1: Curves in the Plane

1.1 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that “fail the vertical line test” cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

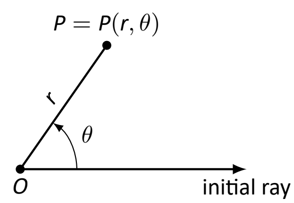


Figure 1.1: Illustrating polar coordinates.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 1.1

Practice will make this process more clear.

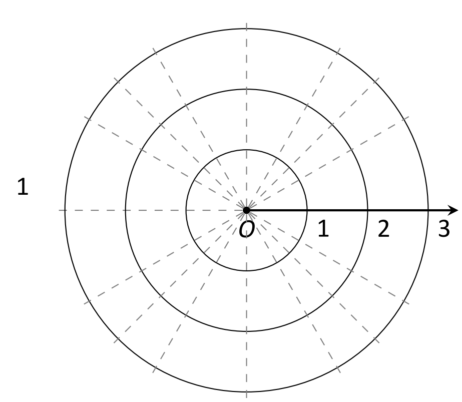
Example 1 **Plotting Polar Coordinates**

Plot the following polar coordinates:

$$A = P(1, \pi/4) \quad B = P(1.5, \pi) \quad C = P(2, -\pi/3) \quad D = P(-1, \pi/4)$$

Solution To aid in the drawing, a polar grid is provided at the bottom of this page. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).



To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units – in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 1.2.

Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 1.3 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

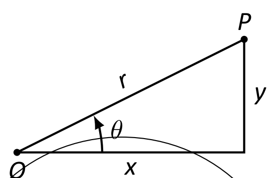


Figure 1.3: Converting between rectangular and polar coordinates.

Key Idea 1 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

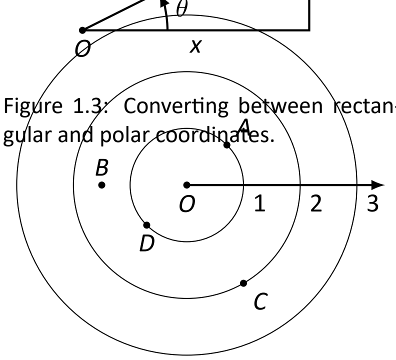


Figure 1.2: Plotting polar points in Example 1.

Example 2 Converting Between Polar and Rectangular Coordinates

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.

2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

Notes:

Solution

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 1, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 1.4 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $P(\sqrt{5}, 1.11)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

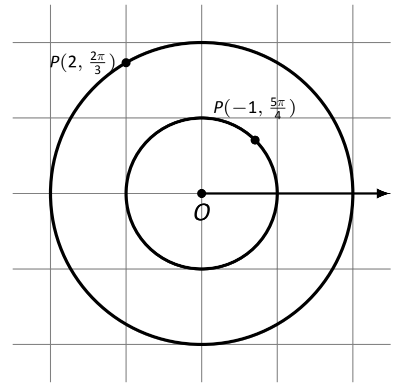
Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4 = -45^\circ.$$

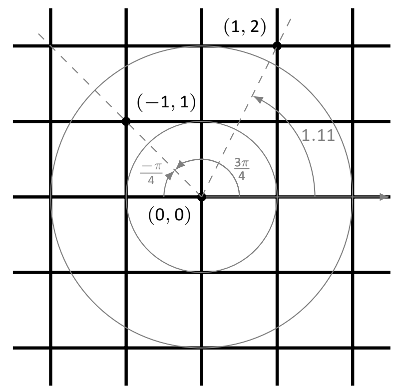
This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1st and 4th quadrants. To find locations in the 2nd and 3rd quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P(\sqrt{2}, 3\pi/4)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $P(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 1.4 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.



(a)



(b)

Figure 1.4: Plotting rectangular and polar points in Example 2.

Notes:

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 3 Introduction to Graphing Polar Functions

Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centered at $(0, 0)$ with radius 1.5. This is sketched in Figure 1.5.

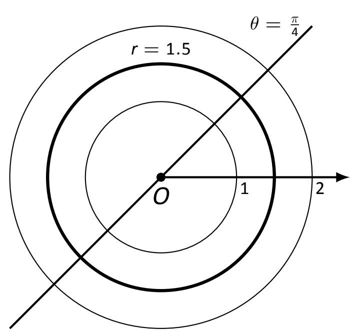


Figure 1.5: Plotting standard polar plots.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45^\circ$ with the initial ray.

   We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

$$\tan\pi/4 = y/x \quad \Rightarrow x\tan\pi/4 = y \quad \Rightarrow y = x.$$

   This graph is also plotted in Figure 1.5.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

Notes:

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 4 **Sketching Polar Functions**
Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

Solution A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values – integers, then add more if needed. When plotting polar equations, start with the "common" angles – multiples of $\pi/6$ and $\pi/4$. Figure 1.6 gives a table of just a few values of θ in $[0, \pi]$.

Consider the point $P(2, 0)$ determined by the first line of the table. The angle is 0 radians – we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is $1 + \sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.

The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

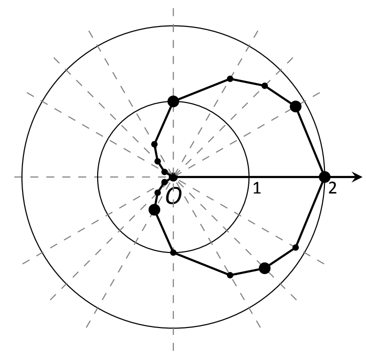


Figure 1.6: Graphing a polar function in Example 4 by plotting points.

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the *x*-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called $\theta_{\texttt{min}}$ and $\theta_{\texttt{max}}$) as well as the $\theta_{\texttt{step}}$ – that is, how far apart the θ values are spaced. The smaller the $\theta_{\texttt{step}}$ value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1 + \cos\theta$ from Example 4 in Figure 1.7.

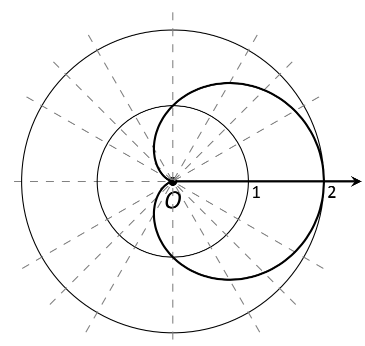
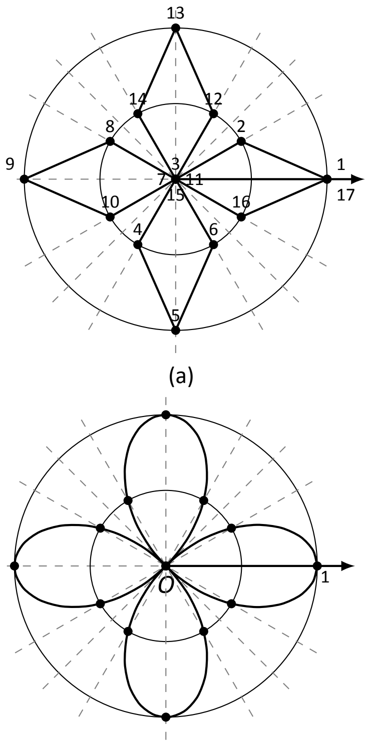


Figure 1.7: Using technology to graph a polar function.

Example 5 **Sketching Polar Functions**
Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 1.8. These points are then plotted in Figure 1.9 (a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.

Notes:



(a)

(b)

Figure 1.9: Polar plots from Example 5.

Pt.	θ	$\cos(2\theta)$
1	0	1.
2	$\pi/6$	0.5
3	$\pi/4$	0.
4	$\pi/3$	-0.5
5	$\pi/2$	$-1.$
6	$2\pi/3$	-0.5
7	$3\pi/4$	0.
8	$5\pi/6$	0.5
9	π	1.

Pt.	θ	$\cos(2\theta)$
10	$7\pi/6$	0.5
11	$5\pi/4$	0.
12	$4\pi/3$	-0.5
13	$3\pi/2$	$-1.$
14	$5\pi/3$	-0.5
15	$7\pi/4$	0.
16	$11\pi/6$	0.5
17	2π	1.

Figure 1.8: Tables of points for plotting a polar curve.

Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 1.9 (b). This plot is an example of a *rose curve*.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 1.

Example 6 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

Solution

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

Notes:

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta \cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

   This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

   We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 1.10; note how it only exists in the first and third quadrants.

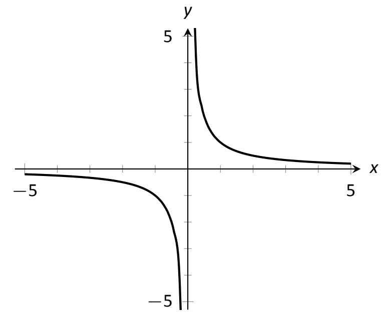


Figure 1.10: Graphing $xy = 1$ from Example 6.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

   The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

Notes:

We recognize this as a circle; by completing the square we can find its radius and center.

$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1.$$

The circle is centered at $(1,0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardioid* (a shape important to the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

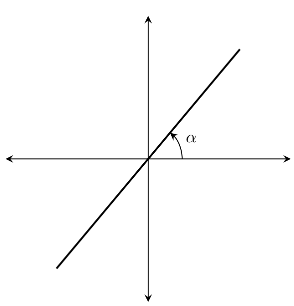
Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

Lines

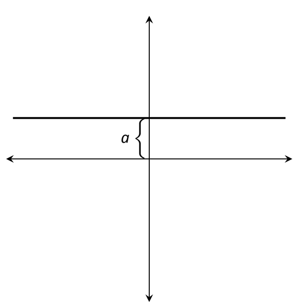
Through the origin:

$\theta = \alpha$



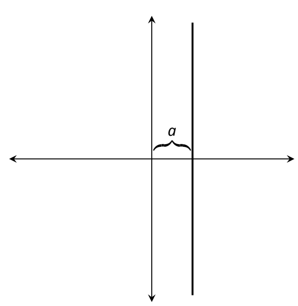
Horizontal line:

$r = a\csc\theta$



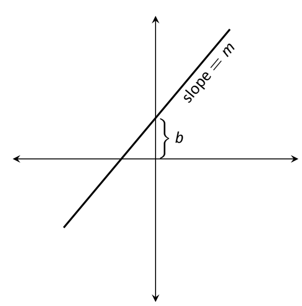
Vertical line:

$r = a\sec\theta$



Not through origin:

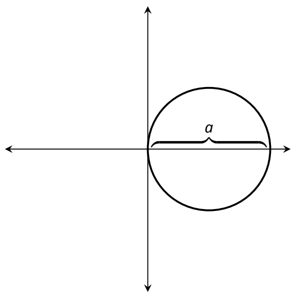
$$r = \frac{b}{\sin\theta - m\cos\theta}$$



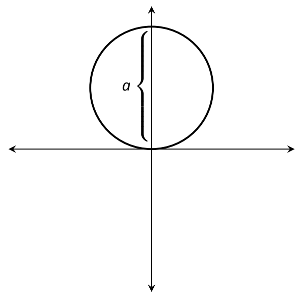
Notes:

Circles

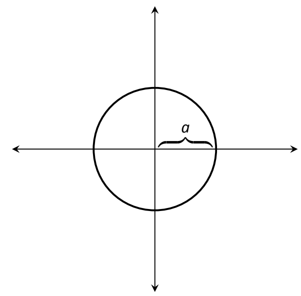
Centered on *x*-axis:
$r = a\cos\theta$



Centered on *y*-axis:
$r = a\sin\theta$

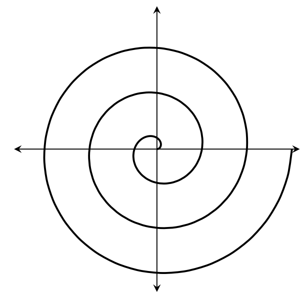


Centered on origin:
$r = a$



Spiral

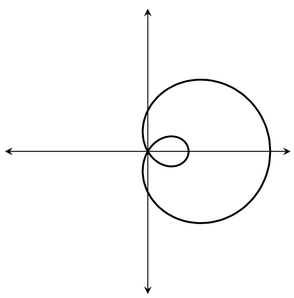
Archimedean spiral
$r = \theta$



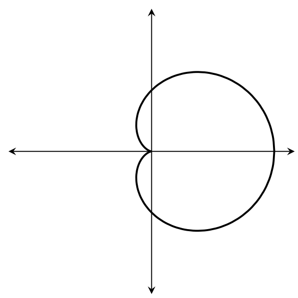
Limaçons

Symmetric about *x*-axis: $r = a \pm b\cos\theta$;   Symmetric about *y*-axis: $r = a \pm b\sin\theta$; $a, b > 0$

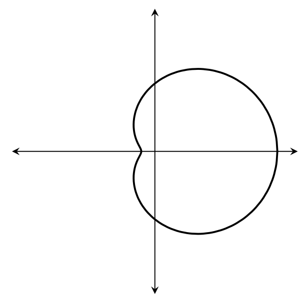
With inner loop:
$\dfrac{a}{b} < 1$



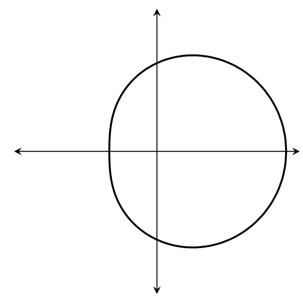
Cardioid:
$\dfrac{a}{b} = 1$



Dimpled:
$1 < \dfrac{a}{b} < 2$



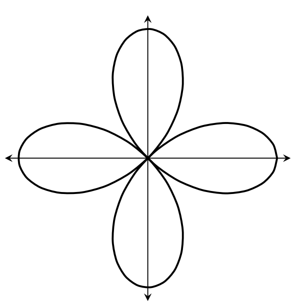
Convex:
$\dfrac{a}{b} > 2$



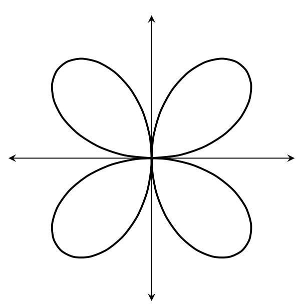
Rose Curves

Symmetric about *x*-axis: $r = a\cos(n\theta)$;   Symmetric about *y*-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

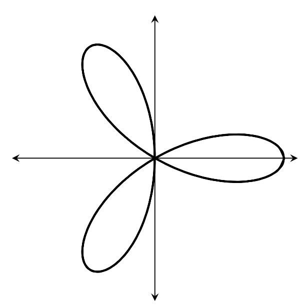
$r = a\cos(2\theta)$



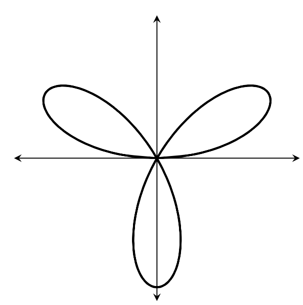
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



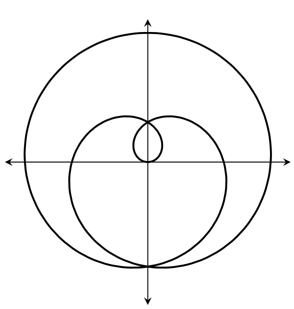
$r = a\sin(3\theta)$



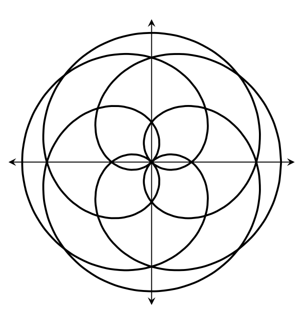
Special Curves

Rose curves

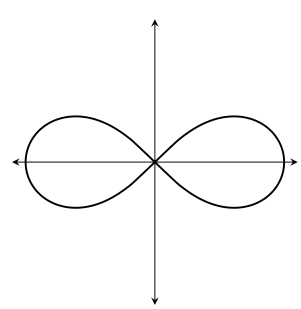
$r = a\sin(\theta/5)$



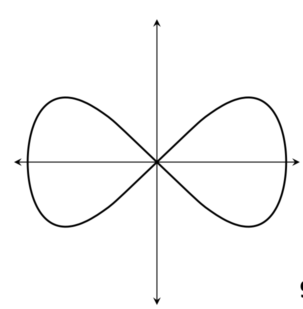
$r = a\sin(2\theta/5)$



Lemniscate:
$r^2 = a^2\cos(2\theta)$



Eight Curve:
$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 7 **Finding points of intersection with polar curves**

Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 1.11(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1+3\cos\theta &= \cos\theta \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4th quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 2nd quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly

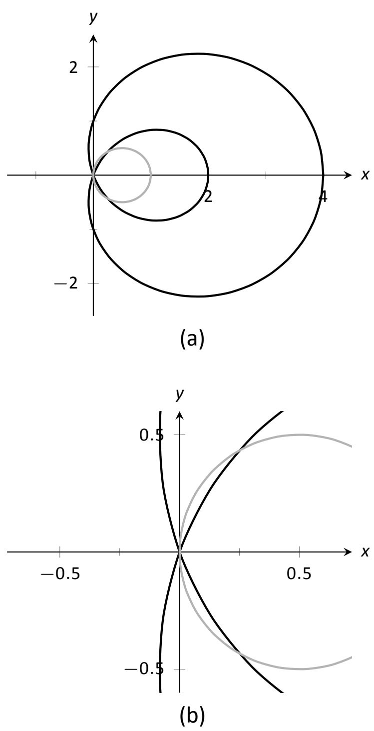


Figure 1.11: Graphs to help determine the points of intersection of the polar functions given in Example 7.

Notes:

traced). The limaçon intersects the pole when $1 + 3\cos\theta = 0$; this occurs when $\cos\theta = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106 = 109.47°$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the *x*-axis. That is, $\theta = 4.3726 = 250.53°$.

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Notes:

Exercises 1.1

Terms and Concepts

09 04 ex 01 1. In your own words, describe how to plot the polar point $P(r,\theta)$.

09 04 ex 02 2. T/F: When plotting a point with polar coordinate $P(r,\theta)$, r must be positive.

09 04 ex 03 3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.

09 04 ex 04 4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

09 04 ex 05 5. Plot the points with the given polar coordinates.

(a) $A = P(2,0)$ (c) $C = P(-2,\pi/2)$

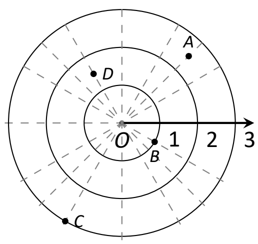
(b) $B = P(1,\pi)$ (d) $D = P(1,\pi/4)$

09 04 ex 06 6. Plot the points with the given polar coordinates.

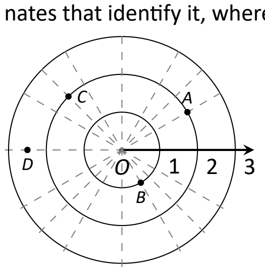
(a) $A = P(2,3\pi)$ (c) $C = P(1,2)$

(b) $B = P(1,-\pi)$ (d) $D = P(1/2,5\pi/6)$

09 04 ex 07 7. For each of the given points give two sets of polar coordinates that identify it, where $0 \le \theta \le 2\pi$.



09 04 ex 08 8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \le \theta \le \pi$.



09 04 ex 09 9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

(a) $A = P(2,\pi/4)$ (c) $C = (2,-1)$

(b) $B = P(2,-\pi/4)$ (d) $D = (-2,1)$

09 04 ex 10 10. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

(a) $A = P(3,\pi)$ (c) $C = (0,4)$

(b) $B = P(1,2\pi/3)$ (d) $D = (1,-\sqrt{3})$

09 04 exset 01 **In Exercises 11 – 29, graph the polar function on the given interval.**

09 04 ex 11 11. $r = 2, \quad 0 \le \theta \le \pi/2$

09 04 ex 12 12. $\theta = \pi/6, \quad -1 \le r \le 2$

09 04 ex 13 13. $r = 1 - \cos\theta, \quad [0,2\pi]$

09 04 ex 14 14. $r = 2 + \sin\theta, \quad [0,2\pi]$

09 04 ex 15 15. $r = 2 - \sin\theta, \quad [0,2\pi]$

09 04 ex 16 16. $r = 1 - 2\sin\theta, \quad [0,2\pi]$

09 04 ex 17 17. $r = 1 + 2\sin\theta, \quad [0,2\pi]$

09 04 ex 18 18. $r = \cos(2\theta), \quad [0,2\pi]$

09 04 ex 19 19. $r = \sin(3\theta), \quad [0,\pi]$

09 04 ex 20 20. $r = \cos(\theta/3), \quad [0,3\pi]$

09 04 ex 21 21. $r = \cos(2\theta/3), \quad [0,6\pi]$

09 04 ex 22 22. $r = \theta/2, \quad [0,4\pi]$

09 04 ex 23 23. $r = 3\sin(\theta), \quad [0,\pi]$

09 04 ex 24 24. $r = \cos\theta\sin\theta, \quad [0,2\pi]$

09 04 ex 25 25. $r = \theta^2 - (\pi/2)^2, \quad [-\pi,\pi]$

09 04 ex 26 26. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0,2\pi]$

09 04 ex 27 27. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0,2\pi]$

09 04 ex 28 28. $r = 3\sec\theta, \quad (-\pi/2,\pi/2)$

09 04 ex 29 29. $r = 3\csc\theta, \quad (0,\pi)$

09 04 exset 02 **In Exercises 30 – 38, convert the polar equation to a rectangular equation.**

09 04 ex 30 30. $r = 2\cos\theta$

09 04 ex 31 31. $r = -4\sin\theta$

09 04 ex 32 32. $r = \cos\theta + \sin\theta$

09 04 ex 33 33. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$

09 04 ex 34 34. $r = \dfrac{3}{\cos\theta}$

09 04 ex 35 35. $r = \dfrac{4}{\sin\theta}$

09 04 ex 36 36. $r = \tan\theta$

09 04 ex 37 37. $r = 2$

09 04 ex 38 38. $\theta = \pi/6$

09 04 exset 03 **In Exercises 39 – 46, convert the rectangular equation to a polar equation.**

09 04 ex 39 39. $y = x$

09 04 ex 40 40. $y = 4x + 7$

09 04 ex 41 41. $x = 5$

09 04 ex 42 42. $y = 5$

09 04 ex 43 43. $x = y^2$

09 04 ex 44 44. $x^2 y = 1$

09 04 ex 45 45. $x^2 + y^2 = 7$

09 04 ex 46 46. $(x + 1)^2 + y^2 = 1$

09 04 exset 04 **In Exercises 47 – 54, find the points of intersection of the polar graphs.**

09 04 ex 47 47. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

09 04 ex 48 48. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

09 04 ex 49 49. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

09 04 ex 50 50. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

09 04 ex 51 51. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

09 04 ex 52 52. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

09 04 ex 53 53. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

09 04 ex 54 54. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

09 04 ex 55 55. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\frac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

09 04 ex 56 56. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

1.2 Calculus and Polar Functions

The previous section defined polar coordinates, leading to polar functions. We investigated plotting these functions and solving a fundamental question about their graphs, namely, where do two polar graphs intersect?

We now turn our attention to answering other questions, whose solutions require the use of calculus. A basis for much of what is done in this section is the ability to turn a polar function $r = f(\theta)$ into a set of parametric equations. Using the identities $x = r\cos\theta$ and $y = r\sin\theta$, we can create the parametric equations $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ and apply the concepts of Section 9.3.

Polar Functions and $\dfrac{dy}{dx}$

We are interested in the lines tangent to a given graph, regardless of whether that graph is produced by rectangular, parametric, or polar equations. In each of these contexts, the slope of the tangent line is $\frac{dy}{dx}$. Given $r = f(\theta)$, we are generally *not* concerned with $r' = f'(\theta)$; that describes how fast r changes with respect to θ. Instead, we will use $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ to compute $\frac{dy}{dx}$.

Using Key Idea 37 we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}.$$

Each of the two derivatives on the right hand side of the equality requires the use of the Product Rule. We state the important result as a Key Idea.

Key Idea 2 Finding $\frac{dy}{dx}$ with Polar Functions

Let $r = f(\theta)$ be a polar function. With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$,

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 8 Finding $\frac{dy}{dx}$ with polar functions.

Consider the limaçon $r = 1 + 2\sin\theta$ on $[0, 2\pi]$.

1. Find the equations of the tangent and normal lines to the graph at $\theta = \pi/4$.

2. Find where the graph has vertical and horizontal tangent lines.

Notes:

Solution

1. We start by computing $\frac{dy}{dx}$. With $f'(\theta) = 2\cos\theta$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos\theta\sin\theta + \cos\theta(1+2\sin\theta)}{2\cos^2\theta - \sin\theta(1+2\sin\theta)}\\ &= \frac{\cos\theta(4\sin\theta+1)}{2(\cos^2\theta-\sin^2\theta)-\sin\theta}.\end{aligned}$$

   When $\theta = \pi/4$, $\frac{dy}{dx} = -2\sqrt{2}-1$ (this requires a bit of simplification). In rectangular coordinates, the point on the graph at $\theta = \pi/4$ is $(1+\sqrt{2}/2, 1+\sqrt{2}/2)$. Thus the rectangular equation of the line tangent to the limaçon at $\theta = \pi/4$ is

$$y = (-2\sqrt{2}-1)\big(x-(1+\sqrt{2}/2)\big)+1+\sqrt{2}/2 \approx -3.83x+8.24.$$

   The limaçon and the tangent line are graphed in Figure 1.12.

   The normal line has the opposite–reciprocal slope as the tangent line, so its equation is

$$y \approx \frac{1}{3.83}x + 1.26.$$

2. To find the horizontal lines of tangency, we find where $\frac{dy}{dx} = 0$; thus we find where the numerator of our equation for $\frac{dy}{dx}$ is 0.

$$\cos\theta(4\sin\theta+1) = 0 \quad\Rightarrow\quad \cos\theta = 0 \quad\text{or}\quad 4\sin\theta+1=0.$$

   On $[0, 2\pi]$, $\cos\theta = 0$ when $\theta = \pi/2,\ 3\pi/2$.
   Setting $4\sin\theta + 1 = 0$ gives $\theta = \sin^{-1}(-1/4) \approx -0.2527 = -14.48°$. We want the results in $[0, 2\pi]$; we also recognize there are two solutions, one in the 3rd quadrant and one in the 4th. Using reference angles, we have our two solutions as $\theta = 3.39$ and 6.03 radians. The four points we obtained where the limaçon has a horizontal tangent line are given in Figure 1.12 with black–filled dots.

   To find the vertical lines of tangency, we set the denominator of $\frac{dy}{dx} = 0$.

$$2(\cos^2\theta - \sin^2\theta) - \sin\theta = 0.$$

   Convert the $\cos^2\theta$ term to $1-\sin^2\theta$:

$$\begin{aligned}2(1-\sin^2\theta-\sin^2\theta)-\sin\theta &= 0\\ 4\sin^2\theta+\sin\theta-2 &= 0.\end{aligned}$$

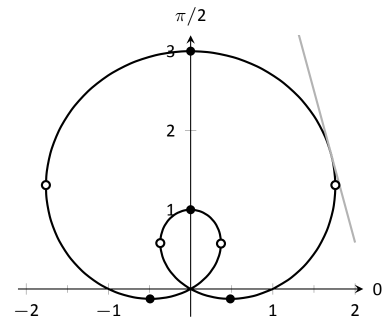


Figure 1.12: The limaçon in Example 8 with its tangent line at $\theta = \pi/4$ and points of vertical and horizontal tangency.

Notes:

Recognize this as a quadratic in the variable $\sin\theta$. Using the quadratic formula, we have

$$\sin\theta = \frac{-1 \pm \sqrt{33}}{8}.$$

We solve $\sin\theta = \frac{-1+\sqrt{33}}{8}$ and $\sin\theta = \frac{-1-\sqrt{33}}{8}$:

$$\begin{aligned} \sin\theta &= \frac{-1+\sqrt{33}}{8} & \sin\theta &= \frac{-1-\sqrt{33}}{8} \\ \theta &= \sin^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) & \theta &= \sin^{-1}\left(\frac{-1-\sqrt{33}}{8}\right) \\ \theta &= 0.6349 & \theta &= -1.0030 \end{aligned}$$

In each of the solutions above, we only get one of the possible two solutions as $\sin^{-1} x$ only returns solutions in $[-\pi/2, \pi/2]$, the 4$^{\text{th}}$ and 1$^{\text{st}}$ quadrants. Again using reference angles, we have:

$$\sin\theta = \frac{-1+\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 0.6349,\ 2.5067 \text{ radians}$$

and

$$\sin\theta = \frac{-1-\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 4.1446,\ 5.2802 \text{ radians}.$$

These points are also shown in Figure 1.12 with white–filled dots.

When the graph of the polar function $r = f(\theta)$ intersects the pole, it means that $f(\alpha) = 0$ for some angle α. Thus the formula for $\frac{dy}{dx}$ in such instances is very simple, reducing simply to

$$\frac{dy}{dx} = \tan\alpha.$$

This equation makes an interesting point. It tells us the slope of the tangent line at the pole is $\tan\alpha$; some of our previous work (see, for instance, Example 3) shows us that the line through the pole with slope $\tan\alpha$ has polar equation $\theta = \alpha$. Thus when a polar graph touches the pole at $\theta = \alpha$, the equation of the tangent line at the pole is $\theta = \alpha$.

Example 9 Finding tangent lines at the pole.

Let $r = 1 + 2\sin\theta$, a limaçon. Find the equations of the lines tangent to the graph at the pole.

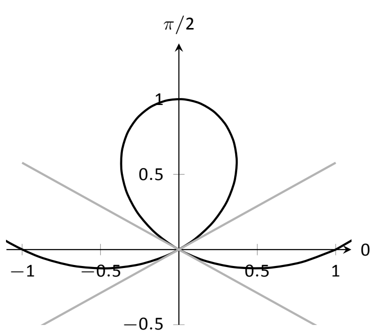


Figure 1.13: Graphing the tangent lines at the pole in Example 9.

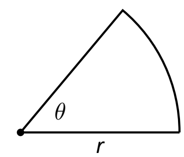
Notes:

Solution We need to know when $r = 0$.

$$\begin{aligned} 1 + 2\sin\theta &= 0 \\ \sin\theta &= -1/2 \\ \theta &= \frac{7\pi}{6},\ \frac{11\pi}{6}. \end{aligned}$$

Thus the equations of the tangent lines, in polar, are $\theta = 7\pi/6$ and $\theta = 11\pi/6$. In rectangular form, the tangent lines are $y = \tan(7\pi/6)x$ and $y = \tan(11\pi/6)x$. The full limaçon can be seen in Figure 1.12; we zoom in on the tangent lines in Figure 1.13.

Note: Recall that the area of a sector of a circle with radius r subtended by an angle θ is $A = \frac{1}{2}\theta r^2$.



Area

When using rectangular coordinates, the equations $x = h$ and $y = k$ defined vertical and horizontal lines, respectively, and combinations of these lines create rectangles (hence the name "rectangular coordinates"). It is then somewhat natural to use rectangles to approximate area as we did when learning about the definite integral.

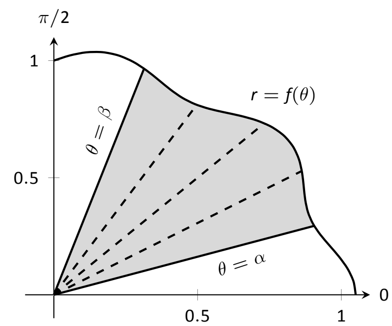
When using polar coordinates, the equations $\theta = \alpha$ and $r = c$ form lines through the origin and circles centered at the origin, respectively, and combinations of these curves form sectors of circles. It is then somewhat natural to calculate the area of regions defined by polar functions by first approximating with sectors of circles.

Consider Figure 1.14 (a) where a region defined by $r = f(\theta)$ on $[\alpha, \beta]$ is given. (Note how the "sides" of the region are the lines $\theta = \alpha$ and $\theta = \beta$, whereas in rectangular coordinates the "sides" of regions were often the vertical lines $x = a$ and $x = b$.)

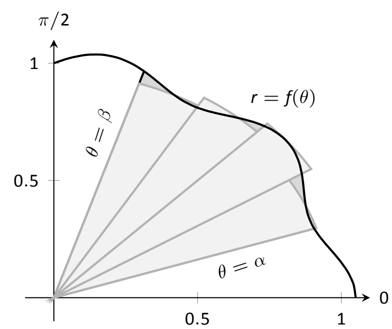
Partition the interval $[\alpha, \beta]$ into n equally spaced subintervals as $\alpha = \theta_1 < \theta_2 < \cdots < \theta_{n+1} = \beta$. The length of each subinterval is $\Delta\theta = (\beta - \alpha)/n$, representing a small change in angle. The area of the region defined by the i^{th} subinterval $[\theta_i, \theta_{i+1}]$ can be approximated with a sector of a circle with radius $f(c_i)$, for some c_i in $[\theta_i, \theta_{i+1}]$. The area of this sector is $\frac{1}{2}f(c_i)^2\Delta\theta$. This is shown in part (b) of the figure, where $[\alpha, \beta]$ has been divided into 4 subintervals. We approximate the area of the whole region by summing the areas of all sectors:

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2} f(c_i)^2 \Delta\theta.$$

This is a Riemann sum. By taking the limit of the sum as $n \to \infty$, we find the



(a)



(b)

Figure 1.14: Computing the area of a polar region.

Notes:

exact area of the region in the form of a definite integral.

Theorem 1 Area of a Polar Region

Let f be continuous and non-negative on $[\alpha, \beta]$, where $0 \leq \beta - \alpha \leq 2\pi$. The area A of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2\,d\theta = \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$$

Note: Example 10 requires the use of the integral $\int \cos^2\theta\,d\theta$. This is handled well by using the power reducing formula as found in the back of this text. Due to the nature of the area formula, integrating $\cos^2\theta$ and $\sin^2\theta$ is required often. We offer here these indefinite integrals as a time–saving measure.

$$\int \cos^2\theta\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$
$$\int \sin^2\theta\,d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$

The theorem states that $0 \leq \beta - \alpha \leq 2\pi$. This ensures that region does not overlap itself, which would give a result that does not correspond directly to the area.

Example 10 Area of a polar region

Find the area of the circle defined by $r = \cos\theta$. (Recall this circle has radius $1/2$.)

Solution This is a direct application of Theorem 1. The circle is traced out on $[0, \pi]$, leading to the integral

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^\pi \cos^2\theta\,d\theta\\ &= \frac{1}{2}\int_0^\pi \frac{1+\cos(2\theta)}{2}\,d\theta\\ &= \frac{1}{4}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_0^\pi\\ &= \frac{1}{4}\pi.\end{aligned}$$

Of course, we already knew the area of a circle with radius $1/2$. We did this example to demonstrate that the area formula is correct.

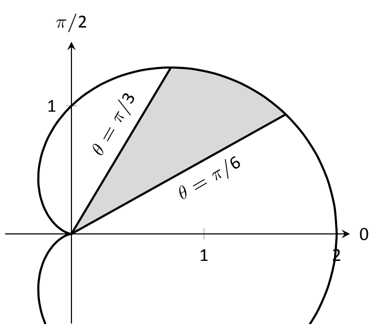


Figure 1.15: Finding the area of the shaded region of a cardioid in Example 11.

Example 11 Area of a polar region

Find the area of the cardioid $r = 1+\cos\theta$ bound between $\theta = \pi/6$ and $\theta = \pi/3$, as shown in Figure 1.15.

Notes:

Solution This is again a direct application of Theorem 1.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+\cos\theta)^2\,d\theta\\&= \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+2\cos\theta+\cos^2\theta)\,d\theta\\&= \frac{1}{2}\left(\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_{\pi/6}^{\pi/3}\\&= \frac{1}{8}\left(\pi+4\sqrt{3}-4\right)\approx 0.7587.\end{aligned}$$

Area Between Curves

Our study of area in the context of rectangular functions led naturally to finding area bounded between curves. We consider the same in the context of polar functions.

Consider the shaded region shown in Figure 1.16. We can find the area of this region by computing the area bounded by $r_2 = f_2(\theta)$ and subtracting the area bounded by $r_1 = f_1(\theta)$ on $[\alpha, \beta]$. Thus

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r_2^2\,d\theta - \frac{1}{2}\int_\alpha^\beta r_1^2\,d\theta = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

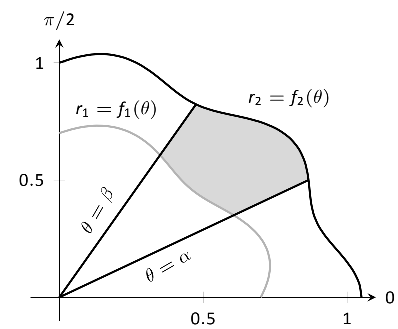


Figure 1.16: Illustrating area bound between two polar curves.

Key Idea 3 Area Between Polar Curves

The area A of the region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$, where $f_1(\theta) \leq f_2(\theta)$ on $[\alpha, \beta]$, is

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

Example 12 Area between polar curves

Find the area bounded between the curves $r = 1+\cos\theta$ and $r = 3\cos\theta$, as shown in Figure 1.17.

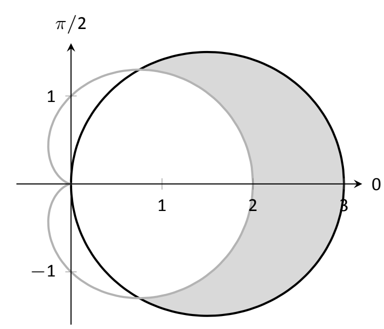


Figure 1.17: Finding the area between polar curves in Example 12.

Solution We need to find the points of intersection between these

Notes:

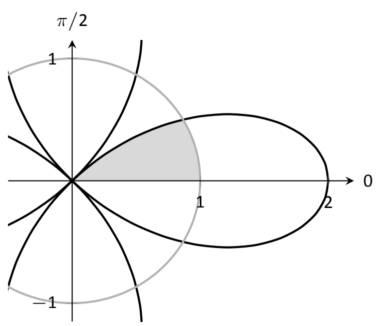
two functions. Setting them equal to each other, we find:

$$\begin{aligned} 1+\cos\theta &= 3\cos\theta \\ \cos\theta &= 1/2 \\ \theta &= \pm\pi/3 \end{aligned}$$

Thus we integrate $\frac{1}{2}\big((3\cos\theta)^2-(1+\cos\theta)^2\big)$ on $[-\pi/3,\pi/3]$.

$$\begin{aligned} \text{Area} &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\big((3\cos\theta)^2-(1+\cos\theta)^2\big)\,d\theta \\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\big(8\cos^2\theta-2\cos\theta-1\big)\,d\theta \\ &= \frac{1}{2}\big(2\sin(2\theta)-2\sin\theta+3\theta\big)\Big|_{-\pi/3}^{\pi/3} \\ &= \pi. \end{aligned}$$

Amazingly enough, the area between these curves has a “nice” value.



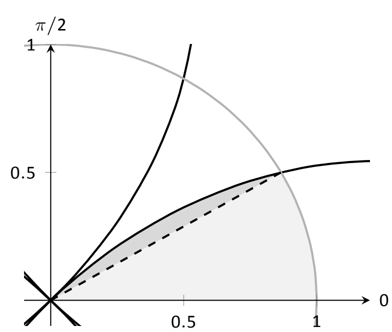


Figure 1.18: Graphing the region bounded by the functions in Example 13.

Example 13 Area defined by polar curves

Find the area bounded between the polar curves $r=1$ and $r=2\cos(2\theta)$, as shown in Figure 1.18 (a).

Solution We need to find the point of intersection between the two curves. Setting the two functions equal to each other, we have

$$2\cos(2\theta)=1 \quad\Rightarrow\quad \cos(2\theta)=\frac{1}{2} \quad\Rightarrow\quad 2\theta=\pi/3 \quad\Rightarrow\quad \theta=\pi/6.$$

In part (b) of the figure, we zoom in on the region and note that it is not really bounded *between* two polar curves, but rather *by* two polar curves, along with $\theta=0$. The dashed line breaks the region into its component parts. Below the dashed line, the region is defined by $r=1$, $\theta=0$ and $\theta=\pi/6$. (Note: the dashed line lies on the line $\theta=\pi/6$.) Above the dashed line the region is bounded by $r=2\cos(2\theta)$ and $\theta=\pi/6$. Since we have two separate regions, we find the area using two separate integrals.

Call the area below the dashed line A_1 and the area above the dashed line A_2. They are determined by the following integrals:

$$A_1=\frac{1}{2}\int_0^{\pi/6}(1)^2\,d\theta \qquad A_2=\frac{1}{2}\int_{\pi/6}^{\pi/4}\big(2\cos(2\theta)\big)^2\,d\theta.$$

Notes:

(The upper bound of the integral computing A_2 is $\pi/4$ as $r = 2\cos(2\theta)$ is at the pole when $\theta = \pi/4$.)

We omit the integration details and let the reader verify that $A_1 = \pi/12$ and $A_2 = \pi/12 - \sqrt{3}/8$; the total area is $A = \pi/6 - \sqrt{3}/8$.

Arc Length

As we have already considered the arc length of curves defined by rectangular and parametric equations, we now consider it in the context of polar equations. Recall that the arc length L of the graph defined by the parametric equations $x = f(t)$, $y = g(t)$ on $[a, b]$ is

$$L = \int_a^b \sqrt{f'(t)^2 + g'(t)^2}\,dt = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt. \tag{1.1}$$

Now consider the polar function $r = f(\theta)$. We again use the identities $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$ to create parametric equations based on the polar function. We compute $x'(\theta)$ and $y'(\theta)$ as done before when computing $\frac{dy}{dx}$, then apply Equation (1.1).

The expression $x'(\theta)^2 + y'(\theta)^2$ can be simplified a great deal; we leave this as an exercise and state that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

This leads us to the arc length formula.

Key Idea 4 Arc Length of Polar Curves

Let $r = f(\theta)$ be a polar function with f' continuous on an open interval I containing $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\,d\theta.$$

Example 14 Arc length of a limaçon

Find the arc length of the limaçon $r = 1 + 2\sin t$.

Solution With $r = 1 + 2\sin t$, we have $r' = 2\cos t$. The limaçon is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Key

Notes:

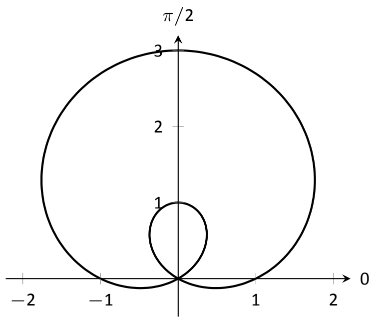


Figure 1.19: The limaçon in Example 14 whose arc length is measured.

Idea 4, we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1+2\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta + 4\sin\theta + 1}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin\theta + 5}\, d\theta \\ &\approx 13.3649. \end{aligned}$$

The final integral cannot be solved in terms of elementary functions, so we resorted to a numerical approximation. (Simpson's Rule, with $n = 4$, approximates the value with 13.0608. Using $n = 22$ gives the value above, which is accurate to 4 places after the decimal.)

Surface Area

The formula for arc length leads us to a formula for surface area. The following Key Idea is based on Key Idea 39.

Key Idea 5 Surface Area of a Solid of Revolution

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on an open interval containing $[\alpha, \beta]$ on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

2. The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:

$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta.$$

Notes:

Example 15 Surface area determined by a polar curve

Find the surface area formed by revolving one petal of the rose curve $r = \cos(2\theta)$ about its central axis (see Figure 1.20).

Solution We choose, as implied by the figure, to revolve the portion of the curve that lies on $[0, \pi/4]$ about the initial ray. Using Key Idea 5 and the fact that $f'(\theta) = -2\sin(2\theta)$, we have

$$\begin{aligned} \text{Surface Area} &= 2\pi\int_0^{\pi/4} \cos(2\theta)\sin(\theta)\sqrt{\big(-2\sin(2\theta)\big)^2 + \big(\cos(2\theta)\big)^2}\,d\theta \\ &\approx 1.36707. \end{aligned}$$

The integral is another that cannot be evaluated in terms of elementary functions. Simpson's Rule, with $n = 4$, approximates the value at 1.36751.

This chapter has been about curves in the plane. While there is great mathematics to be discovered in the two dimensions of a plane, we live in a three dimensional world and hence we should also look to do mathematics in 3D – that is, in *space*. The next chapter begins our exploration into space by introducing the topic of *vectors*, which are incredibly useful and powerful mathematical objects.

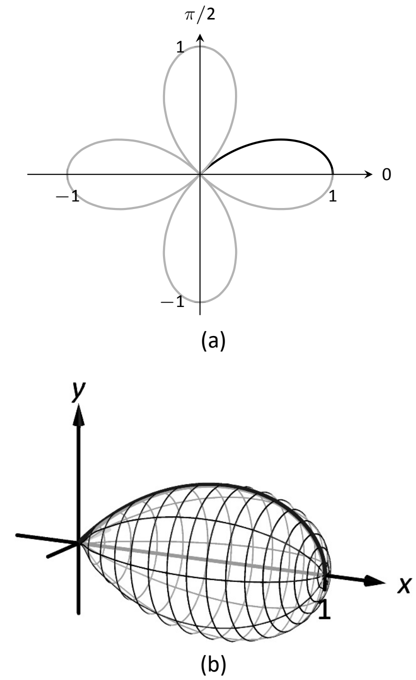


Figure 1.20: Finding the surface area of a rose–curve petal that is revolved around its central axis.

Notes:

Exercises 1.2

Terms and Concepts

09 05 ex 01 1. Given polar equation $r = f(\theta)$, how can one create parametric equations of the same curve?

09 05 ex 02 2. With rectangular coordinates, it is natural to approximate area with ________; with polar coordinates, it is natural to approximate area with ________.

Problems

09 05 exset 01 **In Exercises 3 – 10, find:**

(a) $\dfrac{dy}{dx}$

(b) the equation of the tangent and normal lines to the curve at the indicated θ–value.

09 05 ex 03 3. $r = 1; \quad \theta = \pi/4$

09 05 ex 04 4. $r = \cos\theta; \quad \theta = \pi/4$

09 05 ex 05 5. $r = 1 + \sin\theta; \quad \theta = \pi/6$

09 05 ex 10 6. $r = 1 - 3\cos\theta; \quad \theta = 3\pi/4$

09 05 ex 06 7. $r = \theta; \quad \theta = \pi/2$

09 05 ex 07 8. $r = \cos(3\theta); \quad \theta = \pi/6$

09 05 ex 08 9. $r = \sin(4\theta); \quad \theta = \pi/3$

09 05 ex 09 10. $r = \dfrac{1}{\sin\theta - \cos\theta}; \quad \theta = \pi$

09 05 exset 02 **In Exercises 11 – 14, find the values of θ in the given interval where the graph of the polar function has horizontal and vertical tangent lines.**

09 05 ex 11 11. $r = 3; \quad [0, 2\pi]$

09 05 ex 12 12. $r = 2\sin\theta; \quad [0, \pi]$

09 05 ex 13 13. $r = \cos(2\theta); \quad [0, 2\pi]$

09 05 ex 14 14. $r = 1 + \cos\theta; \quad [0, 2\pi]$

09 05 exset 03 **In Exercises 15 – 16, find the equation of the lines tangent to the graph at the pole.**

09 05 ex 15 15. $r = \sin\theta; \quad [0, \pi]$

09 05 ex 16 16. $r = \sin(3\theta); \quad [0, \pi]$

09 05 exset 04 **In Exercises 17 – 27, find the area of the described region.**

09 05 ex 17 17. Enclosed by the circle: $r = 4\sin\theta$

09 05 ex 20 18. Enclosed by the circle $r = 5$

09 05 ex 18 19. Enclosed by one petal of $r = \sin(3\theta)$

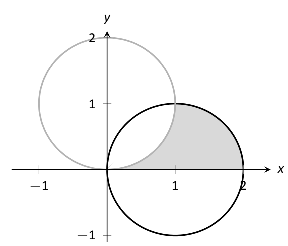
09 05 ex 19 20. Enclosed by the cardioid $r = 1 - \sin\theta$

09 05 ex 21 21. Enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$

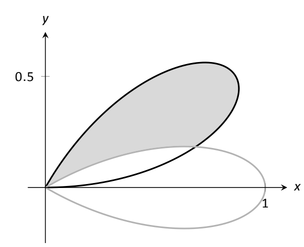
09 05 ex 22 22. Enclosed by the outer loop of the limaçon $r = 1 + 2\cos\theta$ (including area enclosed by the inner loop)

09 05 ex 23 23. Enclosed between the inner and outer loop of the limaçon $r = 1 + 2\cos\theta$

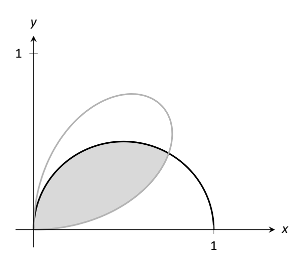
09 05 ex 25 24. Enclosed by $r = 2\cos\theta$ and $r = 2\sin\theta$, as shown:



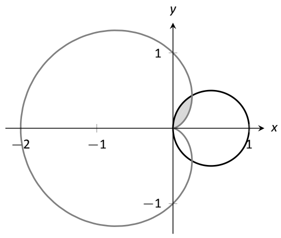
09 05 ex 26 25. Enclosed by $r = \cos(3\theta)$ and $r = \sin(3\theta)$, as shown:



09 05 ex 24 26. Enclosed by $r = \cos\theta$ and $r = \sin(2\theta)$, as shown:



09 05 ex 27 27. Enclosed by $r = \cos\theta$ and $r = 1 - \cos\theta$, as shown.



09 05 exset 05 **In Exercises 28 – 32, answer the questions involving arc length.**

09 05 ex 28 28. Let $x(\theta) = f(\theta)\cos\theta$ and $y(\theta) = f(\theta)\sin\theta$. Show, as suggested by the text, that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

09 05 ex 29 29. Use the arc length formula to compute the arc length of the circle $r = 2$.

09 05 ex 30 30. Use the arc length formula to compute the arc length of the circle $r = 4\sin\theta$.

09 05 ex 31 31. Approximate the arc length of one petal of the rose curve $r = \sin(3\theta)$ with Simpson's Rule and $n = 4$.

09 05 ex 32 32. Approximate the arc length of the cardioid $r = 1 + \cos\theta$ with Simpson's Rule and $n = 6$.

09 05 exset 06 **In Exercises 33 – 37, answer the questions involving surface area.**

09 05 ex 33 33. Use Key Idea 5 to find the surface area of the sphere formed by revolving the circle $r = 2$ about the initial ray.

09 05 ex 34 34. Use Key Idea 5 to find the surface area of the sphere formed by revolving the circle $r = 2\cos\theta$ about the initial ray.

09 05 ex 35 35. Find the surface area of the solid formed by revolving the cardioid $r = 1 + \cos\theta$ about the initial ray.

09 05 ex 36 36. Find the surface area of the solid formed by revolving the circle $r = 2\cos\theta$ about the line $\theta = \pi/2$.

09 05 ex 37 37. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $-\pi/4 \le \theta \le \pi/4$, about the line $\theta = \pi/2$.

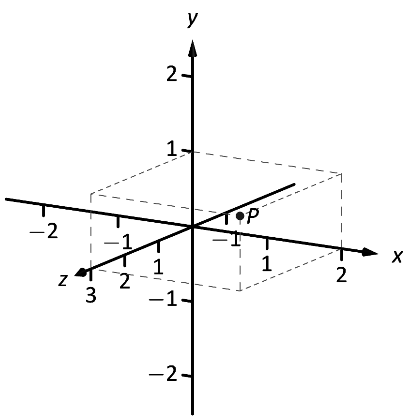


Figure 1.21: Plotting the point $P = (2, 1, 3)$ in space.

This chapter introduces a new mathematical object, the **vector**. Defined in Section 10.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

1.3 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the *x*-*y* plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive *x*-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive *y*-axis, then the extended thumb will point in the direction of the positive *z*-axis. (It may take some thought to verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule,

Notes:

it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 1.21 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the *x*-*y* plane is drawn in its standard way, with the *z*-axis down to the left. The perspective is that the paper represents the *x*-*y* plane and the positive *z* axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the *x*-*y* plane as being a horizontal plane in, say, a room, where the positive *z*-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 1.22. The same point *P* is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

Definition 1 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

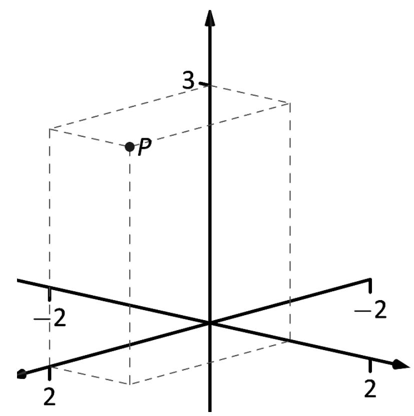


Figure 1.22: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $||\overline{PQ}||$. The above distance formula allows us to compute the length of this segment.

Example 16 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Solution The points P and Q are plotted in Figure 1.23; no special consideration need be made to draw the line segment connecting these two points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured

Notes:

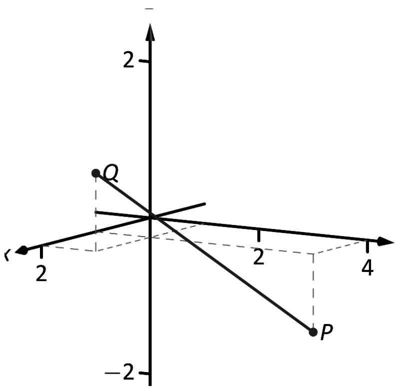


Figure 1.23: Plotting points P and Q in Example 16.

analytically. Applying Definition 1, we have

$$||\overline{PQ}|| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 1 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$||\overline{PC}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 6 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 17 Equation of a sphere

Find the center and radius of the sphere defined by $x^2 + 2x + y^2 - 4y + z^2 - 6z = 2$.

Solution To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

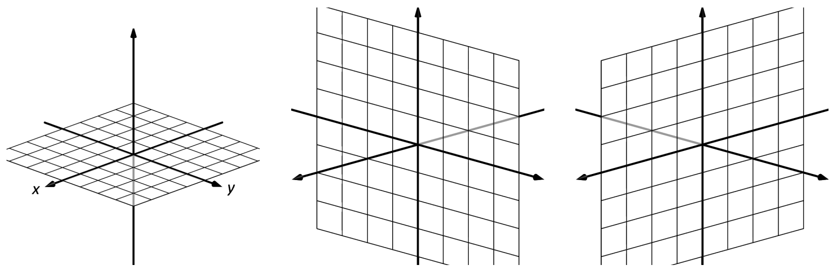
The sphere is centered at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We now consider situations where surfaces are defined where one or two of these variables are absent.

Notes:

Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 1.24), the **coordinate planes**: the *x*-*y* plane, the *y*-*z* plane and the *x*-*z* plane. The *x*-*y* plane is characterized as the set of all points in space where the *z*-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the *x*-*z* plane is all points where the *y*-value is 0, characterized by $y = 0$.



the *x*-*y* plane　　the *y*-*z* plane　　the *x*-*z* plane

Figure 1.24: The coordinate planes.

The equation $x = 2$ describes all points in space where the *x*-value is 2. This is a plane, parallel to the *y*-*z* coordinate plane, shown in Figure 1.25.

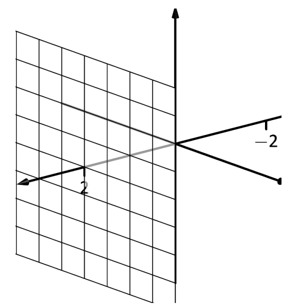


Figure 1.25: The plane $x = 2$.

Example 18　　Regions defined by planes
Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

Solution The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 1.26, which are parallel to the *x*-*z* plane. Thus the region extends infinitely in the *x* and *z* directions, and is bounded by planes in the *y* direction.

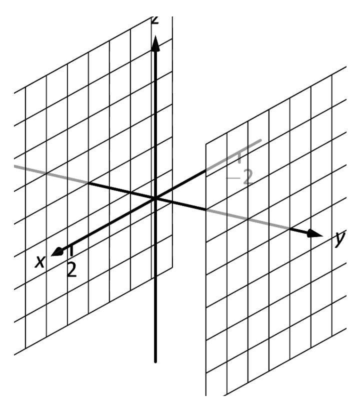


Figure 1.26: Sketching the boundaries of a region in Example 18.

Cylinders

The equation $x = 1$ obviously lacks the *y* and *z* variables, meaning it defines points where the *y* and *z* coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space.* In *the plane*, this equation describes a circle of radius 1, centered at the origin. In space, the *z* coordinate is not specified, meaning it can take on any value. In Figure 1.28 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant *z*-value of -1.5, the middle one has a *z*-value of 0 and the top circle has a *z*-value of 1. By plotting *all* possible *z*-values, we get the surface shown in Figure 1.28 (b). This surface looks like a "tube," or a "cylinder"; mathematicians call

Notes:

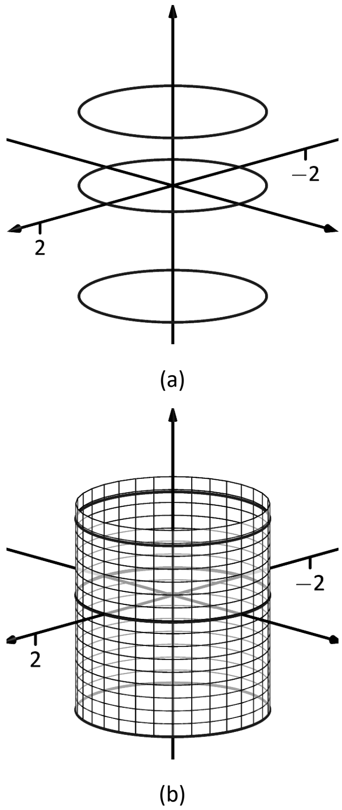


Figure 1.28: Sketching $x^2 + y^2 = 1$.

this surface a **cylinder** for an entirely different reason.

> **Definition 2 Cylinder**
>
> Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3$^{\text{rd}}$ variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 1.28 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

Example 19 Graphing cylinders

Graph the cylinder following cylinders.

1. $z = y^2$ 2. $x = \sin z$

Solution

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 1.27 (a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These rulings give a general idea as to what the surface looks like, drawn in (c).

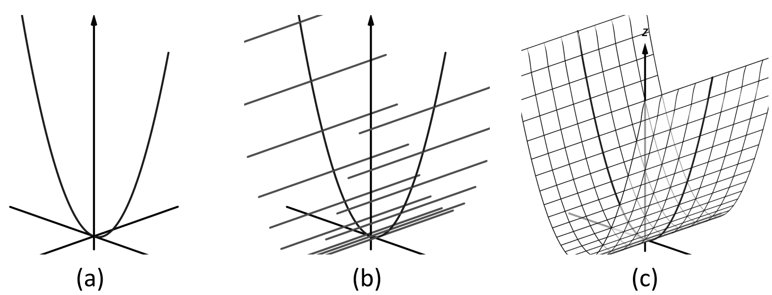


Figure 1.27: Sketching the cylinder defined by $z = y^2$.

2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 1.29 (a). The rules are parallel to the y axis as

Notes:

the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

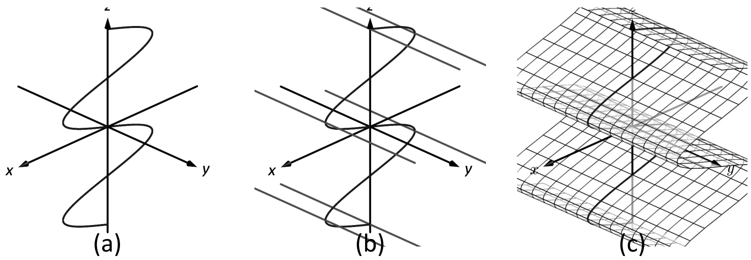


Figure 1.29: Sketching the cylinder defined by $x = \sin z$.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 1.30(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 1.30b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

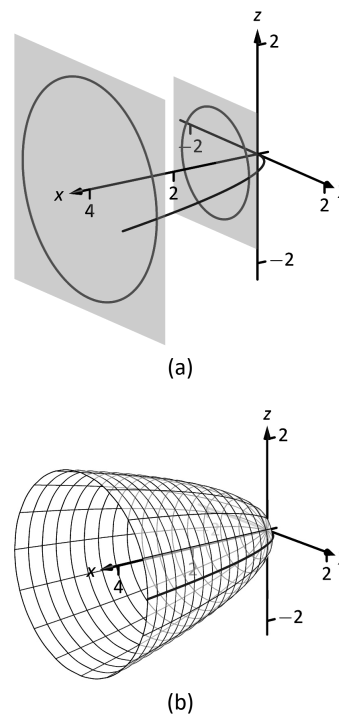


Figure 1.30: Introducing surfaces of revolution.

Key Idea 7 Surfaces of Revolution, Part 1

Let r be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

Notes:

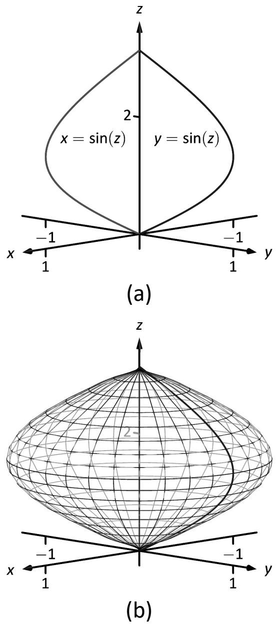


Figure 1.31: Revolving $y = \sin z$ about the z-axis in Example 20.

Example 20 Finding equation of a surface of revolution

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 7, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 1.31(a) and the surface is drawn in Figure 1.31(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 1.31(a).

This particular method of creating surfaces of revolution is limited. For instance, in Example 212 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

> **Key Idea 8 Surfaces of Revolution, Part 2**
>
> Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\big(\sqrt{x^2 + y^2}\big)$.

Example 21 Finding equation of surface of revolution

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

Solution Using Key Idea 8, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 1.32.

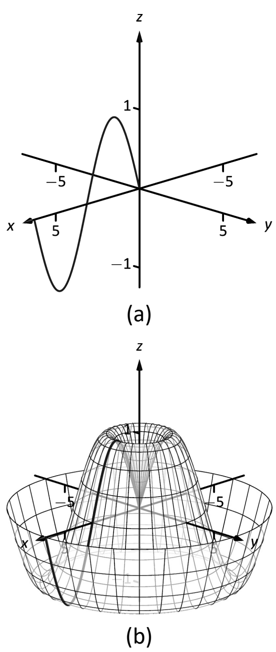


Figure 1.32: Revolving $z = \sin x$ about the z-axis in Example 21.

Notes:

Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

> **Definition 3 Quadric Surface**
>
> A **quadric surface** is the graph of the general second–degree equation in three variables:
>
> $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4+y^2$, shown in Figure 1.33. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

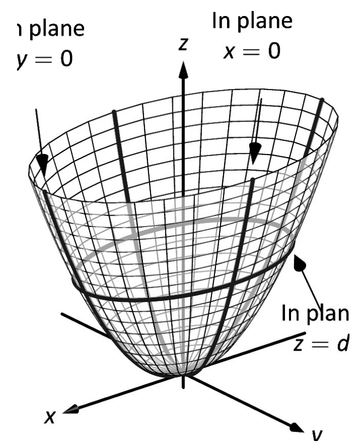
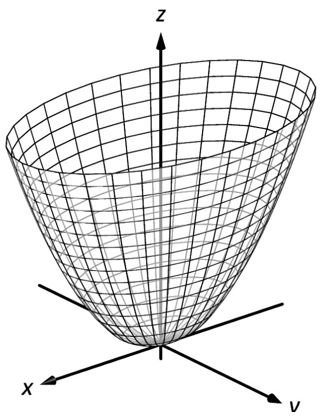


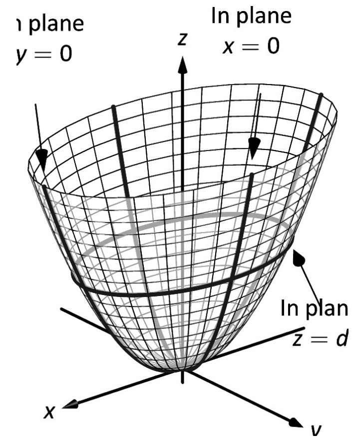
Figure 1.33: The elliptic paraboloid $z = x^2/4 + y^2$.

Notes:

Elliptic Paraboloid, $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



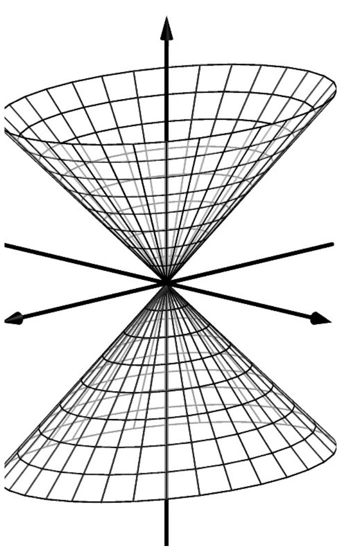
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



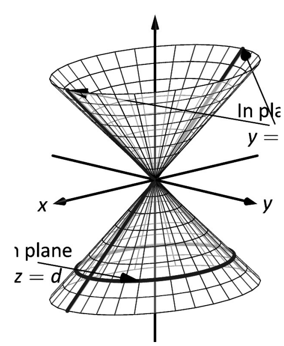
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

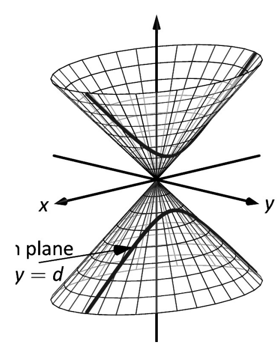
Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



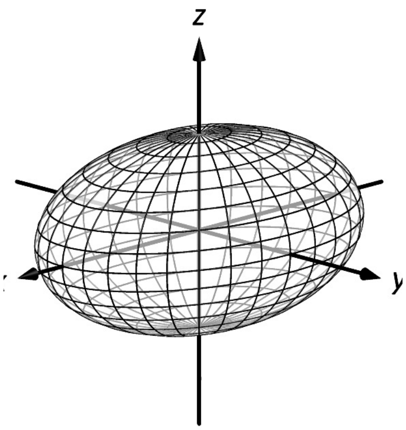
Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse



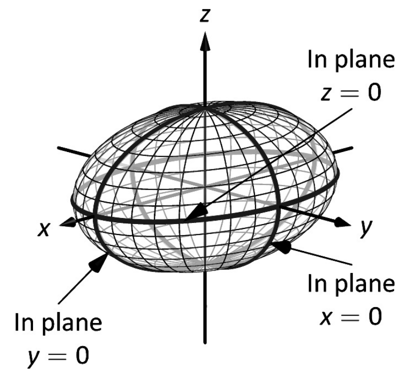


One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

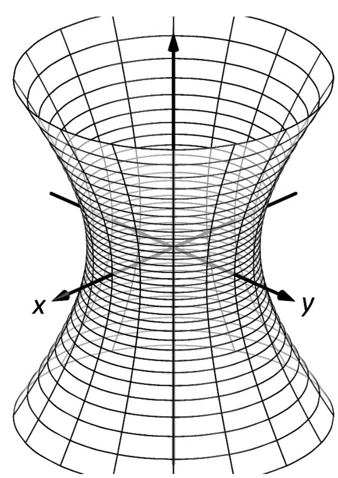


Plane	Trace
$x = d$	Ellipse
$y = d$	Ellipse
$z = d$	Ellipse

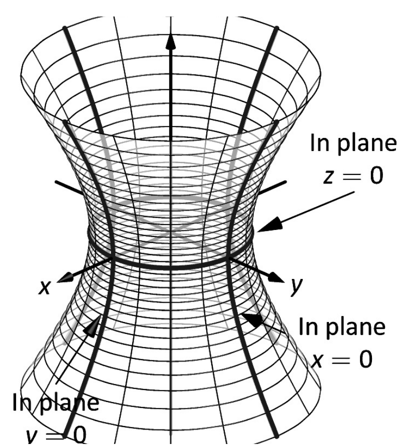


If $a = b = c \neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 6.

Hyperboloid of One Sheet, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

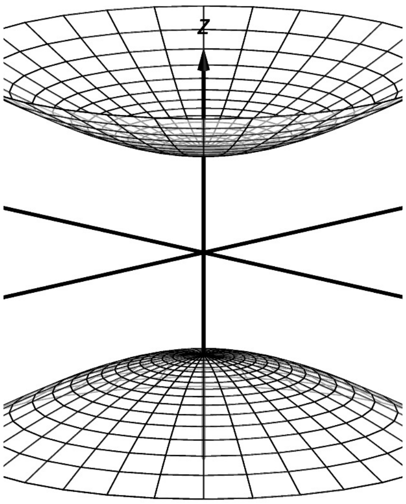


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

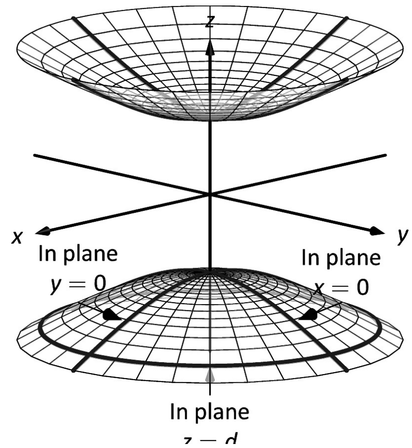


The one variable with a negative coefficient corresponds to the axis that the hyperboloid “opens” along.

Hyperboloid of Two Sheets, $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

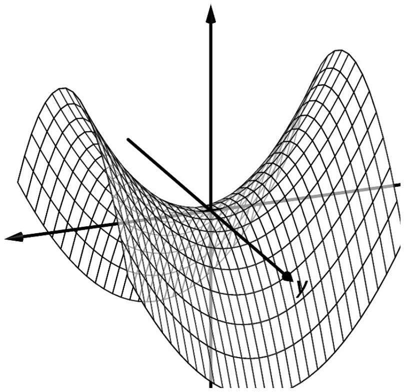


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

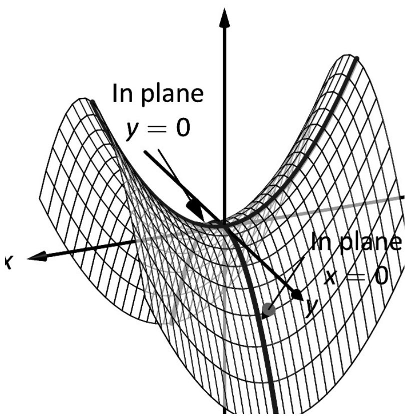


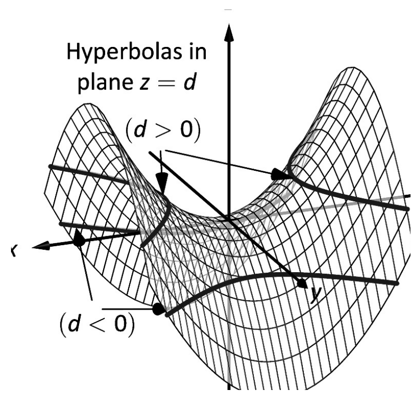
The one variable with a positive coefficient corresponds to the axis that the hyperboloid “opens” along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola





The parabolic traces will open along the axis of the one variable that is raised to the first power.

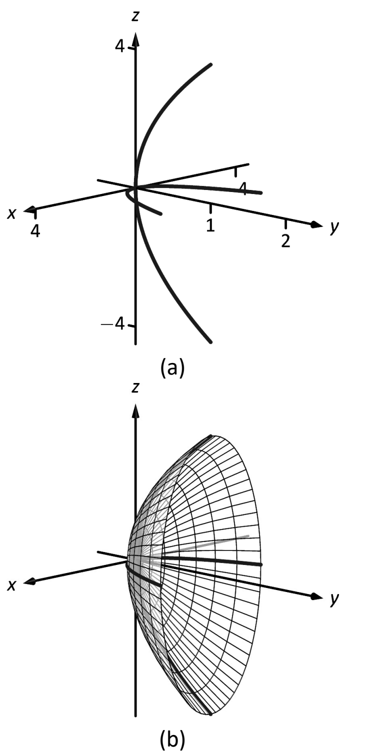


Figure 1.34: Sketching an elliptic paraboloid.

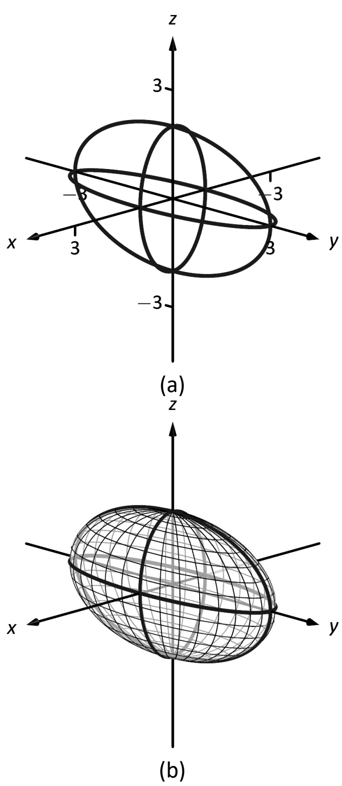


Figure 1.35: Sketching an ellipsoid.

Example 22 **Sketching quadric surfaces**

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$ 3. $z = y^2 - x^2.$

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 1.34(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 1.35(a). Filling in the surface gives Figure 1.35(b).

3. $z = y^2 - x^2$:

Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y-z$ and $x-z$ planes:

$x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

$y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

Sketching these two parabolas gives a sketch like that in Figure 1.36 (a), and filling in the surface gives a sketch like (b).

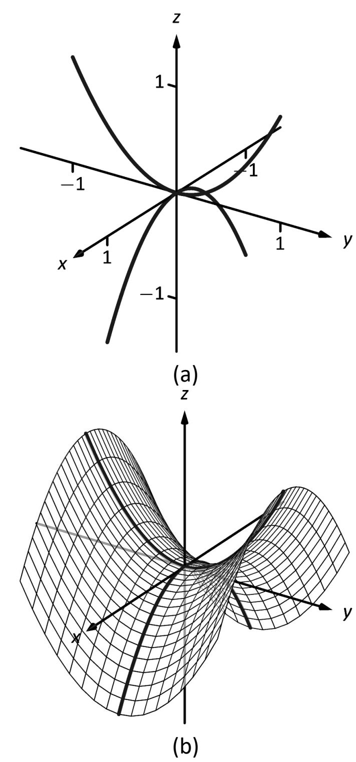


Figure 1.36: Sketching a hyperbolic paraboloid.

Example 23 **Identifying quadric surfaces**

Consider the quadric surface shown in Figure 1.37. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \dfrac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \dfrac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

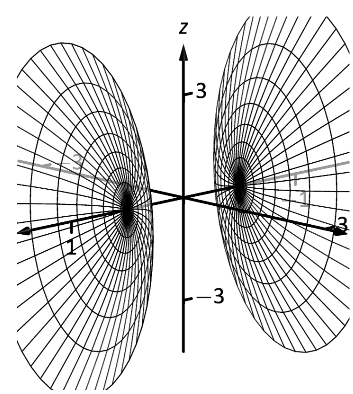


Figure 1.37: A possible equation of this quadric surface is found in Example 23.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Notes:

Exercises 1.3

Terms and Concepts

10 01 ex 08 1. Axes drawn in space must conform to the ________ ________ rule.

10 01 ex 01 2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.

10 01 ex 02 3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.

10 01 ex 03 4. Which quadric surface looks like a Pringles® chip?

10 01 ex 04 5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

10 01 ex 05 6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

10 01 ex 06 7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

10 01 ex 07 8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

10 01 ex 09 9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10 01 ex 10 10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

10 01 exset 01 **In Exercises 11 – 14, describe the region in space defined by the inequalities.**

10 01 ex 11 11. $x^2 + y^2 + z^2 < 1$

10 01 ex 12 12. $0 \le x \le 3$

10 01 ex 13 13. $x \ge 0,\ y \ge 0,\ z \ge 0$

10 01 ex 14 14. $y \ge 3$

10 01 exset 02 **In Exercises 15 – 18, sketch the cylinder in space.**

10 01 ex 15 15. $z = x^3$

10 01 ex 16 16. $y = \cos z$

10 01 ex 17 17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

10 01 ex 18 18. $y = \dfrac{1}{x}$

10 01 exset 05 **In Exercises 19 – 22, give the equation of the surface of revolution described.**

10 01 ex 29 19. Revolve $z = \dfrac{1}{1+y^2}$ about the y-axis.

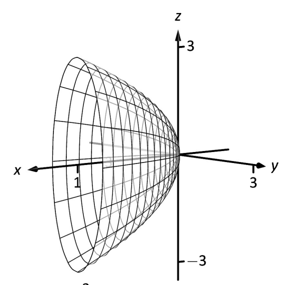
10 01 ex 30 20. Revolve $y = x^2$ about the x-axis.

10 01 ex 31 21. Revolve $z = x^2$ about the z-axis.

10 01 ex 32 22. Revolve $z = 1/x$ about the z-axis.

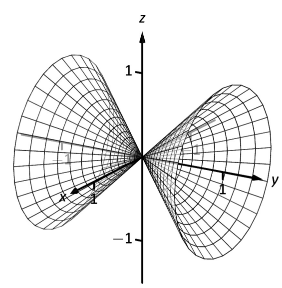
10 01 exset 03 **In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.**

10 01 ex 19 23.



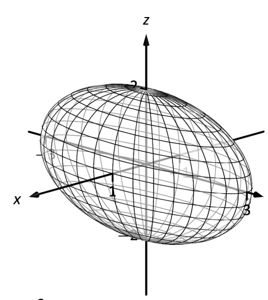
(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

10 01 ex 20 24.



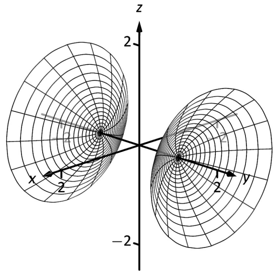
(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

10 01 ex 21 25.



(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

10 01 ex 22 26.



(a) $y^2 - x^2 - z^2 = 1$ (b) $y^2 + x^2 - z^2 = 1$

10 01 exset 04 **In Exercises 27 – 32, sketch the quadric surface.**

10 01 ex 28 27. $z - y^2 + x^2 = 0$

10 01 ex 24 28. $z^2 = x^2 + \dfrac{y^2}{4}$

10 01 ex 23 29. $x = -y^2 - z^2$

10 01 ex 27 30. $16x^2 - 16y^2 - 16z^2 = 1$

10 01 ex 26 31. $\dfrac{x^2}{9} - y^2 + \dfrac{z^2}{25} = 1$

10 01 ex 25 32. $4x^2 + 2y^2 + z^2 = 4$

1.4 Lines

To find the equation of a line in the x-y plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

> To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

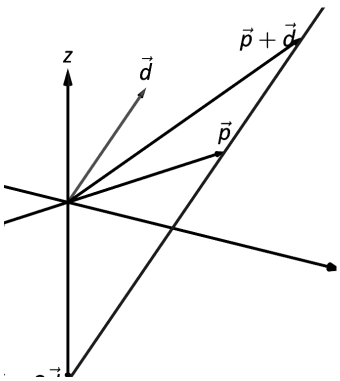


Figure 1.39: Defining a line in space.

Let P be a point in space, let $\vec{p}$ be the vector with initial point at the origin and terminal point at P (i.e., $\vec{p}$ "points" to P), and let $\vec{d}$ be a vector. Consider the points on the line through P in the direction of $\vec{d}$.

Clearly one point on the line is P; we can say that the *vector* $\vec{p}$ lies at this point on the line. To find another point on the line, we can start at $\vec{p}$ and move in a direction parallel to $\vec{d}$. For instance, starting at $\vec{p}$ and traveling one length of $\vec{d}$ places one at another point on the line. Consider Figure 1.39 where certain points along the line are indicated.

The figure illustrates how every point on the line can be obtained by starting with $\vec{p}$ and moving a certain distance in the direction of $\vec{d}$. That is, we can define the line as a function of t:

$$\vec{\ell}(t) = \vec{p} + t\,\vec{d}. \tag{1.2}$$

In many ways, this is *not* a new concept. Compare Equation (1.2) to the familiar "$y = mx + b$" equation of a line:

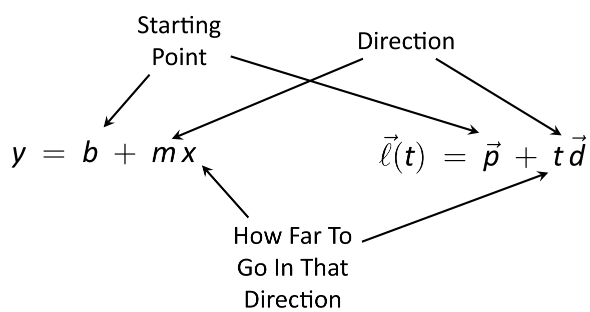


Figure 1.38: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (1.2) is an example of a **vector–valued function**; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

Notes:

There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. Then the equation of the line through $\vec{p}$ in the direction of $\vec{d}$ is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle\,.\end{aligned}$$

The last line states that the x values of the line are given by $x = x_0 + at$, the y values are given by $y = y_0 + bt$, and the z values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$.

Finally, each of the equations for x, y and z above contain the variable t. We can solve for t in each equation:

$$\begin{aligned} x = x_0 + at \quad &\Rightarrow \quad t = \frac{x - x_0}{a}, \\ y = y_0 + bt \quad &\Rightarrow \quad t = \frac{y - y_0}{b}, \\ z = z_0 + ct \quad &\Rightarrow \quad t = \frac{z - z_0}{c}, \end{aligned}$$

assuming $a, b, c \neq 0$. Since t is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:

Definition 4 Equations of Lines in Space

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0 \rangle$ in the direction of $\vec{d} = \langle a, b, c \rangle$.

1. The **vector equation** of the line is
$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The **parametric equations** of the line are
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The **symmetric equations** of the line are
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

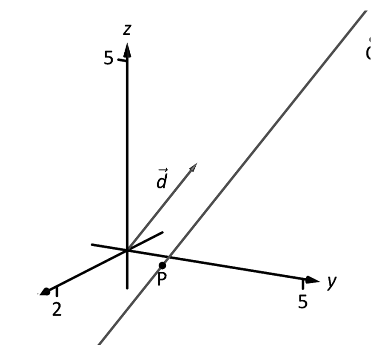


Figure 1.40: Graphing a line in Example 24.

Example 24 Finding the equation of a line

Give all three equations, as given in Definition 4, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2 \rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

Solution We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1 \rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle$;
- the parametric equations of the line are
$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$
- the symmetric equations of the line are
$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a t value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, to make Figure 1.40, a certain graphics program was given the input `(2-x,3+x,1+2*x)`. This particular program requires the variable always be "x" instead of "t").

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 1.40 makes it clear that it does not. We can answer this question without the graph

Notes:

using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of *x*, *y* and *z* and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$

We see that *Q* does not lie on the line as it did not satisfy the symmetric equations.

Example 25 Finding the equation of a line through two points

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

Solution Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point *P* and terminal point *Q*: $\overrightarrow{PQ} = \langle -1, 4, -3\rangle$.

The parametric equations of the line ℓ through *P* in the direction of $\overrightarrow{PQ}$ are:

$$\ell: \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 1.41. Note how in the given parametrization of the line, $t = 0$ corresponds to the point *P*, and $t = 1$ corresponds to the point *Q*. This relates to the understanding of the vector equation of a line described in Figure 1.38. The parametric equations "start" at the point *P*, and *t* determines how far in the direction of $\overrightarrow{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\overrightarrow{PQ}$; when $t = 1$, we travel one length of $\overrightarrow{PQ}$, resulting in the point *Q*.

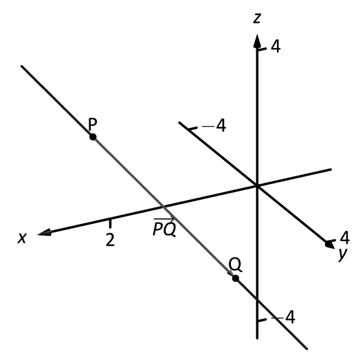


Figure 1.41: A graph of the line in Example 25.

Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\vec{\ell}_1$ and $\vec{\ell}_2$ are

the same line	they share all points;
intersecting lines	share only 1 point;
parallel lines	$\vec{d}_1 \parallel \vec{d}_2$, no points in common; or
skew lines	$\vec{d}_1 \nparallel \vec{d}_2$, no points in common.

Notes:

The next two examples investigate these possibilities.

Example 26 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1 : \begin{array}{rcl} x & = & 1+3t \\ y & = & 2-t \\ z & = & t \end{array} \qquad \ell_2 : \begin{array}{rcl} x & = & -2+4s \\ y & = & 3+s \\ z & = & 5+2s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Solution We start by looking at the directions of each line. Line ℓ_1 has the direction given by $\vec{d}_1 = \langle 3, -1, 1\rangle$ and line ℓ_2 has the direction given by $\vec{d}_2 = \langle 4, 1, 2\rangle$. It should be clear that $\vec{d}_1$ and $\vec{d}_2$ are not parallel, hence ℓ_1 and ℓ_2 are not the same line, nor are they parallel. Figure 1.42 verifies this fact (where the points and directions indicated by the equations of each line are identified).

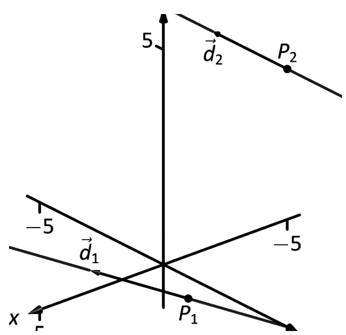


Figure 1.42: Sketching the lines from Example 26.

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for t and s values such that the respective x, y and z values are the same. That is, we want s and t such that:

$$\begin{array}{rcl} 1+3t & = & -2+4s \\ 2-t & = & 3+s \\ t & = & 5+2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for t, substitute that value of t into the equation above it:

$$2-(5+2s) = 3+s \quad \Rightarrow \quad s=-2,\ t=1.$$

A key to remember is that we have *three* equations; we need to check if $s = -2,\ t = 1$ satisfies the first equation as well:

$$1+3(1) \neq -2+4(-2).$$

It does not. Therefore, we conclude that the lines ℓ_1 and ℓ_2 are skew.

Example 27 Comparing lines

Consider lines ℓ_1 and ℓ_2, given in parametric equation form:

$$\ell_1 : \begin{array}{rcl} x & = & -0.7+1.6t \\ y & = & 4.2+2.72t \\ z & = & 2.3-3.36t \end{array} \qquad \ell_2 : \begin{array}{rcl} x & = & 2.8-2.9s \\ y & = & 10.15-4.93s \\ z & = & -5.05+6.09s. \end{array}$$

Determine whether ℓ_1 and ℓ_2 are the same line, intersect, are parallel, or skew.

Notes:

Solution It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the "real world," most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look "messy."

We again start by deciding whether or not each line has the same direction. The direction of ℓ_1 is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36\rangle$ and the direction of ℓ_2 is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09\rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\begin{aligned}\vec{u}_1 &= \frac{\vec{d}_1}{\|\,\vec{d}_1\,\|} = \langle 0.3471, 0.5901, -0.7289\rangle \\ \vec{u}_2 &= \frac{\vec{d}_2}{\|\,\vec{d}_2\,\|} = \langle -0.3471, -0.5901, 0.7289\rangle\,.\end{aligned}$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100}\right\rangle \qquad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100}\right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}}\right\rangle \qquad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}}\right\rangle.$$

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on ℓ_1. To determine if this point also lies on ℓ_2, plug in the x, y and z values of P_1 into the symmetric equations for ℓ_2:

$$\frac{(-0.7)-2.8}{-2.9} \stackrel{?}{=} \frac{(4.2)-10.15}{-4.93} \stackrel{?}{=} \frac{(2.3)-(-5.05)}{6.09} \quad\Rightarrow\quad 1.2069 = 1.2069 = 1.2069.$$

The point P_1 lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 1.43 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

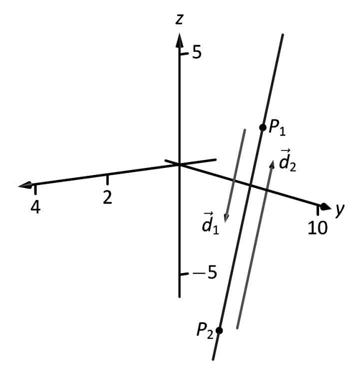


Figure 1.43: Graphing the lines in Example 27.

Notes:

Distances

Given a point Q and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of "distance," i.e., the length of the shortest line segment from the point to the line.) Identifying $\vec{p}$ with the point P, Figure 1.44 will help establish a general method of computing this distance h.

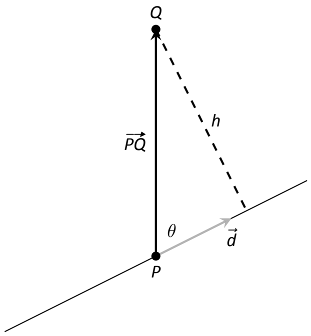


Figure 1.44: Establishing the distance from a point to a line.

From trigonometry, we know $h = \|\overrightarrow{PQ}\| \sin\theta$. We have a similar identity involving the cross product: $\|\overrightarrow{PQ} \times \vec{d}\| = \|\overrightarrow{PQ}\|\,\|\vec{d}\| \sin\theta$. Divide both sides of this latter equation by $\|\vec{d}\|$ to obtain h:

$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\|}{\|\vec{d}\|}. \tag{1.3}$$

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines $\vec{\ell_1}(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell_2}(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 1.45. To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$, we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$ is the distance h we seek:

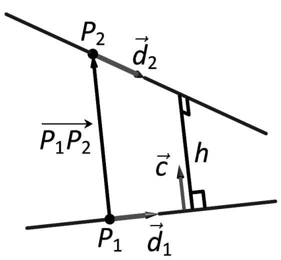


Figure 1.45: Establishing the distance between lines.

$$\begin{aligned} h &= \left\| \operatorname{proj}_{\vec{c}}\, \overrightarrow{P_1P_2} \right\| \\ &= \left\| \frac{\overrightarrow{P_1P_2} \cdot \vec{c}}{\vec{c} \cdot \vec{c}} \vec{c} \right\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|^2} \|\vec{c}\| \\ &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\vec{c}\|}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\overrightarrow{P_1P_2} \cdot c = \overrightarrow{P_1P_2} \cdot (\vec{d}_1 \times \vec{d}_2)$.

The following Key Idea restates these two distance formulas.

Notes:

Key Idea 9 Distances to Lines

1. Let P be a point on a line ℓ that is parallel to $\vec{d}$. The distance h from a point Q to the line ℓ is:

$$h = \frac{||\,\vec{PQ} \times \vec{d}\,||}{||\,\vec{d}\,||}.$$

2. Let P_1 be a point on line ℓ_1 that is parallel to $\vec{d_1}$, and let P_2 be a point on line ℓ_2 parallel to $\vec{d_2}$, and let $\vec{c} = \vec{d_1} \times \vec{d_2}$, where lines ℓ_1 and ℓ_2 are not parallel. The distance h between the two lines is:

$$h = \frac{|\vec{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||}.$$

Example 28 Finding the distance from a point to a line

Find the distance from the point $Q = (1, 1, 3)$ to the line $\vec{\ell}(t) = \langle 1, -1, 1\rangle + t\,\langle 2, 3, 1\rangle$.

Solution The equation of the line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $\vec{PQ} = \langle 0, 2, 2\rangle$. The equation also gives $\vec{d} = \langle 2, 3, 1\rangle$. Following Key Idea 9, we have the distance as

$$\begin{aligned} h &= \frac{||\,\vec{PQ} \times \vec{d}\,||}{||\,\vec{d}\,||} \\ &= \frac{||\ \langle -4, 4, -4\rangle\ ||}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point Q is approximately 1.852 units from the line $\vec{\ell}(t)$.

Example 29 Finding the distance between lines

Find the distance between the lines

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

Solution These are the sames lines as given in Example 26, where we

Notes:

showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3, 1, 5 \rangle .$$

$$\vec{d}_1 = \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7 \rangle .$$

From Key Idea 9 we have the distance h between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Notes:

Exercises 1.4

Terms and Concepts

10 05 ex 01 1. To find an equation of a line, what two pieces of information are needed?

10 05 ex 02 2. Two distinct lines in the plane can intersect or be ________.

10 05 ex 03 3. Two distinct lines in space can intersect, be ________ or be ________.

10 05 ex 04 4. Use your own words to describe what it means for two lines in space to be skew.

Problems

10 05 exset 01 **In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.**

10 05 ex 05 5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5\rangle$.

10 05 ex 06 6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5\rangle$.

10 05 ex 07 7. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.

10 05 ex 08 8. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.

10 05 ex 09 9. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d_1} = \langle 2, -1, 7\rangle$ and $\vec{d_2} = \langle 7, 1, 3\rangle$.

10 05 ex 10 10. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d_1} = \langle 1, 0, 1\rangle$ and $\vec{d_2} = \langle 2, 0, 3\rangle$.

10 05 ex 11 11. Passes through the point of intersection of $\vec{\ell_1}(t)$ and $\vec{\ell_2}(t)$ and orthogonal to both lines, where
$\vec{\ell_1}(t) = \langle 2, 1, 1\rangle + t\,\langle 5, 1, -2\rangle$ and
$\vec{\ell_2}(t) = \langle -2, -1, 2\rangle + t\,\langle 3, 1, -1\rangle$.

10 05 ex 12 12. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where
$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

10 05 ex 13 13. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3\rangle$.

10 05 ex 14 14. Passes through $P = (-2, 5)$, parallel to $\vec{d} = \langle 0, 1\rangle$.

10 05 exset 02 **In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.**

10 05 ex 15 15. $\vec{\ell_1}(t) = \langle 1, 2, 1\rangle + t\,\langle 2, -1, 1\rangle$,
$\vec{\ell_2}(t) = \langle 3, 3, 3\rangle + t\,\langle -4, 2, -2\rangle$.

10 05 ex 16 16. $\vec{\ell_1}(t) = \langle 2, 1, 1\rangle + t\,\langle 5, 1, 3\rangle$,
$\vec{\ell_2}(t) = \langle 14, 5, 9\rangle + t\,\langle 1, 1, 1\rangle$.

10 05 ex 17 17. $\vec{\ell_1}(t) = \langle 3, 4, 1\rangle + t\,\langle 2, -3, 4\rangle$,
$\vec{\ell_2}(t) = \langle -3, 3, -3\rangle + t\,\langle 3, -2, 4\rangle$.

10 05 ex 18 18. $\vec{\ell_1}(t) = \langle 1, 1, 1\rangle + t\,\langle 3, 1, 3\rangle$,
$\vec{\ell_2}(t) = \langle 7, 3, 7\rangle + t\,\langle 6, 2, 6\rangle$.

10 05 ex 19 19. $$\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$$

10 05 ex 20 20. $$\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$$

10 05 ex 21 21. $$\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$$

10 05 ex 22 22. $$\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$$

10 05 exset 03 **In Exercises 23 – 26, find the distance from the point to the line.**

10 05 ex 23 23. $P = (1, 1, 1)$, $\quad \vec{\ell}(t) = \langle 2, 1, 3\rangle + t\,\langle 2, 1, -2\rangle$

10 05 ex 24 24. $P = (2, 5, 6)$, $\quad \vec{\ell}(t) = \langle -1, 1, 1\rangle + t\,\langle 1, 0, 1\rangle$

10 05 ex 25 25. $P = (0, 3)$, $\quad \vec{\ell}(t) = \langle 2, 0\rangle + t\,\langle 1, 1\rangle$

10 05 ex 26 26. $P = (1, 1)$, $\quad \vec{\ell}(t) = \langle 4, 5\rangle + t\,\langle -4, 3\rangle$

10 05 exset 04 **In Exercises 27 – 28, find the distance between the two lines.**

10 05 ex 27 27. $\vec{\ell_1}(t) = \langle 1, 2, 1\rangle + t\,\langle 2, -1, 1\rangle$,
$\vec{\ell_2}(t) = \langle 3, 3, 3\rangle + t\,\langle 4, 2, -2\rangle$.

10 05 ex 28 28. $\vec{\ell_1}(t) = \langle 0, 0, 1\rangle + t\,\langle 1, 0, 0\rangle$,
$\vec{\ell_2}(t) = \langle 0, 0, 3\rangle + t\,\langle 0, 1, 0\rangle$.

10 05 exset 05 **Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 9.**

10 05 ex 29 29. Let Q be a point on the line $\ell(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

10 05 ex 30 30. Let lines $\ell_1(t)$ and $\ell_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

10 05 ex 31 31. Let lines $\ell_1(t)$ and $\ell_2(t)$ be parallel.

(a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

(b) Show why letting $c = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to the use the formula.

(c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

1.5 Planes

Any flat surface, such as a wall, table top or stiff piece of cardboard can be thought of as representing part of a plane. Consider a piece of cardboard with a point P marked on it. One can take a nail and stick it into the cardboard at P such that the nail is perpendicular to the cardboard; see Figure 1.46.

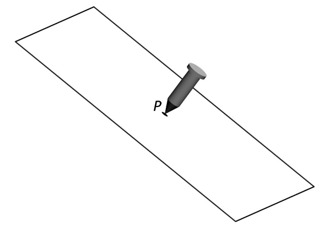


Figure 1.46: Illustrating defining a plane with a sheet of cardboard and a nail.

This nail provides a "handle" for the cardboard. Moving the cardboard around moves P to different locations in space. Tilting the nail (but keeping P fixed) tilts the cardboard. Both moving and tilting the cardboard defines a different plane in space. In fact, we can define a plane by: 1) the location of P in space, and 2) the direction of the nail.

The previous section showed that one can define a line given a point on the line and the direction of the line (usually given by a vector). One can make a similar statement about planes: we can define a plane in space given a point on the plane and the direction the plane "faces" (using the description above, the direction of the nail). Once again, the direction information will be supplied by a vector, called a **normal vector**, that is orthogonal to the plane.

What exactly does "orthogonal to the plane" mean? Choose any two points P and Q in the plane, and consider the vector $\overrightarrow{PQ}$. We say a vector $\vec{n}$ is orthogonal to the plane if $\vec{n}$ is perpendicular to $\overrightarrow{PQ}$ for all choices of P and Q; that is, if $\vec{n}\cdot\overrightarrow{PQ} = 0$ for all P and Q.

This gives us way of writing an equation describing the plane. Let $P = (x_0, y_0, z_0)$ be a point in the plane and let $\vec{n} = \langle a, b, c\rangle$ be a normal vector to the plane. A point $Q = (x, y, z)$ lies in the plane defined by P and $\vec{n}$ if, and only if, $\overrightarrow{PQ}$ is orthogonal to $\vec{n}$. Knowing $\overrightarrow{PQ} = \langle x - x_0, y - y_0, z - z_0\rangle$, consider:

$$\begin{aligned}
\overrightarrow{PQ}\cdot\vec{n} &= 0\\
\langle x - x_0, y - y_0, z - z_0\rangle\cdot\langle a, b, c\rangle &= 0\\
a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0 \qquad (1.4)
\end{aligned}$$

Equation (1.4) defines an *implicit* function describing the plane. More algebra produces:

$$ax + by + cz = ax_0 + by_0 + cz_0.$$

The right hand side is just a number, so we replace it with d:

$$ax + by + cz = d. \qquad (1.5)$$

As long as $c \neq 0$, we can solve for z:

$$z = \frac{1}{c}(d - ax - by). \qquad (1.6)$$

Notes:

Equation (1.6) is especially useful as many computer programs can graph functions in this form. Equations (1.4) and (1.5) have specific names, given next.

Definition 5 Equations of a Plane in Standard and General Forms

The plane passing through the point $P = (x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ can be described by an equation with **standard form**

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0;$$

the equation's **general form** is

$$ax + by + cz = d.$$

A key to remember throughout this section is this: to find the equation of a plane, we need a point and a normal vector. We will give several examples of finding the equation of a plane, and in each one different types of information are given. In each case, we need to use the given information to find a point on the plane and a normal vector.

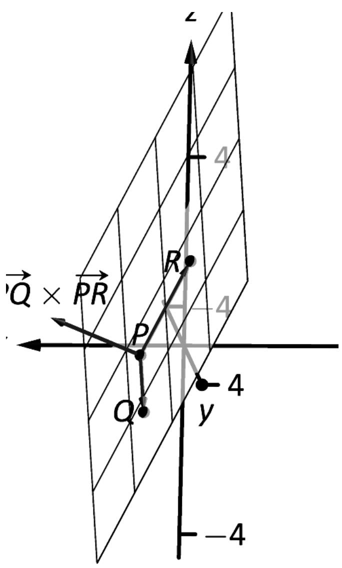


Figure 1.47: Sketching the plane in Example 30.

Example 30 Finding the equation of a plane.

Write the equation of the plane that passes through the points $P = (1, 1, 0)$, $Q = (1, 2, -1)$ and $R = (0, 1, 2)$ in standard form.

Solution We need a vector $\vec{n}$ that is orthogonal to the plane. Since P, Q and R are in the plane, so are the vectors $\vec{PQ}$ and $\vec{PR}$; $\vec{PQ} \times \vec{PR}$ is orthogonal to $\vec{PQ}$ and $\vec{PR}$ and hence the plane itself.

It is straightforward to compute $\vec{n} = \vec{PQ} \times \vec{PR} = \langle 2, 1, 1\rangle$. We can use any point we wish in the plane (any of P, Q or R will do) and we arbitrarily choose P. Following Definition 5, the equation of the plane in standard form is

$$2(x - 1) + (y - 1) + z = 0.$$

The plane is sketched in Figure 1.47.

We have just demonstrated the fact that any three non-collinear points define a plane. (This is why a three-legged stool does not "rock;" it's three feet always lie in a plane. A four-legged stool will rock unless all four feet lie in the same plane.)

Example 31 Finding the equation of a plane.

Verify that lines ℓ_1 and ℓ_2, whose parametric equations are given below, inter-

Notes:

sect, then give the equation of the plane that contains these two lines in general form.

$$\ell_1: \begin{array}{rcl} x &=& -5+2s \\ y &=& 1+s \\ z &=& -4+2s \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& 2+3t \\ y &=& 1-2t \\ z &=& 1+t \end{array}$$

Solution The lines clearly are not parallel. If they do not intersect, they are skew, meaning there is not a plane that contains them both. If they do intersect, there is such a plane.

To find their point of intersection, we set the x, y and z equations equal to each other and solve for s and t:

$$\begin{array}{rcl} -5+2s &=& 2+3t \\ 1+s &=& 1-2t \\ -4+2s &=& 1+t \end{array} \quad \Rightarrow \quad s=2, \quad t=-1.$$

When $s=2$ and $t=-1$, the lines intersect at the point $P=(-1,3,0)$.

Let $\vec{d}_1 = \langle 2,1,2\rangle$ and $\vec{d}_2 = \langle 3,-2,1\rangle$ be the directions of lines ℓ_1 and ℓ_2, respectively. A normal vector to the plane containing these the two lines will also be orthogonal to $\vec{d}_1$ and $\vec{d}_2$. Thus we find a normal vector $\vec{n}$ by computing $\vec{n} = \vec{d}_1 \times \vec{d}_2 = \langle 5, 4-7\rangle$.

We can pick any point in the plane with which to write our equation; each line gives us infinite choices of points. We choose P, the point of intersection. We follow Definition 5 to write the plane's equation in general form:

$$\begin{aligned} 5(x+1)+4(y-3)-7z &= 0 \\ 5x+5+4y-12-7z &= 0 \\ 5x+4y-7z &= 7. \end{aligned}$$

The plane's equation in general form is $5x+4y-7z=7$; it is sketched in Figure 1.48.

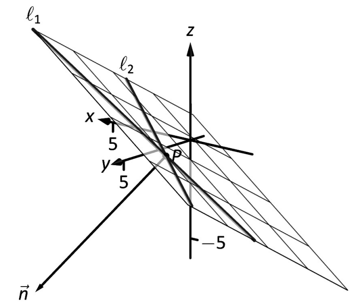


Figure 1.48: Sketching the plane in Example 31.

Example 32 **Finding the equation of a plane**

Give the equation, in standard form, of the plane that passes through the point $P=(-1,0,1)$ and is orthogonal to the line with vector equation $\vec{\ell}(t) = \langle -1,0,1\rangle + t\,\langle 1,2,2\rangle$.

Solution As the plane is to be orthogonal to the line, the plane must be orthogonal to the direction of the line given by $\vec{d} = \langle 1,2,2\rangle$. We use this as our normal vector. Thus the plane's equation, in standard form, is

$$(x+1)+2y+2(z-1)=0.$$

The line and plane are sketched in Figure 1.49.

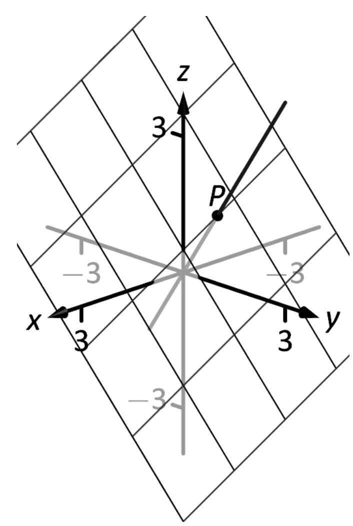


Figure 1.49: The line and plane in Example 32.

Notes:

Example 33 Finding the intersection of two planes

Give the parametric equations of the line that is the intersection of the planes p_1 and p_2, where:

$$p_1 : x - (y - 2) + (z - 1) = 0$$
$$p_2 : -2(x - 2) + (y + 1) + (z - 3) = 0$$

Solution To find an equation of a line, we need a point on the line and the direction of the line.

We can find a point on the line by solving each equation of the planes for z:

$$p_1 : z = -x + y - 1$$
$$p_2 : z = 2x - y - 2$$

We can now set these two equations equal to each other (i.e., we are finding values of x and y where the planes have the same z value):

$$-x + y - 1 = 2x - y - 2$$
$$2y = 3x - 1$$
$$y = \frac{1}{2}(3x - 1)$$

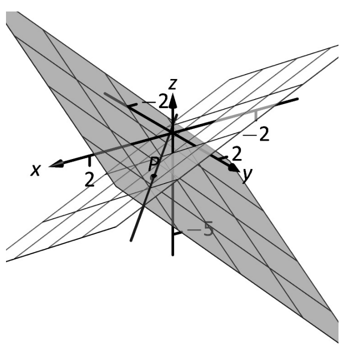


Figure 1.50: Graphing the planes and their line of intersection in Example 33.

We can choose any value for x; we choose $x = 1$. This determines that $y = 1$. We can now use the equations of either plane to find z: when $x = 1$ and $y = 1$, $z = -1$ on both planes. We have found a point P on the line: $P = (1, 1, -1)$.

We now need the direction of the line. Since the line lies in each plane, its direction is orthogonal to a normal vector for each plane. Considering the equations for p_1 and p_2, we can quickly determine their normal vectors. For p_1, $\vec{n}_1 = \langle 1, -1, 1 \rangle$ and for p_2, $\vec{n}_2 = \langle -2, 1, 1 \rangle$. A direction orthogonal to both of these directions is their cross product: $\vec{d} = \vec{n}_1 \times \vec{n}_2 = \langle -2, -3, -1 \rangle$.

The parametric equations of the line through $P = (1, 1, -1)$ in the direction of $d = \langle -2, -3, -1 \rangle$ is:

$$\ell : \quad x = -2t + 1 \quad y = -3t + 1 \quad z = -t - 1.$$

The planes and line are graphed in Figure 1.50.

Example 34 Finding the intersection of a plane and a line

Find the point of intersection, if any, of the line $\ell(t) = \langle 3, -3, -1 \rangle + t \langle -1, 2, 1 \rangle$ and the plane with equation in general form $2x + y + z = 4$.

Solution The equation of the plane shows that the vector $\vec{n} = \langle 2, 1, 1 \rangle$ is a normal vector to the plane, and the equation of the line shows that the line

Notes:

moves parallel to $\vec{d} = \langle -1, 2, 1\rangle$. Since these are not orthogonal, we know there is a point of intersection. (If there were orthogonal, it would mean that the plane and line were parallel to each other, either never intersecting or the line was in the plane itself.)

To find the point of intersection, we need to find a t value such that $\ell(t)$ satisfies the equation of the plane. Rewriting the equation of the line with parametric equations will help:

$$\ell(t) = \begin{cases} x = 3 - t \\ y = -3 + 2t \\ z = -1 + t \end{cases}.$$

Replacing x, y and z in the equation of the plane with the expressions containing t found in the equation of the line allows us to determine a t value that indicates the point of intersection:

$$\begin{aligned} 2x + y + z &= 4 \\ 2(3 - t) + (-3 + 2t) + (-1 + t) &= 4 \\ t &= 2. \end{aligned}$$

When $t = 2$, the point on the line satisfies the equation of the plane; that point is $\ell(2) = \langle 1, 1, 1\rangle$. Thus the point $(1, 1, 1)$ is the point of intersection between the plane and the line, illustrated in Figure 1.51.

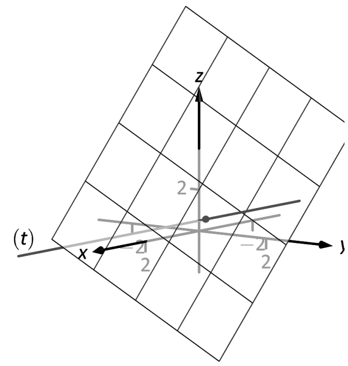


Figure 1.51: Illustrating the intersection of a line and a plane in Example 34.

Distances

Just as it was useful to find distances between points and lines in the previous section, it is also often necessary to find the distance from a point to a plane.

Consider Figure 1.52, where a plane with normal vector $\vec{n}$ is sketched containing a point P and a point Q, not on the plane, is given. We measure the distance from Q to the plane by measuring the length of the projection of $\overrightarrow{PQ}$ onto $\vec{n}$. That is, we want:

$$\left|\left| \operatorname{proj}_{\vec{n}} \overrightarrow{PQ} \right|\right| = \left|\left| \frac{\vec{n} \cdot \overrightarrow{PQ}}{||\,\vec{n}\,||^2}\vec{n} \right|\right| = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||} \tag{1.7}$$

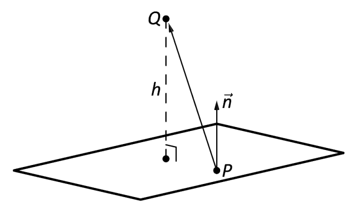


Figure 1.52: Illustrating finding the distance from a point to a plane.

Equation (1.7) is important as it does more than just give the distance between a point and a plane. We will see how it allows us to find several other distances as well: the distance between parallel planes and the distance from a line and a plane. Because Equation (1.7) is important, we restate it as a Key Idea.

Notes:

Key Idea 10 Distance from a Point to a Plane

Let a plane with normal vector $\vec{n}$ be given, and let Q be a point. The distance h from Q to the plane is

$$h = \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||},$$

where P is any point in the plane.

Example 35 Distance between a point and a plane

Find the distance bewteen the point $Q = (2, 1, 4)$ and the plane with equation $2x - 5y + 6z = 9$.

Solution Using the equation of the plane, we find the normal vector $\vec{n} = \langle 2, -5, 6\rangle$. To find a point on the plane, we can let x and y be anything we choose, then let z be whatever satisfies the equation. Letting x and y be 0 seems simple; this makes $z = 1.5$. Thus we let $P = \langle 0, 0, 1.5\rangle$, and $\overrightarrow{PQ} = \langle 2, 1, 2.5\rangle$.

The distance h from Q to the plane is given by Key Idea 10:

$$\begin{aligned} h &= \frac{|\vec{n} \cdot \overrightarrow{PQ}|}{||\,\vec{n}\,||} \\ &= \frac{|\,\langle 2, -5, 6\rangle \cdot \langle 2, 1, 2.5\rangle\,|}{||\;\langle 2, -5, 6\rangle\;||} \\ &= \frac{|14|}{\sqrt{65}} \\ &\approx 1.74. \end{aligned}$$

We can use Key Idea 10 to find other distances. Given two parallel planes, we can find the distance between these planes by letting P be a point on one plane and Q a point on the other. If ℓ is a line parallel to a plane, we can use the Key Idea to find the distance between them as well: again, let P be a point in the plane and let Q be any point on the line. (One can also use Key Idea 9.) The Exercise section contains problems of these types.

These past two sections have not explored lines and planes in space as an exercise of mathematical curiosity. However, there are many, many applications of these fundamental concepts. Complex shapes can be modeled (or, *approximated*) using planes. For instance, part of the exterior of an aircraft may have a complex, yet smooth, shape, and engineers will want to know how air flows across this piece as well as how heat might build up due to air friction. Many equations that help determine air flow and heat dissipation are difficult to apply to arbitrary surfaces, but simple to apply to planes. By approximating a surface with millions of small planes one can more readily model the needed behavior.

Notes:

Exercises 1.5

Terms and Concepts

10 06 ex 01 1. In order to find the equation of a plane, what two pieces of information must one have?

10 06 ex 02 2. What is the relationship between a plane and one of its normal vectors?

Problems

10 06 exset 01 **In Exercises 3 – 6, give any two points in the given plane.**

10 06 ex 03 3. $2x - 4y + 7z = 2$

10 06 ex 04 4. $3(x + 2) + 5(y - 9) - 4z = 0$

10 06 ex 05 5. $x = 2$

10 06 ex 06 6. $4(y + 2) - (z - 6) = 0$

10 06 exset 02 **In Exercises 7 – 20, give the equation of the described plane in standard and general forms.**

10 06 ex 07 7. Passes through $(2, 3, 4)$ and has normal vector $\vec{n} = \langle 3, -1, 7\rangle$.

10 06 ex 08 8. Passes through $(1, 3, 5)$ and has normal vector $\vec{n} = \langle 0, 2, 4\rangle$.

10 06 ex 09 9. Passes through the points $(1, 2, 3)$, $(3, -1, 4)$ and $(1, 0, 1)$.

10 06 ex 10 10. Passes through the points $(5, 3, 8)$, $(6, 4, 9)$ and $(3, 3, 3)$.

10 06 ex 11 11. Contains the intersecting lines $\ell_1(t) = \langle 2, 1, 2\rangle + t\,\langle 1, 2, 3\rangle$ and $\ell_2(t) = \langle 2, 1, 2\rangle + t\,\langle 2, 5, 4\rangle$.

10 06 ex 12 12. Contains the intersecting lines $\ell_1(t) = \langle 5, 0, 3\rangle + t\,\langle -1, 1, 1\rangle$ and $\ell_2(t) = \langle 1, 4, 7\rangle + t\,\langle 3, 0, -3\rangle$.

10 06 ex 13 13. Contains the parallel lines $\ell_1(t) = \langle 1, 1, 1\rangle + t\,\langle 1, 2, 3\rangle$ and $\ell_2(t) = \langle 1, 1, 2\rangle + t\,\langle 1, 2, 3\rangle$.

10 06 ex 14 14. Contains the parallel lines $\ell_1(t) = \langle 1, 1, 1\rangle + t\,\langle 4, 1, 3\rangle$ and $\ell_2(t) = \langle 2, 2, 2\rangle + t\,\langle 4, 1, 3\rangle$.

10 06 ex 15 15. Contains the point $(2, -6, 1)$ and the line
$$\ell(t) = \begin{cases} x = 2 + 5t \\ y = 2 + 2t \\ z = -1 + 2t \end{cases}$$

10 06 ex 16 16. Contains the point $(5, 7, 3)$ and the line
$$\ell(t) = \begin{cases} x = t \\ y = t \\ z = t \end{cases}$$

10 06 ex 17 17. Contains the point $(5, 7, 3)$ and is orthogonal to the line $\ell(t) = \langle 4, 5, 6\rangle + t\,\langle 1, 1, 1\rangle$.

10 06 ex 18 18. Contains the point $(4, 1, 1)$ and is orthogonal to the line
$$\ell(t) = \begin{cases} x = 4 + 4t \\ y = 1 + 1t \\ z = 1 + 1t \end{cases}$$

10 06 ex 19 19. Contains the point $(-4, 7, 2)$ and is parallel to the plane $3(x - 2) + 8(y + 1) - 10z = 0$.

10 06 ex 20 20. Contains the point $(1, 2, 3)$ and is parallel to the plane $x = 5$.

10 06 exset 03 **In Exercises 21 – 22, give the equation of the line that is the intersection of the given planes.**

10 06 ex 21 21. $p1 :\ 3(x - 2) + (y - 1) + 4z = 0$, and $p2 :\ 2(x - 1) - 2(y + 3) + 6(z - 1) = 0$.

10 06 ex 22 22. $p1 :\ 5(x - 5) + 2(y + 2) + 4(z - 1) = 0$, and $p2 :\ 3x - 4(y - 1) + 2(z - 1) = 0$.

10 06 exset 05 **In Exercises 23 – 26, find the point of intersection between the line and the plane.**

10 06 ex 30 23. line: $\langle 5, 1, -1\rangle + t\,\langle 2, 2, 1\rangle$, plane: $5x - y - z = -3$

10 06 ex 31 24. line: $\langle 4, 1, 0\rangle + t\,\langle 1, 0, -1\rangle$, plane: $3x + y - 2z = 8$

10 06 ex 28 25. line: $\langle 1, 2, 3\rangle + t\,\langle 3, 5, -1\rangle$, plane: $3x - 2y - z = 4$

10 06 ex 29 26. line: $\langle 1, 2, 3\rangle + t\,\langle 3, 5, -1\rangle$, plane: $3x - 2y - z = -4$

10 06 exset 04 **In Exercises 27 – 30, find the given distances.**

10 06 ex 23 27. The distance from the point $(1, 2, 3)$ to the plane $3(x - 1) + (y - 2) + 5(z - 2) = 0$.

10 06 ex 24 28. The distance from the point $(2, 6, 2)$ to the plane $2(x - 1) - y + 4(z + 1) = 0$.

10 06 ex 25 29. The distance between the parallel planes $x + y + z = 0$ and $(x - 2) + (y - 3) + (z + 4) = 0$

10 06 ex 26 30. The distance between the parallel planes
$2(x-1)+2(y+1)+(z-2)=0$ and
$2(x-3)+2(y-1)+(z-3)=0$

10 06 ex 27 31. Show why if the point Q lies in a plane, then the distance formula correctly gives the distance from the point to the plane as 0.

10 06 ex 32 32. How is Exercise 30 in Section 1.4 easier to answer once we have an understanding of planes?

A: Solutions To Selected Problems

Chapter 1

Section 1.1

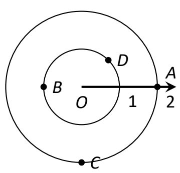
09 04 ex 01 1. Answers will vary.

09 04 ex 02 2. F

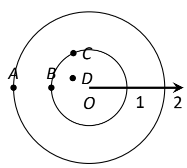
09 04 ex 03 3. T

09 04 ex 04 4. F

09 04 ex 05 5.



09 04 ex 06 6.



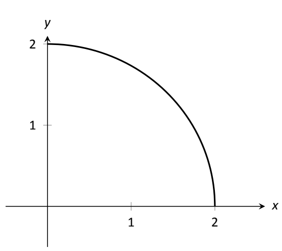
09 04 ex 07 7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
$B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
$C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
$D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

09 04 ex 08 8. $A = P(2, \pi/6)$ and $P(-2, -5\pi/6)$;
$B = P(1, -\pi/3)$ and $P(-1, 2\pi/3)$;
$C = P(2, 3\pi/4)$ and $P(-2, -\pi/4)$;
$D = P(2.5, \pi)$ and $P(2.5, -\pi)$;

09 04 ex 09 9. $A = (\sqrt{2}, \sqrt{2})$
$B = (\sqrt{2}, -\sqrt{2})$
$C = P(\sqrt{5}, -0.46)$
$D = P(\sqrt{5}, 2.68)$

09 04 ex 10 10. $A = (-3, 0)$
$B = (-1/2, \sqrt{3}/2)$
$C = P(4, \pi/2)$
$D = P(2, -\pi/3)$

09 04 ex 11 11.



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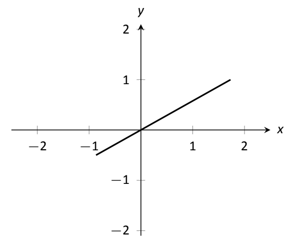

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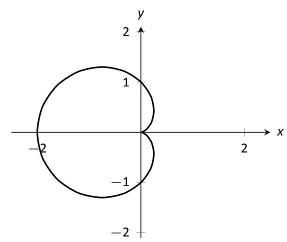

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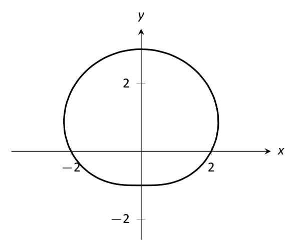

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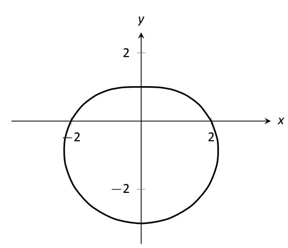

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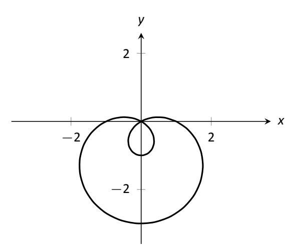

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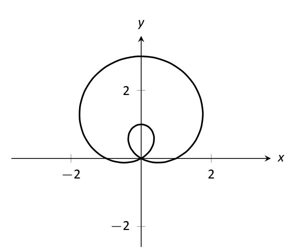

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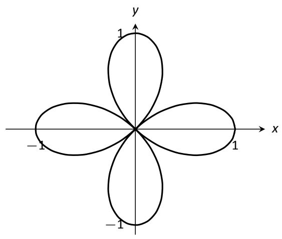

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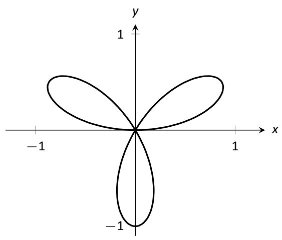
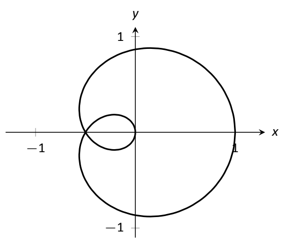
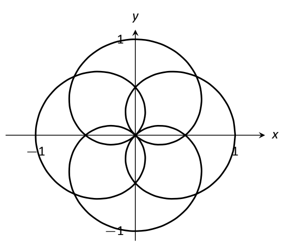


09 04 ex 20 20.



09 04 ex 21 21.



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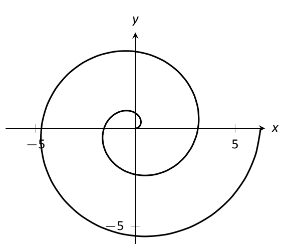

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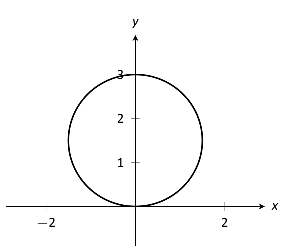
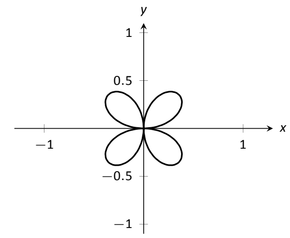


09 04 ex 24 24.



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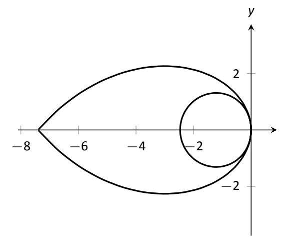
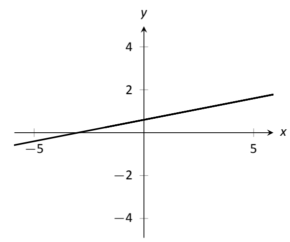
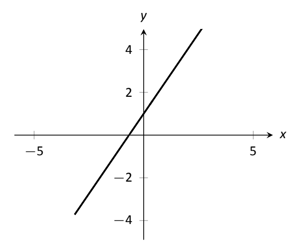


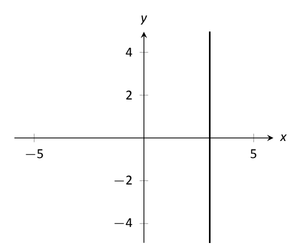
09 04 ex 26 26.



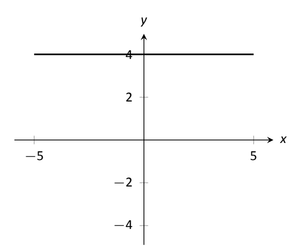
09 04 ex 27 27.



09 04 ex 28 28.



09 04 ex 29 29.



09 04 ex 30 30. $(x-1)^2+y^2=1$

09 04 ex 31 31. $x^2+(y+2)^2=4$

09 04 ex 32 32. $(x-1/2)^2+(y-1/2)^2=1/2$

09 04 ex 33 33. $y = 2/5x + 7/5$

09 04 ex 34 34. $x = 3$

09 04 ex 35 35. $y = 4$

09 04 ex 36 36. $x^4 + x^2y^2x^2 - y^2 = 0$

09 04 ex 37 37. $x^2 + y^2 = 4$

09 04 ex 38 38. $y = x/\sqrt{3}$

09 04 ex 39 39. $\theta = \pi/4$

09 04 ex 40 40. $r = 7/(\sin\theta - 4\cos\theta)$

09 04 ex 41 41. $r = 5\sec\theta$

09 04 ex 42 42. $r = 5\csc\theta$

09 04 ex 43 43. $r = \cos\theta/\sin^2\theta$

09 04 ex 44 44. $r = 1/\sqrt[3]{\cos^2\theta\sin\theta}$

09 04 ex 45 45. $r = \sqrt{7}$

09 04 ex 46 46. $r = -2\cos\theta$

09 04 ex 47 47. $P(\sqrt{3}/2, \pi/6)$, $P(0, \pi/2)$, $P(-\sqrt{3}/2, 5\pi/6)$

09 04 ex 48 48. $P(1, 0)$, $P(0, \pi/2) = P(0, \pi/4)$, $P(-1/2, \pi/3)$

09 04 ex 49 49. $P(0, 0) = P(0, \pi/2)$, $P(\sqrt{2}, \pi/4)$

09 04 ex 50 50. $P(\sqrt{3}/2, \pi/3) = P(-\sqrt{3}/2, 4\pi/3)$, $P(\sqrt{3}/2, 2\pi/3) = P(-\sqrt{3}/2, 5\pi/3)$, $P(0, \pi/2)$

09 04 ex 51 51. $P(\sqrt{2}/2, \pi/12)$, $P(-\sqrt{2}/2, 5\pi/12)$, $P(\sqrt{2}/2, 3\pi/4)$

09 04 ex 52 52. $P(3/2, \pi/3)$, $P(3/2, -\pi/3)$

09 04 ex 53 53. For all points, $r = 1$; $\theta = \pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12,\ 13\pi/12,\ 17\pi/12,\ 19\pi/12,\ 23\pi/12.$

09 04 ex 54 54. $P(0, 0) = P(0, 3\pi/2)$, $P(1 + \sqrt{2}/2, 3\pi/4)$, $P(1 - \sqrt{2}/2, 7\pi/4)$

09 04 ex 55 55. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

09 04 ex 56 56. Answers will vary.

Section 1.2

09 05 ex 01 1. Using $x = r\cos\theta$ and $y = r\sin\theta$, we can write $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

09 05 ex 02 2. rectangles; sectors of circles

09 05 ex 03 3.

(a) $\frac{dy}{dx} = -\cot\theta$

(b) tangent line: $y = -(x - \sqrt{2}/2) + \sqrt{2}/2$; normal line: $y = x$

09 05 ex 04 4.

(a) $\frac{dy}{dx} = 1/2(\tan\theta - \cot\theta)$

(b) tangent line: $y = 1/2$; normal line: $x = 1/2$

09 05 ex 05 5.

(a) $\frac{dy}{dx} = \frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta - \sin\theta(1+\sin\theta)}$

(b) tangent line: $x = 3\sqrt{3}/4$; normal line: $y = 3/4$

09 05 ex 10 6.

(a) $\frac{dy}{dx} = \frac{3\sin^2(t) + (1-3\cos(t))\cos(t)}{3\sin(t)\cos(t) - \sin(t)(1-3\cos(t))}$

(b) tangent line: $y = \frac{1}{1+3\sqrt{2}}(x + (1/\sqrt{2} + 3/2)) + 1/\sqrt{2} + 3/2 \approx$ $y = 0.19(x + 2.21) + 2.21$; normal line: $y = -(1 + 3\sqrt{2})(x + (1/\sqrt{2} + 3/2)) + 1/\sqrt{2} + 3/2$

09 05 ex 06 7.

(a) $\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{\cos\theta - \theta\sin\theta}$

(b) tangent line: $y = -2/\pi x + \pi/2$; normal line: $y = \pi/2x + \pi/2$

09 05 ex 07 8.

(a) $\frac{dy}{dx} = \frac{\cos\theta\cos(3\theta) - 3\sin\theta\sin(3\theta)}{-\cos(3\theta)\sin\theta - 3\cos\theta\sin(3\theta)}$

(b) tangent line: $y = x/\sqrt{3}$; normal line: $y = -\sqrt{3}x$

09 05 ex 08 9.

(a) $\frac{dy}{dx} = \frac{4\sin(t)\cos(4t) + \sin(4t)\cos(t)}{4\cos(t)\cos(4t) - \sin(t)\sin(4t)}$

(b) tangent line: $y = 5\sqrt{3}(x + \sqrt{3}/4) - 3/4$; normal line: $y = -1/5\sqrt{3}(x + \sqrt{3}/4) - 3/4$

09 05 ex 09 10.

(a) $\frac{dy}{dx} = 1$

(b) tangent line: $y = x + 1$; normal line: $y = -x - 1$

09 05 ex 11 11. horizontal: $\theta = \pi/2, 3\pi/2$;
vertical: $\theta = 0, \pi, 2\pi$

09 05 ex 12 12. horizontal: $\theta = 0, \pi/2, \pi$;
vertical: $\theta = \pi/4, 3\pi/4$

09 05 ex 13 13. horizontal:
$\theta = \tan^{-1}(1/\sqrt{5}),\ \pi/2,\ \pi - \tan^{-1}(1/\sqrt{5}),\ \pi + \tan^{-1}(1/\sqrt{5}),\ 3\pi/2,\ 2\pi - \tan^{-1}(1/\sqrt{5})$;
vertical: $\theta = 0,\ \tan^{-1}(\sqrt{5}),\ \pi - \tan^{-1}(\sqrt{5}),\ \pi,\ \pi + \tan^{-1}(\sqrt{5}),\ 2\pi - \tan^{-1}(\sqrt{5})$

09 05 ex 14 14. horizontal: $\theta = \pi/3,\ 5\pi/3$;
vertical: $\theta = 0,\ 2\pi/3,\ 4\pi/3,\ 2\pi$
At $\theta = \pi$, $\frac{dy}{dx} = 0/0$; apply L'Hopital's Rule to find that $\frac{dy}{dx} \to 0$ as $\theta \to \pi$.

09 05 ex 15 15. In polar: $\theta = 0 \cong \theta = \pi$
In rectangular: $y = 0$

09 05 ex 16 16. In polar: $\theta = 0,\ \theta = \pi/3,\ \theta = 2\pi/3$.
In rectangular: $y = 0$, $y = \sqrt{3}x$, and $y = -\sqrt{3}x$.

09 05 ex 17 17. area = 4π

09 05 ex 20 18. area = 25π

09 05 ex 18 19. area = $\pi/12$

09 05 ex 19 20. area = $3\pi/2$

09 05 ex 21 21. area = $\pi - 3\sqrt{3}/2$

09 05 ex 22 22. area = $2\pi + 3\sqrt{3}/2$

09 05 ex 23 23. area = $\pi + 3\sqrt{3}$

09 05 ex 25 24. area = 1

09 05 ex 26 25. area =
$$\int_{\pi/12}^{\pi/3} \frac{1}{2}\sin^2(3\theta)\,d\theta - \int_{\pi/12}^{\pi/6} \frac{1}{2}\cos^2(3\theta)\,d\theta = \frac{1}{12} + \frac{\pi}{24}$$

09 05 ex 24 26. area = $\frac{1}{32}(4\pi - 3\sqrt{3})$

09 05 ex 27 27. area = $\displaystyle\int_0^{\pi/3} \frac{1}{2}(1-\cos\theta)^2\,d\theta + \int_{\pi/3}^{\pi/2} \frac{1}{2}(\cos\theta)^2\,d\theta = \frac{7\pi}{24} - \frac{\sqrt{3}}{2} \approx 0.0503$

09 05 ex 28 28. $x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta$, $y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta$. Square each and add; applying the Pythagoran Theorem twice achieves the result.

09 05 ex 29 29. 4π

09 05 ex 30 30. 4π

09 05 ex 31 31. $L \approx 2.2592$; (actual value $L = 2.22748$)

09 05 ex 32 32. $L \approx 7.62933$; (actual value $L = 8$)

09 05 ex 33 33. $SA = 16\pi$

09 05 ex 34 34. $SA = 4\pi$

09 05 ex 35 35. $SA = 32\pi/5$

09 05 ex 36 36. $SA = 4\pi^2$

09 05 ex 37 37. $SA = 36\pi$

Section 1.3

10 01 ex 08 1. right hand

10 01 ex 01 2. line; plane

10 01 ex 02 3. curve (a parabola); surface (a cylinder)

10 01 ex 03 4. a hyperbolic paraboloid

10 01 ex 04 5. a hyperboloid of two sheets

10 01 ex 05 6. a hyperboloid of one sheet

10 01 ex 06 7. $||\,\overline{AB}\,|| = \sqrt{6}$; $||\,\overline{BC}\,|| = \sqrt{17}$; $||\,\overline{AC}\,|| = \sqrt{11}$. Yes, it is a right triangle as $||\,\overline{AB}\,||^2 + ||\,\overline{AC}\,||^2 = ||\,\overline{BC}\,||^2$.

10 01 ex 07 8. Yes, as opposite sides have equal length. $||\,\overline{AB}\,|| = \sqrt{21} = ||\,\overline{CD}\,||$; $||\,\overline{BC}\,|| = \sqrt{6} = ||\,\overline{AD}\,||$.

10 01 ex 09 9. Center at $(4, -1, 0)$; radius = 3

10 01 ex 10 10. Center at $(-2, 1, 2)$; radius = $\sqrt{5}$

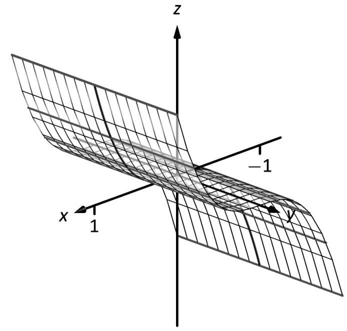
10 01 ex 11 11. Interior of a sphere with radius 1 centered at the origin.

10 01 ex 12 12. Region bounded between the planes $x = 0$ (the $y - z$ coordinate plane) and $x = 3$.

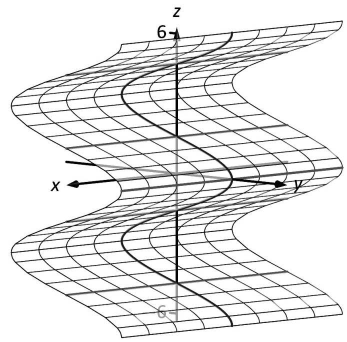
10 01 ex 13 13. The first octant of space; all points (x, y, z) where each of x, y and z are positive. (Analogous to the first quadrant in the plane.)

10 01 ex 14 14. All points in space where the y value is greater than 3; viewing space as often depicted in this text, this is the region "to the right" of the plane $y = 3$ (which is parallel to the $x - z$ coordinate plane.)

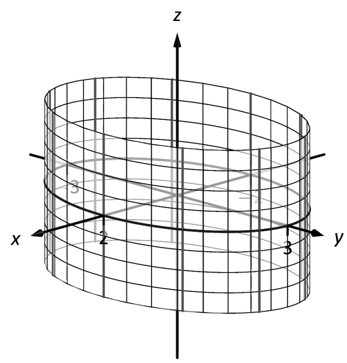
10 01 ex 15 15.



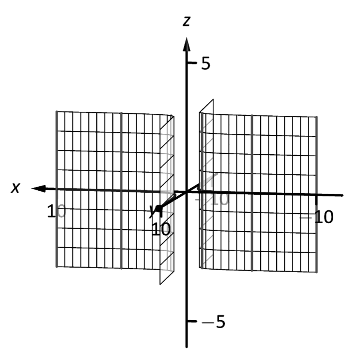
10 01 ex 16 16.



10 01 ex 17 17.



10 01 ex 18 18.



10 01 ex 29 19. $x^2 + z^2 = \frac{1}{(1+y^2)^2}$

10 01 ex 30 20. $y^2 + z^2 = x^4$

10 01 ex 31 21. $z = (\sqrt{x^2+y^2})^2 = x^2 + y^2$

10 01 ex 32 22. $z = \frac{1}{\sqrt{x^2+y^2}}$

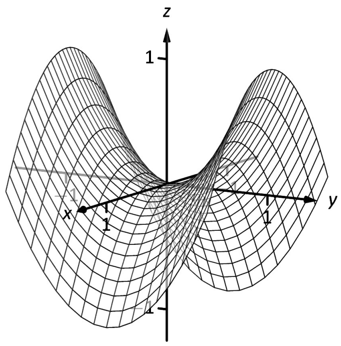
10 01 ex 19 23. (a) $x = y^2 + \dfrac{z^2}{9}$

10 01 ex 20 24. (b) $x^2 - y^2 + z^2 = 0$

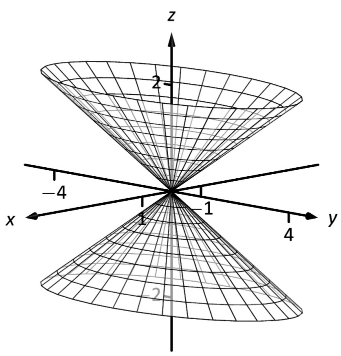
10 01 ex 21 25. (b) $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$

10 01 ex 22 26. (a) $y^2 - x^2 - z^2 = 1$

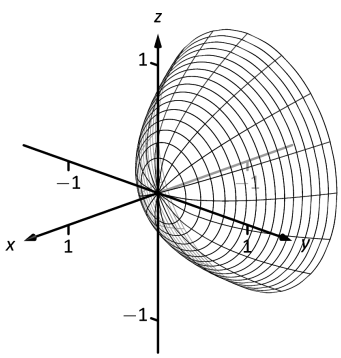
10 01 ex 28 27.



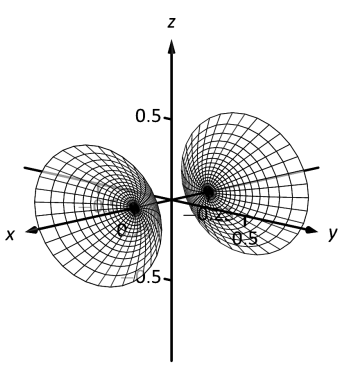
10 01 ex 24 28.



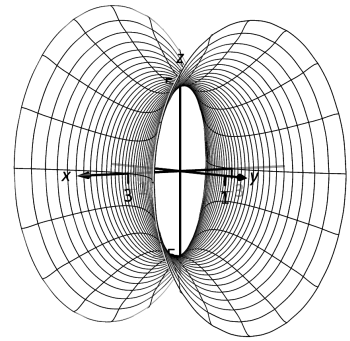
10 01 ex 23 29.



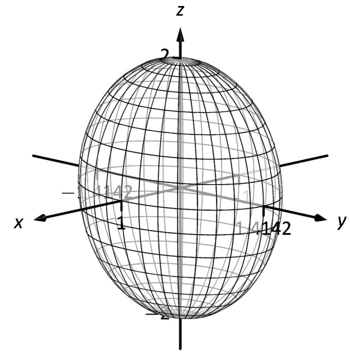
10 01 ex 27 30.



10 01 ex 26 31.



10 01 ex 25 32.



Section 1.4

10 05 ex 01 1. A point on the line and the direction of the line.

10 05 ex 02 2. parallel

10 05 ex 03 3. parallel, skew

10 05 ex 04 4. Answers will vary

10 05 ex 05 5. vector: $\ell(t) = \langle 2, -4, 1\rangle + t\,\langle 9, 2, 5\rangle$
parametric: $x = 2 + 9t$, $y = -4 + 2t$, $z = 1 + 5t$
symmetric: $(x - 2)/9 = (y + 4)/2 = (z - 1)/5$

10 05 ex 06 6. vector: $\ell(t) = \langle 6, 1, 7\rangle + t\,\langle -3, 2, 5\rangle$
parametric: $x = 6 - 3t$, $y = 1 + 2t$, $z = 7 + 5t$
symmetric: $-(x - 6)/3 = (y - 1)/2 = (z - 7)/5$

10 05 ex 07 7. Answers can vary: vector: $\ell(t) = \langle 2, 1, 5\rangle + t\,\langle 5, -3, -1\rangle$
parametric: $x = 2 + 5t$, $y = 1 - 3t$, $z = 5 - t$
symmetric: $(x - 2)/5 = -(y - 1)/3 = -(z - 5)$

10 05 ex 08 8. Answers can vary: vector: $\ell(t) = \langle 1, -2, 3\rangle + t\,\langle 4, 7, 2\rangle$
parametric: $x = 1 + 4t$, $y = -2 + 7t$, $z = 3 + 2t$
symmetric: $(x - 1)/4 = (y + 2)/7 = (z - 3)/2$

10 05 ex 09 9. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 0, 1, 2\rangle + t\,\langle -10, 43, 9\rangle$
parametric: $x = -10t$, $y = 1 + 43t$, $z = 2 + 9t$
symmetric: $-x/10 = (y - 1)/43 = (z - 2)/9$

10 05 ex 10 10. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 5, 1, 9\rangle + t\,\langle 0, -1, 0\rangle$
parametric: $x = 5$, $y = 1 - t$, $z = 9$
symmetric: not defined, as some components of the direction are 0.

10 05 ex 11 11. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 7, 2, -1\rangle + t\,\langle 1, -1, 2\rangle$
parametric: $x = 7 + t$, $y = 2 - t$, $z = -1 + 2t$
symmetric: $x - 7 = 2 - y = (z + 1)/2$

10 05 ex 12 12. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 2,2,3\rangle + t\langle 5,-1,-3\rangle$
parametric: $x = 2+5t$, $y = 2-t$, $z = 3-3t$
symmetric: $(x-2)/5 = -(y-2) = -(z-3)/3$

10 05 ex 13 13. vector: $\ell(t) = \langle 1,1\rangle + t\langle 2,3\rangle$
parametric: $x = 1+2t$, $y = 1+3t$
symmetric: $(x-1)/2 = (y-1)/3$

10 05 ex 14 14. vector: $\ell(t) = \langle -2,5\rangle + t\langle 0,1\rangle$
parametric: $x = -2$, $y = 5+t$
symmetric: not defined

10 05 ex 15 15. parallel

10 05 ex 16 16. intersecting; $\ell_1(2) = \ell_2(-2) = \langle 12,3,7\rangle$

10 05 ex 17 17. intersecting; $\vec{\ell}_1(3) = \vec{\ell}_2(4) = \langle 9,-5,13\rangle$

10 05 ex 18 18. same

10 05 ex 19 19. skew

10 05 ex 20 20. parallel

10 05 ex 21 21. same

10 05 ex 22 22. skew

10 05 ex 23 23. $\sqrt{41}/3$

10 05 ex 24 24. $3\sqrt{2}$

10 05 ex 25 25. $5\sqrt{2}/2$

10 05 ex 26 26. 5

10 05 ex 27 27. $3/\sqrt{2}$

10 05 ex 28 28. 2

10 05 ex 29 29. Since both P and Q are on the line, $\overrightarrow{PQ}$ is parallel to $\vec{d}$. Thus $\overrightarrow{PQ} \times \vec{d} = \vec{0}$, giving a distance of 0.

10 05 ex 30 30. (Note: this solution is easier once one has studied Section 1.5.) Since the two lines intersect, we can state $P_2 = P_1 + a\vec{d}_1 + b\vec{d}_2$ for some scalars a and b. (Here we abuse notation slightly and add points to vectors.) Thus $\overrightarrow{P_1P_2} = a\vec{d}_1 + b\vec{d}_2$. Vector $\vec{c}$ is the cross product of $\vec{d}_1$ and $\vec{d}_2$, hence is orthogonal to both, and hence is orthogonal to $\overrightarrow{P_1P_2}$. Thus $\overrightarrow{P_1P_2} \cdot \vec{c} = 0$, and the distance between lines is 0.

10 05 ex 31 31.

(a) The distance formula cannot be used because since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\vec{c}$ is $\vec{0}$ and we cannot divide by $\|\,\vec{0}\,\|$.

(b) Since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\overrightarrow{P_1P_2}$ lies in the plane formed by the two lines. Thus $\overrightarrow{P_1P_2} \times \vec{d}_2$ is orthogonal to this plane, and $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ is parallel to the plane, but still orthogonal to both $\vec{d}_1$ and $\vec{d}_2$. We desire the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$, which is what the formula provides.

(c) Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can use a measuring tape anywhere along the track, not just at one specific place.) Let $P = P_1$ and $Q = P_2$ as given by the equations of the lines, and apply the formula for distance between a point and a line.

Section 1.5

10 06 ex 01 1. A point in the plane and a normal vector (i.e., a direction orthogonal to the plane).

10 06 ex 02 2. A normal vector is orthogonal to the plane.

10 06 ex 03 3. Answers will vary.

10 06 ex 04 4. Answers will vary.

10 06 ex 05 5. Answers will vary.

10 06 ex 06 6. Answers will vary.

10 06 ex 07 7. Standard form: $3(x-2) - (y-3) + 7(z-4) = 0$
general form: $3x - y + 7z = 31$

10 06 ex 08 8. Standard form: $2(y-3) + 4(z-5) = 0$
general form: $2y + 4z = 26$

10 06 ex 09 9. Answers may vary;
Standard form: $8(x-1) + 4(y-2) - 4(z-3) = 0$
general form: $8x + 4y - 4z = 4$

10 06 ex 10 10. Answers may vary;
Standard form: $-5(x-5) + 3(y-3) + 2(z-8) = 0$
general form: $-5x + 3y + 2z = 0$

10 06 ex 11 11. Answers may vary;
Standard form: $-7(x-2) + 2(y-1) + (z-2) = 0$
general form: $-7x + 2y + z = -10$

10 06 ex 12 12. Answers may vary;
Standard form: $3(x-5) + 3(z-3) = 0$
general form: $3x + 3z = 24$

10 06 ex 13 13. Answers may vary;
Standard form: $2(x-1) - (y-1) = 0$
general form: $2x - y = 1$

10 06 ex 14 14. Answers may vary;
Standard form: $2(x-1) + (y-1) - 3(z-1) = 0$
general form: $2x + y - 3z = 0$

10 06 ex 15 15. Answers may vary;
Standard form: $2(x-2) - (y+6) - 4(z-1) = 0$
general form: $2x - y - 4z = 6$

10 06 ex 16 16. Answers may vary;
Standard form: $4(x-5) - 2(y-7) - 2(z-3) = 0$
general form: $4x - 2y - 2z = 0$

10 06 ex 17 17. Answers may vary;
Standard form: $(x-5) + (y-7) + (z-3) = 0$
general form: $x + y + z = 15$

10 06 ex 18 18. Answers may vary;
Standard form: $4(x-4) + (y-1) + (z-1) = 0$
general form: $4x + y + z = 18$

10 06 ex 19 19. Answers may vary;
Standard form: $3(x+4) + 8(y-7) - 10(z-2) = 0$
general form: $3x + 8y - 10z = 24$

10 06 ex 20 20. Standard form: $x - 1 = 0$
general form: $x = 1$

10 06 ex 21 21. Answers may vary:
$$\ell = \begin{cases} x = 14t \\ y = -1 - 10t \\ z = 2 - 8t \end{cases}$$

10 06 ex 22 22. Answers may vary:

$$\ell = \begin{cases} x = 1 + 20t \\ y = 3 + 2t \\ z = 3.5 - 26t \end{cases}$$

10 06 ex 30 23. $(-3, -7, -5)$

10 06 ex 31 24. $(3, 1, 1)$

10 06 ex 28 25. No point of intersection; the plane and line are parallel.

10 06 ex 29 26. The plane contains the line, so every point on the line is a "point of intersection."

10 06 ex 23 27. $\sqrt{5/7}$

10 06 ex 24 28. $8/\sqrt{21}$

10 06 ex 25 29. $1/\sqrt{3}$

10 06 ex 26 30. 3

10 06 ex 27 31. If P is any point in the plane, and Q is also in the plane, then $\overrightarrow{PQ}$ lies parallel to the plane and is orthogonal to $\vec{n}$, the normal vector. Thus $\vec{n} \cdot \overrightarrow{PQ} = 0$, giving the distance as 0.

10 06 ex 32 32. The intersecting lines define a plane with normal vector $\vec{n} = \vec{c} = \vec{d}_1 \times \vec{d}_2$. Since points P_1 and P_2 lie in the plane, $\vec{c}$ is orthogonal to $\overrightarrow{P_1P_2}$, hence $\overrightarrow{P_1P_2} \cdot \vec{c} = 0$, giving a distance of 0. Knowing the principles of planes, especially their normal vectors, makes this simpler.