=2$.

   V=14.676

9. $y=x^2, y=2^x$; about the line $x=-1$.

   V=55.428

Appendix A: Synthesis of Precalculus

Analytic Geometry and Calculus are closely related subjects. Analytic Geometry is the study of functions and relations as to how their graphs on the Cartesian System relate to the algebra of their equations. Traditionally, Calculus has been the study of functions with a particular interest in tangent lines, maximum/minimum points, and area under the curve. (The Reform Calculus movement places an emphasis on how functions change, rather than the static graph, and consider Calculus to be a study of change and of motion.) Consequently, there is a great deal of overlap between the subjects. The advent of graphing calculators has blurred the distinctions between these fields and made subjects that had previously been strictly Calculus topics easily accessible at the lower level. The point of this course is to thoroughly discuss the subjects of Analytic Geometry that directly pertain to entry-level Calculus and to introduce the concepts and algebraic processes of first semester Calculus. ***Because we are considering these topics on the Cartesian Coordinate System, it is assumed that we are using Real Numbers, unless the directions state otherwise.***

Last year's text was designed to study the various families of functions in light of the main characteristics--or TRAITS-- that the graphs of each family possess. Each chapter a different family and a) review what is known about that family from Algebra 2, b) investigate the analytic traits, c) introduce the Calculus Rule that most applies to that family, d) put it all together in full sketches, and e) take one step beyond. In Chapter 0 of this text, we will review the traits that do not involve the derivative before going into the material that is properly part of Calculus. Part of what will make the Calculus much easier is if we have an arsenal of basic facts that we can bring to any problem. Many of these facts concern the graphs that we learned about individually in Precalculus. We need to synthesize all of this material into a cohesive body of information so that we can make the best use of it in calculus.

Basic Concepts and Definitions

Before we can start, we have to develop a common vocabulary that will be used throughout this course. Much of it comes directly from Algebra 1 and 2.

Domain-- Defn: "The set of values of the independent variable."
Means: the set of *x*-values that can be substituted into the equation to get a Real *y*-value (i.e., *no zero denominator, no negative under an even radical, and no negatives or zero in a logarithm*).

Range-- Defn: "The set of values of the dependent variable."
Means: the set of *y*-values that can come from the equation.

Relation-- Defn: "A set of ordered pairs."
Means: the equation that creates/defines the pairs.

Function-- Defn: "A relation for which there is exactly one value of the dependent variable for each value of the independent variable."
Means: an equation where every *x* gets only one *y*.

Degree-- Defn: "The maximum number of variables that are multiplied together in any one term of the polynomial."
Means: Usually, the highest exponent on a variable.

Families of Functions--Algebraic

1) *Polynomial*--Defn: "An expression containing no other operations than addition, subtraction, and multiplication performed on the variable."
Means: any equation of the form $y = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$, where *n* is a non-negative integer.

2) *Rational*--Defn: "An expression that can be written as the ratio of one polynomial to another."
Means: an equation with an *x* in the denominator.

3) *Radical* (*Irrational*)--Defn: "An expression whose general equation contains a root of a variable and possibly addition, subtraction, multiplication, and/or division."
Means: An equation with an *x* in a radical.

Families of Functions--Transcendental

1) *Exponential*--Defn: "A function whose general equation is of the form $y = a \cdot b^x$."
   Means: there is an x in the exponent.

2) *Logarithmic*--Defn: "The inverse of an exponential function."
   Means: there is a Log or Ln in the equation.

3) *Trigonometric*--Defn: "A function (sin, cos, tan, sec, csc, or cot) whose independent variable represents an angle measure."
   Means: an equation with sine, cosine, tangent, secant, cosecant, or cotangent in it.

4) *Trigonometric Inverse (or ArcTrig)*--Defn: "A function ($\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\sec^{-1}$, $\csc^{-1}$, or $\cot^{-1}$) whose <u>dependent</u> variable represents an angle measure."
   Means: an equation with Arcsine, Arccosine, Arctangent, Arcsecant, Arccosecant, or Arccotangent in it.

Families of Functions --Other

1) *Piece-wise Defined*--A function that is defined by different equations for different parts of its domain.

2) *Inverse*--Two functions or relations that cancel/reverse each other (e.g., $y = x^2$ and $y = \sqrt{x}$).

Much of the graphical work done last year involved finding and/or graphing traits. Traits are the characteristics or defining aspects of a graph and its equation. The following are the Eleven Traits.

TRAITS

1. Domain
2. Range
3. Y-intercept
4. Zeros
5. Vertical Asymptotes
6. Points of Exclusion
7. End Behavior, in particular:
   - 7a. Horizontal Asymptotes
   - 7b. Slant Asymptotes
8. Extremes
9. Intervals of Increasing and Decreasing
10. Points of Inflection
11. Intervals of Concavity

This next table reminds us of what traits each kind of function has.

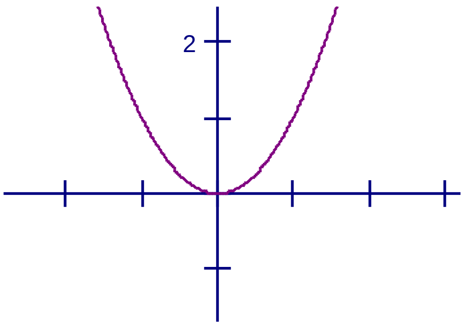
Traits	1	2	3	4	5	6	7	8/9	10/11
Polynomials	X	X	X	X			X	X	X
Rationals	X	X	X	X	X	X	X	X	X
Radicals	X	X	X	X			X	X	X
Sinusoidals	X	X	X	X				X	X
Other Trigs	X	X	X	X	X		X	X	X
Trig Inverses	X	X	X	X	X			X	X
Exponentials	X	X	X				X	X	X
Logs	X	X		X	X		X	X	X

PRODUCTS AND COMPOSITES WILL HAVE ALL THE TRAITS OF THE FUNCTIONS THEY ARE MADE OF.

A.1: Recognizing and Identifying Graphs, Graphing from Traits

Key Concept: Graph, Equations and numerical values are interrelated in that they are different representations of the same concept--the function.

Illustration:



$y = x^2$

x	y
-2	4
-1	1
0	0
1	1
2	4

Each of these represents a set of ordered pairs that make up this function. The equation and graph represent all the ordered pairs, while the table only lists a few pairs, but the basic idea is the same in each case.

OBJECTIVES

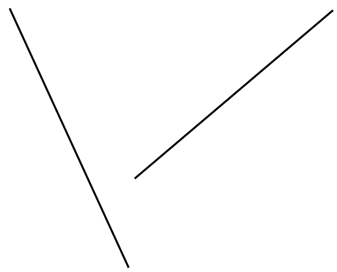
Memorize graphs that are common in algebra and Calculus.
Classify graphs into families of functions.
Sketch graphs from traits.

In this section, we will review the families (or kinds) of functions and their general graphs. The traits will be explored and reviewed more in depth in various sections of this chapter and the traits related to the derivative will be explored in other chapters.

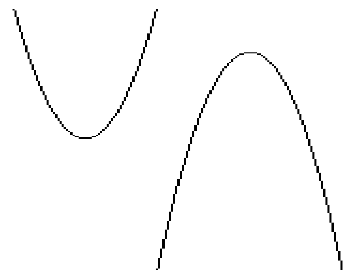
As noted in the Overview, there are nine kinds of functions in which we are interested and that fall into three categories.

Algebraic Functions

1) Polynomial: $y = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$

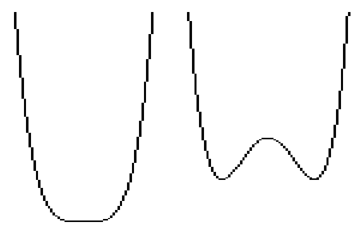


Linear polynomials



Quadratic polynomials

Cubic polynomials

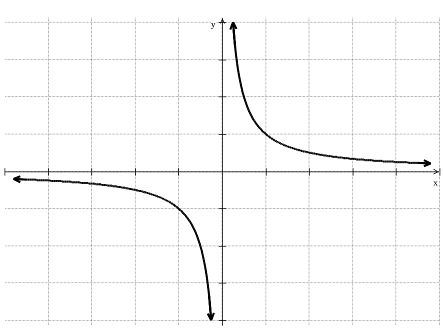


Quartic polynomials

Polynomials are among the most basic of graphs – you can recognize them by the fact that they have no horizontal or vertical asymptotes. They generally look like parabolas or cubics but sometimes with more "turns" or "bumps" (the maxima and mininma.

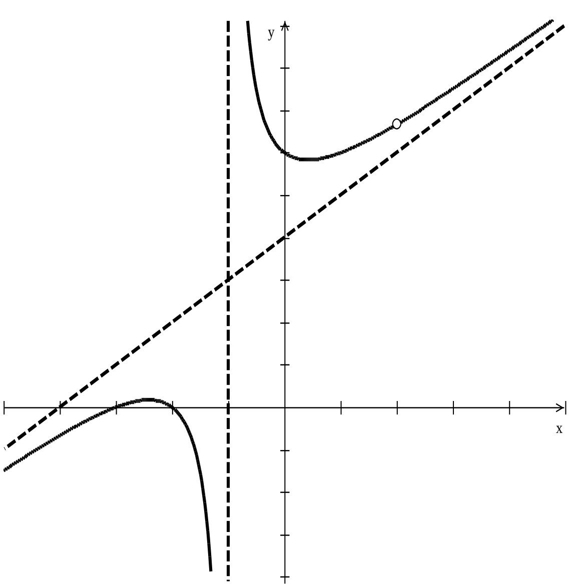
2) Rational: $$y = \frac{A_n x^n + A_{n-1}x^{n-1}...}{B_m x^m + B_{m-1}x^{m-1}...}$$

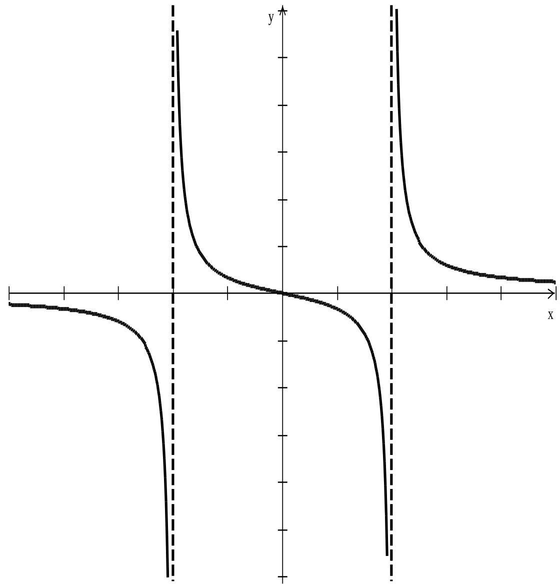
The most basic rational function is $y = \frac{1}{x}$, pictured below.

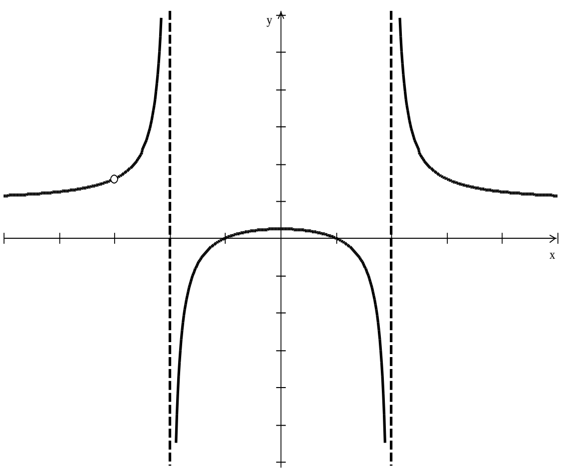


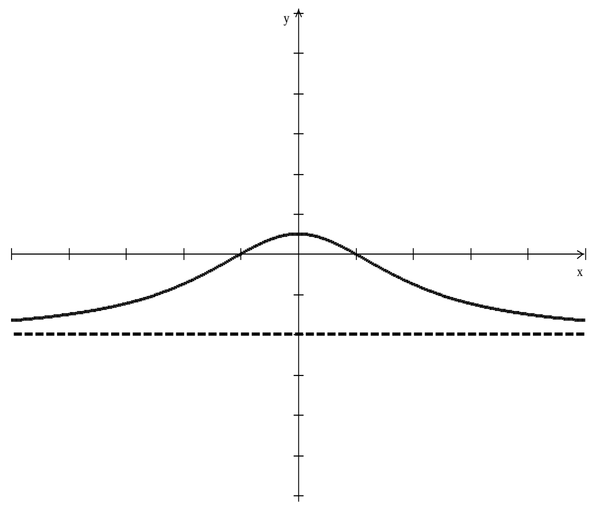
From this graph, you can see the two most common features – the horizontal and the vertical asymptotes. They also generally (though not always) alternate directions around the vertical asymptote. Points of exclusion (POE) also occur on rational function, though not on this particular one.

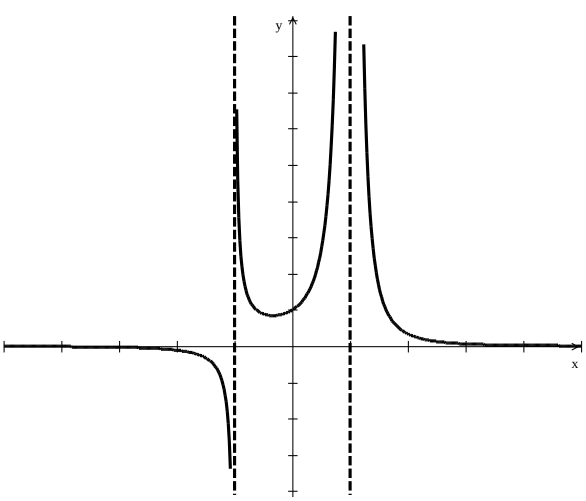
More varied rational functions:







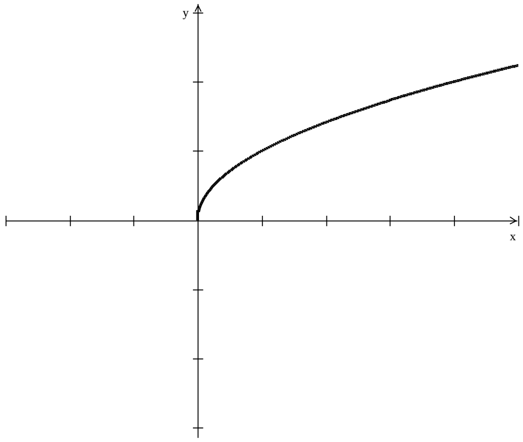




You should be able to see some common features in the VAs, HAs, and POEs. Not all of the graphs have all of these features, but they give you a guide as to what rational functions all generally look like. These are also functions that may have a Slant Asymptote (SA) instead of a HA.

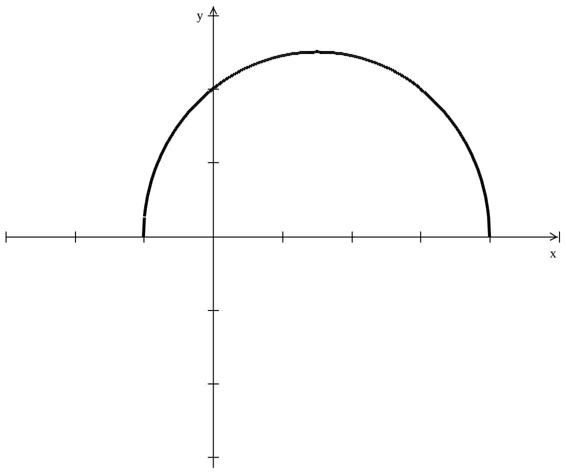
3) Radical: $y=\sqrt[b]{a_n x^n+\ldots+a_0}$

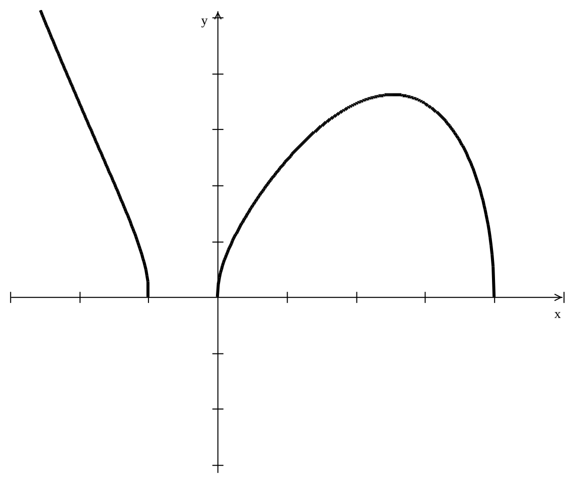
The most basic radical function is $y=\sqrt{x}$, pictured below.

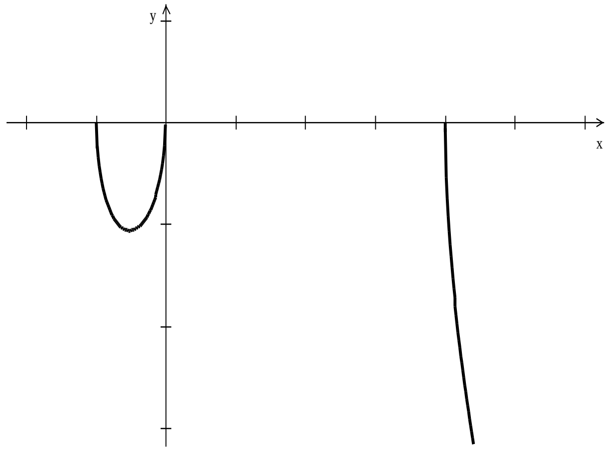


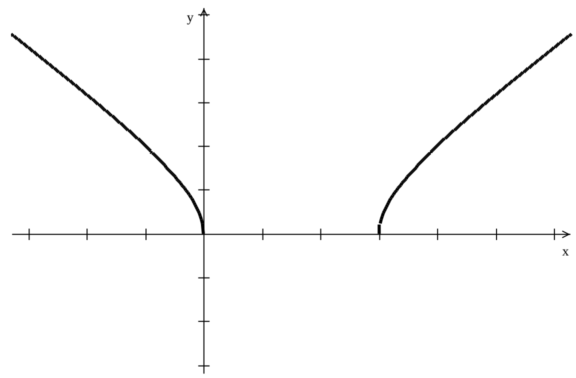
Radicals are easiest to recognize by their profoundly limited domains and ranges. Generally, the range is either $y\in[0,\infty)$ or $y\in(-\infty,0]$ (entirely above or below the *x*-axis).

More radical functions:





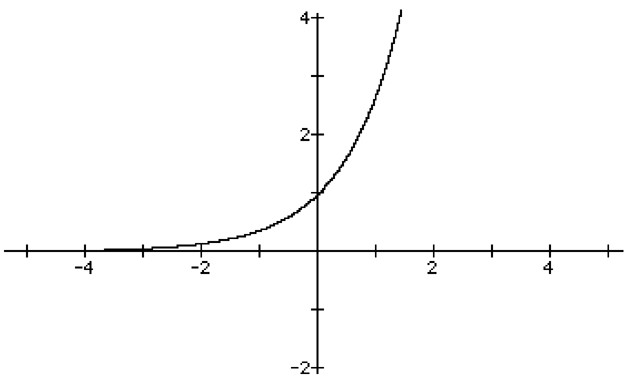




Note that all of them have very restricted domains and ranges.

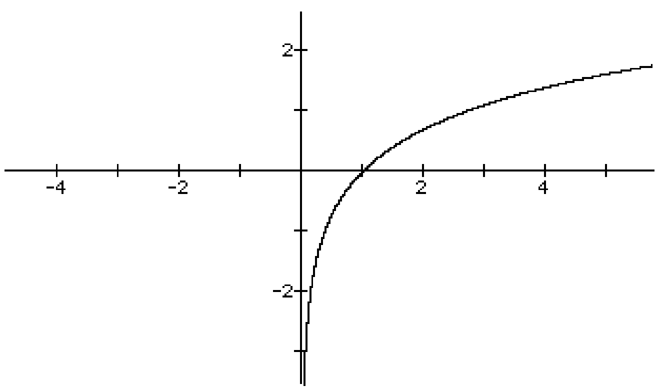
Transcendental Functions

4) Exponential: $y = k + a \cdot e^{cx}$



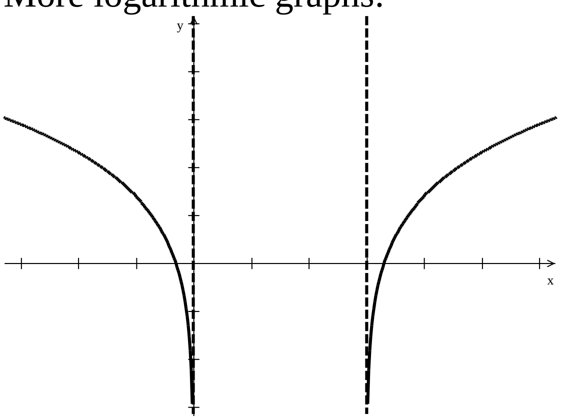
Exponentials tend to flatten on one end and head rapidly up (or down) on the other. There are exceptions to this, but recognizing the basic exponential above is most important. On their own, they do not have zeroes – they must be combined with another function to have zeroes. Their domain is not limited.

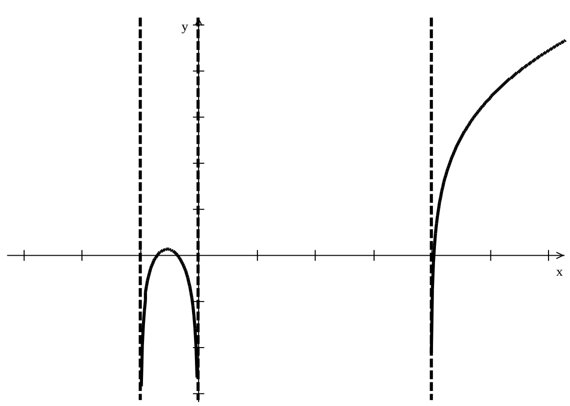
5) Logarithmic: $y = k + a \cdot \ln(x - h)$



Logarithmic graphs have limited domains. They also have vertical asymptotes and usually only head "downwards" to their vertical asymptotes.

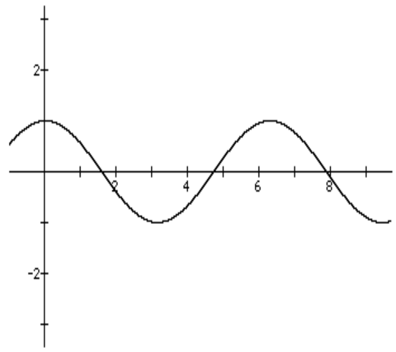
More logarithmic graphs:



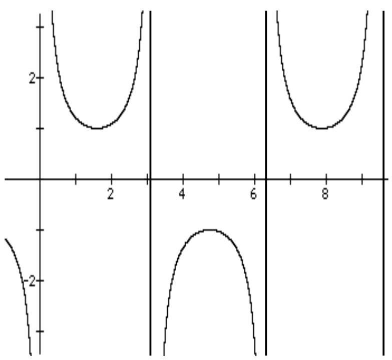


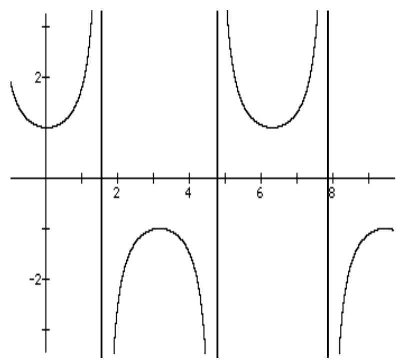
6) Trigonometric: $y = k + A \cdot f\left[B(x-h)\right]$

The graphs of trigonometric functions are widely varied – each pair of functions (sine and cosine, secant and cosecant, tangent and cotangent) have very similar graphs. All of them repeat themselves periodically – that is the key to recognizing them.

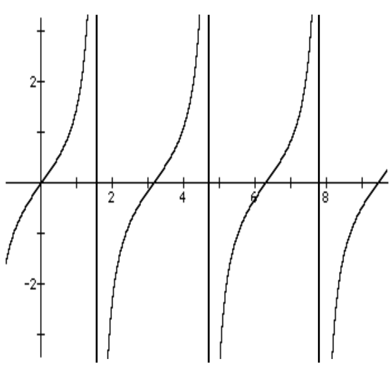


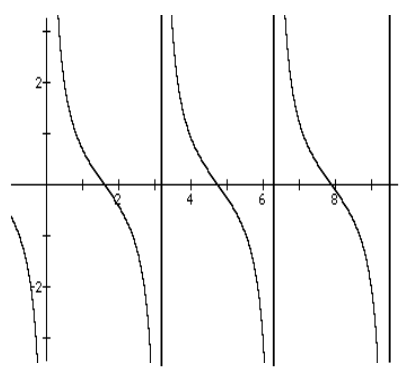
$y = \sin x$ $y = \cos x$





$y = \csc x$ $y = \sec x$



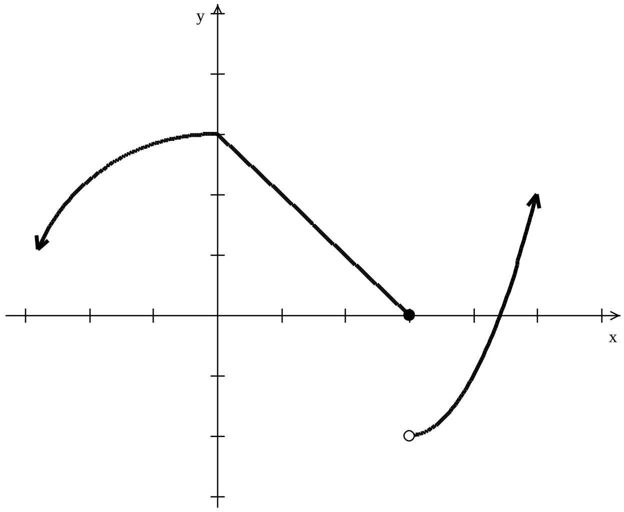


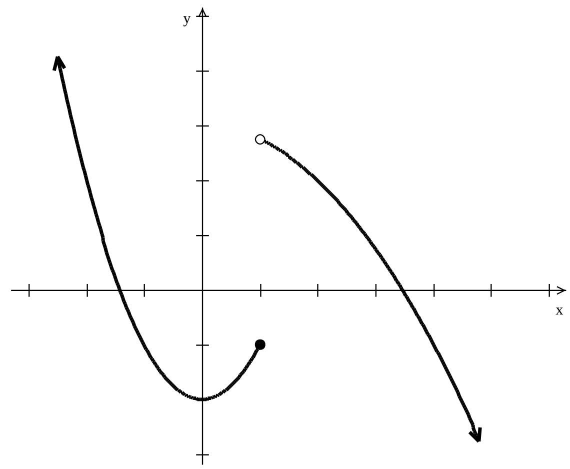
$y = \tan x$ $y = \cot x$

Other

8) Piece-wise Defined: $y = \begin{cases} f(x), & if\ x \le a \\ g(x), & if\ x > a \end{cases}$

They are made up of two or more pieces of other functions. They may or may not be continuous, and combine traits and characteristics of other functions.



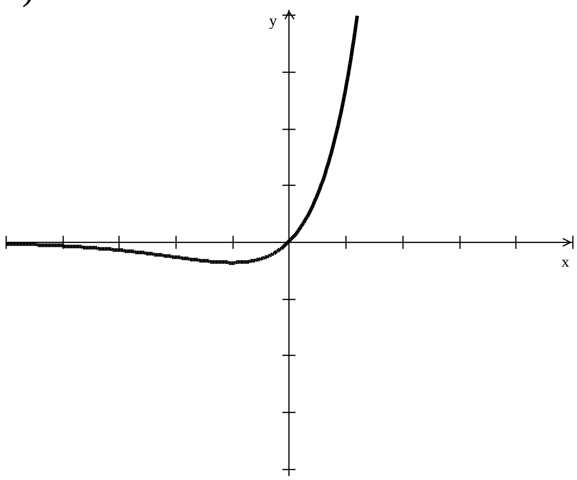


9) Inverses: $y = f^{-1}(x)$

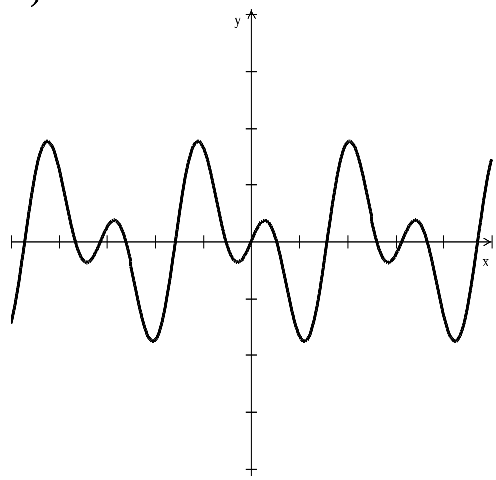
These graphs look like their respective functions, reflected across the line *y* = *x*. We have already seen the most common ones – the exponential and the natural logarithm are inverses of one another. Any function can have an inverse (though not all inverses are still functions, some are relations). We will explore them in more depth in a later section.

Ex 1 Classify each of these graphs into a family of functions.

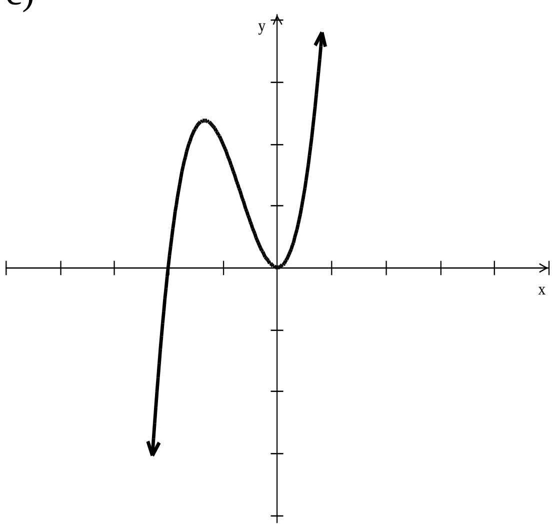
a)



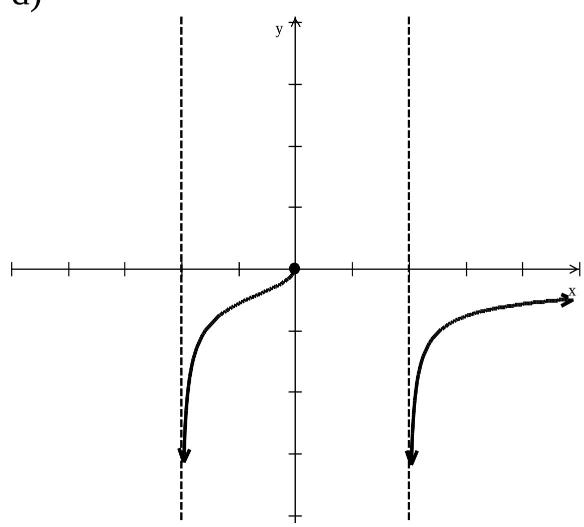
b)



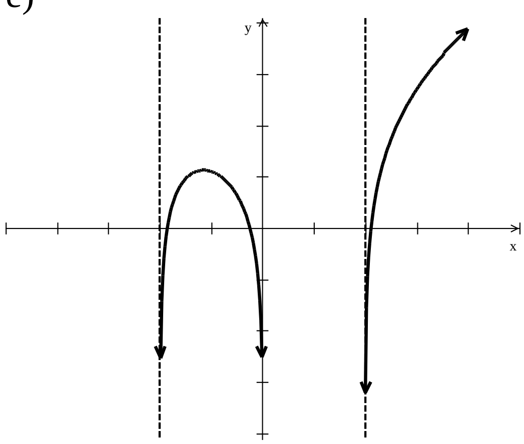
c)



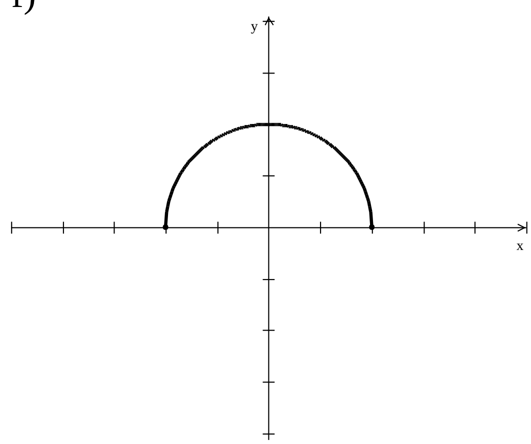
d)



e)



f)



a) Exponential

b) Sinusoidal

c) Polynomial

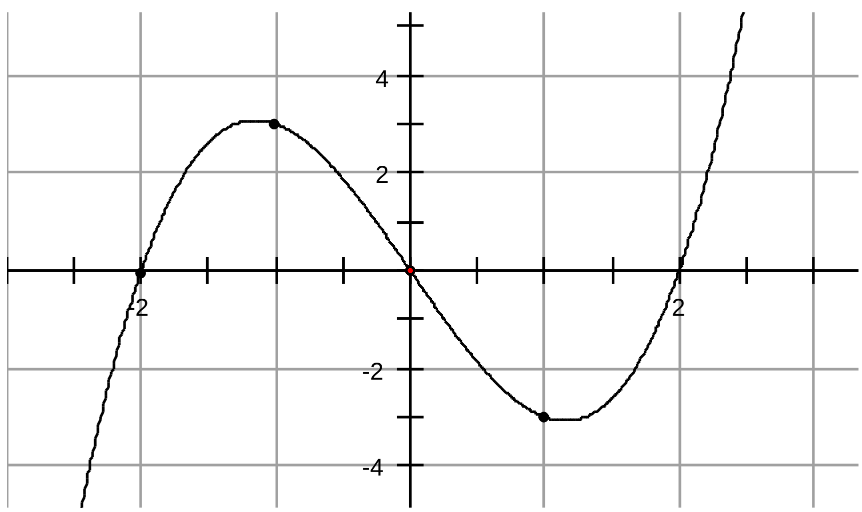
d) Rational and Radical

e) Logarithmic

f) Radical

EX 2 Plot the points in this table, classify the function and make an educated guess about the equation.

x	-2	-1	0	1	2	3
y	0	3	0	-3	0	15



$$y = x^3 - 4x$$

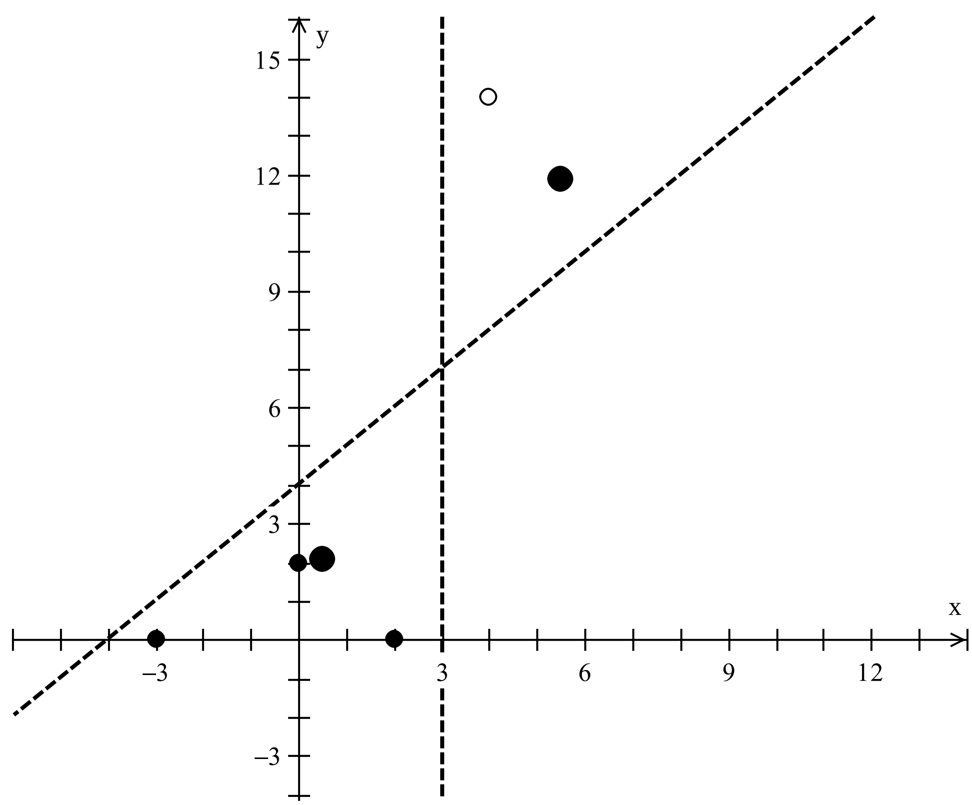
Many times we have the traits of a particular kind of graph, and we have to figure out what the graph looks like. This is simply a matter of plotting all of the traits and sketching a graph based on those traits. Sometimes we are told what kind of graph we have, other times we might have to guess at what kind of function we have graphed.

Ex 3: Sketch the graph of the function that has the following traits:

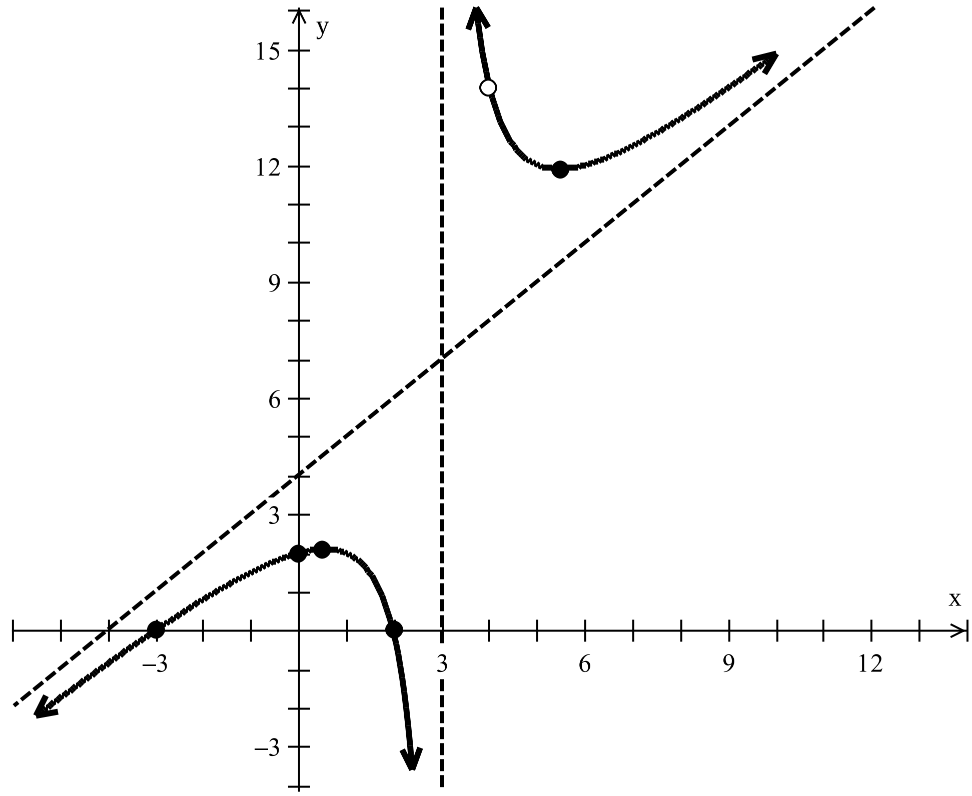
y-intercept: (0,2)
x-intercepts: (–3,0) (2,0)
VA: $x = 3$
Extreme Points: (0.551,2.101)
(5.499,11.899)

POE: (4,14)
SA: $y = x + 4$
Domain: All Reals except $x = 3$ and 4
Range: $y \in \left(-\infty, 0.551\right] \cup \left[11.899, \infty\right)$

Start by plotting the points and the asymptotes



I plotted the extremes as larger dots so that they would stand out as high or low points. From there, it is easy to see how the dots connect and how I have to graph the function.

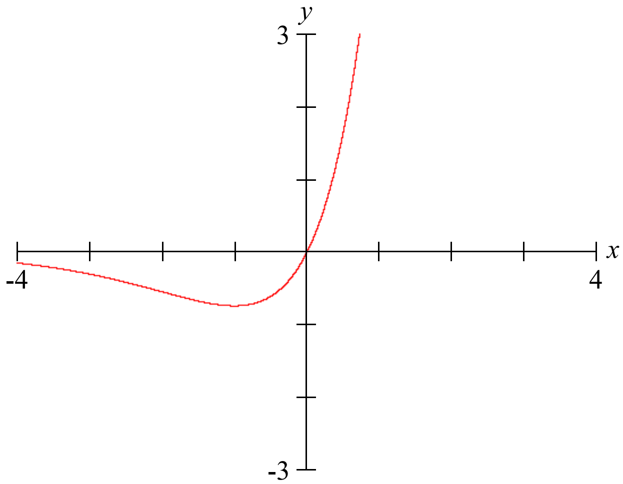


This appears to be a rational function, particularly because of the Slant Asymptote.

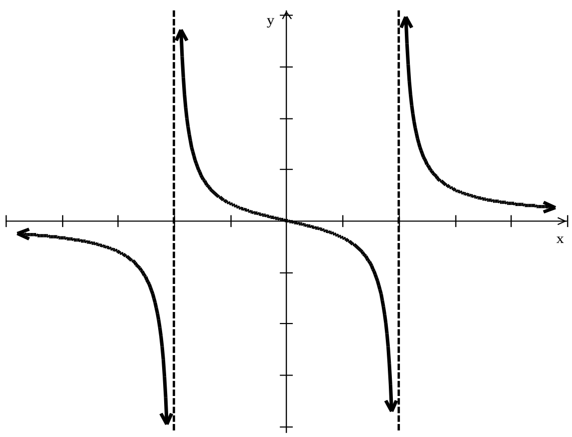
A.1 Homework

Classify these graphs.

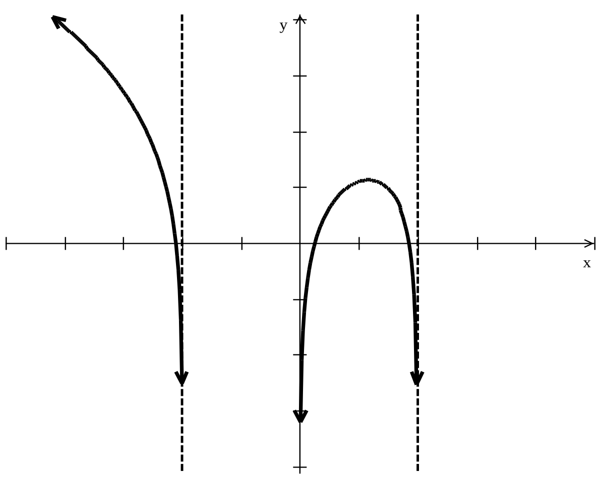
1.



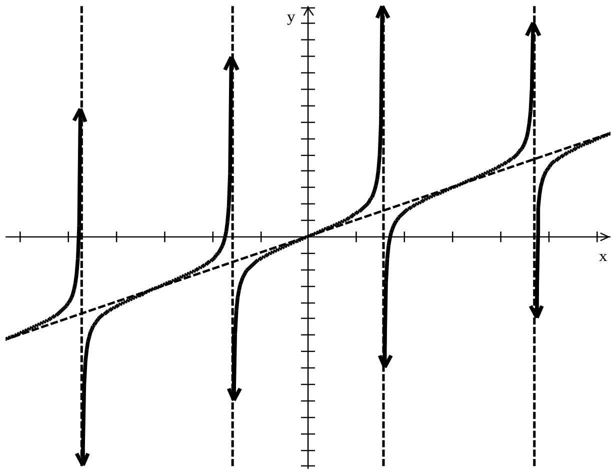
2.



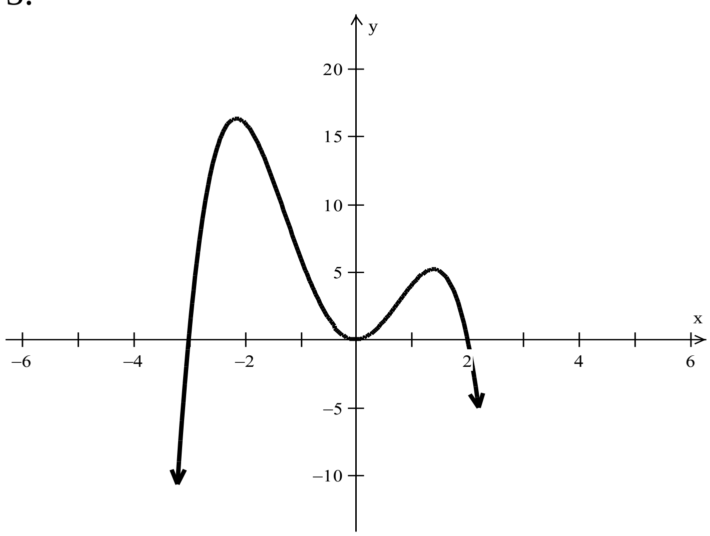
3.



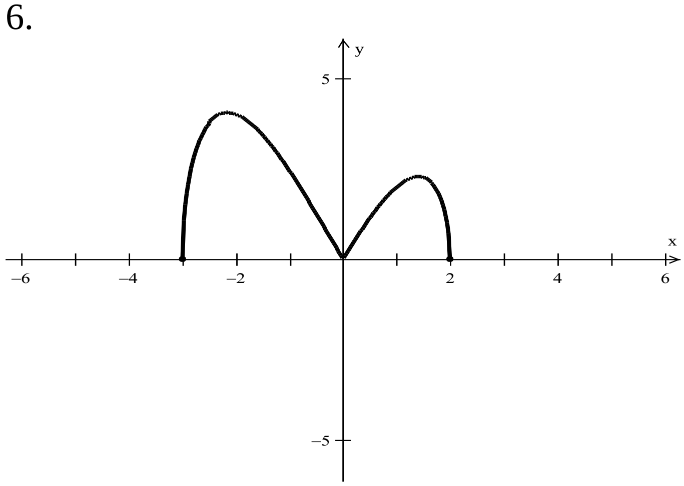
4.



5.



6.



7. A rational function that has x-intercepts at (–2, 0) and (19, 0), VAs at $x = -1$ and $x = 6$, POE at (–7, –5/13), y-intercept at (0, –3), and HA $y = -3/13$

8. A sinusoidal with Domain: $x \in [0,\ 2\pi]$,

Axis Points:

$(0,0),\ \left(\frac{\pi}{3},-\frac{\pi}{3}\right),\ \left(\frac{2\pi}{3},-\frac{2\pi}{3}\right),\ (\pi,-\pi),\ \left(\frac{4\pi}{3},-\frac{4\pi}{3}\right),\ \left(\frac{5\pi}{3},-\frac{5\pi}{3}\right),\ (2\pi,-2\pi)$

Extremes: $(3.778,-4.721),\ (4.599,-3.656),\ (5.723,-6.816),\ (2\pi,-2\pi)$

$(0,0),\ (.410,\ .532),\ (1.684,-2.627),$ and $(2.504,-1.562)$

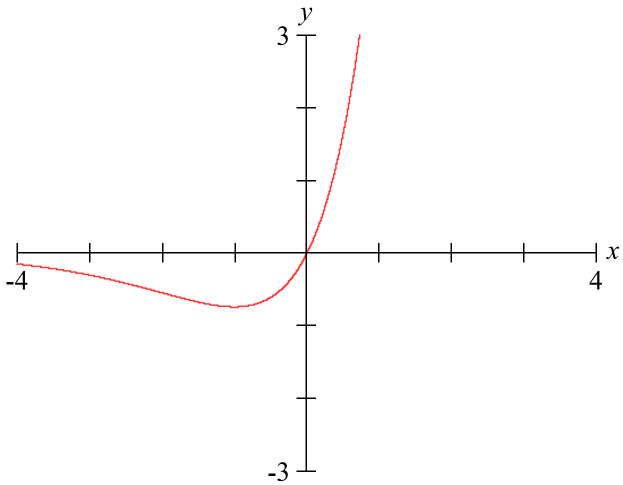
9. An exponential with Domain: $x \in All\ Reals$, Range: $y > 0$, x-intercept: None, End Behavior: Up on the left and $y = 0$ on the right and Extremes: $(0, 0.530)$ and $(2, 2)$

10. The radical function that has Domain: $x \in (-\infty, -1] \cup [2, 5)$, Range: $y \in [0, \infty)$, Critical Values: x = –1, 2, VAs: x = 5, POEs: $(-3, 1)$ and Extremes: y = 0.

11. The logarithmic function with domain $x \in (-\infty, -4) \cup (0, 4)$, VAs at $x = 4$, $x = 0$, and $x = -4$, x-intercepts = –4.031, 0.063, and 3.968, no end behavior on the right and up on the left.

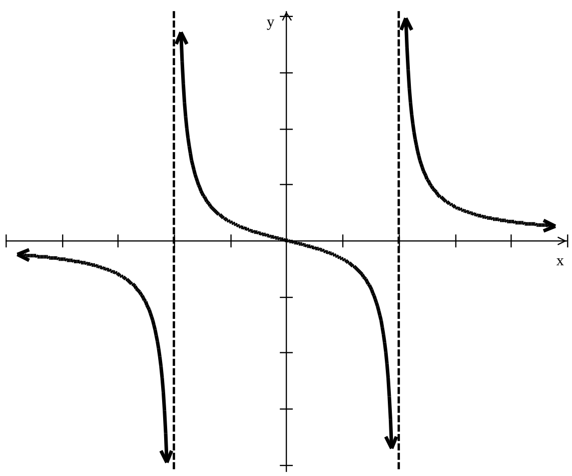
Answers: A.1 Homework

1.



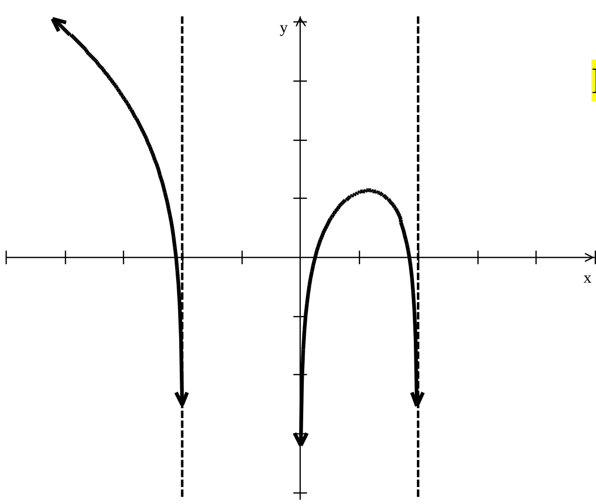
Exponential

2.



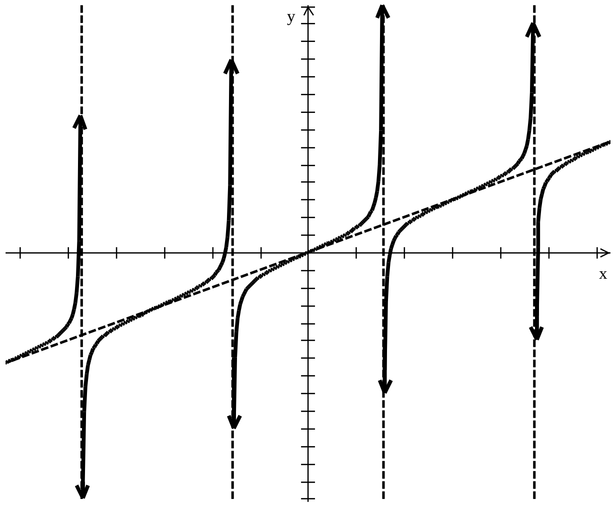
Rational

3.



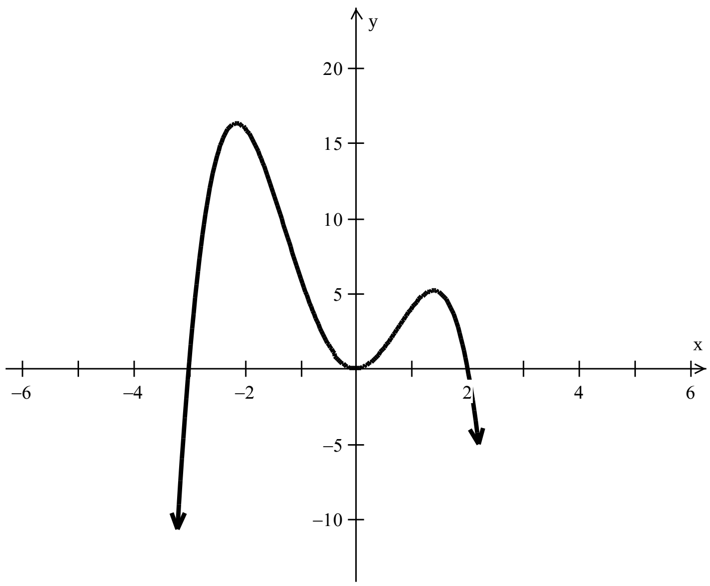
Logarithmic

4.



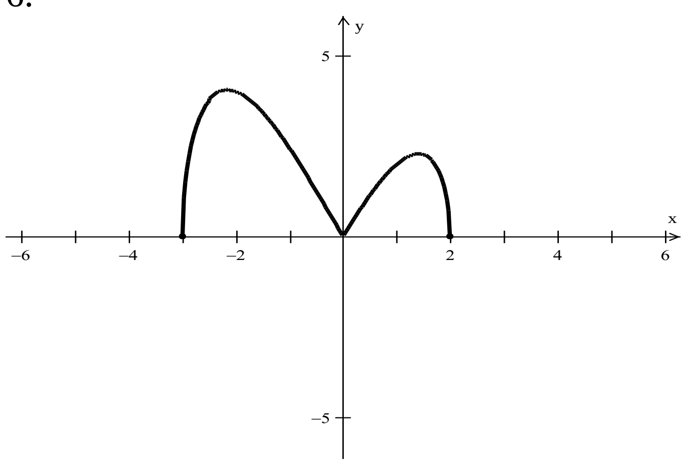
Trigonometric

5.



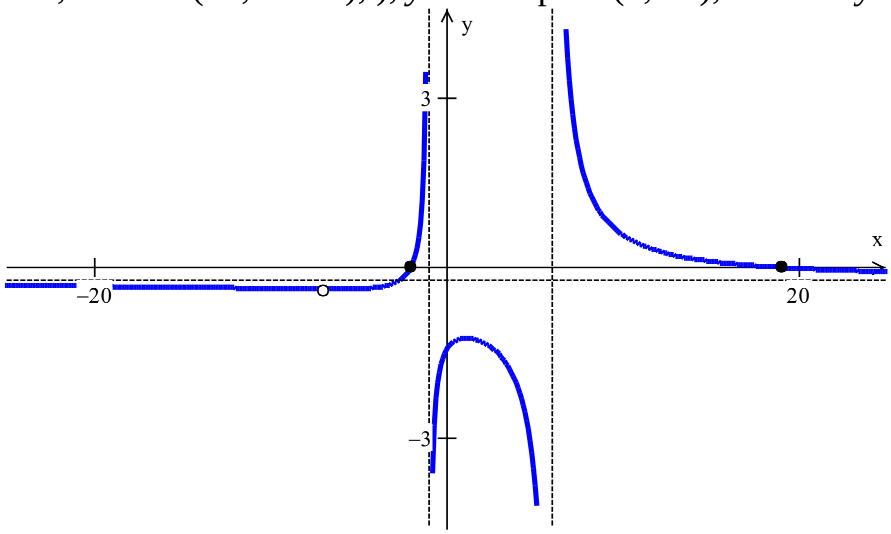
Polynomial

6.



Radical

7. A rational function that has x-intercepts at (–2, 0) and (19, 0), VAs at $x = –1$ and $x = 6$, POE at (–7, –5/13),), y-intercept at (0, –3), and HA $y = –3/13$



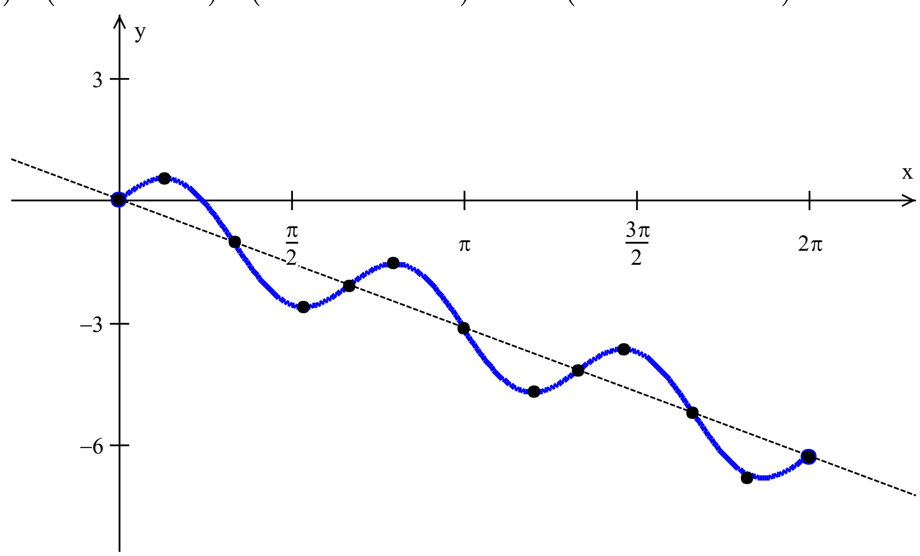
8. A sinusoidal with Domain: $x \in [0,\ 2\pi]$,

Axis Points:

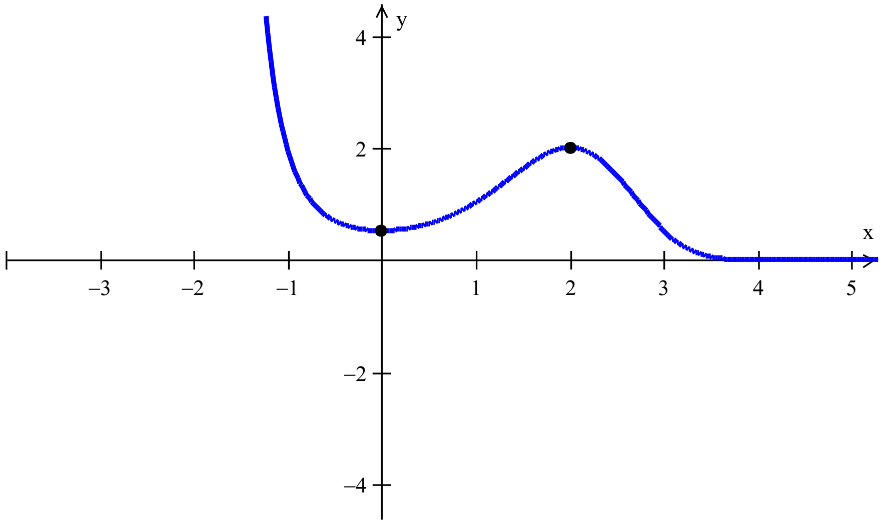
$(0,0),\ \left(\frac{\pi}{3},-\frac{\pi}{3}\right),\ \left(\frac{2\pi}{3},-\frac{2\pi}{3}\right),\ (\pi,-\pi),\ \left(\frac{4\pi}{3},-\frac{4\pi}{3}\right),\ \left(\frac{5\pi}{3},-\frac{5\pi}{3}\right),\ (2\pi,-2\pi)$

Extremes: $(3.778,-4.721),\ (4.599,-3.656),\ (5.723,-6.816),\ (2\pi,-2\pi)$

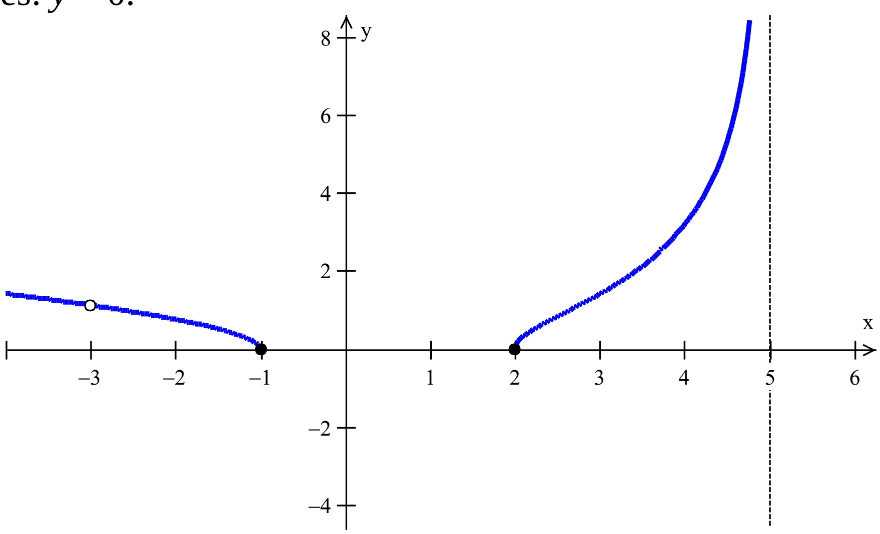
$(0,0),\ (.410,\ .532),\ (1.684,-2.627),$ and $(2.504,\ -1.562)$



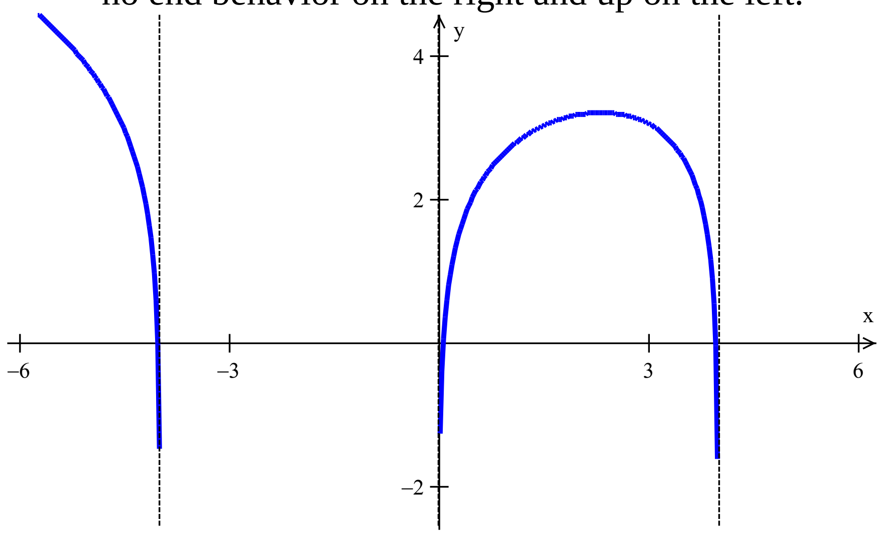
9. An exponential with Domain: $x \in All\ Reals$, Range: $y > 0$, x-intercept: None, End Behavior: Up on the left and $y = 0$ on the right and Extremes: $(0,0.530)$ and $(2,2)$



10. The radical function that has Domain: $x \in (-\infty,-1] \cup [2,5)$, Range: $y \in (-\infty,0]$, Critical Values: $x = -1$, 2, VAs: $x = 5$, POEs: $(-3,1)$ and Extremes: $y = 0$.



11. The logarithmic function with domain $x \in (-\infty, -4) \cup (0, 4)$,
x-intercepts = –4.031, 0.063, and 3.968,
no end behavior on the right and up on the left.



A.2: Factoring Review and Zeroes of Functions

One of the primary concerns of Precalculus was finding the *x*-intercepts of functions; in Calculus finding zeroes is our main concern. There is no fundamental difference in the "how" of finding these things, but there is a slight distinction in the meaning.

Vocabulary:

1. *x-intercept*—The point(s) at which a function crosses the *x*-axis.
2. *Zero*—The *x*-value(s) that makes a function value (*y*-value) zero.

Since where a function crosses the *x*-axis makes the *y*-value zero, these seem to be the same thing. The key difference is that the *x*-intercept is graphical – that is, it is a point on the graph. The zero is simply a number – it could even be imaginary (and therefore not actually on the *x*-axis). Since this is Calculus of the Real Number, we won't even be touching imaginary solutions, so the distinction is even more subtle.

Either way, however, we simply find the numeric value of the zero or the *x*-intercept by setting the *y*-value of a function equal to zero.

Ex 1 Find the zeroes of $y = x^4 - 10x^2 + 9$

$$x^4 - 10x^2 + 9 = 0$$
$$\left(x^2 - 9\right)\left(x^2 - 1\right) = 0$$
$$(x-3)(x+3)(x-1)(x+1) = 0$$

Therefore the zeroes of the function are ± 1 and ± 3

In the most basic cases, this is a very easy process, but we can get much more complicated with factoring. In practice, last year we encountered a number of situations where the factoring was quite a bit messier, so it would be good to remember some of the strategies.

Factoring Strategies

1. Take out the Greatest Common Factor
2. Look for a pattern (Reverse FOIL, Difference of Squares, Sum or Difference of Cubes, etc.)
3. Try factoring by grouping.

While factoring out a common factor seems basic, it is often what happens in Calculus with stranger functions than in basic Algebra.

Ex 2: Factor $x^2e^{x^2}+5xe^{x^2}$

$x^2e^{x^2}+5xe^{x^2}$	Note that xe^{x^2} is in to both terms, we can take it out.
$=xe^{x^2}(x+5)$	This is as factored as it gets.

Notice that we did not set it equal to zero or solve for *x* as we were not asked to.

Ex 3: Factor $\left(x^2+3\right)^{-4}(2x+5)^6+5x\left(x^2+3\right)^{-3}(2x+5)^5$

Notice that there are actually only 2 terms – and there are common factors within the terms, namely the $\left(x^2+3\right)$ and $(2x+5)$, but how many of each do we take out?

Just like with simple "common factors", we take out the least value of the exponent (look at example 2 – with an x^2 and an x, we took out just an x).

$$\left(x^2+3\right)^{-4}(2x+5)^6+5x\left(x^2+3\right)^{-3}(2x+5)^5$$
$$=\left(x^2+3\right)^{-4}(2x+5)^5\left[(2x+5)+5x\left(x^2+3\right)^1\right]$$

Since factoring is undoing multiplication, it is essentially division, so when we take out the negative exponent, **we subtract the exponent value to get what is left.** This is where people frequently misunderstand the process. But we could still simplify what is in the brackets.

$$=\left(x^2+3\right)^{-4}\left(2x+5\right)^5\left[2x+5+5x^3+15x\right]$$
$$=\left(x^2+3\right)^{-4}\left(2x+5\right)^5\left[5x^3+17x+5\right]$$

Ex 4: Find the zeroes of $y=\left(x^2+3\right)^{-4}\left(2x+5\right)^6+5x\left(x^2+3\right)^{-3}\left(2x+5\right)^5$

Since we have already factored this in the previous example, that is where we start.

$$y=\left(x^2+3\right)^{-4}\left(2x+5\right)^5\left[5x^3+17x+5\right]=0$$

So we set each piece equal to zero, except the negative exponent. This factor could not equal zero, because *it actually is in the denominator* (remember, that is what the negative exponent means).

$\left(2x+5\right)^5=0$ so $x=-2.5$

$\left[5x^3+17x+5\right]=0$ We would need to find these zeroes on a calculator
$x=-0.287$

So our zeroes are –2.5 and –0.287

Ex 5 Factor $4x\sec^5\left(x+1\right)\tan\left(5x\right)-20\sec^4\left(x+1\right)$

$$4x\sec^5\left(x+1\right)\tan\left(5x\right)-20\sec^4\left(x+1\right)$$
$$=4\sec^4\left(x+1\right)\left[x\tan\left(5x\right)-5\sec\left(x+1\right)\right]$$

Ex 6 Factor $5x^2e^{2x+1} - 80e^{2x+1}$

$$
\begin{aligned}
&5x^2e^{2x+1} - 80e^{2x+1} \\
&= 5e^{2x+1}\left(x^2 - 16\right) \\
&= 5e^{2x+1}(x-4)(x+4)
\end{aligned}
$$

Often, when we find zeroes or *x*-intercepts, we want the *y*-intercept as well. Finding the *y*-intercept is quite simple. Just plug in 0 for the *x*, and you have the *y*-intercept. This should be second nature to you by now, and it is hardly worth mentioning, but you definitely need to remember how to do it.

Of course, we could also be asked for zeroes of functions that are not basic polynomials.

Ex 7 Find the zeroes of $y = \ln\left(x^2 - 5x + 7\right)$

$$
\begin{aligned}
&0 = \ln\left(x^2 - 5x + 7\right) \\
&1 = x^2 - 5x + 7 \\
&0 = x^2 - 5x + 6 \\
&0 = (x-2)(x-3) \\
&x = 2 \text{ and } 3
\end{aligned}
$$

Get rid of the natural log by using both sides as a power of *e* ($e^0 = 1$ and $e^{\ln(a)} = a$)

Of course, not every answer will come out to nice whole numbers. It would serve us well to remember a number of rules from Precalculus for zeroes of the transcendental functions.

1) $\ln(x) = 0$ when $x = 1$
2) $e^x \neq 0$
3) $\cos(x) = 0$ when $x = \dfrac{\pi}{2} \pm \pi n$
4) $\sin(x) = 0$ when $x = 0 \pm \pi n$
5) $\tan(x) = 0$ when $x = 0 \pm \pi n$

Since there are also times when we might have to take multiple steps in solving for the zeroes, it is also helpful to remember the more general trig inverse rules.

$$\cos^{-1}\frac{x}{r} = \pm\theta \pm 2\pi n$$

$$\sin^{-1}\frac{y}{r} = \begin{cases} \theta \pm 2\pi n \\ (\pi - \theta) \pm 2\pi n \end{cases}$$

$$\tan^{-1}\frac{y}{x} = \theta \pm \pi n$$

Ex 8   Find the zeroes of $y = \left(x^2 - 4\right)\left(1 - \tan(x)\right)$

$$0 = \left(x^2 - 4\right)\left(1 - \tan(x)\right)$$
$$0 = (x-2)(x+2)\left(1 - \tan(x)\right)$$

$$(x-2) = 0 \qquad (x+2) = 0 \qquad 1 - \tan(x) = 0$$
$$x = 2 \qquad x = -2 \qquad \tan(x) = 1$$
$$x = \tan^{-1}(1) = \frac{\pi}{4} \pm \pi n$$

A.2 Homework

Factor each of the following:

1. $2x\cos x\left(x^2+1\right)-4\sin x\left(x^2+1\right)^4$

2. $5e^x-6xe^x+x^2e^x$

3. $e^x\sec^3 e^x+\tan e^x\sec^2 e^x$

4. $8x\left(4x-5\right)^{-2}\left(x^2+1\right)^3-8\left(4x-5\right)^{-3}\left(x^2+1\right)^4$

5. $\cos^3\theta\sin\theta+\sin^2\theta\cos^2\theta$

6. $x^4+27x\ln\left(x^2+1\right)$

Find zeroes for each of the following functions:

7. $f(x)=5e^x-6xe^x+x^2e^x$

8. $y=x^2\cos(\pi x)+\cos(\pi x)$

9. $f(t) = 2e^t \cos e^t - e^t \sqrt{3}$

10. $\frac{dy}{dx} = \sec^3 x + x \sec^2 x$ on $x \in [0, 2\pi]$

A.2 Homework

1. $2x\cos x\left(x^2+1\right)-4\sin x\left(x^2+1\right)^4$

$=2\left(x^2+1\right)\left(x\cos x-2\sin x\left(x^2+1\right)^3\right)$

2. $5e^x-6xe^x+x^2e^x$

$=e^x\left(x-1\right)\left(x-5\right)$

3. $e^x\sec^3 e^x+\tan e^x\sec^2 e^x$

$=\sec^2 e^x\left(e^x\sec e^x+\tan e^x\right)$

4. $8x\left(4x-5\right)^{-2}\left(x^2+1\right)^3-8\left(4x-5\right)^{-3}\left(x^2+1\right)^4$

$=8\left(4x-5\right)^{-3}\left(x^2+1\right)^3\left(3x^2-5x-1\right)$

5. $\cos^3\theta\sin\theta+\sin^2\theta\cos^2\theta$

$=\cos^2\theta\sin\theta\left(\cos\theta+\sin\theta\right)$

6. $x^4+27x\ln\left(x^2+1\right)$

$=x\left(x^3+27\ln\left(x^2+1\right)\right)$

7. $f(x)=5e^x-6xe^x+x^2e^x$

$(1,0);(5,0)$

8. $y=x^2\cos(\pi x)+\cos(\pi x)$

$\left(\frac{1}{2}\pm n,0\right)$

9. $f(t)=2e^t\cos e^t-e^t\sqrt{3}$

$\left(\ln\left(\frac{\pi}{6}+2\pi n\right),0\right);\left(\ln\left(\frac{11\pi}{6}+2\pi n\right),0\right)$

10. $\frac{dy}{dx}=\sec^3 x+x\sec^2 x$ on

$x\in\left[0,2\pi\right]$

$(2.074,0);(4.488,0)$

A.3: Sign Patterns and Domains

Vocabulary

Indeterminate Form of a Number--Defn: "A number for which further analysis is necessary to determine its value."

Means: the y-value equals $\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, or other strange things.

Infinite Number--Defn: "A number of the form $\frac{non-zero}{0}$."

Means: The denominator = 0, but the numerator does not = 0.

Vertical Asymptote--Defn: "The line x = a for which $\lim\limits_{x \to a^+} f(x) = \pm\infty$ or

$\lim\limits_{x \to a^-} f(x) = \pm\infty$."

Means: a vertical boundary line (drawn--not graphed--as a dotted line) on the graph at the x-value where the y-value is infinity. This happens one of two ways: (a) at $\frac{non-zero}{0}$ and (b) at Ln 0.

Your calculator may or may not show them, depending on the window used.

OBJECTIVE
Find Sign Patterns and use them to interpret domains of functions. Find the domain of different functions. Find Points of Exclusion and Vertical Asymptotes.

Remember, the domain is the set of x-values that work in a given function. We have a number of rules for domain from last year that we would do well to remember.

Since domain hinges on what we can plug into a function, all of the rules are based on old algebraic concepts that tell us what we can and cannot do with real numbers. For example, we cannot divide by 0, so anything that would make the denominator of a fraction 0 would be outside the domain of a function.

This gives us three main domain violations from algebra.

Domain:

1) The denominator of a fraction cannot be zero (this includes negative exponents.

2) We cannot take the square root (or any even root) of a negative.

3) We cannot take the log of a negative or zero.

You may notice that two of the three rules deal with a piece of a function equaling zero, while two of the three rules also deal with the sign of something being negative.

We have already dealt with how to find when something is zero, but we need another tool to find whether something is negative.

This tool is the **Sign Pattern**. *A sign pattern for a function is simply telling us when values of the function are positive, zero, or negative (or nonexistent).*

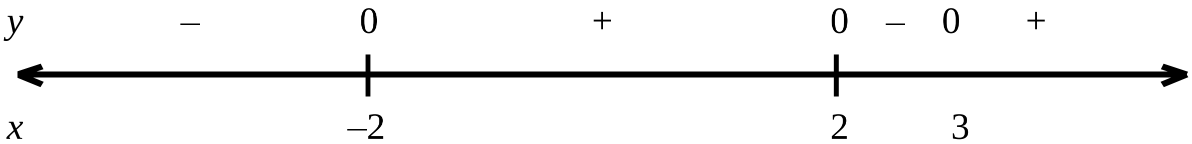
Put another way, a sign pattern can show us a one dimensional representation of the two dimensional graph of a function.

Ex 1 Find the sign pattern for $y = x^3 - 3x^2 - 4x + 12$

The first part of finding a sign pattern involves finding the zeroes. Since the zeroes are neither positive nor negative, they are the boundaries on our sign pattern.

$$0 = x^3 - 3x^2 - 4x + 12$$
$$0 = x^2(x-3) - 4(x-3)$$
$$0 = (x^2 - 4)(x-3)$$
$$0 = (x-2)(x+2)(x-3)$$
zeroes at $x = 2, -2,$ and 3

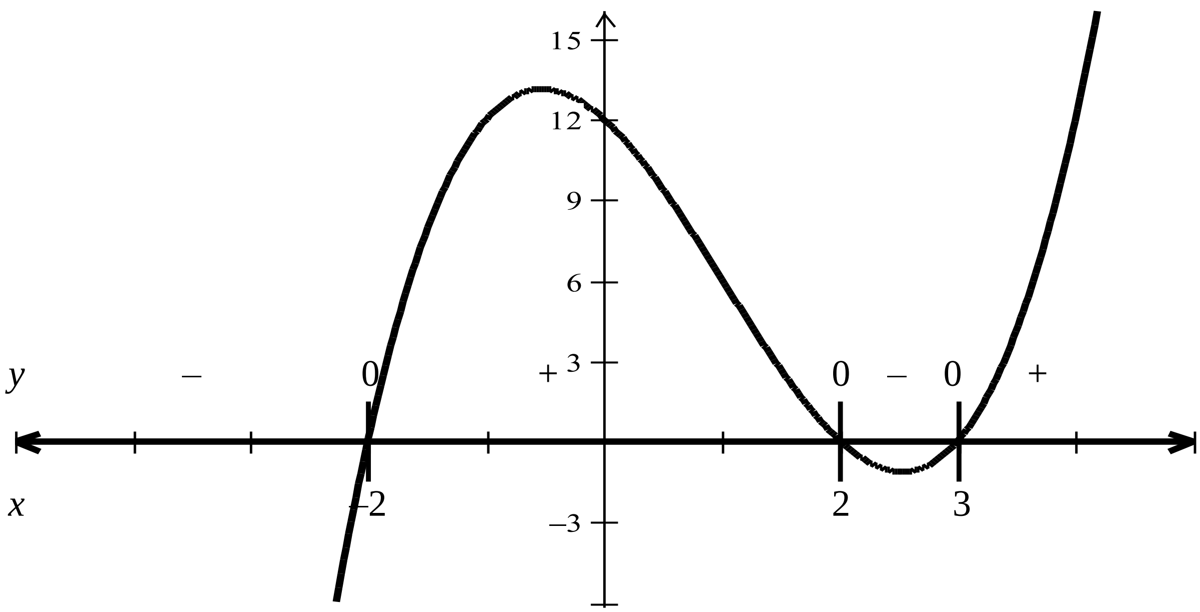
We now place the numbers on a number line:



We can determine the signs by plugging in values into our function and finding out whether we get positive or negative values out for *y*.

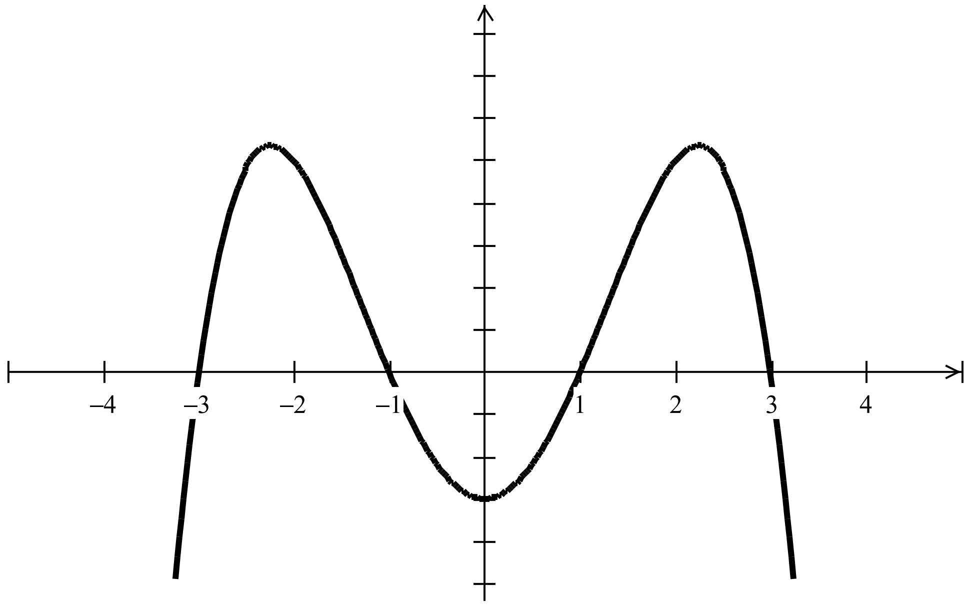
Since we know that the only way to get from positive to negative (or vice versa) on a continuous function is by passing through 0, if we know one value in a region, we know all of the values.

Below is an actual graph of the function with the sign pattern overlaid on it.



Hopefully, you can see that the negative regions on the sign pattern correspond to the negative portions of the graph (below the *x*-axis), and the positive portions correspond to the positive portions of the graph (above the *x*-axis).

Ex 2   Find the sign pattern that corresponds to the graph below:



Ex 3   Find the sign pattern for $y = \dfrac{x^2 - 4}{x^2 + 4x}$

$$y = \frac{(x-2)(x+2)}{x(x+4)}$$

The denominator still affects the sign, so we have find its zeroes as well.

y	+	D.N.E.	–	0	+	D.N.E.	–	0	+
x		–4		–2		0		3	

Notice that we put D.N.E. (for Does Not Exist) over the values from the denominator. The zeroes of the denominator are outside the domain.

Of course, we said that we wanted to uses sign patterns to find domains of functions, so that is what we will look at now.

Ex 4 Find the domain of $y=\sqrt{9x-x^3}$

Since we know that we cannot have a negative under the radical, we only do a sign pattern for the expression $9x-x^3$.

In fact, we want $9x-x^3 \geq 0$

$$x\left(9-x^2\right)=0$$

$$x(3-x)(3+x)=0$$
$$x=\pm 3,0$$

$9x-x^3$	+	0	–	0	+	0	–
x		–3		0		3	

Since the negative numbers are not in our domain, our domain is

$$x \in \left(-\infty,-3\right] \cup \left[0,3\right]$$

Those are the *x*-values that give us positive numbers under the radical.

Ex 5 Find the domain of $y=\ln\left(9x-x^3\right)$

This will be the same sign pattern from the previous problem, because we cannot take the log of a negative number; we also cannot log a 0, so our interpretation is a little different.

$$x \in \left(-\infty,-3\right) \cup \left(0,3\right)$$

Remember, brackets on an interval and the number is included in the region. Parentheses means the number is not included.

Apart from places where large regions of a function don't exist (because of a negative under a radical or in a logarithm), interesting things happen where the domain violation happens because of a 0 (in the denominator or in a logarithm).

Two things can happen in this case:

1) You get a Point of Exclusion (POE) when a rational function yields $\frac{0}{0}$. This happens when the same factor appears in the numerator and the denominator. The *x*-value of the POE is what you get when you set the repeated factor to zero. You get the *y*-value from plugging that *x* into the function after cancelling the repeated factor.

2) You get a Vertical Asymptote (VA) when the denominator of a rational function (that does not have an identical factor in the numerator) equals zero. You will also get a vertical asymptote when the argument of a logarithm equals zero. They will be vertical lines, so you must express them as $x = k$.

Ex 6 Find the vertical asymptotes and points of exclusion for $y = \frac{x+6}{x^2-36}$

$$y = \frac{x+6}{(x-6)(x+6)}$$

The $(x + 6)$ cancels, so that is what gives us the POE.

$$x+6=0$$
$$x=-6$$

$y = \frac{\cancel{x+6}}{(x-6)\cancel{(x+6)}}$ Plugging the -6 for the x gets us the POE $\left(-6, -\frac{1}{12}\right)$

Setting the rest of the denominator equal to 0 gets us a VA at $x = 6$

Ex 7 Find the VAs for the function $y=\ln\left(2x^3+5x^2+2x\right)$

VAs occur where the argument of a logarithm equals 0.

$$2x^3+5x^2+2x=0$$
$$x\left(2x+1\right)\left(x+2\right)=0$$

So the VAs are $x=0$, $x=-\frac{1}{2}$, and $x=-2$

Ex 8 Find the vertical asymptotes of $y=\frac{x+1}{x^2-4x-5}$

$$y=\frac{x+1}{(x-5)(x+1)}$$

so the VA is $x=5$. x = –1 is not a VA because –1 makes y = $\frac{0}{0}$. x = –1 is the *x*-coordinate of the POE.

Ex 9 Find the vertical asymptotes of $y=\frac{2x^3-7x^2+2x+3}{x^3-3x^2+5x-15}$

$$y=\frac{(2x+1)(x-1)(x-3)}{(x-3)(x^2+5)}$$

x = 3 makes $y=\frac{0}{0}$, and x^2+5 cannot equal zero in Real numbers. So

This function has no vertical asymptotes.

Ex 10 Find the zeros, vertical asymptotes and POE of $y=\dfrac{x^2-1}{x^2-4x-5}$

$$y=\frac{(x-1)(x+1)}{(x+1)(x-5)}$$

From this we can see: Zero: $(1\ ,\ 0)$

VA: $x=5$

POE: $x=-1$

Plugging into the cancelled function gets us:

$\left(-1,\dfrac{1}{3}\right)$

Ex 11 Find the zeros and domain of $y=\sqrt{16x-x^3}$.

$$16x-x^3\geq 0$$
$$x(4-x)(4+x)\geq 0$$

y^2	+	0	−	0	+	0	−
x		-4		0		4	

So the zeros are x = 0, 4 and –4, and the domain is $x\in(\infty,-4]\ \cup\ [\,0\,,4\,]$.

$$y=\sqrt{16x-x^3}$$

Ex 12 Find the domain, zeros, VAs and POE and sketch $y = \ln\left(9 - x^2\right)$

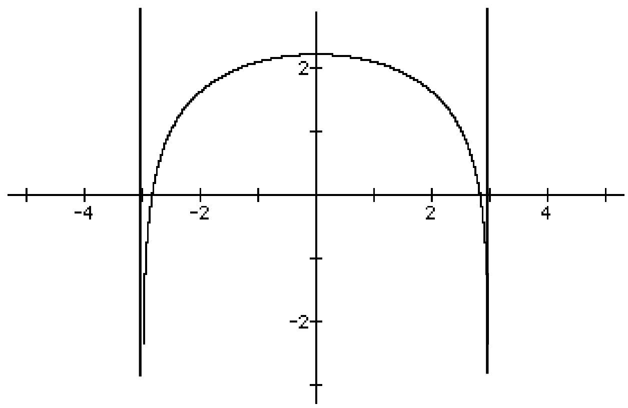
Domain: $9 - x^2 > 0$, so $-3 < x < 3$ or $x \in (-3, 3)$

VA: $9 - x^2 = 0$. So $x = \pm 3$.

Zeros: $\ln\left(9 - x^2\right) = 0$

$$9 - x^2 = e^0 = 1$$
$$8 = x^2$$
$$x = \pm 2\sqrt{2} = \pm 2.828$$

Sketch



Notice that, because of the VAs, ***this curve does not stop as our calculator seems to indicate.*** Remember: **Your Calculator is not infallible!!**

A.3 Homework:

Find the Domain, VAs, and POEs

1. $y=\dfrac{5}{x^2-2x-8}$

2. $y=\dfrac{x^2+2x-15}{x-3}$

3. $y=\ln\left(x^2+2x-15\right)$

4. $y=\dfrac{(x+3)(x-1)}{(x-2)(x+3)(x+1)}$

5. $y=\dfrac{x^2+x-30}{x^3+x^2-x-1}$

6. $y=\sqrt{9-x^2}$

7. $y=\sqrt{x^2-4x-21}$

8. $y=\dfrac{x^3-2x}{x^4-5x^2+6}$

9. $y = \ln\left(x - x^2\right)$

10. $y = \dfrac{2}{x^2 - 2}$

11. $y = \ln\left(\dfrac{-3x}{x^2 - 9}\right)$

12. $y = \dfrac{24}{x^2 + 2x - 15}$

13. $y = \sqrt{x^3 - 5x^2 + 5x - 15}$

14. $y = \ln\left(x^3 - 4x\right)$

15. $y = \dfrac{(x+2)(x-3)}{(x-2)(x-3)(x-1)}$

16. $y = \sqrt{\dfrac{x}{x^2 - 4}}$

17. $y=\sqrt{\dfrac{x^2+3x}{x^2-2x-3}}$

18. $y=\sqrt{2x^3+3x^2-18x-27}$

A.3 Homework:

1. $y=\dfrac{5}{x^2-2x-8}$

Domain: $x\neq 4,-2$

VA: $x=4, x=-2$

POE: None

2. $y=\dfrac{x^2+2x-15}{x-3}$

Domain: $x\neq 3$

VA: None

POE: $(3,8)$

3. $y=\ln\left(x^2+2x-15\right)$

Domain: $x\in(-\infty,-5)\cup(3,\infty)$

VA: $x=3, x=-5$

POE: None

4. $y=\dfrac{(x+3)(x-1)}{(x-2)(x+3)(x+1)}$

Domain: $x\neq -3,-1,2$

VA: $x=2, x=-1$

POE: $\left(-3,-\frac{2}{5}\right)$

5. $y=\dfrac{x^2+x-30}{x^3+x^2-x-1}$

Domain: $x\neq 1,-1$

VA: $x=1, x=-1$

POE: None

6. $y=\sqrt{9-x^2}$

Domain: $x\in[-3,3]$

VA: None

POE: None

7. $y=\sqrt{x^2-4x-21}$

Domain: $x\in(-\infty,-3]\cup[7,\infty)$

VA: None

POE: None

8. $y=\dfrac{x^3-2x}{x^4-5x^2+6}$

Domain: $x\neq \pm\sqrt{2},\pm\sqrt{3}$

VA: $x=\pm\sqrt{3}$

POE: $\left(\sqrt{2},\sqrt{2}\right);\left(-\sqrt{2},-\sqrt{2}\right)$

9. $y=\ln\left(x-x^2\right)$

Domain: $x\in(0,1)$

VA: $x=0, x=1$

POE: None

10. $y=\dfrac{2}{x^2-2}$

Domain: $x\neq\pm\sqrt{2}$

VA: $x=\pm\sqrt{2}$

POE: None

11. $y=\ln\left(\dfrac{-3x}{x^2-9}\right)$

Domain: $x\neq\pm 3,0$

VA: $x=\pm 3, x=0$

POE: None

12. $y=\dfrac{24}{x^2+2x-15}$

Domain: $x\neq 3,-5$

VA: $x=3, x=-5$

POE: None

13. $y=\sqrt{x^3-5x^2+5x-15}$

Domain: $x\in[4.620,\infty)$

VA: None

POE: None

14. $y=\ln\left(x^3-4x\right)$

Domain: $x\in(-2,0)\cup[2,\infty)$

VA: $x=\pm 2, x=0$

POE: None

15. $y=\dfrac{(x+2)(x-3)}{(x-2)(x-3)(x-1)}$

Domain: $x\neq 1,2,3$

VA: $x=1, x=2$

POE: $\left(3,\dfrac{5}{2}\right)$

16. $y=\sqrt{\dfrac{x}{x^2-4}}$

Domain: $x\in(-2,0]\cup(2,\infty)$

VA: $x=\pm 2$

POE: None

17. $y=\sqrt{\dfrac{x^2+3x}{x^2-2x-3}}$

Domain: $x\in(-\infty,-3]\cup(-1,0]\cup(3,\infty)$

VA: $x=-1, x=3$

POE: None

18. $y=\sqrt{2x^3+3x^2-18x-27}$

Domain: $x\in\left[-3,\dfrac{3}{2}\right]\cup[3,\infty)$

VA: None

POE: None

A.4: Calculator Review

Vocabulary:

1. *Extremes*--This is the collective word for maximums and minimums.
2. *Critical Value*--The *x*-coordinate of the extreme
3. *Maximum Value*--The *y*-coordinate of the high point.
4. *Minimum Value*--The *y*-coordinate of the low point.
5. *Relative Extremes*--the points at the crests and troughs of the curve.
6. *Absolute Extremes*--the highest or lowest points of the whole curve.

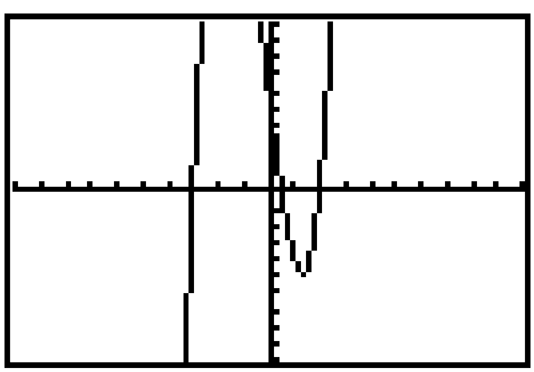
Finding the extremes of a curve algebraically is one of the fundamental subjects of Calculus. We know that lines have no extremes and that quadratics have only one extreme--namely, the vertex. We will leave the subject of algebraic higher order extremes until Chapter 1. But we can easily find them graphically with 2nd Trace 3 or 4.

OBJECTIVE
Find Zeros and extremes by grapher.

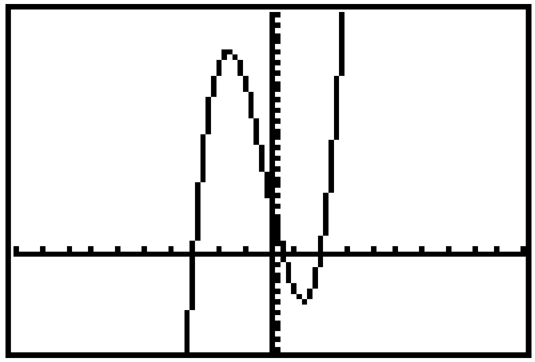
1. Grapher: "2nd Trace Root/Zero" on your calculator will give you approximate (and sometimes exact) zeros, just as we did with the quadratics in the last section. Be aware that the window used and the boundary points picked can effect the outcome.

EX 1 Find the critical values of $y = 2x^3 + x^2 - 13x + 6$

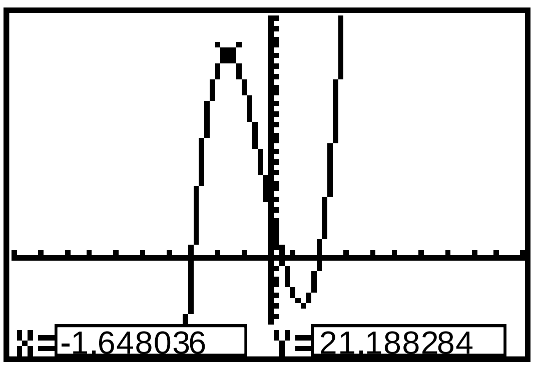
Graphing on Zoom 6, we see

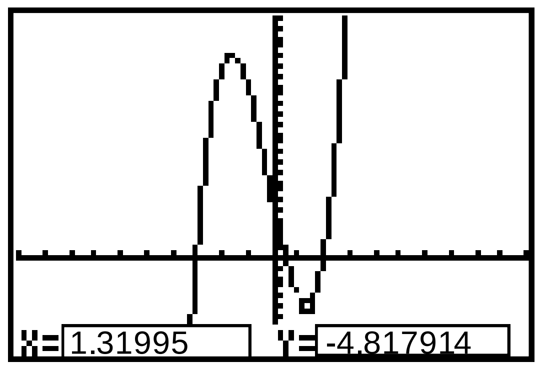


We need to open the y values higher. Try $Y_{max} = 25$



Using 2nd Trace, we find:





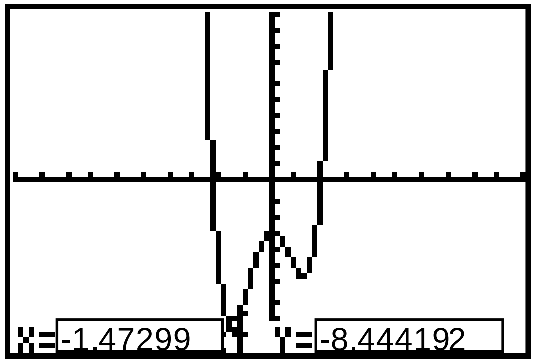
Since the question asked for critical values, the answer is

$$\boxed{x = -1.648 \text{ and } 1.320}$$

Note that the critical value answers do not distinguish between which is for a maximum and which for a minimum.

EX 2 Find the range of $y = x^4 - 4x^2 + x - 3$

Remember that the range (or set of y-values used) is closely associated with the extremes. Graphing and using 2nd trace, we find

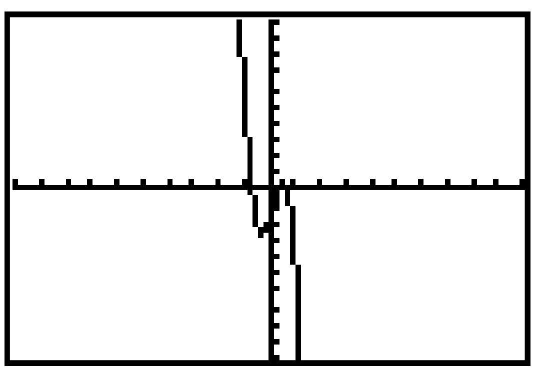


Y = -8.444 is the absolute minimum since the other relative minimum is higher. Therefore, the range is

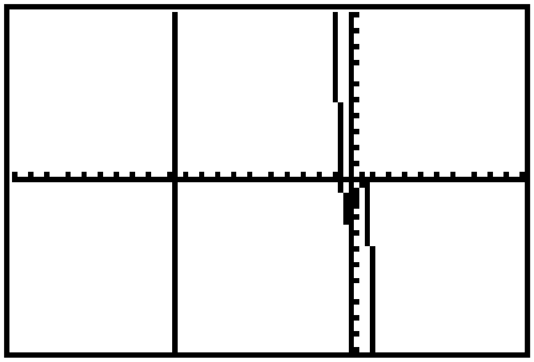
$$y \in [-8.444, \infty)$$

EX 3: Find and draw the complete graph of f(x), state the window used, and identify extremes and zeros, where $f(x) = -x^4 - 10x^3 + 5x^2 + 5x - 2$.

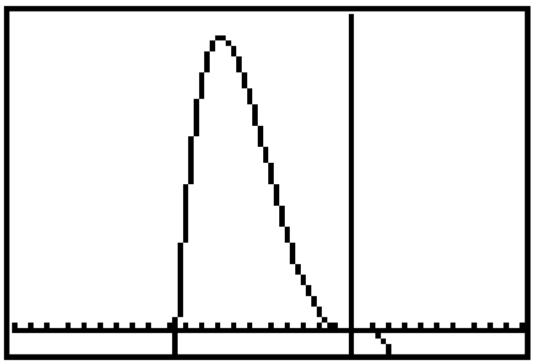
At first look, in Zoom 6, we see



But we know this is not the whole graph because a fourth degree equation with a negative coefficient should look something like an “m”. So we know there is more of a curve to the left. Widening our screen to x = -20, we see



We cannot see the high point, so we raise the y max to 1400 and y min to -100 and see

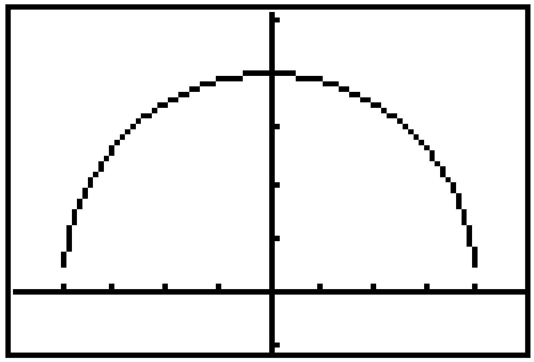


This shows us the complete graph, though the relative minimum and maximum are not readily visible. We will need to use different windows to find the various zeros and extremes. The first window we viewed is good for three of the zeros and two of the extremes. The third window is good for the other zero and the other extreme. The final answers are:

Zeros: $(-10.432,\ 0)$, $(-.691,\ 0)$, $(.368,\ 0)$, $(.754,\ 0)$

Extremes: $(-7.900,\ 1307.214)$, $(-.278,\ -2.795)$, $(.577,\ .518)$

EX 4 Find the complete graph of $y=\sqrt{16-x^2}$.



Notice how the curve looks like it does not touch the x-axis. But we know it must, because we can find zeros.

$$16-x^2 \geq 0$$
$$(4-x)(4+x) \geq 0$$

$$\begin{array}{c|ccccc} y^2 & - & 0 & + & 0 & - \\ \hline x & & -4 & & 4 & \end{array}$$

So the zeros are x = 4 and -4, and the domain is $x \in [-4, 4]$.

As the technology gets better, this misrepresentation of the graph may improve, but we must recognize the limitations of the machine.

A.4 Homework:

a) Find and draw the complete graph of $f(x)$, stating the window used,

b) Find the zeros,

c) Find the extremes and ranges

1. $y = -2x^3 + 2x^2 + 7x - 10$

2. $y = \begin{cases} x^2 - x, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$

3. $y = \dfrac{4 - x^2}{x^2 - 1}$

4. $x^2 - xy + 2y - 4x = 0$ (Hint: Isolate y first.)

5. $y=\sqrt{x^2-4x-5}$

6. $y=\sqrt{x^3+3x^2-6x-8}$

7. $y = -\sqrt{\dfrac{x}{x^2 - 4}}$

8. $y = x\sqrt{36 - x^2}$

9. $y=\ln\left(x^3-4x\right)$

10. $y=2^x\sqrt{7-x}$

11. $y = \ln\left(\dfrac{-3x}{x^2 - 9}\right)$

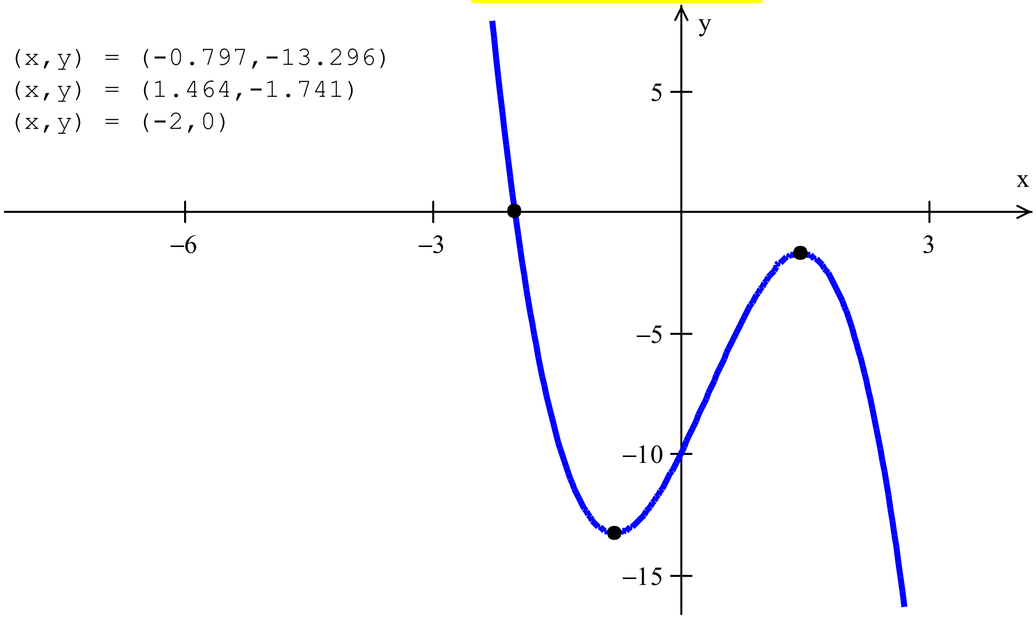
12. $y = e^{\sqrt{4-x^2}}$

13. $y = \dfrac{1}{1 + e^{-x}}$

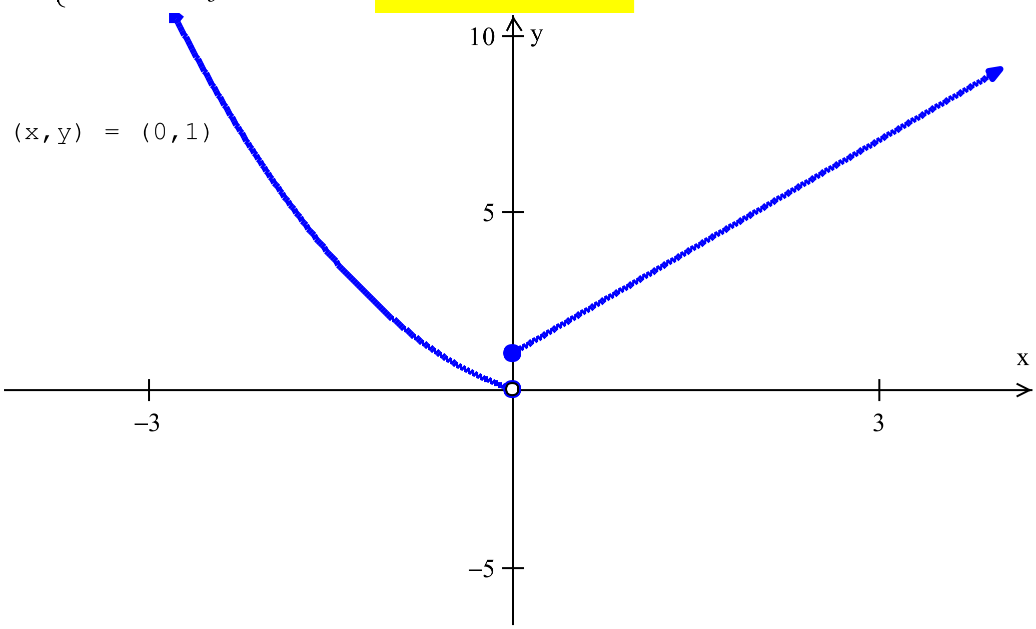
14. $y = \dfrac{\ln\left(e^{x} + 1\right)}{e^{x}}$

A.4 Homework:

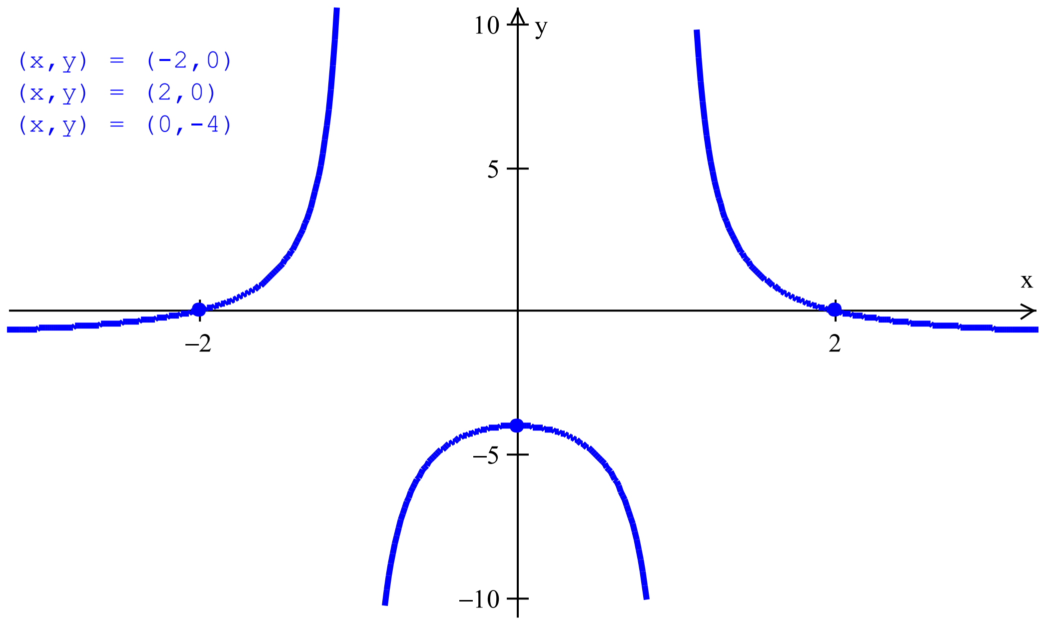
1. $y = -2x^3 + 2x^2 + 7x - 10$ Range: $y \in (-\infty, \infty)$



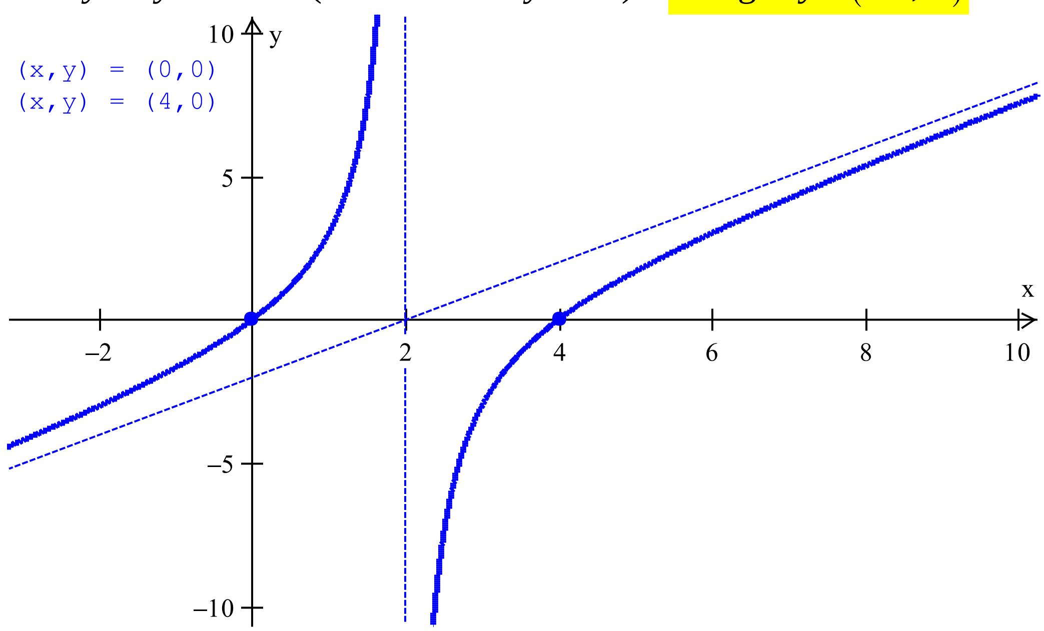
2. $y = \begin{cases} x^2 - x, & \text{if } x < 0 \\ 2x + 1, & \text{if } x \geq 0 \end{cases}$ Range: $y \in (0, \infty)$



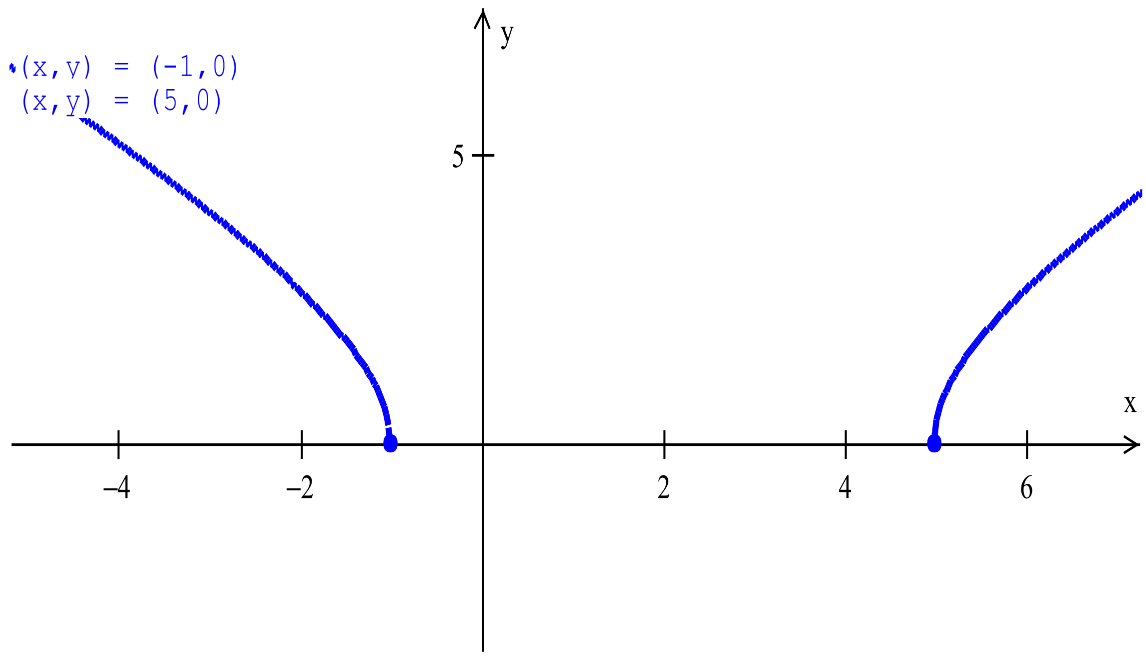
3. $y = \dfrac{4 - x^2}{x^2 - 1}$ Range: $y \in (-\infty, -4] \cup (-1, \infty)$



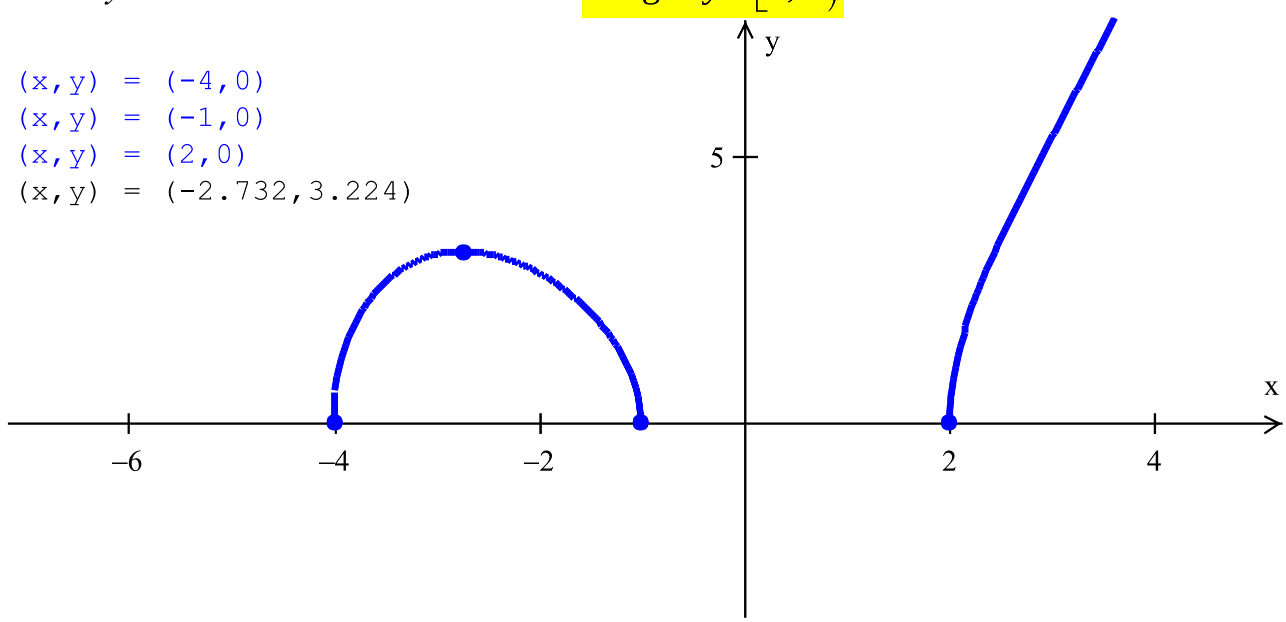
4. $x^2 - xy + 2y - 4x = 0$ (Hint: Isolate y first.) Range: $y \in (-\infty, \infty)$



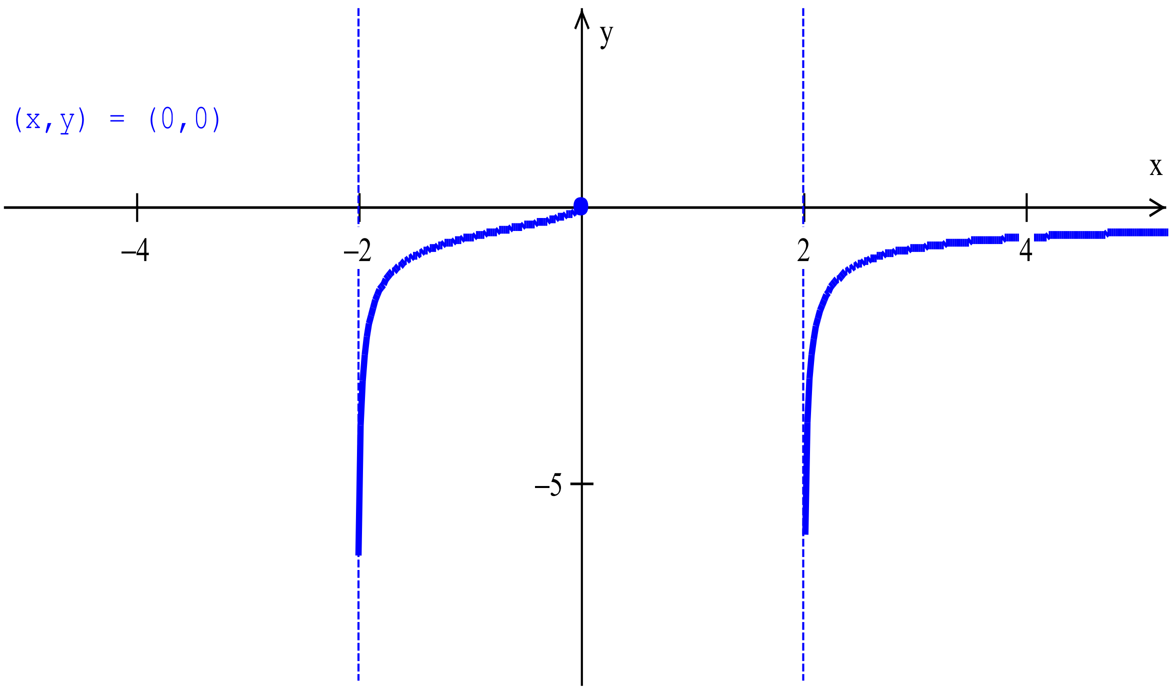
5. $y = \sqrt{x^2 - 4x - 5}$ Range: $y \in [0, \infty)$



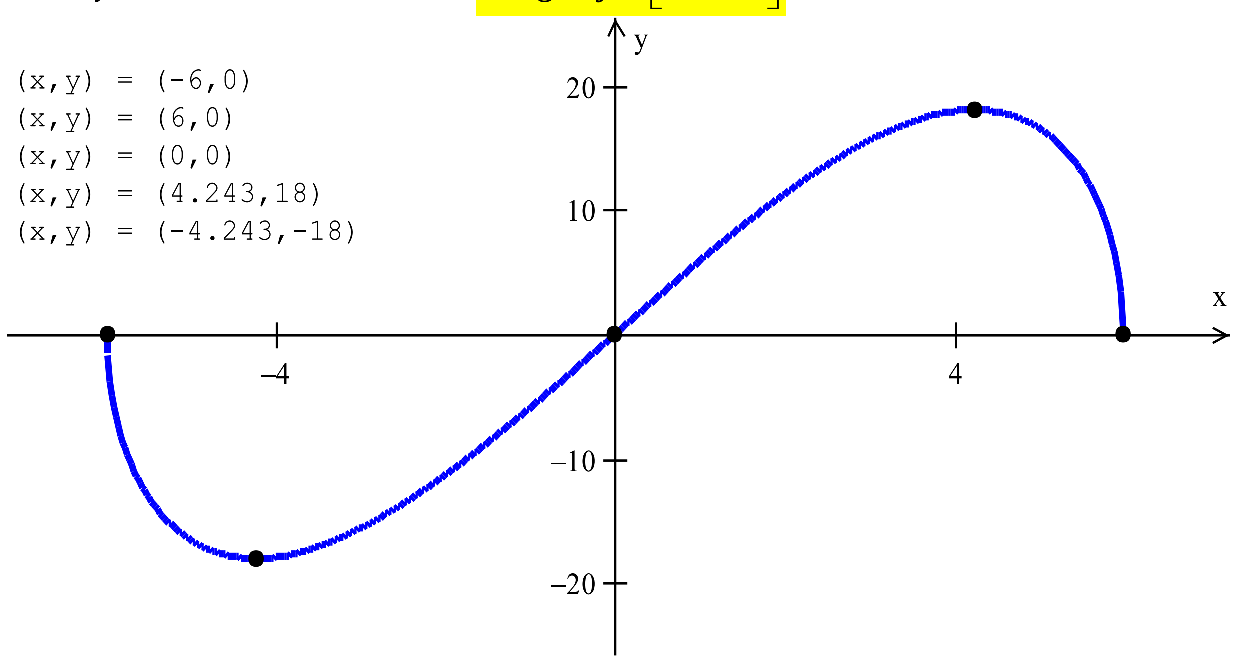
6. $y = \sqrt{x^3 + 3x^2 - 6x - 8}$ Range: $y \in [0, \infty)$



7. $y = -\sqrt{\dfrac{x}{x^2 - 4}}$ Range: $y \in (-\infty, 0]$

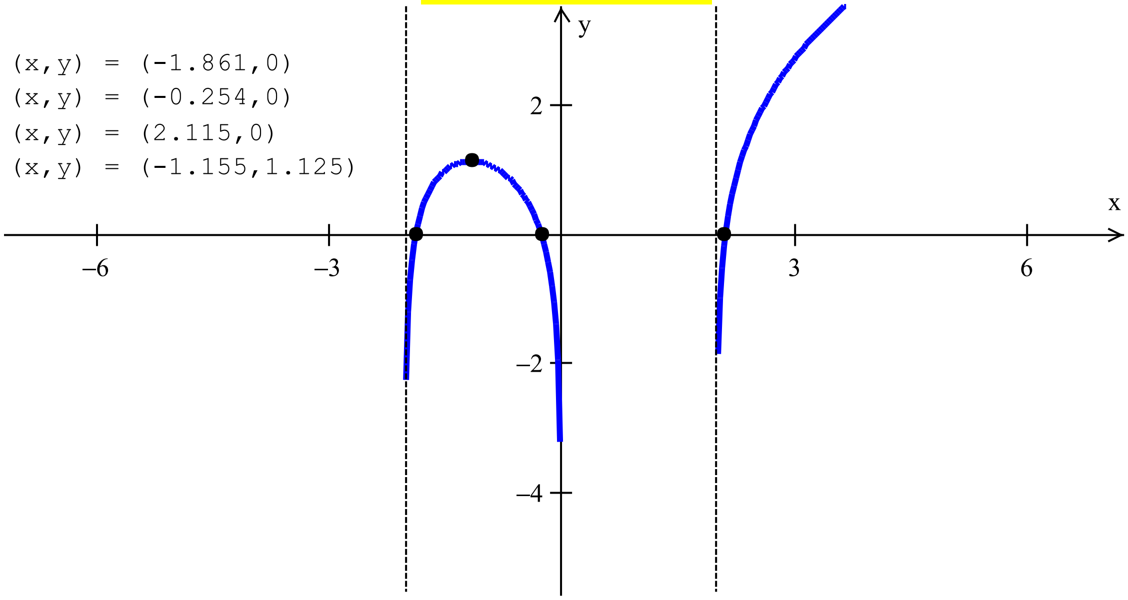


8. $y = x\sqrt{36 - x^2}$ Range: $y \in [-18, 18]$



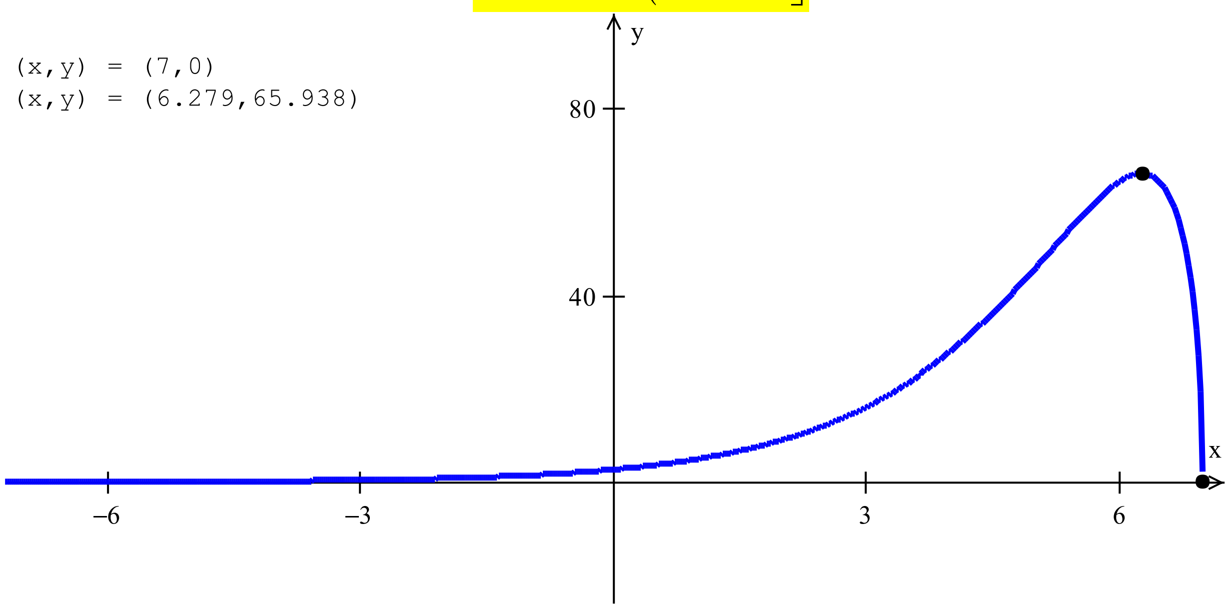
9. $y = \ln\left(x^3 - 4x\right)$ Range: $y \in (-\infty, \infty)$

```
(x,y) = (-1.861,0)
(x,y) = (-0.254,0)
(x,y) = (2.115,0)
(x,y) = (-1.155,1.125)
```

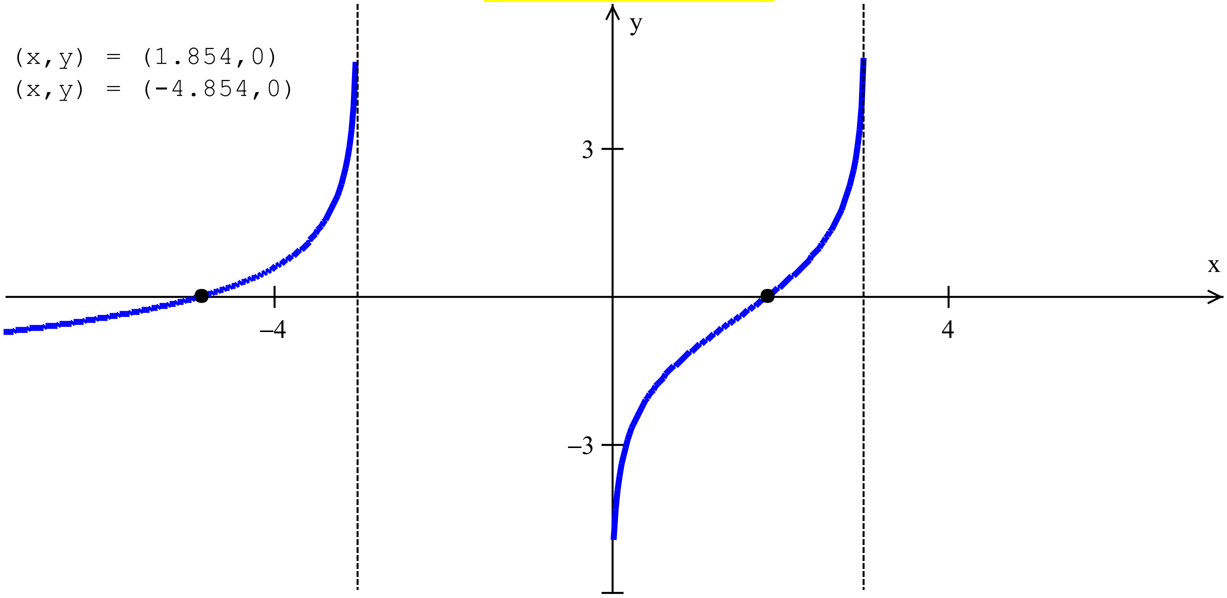


10. $y = 2^x\sqrt{7-x}$ Range: $y \in (0, 65.938]$

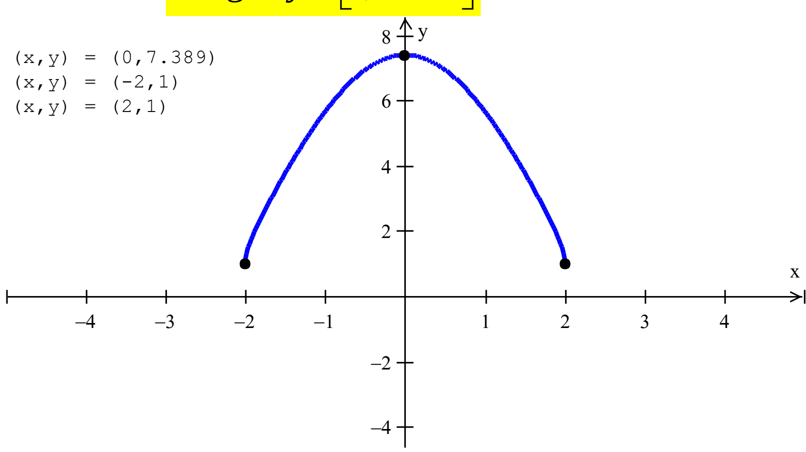
```
(x,y) = (7,0)
(x,y) = (6.279,65.938)
```



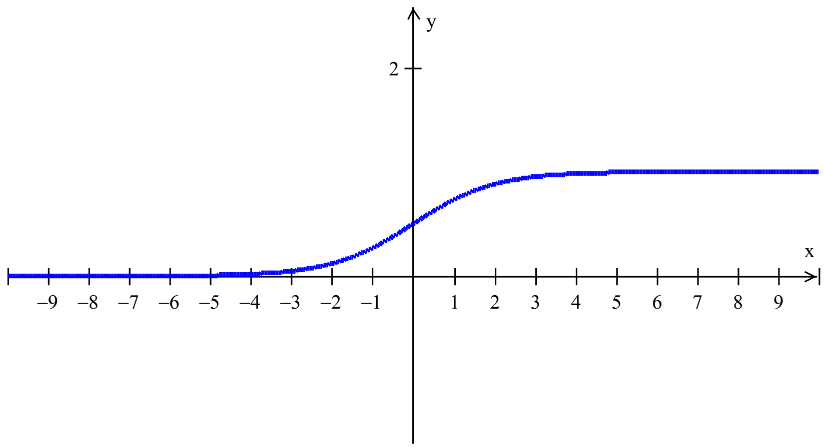
11. $y = \ln\left(\dfrac{-3x}{x^2-9}\right)$ Range: $y \in (-\infty, \infty)$



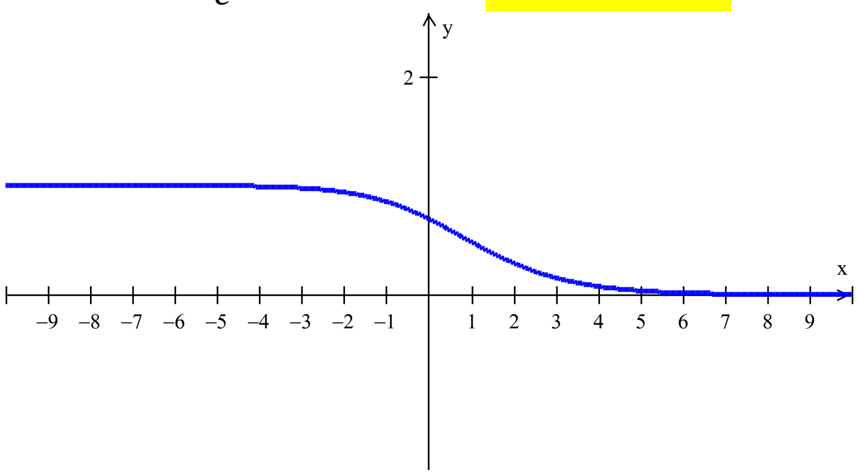
12. $y = e^{\sqrt{4-x^2}}$ Range: $y \in [1, 7.389]$



13. $y = \dfrac{1}{1+e^{-x}}$ Range: $y \in (0,1)$



14. $y = \dfrac{\ln\left(e^x + 1\right)}{e^x}$ Range: $y \in (0,1)$



A.5: Important Facts for Calculus:

When taking derivatives (or anti-derivatives, for that matter) it is usually the best policy to simplify everything as much as possible before taking the derivative. There are 3 basic categories of simplification: Algebraic, Logarithmic/Exponential, and Trigonometric.

Algebraic:

Simplify any algebraic expression to make it as much like a polynomial as possible.

- Distribute/FOIL: If there are multiple factors that are simple to multiply, do it.
- Distribute division by a common **monomial**: Eliminate the need for the quotient rule when dividing by a monomial by dividing each term by the monomial
- Write out everything in terms of fractional or negative exponents – you cannot use the power rule without doing this.

Logarithmic/Exponential

- Remember that natural logarithms and base e functions cancel one another:
  - $\ln\left(e^{u}\right)=u$
  - $e^{\ln(u)}=u$
- Any power on **the entire argument** of a logarithm can come out as multiplication. This includes fractional exponents
  - Example: $\ln\left(\sqrt{x^2+5x+6}\right)$; since there is a ½ power on the entire argument, it simplifies to $\frac{1}{2}\ln\left(x^2+5x+6\right)$
  - $\ln\left(u^{a}\right)=a\ln\left(u\right)$

Trigonometric:

- Remember the quotient identities, they can make your life much easier
  - $\dfrac{\sin u}{\cos u} = \tan u$
  - $\dfrac{\cos u}{\sin u} = \cot u$
- The double angle for sine, and the basic Pythagorean identities come up fairly often:
  - $2\sin A\cos A = \sin(2A)$
  - $\cos^2 A + \sin^2 A = 1$

All these are simplifications, done <u>before</u> you take a derivative – they are not derivative rules. Make sure you finish the problem by taking the derivative after you simplify.

More simplification rules:

Rules of exponents

1. $x^a x^b = x^{a+b}$
2. $\dfrac{x^a}{x^b} = x^{a-b}$
3. $\left(x^a\right)^b = x^{ab}$
4. $x^{-n} = \dfrac{1}{x^n}$

Rules of Radicals

$\sqrt[n]{x} = x^{1/n}$

$\sqrt[n]{x^a} = x^{a/n}$

Rules of Logarithms

$$\log_a x + \log_a y = \log_a (xy)$$

$$\log_a x - \log_a y = \log_a \frac{x}{y}$$

$$\log_a x^n = n \log_a x$$

Some Facts to Remember:

$$Ln\,1 = 0 \qquad Ln\,e = 1$$

$$Ln\,e^u = u \qquad e^{Ln\,u} = u$$

Change of Base Rules:

$$Log_a\,u = \frac{Log\,u}{Log\,a} \quad or \quad = \frac{Ln\,u}{Ln\,a}$$

$$a^u = e^{u \cdot Ln\,a}$$

The Table

Rads	Deg	Cos x	Sin x	Tan x
0	0	1	0	0
$\frac{\pi}{6}$	30	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90	0	1	DNE
$\frac{2\pi}{3}$	120	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	135	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	–1
$\frac{5\pi}{6}$	150	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
π	180	–1	0	0
$\frac{7\pi}{6}$	210	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{5\pi}{4}$	225	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
$\frac{4\pi}{3}$	240	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\sqrt{3}$
$\frac{3\pi}{2}$	270	0	–1	DNE
$\frac{5\pi}{3}$	300	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\sqrt{3}$
$\frac{7\pi}{4}$	315	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	–1
$\frac{11\pi}{6}$	330	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\frac{1}{\sqrt{3}}$
2π	360	1	0	0

<u>Unless stated otherwise, the problems are always in Radian mode.</u>