14: Vector Analysis

In previous chapters we have explored a relationship between vectors and integration. Our most tangible result: if $\vec{v}(t)$ is the vector–valued velocity function of a moving object, then integrating $\vec{v}(t)$ from $t = a$ to $t = b$ gives the displacement of that object over that time interval.

This chapter explores completely different relationships between vectors and integration. These relationships will enable us to compute the work done by a magnetic field in moving an object along a path and find how much air moves through an oddly–shaped screen in space, among other things.

Our upcoming work with integration will benefit from a review. We are not concerned here with techniques of integration, but rather what an integral "does" and how that relates to the notation we use to describe it.

Integration Review

Recall from Section 13.1 that when R is a region in the x-y plane, $\iint_R dA$ gives the area of the region R. The integral symbols are "elongated esses" meaning "sum" and dA represents "a small amount of area." Taken together, $\iint_R dA$ means "sum up, over R, small amounts of area." This sum then gives the total area of R. We use two integral symbols since R is a two–dimensional region.

Now let $z = f(x, y)$ represent a surface. The double integral $\iint_R f(x, y)\ dA$ means "sum up, over R, function values (heights) given by f times small amounts of area." Since "height $\times$ area = volume," we are summing small amounts of volume over R, giving the total signed volume under the surface $z = f(x, y)$ and above the x-y plane.

This notation does not directly inform us *how* to evaluate the double integrals to find an area or a volume. With additional work, we recognize that a small amount of area dA can be measured as the area of a small rectangle, with one side length a small change in x and the other side length a small change in y. That is, $dA = dx\,dy$ or $dA = dy\,dx$. We could also compute a small amount of area by thinking in terms of polar coordinates, where $dA = r\,dr\,d\theta$. These understandings lead us to the iterated integrals we used in Chapter 13.

Let us back our review up farther. Note that $\int_1^3\ dx = x\Big|_1^3 = 3 - 1 = 2$. We have simply measured the length of the interval $[1, 3]$. We could rewrite the above integral using syntax similar to the double integral syntax above:

$$\int_1^3\ dx = \int_I dx, \quad \text{where } I = [1, 3].$$

We interpret "$\int_I dx$" as meaning "sum up, over the interval I, small changes in x." A change in x is a length along the x-axis, so we are adding up along I small lengths, giving the total length of I.

We could also write $\int_1^3 f(x)\ dx$ as $\int_I f(x)\ dx$, interpreted as "sum up, over I, heights given by $y = f(x)$ times small changes in x." Since "height$\times$length = area," we are summing up areas and finding the total signed area between $y = f(x)$ and the x-axis.

This method of referring to the process of integration can be very powerful. It is the core of our notion of the Riemann Sum. When faced with a quantity to compute, if one can think of a way to approximate its value through a sum, the one is well on their way to constructing an integral (or, double or triple integral) that computes the desired quantity. We will demonstrate this process throughout this chapter, starting with the next section.

14.1 Introduction to Line Integrals

We first used integration to find "area under a curve." In this section, we learn to do this (again), but in a different context.

Consider the surface and curve shown in Figure 14.1.1(a). The surface is given by $f(x, y) = 1 - \cos(x)\sin(y)$. The dashed curve lies in the x-y plane and is the familiar $y = x^2$ parabola from $-1 \leq x \leq 1$; we'll call this curve C. The curve drawn with a solid line in the graph is the curve in space that lies on our surface with x and y values that lie on C.

The question we want to answer is this: what is the area that lies below the curve drawn with the solid line? In other words, what is the area of the region above C and under the the surface f? This region is shown in Figure 14.1.1(b).

We suspect the answer can be found using an integral, but before trying to figure out what that integral is, let us first try to approximate its value.

In Figure 14.1.1(c), four rectangles have been drawn over the curve C. The bottom corners of each rectangle lie on C, and each rectangle has a height given by the function $f(x, y)$ for some (x, y) pair along C between the rectangle's bottom corners.

As we know how to find the area of each rectangle, we are able to approximate the area above C and under f. Clearly, our approximation will be *an approximation*. The heights of the rectangles do not match exactly with the surface f, nor does the base of each rectangle follow perfectly the path of C.

In typical calculus fashion, our approximation can be improved by using more rectangles. The sum of the areas of these rectangles gives an approximate value of the true area above C and under f. As the area of each rectangle is "height$\times$width", we assert that the

$$\text{area above } C \approx \sum(\text{heights} \times \text{widths}).$$

When first learning of the integral, and approximating areas with "heights$\times$

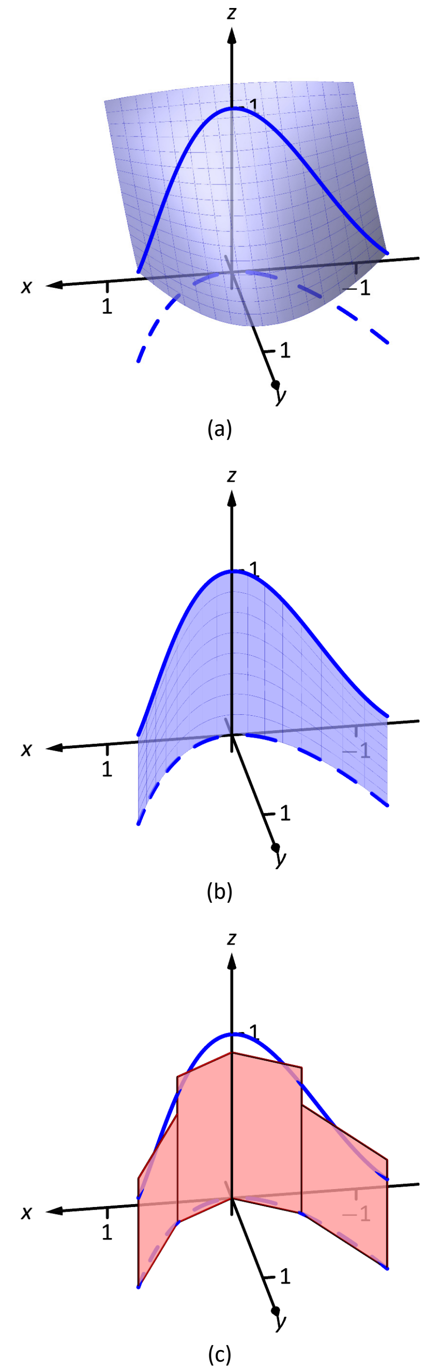


Figure 14.1.1: Finding area under a curve in space.

Notes:

widths", the width was a small change in x: dx. That will not suffice in this context. Rather, each width of a rectangle is actually approximating the arc length of a small portion of C. In Section 11.5, we used s to represent the arc–length parameter of a curve. A small amount of arc length will thus be represented by ds.

The height of each rectangle will be determined in some way by the surface f. If we parametrize C by s, an s-value corresponds to an (x, y) pair that lies on the parabola C. Since f is a function of x and y, and x and y are functions of s, we can say that f is a function of s. Given a value s, we can compute $f(s)$ and find a height. Thus

$$\begin{aligned}\text{area under } f \text{ and above } C &\approx \sum(\text{heights} \times \text{widths});\\ \text{area under } f \text{ and above } C &= \lim_{||\Delta s||\to 0}\sum f(c_i)\Delta s_i\\ &= \int_C f(s)\, ds. \qquad (14.1)\end{aligned}$$

Here we have introduce a new notation, the integral symbol with a subscript of C. It is reminiscent of our usage of $\iint_R$. Using the train of thought found in the Integration Review preceding this section, we interpret "$\int_C f(s)\, ds$" as meaning "sum up, along a curve C, function values $f(s)\times$small arc lengths." It is understood here that s represents the arc–length parameter.

All this leads us to a definition. The integral found in Equation 14.1 is called a **line integral**. We formally define it below, but note that the definition is very abstract. On one hand, one is apt to say "the defintion makes sense," while on the other, one is equally apt to say "but I don't know what I'm supposed to do with this definition." We'll address that after the definition, and actually find an answer to the area problem we posed at the beginning of this section.

Definition 14.1.1 Line Integral Over A Scalar Field

Let C be a smooth curve parametrized by s, the arc–length parameter, and let f be a continuous function of s. A **line integral** is an integral of the form

$$\int_C f(s)\, ds = \lim_{||\Delta s||\to 0}\sum_{i=1}^{n} f(c_i)\Delta s_i,$$

where $s_1 < s_2 < \ldots < s_n$ is any partition of the s-interval over which C is defined, c_i is any value in the i^{th} subinterval, Δs_i is the width of the i^{th} subinterval, and $||\Delta s||$ is the length of the longest subinterval in the partition.

Note: Definition 14.1.1 uses the term **scalar field** which has not yet been defined. Its meaning is discussed in the paragraph preceding Definition 14.3.1 when it is compared to a **vector field**.

Notes:

When C is a **closed** curve, i.e., a curve that ends at the same point at which it starts, we use

$$\oint_C f(s)\,ds \quad \text{instead of} \quad \int_C f(s)\,ds.$$

The definition of the line integral does not specify whether C is a curve in the plane or space (or hyperspace), as the definition holds regardless. For now, we'll assume C lies in the x-y plane.

This definition of the line integral doesn't really say anything new. If C is a curve and s is the arc–length parameter of C on $a \leq s \leq b$, then

$$\int_C f(s)\,ds = \int_a^b f(s)\,ds.$$

The real difference with this integral from the standard "$\int_a^b f(x)\,dx$" we used in the past is that of context. Our previous integrals naturally summed up values over an interval on the x-axis, whereas now we are summing up values over a curve. *If* we can parametrize the curve with the arc–length parameter, we can evaluate the line integral just as before. Unfortunately, parametrizing a curve in terms of the arc–length parameter is usually very difficult, so we must develop a method of evaluating line integrals using a different parametrization.

Given a curve C, find any parametrization of C: $x = g(t)$ and $y = h(t)$, for continuous functions g and h, where $a \leq t \leq b$. We can represent this parametrization with a vector–valued function, $\vec{r}(t) = \langle g(t), h(t)\rangle$.

In Section 11.5, we defined the arc–length parameter in Equation 11.1 as

$$s(t) = \int_0^t ||\,\vec{r}\,'(u)\,||\,du.$$

By the Fundamental Theorem of Calculus, $ds = ||\,\vec{r}\,'(t)\,||\,dt$. We can substitute the right hand side of this equation for ds in the line integral definition.

We can view f as being a function of x and y since it is a function of s. Thus $f(s) = f(x,y) = f\big(g(t), h(t)\big)$. This gives us a concrete way to evaluate a line integral:

$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\,||\,\vec{r}\,'(t)\,||\,dt.$$

We write this as a Key Idea, along with its three–dimensional analogue, followed by an example where we finally evaluate an integral and find an area.

Notes:

Key Idea 14.1.1 Evaluating a Line Integral

- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t)\rangle$, $a \leq t \leq b$, where g and h are continuously differentiable, and let $z = f(x, y)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t)\big)\,||\,\vec{r}\,'(t)\,||\,dt.$$
- Let C be a curve parametrized by $\vec{r}(t) = \langle g(t), h(t), k(t)\rangle$, $a \leq t \leq b$, where g, h and k are continuously differentiable, and let $w = f(x, y, z)$. Then
$$\int_C f(s)\,ds = \int_a^b f\big(g(t), h(t), k(t)\big)\,||\,\vec{r}\,'(t)\,||\,dt.$$

To be clear, the first point of Key Idea 14.1.1 can be used to find the area under a surface $z = f(x, y)$ and above a curve C. We will later give an understanding of the line integral when C is a curve in space.

Let's do an example where we actually compute an area.

Example 14.1.1 Evaluating a line integral: area under a surface over a curve.

Find the area under the surface $f(x, y) = \cos(x) + \sin(y) + 2$ over the curve C, which is the segment of the line $y = 2x + 1$ on $-1 \leq x \leq 1$, as shown in Figure 14.1.2.

Solution Our first step is to represent C with a vector–valued function. Since C is a simple line, and we have a explicit relationship between y and x (namely, that y is $2x+1$), we can let $x = t$, $y = 2t+1$, and write $\vec{r}(t) = \langle t, 2t+1\rangle$ for $-1 \leq t \leq 1$.

We find the values of f over C as $f(x, y) = f(t, 2t+1) = \cos(t)+\sin(2t+1)+2$. We also need $||\,\vec{r}\,'(t)\,||$; with $\vec{r}\,'(t) = \langle 1, 2\rangle$, we have $||\,\vec{r}\,'(t)\,|| = \sqrt{5}$. Thus $ds = \sqrt{5}\,dt$.

The area we seek is
$$\begin{aligned}\int_C f(s)\,ds &= \int_{-1}^{1} \big(\cos(t) + \sin(2t + 1) + 2\big)\sqrt{5}\,dt\\ &= \sqrt{5}\Big(\sin(t) - \frac{1}{2}\cos(2t + 1) + 2t\Big)\Big|_{-1}^{1}\\ &\approx 14.418 \text{ units}^2.\end{aligned}$$

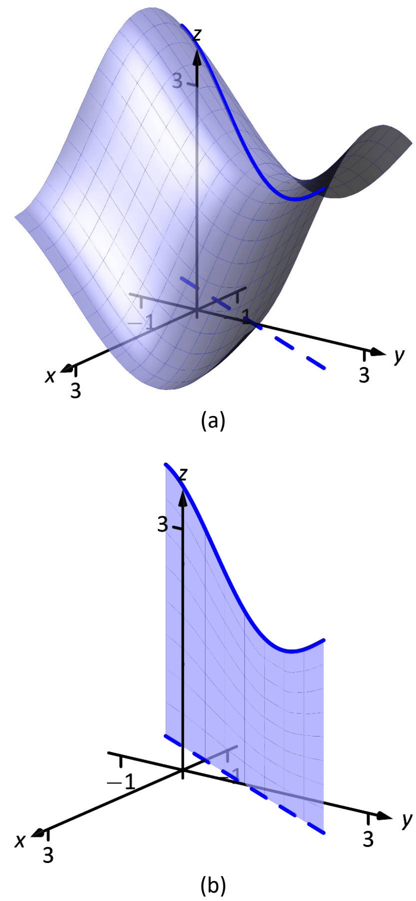


Figure 14.1.2: Finding area under a curve in Example 14.1.1.

Notes:

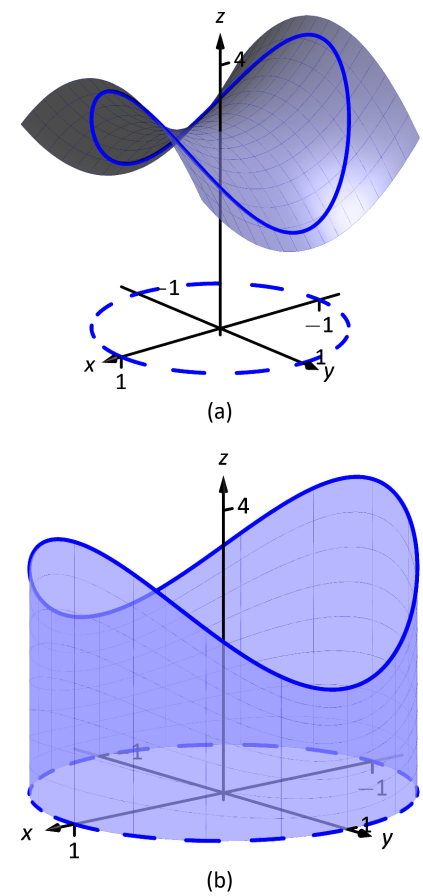


Figure 14.1.3: Finding area under a curve in Example 14.1.2.

We will practice setting up and evaluating a line integral in another example, then find the area described at the beginning of this section.

Example 14.1.2 Evaluating a line integral: area under a surface over a curve.
Find the area over the unit circle in the x-y plane and under the surface $f(x,y) = x^2 - y^2 + 3$, shown in Figure 14.1.3.

Solution The curve C is the unit circle, which we will describe with the parametrization $\vec{r}(t) = \langle \cos t, \sin t\rangle$ for $0 \leq t \leq 2\pi$. We find $||\,\vec{r}\,'(t)\,|| = 1$, so $ds = 1dt$.

We find the values of f over C as $f(x,y) = f(\cos t, \sin t) = \cos^2 t - \sin^2 t + 3$. Thus the area we seek is (note the use of the $\oint f(s)ds$ notation):

$$\begin{aligned}\oint_C f(s)\ ds &= \int_0^{2\pi} \big(\cos^2 t - \sin^2 t + 3\big)\ dt\\ &= 6\pi.\end{aligned}$$

(Note: we may have approximated this answer from the start. The unit circle has a circumference of 2π, and we may have guessed that due to the apparent symmetry of our surface, the average height of the surface is 3.)

We now consider the example that introduced this section.

Example 14.1.3 Evaluating a line integral: area under a surface over a curve.
Find the area under $f(x,y) = 1 - \cos(x)\sin(y)$ and over the parabola $y = x^2$, from $-1 \leq x \leq 1$.

Solution We parametrize our curve C as $\vec{r}(t) = \langle t, t^2\rangle$ for $-1 \leq t \leq 1$; we find $||\,\vec{r}\,'(t)\,|| = \sqrt{1+4t^2}$, so $ds = \sqrt{1+4t^2}\ dt$.

Replacing x and y with their respective functions of t, we have $f(x,y) = f(t,t^2) = 1 - \cos(t)\sin(t^2)$. Thus the area under f and over C is found to be

$$\int_C f(s)\ ds = \int_{-1}^{1} \Big(1 - \cos(t)\sin\big(t^2\big)\Big)\sqrt{1+t^2}\ dt.$$

This integral is impossible to evaluate using the techniques developed in this text. We resort to a numerical approximation; accurate to two places after the decimal, we find the area is

$$= 2.17.$$

Notes:

We give one more example of finding area.

Example 14.1.4 Evaluating a line integral: area under a curve in space.
Find the area above the x-y plane and below the helix parametrized by $\vec{r}(t) = \langle \cos t, 2\sin t, t/\pi\rangle$, for $0 \le t \le 2\pi$, as shown in Figure 14.1.4.

SOLUTION Note how this is problem is different than the previous examples: here, the height is not given by a surface, but by the curve itself.

We use the given vector-valued function $\vec{r}(t)$ to determine the curve C in the x-y plane by simply using the first two components of $\vec{r}(t)$: $\vec{c}(t) = \langle \cos t, 2\sin t\rangle$. Thus $ds = ||\,\vec{c}\,'(t)\,||\,dt = \sqrt{\sin^2 t + 4\cos^2 t}\,dt$.

The height is not found by evaluating a surface over C, but rather it is given directly by the third component of $\vec{r}(t)$: t/π. Thus

$$\oint_C f(s)\,ds = \int_0^{2\pi} \frac{t}{\pi}\sqrt{\sin^2 t + 4\cos^2 t}\,dt \approx 9.69,$$

where the approximation was obtained using numerical methods.

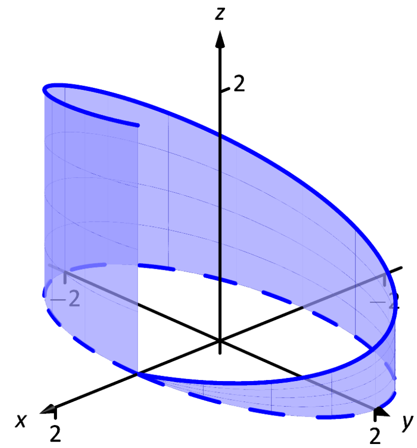


Figure 14.1.4: Finding area under a curve in Example 14.1.4.

Note how in each of the previous examples we are effectively finding "area under a curve", just as we did when first learning of integration. We have used the phrase "area *over* a curve C and under a surface," but that is because of the important role C plays in the integral. The figures show how the curve C defines another curve on the surface $z = f(x, y)$, and we are finding the area under that curve.

Properties of Line Integrals

Many properties of line integrals can be inferred from general integration properties. For instance, if k is a scalar, then $\int_C k f(s)ds = k\int_C f(s)ds$.

One property in particular of line integrals is worth noting. If C is a curve composed of subcurves C_1 and C_2, where they share only one point in common (see Figure 14.1.5(a)), then the line integral over C is the sum of the line integrals over C_1 and C_2:

$$\int_C f(s)\,ds = \int_{C_1} f(s)\,ds + \int_{C_2} f(s)\,ds.$$

This property allows us to evaluate line integrals over some curves C that are not smooth. Note how in Figure 14.1.5(b) the curve is not smooth at D, so by our definition of the line integral we cannot evaluate $\int_C f(s)ds$. However, one can evaluate line integrals over C_1 and C_2 and their sum will be the desired quantity.

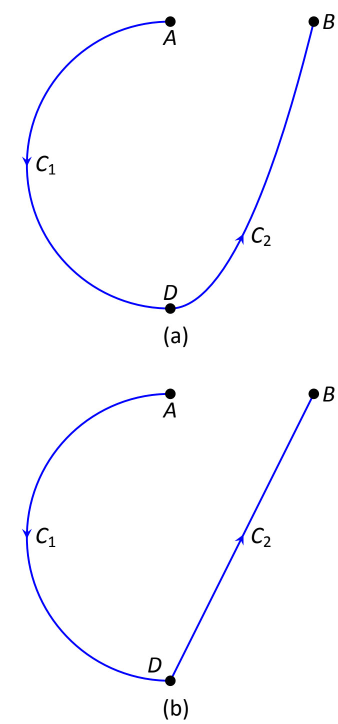


Figure 14.1.5: Illustrating properties of line integrals.

Notes:

A curve C that is composed of two or more smooth curves is said to be **piecewise smooth**. In this chapter, any statement that is made about smooth curves also holds for piecewise smooth curves.

We state these properties as a theorem.

Theorem 14.1.1 Properties of Line Integrals Over Scalar Fields

1. Let C be a smooth curve parametrized by the arc–length parameter s, let f and g be continuous functions of s, and let k_1 and k_2 be scalars. Then

$$\int_C \big(k_1 f(s) + k_2 g(s)\big)\, ds = k_1 \int_C f(s)\, ds + k_2 \int_C g(s)\, ds.$$

2. Let C be piecewise smooth, composed of smooth components C_1 and C_2. Then

$$\int_C f(s)\, ds = \int_{C_1} f(s)\, ds + \int_{C_2} f(s)\, ds.$$

Mass and Center of Mass

We first learned integration as a method to find area under a curve, then later used integration to compute a variety of other quantities, such as arc length, volume, force, etc. In this section, we also introduced line integrals as a method to find area under a curve, and now we explore one more application.

Let a curve C (either in the plane or in space) represent a thin wire with variable density $\delta(s)$. We can approximate the mass of the wire by dividing the wire (i.e., the curve) into small segments of length Δs_i and assume the density is constant across these small segments. The mass of each segment is density of the segment $\times$ its length; by summing up the approximate mass of each segment we can approximate the total mass:

$$\text{Total Mass of Wire } = \sum \delta(s_i)\Delta s_i.$$

By taking the limit as the length of the segments approaches 0, we have the definition of the line integral as seen in Definition 14.1.1. When learning of the line integral, we let $f(s)$ represent a height; now we let $f(s) = \delta(s)$ represent a density.

We can extend this understanding of computing mass to also compute the center of mass of a thin wire. (As a reminder, the center of mass can be a useful

Notes:

piece of information as objects rotate about that center.) We give the relevant formulas in the next definition, followed by an example. Note the similarities between this definition and Definition 13.6.4, which gives similar properties of solids in space.

Definition 14.1.2 Mass, Center of Mass of Thin Wire

Let a thin wire lie along a smooth curve C with continuous density function $\delta(s)$, where s is the arc length parameter.

1. The **mass** of the thin wire is $M = \int_C \delta(s)\,ds$.
2. The **moment about the y-z plane** is $M_{yz} = \int_C x\delta(s)\,ds$.
3. The **moment about the x-z plane** is $M_{xz} = \int_C y\delta(s)\,ds$.
4. The **moment about the x-y plane** is $M_{xy} = \int_C z\delta(s)\,ds$.
5. The **center of mass** of the wire is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

Example 14.1.5 Evaluating a line integral: calculating mass.

A thin wire follows the path $\vec{r}(t) = \langle 1+\cos t, 1+\sin t, 1+\sin(2t)\rangle$, $0 \leq t \leq 2\pi$. The density of the wire is determined by its position in space: $\delta(x,y,z) = y+z$ gm/cm. The wire is shown in Figure 14.1.6, where a light color indicates low density and a dark color represents high density. Find the mass and center of mass of the wire.

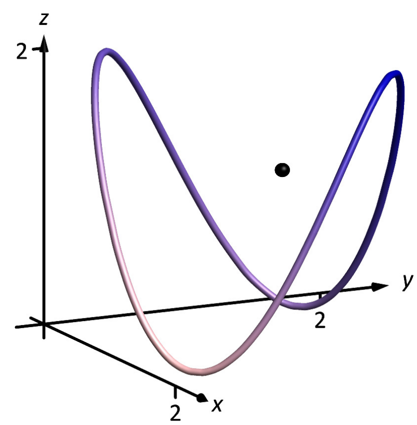


Figure 14.1.6: Finding the mass of a thin wire in Example 14.1.5.

Solution We compute the density of the wire as

$$\delta(x,y,z) = \delta\big(1+\cos t, 1+\sin t, 1+\sin(2t)\big) = 2+\sin t+\sin(2t).$$

We compute ds as

$$ds = ||\,\vec{r}\,'(t)\,||\,dt = \sqrt{\sin^2 t+\cos^2 t+4\cos^2(2t)}\,dt = \sqrt{1+4\cos^2(2t)}\,dt.$$

Thus the mass is

$$M = \oint_C \delta(s)\,ds = \int_0^{2\pi} \big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 21.08\text{gm}.$$

Notes:

We compute the moments about the coordinate planes:

$$M_{yz} = \oint_C x\delta(s)\,ds = \int_0^{2\pi} (1+\cos t)\big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 21.08.$$
$$M_{xz} = \oint_C y\delta(s)\,ds = \int_0^{2\pi} (1+\sin t)\big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 26.35$$
$$M_{xy} = \oint_C z\delta(s)\,ds = \int_0^{2\pi} \big(1+\sin(2t)\big)\big(2+\sin t+\sin(2t)\big)\sqrt{1+4\cos^2(2t)}\,dt \approx 25.40$$

Thus the center of mass of the wire is located at

$$(\overline{x},\overline{y},\overline{z}) = \left(\frac{M_{yz}}{M},\frac{M_{xz}}{M},\frac{M_{xy}}{M}\right) \approx (1, 1.25, 1.20),$$

as indicated by the dot in Figure 14.1.6. Note how in this example, the curve C is "centered" about the point $(1,1,1)$, though the variable density of the wire pulls the center of mass out along the y and z axes.

We end this section with a callback to the Integration Review that preceded this section. A line integral looks like: $\int_C f(s)\,ds$. As stated before the definition of the line integral, this means "sum up, along a curve C, function values $f(s)\times$ small arc lengths." When $f(s)$ represents a height, we have "height $\times$ length = area." When $f(s)$ is a density (and we use $\delta(s)$ by convention), we have "density (mass per unit length) $\times$ length = mass."

In the next section, we investigate a new mathematical object, the *vector field*. The remaining sections of this chapter are devoted to understanding integration in the context of vector fields.

Notes:

Exercises 14.1

Terms and Concepts

1. Explain how a line integral can be used to find the area under a curve.

2. How does the evaluation of a line integral given as $\int_C f(s)\, ds$ differ from a line integral given as $\oint_C f(s)\, ds$?

3. Why are most line integrals evaluated using Key Idea 14.1.1 instead of "directly" as $\int_C f(s)\, ds$?

4. Sketch a closed, piecewise smooth curve composed of three subcurves.

Problems

In Exercises 5 – 10, a planar curve C is given along with a surface f that is defined over C. Evaluate the line integral $\displaystyle\int_C f(s)\, ds$.

5. C is the line segment joining the points $(-2,-1)$ and $(1,2)$; the surface is $f(x,y)=x^2+y^2+2$.

6. C is the segment of $y=3x+2$ on $[1,2]$; the surface is $f(x,y)=5x+2y$.

7. C is the circle with radius 2 centered at the point $(4,2)$; the surface is $f(x,y)=3x-y$.

8. C is the curve given by $\vec{r}(t)=\langle \cos t+t\sin t, \sin t-t\cos t\rangle$ on $[0,2\pi]$; the surface is $f(x,y)=5$.

9. C is the piecewise curve composed of the line segments that connect $(0,1)$ to $(1,1)$, then connect $(1,1)$ to $(1,0)$; the surface is $f(x,y)=x+y^2$.

10. C is the piecewise curve composed of the line segment joining the points $(0,0)$ and $(1,1)$, along with the quarter–circle parametrized by $\langle \cos t, -\sin t+1\rangle$ on $[0,\pi/2]$(which starts at the point $(1,1)$ and ends at $(0,0)$; the surface is $f(x,y)=x^2+y^2$.

In Exercises 11 – 14, a planar curve C is given along with a surface f that is defined over C. Set up the line integral $\displaystyle\int_C f(s)\, ds$, then approximate its value using technology.

11. C is the portion of the parabola $y=2x^2+x+1$ on $[0,1]$; the surface is $f(x,y)=x^2+2y$.

12. C is the portion of the curve $y=\sin x$ on $[0,\pi]$; the surface is $f(x,y)=x$.

13. C is the ellipse given by $\vec{r}(t)=\langle 2\cos t, \sin t\rangle$ on $[0,2\pi]$; the surface is $f(x,y)=10-x^2-y^2$.

14. C is the portion of $y=x^3$ on $[-1,1]$; the surface is $f(x,y)=2x+3y+5$.

In Exercises 15 – 18, a parametrized curve C in space is given. Find the area above the x-y plane that is under C.

15. C: $\vec{r}(t)=\langle 5t, t, t^2\rangle$ for $1\le t\le 2$.

16. C: $\vec{r}(t)=\langle \cos t, \sin t, \sin(2t)+1\rangle$ for $0\le t\le 2\pi$.

17. C: $\vec{r}(t)=\langle 3\cos t, 3\sin t, t^2\rangle$ for $0\le t\le 2\pi$.

18. C: $\vec{r}(t)=\langle 3t, 4t, t\rangle$ for $0\le t\le 1$.

In Exercises 19 – 20, a parametrized curve C is given that represents a thin wire with density δ. Find the mass and center of mass of the thin wire.

19. C: $\vec{r}(t)=\langle \cos t, \sin t, t\rangle$ for $0\le t\le 4\pi$; $\delta(x,y,z)=z$.

20. C: $\vec{r}(t)=\langle t-t^2, t^2-t^3, t^3-t^4\rangle$ for $0\le t\le 1$; $\delta(x,y,z)=x+2y+2z$. Use technology to approximate the value of each integral.

14.2 Vector Fields

We have studied functions of two and three variables, where the input of such functions is a point (either a point in the plane or in space) and the output is a number.

We could also create functions where the input is a point (again, either in the plane or in space), but the output is a *vector*. For instance, we could create the following function: $\vec{F}(x,y) = \langle x+y, x-y\rangle$, where $\vec{F}(2,3) = \langle 5,-1\rangle$. We are to think of $\vec{F}$ assigning the vector $\langle 5,-1\rangle$ to the point $(2,3)$; in some sense, the vector $\langle 5,-1\rangle$ lies at the point $(2,3)$.

Such functions are extremely useful in any context where magnitude and direction are important. For instance, we could create a function $\vec{F}$ that represents the electromagnetic force exerted at a point by a electromagnetic field, or the velocity of air as it moves across an airfoil.

Because these functions are so important, we need to formally define them.

Definition 14.2.1 Vector Field

1. A **vector field in the plane** is a function $\vec{F}(x,y)$ whose domain is a subset of $\mathbb{R}^2$ and whose output is a two–dimensional vector:
$$\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle.$$
2. A **vector field in space** is a function $\vec{F}(x,y,z)$ whose domain is a subset of $\mathbb{R}^3$ and whose output is a three–dimensional vector:
$$\vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle.$$

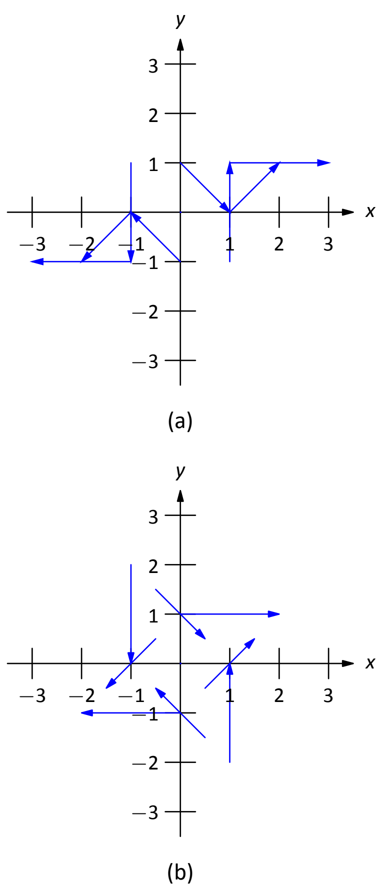


Figure 14.2.1: Demonstrating methods of graphing vector fields.

This definition may seem odd at first, as a special type of function is called a "field." However, as the function determines a "field of vectors", we can say the field is *defined by* the function, and thus the field *is* a function.

Visualizing vector fields helps cement this connection. When graphing a vector field in the plane, the general idea is to draw the vector $\vec{F}(x,y)$ at the point (x,y). For instance, using $\vec{F}(x,y) = \langle x+y, x-y\rangle$ as before, at $(1,1)$ we would draw $\langle 2,0\rangle$.

In Figure 14.2.1(a), one can see that the vector $\langle 2,0\rangle$ is drawn *starting from* the point $(1,1)$. A total of 8 vectors are drawn, with the x- and y-values of $-1,0,1$. In many ways, the resulting graph is a mess; it is hard to tell what this field "looks like."

In Figure 14.2.1(b), the same field is redrawn with each vector $\vec{F}(x,y)$ drawn *centered on* the point (x,y). This makes for a better looking image, though the

Notes:

long vectors can cause confusion: when one vector intersects another, the image looks cluttered.

A common way to address this problem is limit the length of each arrow, and represent long vectors with thick arrows, as done in Figure 14.2.2(a). Usually we do not use a graph of a vector field to determine exactly the magnitude of a particular vector. Rather, we are more concerned with the relative magnitudes of vectors: which are bigger than others? Thus limiting the length of the vectors is not problematic.

Drawing arrows with variable thickness is best done with technology; search the documentation of your favorite graphing program for terms like "vector fields" or "slope fields" to learn how. Technology obviously allows us to plot many vectors in a vector field nicely; in Figure 14.2.2(b), we see the same vector field drawn with many vectors, and finally get a clear picture of how this vector field behaves. (If this vector field represented the velocity of air moving across a flat surface, we could see that the air tends to move either to the upper–right or lower–left, and moves very slowly near the origin.)

We can similarly plot vector fields in space, as shown in Figure 14.2.3, though it is not often done. The plots get very busy very quickly, as there are lots of arrows drawn in a small amount of space. In Figure 14.2.3 the field $\vec{F} = \langle -y, x, z\rangle$ is graphed. If one could view the graph from above, one could see the arrows point in a cirlce about the z-axis. One should also note how the arrows far from the origin are larger than those close to the origin.

It is good practice to try to visualize certain vector fields in one's head. For instance, consider a point mass at the origin and the vector field that represents the gravitational force exerted by the mass at any point in the room. The field would consist of arrows pointing toward the origin, increasing in size as they near the origin (as the gravitational pull is strongest near the point mass).

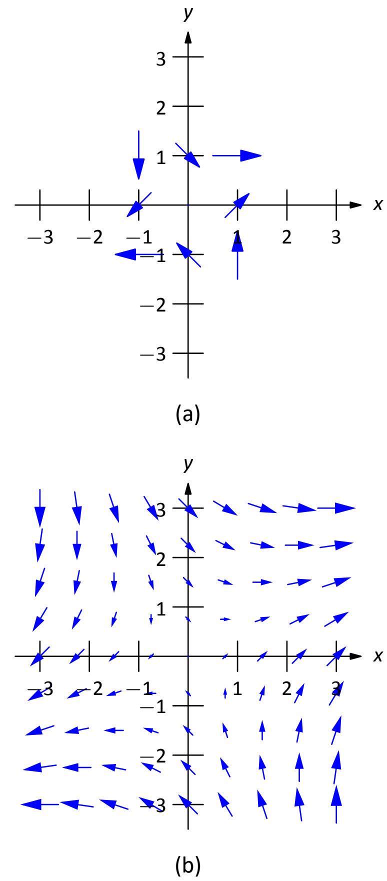


Figure 14.2.2: Demonstrating methods of graphing vector fields.

Vector Field Notation and Del Operator

Definition 14.2.1 defines a vector field $\vec{F}$ using the notation

$$\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle \quad \text{and} \quad \vec{F}(x,y,z) = \langle M(x,y,z), N(x,y,z), P(x,y,z)\rangle.$$

That is, the components of $\vec{F}$ are each functions of x and y (and also z in space). As done in other contexts, we will drop the "of x, y and z" portions of the notation and refer to vector fields in the plane and in space as

$$\vec{F} = \langle M, N\rangle \quad \text{and} \quad \vec{F} = \langle M, N, P\rangle,$$

respectively, as this shorthand is quite convenient.

Another item of notation will become useful: the "del operator." Recall in Section 12.6 how we used the symbol ∇ (pronounced "del") to represent the

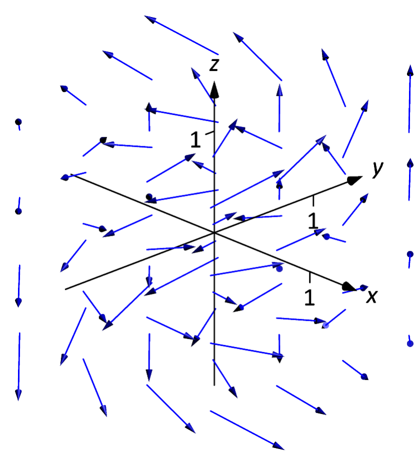


Figure 14.2.3: Graphing a vector field in space.

Notes:

gradient of a function of two variables. That is, if $z = f(x,y)$, then "del f" $= \nabla f = \langle f_x, f_y \rangle$.

We now define ∇ to be the "del operator." It is a vector whose components are partial derivative operations.

In the plane, $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$; in space, $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

With this definition of ∇, we can better understand the gradient ∇f. As f returns a scalar, the properties of scalar and vector multiplication gives

$$\nabla f = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle f = \left\langle \frac{\partial}{\partial x} f, \frac{\partial}{\partial y} f \right\rangle = \langle f_x, f_y \rangle.$$

Now apply the del operator ∇ to vector fields. Let $\vec{F} = \langle x + \sin y, y^2 + z, x^2 \rangle$. We can use vector operations and find the dot product of ∇ and $\vec{F}$:

$$\begin{aligned} \nabla \cdot \vec{F} &= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x + \sin y, y^2 + z, x^2 \rangle \\ &= \frac{\partial}{\partial x}(x + \sin y) + \frac{\partial}{\partial y}(y^2 + z) + \frac{\partial}{\partial z}(x^2) \\ &= 1 + 2y. \end{aligned}$$

We can also compute their cross products:

$$\begin{aligned} \nabla \times \vec{F} &= \left\langle \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial z}(y^2 + z), \frac{\partial}{\partial z}(x + \sin y) - \frac{\partial}{\partial x}(x^2), \frac{\partial}{\partial x}(y^2 + z) - \frac{\partial}{\partial y}(x + \sin y) \right\rangle \\ &= \langle -1, -2x, -\cos y \rangle. \end{aligned}$$

We do not yet know why we would want to compute the above. However, as we next learn about properties of vector fields, we will see how these dot and cross products with the del operator are quite useful.

Divergence and Curl

Two properties of vector fields will prove themselves to be very important: divergence and curl. Each is a special "derivative" of a vector field; that is, each measures an instantaneous rate of change of a vector field.

If the vector field represents the velocity of a fluid or gas, then the **divergence** of the field is a measure of the "compressibility" of the fluid. If the divergence is negative at a point, it means that the fluid is compressing: more fluid is going into the point than is going out. If the divergence is positive, it means the fluid is expanding: more fluid is going out at that point than going in. A divergence of zero means the same amount of fluid is going in as is going out. If the divergence is zero at all points, we say the field is **incompressible**.

Notes:

It turns out that the proper measure of divergence is simply $\nabla \cdot \vec{F}$, as stated in the following definition.

Definition 14.2.2 Divergence of a Vector Field

The **divergence** of a vector field $\vec{F}$ is

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F}.$$

- In the plane, with $\vec{F} = \langle M, N\rangle$, $\text{div}\,\vec{F} = M_x + N_y$.
- In space, with $\vec{F} = \langle M, N, P\rangle$, $\text{div}\,\vec{F} = M_x + N_y + P_z$.

Curl is a measure of the spinning action of the field. Let $\vec{F}$ represent the flow of water over a flat surface. If a small round cork were held in place at a point in the water, would the water cause the cork to spin? No spin corresponds to zero curl; counterclockwise spin corresponds to positive curl and clockwise spin corresponds to negative curl.

In space, things are a bit more complicated. Again let $\vec{F}$ represent the flow of water, and imagine suspending a tennis ball in one location in this flow. The water may cause the ball to spin along an axis. If so, the curl of the vector field is a *vector* (not a *scalar*, as before), parallel to the axis of rotation, following a right hand rule: when the thumb of one's right hand points in the direction of the curl, the ball will spin in the direction of the curling fingers of the hand.

In space, it turns out the proper measure of curl is $\nabla \times \vec{F}$, as stated in the following definition. To find the curl of a planar vector field $\vec{F} = \langle M, N\rangle$, embed it into space as $\vec{F} = \langle M, N, 0\rangle$ and apply the cross product definition. Since M and N are functions of just x and y (and not z), all partial derivatives with respect to z become 0 and the result is simply $\langle 0, 0, N_x - M_y\rangle$. The third component is the measure of curl of a planar vector field.

Definition 14.2.3 Curl of a Vector Field

- Let $\vec{F} = \langle M, N\rangle$ be a vector field in the plane. The **curl** of $\vec{F}$ is $\text{curl}\,\vec{F} = N_x - M_y$.
- Let $\vec{F} = \langle M, N, P\rangle$ be a vector field in space. The **curl** of $\vec{F}$ is $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \langle P_y - N_z, M_z - P_x, N_x - M_y\rangle$.

We adopt the convention of referring to curl as $\nabla \times \vec{F}$, regardless of whether

Notes:

$\vec{F}$ is a vector field in two or three dimensions.

We now practice computing these quantities.

Example 14.2.1 Computing divergence and curl of planar vector fields

For each of the planar vector fields given below, view its graph and try to visually determine if its divergence and curl are 0. Then compute the divergence and curl.

1. $\vec{F} = \langle y, 0\rangle$ (see Figure 14.2.4(a))
2. $\vec{F} = \langle -y, x\rangle$ (see Figure 14.2.4(b))
3. $\vec{F} = \langle x, y\rangle$ (see Figure 14.2.5(a))
4. $\vec{F} = \langle \cos y, \sin x\rangle$ (see Figure 14.2.5(b))

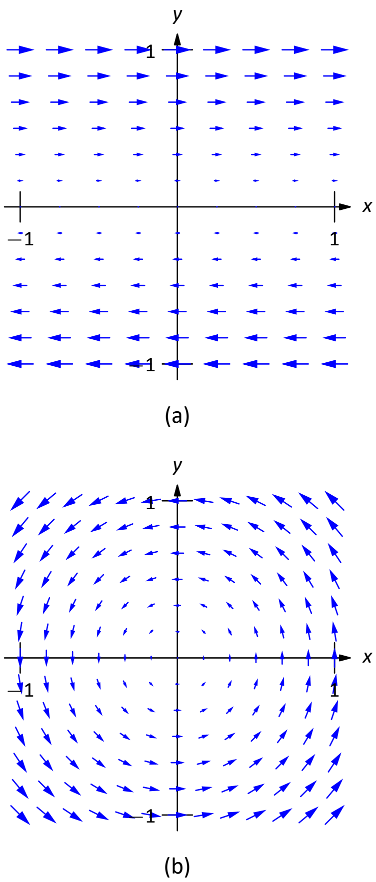


Figure 14.2.4: The vector fields in parts (a) and (b) in Example 14.2.1.

Solution

1. The arrow sizes are constant along any horizontal line, so if one were to draw a small box anywhere on the graph, it would seem that the same amount of fluid would enter the box as exit. Therefore it seems the divergence is zero; it is, as

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = M_x + N_y = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(0) = 0.$$

   At any point on the x-axis, arrows above it move to the right and arrows below it move to the left, indicating that a cork placed on the axis would spin clockwise. A cork placed anywhere above the x-axis would have water above it moving to the right faster than the water below it, also creating a clockwise spin. A clockwise spin also appears to be created at points below the x-axis. Thus it seems the curl should be negative (and not zero). Indeed, it is:

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = N_x - M_y = \frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(y) = -1.$$

2. It appears that all vectors that lie on a circle of radius r, centered at the origin, have the same length (and indeed this is true). That implies that the divergence should be zero: draw any box on the graph, and any fluid coming in will lie along a circle that takes the same amount of fluid out. Indeed, the divergence is zero, as

$$\text{div}\,\vec{F} = \nabla\cdot\vec{F} = M_x + N_y = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

Notes:

Clearly this field moves objects in a circle, but would it induce a cork to spin? It appears that yes, it would: place a cork anywhere in the flow, and the point of the cork closest to the origin would feel less flow than the point on the cork farthest from the origin, which would induce a counterclockwise flow. Indeed, the curl is positive:

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = N_x - M_y = \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y) = 1 - (-1) = 2.$$

Since the curl is constant, we conclude the induced spin is the same no matter where one is in this field.

3. At the origin, there are many arrows pointing out but no arrows pointing in. We conclude that at the origin, the divergence must be positive (and not zero). If one were to draw a box anywhere in the field, the edges farther from the origin would have larger arrows passing through them than the edges close to the origin, indicating that more is going from a point than going in. This indicates a positive (and not zero) divergence. This is correct:

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = M_x + N_y = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) = 1 + 1 = 2.$$

One may find this curl to be harder to determine visually than previous examples. One might note that any arrow that induces a clockwise spin on a cork will have an equally sized arrow inducing a counterclockwise spin on the other side, indicating no spin and no curl. This is correct, as

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = N_x - M_y = \frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x) = 0.$$

4. One might find this divergence hard to determine visually as large arrows appear in close proximity to small arrows, each pointing in different directions. Instead of trying to rationalize a guess, we compute the divergence:

$$\operatorname{div}\vec{F} = \nabla \cdot \vec{F} = M_x + N_y = \frac{\partial}{\partial x}(\cos y) + \frac{\partial}{\partial y}(\sin x) = 0.$$

Perhaps surprisingly, the divergence is 0.

Will all the loops of different directions in the field, one is apt to reason the curl is variable. Indeed, it is:

$$\operatorname{curl}\vec{F} = \nabla \times \vec{F} = N_x - M_y = \frac{\partial}{\partial x}(\sin x) - \frac{\partial}{\partial y}(\cos y) = \cos x + \sin y.$$

Depending on the values of x and y, the curl may be positive, negative, or zero.

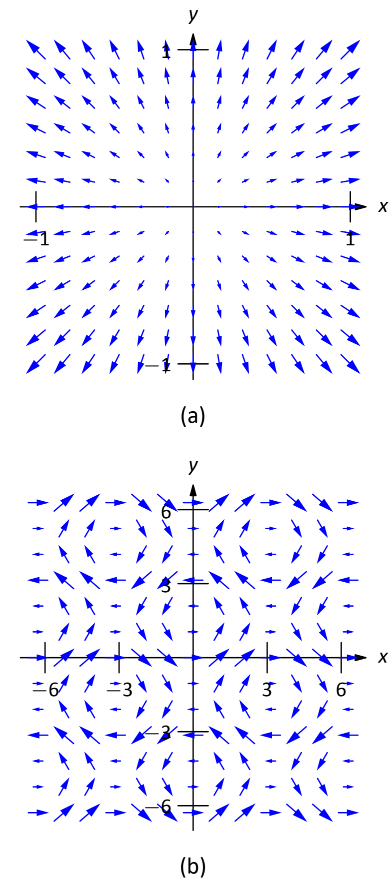


Figure 14.2.5: The vector fields in parts (c) and (d) in Example 14.2.1.

Notes:

Example 14.2.2 Computing divergence and curl of vector fields in space

Compute the divergence and curl of each of the following vector fields.

1. $\vec{F} = \langle x^2 + y + z, -x - z, x + y \rangle$
2. $\vec{F} = \langle e^{xy}, \sin(x+z), x^2 + y \rangle$

SOLUTION We compute the divergence and curl of each field following the definitions.

1. $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = M_x + N_y + P_z = 2x + 0 + 0 = 2x.$

   $$\begin{aligned}\operatorname{curl} \vec{F} = \nabla \times \vec{F} &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 1 - (-1), 1 - 1, -1 - (1) \rangle = \langle 2, 0, -2 \rangle.\end{aligned}$$

   For this particular field, no matter the location in space, a spin is induced with axis parallel to $\langle 2, 0, -2 \rangle$.

2. $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = M_x + N_y + P_z = ye^{xy} + 0 + 0 = ye^{xy}.$

   $$\begin{aligned}\operatorname{curl} \vec{F} = \nabla \times \vec{F} &= \langle P_y - N_z, M_z - P_x, N_x - M_y \rangle \\ &= \langle 1 - \cos(x+z), -2x, \cos(x+z) - xe^{xy} \rangle.\end{aligned}$$

Example 14.2.3 Creating a field representing gravitational force

The force of gravity between two objects is inversely proportional to the square of the distance between the objects. Locate a point mass at the origin. Create a vector field $\vec{F}$ that represents the gravitational pull of the point mass at any point (x, y, z). Find the divergence and curl of this field.

SOLUTION The point mass pulls toward the origin, so at (x, y, z), the force will pull in the direction of $\langle -x, -y, -z \rangle$. To get the proper magnitude, it will be useful to find the unit vector in this direction. Dividing by its magnitude, we have

$$\vec{u} = \left\langle \frac{-x}{\sqrt{x^2 + y^2 + z^2}}, \frac{-y}{\sqrt{x^2 + y^2 + z^2}}, \frac{-z}{\sqrt{x^2 + y^2 + z^2}} \right\rangle.$$

The magnitude of the force is inversely proportional to the square of the distance between the two points. Letting k be the constant of proportionality, we have the magnitude as $\dfrac{k}{x^2 + y^2 + z^2}$. Multiplying this magnitude by the unit vector above, we have the desired vector field:

$$\vec{F} = \left\langle \frac{-kx}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-ky}{(x^2 + y^2 + z^2)^{3/2}}, \frac{-kz}{(x^2 + y^2 + z^2)^{3/2}} \right\rangle.$$

Notes:

We leave it to the reader to confirm that div $\vec{F} = 0$ and curl $\vec{F} = \vec{0}$.

The analogous planar vector field is given in Figure 14.2.6. Note how all arrows point to the origin, and the magnitude gets very small when "far" from the origin.

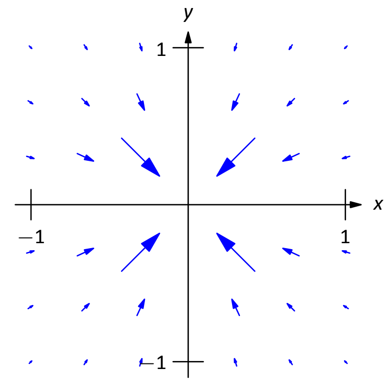


Figure 14.2.6: A vector field representing a planar gravitational force.

A function $z = f(x,y)$ naturally induces a vector field, $\vec{F} = \nabla f = \langle f_x, f_y \rangle$. Given what we learned of the gradient in Section 12.6, we know that the vectors of $\vec{F}$ point in the direction of greatest increase of f. Because of this, f is said to be the **potential function** of $\vec{F}$. Vector fields that are the gradient of potential functions will play an important role in the next section.

Example 14.2.4 A vector field that is the gradient of a potential function

Let $f(x,y) = 3 - x^2 - 2y^2$ and let $\vec{F} = \nabla f$. Graph $\vec{F}$, and find the divergence and curl of $\vec{F}$.

Solution Given f, we find $\vec{F} = \nabla f = \langle -2x, -4y \rangle$. A graph of $\vec{F}$ is given in Figure 14.2.7(a). In part (b) of the figure, the vector field is given along with a graph of the surface itself; one can see how each vector is pointing in the direction of "steepest uphill", which, in this case, is not simply just "toward the origin."

We leave it to the reader to confirm that div $\vec{F} = -6$ and curl $\vec{F} = 0$.

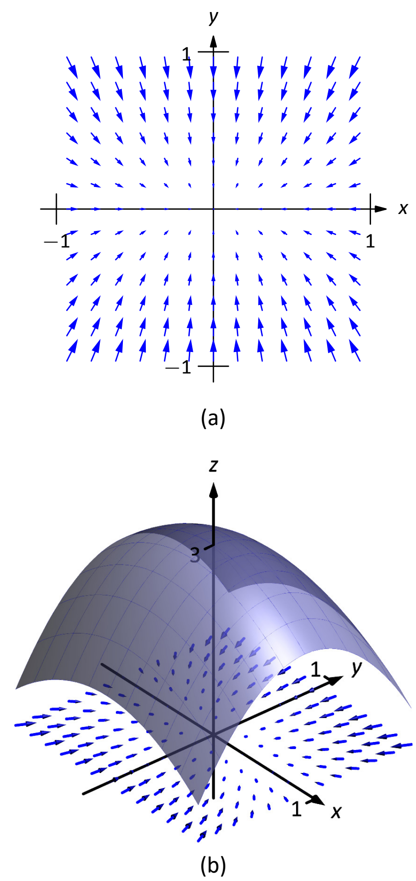


Figure 14.2.7: A graph of a function $z = f(x,y)$ and the vector field $\vec{F} = \nabla f$ in Example 14.2.4.

There are some important concepts visited in this section that will be revisited in subsequent sections and again at the very end of this chapter. One is: given a vector field $\vec{F}$, both div $\vec{F}$ and curl $\vec{F}$ are measures of rates of change of $\vec{F}$. The divergence measures how much the field spreads (diverges) at a point, and the curl measures how much the field twists (curls) at a point. Another important concept is this: given $z = f(x,y)$, the gradient ∇f is also a measure of a rate of change of f. We will see how the integrals of these rates of change produce meaningful results.

This section introduces the concept of a vector field. The next section "applies calculus" to vector fields. A common application is this: let $\vec{F}$ be a vector field representing a force (hence it is called a "force field," though this name has a decidedly comic-book feel) and let a particle move along a curve C under the influence of this force. What work is performed by the field on this particle? The solution lies in correctly applying the concepts of line integrals in the context of vector fields.

Notes:

Exercises 14.2

Terms and Concepts

14 02 ex 01 1. Give two quantities that can be represented by a vector field in the plane or in space.

14 02 ex 02 2. In your own words, describe what it means for a vector field to have a negative divergence at a point.

14 02 ex 03 3. In your own words, describe what it means for a vector field to have a negative curl at a point.

14 02 ex 04 4. The divergence of a vector field $\vec{F}$ at a particular point is 0. Does this mean that $\vec{F}$ is incompressible? Why/why not?

Problems

14 02 exset 01 **In Exercises 5 – 8, sketch the given vector field over the rectangle with opposing corners $(-2,-2)$ and $(2,2)$, sketching one vector for every point with integer coordinates (i.e., at $(0,0)$, $(1,2)$, etc.).**

14 02 ex 05 5. $\vec{F}=\langle x, 0\rangle$

14 02 ex 06 6. $\vec{F}=\langle 0, x\rangle$

14 02 ex 07 7. $\vec{F}=\langle 1, -1\rangle$

14 02 ex 08 8. $\vec{F}=\langle y^2, 1\rangle$

14 02 exset 02 **In Exercises 9 – 18, find the divergence and curl of the given vector field.**

14 02 ex 09 9. $\vec{F}=\langle x, y^2\rangle$

14 02 ex 10 10. $\vec{F}=\langle -y^2, x\rangle$

14 02 ex 11 11. $\vec{F}=\langle \cos(xy), \sin(xy)\rangle$

14 02 ex 12 12. $\vec{F}=\left\langle \dfrac{-2x}{(x^2+y^2)^2}, \dfrac{-2y}{(x^2+y^2)^2}\right\rangle$

14 02 ex 13 13. $\vec{F}=\langle x+y, y+z, x+z\rangle$

14 02 ex 14 14. $\vec{F}=\left\langle x^2+z^2, x^2+y^2, y^2+z^2\right\rangle$

14 02 ex 15 15. $\vec{F}=\nabla f$, where $f(x,y)=\frac{1}{2}x^2+\frac{1}{3}y^3$.

14 02 ex 16 16. $\vec{F}=\nabla f$, where $f(x,y)=x^2y$.

14 02 ex 17 17. $\vec{F}=\nabla f$, where $f(x,y,z)=x^2y+\sin z$.

14 02 ex 18 18. $\vec{F}=\nabla f$, where $f(x,y,z)=\dfrac{1}{x^2+y^2+z^2}$.

A: Solutions To Selected Problems

Chapter 14

Section 14.1

14 01 ex 01 1. When C is a curve in the plane and f is a surface defined over C, then $\int_C f(s)\,ds$ describes the area under the spatial curve that lies on f, over C.

14 01 ex 02 2. The evaluation is the same. The $\oint$ notation signifies that the curve C is a closed curve, though the evaluation is the same.

14 01 ex 03 3. The variable s denotes the arc-length parameter, which is generally difficult to use. The Key Idea allows one to parametrize a curve using another, ideally easier-to-use, parameter.

14 01 ex 04 4. Answers will vary.

14 01 ex 05 5. $12\sqrt{2}$

14 01 ex 06 6. $41\sqrt{10}/2$

14 01 ex 07 7. 40π

14 01 ex 08 8. $10\pi^2$

14 01 ex 14 9. Over the first subcurve of C, the line integral has a value of $3/2$; over the second subcurve, the line integral has a value of $4/3$. The total value of the line integral is thus $17/6$.

14 01 ex 13 10. Over the first subcurve of C, the line integral has a value of $2\sqrt{2}/3$; over the second subcurve, the line integral has a value of $\pi - 2$. The total value of the line integral is thus $\pi + 2\sqrt{2}/3 - 2$.

14 01 ex 09 11. $\int_0^1 (5t^2 +_2 t + 2)\sqrt{(4t+1)^2+1}\,dt \approx 17.071$

14 01 ex 10 12. $\int_0^\pi t\sqrt{1+\cos^2 t}\,dt \approx 6.001$

14 01 ex 11 13. $\oint_0^{2\pi} \left(10 - 4\cos^2 t - \sin^2 t\right)\sqrt{\cos^2 t + 4\sin^2 t}\,dt \approx 74.986$

14 01 ex 12 14. $\int_{-1}^1 \left(3t^3 + 2t + 5\right)\sqrt{9t^4+1}\,dt \approx 15.479$

14 01 ex 15 15. $7\sqrt{26}/3$

14 01 ex 16 16. 2π

14 01 ex 17 17. $8\pi^3$

14 01 ex 18 18. $5/2$

14 01 ex 19 19. $M = 8\sqrt{2}\pi^2$; center of mass is $(0, -1/(2\pi), 8\pi/3)$.

14 01 ex 20 20. $M \approx 0.237$; center of mass is approximately $(0.173, 0.099, 0.065)$.

Section 14.2

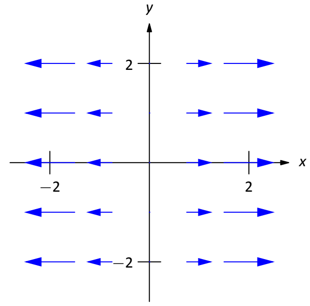
14 02 ex 01 1. Answers will vary. Appropriate answers include velocities of moving particles (air, water, etc.); gravitational or electromagnetic forces.

14 02 ex 02 2. Specific answers will vary, though should relate to the idea that "more of the vector field is moving into that point than out of that point."

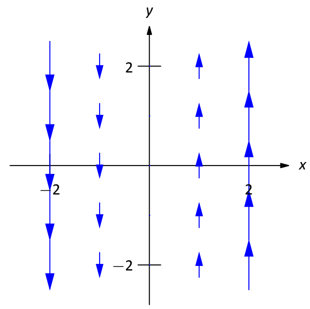
14 02 ex 03 3. Specific answers will vary, though should relate to the idea that the vector field is spinning clockwise at that point.

14 02 ex 04 4. No; to be incompressible, the divergence needs to be 0 everywhere, not just at one point.

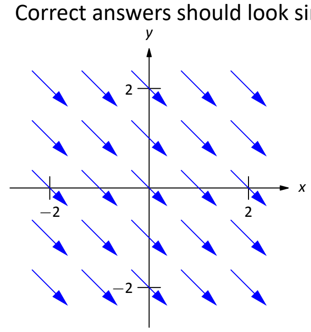
14 02 ex 05 5. Correct answers should look similar to



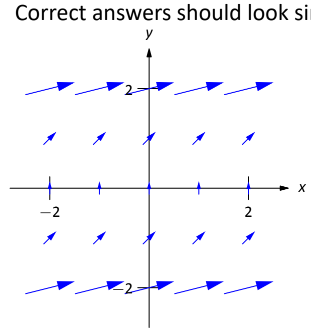
14 02 ex 06 6. Correct answers should look similar to



14 02 ex 07 7. Correct answers should look similar to



14 02 ex 08 8. Correct answers should look similar to



14 02 ex 09 9. $\text{div}\,\vec{F} = 1 + 2y$
$\text{curl}\,\vec{F} = 0$

14 02 ex 10 10. $\text{div}\,\vec{F} = 0$
$\text{curl}\,\vec{F} = 1 + 2y$

11. $\text{div}\,\vec{F} = x\cos(xy) - y\sin(xy)$
$\text{curl}\,\vec{F} = y\cos(xy) + x\sin(xy)$

14 02 ex 12 12. $\text{div}\,\vec{F} = \frac{4}{(x^2+y^2)^2}$

$\text{curl}\,\vec{F} = 0$

14 02 ex 13 13. $\text{div}\,\vec{F} = 3$

$\text{curl}\,\vec{F} = \langle -1, -1, -1 \rangle$

14 02 ex 14 14. $\text{div}\,\vec{F} = 2x + 2y + 2z$

$\text{curl}\,\vec{F} = \langle 2y, 2z, 2x \rangle$

14 02 ex 15 15. $\text{div}\,\vec{F} = 1 + 2y$

$\text{curl}\,\vec{F} = 0$

14 02 ex 16 16. $\text{div}\,\vec{F} = 2y$

$\text{curl}\,\vec{F} = 0$

14 02 ex 17 17. $\text{div}\,\vec{F} = 2y - \sin z$

$\text{curl}\,\vec{F} = \vec{0}$

14 02 ex 18 18. $\text{div}\,\vec{F} = \frac{2}{(x^2+y^2+z^2)^2}$

$\text{curl}\,\vec{F} = \vec{0}$