1: Multiple Integration

The previous chapter introduced multivariable functions and we applied concepts of differential calculus to these functions. We learned how we can view a function of two variables as a surface in space, and learned how partial derivatives convey information about how the surface is changing in any direction.

In this chapter we apply techniques of integral calculus to multivariable functions. In Chapter 5 we learned how the definite integral of a single variable function gave us "area under the curve." In this chapter we will see that integration applied to a multivariable function gives us "volume under a surface." And just as we learned applications of integration beyond finding areas, we will find applications of integration in this chapter beyond finding volume.

1.1 Iterated Integrals and Area

In Chapter 12 we found that it was useful to differentiate functions of several variables with respect to one variable, while treating all the other variables as constants or coefficients. We can integrate functions of several variables in a similar way. For instance, if we are told that $f_x(x,y) = 2xy$, we can treat y as staying constant and integrate to obtain $f(x,y)$:

$$\begin{aligned} f(x,y) &= \int f_x(x,y)\,dx \\ &= \int 2xy\,dx \\ &= x^2y + C. \end{aligned}$$

Make a careful note about the constant of integration, C. This "constant" is something with a derivative of 0 with respect to x, so it could be any expression that contains only constants and functions of y. For instance, if $f(x,y) = x^2y + \sin y + y^3 + 17$, then $f_x(x,y) = 2xy$. To signify that C is actually a function of y, we write:

$$f(x,y) = \int f_x(x,y)\,dx = x^2y + C(y).$$

Using this process we can even evaluate definite integrals.

Example 1.1.1 Integrating functions of more than one variable

Evaluate the integral $\displaystyle\int_1^{2y} 2xy\,dx$.

Solution We find the indefinite integral as before, then apply the Fundamental Theorem of Calculus to evaluate the definite integral:

$$\begin{aligned} \int_1^{2y} 2xy\,dx &= x^2y\Big|_1^{2y} \\ &= (2y)^2y - (1)^2y \\ &= 4y^3 - y. \end{aligned}$$

We can also integrate with respect to y. In general,

$$\int_{h_1(y)}^{h_2(y)} f_x(x,y)\,dx = f(x,y)\Big|_{h_1(y)}^{h_2(y)} = f\big(h_2(y),y\big) - f\big(h_1(y),y\big),$$

and

$$\int_{g_1(x)}^{g_2(x)} f_y(x,y)\,dy = f(x,y)\Big|_{g_1(x)}^{g_2(x)} = f\big(x,g_2(x)\big) - f\big(x,g_1(x)\big).$$

Note that when integrating with respect to x, the bounds are functions of y (of the form $x = h_1(y)$ and $x = h_2(y)$) and the final result is also a function of y. When integrating with respect to y, the bounds are functions of x (of the form $y = g_1(x)$ and $y = g_2(x)$) and the final result is a function of x. Another example will help us understand this.

Example 1.1.2 Integrating functions of more than one variable

Evaluate $\displaystyle\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy$.

Solution We consider x as staying constant and integrate with respect to y:

$$\begin{aligned}\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy &= \left(\frac{5x^3y^{-2}}{-2} + \frac{6y^3}{3}\right)\Bigg|_1^x \\ &= \left(-\frac{5}{2}x^3x^{-2} + 2x^3\right) - \left(-\frac{5}{2}x^3 + 2\right) \\ &= \frac{9}{2}x^3 - \frac{5}{2}x - 2.\end{aligned}$$

Note how the bounds of the integral are from $y = 1$ to $y = x$ and that the final answer is a function of x.

In the previous example, we integrated a function with respect to y and ended up with a function of x. We can integrate this as well. This process is known as **iterated integration**, or **multiple integration.**

Example 1.1.3 Integrating an integral

Evaluate $\displaystyle\int_1^2 \left(\int_1^x \left(5x^3y^{-3} + 6y^2\right)\,dy\right)\,dx$.

Solution We follow a standard "order of operations" and perform the operations inside parentheses first (which is the integral evaluated in Example

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1.1.2.)

$$\begin{aligned}\int_1^2 \left(\int_1^x \left(5x^3y^{-3}+6y^2\right)\,dy\right)\,dx &= \int_1^2 \left(\left[\frac{5x^3y^{-2}}{-2}+\frac{6y^3}{3}\right]\Bigg|_1^x\right)\,dx\\ &= \int_1^2 \left(\frac{9}{2}x^3-\frac{5}{2}x-2\right)\,dx\\ &= \left(\frac{9}{8}x^4-\frac{5}{4}x^2-2x\right)\Bigg|_1^2\\ &= \frac{89}{8}.\end{aligned}$$

Note how the bounds of x were $x=1$ to $x=2$ and the final result was a number.

The previous example showed how we could perform something called an iterated integral; we do not yet know *why* we would be interested in doing so nor what the result, such as the number 89/8, *means*. Before we investigate these questions, we offer some definitions.

Definition 1.1.1 Iterated Integration

Iterated integration is the process of repeatedly integrating the results of previous integrations. Integrating one integral is denoted as follows.

Let a, b, c and d be numbers and let $g_1(x)$, $g_2(x)$, $h_1(y)$ and $h_2(y)$ be functions of x and y, respectively. Then:

1. $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right)\,dy.$
2. $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right)\,dx.$

Again make note of the bounds of these iterated integrals.

With $\displaystyle\int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$, x varies from $h_1(y)$ to $h_2(y)$, whereas y varies from c to d. That is, the bounds of x are *curves*, the curves $x=h_1(y)$ and $x=h_2(y)$, whereas the bounds of y are *constants*, $y=c$ and $y=d$. It is useful to remember that when setting up and evaluating such iterated integrals, we integrate "from

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curve to curve, then from point to point."

We now begin to investigate *why* we are interested in iterated integrals and *what* they mean.

Area of a plane region

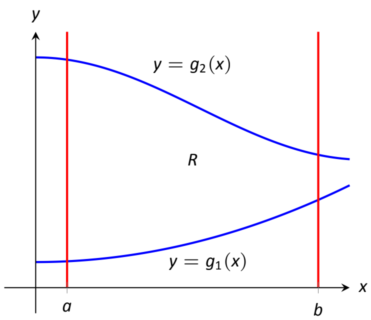

Figure 1.1.1: Calculating the area of a plane region R with an iterated integral.

Consider the plane region R bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, shown in Figure 1.1.1. We learned in Section 7.1 that the area of R is given by

$$\int_a^b \big(g_2(x) - g_1(x)\big)\,dx.$$

We can view the expression $\big(g_2(x) - g_1(x)\big)$ as

$$\big(g_2(x) - g_1(x)\big) = \int_{g_1(x)}^{g_2(x)} 1\,dy = \int_{g_1(x)}^{g_2(x)} \,dy,$$

meaning we can express the area of R as an iterated integral:

$$\text{area of } R = \int_a^b \big(g_2(x) - g_1(x)\big)\,dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} \,dy\right)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \,dy\,dx.$$

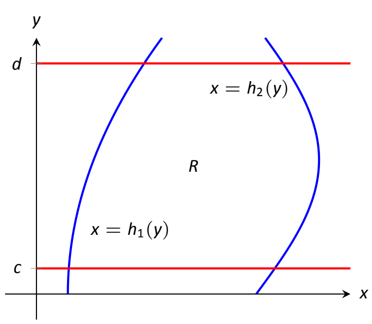

Figure 1.1.2: Calculating the area of a plane region R with an iterated integral.

In short: a certain iterated integral can be viewed as giving the area of a plane region.

A region R could also be defined by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, as shown in Figure 1.1.2. Using a process similar to that above, we have

$$\text{the area of } R = \int_c^d \int_{h_1(y)}^{h_2(y)} \,dx\,dy.$$

We state this formally in a theorem.

Notes:

Theorem 1.1.1 Area of a plane region

1. Let R be a plane region bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a,b]$. The area A of R is
$$A = \int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx.$$
2. Let R be a plane region bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c,d]$. The area A of R is
$$A = \int_c^d \int_{h_1(y)}^{h_2(y)} dx\,dy.$$

The following examples should help us understand this theorem.

Example 1.1.4 Area of a rectangle

Find the area A of the rectangle with corners $(-1,1)$ and $(3,3)$, as shown in Figure 1.1.3.

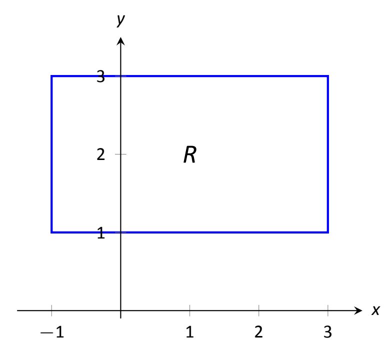

Figure 1.1.3: Calculating the area of a rectangle with an iterated integral in Example 1.1.4.

Solution Multiple integration is obviously overkill in this situation, but we proceed to establish its use.

The region R is bounded by $x=-1$, $x=3$, $y=1$ and $y=3$. Choosing to integrate with respect to y first, we have
$$A = \int_{-1}^{3}\int_{1}^{3} 1\,dy\,dx = \int_{-1}^{3}\left(y\Big|_1^3\right)\,dx = \int_{-1}^{3} 2\,dx = 2x\Big|_{-1}^{3} = 8.$$

We could also integrate with respect to x first, giving:
$$A = \int_{1}^{3}\int_{-1}^{3} 1\,dx\,dy = \int_{1}^{3}\left(x\Big|_{-1}^3\right)\,dy = \int_{1}^{3} 4\,dy = 4y\Big|_{1}^{3} = 8.$$

Clearly there are simpler ways to find this area, but it is interesting to note that this method works.

Example 1.1.5 Area of a triangle

Find the area A of the triangle with vertices at $(1,1)$, $(3,1)$ and $(5,5)$, as shown in Figure 1.1.4.

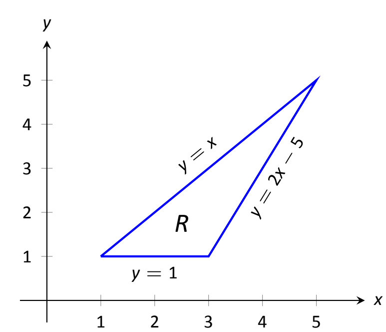

Figure 1.1.4: Calculating the area of a triangle with iterated integrals in Example 1.1.5.

Solution The triangle is bounded by the lines as shown in the figure. Choosing to integrate with respect to x first gives that x is bounded by $x = y$

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to $x = \frac{y+5}{2}$, while y is bounded by $y = 1$ to $y = 5$. (Recall that since x-values increase from left to right, the leftmost curve, $x = y$, is the lower bound and the rightmost curve, $x = (y+5)/2$, is the upper bound.) The area is

$$\begin{aligned} A &= \int_1^5 \int_y^{\frac{y+5}{2}} dx\, dy \\ &= \int_1^5 \left(x \Big|_y^{\frac{y+5}{2}} \right) dy \\ &= \int_1^5 \left(-\frac{1}{2}y + \frac{5}{2} \right) dy \\ &= \left(-\frac{1}{4}y^2 + \frac{5}{2}y \right) \Big|_1^5 \\ &= 4. \end{aligned}$$

We can also find the area by integrating with respect to y first. In this situation, though, we have two functions that act as the lower bound for the region R, $y = 1$ and $y = 2x - 5$. This requires us to use two iterated integrals. Note how the x-bounds are different for each integral:

$$\begin{aligned} A &= \int_1^3 \int_1^x 1\, dy\, dx && + && \int_3^5 \int_{2x-5}^x 1\, dy\, dx \\ &= \int_1^3 \big(y\big)\Big|_1^x dx && + && \int_3^5 \big(y\big)\Big|_{2x-5}^x dx \\ &= \int_1^3 \big(x-1\big)\, dx && + && \int_3^5 \big(-x+5\big)\, dx \\ &= 2 && + && 2 \\ &= 4. \end{aligned}$$

As expected, we get the same answer both ways.

Example 1.1.6 Area of a plane region

Find the area of the region enclosed by $y = 2x$ and $y = x^2$, as shown in Figure 1.1.5.

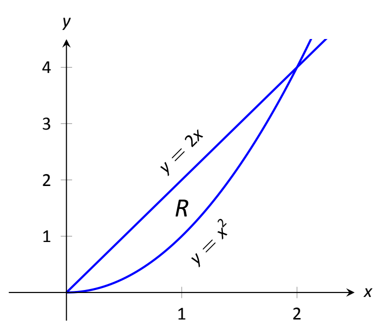

Figure 1.1.5: Calculating the area of a plane region with iterated integrals in Example 1.1.6.

Solution Once again we'll find the area of the region using both orders of integration.

Using $dy\, dx$:

$$\int_0^2 \int_{x^2}^{2x} 1\, dy\, dx = \int_0^2 (2x - x^2)\, dx = \big(x^2 - \frac{1}{3}x^3\big)\Big|_0^2 = \frac{4}{3}.$$

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Using $dx\,dy$:

$$\int_0^4\int_{y/2}^{\sqrt{y}} 1\,dx\,dy = \int_0^4 (\sqrt{y}-y/2)\,dy = \left(\frac{2}{3}y^{3/2}-\frac{1}{4}y^2\right)\Big|_0^4 = \frac{4}{3}.$$

Changing Order of Integration

In each of the previous examples, we have been given a region R and found the bounds needed to find the area of R using both orders of integration. We integrated using both orders of integration to demonstrate their equality.

We now approach the skill of describing a region using both orders of integration from a different perspective. Instead of starting with a region and creating iterated integrals, we will start with an iterated integral and rewrite it in the other integration order. To do so, we'll need to understand the region over which we are integrating.

The simplest of all cases is when both integrals are bound by constants. The region described by these bounds is a rectangle (see Example 1.1.4), and so:

$$\int_a^b\int_c^d 1\,dy\,dx = \int_c^d\int_a^b 1\,dx\,dy.$$

When the inner integral's bounds are not constants, it is generally very useful to sketch the bounds to determine what the region we are integrating over looks like. From the sketch we can then rewrite the integral with the other order of integration.

Examples will help us develop this skill.

Example 1.1.7 Changing the order of integration

Rewrite the iterated integral $\displaystyle\int_0^6\int_0^{x/3} 1\,dy\,dx$ with the order of integration $dx\,dy$.

SOLUTION We need to use the bounds of integration to determine the region we are integrating over.

The bounds tell us that y is bounded by 0 and $x/3$; x is bounded by 0 and 6. We plot these four curves: $y=0$, $y=x/3$, $x=0$ and $x=6$ to find the region described by the bounds. Figure 1.1.6 shows these curves, indicating that R is a triangle.

To change the order of integration, we need to consider the curves that bound the x-values. We see that the lower bound is $x=3y$ and the upper bound is $x=6$. The bounds on y are 0 to 2. Thus we can rewrite the integral as $\displaystyle\int_0^2\int_{3y}^6 1\,dx\,dy.$

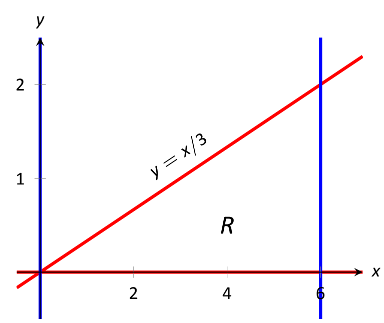

Figure 1.1.6: Sketching the region R described by the iterated integral in Example 1.1.7.

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Example 1.1.8 Changing the order of integration

Change the order of integration of $\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy$.

Solution We sketch the region described by the bounds to help us change the integration order. x is bounded below and above (i.e., to the left and right) by $x = y^2/4$ and $x = (y+4)/2$ respectively, and y is bounded between 0 and 4. Graphing the previous curves, we find the region R to be that shown in Figure 1.1.7.

To change the order of integration, we need to establish curves that bound y. The figure makes it clear that there are two lower bounds for y: $y = 0$ on $0 \leq x \leq 2$, and $y = 2x - 4$ on $2 \leq x \leq 4$. Thus we need two double integrals. The upper bound for each is $y = 2\sqrt{x}$. Thus we have

$$\int_0^4 \int_{y^2/4}^{(y+4)/2} 1\,dx\,dy = \int_0^2 \int_0^{2\sqrt{x}} 1\,dy\,dx + \int_2^4 \int_{2x-4}^{2\sqrt{x}} 1\,dy\,dx.$$

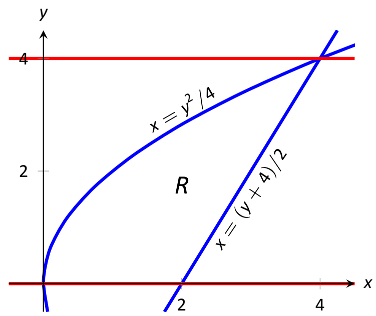


Figure 1.1.7: Drawing the region determined by the bounds of integration in Example 1.1.8.

This section has introduced a new concept, the iterated integral. We developed one application for iterated integration: area between curves. However, this is not new, for we already know how to find areas bounded by curves.

In the next section we apply iterated integration to solve problems we currently do not know how to handle. The "real" goal of this section was not to learn a new way of computing area. Rather, our goal was to learn how to define a region in the plane using the bounds of an iterated integral. That skill is very important in the following sections.

Notes:

Exercises 1.1

Terms and Concepts

13 01 ex 01 1. When integrating $f_x(x,y)$ with respect to x, the constant of integration C is really which: $C(x)$ or $C(y)$? What does this mean?

13 01 ex 02 2. Integrating an integral is called ________ ________.

13 01 ex 03 3. When evaluating an iterated integral, we integrate from ________ to ________, then from ________ to ________.

13 01 ex 04 4. One understanding of an iterated integral is that $\displaystyle\int_a^b \int_{g_1(x)}^{g_2(x)} dy\,dx$ gives the ________ of a plane region.

Problems

13 01 exset 01 **In Exercises 5 – 10, evaluate the integral and subsequent iterated integral.**

13 01 ex 09 5.

(a) $\displaystyle\int_2^5 \left(6x^2+4xy-3y^2\right) dy$

(b) $\displaystyle\int_{-3}^{-2} \int_2^5 \left(6x^2+4xy-3y^2\right) dy\,dx$

13 01 ex 10 6.

(a) $\displaystyle\int_0^{\pi} \left(2x\cos y+\sin x\right) dx$

(b) $\displaystyle\int_0^{\pi/2} \int_0^{\pi} \left(2x\cos y+\sin x\right) dx\,dy$

13 01 ex 05 7.

(a) $\displaystyle\int_1^{x} \left(x^2y-y+2\right) dy$

(b) $\displaystyle\int_0^{2} \int_1^{x} \left(x^2y-y+2\right) dy\,dx$

13 01 ex 06 8.

(a) $\displaystyle\int_y^{y^2} \left(x-y\right) dx$

(b) $\displaystyle\int_{-1}^{1} \int_y^{y^2} \left(x-y\right) dx\,dy$

13 01 ex 07 9.

(a) $\displaystyle\int_0^{y} \left(\cos x\sin y\right) dx$

(b) $\displaystyle\int_0^{\pi} \int_0^{y} \left(\cos x\sin y\right) dx\,dy$

13 01 ex 08 10.

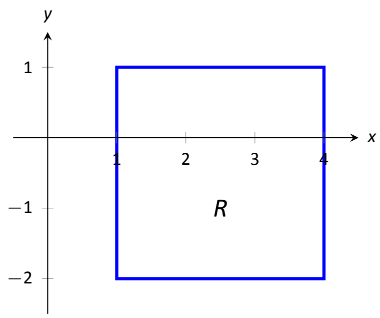
(a) $\displaystyle\int_0^{x} \left(\frac{1}{1+x^2}\right) dy$

(b) $\displaystyle\int_1^{2} \int_0^{x} \left(\frac{1}{1+x^2}\right) dy\,dx$

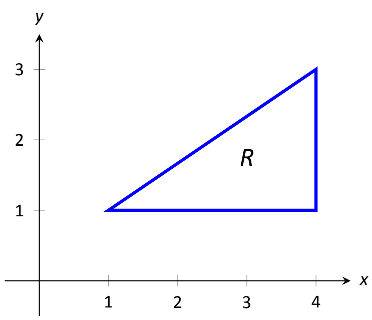
13 01 exset 02

In Exercises 11 – 16, a graph of a planar region R is given. Give the iterated integrals, with both orders of integration $dy\,dx$ and $dx\,dy$, that give the area of R. Evaluate one of the iterated integrals to find the area.

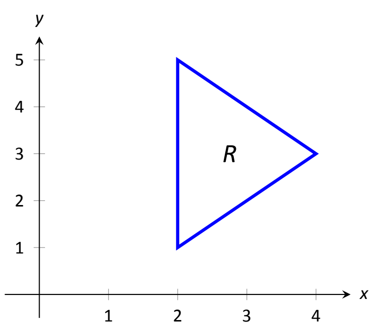
13 01 ex 11

11. 

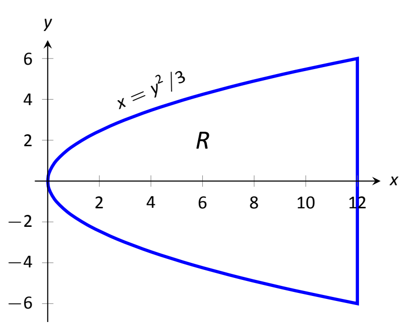
13 01 ex 12

12. 

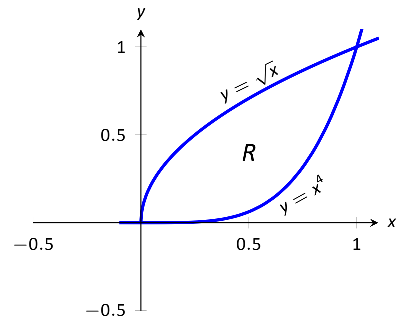
13 01 ex 13

13. 

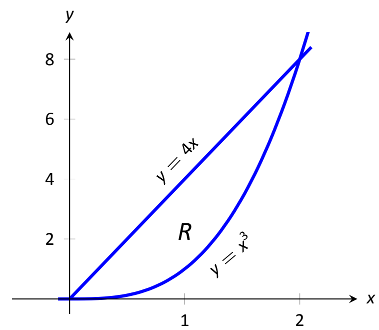
13 01 ex 14

14. 

13 01 ex 15

15. 

13 01 ex 16

16. 

13 01 exset 03

In Exercises 17 – 22, iterated integrals are given that compute the area of a region R in the x-y plane. Sketch the region R, and give the iterated integral(s) that give the area of R with the opposite order of integration.

13 01 ex 18

17. $\displaystyle\int_{-2}^{2}\int_{0}^{4-x^2} dy\,dx$

13 01 ex 20

18. $\displaystyle\int_{0}^{1}\int_{5-5x}^{5-5x^2} dy\,dx$

13 01 ex 19

19. $\displaystyle\int_{-2}^{2}\int_{0}^{2\sqrt{4-y^2}} dx\,dy$

13 01 ex 17

20. $\displaystyle\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} dy\,dx$

13 01 ex 22

21. $\displaystyle\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy + \int_{1}^{4}\int_{y-2}^{\sqrt{y}} dx\,dy$

13 01 ex 21

22. $\displaystyle\int_{-1}^{1}\int_{(x-1)/2}^{(1-x)/2} dy\,dx$

1.2 Double Integration and Volume

The definite integral of f over $[a, b]$, $\int_a^b f(x)\,dx$, was introduced as "the signed area under the curve." We approximated the value of this area by first subdividing $[a, b]$ into n subintervals, where the i^{th} subinterval has length Δx_i, and letting c_i be any value in the i^{th} subinterval. We formed rectangles that approximated part of the region under the curve with width Δx_i, height $f(c_i)$, and hence with area $f(c_i)\Delta x_i$. Summing all the rectangle's areas gave an approximation of the definite integral, and Theorem 5.3.2 stated that

$$\int_a^b f(x)\,dx = \lim_{\|\Delta x\| \to 0} \sum f(c_i)\Delta x_i,$$

connecting the area under the curve with sums of the areas of rectangles.

We use a similar approach in this section to find volume under a surface.

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function defined on R. We wish to find the signed volume under the surface of f over R. (We use the term "signed volume" to denote that space above the x-y plane, under f, will have a positive volume; space above f and under the x-y plane will have a "negative" volume, similar to the notion of signed area used before.)

We start by partitioning R into n rectangular subregions as shown in Figure 1.2.1(a). For simplicity's sake, we let all widths be Δx and all heights be Δy. Note that the sum of the areas of the rectangles is not equal to the area of R, but rather is a close approximation. Arbitrarily number the rectangles 1 through n, and pick a point (x_i, y_i) in the i^{th} subregion.

The volume of the rectangular solid whose base is the i^{th} subregion and whose height is $f(x_i, y_i)$ is $V_i = f(x_i, y_i)\Delta x\Delta y$. Such a solid is shown in Figure 1.2.1(b). Note how this rectangular solid only approximates the true volume under the surface; part of the solid is above the surface and part is below.

For each subregion R_i used to approximate R, create the rectangular solid with base area $\Delta x\Delta y$ and height $f(x_i, y_i)$. The sum of all rectangular solids is

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta x\Delta y.$$

This approximates the signed volume under f over R. As we have done before, to get a better approximation we can use more rectangles to approximate the region R.

In general, each rectangle could have a different width Δx_j and height Δy_k, giving the i^{th} rectangle an area $\Delta A_i = \Delta x_j \Delta y_k$ and the i^{th} rectangular solid a

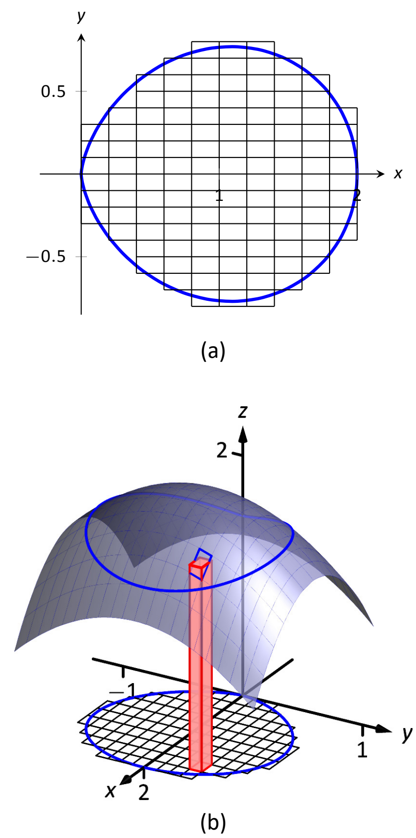


Figure 1.2.1: Developing a method for finding signed volume under a surface.

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volume of $f(x_i, y_i)\Delta A_i$. Let $||\Delta A||$ denote the length of the longest diagonal of all rectangles in the subdivision of R; $||\Delta A|| \to 0$ means each rectangle's width and height are both approaching 0. If f is a continuous function, as $||\Delta A||$ shrinks (and hence $n \to \infty$) the summation $\displaystyle\sum_{i=1}^{n} f(x_i, y_i)\Delta A_i$ approximates the signed volume better and better. This leads to a definition.

Note: Recall that the integration symbol "$\int$" is an "elongated S," representing the word "sum." We interpreted $\int_a^b f(x)\,dx$ as "take the *sum* of the areas of rectangles over the interval $[a, b]$." The double integral uses two integration symbols to represent a "double sum." When adding up the volumes of rectangular solids over a partition of a region R, as done in Figure 1.2.1, one could first add up the volumes across each row (one type of sum), then add these totals together (another sum), as in

$$\sum_{j=1}^{n}\sum_{i=1}^{m} f(x_i, y_j)\Delta x_i \Delta y_j.$$

One can rewrite this as

$$\sum_{j=1}^{n}\left(\sum_{i=1}^{m} f(x_i, y_j)\Delta x_i\right)\Delta y_j.$$

The summation inside the parenthesis indicates the sum of heights $\times$ widths, which gives an area; multiplying these areas by the thickness Δy_j gives a volume. The illustration in Figure 1.2.2 relates to this understanding.

Definition 1.2.1 Double Integral, Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. The **signed volume** V under f over R is denoted by the **double integral**

$$V = \iint_R f(x, y)\, dA.$$

Alternate notations for the double integral are

$$\iint_R f(x, y)\, dA = \iint_R f(x, y)\, dx\, dy = \iint_R f(x, y)\, dy\, dx.$$

The definition above does not state how to find the signed volume, though the notation offers a hint. We need the next two theorems to evaluate double integrals to find volume.

Theorem 1.2.1 Double Integrals and Signed Volume

Let $z = f(x, y)$ be a continuous function defined over a closed region R in the x-y plane. Then the signed volume V under f over R is

$$V = \iint_R f(x, y)\, dA = \lim_{||\Delta A|| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta A_i.$$

This theorem states that we can find the exact signed volume using a limit of sums. The partition of the region R is not specified, so any partitioning where the diagonal of each rectangle shrinks to 0 results in the same answer.

This does not offer a very satisfying way of computing volume, though. Our experience has shown that evaluating the limits of sums can be tedious. We seek a more direct method.

Recall Theorem 7.2.1 in Section 7.2. This stated that if $A(x)$ gives the cross-sectional area of a solid at x, then $\int_a^b A(x)\, dx$ gave the volume of that solid over

Notes:

$[a, b]$.

Consider Figure 1.2.2, where a surface $z = f(x, y)$ is drawn over a region R. Fixing a particular x value, we can consider the area under f over R where x has that fixed value. That area can be found with a definite integral, namely

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

Remember that though the integrand contains x, we are viewing x as fixed. Also note that the bounds of integration are functions of x: the bounds depend on the value of x.

As $A(x)$ is a cross-sectional area function, we can find the signed volume V under f by integrating it:

$$V = \int_a^b A(x)\, dx = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x, y)\, dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

This gives a concrete method for finding signed volume under a surface. We could do a similar procedure where we started with y fixed, resulting in a iterated integral with the order of integration $dx\, dy$. The following theorem states that both methods give the same result, which is the value of the double integral. It is such an important theorem it has a name associated with it.

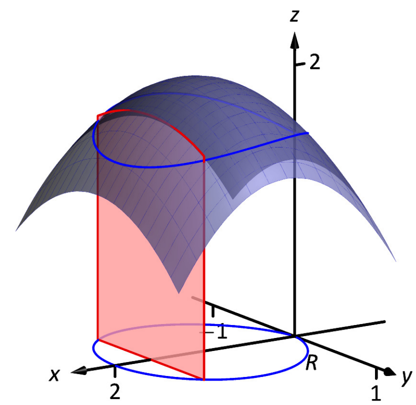

Figure 1.2.2: Finding volume under a surface by sweeping out a cross–sectional area.

Theorem 1.2.2 Fubini's Theorem

Let R be a closed, bounded region in the x-y plane and let $z = f(x, y)$ be a continuous function on R.

1. If R is bounded by $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$, where g_1 and g_2 are continuous functions on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is bounded by $c \leq y \leq d$ and $h_1(y) \leq x \leq h_2(y)$, where h_1 and h_2 are continuous functions on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

Note that once again the bounds of integration follow the "curve to curve, point to point" pattern discussed in the previous section. In fact, one of the

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main points of the previous section is developing the skill of describing a region R with the bounds of an iterated integral. Once this skill is developed, we can use double integrals to compute many quantities, not just signed volume under a surface.

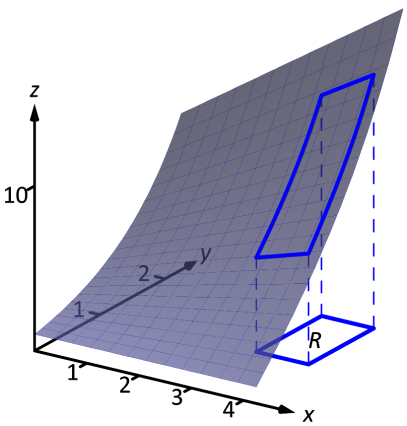


Figure 1.2.3: Finding the signed volume under a surface in Example 1.2.1.

Example 1.2.1 Evaluating a double integral

Let $f(x,y) = xy + e^y$. Find the signed volume under f on the region R, which is the rectangle with corners $(3,1)$ and $(4,2)$ pictured in Figure 1.2.3, using Fubini's Theorem and both orders of integration.

SOLUTION We wish to evaluate $\iint_R \left(xy + e^y\right)\, dA$. As R is a rectangle, the bounds are easily described as $3 \le x \le 4$ and $1 \le y \le 2$.

Using the order $dy\, dx$:

$$\begin{aligned}\iint_R \left(xy + e^y\right) dA &= \int_3^4 \int_1^2 \left(xy + e^y\right) dy\, dx \\ &= \int_3^4 \left(\left[\frac{1}{2}xy^2 + e^y\right]\Big|_1^2 \right) dx \\ &= \int_3^4 \left(\frac{3}{2}x + e^2 - e\right) dx \\ &= \left(\frac{3}{4}x^2 + \left(e^2 - e\right)x\right)\Big|_3^4 \\ &= \frac{21}{4} + e^2 - e \approx 9.92.\end{aligned}$$

Now we check the validity of Fubini's Theorem by using the order $dx\, dy$:

$$\begin{aligned}\iint_R \left(xy + e^y\right) dA &= \int_1^2 \int_3^4 \left(xy + e^y\right) dx\, dy \\ &= \int_1^2 \left(\left[\frac{1}{2}x^2y + xe^y\right]\Big|_3^4 \right) dy \\ &= \int_1^2 \left(\frac{7}{2}y + e^y\right) dy \\ &= \left(\frac{7}{4}y^2 + e^y\right)\Big|_1^2 \\ &= \frac{21}{4} + e^2 - e \approx 9.92.\end{aligned}$$

Both orders of integration return the same result, as expected.

Notes:

Example 1.2.2 Evaluating a double integral

Evaluate $\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA$, where R is the triangle bounded by $x = 0$, $y = 0$ and $x/2 + y = 1$, as shown in Figure 1.2.4.

Solution While it is not specified which order we are to use, we will evaluate the double integral using both orders to help drive home the point that it does not matter which order we use.

Using the order $dy\, dx$: The bounds on y go from "curve to curve," i.e., $0 \le y \le 1 - x/2$, and the bounds on x go from "point to point," i.e., $0 \le x \le 2$.

$$\begin{aligned}
\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA &= \int_0^2 \int_0^{-\frac{x}{2}+1} \left(3xy - x^2 - y^2 + 6\right)\, dy\, dx \\
&= \int_0^2 \left(\frac{3}{2}xy^2 - x^2y - \frac{1}{3}y^3 + 6y\right)\Big|_0^{-\frac{x}{2}+1} dx \\
&= \int_0^2 \left(\frac{11}{12}x^3 - \frac{11}{4}x^2 - x + \frac{17}{3}\right) dx \\
&= \left(\frac{11}{48}x^4 - \frac{11}{12}x^3 - \frac{1}{2}x^2 + \frac{17}{3}x\right)\Big|_0^2 \\
&= \frac{17}{3} = 5.\overline{6}.
\end{aligned}$$

Now lets consider the order $dx\, dy$. Here x goes from "curve to curve," $0 \le x \le 2 - 2y$, and y goes from "point to point," $0 \le y \le 1$:

$$\begin{aligned}
\iint_R \left(3xy - x^2 - y^2 + 6\right)\, dA &= \int_0^1 \int_0^{2-2y} \left(3xy - x^2 - y^2 + 6\right)\, dx\, dy \\
&= \int_0^1 \left(\frac{3}{2}x^2y - \frac{1}{3}x^3 - xy^2 + 6x\right)\Big|_0^{2-2y} dy \\
&= \int_0^1 \left(\frac{32}{3}y^3 - 22y^2 + 2y + \frac{28}{3}\right) dy \\
&= \left(\frac{8}{3}y^4 - \frac{22}{3}y^3 + y^2 + \frac{28}{3}y\right)\Big|_0^1 \\
&= \frac{17}{3} = 5.\overline{6}.
\end{aligned}$$

We obtained the same result using both orders of integration.

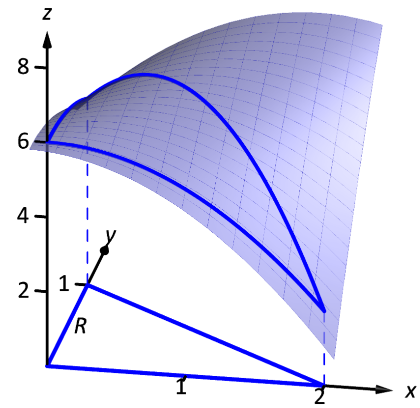


Figure 1.2.4: Finding the signed volume under the surface in Example 1.2.2.

Note how in these two examples that the bounds of integration depend only on R; the bounds of integration have nothing to do with $f(x, y)$. This is an important concept, so we include it as a Key Idea.

Notes:

Key Idea 1.2.1 Double Integration Bounds

When evaluating $\iint_R f(x,y)\, dA$ using an iterated integral, the bounds of integration depend only on R. The surface f does not determine the bounds of integration.

Before doing another example, we give some properties of double integrals. Each should make sense if we view them in the context of finding signed volume under a surface, over a region.

Theorem 1.2.3 Properties of Double Integrals

Let f and g be continuous functions over a closed, bounded plane region R, and let c be a constant.

1. $\displaystyle \iint_R c\,f(x,y)\, dA = c\iint_R f(x,y)\, dA.$
2. $\displaystyle \iint_R \big(f(x,y)\pm g(x,y)\big)\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$
3. If $f(x,y)\geq 0$ on R, then $\displaystyle \iint_R f(x,y)\, dA \geq 0.$
4. If $f(x,y)\geq g(x,y)$ on R, then $\displaystyle \iint_R f(x,y)\, dA \geq \iint_R g(x,y)\, dA.$
5. Let R be the union of two nonoverlapping regions, $R = R_1 \bigcup R_2$ (see Figure 1.2.5). Then
$$\iint_R f(x,y)\, dA = \iint_{R_1} f(x,y)\, dA + \iint_{R_2} f(x,y)\, dA.$$

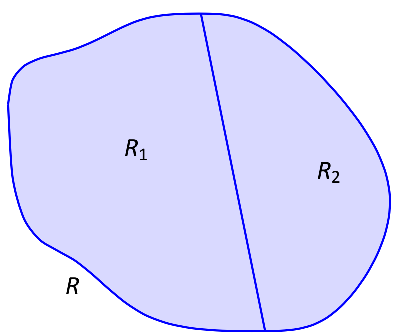


Figure 1.2.5: R is the union of two nonoverlapping regions, R_1 and R_2.

Example 1.2.3 Evaluating a double integral

Let $f(x,y) = \sin x\cos y$ and R be the triangle with vertices $(-1,0)$, $(1,0)$ and $(0,1)$ (see Figure 1.2.6). Evaluate the double integral $\iint_R f(x,y)\, dA$.

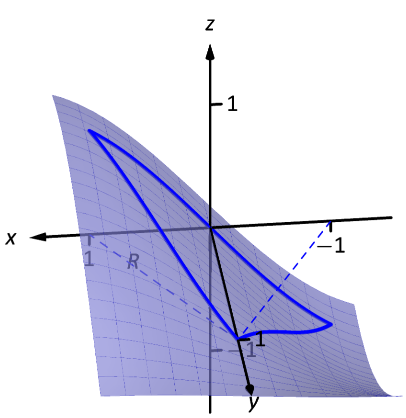


Figure 1.2.6: Finding the signed volume under a surface in Example 1.2.3.

Solution If we attempt to integrate using an iterated integral with the order $dy\,dx$, note how there are two upper bounds on R meaning we'll need to use two iterated integrals. We would need to split the triangle into two regions along the y-axis, then use Theorem 1.2.3, part 5.

Instead, let's use the order $dx\,dy$. The curves bounding x are $y-1\leq x\leq$

Notes:

$1 - y$; the bounds on y are $0 \leq y \leq 1$. This gives us:

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 \int_{y-1}^{1-y} \sin x \cos y \,dx\,dy \\ &= \int_0^1 \Big(-\cos x \cos y\Big)\Big|_{y-1}^{1-y}\,dy \\ &= \int_0^1 \cos y\Big(-\cos(1-y) + \cos(y-1)\Big)\,dy.\end{aligned}$$

Recall that the cosine function is an even function; that is, $\cos x = \cos(-x)$. Therefore, from the last integral above, we have $\cos(y-1) = \cos(1-y)$. Thus the integrand simplifies to 0, and we have

$$\begin{aligned}\iint_R f(x,y)\,dA &= \int_0^1 0\,dy \\ &= 0.\end{aligned}$$

It turns out that over R, there is just as much volume above the x-y plane as below (look again at Figure 1.2.6), giving a final signed volume of 0.

Example 1.2.4 Evaluating a double integral

Evaluate $\iint_R (4-y)\,dA$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$, graphed in Figure 1.2.7.

Solution Graphing each curve can help us find their points of intersection. Solving analytically, the second equation tells us that $y = x^2/4$. Substituting this value in for y in the first equation gives us $x^4/16 = 4x$. Solving for x:

$$\begin{aligned}\frac{x^4}{16} &= 4x \\ x^4 - 64x &= 0 \\ x(x^3 - 64) &= 0 \\ x &= 0,\ 4.\end{aligned}$$

Thus we've found analytically what was easy to approximate graphically: the regions intersect at $(0,0)$ and $(4,4)$, as shown in Figure 1.2.7.

We now choose an order of integration: $dy\,dx$ or $dx\,dy$? Either order works; since the integrand does not contain x, choosing $dx\,dy$ might be simpler – at least, the first integral is very simple.

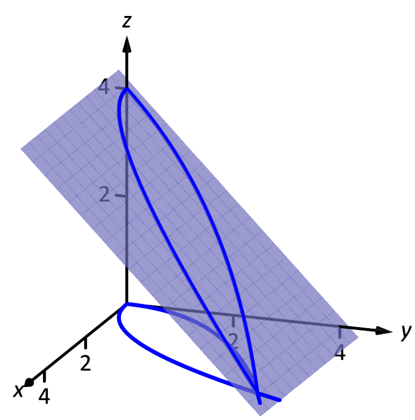


Figure 1.2.7: Finding the volume under the surface in Example 1.2.4.

Notes:

Thus we have the following "curve to curve, point to point" bounds: $y^2/4 \leq x \leq 2\sqrt{y}$, and $0 \leq y \leq 4$.

$$\begin{aligned}\iint_R (4-y)\,dA &= \int_0^4 \int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy \\ &= \int_0^4 \Big(x(4-y)\Big)\Big|_{y^2/4}^{2\sqrt{y}} dy \\ &= \int_0^4 \Big(\big(2\sqrt{y}-\frac{y^2}{4}\big)(4-y)\Big)\,dy = \int_0^4 \Big(\frac{y^3}{4}-y^2-2y^{3/2}+8y^{1/2}\Big)\,dy \\ &= \left(\frac{y^4}{16}-\frac{y^3}{3}-\frac{4y^{5/2}}{5}+\frac{16y^{3/2}}{3}\right)\Bigg|_0^4 \\ &= \frac{176}{15} = 11.7\overline{3}.\end{aligned}$$

The signed volume under the surface f is about 11.7 cubic units.

In the previous section we practiced changing the order of integration of a given iterated integral, where the region R was not explicitly given. Changing the bounds of an integral is more than just an test of understanding. Rather, there are cases where integrating in one order is really hard, if not impossible, whereas integrating with the other order is feasible.

Example 1.2.5 Changing the order of integration

Rewrite the iterated integral $\int_0^3 \int_y^3 e^{-x^2}\,dx\,dy$ with the order $dy\,dx$. Comment on the feasibility to evaluate each integral.

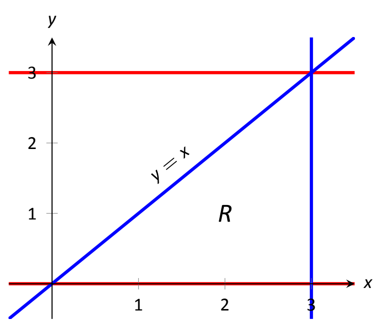


Figure 1.2.8: Determining the region R determined by the bounds of integration in Example 1.2.5.

Solution Once again we make a sketch of the region over which we are integrating to facilitate changing the order. The bounds on x are from $x = y$ to $x = 3$; the bounds on y are from $y = 0$ to $y = 3$. These curves are sketched in Figure 1.2.8, enclosing the region R.

To change the bounds, note that the curves bounding y are $y = 0$ up to $y = x$; the triangle is enclosed between $x = 0$ and $x = 3$. Thus the new bounds of integration are $0 \leq y \leq x$ and $0 \leq x \leq 3$, giving the iterated integral $\int_0^3 \int_0^x e^{-x^2}\,dy\,dx$.

How easy is it to evaluate each iterated integral? Consider the order of integrating $dx\,dy$, as given in the original problem. The first indefinite integral we need to evaluate is $\int e^{-x^2}\,dx$; we have stated before (see Section 5.5) that this integral cannot be evaluated in terms of elementary functions. We are stuck.

Changing the order of integration makes a big difference here. In the second

Notes:

iterated integral, we are faced with $\int e^{-x^2}\,dy$; integrating with respect to y gives us $ye^{-x^2}+C$, and the first definite integral evaluates to

$$\int_0^x e^{-x^2}\,dy = xe^{-x^2}.$$

Thus

$$\int_0^3\int_0^x e^{-x^2}\,dy\,dx = \int_0^3 \left(xe^{-x^2}\right)dx.$$

This last integral is easy to evaluate with substitution, giving a final answer of $\frac{1}{2}(1-e^{-9})\approx 0.5$. Figure 1.2.9 shows the surface over R.

In short, evaluating one iterated integral is impossible; the other iterated integral is relatively simple.

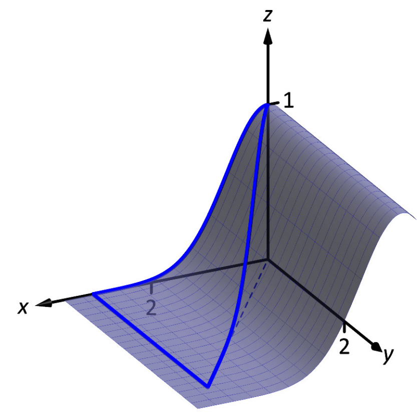


Figure 1.2.9: Showing the surface f defined in Example 1.2.5 over its region R.

Definition 5.4.1 defines the average value of a single–variable function $f(x)$ on the interval $[a,b]$ as

$$\text{average value of } f(x) \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx;$$

that is, it is the "area under f over an interval divided by the length of the interval." We make an analogous statement here: the average value of $z=f(x,y)$ over a region R is the volume under f over R divided by the area of R.

Definition 1.2.2 The Average Value of f on R

Let $z=f(x,y)$ be a continuous function defined over a closed region R in the x-y plane. The **average value of f on R** is

$$\text{average value of } f \text{ on } R = \frac{\displaystyle\iint_R f(x,y)\,dA}{\displaystyle\iint_R dA}.$$

Example 1.2.6 Finding average value of a function over a region R

Find the average value of $f(x,y)=4-y$ over the region R, which is bounded by the parabolas $y^2=4x$ and $x^2=4y$. Note: this is the same function and region as used in Example 1.2.4.

SOLUTION In Example 1.2.4 we found

$$\iint_R f(x,y)\,dA = \int_0^4\int_{y^2/4}^{2\sqrt{y}} (4-y)\,dx\,dy = \frac{176}{15}.$$

Notes:

We find the area of R by computing $\iint_R dA$:

$$\iint_R dA = \int_0^4 \int_{y^2/4}^{2\sqrt{y}} dx\,dy = \frac{16}{3}.$$

Dividing the volume under the surface by the area gives the average value:

$$\text{average value of } f \text{ on } R = \frac{176/15}{16/3} = \frac{11}{5} = 2.2.$$

While the surface, as shown in Figure 1.2.10, covers z-values from $z = 0$ to $z = 4$, the "average" z-value on R is 2.2.

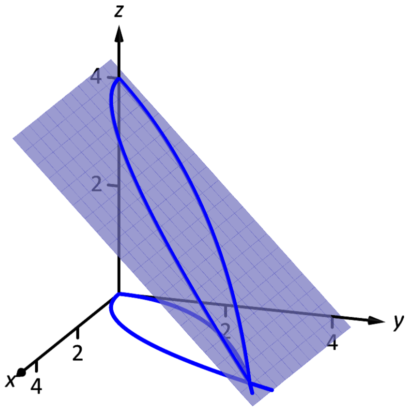


Figure 1.2.10: Finding the average value of f in Example 1.2.6.

The previous section introduced the iterated integral in the context of finding the area of plane regions. This section has extended our understanding of iterated integrals; now we see they can be used to find the signed volume under a surface.

This new understanding allows us to revisit what we did in the previous section. Given a region R in the plane, we computed $\iint_R 1\,dA$; again, our understanding at the time was that we were finding the area of R. However, we can now view the function $z = 1$ as a surface, a flat surface with constant z-value of 1. The double integral $\iint_R 1\,dA$ finds the volume, under $z = 1$, over R, as shown in Figure 1.2.11. Basic geometry tells us that if the base of a general right cylinder has area A, its volume is $A \cdot h$, where h is the height. In our case, the height is 1. We were "actually" computing the volume of a solid, though we interpreted the number as an area.

The next section extends our abilities to find "volumes under surfaces." Currently, some integrals are hard to compute because either the region R we are integrating over is hard to define with rectangular curves, or the integrand itself is hard to deal with. Some of these problems can be solved by converting everything into polar coordinates.

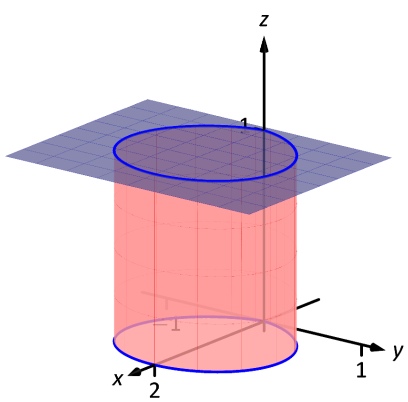


Figure 1.2.11: Showing how an iterated integral used to find area also finds a certain volume.

Notes:

Exercises 1.2

Terms and Concepts

13 02 ex 01 1. An integral can be interpreted as giving the signed area over an interval; a double integral can be interpreted as giving the signed ________ over a region.

13 02 ex 02 2. Explain why the following statement is false: "Fubini's Theorem states that $\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\; dy\; dx = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)\, dx\, dy$."

13 02 ex 03 3. Explain why if $f(x,y) > 0$ over a region R, then $\iint_R f(x,y)\, dA > 0$.

13 02 ex 04 4. If $\iint_R f(x,y)\, dA = \iint_R g(x,y)\, dA$, does this imply $f(x,y) = g(x,y)$?

Problems

13 02 exset 01 **In Exercises 5 – 10,**

(a) Evaluate the given iterated integral, and

(b) rewrite the integral using the other order of integration.

13 02 ex 05 5. $\displaystyle\int_1^2 \int_{-1}^1 \left(\frac{x}{y} + 3\right)\; dx\; dy$

13 02 ex 06 6. $\displaystyle\int_{-\pi/2}^{\pi/2} \int_0^{\pi} (\sin x \cos y)\; dx\; dy$

13 02 ex 07 7. $\displaystyle\int_0^4 \int_0^{-x/2+2} \left(3x^2 - y + 2\right)\; dy\; dx$

13 02 ex 08 8. $\displaystyle\int_1^3 \int_y^3 \left(x^2y - xy^2\right)\; dx\; dy$

13 02 ex 09 9. $\displaystyle\int_0^1 \int_{-\sqrt{1-y}}^{\sqrt{1-y}} (x + y + 2)\; dx\; dy$

13 02 ex 10 10. $\displaystyle\int_0^9 \int_{y/3}^{\sqrt{y}} \left(xy^2\right)\; dx\; dy$

13 02 exset 02 **In Exercises 11 – 18:**

(a) Sketch the region R given by the problem.

(b) Set up the iterated integrals, in both orders, that evaluate the given double integral for the described region R.

(c) Evaluate one of the iterated integrals to find the signed volume under the surface $z = f(x,y)$ over the region R.

13 02 ex 11 11. $\displaystyle\iint_R x^2y\, dA$, where R is bounded by $y = \sqrt{x}$ and $y = x^2$.

13 02 ex 12 12. $\displaystyle\iint_R x^2y\, dA$, where R is bounded by $y = \sqrt[3]{x}$ and $y = x^3$.

13 02 ex 13 13. $\displaystyle\iint_R x^2 - y^2\, dA$, where R is the rectangle with corners $(-1,-1)$, $(1,-1)$, $(1,1)$ and $(-1,1)$.

13 02 ex 14 14. $\displaystyle\iint_R ye^x\, dA$, where R is bounded by $x = 0$, $x = y^2$ and $y = 1$.

13 02 ex 15 15. $\displaystyle\iint_R \left(6 - 3x - 2y\right) dA$, where R is bounded by $x = 0$, $y = 0$ and $3x + 2y = 6$.

13 02 ex 16 16. $\displaystyle\iint_R e^y\, dA$, where R is bounded by $y = \ln x$ and $y = \dfrac{1}{e-1}(x-1)$.

13 02 ex 17 17. $\displaystyle\iint_R \left(x^3y - x\right) dA$, where R is the half of the circle $x^2 + y^2 = 9$ in the first and second quadrants.

13 02 ex 18 18. $\displaystyle\iint_R \left(4 - 3y\right) dA$, where R is bounded by $y = 0$, $y = x/e$ and $y = \ln x$.

13 02 exset 03 **In Exercises 19 – 22, state why it is difficult/impossible to integrate the iterated integral in the given order of integration. Change the order of integration and evaluate the new iterated integral.**

13 02 ex 19 19. $\displaystyle\int_0^4 \int_{y/2}^2 e^{x^2}\; dx\; dy$

13 02 ex 20 20. $\displaystyle\int_0^{\sqrt{\pi/2}} \int_x^{\sqrt{\pi/2}} \cos\left(y^2\right)\, dy\, dx$

13 02 ex 21 21. $\displaystyle\int_0^1 \int_y^1 \frac{2y}{x^2 + y^2}\; dx\; dy$

13 02 ex 22 22. $\displaystyle\int_{-1}^1 \int_1^2 \frac{x \tan^2 y}{1 + \ln y}\; dy\; dx$

13 02 exset 04 **In Exercises 23 – 26, find the average value of f over the region R. Notice how these functions and regions are related to the iterated integrals given in Exercises 5 – 8.**

13 02 ex 23 23. $f(x,y) = \dfrac{x}{y} + 3$; R is the rectangle with opposite corners $(-1,1)$ and $(1,2)$.

13 02 ex 24 24. $f(x,y) = \sin x \cos y$; R is bounded by $x = 0$, $x = \pi$, $y = -\pi/2$ and $y = \pi/2$.

13 02 ex 25 25. $f(x,y) = 3x^2 - y + 2$; R is bounded by the lines $y = 0$, $y = 2 - x/2$ and $x = 0$.

13 02 ex 26 26. $f(x,y) = x^2y - xy^2$; R is bounded by $y = x$, $y = 1$ and $x = 3$.

1.3 Double Integration with Polar Coordinates

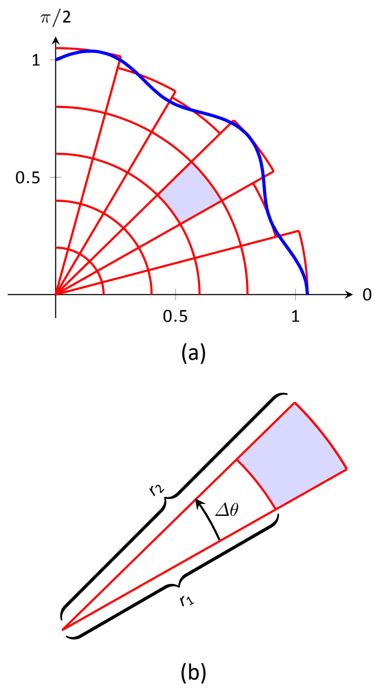


Figure 1.3.1: Approximating a region R with portions of sectors of circles.

We have used iterated integrals to evaluate double integrals, which give the signed volume under a surface, $z = f(x, y)$, over a region R of the x-y plane. The integrand is simply $f(x, y)$, and the bounds of the integrals are determined by the region R.

Some regions R are easy to describe using rectangular coordinates – that is, with equations of the form $y = f(x)$, $x = a$, etc. However, some regions are easier to handle if we represent their boundaries with polar equations of the form $r = f(\theta)$, $\theta = \alpha$, etc.

The basic form of the double integral is $\iint_R f(x, y)\, dA$. We interpret this integral as follows: over the region R, sum up lots of products of heights (given by $f(x_i, y_i)$) and areas (given by ΔA_i). That is, dA represents "a little bit of area." In rectangular coordinates, we can describe a small rectangle as having area $dx\,dy$ or $dy\,dx$ – the area of a rectangle is simply length$\times$width – a small change in x times a small change in y. Thus we replace dA in the double integral with $dx\,dy$ or $dy\,dx$.

Now consider representing a region R with polar coordinates. Consider Figure 1.3.1(a). Let R be the region in the first quadrant bounded by the curve. We can approximate this region using the natural shape of polar coordinates: portions of sectors of circles. In the figure, one such region is shaded, shown again in part (b) of the figure.

As the area of a sector of a circle with radius r, subtended by an angle θ, is $A = \frac{1}{2}r^2\theta$, we can find the area of the shaded region. The whole sector has area $\frac{1}{2}r_2^2\Delta\theta$, whereas the smaller, unshaded sector has area $\frac{1}{2}r_1^2\Delta\theta$. The area of the shaded region is the difference of these areas:

$$\Delta A_i = \frac{1}{2}r_2^2\Delta\theta - \frac{1}{2}r_1^2\Delta\theta = \frac{1}{2}\left(r_2^2 - r_1^2\right)\left(\Delta\theta\right) = \frac{r_2 + r_1}{2}\left(r_2 - r_1\right)\Delta\theta.$$

Note that $(r_2 + r_1)/2$ is just the average of the two radii.

To approximate the region R, we use many such subregions; doing so shrinks the difference $r_2 - r_1$ between radii to 0 and shrinks the change in angle $\Delta\theta$ also to 0. We represent these infinitesimal changes in radius and angle as dr and $d\theta$, respectively. Finally, as dr is small, $r_2 \approx r_1$, and so $(r_2 + r_1)/2 \approx r_1$. Thus, when dr and $d\theta$ are small,

$$\Delta A_i \approx r_i\, dr\, d\theta.$$

Taking a limit, where the number of subregions goes to infinity and both $r_2 - r_1$ and $\Delta\theta$ go to 0, we get

$$dA = r\, dr\, d\theta.$$

So to evaluate $\iint_R f(x, y)\, dA$, replace dA with $r\,dr\,d\theta$. Convert the function $z = f(x, y)$ to a function with polar coordinates with the substitutions $x = r\cos\theta$,

Notes:

$y = r\sin\theta$. Finally, find bounds $g_1(\theta) \le r \le g_2(\theta)$ and $\alpha \le \theta \le \beta$ that describe R. This is the key principle of this section, so we restate it here as a Key Idea.

Key Idea 1.3.1 Evaluating Double Integrals with Polar Coordinates

Let R be a plane region bounded by the polar equations $\alpha \le \theta \le \beta$ and $g_1(\theta) \le r \le g_2(\theta)$. Then

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f\big(r\cos\theta, r\sin\theta\big)\,r\,dr\,d\theta.$$

Examples will help us understand this Key Idea.

Example 1.3.1 Evaluating a double integral with polar coordinates

Find the signed volume under the plane $z = 4 - x - 2y$ over the disk bounded by the circle with equation $x^2 + y^2 = 1$.

Solution The bounds of the integral are determined solely by the region R over which we are integrating. In this case, it is a disk with boundary $x^2 + y^2 = 1$. We need to find polar bounds for this region. It may help to review Section 9.4; bounds for this disk are $0 \le r \le 1$ and $0 \le \theta \le 2\pi$.

We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$. That means we make the following substitutions:

$$4 - x - 2y \quad\Rightarrow\quad 4 - r\cos\theta - 2r\sin\theta.$$

Finally, we replace dA in the double integral with $r\,dr\,d\theta$. This gives the final iterated integral, which we evaluate:

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{2\pi}\int_0^1 \big(4 - r\cos\theta - 2r\sin\theta\big)r\,dr\,d\theta\\
&= \int_0^{2\pi}\int_0^1 \big(4r - r^2(\cos\theta - 2\sin\theta)\big)\,dr\,d\theta\\
&= \int_0^{2\pi}\left(2r^2 - \frac{1}{3}r^3(\cos\theta - 2\sin\theta)\right)\Bigg|_0^1\,d\theta\\
&= \int_0^{2\pi}\left(2 - \frac{1}{3}\big(\cos\theta - 2\sin\theta\big)\right)\,d\theta\\
&= \left(2\theta - \frac{1}{3}\big(\sin\theta + 2\cos\theta\big)\right)\Bigg|_0^{2\pi}\\
&= 4\pi \approx 12.566.
\end{aligned}$$

The surface and region R are shown in Figure 1.3.2.

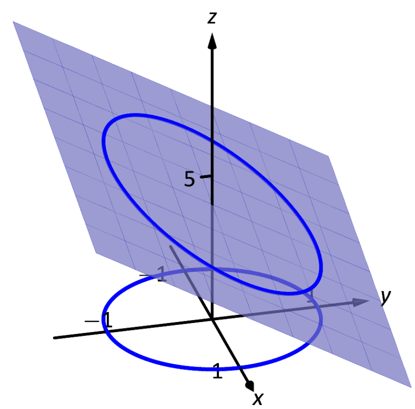


Figure 1.3.2: Evaluating a double integral with polar coordinates in Example 1.3.1.

Notes:

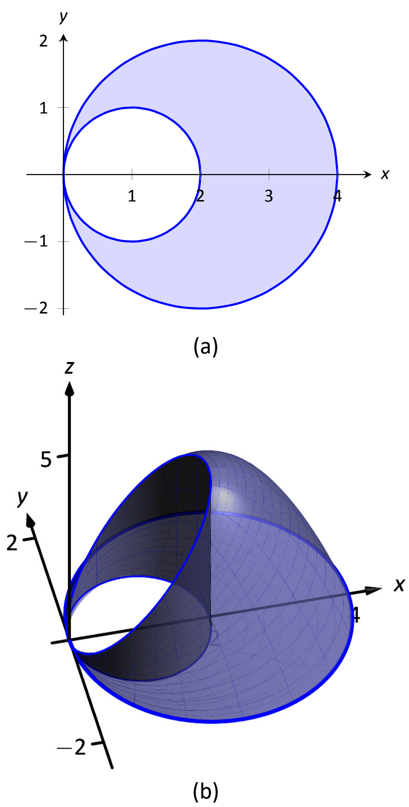


Figure 1.3.3: Showing the region R and surface used in Example 1.3.2.

Example 1.3.2 Evaluating a double integral with polar coordinates

Find the volume under the paraboloid $z = 4 - (x-2)^2 - y^2$ over the region bounded by the circles $(x-1)^2 + y^2 = 1$ and $(x-2)^2 + y^2 = 4$.

Solution At first glance, this seems like a very hard volume to compute as the region R (shown in Figure 1.3.3(a)) has a hole in it, cutting out a strange portion of the surface, as shown in part (b) of the figure. However, by describing R in terms of polar equations, the volume is not very difficult to compute. It is straightforward to show that the circle $(x-1)^2 + y^2 = 1$ has polar equation $r = 2\cos\theta$, and that the circle $(x-2)^2 + y^2 = 4$ has polar equation $r = 4\cos\theta$. Each of these circles is traced out on the interval $0 \le \theta \le \pi$. The bounds on r are $2\cos\theta \le r \le 4\cos\theta$.

Replacing x with $r\cos\theta$ in the integrand, along with replacing y with $r\sin\theta$, prepares us to evaluate the double integral $\iint_R f(x,y)\, dA$:

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta} \Big(4 - \big(r\cos\theta - 2\big)^2 - \big(r\sin\theta\big)^2\Big) r\, dr\, d\theta \\
&= \int_0^{\pi}\int_{2\cos\theta}^{4\cos\theta} \big(-r^3 + 4r^2\cos\theta\big)\, dr\, d\theta \\
&= \int_0^{\pi} \left(-\frac{1}{4}r^4 + \frac{4}{3}r^3\cos\theta\right)\Bigg|_{2\cos\theta}^{4\cos\theta} d\theta \\
&= \int_0^{\pi} \left(\left[-\frac{1}{4}(256\cos^4\theta) + \frac{4}{3}(64\cos^4\theta)\right] - \right. \\
&\qquad \left.\left[-\frac{1}{4}(16\cos^4\theta) + \frac{4}{3}(8\cos^4\theta)\right]\right) d\theta \\
&= \int_0^{\pi} \frac{44}{3}\cos^4\theta\, d\theta.
\end{aligned}$$

To integrate $\cos^4\theta$, rewrite it as $\cos^2\theta\cos^2\theta$ and employ the power-reducing formula twice:

$$\begin{aligned}
\cos^4\theta &= \cos^2\theta\cos^2\theta \\
&= \frac{1}{2}\big(1 + \cos(2\theta)\big)\frac{1}{2}\big(1 + \cos(2\theta)\big) \\
&= \frac{1}{4}\big(1 + 2\cos(2\theta) + \cos^2(2\theta)\big) \\
&= \frac{1}{4}\left(1 + 2\cos(2\theta) + \frac{1}{2}\big(1 + \cos(4\theta)\big)\right) \\
&= \frac{3}{8} + \frac{1}{2}\cos(2\theta) + \frac{1}{8}\cos(4\theta).
\end{aligned}$$

Notes:

Picking up from where we left off above, we have

$$\begin{aligned}
&= \int_0^{\pi} \frac{44}{3}\cos^4\theta\, d\theta \\
&= \int_0^{\pi} \frac{44}{3}\left(\frac{3}{8}+\frac{1}{2}\cos(2\theta)+\frac{1}{8}\cos(4\theta)\right) d\theta \\
&= \frac{44}{3}\left(\frac{3}{8}\theta+\frac{1}{4}\sin(2\theta)+\frac{1}{32}\sin(4\theta)\right)\Bigg|_0^{\pi} \\
&= \frac{11}{2}\pi \approx 17.279.
\end{aligned}$$

While this example was not trivial, the double integral would have been *much* harder to evaluate had we used rectangular coordinates.

Example 1.3.3 Evaluating a double integral with polar coordinates

Find the volume under the surface $f(x,y) = \dfrac{1}{x^2+y^2+1}$ over the sector of the circle with radius a centered at the origin in the first quadrant, as shown in Figure 1.3.4.

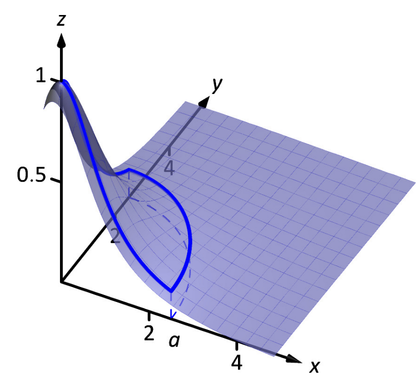


Figure 1.3.4: The surface and region R used in Example 1.3.3.

SOLUTION The region R we are integrating over is a circle with radius a, restricted to the first quadrant. Thus, in polar, the bounds on R are $0 \le r \le a$, $0 \le \theta \le \pi/2$. The integrand is rewritten in polar as

$$\frac{1}{x^2+y^2+1} \Rightarrow \frac{1}{r^2\cos^2\theta + r^2\sin^2\theta + 1} = \frac{1}{r^2+1}.$$

We find the volume as follows:

$$\begin{aligned}
\iint_R f(x,y)\, dA &= \int_0^{\pi/2}\int_0^a \frac{r}{r^2+1}\, dr\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\Big(\ln|r^2+1|\Big)\Big|_0^a\, d\theta \\
&= \int_0^{\pi/2} \frac{1}{2}\ln(a^2+1)\, d\theta \\
&= \left(\frac{1}{2}\ln(a^2+1)\theta\right)\Bigg|_0^{\pi/2} \\
&= \frac{\pi}{4}\ln(a^2+1).
\end{aligned}$$

Figure 1.3.4 shows that f shrinks to near 0 very quickly. Regardless, as a grows, so does the volume, without bound.

Note: Previous work has shown that there is finite *area* under $\frac{1}{x^2+1}$ over the entire x-axis. However, Example 1.3.3 shows that there is infinite *volume* under $\frac{1}{x^2+y^2+1}$ over the entire x-y plane.

Notes:

Example 1.3.4 Finding the volume of a sphere

Find the volume of a sphere with radius a.

Solution The sphere of radius a, centered at the origin, has equation $x^2+y^2+z^2=a^2$; solving for z, we have $z=\sqrt{a^2-x^2-y^2}$. This gives the upper half of a sphere. We wish to find the volume under this top half, then double it to find the total volume.

The region we need to integrate over is the disk of radius a, centered at the origin. Polar bounds for this equation are $0 \le r \le a$, $0 \le \theta \le 2\pi$.

All together, the volume of a sphere with radius a is:

$$2\iint_R \sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a \sqrt{a^2-(r\cos\theta)^2-(r\sin\theta)^2}\,r\,dr\,d\theta$$
$$= 2\int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta.$$

We can evaluate this inner integral with substitution. With $u=a^2-r^2$, $du=-2r\,dr$. The new bounds of integration are $u(0)=a^2$ to $u(a)=0$. Thus we have:

$$\begin{aligned}
&= \int_0^{2\pi}\int_{a^2}^{0}\left(-u^{1/2}\right)du\,d\theta\\
&= \int_0^{2\pi}\left(-\frac{2}{3}u^{3/2}\right)\Big|_{a^2}^{0}\,d\theta\\
&= \int_0^{2\pi}\left(\frac{2}{3}a^3\right)d\theta\\
&= \left(\frac{2}{3}a^3\theta\right)\Big|_0^{2\pi}\\
&= \frac{4}{3}\pi a^3.
\end{aligned}$$

Generally, the formula for the volume of a sphere with radius r is given as $4/3\pi r^3$; we have justified this formula with our calculation.

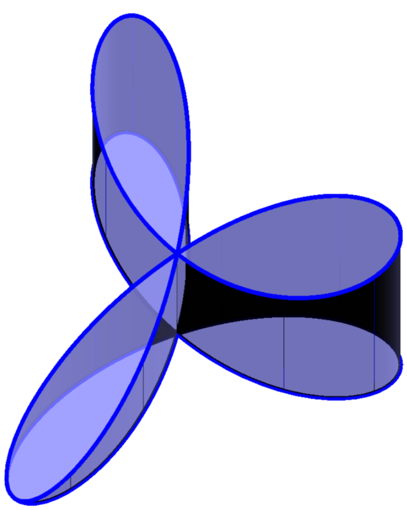


Figure 1.3.5: Visualizing the solid used in Example 1.3.5.

Example 1.3.5 Finding the volume of a solid

A sculptor wants to make a solid bronze cast of the solid shown in Figure 1.3.5, where the base of the solid has boundary, in polar coordinates, $r=\cos(3\theta)$, and the top is defined by the plane $z=1-x+0.1y$. Find the volume of the solid.

Solution From the outset, we should recognize that knowing *how to set up* this problem is probably more important than knowing *how to compute*

Notes:

the integrals. The iterated integral to come is not "hard" to evaluate, though it is long, requiring lots of algebra. Once the proper iterated integral is determined, one can use readily–available technology to help compute the final answer.

The region R that we are integrating over is bound by $0 \leq r \leq \cos(3\theta)$, for $0 \leq \theta \leq \pi$ (note that this rose curve is traced out on the interval $[0, \pi]$, not $[0, 2\pi]$). This gives us our bounds of integration. The integrand is $z = 1 - x + 0.1y$; converting to polar, we have that the volume V is:

$$V = \iint_R f(x,y)\, dA = \int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta.$$

Distributing the r, the inner integral is easy to evaluate, leading to

$$\int_0^{\pi} \left(\frac{1}{2}\cos^2(3\theta) - \frac{1}{3}\cos^3(3\theta)\cos\theta + \frac{0.1}{3}\cos^3(3\theta)\sin\theta\right)\, d\theta.$$

This integral takes time to compute by hand; it is rather long and cumbersome. The powers of cosine need to be reduced, and products like $\cos(3\theta)\cos\theta$ need to be turned to sums using the Product To Sum formulas in the back cover of this text.

We rewrite $\frac{1}{2}\cos^2(3\theta)$ as $\frac{1}{4}\big(1+\cos(6\theta)\big)$. We can also rewrite $\frac{1}{3}\cos^3(3\theta)\cos\theta$ as:

$$\frac{1}{3}\cos^3(3\theta)\cos\theta = \frac{1}{3}\cos^2(3\theta)\cos(3\theta)\cos\theta = \frac{1}{3}\frac{1+\cos(6\theta)}{2}\big(\cos(4\theta)+\cos(2\theta)\big).$$

This last expression still needs simplification, but eventually all terms can be reduced to the form $a\cos(m\theta)$ or $a\sin(m\theta)$ for various values of a and m.

We forgo the algebra and recommend the reader employ technology, such as WolframAlpha®, to compute the numeric answer. Such technology gives:

$$\int_0^{\pi} \int_0^{\cos(3\theta)} \big(1 - r\cos\theta + 0.1r\sin\theta\big) r\, dr\, d\theta = \frac{\pi}{4} \approx 0.785u^3.$$

Since the units were not specified, we leave the result as almost 0.8 cubic units (meters, feet, etc.) Should the artist want to scale the piece uniformly, so that each rose petal had a length other than 1, she should keep in mind that scaling by a factor of k scales the volume by a factor of k^3.

We have used iterated integrals to find areas of plane regions and volumes under surfaces. Just as a single integral can be used to compute much more than "area under the curve," iterated integrals can be used to compute much more than we have thus far seen. The next two sections show two, among many, applications of iterated integrals.

Notes:

Exercises 1.3

Terms and Concepts

13 03 ex 01 1. When evaluating $\iint_R f(x,y)\ dA$ using polar coordinates, $f(x,y)$ is replaced with ________ and dA is replaced with ________.

13 03 ex 02 2. Why would one be interested in evaluating a double integral with polar coordinates?

Problems

13 03 exset 01 **In Exercises 3 – 10, a function $f(x,y)$ is given and a region R of the x-y plane is described. Set up and evaluate $\iint_R f(x,y)\ dA$ using polar coordinates.**

13 03 ex 03 3. $f(x,y) = 3x - y + 4$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

13 03 ex 04 4. $f(x,y) = 4x + 4y$; R is the region enclosed by the circle $x^2 + y^2 = 4$.

13 03 ex 05 5. $f(x,y) = 8 - y$; R is the region enclosed by the circles with polar equations $r = \cos\theta$ and $r = 3\cos\theta$.

13 03 ex 06 6. $f(x,y) = 4$; R is the region enclosed by the petal of the rose curve $r = \sin(2\theta)$ in the first quadrant.

13 03 ex 07 7. $f(x,y) = \ln\left(x^2 + y^2\right)$; R is the annulus enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

13 03 ex 08 8. $f(x,y) = 1 - x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 1$.

13 03 ex 09 9. $f(x,y) = x^2 - y^2$; R is the region enclosed by the circle $x^2 + y^2 = 36$ in the first and fourth quadrants.

13 03 ex 10 10. $f(x,y) = (x-y)/(x+y)$; R is the region enclosed by the lines $y = x$, $y = 0$ and the circle $x^2 + y^2 = 1$ in the first quadrant.

13 03 exset 02 **In Exercises 11 – 14, an iterated integral in rectangular coordinates is given. Rewrite the integral using polar coordinates and evaluate the new double integral.**

13 03 ex 11 11. $\displaystyle\int_0^5 \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}\ dy\,dx$

13 03 ex 12 12. $\displaystyle\int_{-4}^4 \int_{-\sqrt{16-y^2}}^{0} \left(2y - x\right)\ dx\,dy$

13 03 ex 13 13. $\displaystyle\int_0^2 \int_y^{\sqrt{8-y^2}} \left(x + y\right)\ dx\,dy$

13 03 ex 14 14. $\displaystyle\int_{-2}^{-1} \int_0^{\sqrt{4-x^2}} \left(x+5\right)\ dy\,dx + \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \left(x+5\right)\ dy\,dx + \int_1^2 \int_0^{\sqrt{4-x^2}} \left(x+5\right)\ dy\,dx$

Hint: draw the region of each integral carefully and see how they all connect.

13 03 exset 03 **In Exercises 15 – 16, special double integrals are presented that are especially well suited for evaluation in polar coordinates.**

13 03 ex 15 15. Consider $\displaystyle\iint_R e^{-(x^2+y^2)}\ dA$.

(a) Why is this integral difficult to evaluate in rectangular coordinates, regardless of the region R?

(b) Let R be the region bounded by the circle of radius a centered at the origin. Evaluate the double integral using polar coordinates.

(c) Take the limit of your answer from (b), as $a \to \infty$. What does this imply about the volume under the surface of $e^{-(x^2+y^2)}$ over the entire x-y plane?

13 03 ex 16 16. The surface of a right circular cone with height h and base radius a can be described by the equation $f(x,y) = h - h\sqrt{\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2}}$, where the tip of the cone lies at $(0,0,h)$ and the circular base lies in the x-y plane, centered at the origin.

Confirm that the volume of a right circular cone with height h and base radius a is $V = \dfrac{1}{3}\pi a^2 h$ by evaluating $\displaystyle\iint_R f(x,y)\ dA$ in polar coordinates.

1.4 Center of Mass

We have used iterated integrals to find areas of plane regions and signed volumes under surfaces. A brief recap of these uses will be useful in this section as we apply iterated integrals to compute the **mass** and **center of mass** of planar regions.

To find the area of a planar region, we evaluated the double integral $\iint_R \, dA$. That is, summing up the areas of lots of little subregions of R gave us the total area. Informally, we think of $\iint_R \, dA$ as meaning "sum up lots of little areas over R."

To find the signed volume under a surface, we evaluated the double integral $\iint_R f(x,y) \, dA$. Recall that the "dA" is not just a "bookend" at the end of an integral; rather, it is multiplied by $f(x,y)$. We regard $f(x,y)$ as giving a height, and dA still giving an area: $f(x,y) \, dA$ gives a volume. Thus, informally, $\iint_R f(x,y) \, dA$ means "sum up lots of little volumes over R."

We now extend these ideas to other contexts.

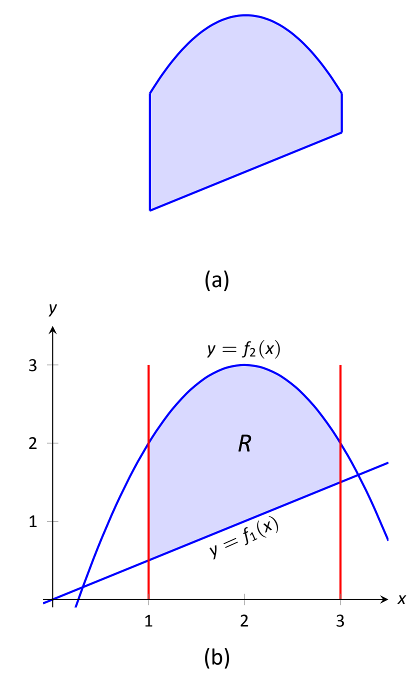


Figure 1.4.1: Illustrating the concept of a lamina.

Mass and Weight

Consider a thin sheet of material with constant thickness and finite area. Mathematicians (and physicists and engineers) call such a sheet a **lamina**. So consider a lamina, as shown in Figure 1.4.1(a), with the shape of some planar region R, as shown in part (b).

We can write a simple double integral that represents the mass of the lamina: $\iint_R \, dm$, where "dm" means "a little mass." That is, the double integral states the total mass of the lamina can be found by "summing up lots of little masses over R."

To evaluate this double integral, partition R into n subregions as we have done in the past. The i^{th} subregion has area ΔA_i. A fundamental property of mass is that "mass=density×area." If the lamina has a constant density δ, then the mass of this i^{th} subregion is $\Delta m_i = \delta \Delta A_i$. That is, we can compute a small amount of mass by multiplying a small amount of area by the density.

If density is variable, with density function $\delta = \delta(x,y)$, then we can approximate the mass of the i^{th} subregion of R by multiplying ΔA_i by $\delta(x_i, y_i)$, where (x_i, y_i) is a point in that subregion. That is, for a small enough subregion of R, the density across that region is almost constant.

Note: *Mass* and *weight* are different measures. Since they are scalar multiples of each other, it is often easy to treat them as the same measure. In this section we effectively treat them as the same, as our technique for finding mass is the same as for finding weight. The density functions used will simply have different units.

The total mass M of the lamina is approximately the sum of approximate masses of subregions:

$$M \approx \sum_{i=1}^{n} \Delta m_i = \sum_{i=1}^{n} \delta(x_i, y_i) \Delta A_i.$$

Notes:

Taking the limit as the size of the subregions shrinks to 0 gives us the actual mass; that is, integrating $\delta(x, y)$ over R gives the mass of the lamina.

Definition 1.4.1 Mass of a Lamina with Vairable Density

Let $\delta(x, y)$ be a continuous density function of a lamina corresponding to a plane region R. The mass M of the lamina is

$$\text{mass } M = \iint_R \, dm = \iint_R \delta(x, y) \, dA.$$

Example 1.4.1 Finding the mass of a lamina with constant density

Find the mass of a square lamina, with side length 1, with a density of $\delta = 3\text{gm/cm}^2$.

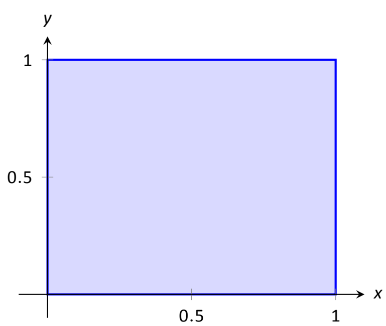


Figure 1.4.2: A region R representing a lamina in Example 1.4.1.

Solution We represent the lamina with a square region in the plane as shown in Figure 1.4.2. As the density is constant, it does not matter where we place the square.

Following Definition 1.4.1, the mass M of the lamina is

$$M = \iint_R 3 \, dA = \int_0^1 \int_0^1 3 \, dx \, dy = 3 \int_0^1 \int_0^1 \, dx \, dy = 3\text{gm}.$$

This is all very straightforward; note that all we really did was find the area of the lamina and multiply it by the constant density of 3gm/cm^2.

Example 1.4.2 Finding the mass of a lamina with variable density

Find the mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 1.4.2), with variable density $\delta(x, y) = (x + y + 2)\text{gm/cm}^2$.

Solution The variable density δ, in this example, is very uniform, giving a density of 3 in the center of the square and changing linearly. A graph of $\delta(x, y)$ can be seen in Figure 1.4.3; notice how "same amount" of density is above $z = 3$ as below. We'll comment on the significance of this momentarily.

The mass M is found by integrating $\delta(x, y)$ over R. The order of integration

Notes:

is not important; we choose $dx\,dy$ arbitrarily. Thus:

$$\begin{aligned}
M = \iint_R (x+y+2)\,dA &= \int_0^1 \int_0^1 (x+y+2)\,dx\,dy \\
&= \int_0^1 \left(\frac{1}{2}x^2 + x(y+2)\right)\Big|_0^1\,dy \\
&= \int_0^1 \left(\frac{5}{2}+y\right)\,dy \\
&= \left(\frac{5}{2}y + \frac{1}{2}y^2\right)\Big|_0^1 \\
&= 3\text{gm}.
\end{aligned}$$

It turns out that since since the density of the lamina is so uniformly distributed "above and below" $z = 3$ that the mass of the lamina is the same as if it had a constant density of 3. The density functions in Examples 1.4.1 and 1.4.2 are graphed in Figure 1.4.3, which illustrates this concept.

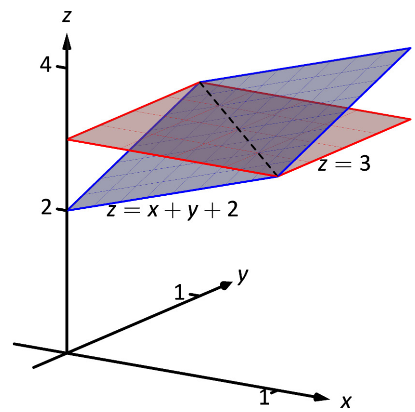


Figure 1.4.3: Graphing the density functions in Examples 1.4.1 and 1.4.2.

Example 1.4.3 Finding the weight of a lamina with variable density

Find the weight of the lamina represented by the disk with radius 2ft, centered at the origin, with density function $\delta(x,y) = (x^2+y^2+1)\text{lb/ft}^2$. Compare this to the weight of the lamina with the same shape and density $\delta(x,y) = (2\sqrt{x^2+y^2}+1)\text{lb/ft}^2$.

Solution A direct application of Definition 1.4.1 states that the weight of the lamina is $\iint_R \delta(x,y)\,dA$. Since our lamina is in the shape of a circle, it makes sense to approach the double integral using polar coordinates.

The density function $\delta(x,y) = x^2+y^2+1$ becomes $\delta(r,\theta) = (r\cos\theta)^2 + (r\sin\theta)^2 + 1 = r^2+1$. The circle is bounded by $0 \le r \le 2$ and $0 \le \theta \le 2\pi$. Thus the weight W is:

$$\begin{aligned}
W &= \int_0^{2\pi} \int_0^2 (r^2+1)r\,dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{4}r^4 + \frac{1}{2}r^2\right)\Big|_0^2\,d\theta \\
&= \int_0^{2\pi} (6)\,d\theta \\
&= 12\pi \approx 37.70\text{lb}.
\end{aligned}$$

Now compare this with the density function $\delta(x,y) = 2\sqrt{x^2+y^2}+1$. Converting this to polar coordinates gives $\delta(r,\theta) = 2\sqrt{(r\cos\theta)^2+(r\sin\theta)^2}+1 =$

Notes:

$2r+1$. Thus the weight W is:

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^2 (2r+1)r\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{2}{3}r^3+\frac{1}{2}r^2\right)\Big|_0^2 d\theta \\ &= \int_0^{2\pi}\left(\frac{22}{3}\right)\,d\theta \\ &= \frac{44}{3}\pi \approx 46.08\text{lb}. \end{aligned}$$

One would expect different density functions to return different weights, as we have here. The density functions were chosen, though, to be similar: each gives a density of 1 at the origin and a density of 5 at the outside edge of the circle, as seen in Figure 1.4.4.

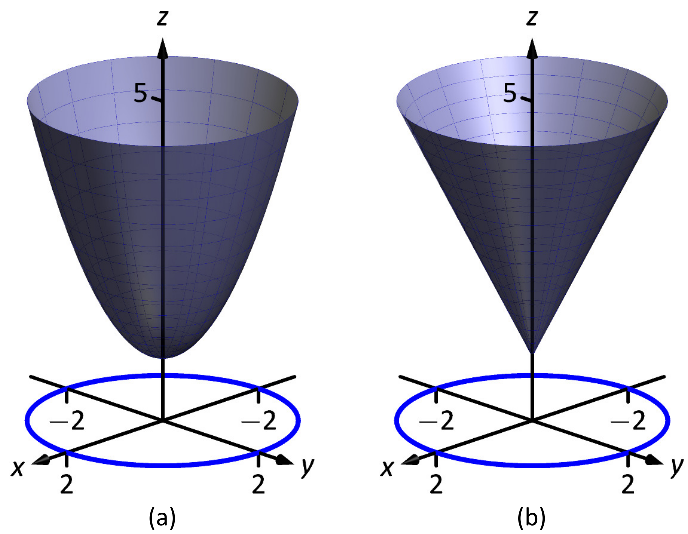


Figure 1.4.4: Graphing the density functions in Example 1.4.3. In (a) is the density function $\delta(x,y)=x^2+y^2+1$; in (b) is $\delta(x,y)=2\sqrt{x^2+y^2}+1$.

Notice how $x^2+y^2+1\leq 2\sqrt{x^2+y^2}+1$ over the circle; this results in less weight.

Plotting the density functions can be useful as our understanding of mass can be related to our understanding of "volume under a surface." We interpreted $\iint_R f(x,y)\,dA$ as giving the volume under f over R; we can understand $\iint_R \delta(x,y)\,dA$ in the same way. The "volume" under δ over R is actually mass;

Notes:

by compressing the "volume" under δ onto the x-y plane, we get "more mass" in some areas than others – i.e., areas of greater density.

Knowing the mass of a lamina is one of several important measures. Another is the **center of mass**, which we discuss next.

Center of Mass

Consider a disk of radius 1 with uniform density. It is common knowledge that the disk will balance on a point if the point is placed at the center of the disk. What if the disk does not have a uniform density? Through trial-and-error, we should still be able to find a spot on the disk at which the disk will balance on a point. This balance point is referred to as the **center of mass**, or **center of gravity**. It is though all the mass is "centered" there. In fact, if the disk has a mass of 3kg, the disk will behave physically as though it were a point-mass of 3kg located at its center of mass. For instance, the disk will naturally spin with an axis through its center of mass (which is why it is important to "balance" the tires of your car: if they are "out of balance", their center of mass will be outside of the axle and it will shake terribly).

We find the center of mass based on the principle of a **weighted average**. Consider a college class in which your homework average is 90%, your test average is 73%, and your final exam grade is an 85%. Experience tells us that our final grade *is not* the *average* of these three grades: that is, it is not:

$$\frac{0.9+0.73+0.85}{3} \approx 0.837 = 83.7\%.$$

That is, you are probably not pulling a B in the course. Rather, your grades are *weighted*. Let's say the homework is worth 10% of the grade, tests are 60% and the exam is 30%. Then your final grade is:

$$(0.1)(0.9)+(0.6)(0.73)+(0.3)(0.85) = 0.783 = 78.3\%.$$

Each grade is multiplied by a **weight**.

In general, given values $x_1, x_2, \ldots, x_n$ and weights $w_1, w_2, \ldots, w_n$, the weighted average of the n values is

$$\sum_{i=1}^{n} w_i x_i \bigg/ \sum_{i=1}^{n} w_i.$$

In the grading example above, the sum of the weights 0.1, 0.6 and 0.3 is 1, so we don't see the division by the sum of weights in that instance.

How this relates to center of mass is given in the following theorem.

Notes:

Theorem 1.4.1 Center of Mass of Discrete Linear System

Let point masses $m_1, m_2, \ldots, m_n$ be distributed along the x-axis at locations $x_1, x_2, \ldots, x_n$, respectively. The center of mass $\overline{x}$ of the system is located at

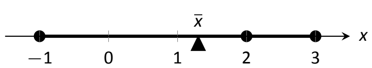
$$\overline{x} = \sum_{i=1}^{n} m_i x_i \Big/ \sum_{i=1}^{n} m_i.$$

Example 1.4.4 Finding the center of mass of a discrete linear system

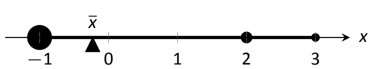
1. Point masses of 2gm are located at $x = -1$, $x = 2$ and $x = 3$ are connected by a thin rod of negligible weight. Find the center of mass of the system.
2. Point masses of 10gm, 2gm and 1gm are located at $x = -1$, $x = 2$ and $x = 3$, respectively, are connected by a thin rod of negligible weight. Find the center of mass of the system.

Solution

1. Following Theorem 1.4.1, we compute the center of mass as:
$$\overline{x} = \frac{2(-1) + 2(2) + 2(3)}{2 + 2 + 2} = \frac{4}{3} = 1.\overline{3}.$$
So the system would balance on a point placed at $x = 4/3$, as illustrated in Figure 1.4.5(a).
2. Again following Theorem 1.4.1, we find:
$$\overline{x} = \frac{10(-1) + 2(2) + 1(3)}{10 + 2 + 1} = \frac{-3}{13} \approx -0.23.$$
Placing a large weight at the left hand side of the system moves the center of mass left, as shown in Figure 1.4.5(b).



(a)



(b)

Figure 1.4.5: Illustrating point masses along a thin rod and the center of mass.

In a discrete system (i.e., mass is located at individual points, not along a continuum) we find the center of mass by dividing the mass into a **moment** of the system. In general, a moment is a weighted measure of distance from a particular point or line. In the case described by Theorem 1.4.1, we are finding a weighted measure of distances from the y-axis, so we refer to this as **the moment about the y-axis**, represented by M_y. Letting M be the total mass of the system, we have $\overline{x} = M_y/M$.

Notes:

We can extend the concept of the center of mass of discrete points along a line to the center of mass of discrete points in the plane rather easily. To do so, we define some terms then give a theorem.

Definition 1.4.2 Moments about the x- and y- Axes.

Let point masses m_1, $m_2, \ldots, m_n$ be located at points (x_1, y_1), $(x_2, y_2) \ldots, (x_n, y_n)$, respectively, in the x-y plane.

1. The **moment about the y-axis**, M_y, is $M_y = \displaystyle\sum_{i=1}^{n} m_i x_i$.

2. The **moment about the x-axis**, M_x, is $M_x = \displaystyle\sum_{i=1}^{n} m_i y_i$.

One can think that these definitions are "backwards" as M_y sums up "x" distances. But remember, "x" distances are measurements of distance from the y-axis, hence defining the moment about the y-axis.

We now define the center of mass of discrete points in the plane.

Theorem 1.4.2 Center of Mass of Discrete Planar System

Let point masses m_1, $m_2, \ldots, m_n$ be located at points (x_1, y_1), $(x_2, y_2) \ldots, (x_n, y_n)$, respectively, in the x-y plane, and let $M = \displaystyle\sum_{i=1}^{n} m_i$.

The center of mass of the system is at $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{M_y}{M} \quad \text{and} \quad \overline{y} = \frac{M_x}{M}.$$

Example 1.4.5 Finding the center of mass of a discrete planar system

Let point masses of 1kg, 2kg and 5kg be located at points $(2, 0)$, $(1, 1)$ and $(3, 1)$, respectively, and are connected by thin rods of negligible weight. Find the center of mass of the system.

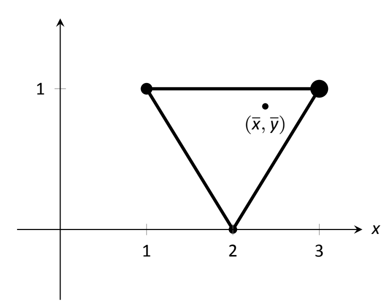


Figure 1.4.6: Illustrating the center of mass of a discrete planar system in Example 1.4.5.

SOLUTION We follow Theorem 1.4.2 and Definition 1.4.2 to find M, M_x and M_y:

$M = 1 + 2 + 5 = 8\text{kg}.$

Notes:

$$M_x = \sum_{i=1}^{n} m_i y_i \qquad\qquad M_y = \sum_{i=1}^{n} m_i x_i$$
$$= 1(0) + 2(1) + 5(1) \qquad\qquad = 1(2) + 2(1) + 5(3)$$
$$= 7. \qquad\qquad = 19.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{19}{8}, \frac{7}{8}\right) = (2.375, 0.875)$, illustrated in Figure 1.4.6.

We finally arrive at our true goal of this section: finding the center of mass of a lamina with variable density. While the above measurement of center of mass is interesting, it does not directly answer more realistic situations where we need to find the center of mass of a contiguous region. However, understanding the discrete case allows us to approximate the center of mass of a planar lamina; using calculus, we can refine the approximation to an exact value.

We begin by representing a planar lamina with a region R in the x-y plane with density function $\delta(x, y)$. Partition R into n subdivisions, each with area ΔA_i. As done before, we can approximate the mass of the i^{th} subregion with $\delta(x_i, y_i)\Delta A_i$, where (x_i, y_i) is a point inside the i^{th} subregion. We can approximate the moment of this subregion about the y-axis with $x_i\delta(x_i, y_i)\Delta A_i$ – that is, by multiplying the approximate mass of the region by its approximate distance from the y-axis. Similarly, we can approximate the moment about the x-axis with $y_i\delta(x_i, y_i)\Delta A_i$. By summing over all subregions, we have:

$$\text{mass: } M \approx \sum_{i=1}^{n} \delta(x_i, y_i)\Delta A_i \quad \text{(as seen before)}$$
$$\text{moment about the } x\text{-axis: } M_x \approx \sum_{i=1}^{n} y_i\delta(x_i, y_i)\Delta A_i$$
$$\text{moment about the } y\text{-axis: } M_y \approx \sum_{i=1}^{n} x_i\delta(x_i, y_i)\Delta A_i$$

By taking limits, where size of each subregion shrinks to 0 in both the x and y directions, we arrive at the double integrals given in the following theorem.

Notes:

Theorem 1.4.3 Center of Mass of a Planar Lamina, Moments

Let a planar lamina be represented by a region R in the x-y plane with density function $\delta(x,y)$.

1. mass: $M = \iint_R \delta(x,y)\,dA$
2. moment about the x-axis: $M_x = \iint_R y\delta(x,y)\,dA$
3. moment about the y-axis: $M_y = \iint_R x\delta(x,y)\,dA$
4. The center of mass of the lamina is

$$(\overline{x},\overline{y}) = \left(\frac{M_y}{M},\frac{M_x}{M}\right).$$

We start our practice of finding centers of mass by revisiting some of the lamina used previously in this section when finding mass. We will just set up the integrals needed to compute M, M_x and M_y and leave the details of the integration to the reader.

Example 1.4.6 Finding the center of mass of a lamina

Find the center mass of a square lamina, with side length 1, with a density of $\delta = 3\text{gm/cm}^2$. (Note: this is the lamina from Example 1.4.1.)

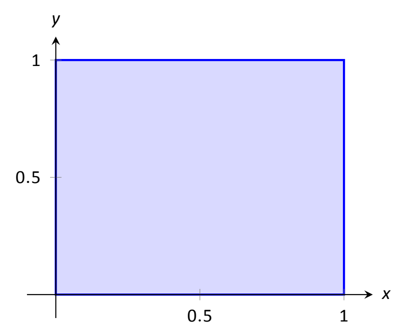


Figure 1.4.7: A region R representing a lamina in Example 1.4.1.

Solution We represent the lamina with a square region in the plane as shown in Figure 1.4.7 as done previously.

Following Theorem 1.4.3, we find M, M_x and M_y:

$$M = \iint_R 3\,dA = \int_0^1\int_0^1 3\,dx\,dy = 3\text{gm}.$$

$$M_x = \iint_R 3y\,dA = \int_0^1\int_0^1 3y\,dx\,dy = 3/2 = 1.5.$$

$$M_y = \iint_R 3x\,dA = \int_0^1\int_0^1 3x\,dx\,dy = 3/2 = 1.5.$$

Thus the center of mass is $(\overline{x},\overline{y}) = \left(\frac{M_y}{M},\frac{M_x}{M}\right) = (1.5/3, 1.5/3) = (0.5, 0.5)$. This is what we should have expected: the center of mass of a square with constant density is the center of the square.

Notes:

Example 1.4.7 Finding the center of mass of a lamina

Find the center of mass of a square lamina, represented by the unit square with lower lefthand corner at the origin (see Figure 1.4.7), with variable density $\delta(x, y) = (x + y + 2)\text{gm/cm}^2$. (Note: this is the lamina from Example 1.4.2.)

Solution We follow Theorem 1.4.3, to find M, M_x and M_y:

$$M = \iint_R (x + y + 2)\, dA = \int_0^1 \int_0^1 (x + y + 2)\, dx\, dy = 3\text{gm}.$$

$$M_x = \iint_R y(x + y + 2)\, dA = \int_0^1 \int_0^1 y(x + y + 2)\, dx\, dy = \frac{19}{12}.$$

$$M_y = \iint_R x(x + y + 2)\, dA = \int_0^1 \int_0^1 x(x + y + 2)\, dx\, dy = \frac{19}{12}.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{19}{36}, \frac{19}{36}\right) \approx (0.528, 0.528)$. While the mass of this lamina is the same as the lamina in the previous example, the greater density found with greater x and y values pulls the center of mass from the center slightly towards the upper righthand corner.

Example 1.4.8 Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the circle with radius 2ft, centered at the origin, with density function $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/ft}^2$. (Note: this is one of the lamina used in Example 1.4.3.)

Solution As done in Example 1.4.3, it is best to describe R using polar coordinates. Thus when we compute M_y, we will integrate not $x\delta(x, y) = x(x^2 + y^2 + 1)$, but rather $\big(r\cos\theta\big)\delta(r\cos\theta, r\sin\theta) = \big(r\cos\theta\big)\big(r^2 + 1\big)$. We compute M, M_x and M_y:

$$M = \int_0^{2\pi} \int_0^2 (r^2 + 1) r\, dr\, d\theta = 12\pi \approx 37.7\text{lb}.$$

$$M_x = \int_0^{2\pi} \int_0^2 (r\sin\theta)(r^2 + 1) r\, dr\, d\theta = 0.$$

$$M_y = \int_0^{2\pi} \int_0^2 (r\cos\theta)(r^2 + 1) r\, dr\, d\theta = 0.$$

Since R and the density of R are both symmetric about the x and y axes, it should come as no big surprise that the moments about each axis is 0. Thus the center

Notes:

of mass is $(\bar{x}, \bar{y}) = (0, 0)$.

Example 1.4.9 Finding the center of mass of a lamina

Find the center of mass of the lamina represented by the region R shown in Figure 1.4.8, half an annulus with outer radius 6 and inner radius 5, with constant density 2lb/ft^2.

Solution Once again it will be useful to represent R in polar coordinates. Using the description of R and/or the illustration, we see that R is bounded by $5 \leq r \leq 6$ and $0 \leq \theta \leq \pi$. As the lamina is symmetric about the y-axis, we should expect $M_y = 0$. We compute M, M_x and M_y:

$$M = \int_0^{\pi} \int_5^6 (2) r \, dr \, d\theta = 11\pi \text{lb}.$$

$$M_x = \int_0^{\pi} \int_5^6 (r \sin\theta)(2) r \, dr \, d\theta = \frac{364}{3} \approx 121.33.$$

$$M_y = \int_0^{\pi} \int_5^6 (r \cos\theta)(2) r \, dr \, d\theta = 0.$$

Thus the center of mass is $(\bar{x}, \bar{y}) = \left(0, \frac{364}{33\pi}\right) \approx (0, 3.51)$. The center of mass is indicated in Figure 1.4.8; note how it lies outside of R!

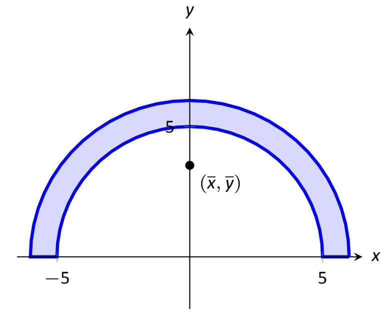


Figure 1.4.8: Illustrating the region R in Example 1.4.9.

This section has shown us another use for iterated integrals beyond finding area or signed volume under the curve. While there are many uses for iterated integrals, we give one more application in the following section: computing surface area.

Notes:

Exercises 1.4

Terms and Concepts

13 04 ex 01 1. Why is it easy to use "mass" and "weight" interchangeably, even though they are different measures?

13 04 ex 02 2. Given a point (x, y), the value of x is a measure of distance from the ________-axis.

13 04 ex 03 3. We can think of $\iint_R \, dm$ as meaning "sum up lots of ________"

13 04 ex 04 4. What is a "discrete planar system?"

13 04 ex 05 5. Why does M_x use $\iint_R y\delta(x, y)\, dA$ instead of $\iint_R x\delta(x, y)\, dA$; that is, why do we use "y" and not "x"?

13 04 ex 06 6. Describe a situation where the center of mass of a lamina does not lie within the region of the lamina itself.

Problems

13 04 exset 01 **In Exercises 7 – 10, point masses are given along a line or in the plane. Find the center of mass $\bar{x}$ or $(\bar{x}, \bar{y})$, as appropriate. (All masses are in grams and distances are in cm.)**

13 04 ex 07 7. $m_1 = 4$ at $x = 1$; $\quad m_2 = 3$ at $x = 3$; $\quad m_3 = 5$ at $x = 10$

13 04 ex 08 8. $m_1 = 2$ at $x = -3$; $\quad m_2 = 2$ at $x = -1$;
$m_3 = 3$ at $x = 0$; $\quad m_4 = 3$ at $x = 7$

13 04 ex 10 9. $m_1 = 2$ at $(-2, -2)$; $\quad m_2 = 2$ at $(2, -2)$;
$m_3 = 20$ at $(0, 4)$

13 04 ex 09 10. $m_1 = 1$ at $(-1, -1)$; $\quad m_2 = 2$ at $(-1, 1)$;
$m_3 = 2$ at $(1, 1)$; $\quad m_4 = 1$ at $(1, -1)$

13 04 exset 02 **In Exercises 11 – 18, find the mass/weight of the lamina described by the region R in the plane and its density function $\delta(x, y)$.**

13 04 ex 11 11. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5\text{gm/cm}^2$

13 04 ex 12 12. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)\text{gm/cm}^2$

13 04 ex 13 13. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2\text{lb/in}^2$

13 04 ex 14 14. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$

13 04 ex 15 15. R is the disk centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)\text{kg/m}^2$

13 04 ex 16 16. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$

13 04 ex 17 17. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4\text{lb/ft}^2$

13 04 ex 18 18. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

13 04 exset 03 **In Exercises 19 – 26, find the center of mass of the lamina described by the region R in the plane and its density function $\delta(x, y)$.**
Note: these are the same lamina as in Exercises 11 – 18.

13 04 ex 19 19. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = 5\text{gm/cm}^2$

13 04 ex 20 20. R is the rectangle with corners $(1, -3)$, $(1, 2)$, $(7, 2)$ and $(7, -3)$; $\delta(x, y) = (x + y^2)\text{gm/cm}^2$

13 04 ex 21 21. R is the triangle with corners $(-1, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = 2\text{lb/in}^2$

13 04 ex 22 22. R is the triangle with corners $(0, 0)$, $(1, 0)$, and $(0, 1)$; $\delta(x, y) = (x^2 + y^2 + 1)\text{lb/in}^2$

13 04 ex 23 23. R is the disk centered at the origin with radius 2; $\delta(x, y) = (x + y + 4)\text{kg/m}^2$

13 04 ex 24 24. R is the circle sector bounded by $x^2 + y^2 = 25$ in the first quadrant; $\delta(x, y) = (\sqrt{x^2 + y^2} + 1)\text{kg/m}^2$

13 04 ex 25 25. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = 4\text{lb/ft}^2$

13 04 ex 26 26. R is the annulus in the first and second quadrants bounded by $x^2 + y^2 = 9$ and $x^2 + y^2 = 36$; $\delta(x, y) = \sqrt{x^2 + y^2}\text{lb/ft}^2$

13 04 exset 04 **The *moment of inertia* I is a measure of the tendency of a lamina to resist rotating about an axis or continue to rotate about an axis. I_x is the moment of inertia about the x-axis, I_y is the moment of inertia about the x-axis, and I_O is the moment of inertia about the origin. These are computed as follows:**

- $I_x = \displaystyle\iint_R y^2 \, dm$
- $I_y = \displaystyle\iint_R x^2 \, dm$
- $I_O = \displaystyle\iint_R \left(x^2 + y^2\right) dm$

In Exercises 27 – 30, a lamina corresponding to a planar region R is given with a mass of 16 units. For each, compute I_x, I_y and I_O.

13 04 ex 27 27. R is the 4×4 square with corners at $(-2, -2)$ and $(2, 2)$ with density $\delta(x, y) = 1$.

13 04 ex 28 28. R is the 8×2 rectangle with corners at $(-4, -1)$ and $(4, 1)$ with density $\delta(x, y) = 1$.

13 04 ex 29 29. R is the 4×2 rectangle with corners at $(-2, -1)$ and $(2, 1)$ with density $\delta(x, y) = 2$.

13 04 ex 30 30. R is the disk with radius 2 centered at the origin with density $\delta(x, y) = 4/\pi$.

1.5 Surface Area

In Section 7.4 we used definite integrals to compute the arc length of plane curves of the form $y = f(x)$. We later extended these ideas to compute the arc length of plane curves defined by parametric or polar equations.

The natural extension of the concept of "arc length over an interval" to surfaces is "surface area over a region."

Consider the surface $z = f(x, y)$ over a region R in the x-y plane, shown in Figure 1.5.1(a). Because of the domed shape of the surface, the surface area will be greater than that of the area of the region R. We can find this area using the same basic technique we have used over and over: we'll make an approximation, then using limits, we'll refine the approximation to the exact value.

As done to find the volume under a surface or the mass of a lamina, we subdivide R into n subregions. Here we subdivide R into rectangles, as shown in the figure. One such subregion is outlined in the figure, where the rectangle has dimensions Δx_i and Δy_i, along with its corresponding region on the surface.

In part (b) of the figure, we zoom in on this portion of the surface. When Δx_i and Δy_i are small, the function is approximated well by the tangent plane at any point (x_i, y_i) in this subregion, which is graphed in part (b). In fact, the tangent plane approximates the function so well that in this figure, it is virtually indistinguishable from the surface itself! Therefore we can approximate the surface area S_i of this region of the surface with the area T_i of the corresponding portion of the tangent plane.

This portion of the tangent plane is a parallelogram, defined by sides $\vec{u}$ and $\vec{v}$, as shown. One of the applications of the cross product from Section 10.4 is that the area of this parallelogram is $||\, \vec{u} \times \vec{v}\, ||$. Once we can determine $\vec{u}$ and $\vec{v}$, we can determine the area.

$\vec{u}$ is tangent to the surface in the direction of x, therefore, from Section 12.7, $\vec{u}$ is parallel to $\langle 1, 0, f_x(x_i, y_i)\rangle$. The x-displacement of $\vec{u}$ is Δx_i, so we know that $\vec{u} = \Delta x_i \langle 1, 0, f_x(x_i, y_i)\rangle$. Similar logic shows that $\vec{v} = \Delta y_i \langle 0, 1, f_y(x_i, y_i)\rangle$. Thus:

$$\begin{aligned}
\text{surface area } S_i &\approx \text{area of } T_i \\
&= ||\, \vec{u} \times \vec{v}\, || \\
&= \big|\big| \Delta x_i \langle 1, 0, f_x(x_i, y_i)\rangle \times \Delta y_i \langle 0, 1, f_y(x_i, y_i)\rangle \big|\big| \\
&= \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2}\, \Delta x_i \Delta y_i.
\end{aligned}$$

Note that $\Delta x_i \Delta y_i = \Delta A_i$, the area of the i^{th} subregion.

Summing up all n of the approximations to the surface area gives

$$\text{surface area over } R \approx \sum_{i=1}^{n} \sqrt{1 + f_x(x_i, y_i)^2 + f_y(x_i, y_i)^2}\, \Delta A_i.$$

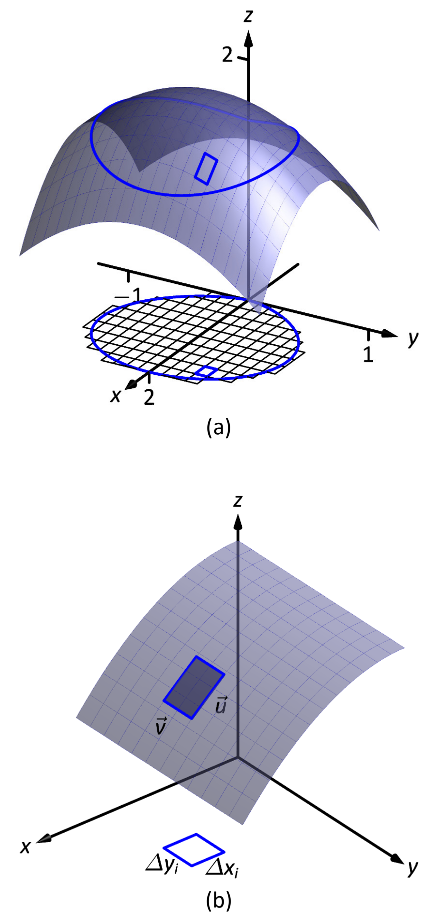


Figure 1.5.1: Developing a method of computing surface area.

Notes:

Once again take a limit as all of the Δx_i and Δy_i shrink to 0; this leads to a double integral.

Note: as done before, we think of "$\iint_R dS$" as meaning "sum up lots of little surface areas over R."

The concept of surface area is *defined* here, for while we already have a notion of the area of a region in the *plane*, we did not yet have a solid grasp of what "the area of a surface in *space*" means.

Definition 1.5.1 Surface Area

Let $z = f(x, y)$ where f_x and f_y are continuous over a closed, bounded region R. The surface area S over R is

$$\begin{aligned} S &= \iint_R dS \\ &= \iint_R \sqrt{1 + f_x(x, y)^2 + f_y(x, y)^2}\, dA. \end{aligned}$$

We test this definition by using it to compute surface areas of known surfaces. We start with a triangle.

Example 1.5.1 Finding the surface area of a plane over a triangle

Let $f(x, y) = 4 - x - 2y$, and let R be the region in the plane bounded by $x = 0$, $y = 0$ and $y = 2 - x/2$, as shown in Figure 1.5.2. Find the surface area of f over R.

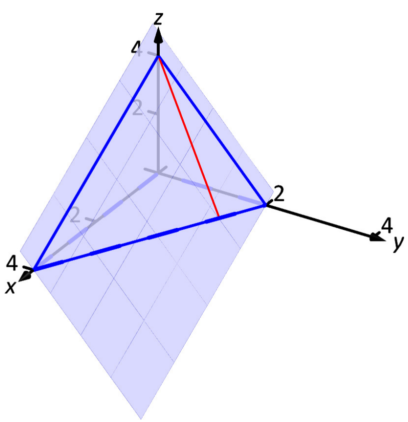


Figure 1.5.2: Finding the area of a triangle in space in Example 1.5.1.

SOLUTION We follow Definition 1.5.1. We start by noting that $f_x(x, y) = -1$ and $f_y(x, y) = -2$. To define R, we use bounds $0 \leq y \leq 2 - x/2$ and $0 \leq x \leq 4$. Therefore

$$\begin{aligned} S &= \iint_R dS \\ &= \int_0^4 \int_0^{2-x/2} \sqrt{1 + (-1)^2 + (-2)^2}\, dy\, dx \\ &= \int_0^4 \sqrt{6}\left(2 - \frac{x}{2}\right) dx \\ &= 4\sqrt{6}. \end{aligned}$$

Because the surface is a triangle, we can figure out the area using geometry. Considering the base of the triangle to be the side in the x-y plane, we find the length of the base to be $\sqrt{20}$. We can find the height using our knowledge of vectors: let $\vec{u}$ be the side in the x-z plane and let $\vec{v}$ be the side in the x-y plane. The height is then $||\, \vec{u} - \text{proj}_{\,\vec{v}}\, \vec{u}\, || = 4\sqrt{6/5}$. Geometry states that the area is thus

$$\frac{1}{2} \cdot 4\sqrt{6/5} \cdot \sqrt{20} = 4\sqrt{6}.$$

We affirm the validity of our formula.

Notes:

It is "common knowledge" that the surface area of a sphere of radius r is $4\pi r^2$. We confirm this in the following example, which involves using our formula with polar coordinates.

Example 1.5.2 The surface area of a sphere.

Find the surface area of the sphere with radius a centered at the origin, whose top hemisphere has equation $f(x,y) = \sqrt{a^2 - x^2 - y^2}$.

Solution We start by computing partial derivatives and find

$$f_x(x,y) = \frac{-x}{\sqrt{a^2-x^2-y^2}} \quad \text{and} \quad f_y(x,y) = \frac{-y}{\sqrt{a^2-x^2-y^2}}.$$

As our function f only defines the top upper hemisphere of the sphere, we double our surface area result to get the total area:

$$\begin{aligned} S &= 2\iint_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA \\ &= 2\iint_R \sqrt{1 + \frac{x^2+y^2}{a^2-x^2-y^2}}\, dA. \end{aligned}$$

The region R that we are integrating over is bounded by the circle, centered at the origin, with radius a: $x^2 + y^2 = a^2$. Because of this region, we are likely to have greater success with our integration by converting to polar coordinates. Using the substitutions $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$ and bounds $0 \le \theta \le 2\pi$ and $0 \le r \le a$, we have:

$$\begin{aligned} S &= 2\int_0^{2\pi}\int_0^a \sqrt{1 + \frac{r^2\cos^2\theta + r^2\sin^2\theta}{a^2 - r^2\cos^2\theta - r^2\sin^2\theta}}\, r\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{1 + \frac{r^2}{a^2-r^2}}\,dr\,d\theta \\ &= 2\int_0^{2\pi}\int_0^a r\sqrt{\frac{a^2}{a^2-r^2}}\,dr\,d\theta. \qquad (1.1) \end{aligned}$$

Apply substitution $u = a^2 - r^2$ and integrate the inner integral, giving

$$\begin{aligned} &= 2\int_0^{2\pi} a^2\,d\theta \\ &= 4\pi a^2. \end{aligned}$$

Our work confirms our previous formula.

Note: The inner integral in Equation (1.1) is an improper integral, as the integrand of $\int_0^a r\sqrt{\frac{a^2}{a^2-r^2}}\,dr$ is not defined at $r = a$. To properly evaluate this integral, one must use the techniques of Section 6.8.

The reason this need arises is that the function $f(x,y) = \sqrt{a^2-x^2-y^2}$ fails the requirements of Definition 1.5.1, as f_x and f_y are not continuous on the boundary of the circle $x^2+y^2 = a^2$.

The computation of the surface area is still valid. The definition makes stronger requirements than necessary in part to avoid the use of improper integration, as when f_x and/or f_y are not continuous, the resulting improper integral may not converge. Since the improper integral does converge in this example, the surface area is accurately computed.

Notes:

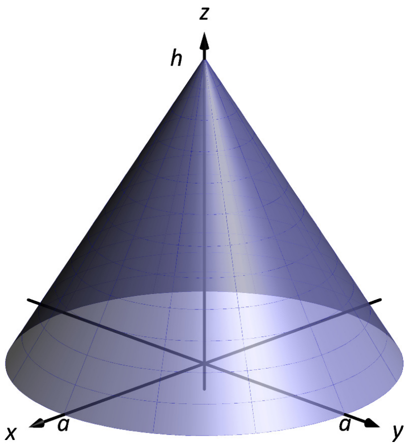


Figure 1.5.3: Finding the surface area of a cone in Example 1.5.3.

Note: Once again f_x and f_y are not continuous on the domain of f, as both are undefined at $(0,0)$. (A similar problem occurred in the previous example.) Once again the resulting improper integral converges and the computation of the surface area is valid.

Example 1.5.3 Finding the surface area of a cone

The general formula for a right cone with height h and base radius a is

$$f(x,y) = h - \frac{h}{a}\sqrt{x^2+y^2},$$

shown in Figure 1.5.3. Find the surface area of this cone.

Solution We begin by computing partial derivatives.

$$f_x(x,y) = -\frac{xh}{a\sqrt{x^2+y^2}} \quad \text{and} \quad f_y(x,y) = -\frac{yh}{a\sqrt{x^2+y^2}}.$$

Since we are integrating over the disk bounded by $x^2+y^2=a^2$, we again use polar coordinates. Using the standard substitutions, our integrand becomes

$$\sqrt{1+\left(\frac{hr\cos\theta}{a\sqrt{r^2}}\right)^2+\left(\frac{hr\sin\theta}{a\sqrt{r^2}}\right)^2}.$$

This may look intimidating at first, but there are lots of simple simplifications to be done. It amazingly reduces to just

$$\sqrt{1+\frac{h^2}{a^2}} = \frac{1}{a}\sqrt{a^2+h^2}.$$

Our polar bounds are $0 \le \theta \le 2\pi$ and $0 \le r \le a$. Thus

$$\begin{aligned} S &= \int_0^{2\pi}\int_0^a r\frac{1}{a}\sqrt{a^2+h^2}\,dr\,d\theta \\ &= \int_0^{2\pi}\left(\frac{1}{2}r^2\frac{1}{a}\sqrt{a^2+h^2}\right)\Bigg|_0^a\,d\theta \\ &= \int_0^{2\pi}\frac{1}{2}a\sqrt{a^2+h^2}\,d\theta \\ &= \pi a\sqrt{a^2+h^2}. \end{aligned}$$

This matches the formula found in the back of this text.

Example 1.5.4 Finding surface area over a region

Find the area of the surface $f(x,y)=x^2-3y+3$ over the region R bounded by $-x \le y \le x$, $0 \le x \le 4$, as pictured in Figure 1.5.4.

Solution It is straightforward to compute $f_x(x,y)=2x$ and $f_y(x,y)=-3$. Thus the surface area is described by the double integral

$$\iint_R \sqrt{1+(2x)^2+(-3)^2}\,dA = \iint_R \sqrt{10+4x^2}\,dA.$$

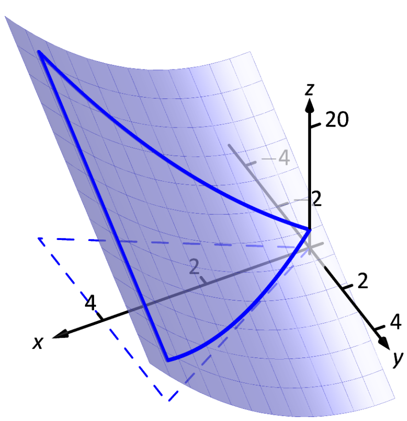


Figure 1.5.4: Graphing the surface in Example 1.5.4.

Notes:

As with integrals describing arc length, double integrals describing surface area are in general hard to evaluate directly because of the square–root. This particular integral can be easily evaluated, though, with judicious choice of our order of integration.

Integrating with order $dx\,dy$ requires us to evaluate $\int \sqrt{10+4x^2}\,dx$. This can be done, though it involves Integration By Parts and $\sinh^{-1} x$. Integrating with order $dy\,dx$ has as its first integral $\int \sqrt{10+4x^2}\,dy$, which is easy to evaluate: it is simply $y\sqrt{10+4x^2}+C$. So we proceed with the order $dy\,dx$; the bounds are already given in the statement of the problem.

$$\begin{aligned}\iint_R \sqrt{10+4x^2}\,dA &= \int_0^4 \int_{-x}^{x} \sqrt{10+4x^2}\,dy\,dx \\ &= \int_0^4 \left(y\sqrt{10+4x^2}\right)\Big|_{-x}^{x}\,dx \\ &= \int_0^4 \left(2x\sqrt{10+4x^2}\right)\,dx.\end{aligned}$$

Apply substitution with $u = 10+4x^2$:

$$\begin{aligned}&= \left(\frac{1}{6}\left(10+4x^2\right)^{3/2}\right)\Big|_0^4 \\ &= \frac{1}{3}\left(37\sqrt{74}-5\sqrt{10}\right) \approx 100.825\text{u}^2.\end{aligned}$$

So while the region R over which we integrate has an area of 16u^2, the surface has a much greater area as its z-values change dramatically over R.

In practice, technology helps greatly in the evaluation of such integrals. High powered computer algebra systems can compute integrals that are difficult, or at least time consuming, by hand, and can at the least produce very accurate approximations with numerical methods. In general, just knowing *how* to set up the proper integrals brings one very close to being able to compute the needed value. Most of the work is actually done in just describing the region R in terms of polar or rectangular coordinates. Once this is done, technology can usually provide a good answer.

We have learned how to integrate integrals; that is, we have learned to evaluate double integrals. In the next section, we learn how to integrate double integrals – that is, we learn to evaluate *triple integrals*, along with learning some uses for this operation.

Notes:

Exercises 1.5

Terms and Concepts

13 05 ex 01 1. “Surface area” is analogous to what previously studied concept?

13 05 ex 02 2. To approximate the area of a small portion of a surface, we computed the area of its ________ plane.

13 05 ex 03 3. We interpret $\iint_R dS$ as “sum up lots of little ________ ________.”

13 05 ex 04 4. Why is it important to know how to set up a double integral to compute surface area, even if the resulting integral is hard to evaluate?

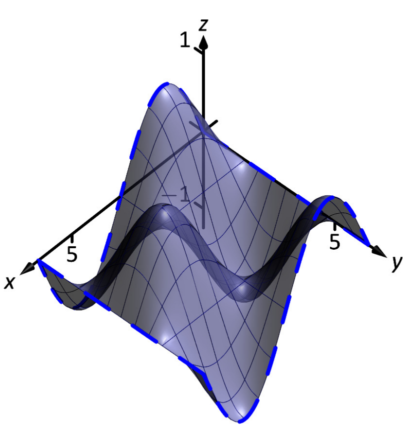
13 05 ex 18 5. Why do $z = f(x, y)$ and $z = g(x, y) = f(x, y) + h$, for some real number h, have the same surface area over a region R?

13 05 ex 19 6. Let $z = f(x, y)$ and $z = g(x, y) = 2f(x, y)$. Why is the surface area of g over a region R not twice the surface area of f over R?

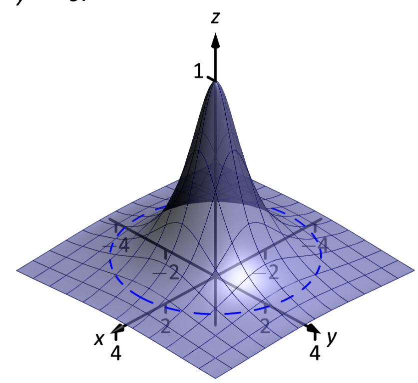
Problems

13 05 exset 01 **In Exercises 7 – 10, *set up* the iterated integral that computes the surface area of the given surface over the region R.**

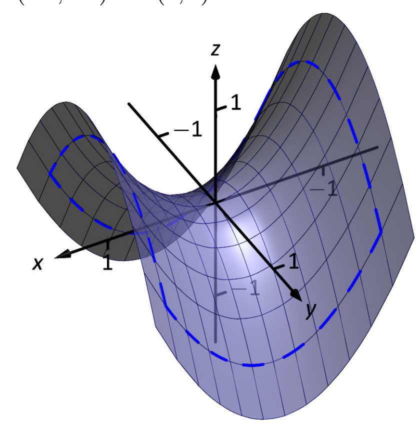
13 05 ex 05 7. $f(x, y) = \sin x \cos y$; R is the rectangle with bounds $0 \leq x \leq 2\pi$, $0 \leq y \leq 2\pi$.



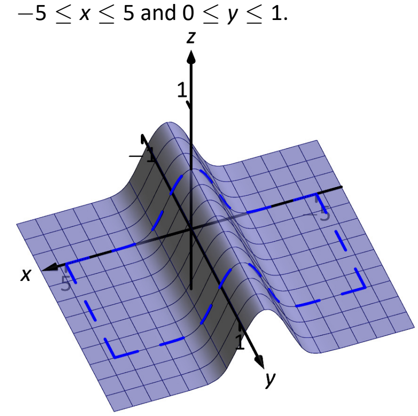
13 05 ex 06 8. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$; R is bounded by the circle $x^2 + y^2 = 9$.



13 05 ex 07 9. $f(x, y) = x^2 - y^2$; R is the rectangle with opposite corners $(-1, -1)$ and $(1, 1)$.



13 05 ex 08 10. $f(x, y) = \dfrac{1}{e^{x^2} + 1}$; R is the rectangle bounded by $-5 \leq x \leq 5$ and $0 \leq y \leq 1$.



13 05 exset 02 **In Exercises 11 – 19, find the area of the given surface over the region R.**

13 05 ex 09 11. $f(x, y) = 3x - 7y + 2$; R is the rectangle with opposite corners $(-1, 0)$ and $(1, 3)$.

13 05 ex 10 12. $f(x, y) = 2x + 2y + 2$; R is the triangle with corners $(0, 0)$,

$(1,0)$ and $(0,1)$.

13 05 ex 11 13. $f(x,y) = x^2 + y^2 + 10$; R is bounded by the circle $x^2 + y^2 = 16$.

13 05 ex 14 14. $f(x,y) = -2x + 4y^2 + 7$ over R, the triangle bounded by $y = -x$, $y = x$, $0 \le y \le 1$.

13 05 ex 15 15. $f(x,y) = x^2 + y$ over R, the triangle bounded by $y = 2x$, $y = 0$ and $x = 2$.

13 05 ex 16 16. $f(x,y) = \frac{2}{3}x^{3/2} + 2y^{3/2}$ over R, the rectangle with opposite corners $(0,0)$ and $(1,1)$.

13 05 ex 12 17. $f(x,y) = 10 - 2\sqrt{x^2 + y^2}$ over R, bounded by the circle $x^2 + y^2 = 25$. (This is the cone with height 10 and base radius 5; be sure to compare your result with the known formula.)

13 05 ex 13 18. Find the surface area of the sphere with radius 5 by doubling the surface area of $f(x,y) = \sqrt{25 - x^2 - y^2}$ over R, bounded by the circle $x^2 + y^2 = 25$. (Be sure to compare your result with the known formula.)

13 05 ex 17 19. Find the surface area of the ellipse formed by restricting the plane $f(x,y) = cx + dy + h$ to the region R, bounded by the circle $x^2 + y^2 = 1$, where c, d and h are some constants. Your answer should be given in terms of c and d; why does the value of h not matter?

1.6 Volume Between Surfaces and Triple Integration

We learned in Section 1.2 how to compute the signed volume V under a surface $z = f(x, y)$ over a region R: $V = \iint_R f(x, y)\, dA$. It follows naturally that if $f(x, y) \geq g(x, y)$ on R, then the **volume between** $f(x, y)$ **and** $g(x, y)$ **on** R is

$$V = \iint_R f(x, y)\, dA - \iint_R g(x, y)\, dA = \iint_R \big(f(x, y) - g(x, y)\big)\, dA.$$

Theorem 1.6.1 Volume Between Surfaces

Let f and g be continuous functions on a closed, bounded region R, where $f(x, y) \geq g(x, y)$ for all (x, y) in R. The volume V between f and g over R is

$$V = \iint_R \big(f(x, y) - g(x, y)\big)\, dA.$$

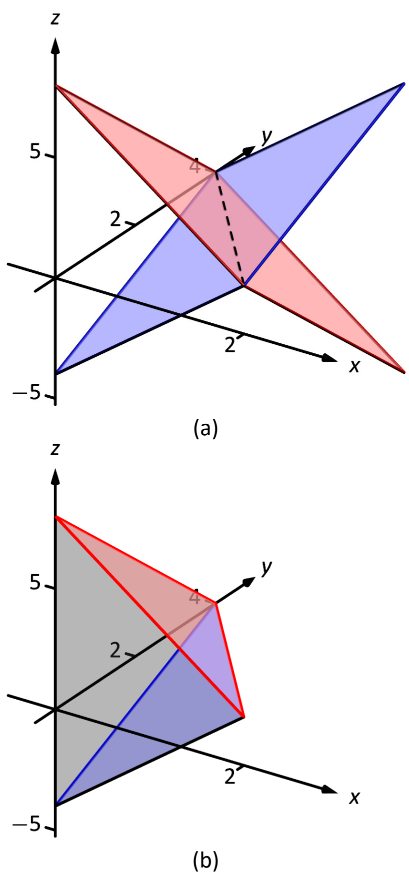


Figure 1.6.1: Finding the volume between the planes given in Example 1.6.1.

Example 1.6.1 Finding volume between surfaces

Find the volume of the space region bounded by the planes $z = 3x + y - 4$, $z = 8 - 3x - 2y$, $x = 0$ and $y = 0$. In Figure 1.6.1(a) the planes are drawn; in (b), only the defined region is given.

Solution We need to determine the region R over which we will integrate. To do so, we need to determine where the planes intersect. They have common z-values when $3x + y - 4 = 8 - 3x - 2y$. Applying a little algebra, we have:

$$\begin{aligned} 3x + y - 4 &= 8 - 3x - 2y \\ 6x + 3y &= 12 \\ 2x + y &= 4 \end{aligned}$$

The planes intersect along the line $2x+y = 4$. Therefore the region R is bounded by $x = 0$, $y = 0$, and $y = 4 - 2x$; we can convert these bounds to integration bounds of $0 \leq x \leq 2$, $0 \leq y \leq 4 - 2x$. Thus

$$\begin{aligned} V &= \iint_R \big(8 - 3x - 2y - (3x + y - 4)\big)\, dA \\ &= \int_0^2 \int_0^{4-2x} \big(12 - 6x - 3y\big)\, dy\, dx \\ &= 16\text{u}^3. \end{aligned}$$

The volume between the surfaces is 16 cubic units.

Notes:

In the preceding example, we found the volume by evaluating the integral

$$\int_0^2 \int_0^{4-2x} \big(8-3x-2y-(3x+y-4)\big)\, dy\, dx.$$

Note how we can rewrite the integrand as an integral, much as we did in Section 1.1:

$$8-3x-2y-(3x+y-4) = \int_{3x+y-4}^{8-3x-2y} dz.$$

Thus we can rewrite the double integral that finds volume as

$$\int_0^2 \int_0^{4-2x} \big(8-3x-2y-(3x+y-4)\big)\, dy\, dx = \int_0^2 \int_0^{4-2x} \left(\int_{3x+y-4}^{8-3x-2y} dz\right) dy\, dx.$$

This no longer looks like a "double integral," but more like a "triple integral." Just as our first introduction to double integrals was in the context of finding the area of a plane region, our introduction into triple integrals will be in the context of finding the volume of a space region.

To formally find the volume of a closed, bounded region D in space, such as the one shown in Figure 1.6.2(a), we start with an approximation. Break D into n rectangular solids; the solids near the boundary of D may possibly not include portions of D and/or include extra space. In Figure 1.6.2(b), we zoom in on a portion of the boundary of D to show a rectangular solid that contains space not in D; as this is an approximation of the volume, this is acceptable and this error will be reduced as we shrink the size of our solids.

The volume ΔV_i of the i^{th} solid D_i is $\Delta V_i = \Delta x_i \Delta y_i \Delta z_i$, where Δx_i, Δy_i and Δz_i give the dimensions of the rectangular solid in the x, y and z directions, respectively. By summing up the volumes of all n solids, we get an approximation of the volume V of D:

$$V \approx \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

Let $||\Delta D||$ represent the length of the longest diagonal of rectangular solids in the subdivision of D. As $||\Delta D|| \to 0$, the volume of each solid goes to 0, as do each of Δx_i, Δy_i and Δz_i, for all i. Our calculus experience tells us that taking a limit as $||\Delta D|| \to 0$ turns our approximation of V into an exact calculation of V. Before we state this result in a theorem, we use a definition to define some terms.

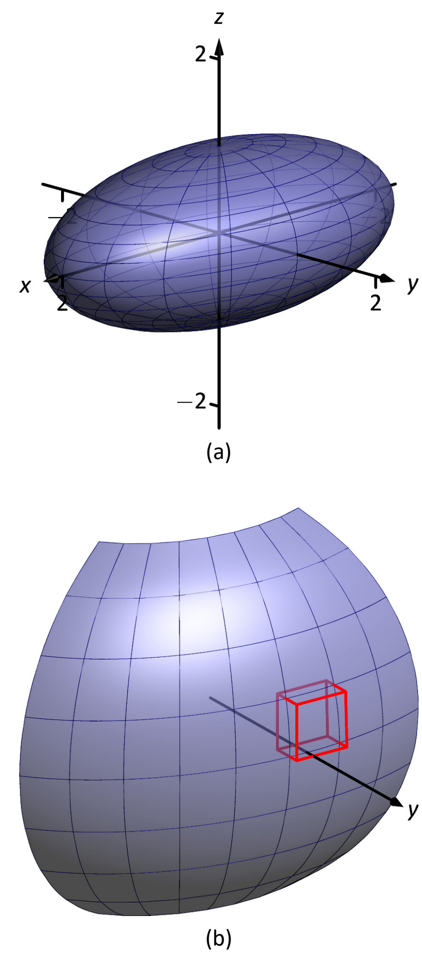


Figure 1.6.2: Approximating the volume of a region D in space.

Notes:

Definition 1.6.1 Triple Integrals, Iterated Integration (Part I)

Let D be a closed, bounded region in space. Let a and b be real numbers, let $g_1(x)$ and $g_2(x)$ be continuous functions of x, and let $f_1(x,y)$ and $f_2(x,y)$ be continuous functions of x and y.

1. The volume V of D is denoted by a **triple integral**,

$$V = \iiint_D dV.$$

2. The iterated integral $\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx$ is evaluated as

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} \left(\int_{f_1(x,y)}^{f_2(x,y)} dz \right) dy\, dx.$$

Evaluating the above iterated integral is **triple integration.**

Our informal understanding of the notation $\iiint_D dV$ is "sum up lots of little volumes over D," analogous to our understanding of $\iint_R dA$ and $\iint_R dm$.

We now state the major theorem of this section.

Theorem 1.6.2 Triple Integration (Part I)

Let D be a closed, bounded region in space and let ΔD be any subdivision of D into n rectangular solids, where the i^{th} subregion D_i has dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$ and volume ΔV_i.

1. The volume V of D is

$$V = \iiint_D dV = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta V_i = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} \Delta x_i \Delta y_i \Delta z_i.$$

2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x,y)$ and $z = f_2(x,y)$, where $a < b$, $g_1(x) \le g_2(x)$ and $f_1(x,y) \le f_2(x,y)$ on D, then

$$\iiint_D dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx.$$

3. V can be determined using iterated integration with other orders of integration (there are 6 total), as long as D is defined by the region enclosed by a pair of planes, a pair of cylinders, and a pair of surfaces.

Notes:

We evaluated the area of a plane region R by iterated integration, where the bounds were "from curve to curve, then from point to point." Theorem 1.6.2 allows us to find the volume of a space region with an iterated integral with bounds "from surface to surface, then from curve to curve, then from point to point." In the iterated integral

$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} dz\, dy\, dx,$$

the bounds $a \leq x \leq b$ and $g_1(x) \leq y \leq g_2(x)$ define a region R in the x-y plane over which the region D exists in space. However, these bounds are also defining surfaces in space; $x = a$ is a plane and $y = g_1(x)$ is a cylinder. The combination of these 6 surfaces enclose, and define, D.

Examples will help us understand triple integration, including integrating with various orders of integration.

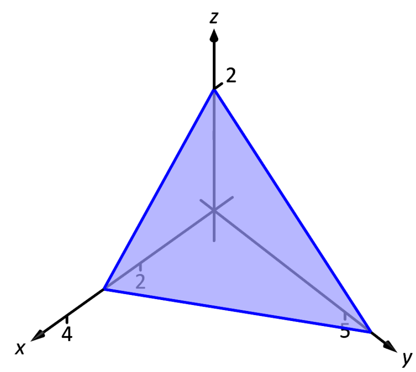
Example 1.6.2 Finding the volume of a space region with triple integration

Find the volume of the space region in the 1^{st} octant bounded by the plane $z = 2 - y/3 - 2x/3$, shown in Figure 1.6.3(a), using the order of integration $dz\, dy\, dx$. Set up the triple integrals that give the volume in the other 5 orders of integration.

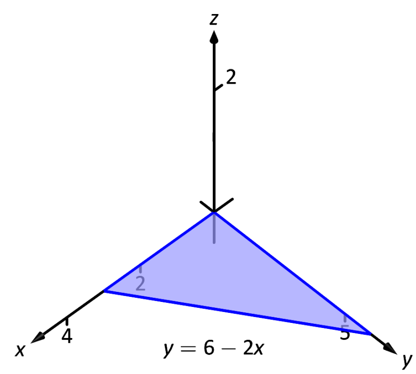
Solution Starting with the order of integration $dz\, dy\, dx$, we need to first find bounds on z. The region D is bounded below by the plane $z = 0$ (because we are restricted to the first octant) and above by $z = 2 - y/3 - 2x/3$; $0 \leq z \leq 2 - y/3 - 2x/3$.

To find the bounds on y and x, we "collapse" the region onto the x-y plane, giving the triangle shown in Figure 1.6.3(b). (We know the equation of the line $y = 6 - 2x$ in two ways. First, by setting $z = 0$, we have $0 = 2 - y/3 - 2x/3 \Rightarrow y = 6 - 2x$. Secondly, we know this is going to be a straight line between the points $(3, 0)$ and $(0, 6)$ in the x-y plane.)

We define that region R, in the integration order of $dy\, dx$, with bounds $0 \leq$



(a)



(b)

Figure 1.6.3: The region D used in Example 1.6.2 in (a); in (b), the region found by collapsing D onto the x-y plane.

Notes:

$y \leq 6-2x$ and $0 \leq x \leq 3$. Thus the volume V of the region D is:

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dy\, dz \\ &= \int_0^3 \int_0^{6-2x} \left(\int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz \right) dy\, dz \\ &= \int_0^3 \int_0^{6-2x} z\Big|_0^{2-\frac{1}{3}y-\frac{2}{3}x} dy\, dz \\ &= \int_0^3 \int_0^{6-2x} \left(2-\frac{1}{3}y-\frac{2}{3}x\right) dy\, dz. \end{aligned}$$

From this step on, we are evaluating a double integral as done many times before. We skip these steps and give the final volume,

$$= 6\text{u}^3.$$

The order *dz dx dy*:

Now consider the volume using the order of integration *dz dx dy*. The bounds on z are the same as before, $0 \leq z \leq 2-y/3-2x/3$. Collapsing the space region on the x-y plane as shown in Figure 1.6.3(b), we now describe this triangle with the order of integration *dx dy*. This gives bounds $0 \leq x \leq 3-y/2$ and $0 \leq y \leq 6$. Thus the volume is given by the triple integral

$$V = \int_0^6 \int_0^{3-\frac{1}{2}y} \int_0^{2-\frac{1}{3}y-\frac{2}{3}x} dz\, dx\, dy.$$

The order *dx dy dz*:

Following our "surface to surface..." strategy, we need to determine the *x-surfaces* that bound our space region. To do so, approach the region "from behind," in the direction of increasing x. The first surface we hit as we enter the region is the y-z plane, defined by $x=0$. We come out of the region at the plane $z = 2-y/3-2x/3$; solving for x, we have $x = 3-y/2-3z/2$. Thus the bounds on x are: $0 \leq x \leq 3-y/2-3z/2$.

Now collapse the space region onto the y-z plane, as shown in Figure 1.6.4(a). (Again, we find the equation of the line $z = 2-y/3$ by setting $x=0$ in the equation $x = 3-y/2-3z/2$.) We need to find bounds on this region with the order

Notes:

$dy\,dz$. The *curves* that bound y are $y = 0$ and $y = 6 - 3z$; the *points* that bound z are 0 and 2. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le y \le 6 - 3z \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{6-3z} \int_0^{3-y/2-3z/2} dx\,dy\,dz.$$

The order $dx\,dz\,dy$:

The x-bounds are the same as the order above. We now consider the triangle in Figure 1.6.4(a) and describe it with the order $dz\,dy$: $0 \le z \le 2 - y/3$ and $0 \le y \le 6$. Thus the volume is given by:

$$\begin{array}{c} 0 \le x \le 3 - y/2 - 3z/2 \\ 0 \le z \le 2 - y/3 \\ 0 \le y \le 6 \end{array} \quad \Rightarrow \quad \int_0^6 \int_0^{2-y/3} \int_0^{3-y/2-3z/2} dx\,dz\,dy.$$

The order $dy\,dz\,dx$:

We now need to determine the y-surfaces that determine our region. Approaching the space region from "behind" and moving in the direction of increasing y, we first enter the region at $y = 0$, and exit along the plane $z = 2 - y/3 - 2x/3$. Solving for y, this plane has equation $y = 6 - 2x - 3z$. Thus y has bounds $0 \le y \le 6 - 2x - 3z$.

Now collapse the region onto the x-z plane, as shown in Figure 1.6.4(b). The curves bounding this triangle are $z = 0$ and $z = 2 - 2x/3$; x is bounded by the points $x = 0$ to $x = 3$. Thus the triple integral giving volume is:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le z \le 2 - 2x/3 \\ 0 \le x \le 3 \end{array} \quad \Rightarrow \quad \int_0^3 \int_0^{2-2x/3} \int_0^{6-2x-3z} dy\,dz\,dx.$$

The order $dy\,dx\,dz$:

The y-bounds are the same as in the order above. We now determine the bounds of the triangle in Figure 1.6.4(b) using the order $dy\,dx\,dz$. x is bounded by $x = 0$ and $x = 3 - 3z/2$; z is bounded between $z = 0$ and $z = 2$. This leads to the triple integral:

$$\begin{array}{c} 0 \le y \le 6 - 2x - 3z \\ 0 \le x \le 3 - 3z/2 \\ 0 \le z \le 2 \end{array} \quad \Rightarrow \quad \int_0^2 \int_0^{3-3z/2} \int_0^{6-2x-3z} dy\,dx\,dz.$$

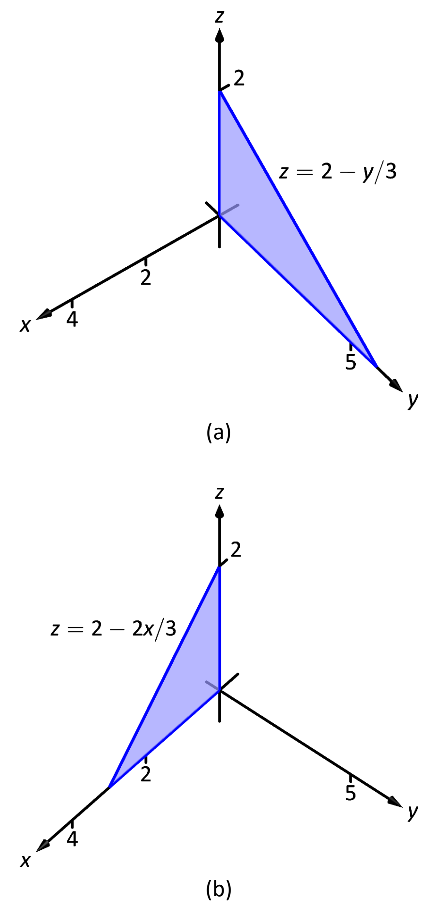


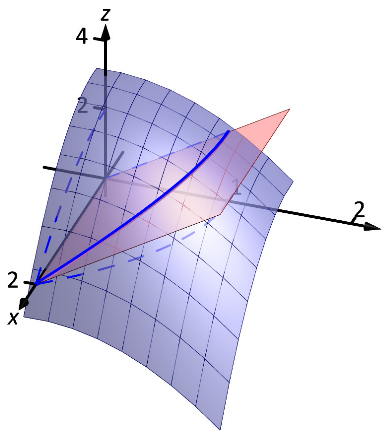
Figure 1.6.4: The region D in Example 1.6.2 is collapsed onto the y-z plane in (a); in (b), the region is collapsed onto the x-z plane.

Notes:

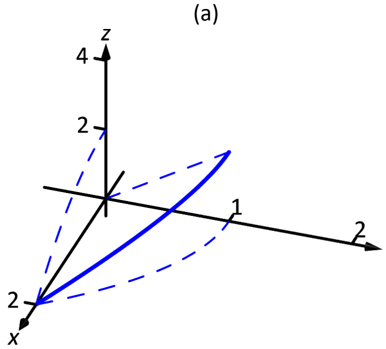
This problem was long, but hopefully useful, demonstrating how to determine bounds with every order of integration to describe the region D. In practice, we only need 1, but being able to do them all gives us flexibility to choose the order that suits us best.

In the previous example, we collapsed the surface into the x-y, x-z, and y-z planes as we determined the "curve to curve, point to point" bounds of integration. Since the surface was a triangular portion of a plane, this collapsing, or *projecting*, was simple: the *projection* of a straight line in space onto a coordinate plane is a line.

The following example shows us how to do this when dealing with more complicated surfaces and curves.



(a)



(b)

Figure 1.6.5: Finding the projections of the curve of intersection in Example 1.6.3.

Example 1.6.3 Finding the projection of a curve in space onto the coordinate planes

Consider the surfaces $z = 3 - x^2 - y^2$ and $z = 2y$, as shown in Figure 1.6.5(a). The curve of their intersection is shown, along with the projection of this curve into the coordinate planes, shown dashed. Find the equations of the projections into the coordinate planes.

Solution The two surfaces are $z = 3 - x^2 - y^2$ and $z = 2y$. To find where they intersect, it is natural to set them equal to each other: $3 - x^2 - y^2 = 2y$. This is an implicit function of x and y that gives all points (x, y) in the x-y plane where the z values of the two surfaces are equal.

We can rewrite this implicit function by completing the square:

$$3 - x^2 - y^2 = 2y \quad \Rightarrow \quad y^2 + 2y + x^2 = 3 \quad \Rightarrow \quad (y+1)^2 + x^2 = 4.$$

Thus in the x-y plane the projection of the intersection is a circle with radius 2, centered at $(0, -1)$.

To project onto the x-z plane, we do a similar procedure: find the x and z values where the y values on the surface are the same. We start by solving the equation of each surface for y. In this particular case, it works well to actually solve for y^2:

$z = 3 - x^2 - y^2 \quad \Rightarrow \quad y^2 = 3 - x^2 - z$

$z = 2y \quad \Rightarrow \quad y^2 = z^2/4.$

Thus we have (after again completing the square):

$$3 - x^2 - z = z^2/4 \quad \Rightarrow \quad \frac{(z+2)^2}{16} + \frac{x^2}{4} = 1,$$

and ellipse centered at $(0, -2)$ in the x-z plane with a major axis of length 8 and a minor axis of length 4.

Notes:

Finally, to project the curve of intersection into the y-z plane, we solve equation for x. Since $z = 2y$ is a cylinder that lacks the variable x, it becomes our equation of the projection in the y-z plane.

All three projections are shown in Figure 1.6.5(b).

Example 1.6.4 Finding the volume of a space region with triple integration

Set up the triple integrals that find the volume of the space region D bounded by the surfaces $x^2+y^2=1$, $z=0$ and $z=-y$, as shown in Figure 1.6.6(a), with the orders of integration $dz\,dy\,dx$, $dy\,dx\,dz$ and $dx\,dz\,dy$.

Solution The order $dz\,dy\,dx$:

The region D is bounded below by the plane $z=0$ and above by the plane $z=-y$. The cylinder $x^2+y^2=1$ does not offer any bounds in the z-direction, as that surface is parallel to the z-axis. Thus $0\le z\le -y$.

Collapsing the region into the x-y plane, we get part of the disk bounded by the circle with equation $x^2+y^2=1$ as shown in Figure 1.6.6(b). As a function of x, this half circle has equation $y=-\sqrt{1-x^2}$. Thus y is bounded below by $-\sqrt{1-x^2}$ and above by $y=0$: $-\sqrt{1-x^2}\le y\le 0$. The x bounds of the half circle are $-1\le x\le 1$. All together, the bounds of integration and triple integral are as follows:

$$\begin{array}{c} 0\le z\le -y \\ -\sqrt{1-x^2}\le y\le 0 \\ -1\le x\le 1 \end{array} \quad\Rightarrow\quad \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx.$$

We evaluate this triple integral:

$$\begin{aligned}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_{0}^{-y} dz\,dy\,dx &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\big(-y\big)\,dy\,dx\\ &= \int_{-1}^{1}\Big(-\frac{1}{2}y^2\Big)\Big|_{-\sqrt{1-x^2}}^{0}\,dx\\ &= \int_{-1}^{1}\frac{1}{2}\big(1-x^2\big)\,dx\\ &= \left(\frac{1}{2}\left(x-\frac{1}{3}x^3\right)\right)\Big|_{-1}^{1}\\ &= \frac{2}{3}\text{units}^3.\end{aligned}$$

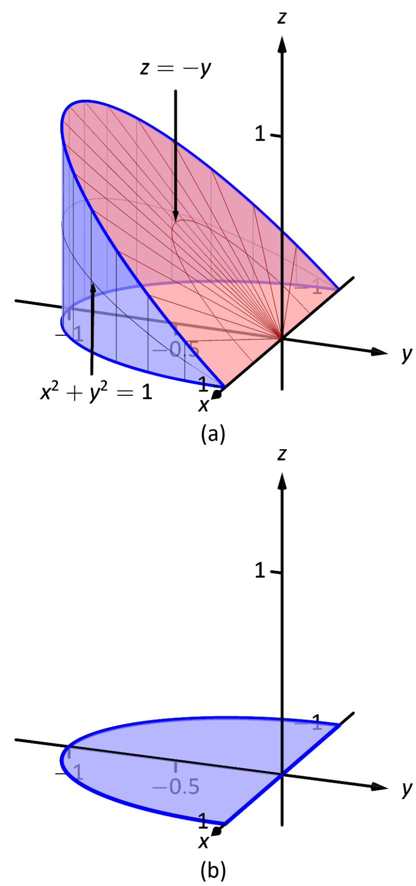


Figure 1.6.6: The region D in Example 1.6.4 is shown in (a); in (b), it is collapsed onto the x-y plane.

Notes:

With the order $dy\,dx\,dz$:

The region is bounded "below" in the y-direction by the surface $x^2+y^2=1 \Rightarrow y=-\sqrt{1-x^2}$ and "above" by the surface $y=-z$. Thus the y bounds are $-\sqrt{1-x^2} \leq y \leq -z$.

Collapsing the region onto the x-z plane gives the region shown in Figure 1.6.7(a); this half disk is bounded by $z=0$ and $x^2+z^2=1$. (We find this curve by solving each surface for y^2, then setting them equal to each other. We have $y^2=1-x^2$ and $y=-z \Rightarrow y^2=z^2$. Thus $x^2+z^2=1$.) It is bounded below by $x=-\sqrt{1-z^2}$ and above by $x=\sqrt{1-z^2}$, where z is bounded by $0 \leq z \leq 1$. All together, we have:

$$\begin{array}{c} -\sqrt{1-x^2} \leq y \leq -z \\ -\sqrt{1-z^2} \leq x \leq \sqrt{1-z^2} \\ 0 \leq z \leq 1 \end{array} \quad \Rightarrow \quad \int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-x^2}}^{-z} dy\,dx\,dz.$$

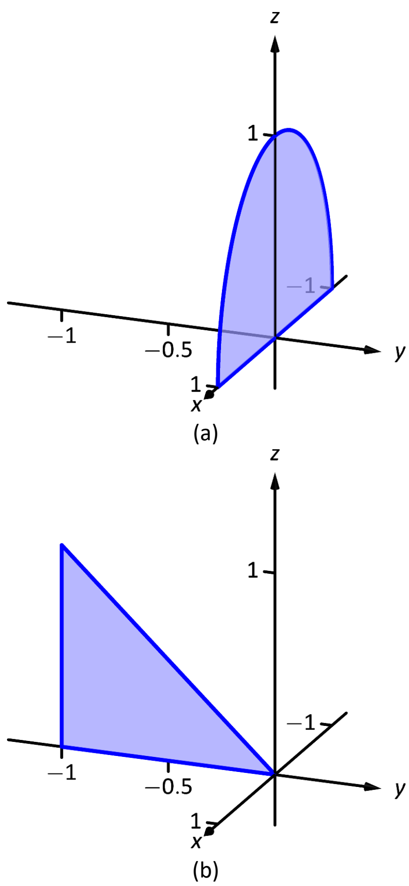


Figure 1.6.7: The region D in Example 1.6.4 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

With the order $dx\,dz\,dy$:

D is bounded below by the surface $x=-\sqrt{1-y^2}$ and above by $\sqrt{1-y^2}$. We then collapse the region onto the y-z plane and get the triangle shown in Figure 1.6.7(b). (The hypotenuse is the line $z=-y$, just as the plane.) Thus z is bounded by $0 \leq z \leq -y$ and y is bounded by $-1 \leq y \leq 0$. This gives:

$$\begin{array}{c} -\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2} \\ 0 \leq z \leq -y \\ -1 \leq y \leq 0 \end{array} \quad \Rightarrow \quad \int_{-1}^0 \int_0^{-y} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dz\,dy.$$

The following theorem states two things that should make "common sense" to us. First, using the triple integral to find volume of a region D should always return a positive number; we are computing *volume* here, not *signed volume*. Secondly, to compute the volume of a "complicated" region, we could break it up into subregions and compute the volumes of each subregion separately, summing them later to find the total volume.

Notes:

Theorem 1.6.3 Properties of Triple Integrals

Let D be a closed, bounded region in space, and let D_1 and D_2 be non-overlapping regions such that $D = D_1 \bigcup D_2$.

1. $\displaystyle\iiint_D dV \geq 0$

2. $\displaystyle\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV.$

We use this latter property in the next example.

Example 1.6.5 Finding the volume of a space region with triple integration

Find the volume of the space region D bounded by the coordinate planes, $z = 1 - x/2$ and $z = 1 - y/4$, as shown in Figure 1.6.8(a). Set up the triple integrals that find the volume of D in all 6 orders of integration.

Solution Following the bounds–determining strategy of "surface to surface, curve to curve, and point to point," we can see that the most difficult orders of integration are the two in which we integrate with respect to z first, for there are two "upper" surfaces that bound D in the z-direction. So we start by noting that we have

$$0 \leq z \leq 1 - \frac{1}{2}x \quad \text{and} \quad 0 \leq z \leq 1 - \frac{1}{4}y.$$

We now collapse the region D onto the x-y axis, as shown in Figure 1.6.8(b). The boundary of D, the line from $(0,0,1)$ to $(2,4,0)$, is shown in part (b) of the figure as a dashed line; it has equation $y = 2x$. (We can recognize this in two ways: one, in collapsing the line from $(0,0,1)$ to $(2,4,0)$ onto the x-y plane, we simply ignore the z-values, meaning the line now goes from $(0,0)$ to $(2,4)$. Secondly, the two surfaces meet where $z = 1 - x/2$ is equal to $z = 1 - y/4$: thus $1 - x/2 = 1 - y/4 \Rightarrow y = 2x$.)

We use the second property of Theorem 1.6.3 to state that

$$\iiint_D dV = \iiint_{D_1} dV + \iiint_{D_2} dV,$$

where D_1 and D_2 are the space regions above the plane regions R_1 and R_2, respectively. Thus we can say

$$\iiint_D dV = \iint_{R_1}\left(\int_0^{1-x/2} dz\right) dA + \iint_{R_2}\left(\int_0^{1-y/4} dz\right) dA.$$

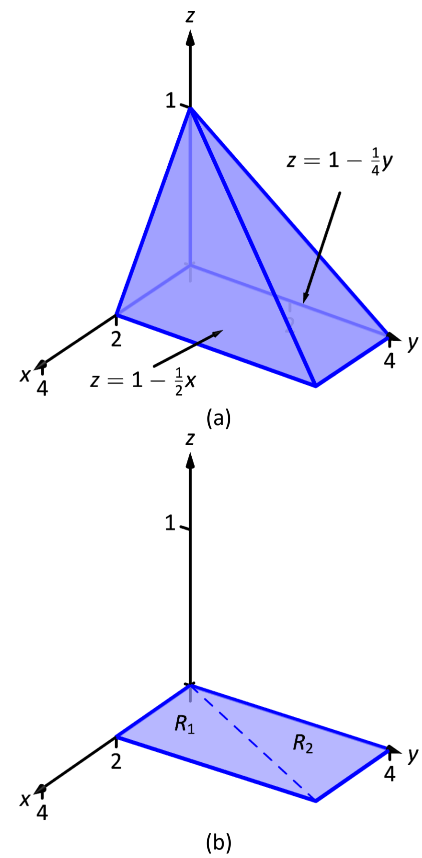


Figure 1.6.8: The region D in Example 1.6.5 is shown in (a); in (b), it is collapsed onto the x-y plane.

Notes:

All that is left is to determine bounds of R_1 and R_2, depending on whether we are integrating with order *dx dy* or *dy dx*. We give the final integrals here, leaving it to the reader to confirm these results.

dz dy dx:

$$\begin{array}{ccc} 0 \leq z \leq 1 - x/2 & \qquad & 0 \leq z \leq 1 - y/4 \\ 0 \leq y \leq 2x & & 2x \leq y \leq 4 \\ 0 \leq x \leq 2 & & 0 \leq x \leq 2 \end{array}$$

$$\iiint_D dV \quad = \quad \int_0^2 \int_0^{2x} \int_0^{1-x/2} dz\,dy\,dx \quad + \quad \int_0^2 \int_{2x}^4 \int_0^{1-y/4} dz\,dy\,dx$$

dz dx dy:

$$\begin{array}{ccc} 0 \leq z \leq 1 - x/2 & \qquad & 0 \leq z \leq 1 - y/4 \\ y/2 \leq x \leq 2 & & 0 \leq x \leq y/2 \\ 0 \leq y \leq 4 & & 0 \leq y \leq 4 \end{array}$$

$$\iiint_D dV \quad = \quad \int_0^4 \int_{y/2}^2 \int_0^{1-x/2} dz\,dx\,dy \quad + \quad \int_0^4 \int_0^{y/2} \int_0^{1-y/4} dz\,dx\,dy$$

The remaining four orders of integration do not require a sum of triple integrals. In Figure 1.6.9 we show D collapsed onto the other two coordinate planes. Using these graphs, we give the final orders of integration here, again leaving it to the reader to confirm these results.

dy dx dz:

$$\begin{array}{c} 0 \leq y \leq 4 - 4z \\ 0 \leq x \leq 2 - 2z \\ 0 \leq z \leq 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{2-2z} \int_0^{4-4z} dy\,dx\,dz$$

dy dz dx:

$$\begin{array}{c} 0 \leq y \leq 4 - 4z \\ 0 \leq z \leq 1 - x/2 \\ 0 \leq x \leq 2 \end{array} \quad \Rightarrow \int_0^2 \int_0^{1-x/2} \int_0^{4-4z} dy\,dx\,dz$$

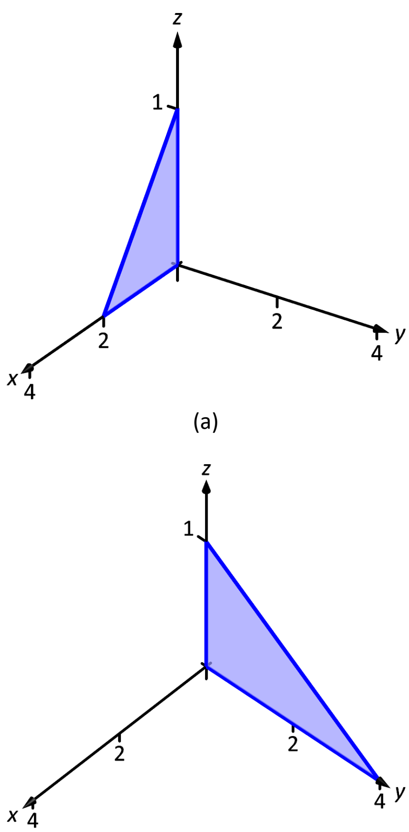


Figure 1.6.9: The region D in Example 1.6.5 is shown collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

Notes:

dx dy dz:

$$\begin{array}{c} 0 \le x \le 2-2z \\ 0 \le y \le 4-4z \\ 0 \le z \le 1 \end{array} \quad \Rightarrow \int_0^1 \int_0^{4-4z} \int_0^{2-2z} dx\, dy\, dz$$

dx dz dy:

$$\begin{array}{c} 0 \le x \le 2-2z \\ 0 \le z \le 1-y/4 \\ 0 \le y \le 4 \end{array} \quad \Rightarrow \int_0^4 \int_0^{1-y/4} \int_0^{2-2z} dx\, dz\, dy$$

We give one more example of finding the volume of a space region.

Example 1.6.6 Finding the volume of a space region

Set up a triple integral that gives the volume of the space region D bounded by $z = 2x^2+2$ and $z = 6-2x^2-y^2$. These surfaces are plotted in Figure 1.6.10(a) and (b), respectively; the region D is shown in part (c) of the figure.

Solution The main point of this example is this: integrating with respect to z first is rather straightforward; integrating with respect to x first is not.

The order *dz dy dx*:

The bounds on z are clearly $2x^2+2 \le z \le 6-2x^2-y^2$. Collapsing D onto the x-y plane gives the ellipse shown in Figure 1.6.10(c). The equation of this ellipse is found by setting the two surfaces equal to each other:

$$2x^2+2 = 6-2x^2-y^2 \quad \Rightarrow \quad 4x^2+y^2 = 4 \quad \Rightarrow \quad x^2+\frac{y^2}{4} = 1.$$

We can describe this ellipse with the bounds

$$-\sqrt{4-4x^2} \le y \le \sqrt{4-4x^2} \quad \text{and} \quad -1 \le x \le 1.$$

Thus we find volume as

$$\begin{array}{c} 2x^2+2 \le z \le 6-2x^2-y^2 \\ -\sqrt{4-4x^2} \le y \le \sqrt{4-4x^2} \\ -1 \le x \le 1 \end{array} \quad \Rightarrow \int_{-1}^1 \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} \int_{2x^2+2}^{6-2x^2-y^2} dz\, dy\, dx\ .$$

The order *dy dz dx*:

Notes:

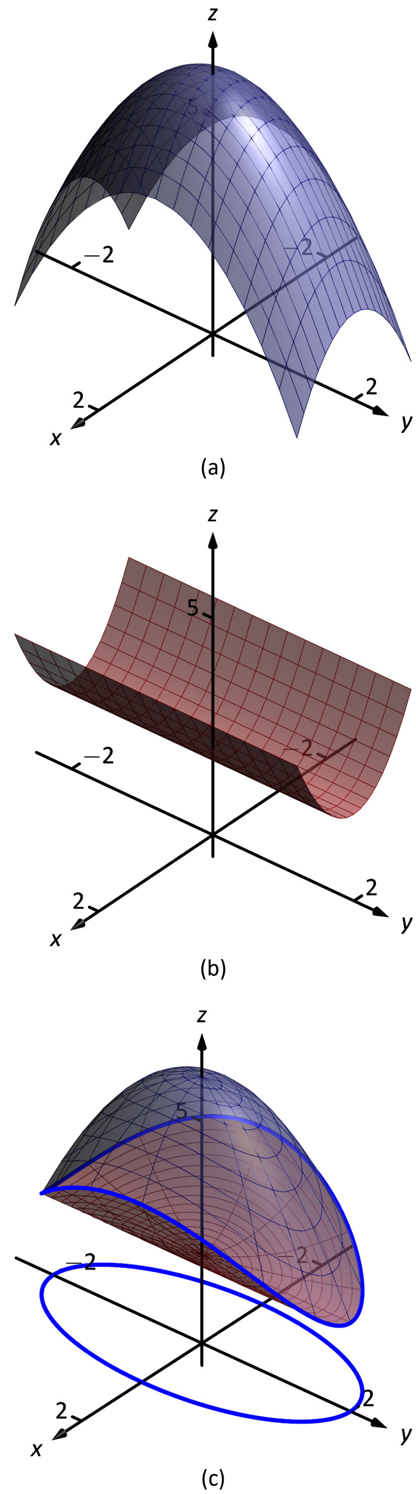


Figure 1.6.10: The region D is bounded by the surfaces shown in (a) and (b); D is shown in (c).

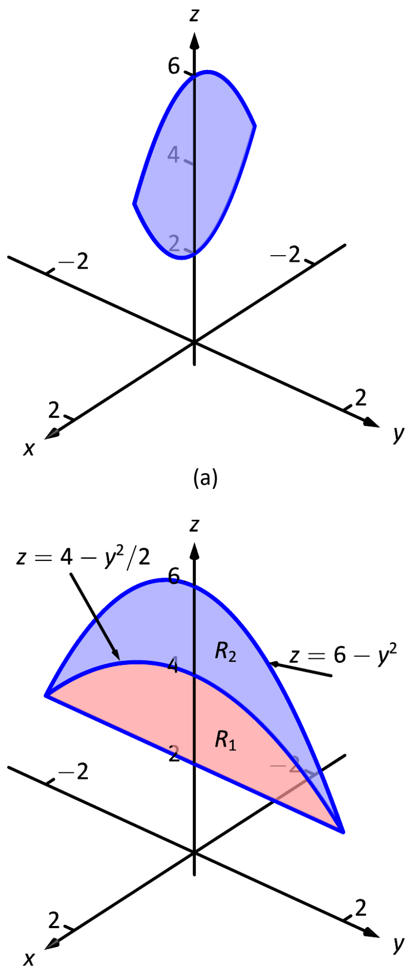


Figure 1.6.11: The region D in Example 1.6.6 is collapsed onto the x-z plane in (a); in (b), it is collapsed onto the y-z plane.

Integrating with respect to y is not too difficult. Since the surface $z = 2x^2 + 2$ is a cylinder whose directrix is the y-axis, it does not create a border for y. The paraboloid $z = 6 - 2x^2 - y^2$ does; solving for y, we get the bounds

$$-\sqrt{6 - 2x^2 - z} \le y \le \sqrt{6 - 2x^2 - z}.$$

Collapsing D onto the x-z axes gives the region shown in Figure 1.6.11(a); the lower curve is from the cylinder, with equation $z = 2x^2 + 2$. The upper curve is from the paraboloid; with $y = 0$, the curve is $z = 6 - 2x^2$. Thus bounds on z are $2x^2 + 2 \le z \le 6 - 2x^2$; the bounds on x are $-1 \le x \le 1$. Thus we have:

$$\begin{array}{c} -\sqrt{6 - 2x^2 - z} \le y \le \sqrt{6 - 2x^2 - z} \\ 2x^2 + 2 \le z \le 6 - 2x^2 \\ -1 \le x \le 1 \end{array} \qquad \Rightarrow \int_{-1}^{1} \int_{2x^2+2}^{6-2x^2} \int_{-\sqrt{6-2x^2-z}}^{\sqrt{6-2x^2-z}} dy\, dz\, dx.$$

The order $dx\, dz\, dy$:

This order takes more effort as D must be split into two subregions. The two surfaces create two sets of upper/lower bounds in terms of x; the cylinder creates bounds

$$-\sqrt{z/2 - 1} \le x \le \sqrt{z/2 - 1}$$

for region D_1 and the paraboloid creates bounds

$$-\sqrt{3 - y^2/2 - z^2/2} \le x \le \sqrt{3 - y^2/2 - z^2/2}$$

for region D_2.

Collapsing D onto the y-z axes gives the regions shown in Figure 1.6.11(b). We find the equation of the curve $z = 4 - y^2/2$ by noting that the equation of the ellipse seen in Figure 1.6.10(c) has equation

$$x^2 + y^2/4 = 1 \quad \Rightarrow \quad x = \sqrt{1 - y^2/4}.$$

Substitute this expression for x in either surface equation, $z = 6 - 2x^2 - y^2$ or $z = 2x^2 + 2$. In both cases, we find

$$z = 4 - \frac{1}{2}y^2.$$

Region R_1, corresponding to D_1, has bounds

$$2 \le z \le 4 - y^2/2, \quad -2 \le y \le 2$$

and region R_2, corresponding to D_2, has bounds

$$4 - y^2/2 \le z \le 6 - y^2, \quad -2 \le y \le 2.$$

Notes:

Thus the volume of D is given by:

$$\int_{-2}^{2}\int_{2}^{4-y^2/2}\int_{-\sqrt{z/2-1}}^{\sqrt{z/2-1}} dx\, dz\, dy \;+\; \int_{-2}^{2}\int_{4-y^2/2}^{6-y^2}\int_{-\sqrt{3-y^2/2-z^2/2}}^{\sqrt{3-y^2/2-z^2/2}} dx\, dz\, dy.$$

If all one wanted to do in Example 1.6.6 was find the volume of the region D, one would have likely stopped at the first integration setup (with order $dz\,dy\,dx$) and computed the volume from there. However, we included the other two methods 1) to show that it could be done, "messy" or not, and 2) because sometimes we "have" to use a less desirable order of integration in order to actually integrate.

Triple Integration and Functions of Three Variables

There are uses for triple integration beyond merely finding volume, just as there are uses for integration beyond "area under the curve." These uses start with understanding how to integrate functions of three variables, which is effectively no different than integrating functions of two variables. This leads us to a definition, followed by an example.

Definition 1.6.2 Iterated Integration, (Part II)

Let D be a closed, bounded region in space, over which $g_1(x)$, $g_2(x)$, $f_1(x,y)$, $f_2(x,y)$ and $h(x,y,z)$ are all continuous, and let a and b be real numbers.

The **iterated integral** $\displaystyle\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\, dz\, dy\, dx$ is evaluated as

$$\int_a^b\int_{g_1(x)}^{g_2(x)}\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\, dz\, dy\, dx = \int_a^b\int_{g_1(x)}^{g_2(x)}\left(\int_{f_1(x,y)}^{f_2(x,y)} h(x,y,z)\, dz\right) dy\, dx.$$

Example 1.6.7 Evaluating a triple integral of a function of three variables

Evaluate $\displaystyle\int_0^1\int_{x^2}^{x}\int_{x^2-y}^{2x+3y} (xy+2xz)\, dz\, dy\, dx.$

SOLUTION We evaluate this integral according to Definition 1.6.2.

Notes:

$$\begin{aligned}
&\int_0^1 \int_{x^2}^{x} \int_{x^2-y}^{2x+3y} \big(xy+2xz\big)\, dz\, dy\, dx\\
&= \int_0^1 \int_{x^2}^{x} \left(\int_{x^2-y}^{2x+3y} \big(xy+2xz\big)\, dz\right)\, dy\, dx\\
&= \int_0^1 \int_{x^2}^{x} \left(\big(xyz+xz^2\big)\Big|_{x^2-y}^{2x+3y}\right)\, dy\, dx\\
&= \int_0^1 \int_{x^2}^{x} \left(xy(2x+3y)+x(2x+3y)^2-\Big(xy(x^2-y)+x(x^2-y)^2\Big)\right)\, dy\, dx\\
&= \int_0^1 \int_{x^2}^{x} \Big(-x^5+x^3y+4x^3+14x^2y+12xy^2\Big)\, dy\, dx.
\end{aligned}$$

We continue as we have in the past, showing fewer steps.

$$\begin{aligned}
&= \int_0^1 \left(-\frac{7}{2}x^7-8x^6-\frac{7}{2}x^5+15x^4\right)\, dx\\
&= \frac{281}{336} \approx 0.836.
\end{aligned}$$

We now know *how* to evaluate a triple integral of a function of three variables; we do not yet understand what it *means*. We build up this understanding in a way very similar to how we have understood integration and double integration.

Let $h(x,y,z)$ be a continuous function of three variables, defined over some space region D. We can partition D into n rectangular–solid subregions, each with dimensions $\Delta x_i \times \Delta y_i \times \Delta z_i$. Let (x_i, y_i, z_i) be some point in the i^{th} subregion, and consider the product $h(x_i,y_i,z_i)\Delta x_i\Delta y_i\Delta z_i$. It is the product of a function value (that's the $h(x_i,y_i,z_i)$ part) and a small volume ΔV_i (that's the $\Delta x_i\Delta y_i\Delta z_i$ part). One of the simplest understanding of this type of product is when h describes the density of an object, for then $h \times \text{volume} = \text{mass}$.

We can sum up all n products over D. Again letting $||\Delta D||$ represent the length of the longest diagonal of the n rectangular solids in the partition, we can take the limit of the sums of products as $||\Delta D|| \to 0$. That is, we can find

$$S = \lim_{||\Delta D||\to 0} \sum_{i=1}^{n} h(x_i,y_i,z_i)\Delta V_i = \lim_{||\Delta D||\to 0} \sum_{i=1}^{n} h(x_i,y_i,z_i)\Delta x_i\Delta y_i\Delta z_i.$$

While this limit has lots of interpretations depending on the function h, in the case where h describes density, S is the total mass of the object described by the region D.

Notes:

We now use the above limit to define the **triple integral**, give a theorem that relates triple integrals to iterated iteration, followed by the application of triple integrals to find the centers of mass of solid objects.

Definition 1.6.3 Triple Integral

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded region D in space, and let ΔD be any partition of D into n rectangular solids with volume ΔV_i. The **triple integral of h over D** is

$$\iiint_D h(x, y, z)\, dV = \lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i.$$

The following theorem assures us that the above limit exists for continuous functions h and gives us a method of evaluating the limit.

Theorem 1.6.4 Triple Integration (Part II)

Let $w = h(x, y, z)$ be a continuous function over a closed, bounded region D in space, and let ΔD be any partition of D into n rectangular solids with volume V_i.

1. The limit $\displaystyle\lim_{||\Delta D|| \to 0} \sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta V_i$ exists.
2. If D is defined as the region bounded by the planes $x = a$ and $x = b$, the cylinders $y = g_1(x)$ and $y = g_2(x)$, and the surfaces $z = f_1(x, y)$ and $z = f_2(x, y)$, where $a < b$, $g_1(x) \le g_2(x)$ and $f_1(x, y) \le f_2(x, y)$ on D, then

$$\iiint_D h(x, y, z)\, dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{f_1(x,y)}^{f_2(x,y)} h(x, y, z)\, dz\, dy\, dx.$$

We now apply triple integration to find the centers of mass of solid objects.

Mass and Center of Mass

One may wish to review Section 1.4 for a reminder of the relevant terms and concepts.

Notes:

Note: In the marginal note on page 12, we showed how the summation of rectangles over a region R in the plane could be viewed as a double sum, leading to the double integral. Likewise, we can view the sum $\displaystyle\sum_{i=1}^{n} h(x_i, y_i, z_i)\Delta x_i \Delta y_i \Delta z_i$ as a triple sum,

$$\sum_{k=1}^{p}\sum_{j=1}^{n}\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i \Delta y_j \Delta z_k,$$

which we evaluate as

$$\sum_{k=1}^{p}\left(\sum_{j=1}^{n}\left(\sum_{i=1}^{m} h(x_i, y_j, z_k)\Delta x_i\right)\Delta y_j\right)\Delta z_k.$$

Here we fix a k value, which establishes the z-height of the rectangular solids on one "level" of all the rectangular solids in the space region D. The inner double summation adds up all the volumes of the rectangular solids on this level, while the outer summation adds up the volumes of each level.
This triple summation understanding leads to the $\iiint_D$ notation of the triple integral, as well as the method of evaluation shown in Theorem 1.6.4.

Definition 1.6.4 Mass, Center of Mass of Solids

Let a solid be represented by a closed, bounded region D in space with variable density function $\delta(x, y, z)$.

1. The **mass** of the object is $M = \iiint_D dm = \iiint_D \delta(x, y, z)\, dV$.
2. The **moment about the y-z plane** is $M_{yz} = \iiint_D x\delta(x, y, z)\, dV$.
3. The **moment about the x-z plane** is $M_{xz} = \iiint_D y\delta(x, y, z)\, dV$.
4. The **moment about the x-y plane** is $M_{xy} = \iiint_D z\delta(x, y, z)\, dV$.
5. The **center of mass** of the object is

$$(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right).$$

Example 1.6.8 Finding the center of mass of a solid

Find the mass and center of mass of the solid represented by the space region bounded by the coordinate planes and $z = 2 - y/3 - 2x/3$, shown in Figure 1.6.12, with constant density $\delta(x, y, z) = 3\text{gm/cm}^3$. (Note: this space region was used in Example 1.6.2.)

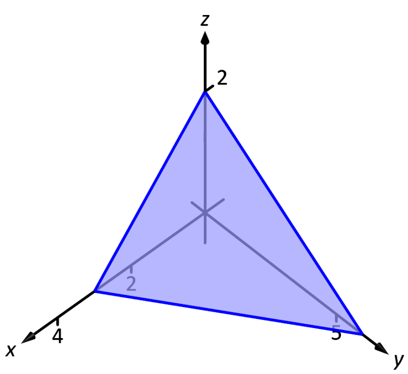


Figure 1.6.12: Finding the center of mass of this solid in Example 1.6.8.

Solution We apply Definition 1.6.4. In Example 1.6.2, we found bounds for the order of integration $dz\,dy\,dx$ to be $0 \le z \le 2-y/3-2x/3$, $0 \le y \le 6-2x$ and $0 \le x \le 3$. We find the mass of the object:

$$\begin{aligned} M &= \iiint_D \delta(x, y, z)\, dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3)\, dz\, dy\, dx \\ &= 3\int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} dz\, dy\, dx \\ &= 3(6) = 18\text{gm}. \end{aligned}$$

The evaluation of the triple integral is done in Example 1.6.2, so we skipped those steps above. Note how the mass of an object with constant density is simply "density$\times$volume."

Notes:

We now find the moments about the planes.

$$\begin{aligned} M_{xy} &= \iiint_D 3z\, dV \\ &= \int_0^3 \int_0^{6-2x} \int_0^{2-y/3-2x/3} (3z)\, dz\, dy\, dx \\ &= \int_0^3 \int_0^{6-2x} \frac{3}{2}\big(2 - y/3 - 2x/3\big)^2\, dy\, dx \\ &= \int_0^3 -\frac{4}{9}\big(x-3\big)^3\, dx \\ &= 9. \end{aligned}$$

We omit the steps of integrating to find the other moments.

$$\begin{aligned} M_{yz} &= \iiint_D 3x\, dV \\ &= \frac{27}{2}. \\ M_{xz} &= \iiint_D 3y\, dV \\ &= 27. \end{aligned}$$

The center of mass is

$$\big(\overline{x}, \overline{y}, \overline{z}\big) = \left(\frac{27/2}{18}, \frac{27}{18}, \frac{9}{18}\right) = \big(0.75, 1.5, 0.5\big).$$

Example 1.6.9 Finding the center of mass of a solid

Find the center of mass of the solid represented by the region bounded by the planes $z = 0$ and $z = -y$ and the cylinder $x^2 + y^2 = 1$, shown in Figure 1.6.13, with density function $\delta(x, y, z) = 10 + x^2 + 5y - 5z$. (Note: this space region was used in Example 1.6.4.)

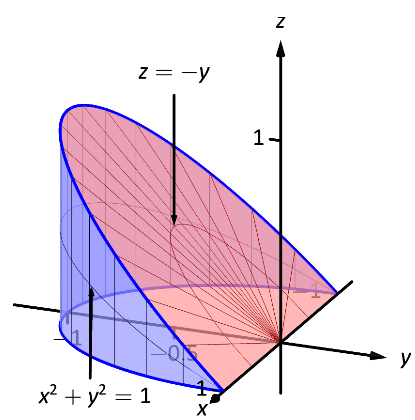


Figure 1.6.13: Finding the center of mass of this solid in Example 1.6.9.

Solution As we start, consider the density function. It is symmetric about the y-z plane, and the farther one moves from this plane, the denser the object is. The symmetry indicates that $\overline{x}$ should be 0.

As one moves away from the origin in the y or z directions, the object becomes less dense, though there is more volume in these regions.

Though none of the integrals needed to compute the center of mass are particularly hard, they do require a number of steps. We emphasize here the

Notes:

importance of knowing how to set up the proper integrals; in complex situations we can appeal to technology for a good approximation, if not the exact answer. We use the order of integration *dz dy dx*, using the bounds found in Example 1.6.4. (As these are the same for all four triple integrals, we explicitly show the bounds only for M.)

$$\begin{aligned}
M &= \iiint_D \big(10 + x^2 + 5y - 5z\big)\, dV \\
&= \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{0} \int_{0}^{-y} \big(10 + x^2 + 5y - 5z\big)\, dV \\
&= \frac{64}{5} - \frac{15\pi}{16} \approx 3.855. \\
M_{yz} &= \iiint_D x\big(10 + x^2 + 5y - 5z\big)\, dV \\
&= 0. \\
M_{xz} &= \iiint_D y\big(10 + x^2 + 5y - 5z\big)\, dV \\
&= 2 - \frac{61\pi}{48} \approx -1.99. \\
M_{xy} &= \iiint_D z\big(10 + x^2 + 5y - 5z\big)\, dV \\
&= \frac{61\pi}{96} - \frac{10}{9} \approx 0.885.
\end{aligned}$$

Note how $M_{yz} = 0$, as expected. The center of mass is

$$\big(\bar{x}, \bar{y}, \bar{z}\big) = \left(0, \frac{-1.99}{3.855}, \frac{0.885}{3.855}\right) \approx \big(0, -0.516, 0.230\big).$$

As stated before, there are many uses for triple integration beyond finding volume. When $h(x, y, z)$ describes a rate of change function over some space region D, then $\iiint_D h(x, y, z)\, dV$ gives the total change over D. Our one specific example of this was computing mass; a density function is simply a "rate of mass change per volume" function. Integrating density gives total mass.

While knowing *how to integrate* is important, it is arguably much more important to know *how to set up* integrals. It takes skill to create a formula that describes a desired quantity; modern technology is very useful in evaluating these formulas quickly and accurately.

Notes:

This chapter investigated the natural follow–on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating "quadruple integrals" whose bounds describe a region in 4–dimensional space (which are very hard to visualize). We can also look back to "regular" integration where we found the area under a curve in the plane. A natural analogue to this is finding the "area under a curve," where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of "integration."

Notes:

Exercises 1.6

Terms and Concepts

13 06 ex 01 1. The strategy for establishing bounds for triple integrals is "________ to ________, ________ to ________ and ________ to ________."

13 06 ex 02 2. Give an informal interpretation of what "$\iiint_D dV$" means.

13 06 ex 23 3. Give two uses of triple integration.

13 06 ex 24 4. If an object has a constant density δ and a volume V, what is its mass?

Problems

13 06 exset 01 **In Exercises 5 – 8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the x, y plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.**

13 06 ex 03 5. $f_1(x,y) = 8 - x^2 - y^2$, $f_2(x,y) = 2x + y$;
R is the square with corners $(-1,-1)$ and $(1,1)$.

13 06 ex 04 6. $f_1(x,y) = x^2 + y^2$, $f_2(x,y) = -x^2 - y^2$;
R is the square with corners $(0,0)$ and $(2,3)$.

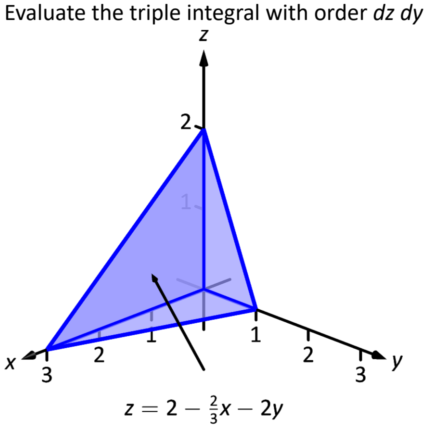
13 06 ex 05 7. $f_1(x,y) = \sin x \cos y$, $f_2(x,y) = \cos x \sin y + 2$;
R is the triangle with corners $(0,0)$, $(\pi,0)$ and (π,π).

13 06 ex 06 8. $f_1(x,y) = 2x^2 + 2y^2 + 3$, $f_2(x,y) = 6 - x^2 - y^2$;
R is the disk bounded by $x^2 + y^2 = 1$.

13 06 exset 02 **In Exercises 9 – 16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.**

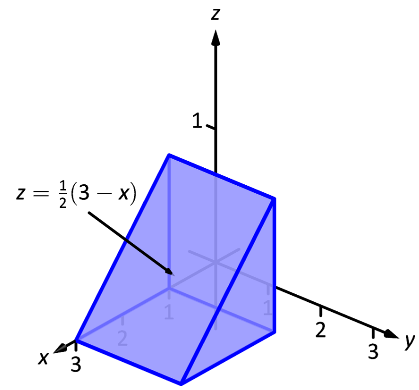
13 06 ex 07 9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.

Evaluate the triple integral with order $dz\,dy\,dx$.



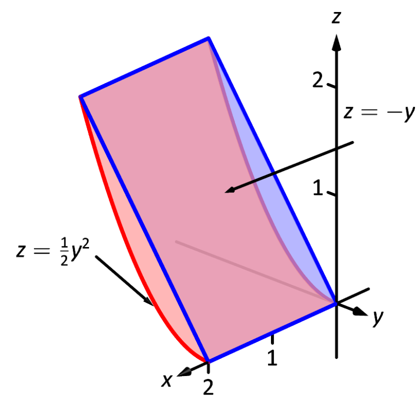
13 06 ex 08 10. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$.

Evaluate the triple integral with order $dx\,dy\,dz$.



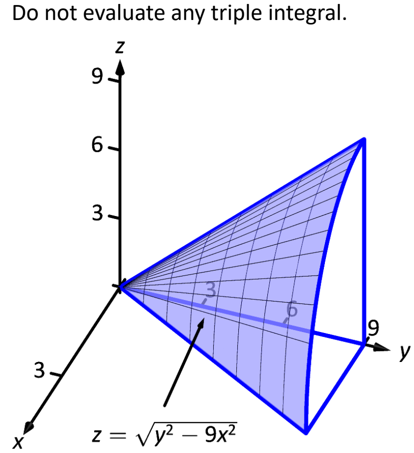
13 06 ex 09 11. D is bounded by the planes $x = 0$, $x = 2$, $z = -y$ and by $z = y^2/2$.

Evaluate the triple integral with the order $dy\,dz\,dx$.



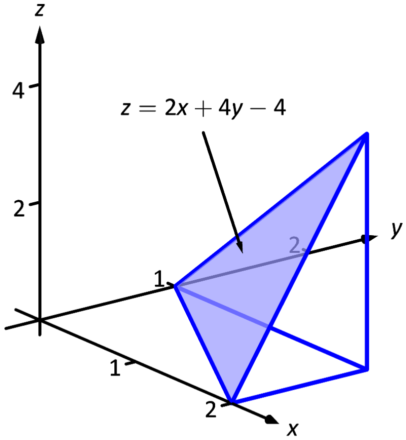
13 06 ex 10 12. D is bounded by the planes $z = 0$, $y = 9$, $x = 0$ and by $z = \sqrt{y^2 - 9x^2}$.

Do not evaluate any triple integral.



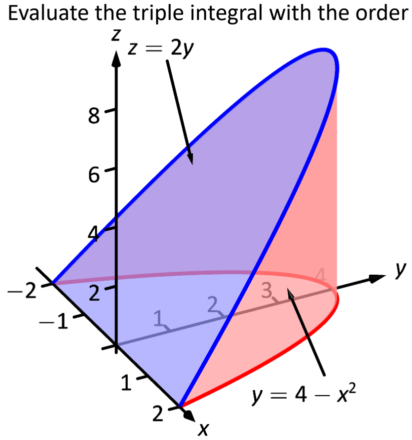
13 06 ex 11 13. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$.

Evaluate the triple integral with the order $dx\,dy\,dz$.

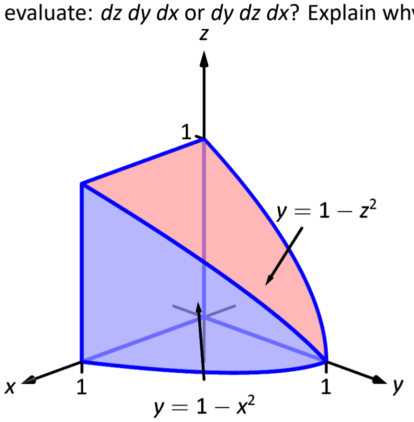


13 06 ex 12 14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.

Evaluate the triple integral with the order $dz\,dy\,dx$.

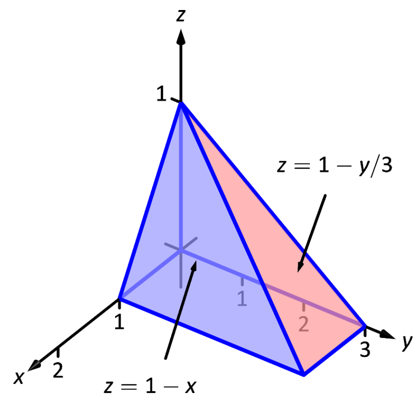


13 06 ex 13 15. D is bounded by the coordinate planes and by $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order is easier to evaluate: $dz\,dy\,dx$ or $dy\,dz\,dx$? Explain why.



13 06 ex 14 16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.

Evaluate the triple integral with order $dx\,dy\,dz$.



13 06 exset 04

In Exercises 17 – 20, evaluate the triple integral.

13 06 ex 19 17. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{\pi}\big(\cos x \sin y \sin z\big)\,dz\,dy\,dx$

13 06 ex 20 18. $\displaystyle\int_0^{1}\int_0^{x}\int_0^{x+y}\big(x + y + z\big)\,dz\,dy\,dx$

13 06 ex 22 19. $\displaystyle\int_0^{\pi}\int_0^{1}\int_0^{z}\big(\sin(yz)\big)\,dx\,dy\,dz$

13 06 ex 21 20. $\displaystyle\int_{\pi}^{\pi^2}\int_x^{x^3}\int_{-y^2}^{y^2}\left(z\frac{x^2y + y^2x}{e^{x^2+y^2}}\right)\,dz\,dy\,dx$

13 06 exset 03

In Exercises 21 – 24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x, y, z)$.

13 06 ex 15 21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\quad \delta(x, y, z) = 10\text{gm/cm}^3$.
(Note: this is the same region as used in Exercise 9.)

13 06 ex 16 22. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$; $\quad \delta(x, y, z) = 2\text{gm/cm}^3$.
(Note: this is the same region as used in Exercise 10.)

13 06 ex 17 23. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$; $\quad \delta(x, y, z) = x^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 13.)

13 06 ex 18 24. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$. $\delta(x, y, z) = y^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 14.)

A: Solutions To Selected Problems

Chapter 1

Section 1.1

13 01 ex 01 1. $C(y)$, meaning that instead of being just a constant, like the number 5, it is a function of y, which acts like a constant when taking derivatives with respect to x.

13 01 ex 02 2. iterated integration

13 01 ex 03 3. curve to curve, then from point to point

13 01 ex 04 4. area

13 01 ex 09 5.

(a) $18x^2 + 42x - 117$

(b) -108

13 01 ex 10 6.

(a) $2 + \pi^2 \cos y$

(b) $\pi^2 + \pi$

13 01 ex 05 7.

(a) $x^4/2 - x^2 + 2x - 3/2$

(b) $23/15$

13 01 ex 06 8.

(a) $y^4/2 - y^3 + y^2/2$

(b) $8/15$

13 01 ex 07 9.

(a) $\sin^2 y$

(b) $\pi/2$

13 01 ex 08 10.

(a) $x/(1 + x^2)$

(b) $\frac{1}{2} \ln \left(\frac{5}{2}\right)$

13 01 ex 11 11. $\int_1^4 \int_{-2}^1 dy\,dx$ and $\int_{-2}^1 \int_1^4 dx\,dy$.
area of $R = 9u^2$

13 01 ex 12 12. $\int_1^4 \int_1^{\frac{2}{3}x+\frac{1}{3}} dy\,dx$ and $\int_1^3 \int_{\frac{3}{2}y-\frac{1}{2}}^4 dx\,dy$.
area of $R = 3u^2$

13 01 ex 13 13. $\int_2^4 \int_{x-1}^{7-x} dy\,dx$. The order $dx\,dy$ needs two iterated integrals as x is bounded above by two different functions. This gives:

$$\int_1^3 \int_2^{y+1} dx\,dy + \int_3^5 \int_2^{7-y} dx\,dy.$$

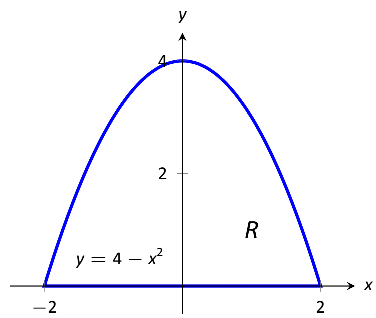
area of $R = 4u^2$

13 01 ex 14 14. $\int_0^{12} \int_{-\sqrt{3x}}^{\sqrt{3x}} dy\,dx$ and $\int_{-6}^6 \int_{y^2/3}^{12} dx\,dy$
area of $R = 96u^2$

13 01 ex 15 15. $\int_0^1 \int_{x^4}^{\sqrt{x}} dy\,dx$ and $\int_0^1 \int_{y^2}^{\sqrt[4]{y}} dx\,dy$
area of $R = 7/15u^2$

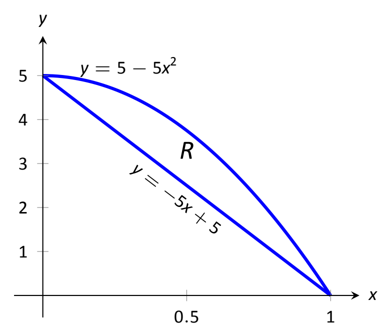
13 01 ex 16 16. $\int_0^2 \int_{x^3}^{4x} dy\,dx$ and $\int_0^8 \int_{y/4}^{\sqrt[3]{y}} dx\,dy$
area of $R = 4u^2$

13 01 ex 18 17.



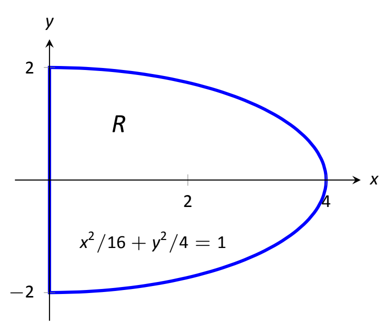
area of $R = \int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy$

13 01 ex 20 18.



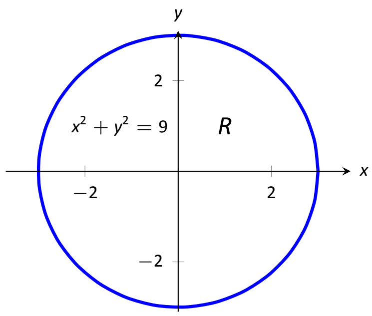
area of $R = \int_0^5 \int_{1-y/5}^{\sqrt{1-y/5}} dx\,dy$

13 01 ex 19 19.



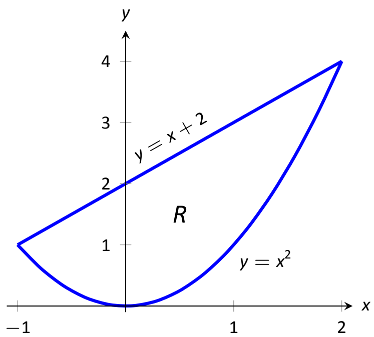
area of $R = \int_0^4 \int_{-\sqrt{4-x^2/4}}^{\sqrt{4-x^2/4}} dy\,dx$

13 01 ex 17 20.



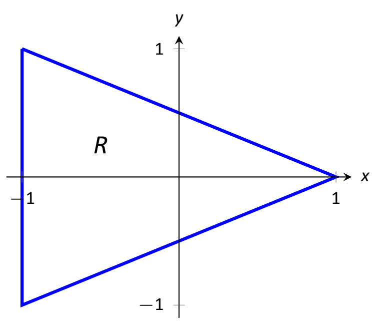
area of $R = \int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dx\,dy$

13 01 ex 22 21.



area of $R = \int_{-1}^{2}\int_{x^2}^{x+2} dy\,dx$

13 01 ex 21 22.



area of $R = \int_{-1}^{0}\int_{0}^{2y+1} dx\,dy + \int_{0}^{1}\int_{0}^{1-2y} dx\,dy$

Section 1.2

13 02 ex 01 1. volume

13 02 ex 02 2. When switching the order of integration, the bounds integrals must change to reflect the bounds of the region of integration. You cannot merely change the letters x and y in a few places.

13 02 ex 03 3. The double integral gives the signed volume under the surface. Since the surface is always positive, it is always above the x-y plane and hence produces only "positive" volume.

13 02 ex 04 4. No. It means that there is the same amount of signed volume under f and g over R, but the functions could be very different.

13 02 ex 05 5. 6; $\int_{-1}^{1}\int_{1}^{2}\left(\frac{x}{y}+3\right) dy\,dx$

13 02 ex 06 6. 4; $\int_{0}^{\pi}\int_{-\pi/2}^{\pi/2}(\sin x\cos y)\ dy\,dx$

13 02 ex 07 7. $112/3$; $\int_{0}^{2}\int_{0}^{4-2y}\left(3x^2-y+2\right)\ dx\,dy$

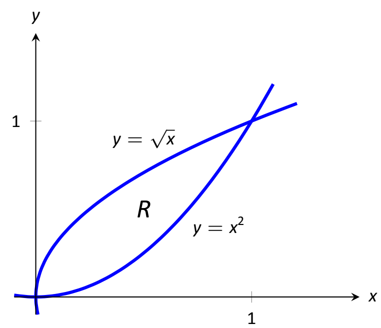
13 02 ex 08 8. $76/15$; $\int_{1}^{3}\int_{1}^{x}\left(x^2y-xy^2\right)\ dy\,dx$

13 02 ex 09 9. $16/5$; $\int_{-1}^{1}\int_{0}^{1-x^2}\left(x+y+2\right)\ dy\,dx$

13 02 ex 10 10. $6561/40$; $\int_{0}^{3}\int_{x^2}^{3x}\left(xy^2\right)\ dy\,dx$

13 02 ex 11 11.

(a)

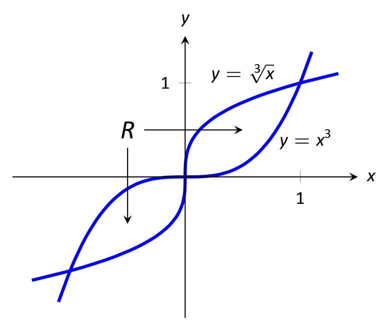


(b) $\int_{0}^{1}\int_{x^2}^{\sqrt{x}} x^2y\,dy\,dx = \int_{0}^{1}\int_{y^2}^{\sqrt{y}} x^2y\,dx\,dy.$

(c) $\frac{3}{56}$

13 02 ex 12 12.

(a)

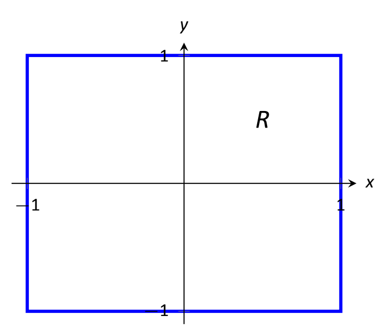


(b) $\int_{0}^{1}\int_{x^3}^{\sqrt[3]{x}} x^2y\,dy\,dx + \int_{-1}^{0}\int_{\sqrt[3]{x}}^{x^3} x^2y\,dy\,dx$
$= \int_{0}^{1}\int_{y^3}^{\sqrt[3]{y}} x^2y\,dx\,dy + \int_{-1}^{0}\int_{\sqrt[3]{y}}^{y^3} x^2y\,dx\,dy.$

(c) 0

13 02 ex 13 13.

(a)

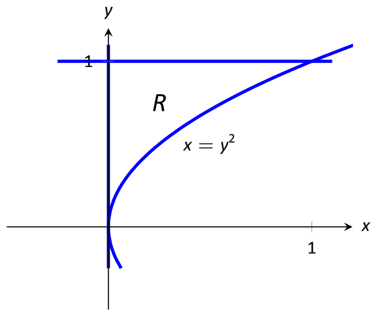


(b) $\int_{-1}^{1}\int_{-1}^{1} x^2-y^2\,dy\,dx = \int_{-1}^{1}\int_{-1}^{1} x^2-y^2\,dx\,dy.$

(c) 0

13 02 ex 14 14.

(a)

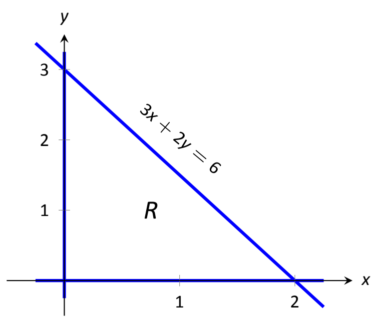


(b) $\displaystyle\int_0^1\int_0^{y^2} ye^x\,dx\,dy = \int_0^1\int_{\sqrt{x}}^1 ye^x\,dy\,dx.$

(c) $e/2 - 1$

13 02 ex 15 15.

(a)



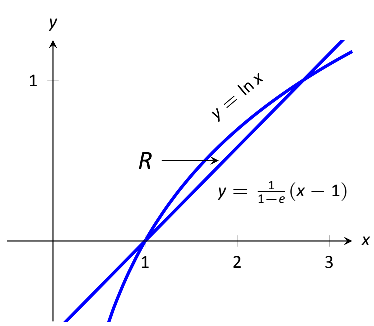
(b)

(c) $\displaystyle\int_0^2\int_0^{3-3/2x} \big(6 - 3x - 2y\big)\,dy\,dx =$
$\displaystyle\int_0^3\int_0^{2-2/3y} \big(6 - 3x - 2y\big)\,dx\,dy.$

(d) 6

13 02 ex 16 16.

(a)



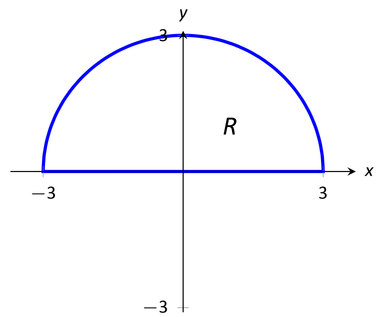
(b)

(c) $\displaystyle\int_1^e\int_{\frac{x-1}{e-1}}^{\ln x} e^y\,dy\,dx = \int_0^1\int_{e^y}^{y(e-1)+1} e^y\,dx\,dy.$

(d) $-\frac{1}{2}e^2 + 2e - \frac{3}{2}$

13 02 ex 17 17.

(a)

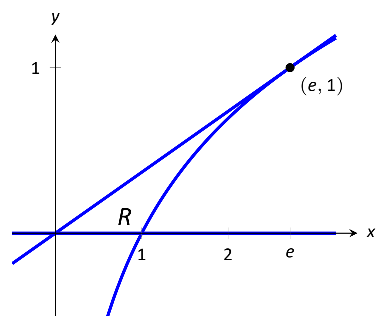


(b) $\displaystyle\int_{-3}^3\int_0^{\sqrt{9-x^2}} \big(x^3y - x\big)\,dy\,dx =$
$\displaystyle\int_0^3\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \big(x^3y - x\big)\,dx\,dy.$

(c) 0

13 02 ex 18 18.

(a)



(b) $\displaystyle\int_0^1\int_{ey}^{e^y} \big(4 - 3y\big)\,dx\,dy =$
$\displaystyle\int_0^1\int_0^{x/e} \big(4 - 3y\big)\,dy\,dx + \int_1^e\int_{\ln x}^{x/e} \big(4 - 3y\big)\,dy\,dx.$

(c) $3e - 7$

13 02 ex 19 19. Integrating e^{x^2} with respect to x is not possible in terms of elementary functions. $\displaystyle\int_0^2\int_0^{2x} e^{x^2}\,dy\,dx = e^4 - 1.$

13 02 ex 20 20. Integrating $\cos(y^2)$ with respect to y is not possible in terms of elementary functions.
$\displaystyle\int_0^{\sqrt{\pi/2}}\int_0^y \cos(y^2)\,dx\,dy = \frac{1}{2}.$

13 02 ex 21 21. Integrating $\displaystyle\int_y^1 \frac{2y}{x^2 + y^2}\,dx$ gives $\tan^{-1}(1/y) - \pi/4$; integrating $\tan^{-1}(1/y)$ is hard.
$\displaystyle\int_0^1\int_0^x \frac{2y}{x^2 + y^2}\,dy\,dx = \ln 2.$

13 02 ex 22 22. Integrating in the order shown is hard/impossible. By changing the order of integration, we have $\displaystyle\int_1^2\int_{-1}^1 \frac{x\tan^2 y}{1 + \ln y}\,dx\,dy = 0$, since the integrand is an odd function with respect to x. Thus the iterated integral evaluates to 0.

13 02 ex 23 23. average value of $f = 6/2 = 3$

13 02 ex 24 24. average value of $f = 4/\pi^2$

13 02 ex 25 25. average value of $f = \frac{112/3}{4} = 28/3$

13 02 ex 26 26. average value of $f = \frac{76/15}{2} = \frac{38}{15} \approx 2.53$

Section 1.3

13 03 ex 01 1. $f\big(r\cos\theta, r\sin\theta\big)$, $r\,dr\,d\theta$

13 03 ex 02 2. Some regions in the x-y plane are easier to describe using polar coordinates than using rectangular coordinates. Also, some integrals are easier to evaluate one the polar substitutions have been made.

13 03 ex 03 3. $\displaystyle\int_0^{2\pi}\int_0^1 \big(3r\cos\theta - r\sin\theta + 4\big) r\,dr\,d\theta = 4\pi$

13 03 ex 04 4. $\displaystyle\int_0^{2\pi}\int_0^2 \big(4r\cos\theta + 4r\sin\theta\big) r\,dr\,d\theta = 0$

13 03 ex 05 5. $\displaystyle\int_0^{\pi}\int_{\cos\theta}^{3\cos\theta} \big(8 - r\sin\theta\big) r\,dr\,d\theta = 16\pi$

13 03 ex 06 6. $\displaystyle\int_0^{\pi/2}\int_0^{\sin(2\theta)} \big(4\big) r\,dr\,d\theta = \pi/2$

13 03 ex 07 7. $\displaystyle\int_0^{2\pi}\int_1^2 \big(\ln(r^2)\big) r\,dr\,d\theta = 2\pi\big(\ln 16 - 3/2\big)$

13 03 ex 08 8. $\displaystyle\int_0^{2\pi}\int_0^1 \big(1 - r^2\big) r\,dr\,d\theta = \pi/2$

13 03 ex 09 9. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^6 \big(r^2\cos^2\theta - r^2\sin^2\theta\big) r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\int_0^6 \big(r^2\cos(2\theta)\big) r\,dr\,d\theta = 0$

13 03 ex 10 10. $\displaystyle\int_0^{\pi/4}\int_0^1 \left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right) r\,dr\,d\theta = \ln 2$

13 03 ex 11 11. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^5 \big(r^2\big)\,dr\,d\theta = 125\pi/3$

13 03 ex 12 12. $\displaystyle\int_{\pi/2}^{3\pi/2}\int_0^4 \big(2r\sin\theta - r\cos\theta\big) r\,dr\,d\theta = 128/3$

13 03 ex 13 13. $\displaystyle\int_0^{\pi/4}\int_0^{\sqrt{8}} \big(r\cos\theta + r\sin\theta\big) r\,dr\,d\theta = 16\sqrt{2}/3$

13 03 ex 14 14. $\displaystyle\int_0^{\pi}\int_1^2 \big(r\cos\theta + 5\big) r\,dr\,d\theta = 15\pi/2$

13 03 ex 15 15.

(a) This is impossible to integrate with rectangular coordinates as $e^{-(x^2+y^2)}$ does not have an antiderivative in terms of elementary functions.

(b) $\displaystyle\int_0^{2\pi}\int_0^a re^{r^2}\,dr\,d\theta = \pi(1 - e^{-a^2})$.

(c) $\displaystyle\lim_{a\to\infty} \pi(1 - e^{-a^2}) = \pi$. This implies that there is a finite volume under the surface $e^{-(x^2+y^2)}$ over the entire x-y plane.

13 03 ex 16 16.

$$\begin{aligned}
\iint_R f(x,y)\,dA &= \int_0^{2\pi}\int_0^a \left(h - h\sqrt{\frac{r^2\cos^2\theta}{a^2} + \frac{r^2\sin^2\theta}{a^2}}\right) r\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^a \left(hr - h\frac{r^2}{a}\right) dr\,d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{2}hr^2 - \frac{h}{3a}r^3\right)\Big|_0^a\,d\theta \\
&= \int_0^{2\pi} \left(\frac{1}{6}a^2h\right)\,d\theta \\
&= \frac{1}{3}\pi a^2 h.
\end{aligned}$$

Section 1.4

13 04 ex 01 1. Because they are scalar multiples of each other.

13 04 ex 02 2. y

13 04 ex 03 3. "little masses"

13 04 ex 04 4. A collection of individual masses in the plane. Each mass is a point mass, i.e., mass located at a point, not across a region.

13 04 ex 05 5. M_x measures the moment about the x-axis, meaning we need to measure distance from the x-axis. Such measurements are measures in the y-direction.

13 04 ex 06 6. If the lamina is an annulus, the center of mass will likely be in the middle, outside of the region. (See Example 1.4.9.)

13 04 ex 07 7. $\overline{x} = 5.25$

13 04 ex 08 8. $\overline{x} = 1.3$

13 04 ex 10 9. $(\overline{x}, \overline{y}) = (0, 3)$

13 04 ex 09 10. $(\overline{x}, \overline{y}) = (0, 1/3)$

13 04 ex 11 11. $M = 150$gm;

13 04 ex 12 12. $M = 190$gm

13 04 ex 13 13. $M = 2$lb

13 04 ex 14 14. $M = 2/3$lb

13 04 ex 15 15. $M = 16\pi \approx 50.27$kg

13 04 ex 16 16. $M = 325\pi/12 \approx 85$kg

13 04 ex 17 17. $M = 54\pi \approx 169.65$lb

13 04 ex 18 18. $M = 63\pi \approx 197.92$lb

13 04 ex 19 19. $M = 150$gm; $M_y = 600$; $M_x = -75$; $(\overline{x}, \overline{y}) = (4, -0.5)$

13 04 ex 20 20. $M = 190$gm; $M_y = 850$; $M_x = -315/2$; $(\overline{x}, \overline{y}) = (4.47, -0.83)$

13 04 ex 21 21. $M = 2$lb; $M_y = 0$; $M_x = 2/3$; $(\overline{x}, \overline{y}) = (0, 1/3)$

13 04 ex 22 22. $M = 2/3$lb; $M_y = 7/30$; $M_x = 7/30$; $(\overline{x}, \overline{y}) = (0.35, 0.35)$

13 04 ex 23 23. $M = 16\pi \approx 50.27$kg; $M_y = 4\pi$; $M_x = 4\pi$; $(\overline{x}, \overline{y}) = (1/4, 1/4)$

13 04 ex 24 24. $M = 325\pi/12 \approx 85$kg; $M_y = 2375/12$; $M_x = 2375/12$; $(\overline{x}, \overline{y}) = (2.33, 2.33)$

13 04 ex 25 25. $M = 54\pi \approx 169.65$lb; $M_y = 0$; $M_x = 504$; $(\overline{x}, \overline{y}) = (0, 2.97)$

13 04 ex 26 26. $M = 63\pi \approx 197.92$lb; $M_y = 0$; $M_x = 1215/2$; $(\overline{x}, \overline{y}) = (0, 3.07)$

13 04 ex 27 27. $I_x = 64/3$; $I_y = 64/3$; $I_O = 128/3$

13 04 ex 28 28. $I_x = 16/3$; $I_y = 256/3$; $I_O = 272/3$

13 04 ex 29 29. $I_x = 16/3$; $I_y = 64/3$; $I_O = 80/3$

13 04 ex 30 30. $I_x = 16$; $I_y = 16$; $I_O = 32$

Section 1.5

13 05 ex 01 1. arc length

13 05 ex 02 2. tangent

13 05 ex 03 3. surface areas

13 05 ex 04 4. Technology makes good approximations accessible, if not exact answers.

13 05 ex 18 5. Intuitively, adding h to f only shifts f up (i.e., parallel to the z-axis) and does not change its shape. Therefore it will not change the surface area over R.

Analytically, $f_x = g_x$ and $f_y = g_y$; therefore, the surface area of each is computed with identical double integrals.

13 05 ex 19 6. Analytically, $g_x = 2f_x$ and $g_y = 2f_y$. The double integral to compute the surface area of f over R is $\iint_R \sqrt{1+f_x^2+f_y^2}\, dA$; the double integral to compute the surface area of g over R is $\iint_R \sqrt{1+4f_x^2+4f_y^2}\, dA$, which is *not* twice the double integral used to calculate the surface area of f.

13 05 ex 05 7. $SA = \int_0^{2\pi}\int_0^{2\pi} \sqrt{1+\cos^2 x\cos^2 y+\sin^2 x\sin^2 y}\, dx\, dy$

13 05 ex 06 8. $SA = \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \sqrt{1+\frac{4x^2+4y^2}{(1+x^2+y^2)^4}}\, dx\, dy$

Polar offers simpler bounds:

$$SA = \int_0^{2\pi}\int_0^{3} r\sqrt{1+\frac{4r^2}{(1+r^2)^4}}\, dr\, d\theta$$

13 05 ex 07 9. $SA = \int_{-1}^{1}\int_{-1}^{1} \sqrt{1+4x^2+4y^2}\, dx\, dy$

13 05 ex 08 10. $SA = \int_{-5}^{5}\int_0^{1} \sqrt{1+\frac{4x^2e^{2x^2}}{\left(1+e^{x^2}\right)^4}}\, dy\, dx$

13 05 ex 09 11. $SA = \int_0^{3}\int_{-1}^{1} \sqrt{1+9+49}\, dx\, dy = 6\sqrt{59} \approx 46.09$

13 05 ex 10 12. $SA = \int_0^{1}\int_0^{1-x} \sqrt{1+4+4}\, dy\, dx = 18$

13 05 ex 11 13. This is easier in polar:

$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^{4} r\sqrt{1+4r^2\cos^2 t+4r^2\sin^2 t}\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^{4} r\sqrt{1+4r^2}\, dr\, d\theta \\ &= \frac{\pi}{6}\left(65\sqrt{65}-1\right) \approx 273.87 \end{aligned}$$

13 05 ex 14 14.

$$\begin{aligned} SA &= \int_0^{1}\int_{-y}^{y} \sqrt{1+4+64y^2}\, dx\, dy \\ &= \int_0^{1} \left(2y\sqrt{5+64y^2}\right) dy \\ &= \frac{1}{96}\left(69\sqrt{69}-5\sqrt{5}\right) \approx 5.85 \end{aligned}$$

13 05 ex 15 15.

$$\begin{aligned} SA &= \int_0^{2}\int_0^{2x} \sqrt{1+1+4x^2}\, dy\, dx \\ &= \int_0^{2} \left(2x\sqrt{2+4x^2}\right) dx \\ &= \frac{26}{3}\sqrt{2} \approx 12.26 \end{aligned}$$

13 05 ex 16 16.

$$\begin{aligned} SA &= \int_0^{1}\int_0^{1} \sqrt{1+x+9y}\, dx\, dy \\ &= \int_0^{1} \frac{2}{3}\left((9y+2)^{3/2}-(9y+1)^{3/2}\right) dy \\ &= \frac{4}{135}\left(121\sqrt{11}-100\sqrt{10}-4\sqrt{2}+1\right) \approx 2.383 \end{aligned}$$

13 05 ex 12 17. This is easier in polar:

$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^{5} r\sqrt{1+\frac{4r^2\cos^2\theta+4r^2\sin^2\theta}{r^2\sin^2\theta+r^2\cos^2\theta}}\, dr\, d\theta \\ &= \int_0^{2\pi}\int_0^{5} r\sqrt{5}\, dr\, d\theta \\ &= 25\pi\sqrt{5} \approx 175.62 \end{aligned}$$

13 05 ex 13 18. This is easier in polar:

$$\begin{aligned} SA &= 2\int_0^{2\pi}\int_0^{5} r\sqrt{1+\frac{r^2\cos^2 t+r^2\sin^2 t}{25-r^2\sin^2 t-r^2\cos^2 t}}\, dr\, d\theta \\ &= 2\int_0^{2\pi}\int_0^{5} r\sqrt{\frac{1}{25-r^2}}\, dr\, d\theta \\ &= 100\pi \approx 314.16 \end{aligned}$$

13 05 ex 17 19. Integrating in polar is easiest considering R:

$$\begin{aligned} SA &= \int_0^{2\pi}\int_0^{1} r\sqrt{1+c^2+d^2}\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{2}\left(\sqrt{1+c^2+d^2}\right) d\theta \\ &= \pi\sqrt{1+c^2+d^2}. \end{aligned}$$

The value of h does not matter as it only shifts the plane vertically (i.e., parallel to the z-axis). Different values of h do not create different ellipses in the plane.

Section 1.6

13 06 ex 01 1. surface to surface, curve to curve and point to point

13 06 ex 02 2. One possible answer is "sum up lots of little volumes over D."

13 06 ex 23 3. Answers can vary. From this section we used triple integration to find the volume of a solid region, the mass of a solid, and the center of mass of a solid.

13 06 ex 24 4. δV.

13 06 ex 03 5. $V = \int_{-1}^{1}\int_{-1}^{1} \left(8-x^2-y^2-(2x+y)\right) dx\, dy = 88/3$

13 06 ex 04 6. $V = \int_0^2 \int_0^3 \left(x^2+y^2-(-x^2-y^2)\right)\, dy\, dx = 52$

13 06 ex 05 7. $V = \int_0^{\pi} \int_0^x \left(\cos x \sin y + 2 - \sin x \cos y\right)\, dy\, dx = \pi^2 - \pi \approx 6.728$

13 06 ex 06 8. $V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(6 - x^2 - y^2 - (2x^2+2y^2+3)\right)\, dy\, dx$.
Integrating in polar is easier, giving
$V = \int_0^{2\pi} \int_0^1 \left(3-3r^2\right) r\, dr\, d\theta = 3\pi/2$.

13 06 ex 07 9. $dz\,dy\,dx$: $\displaystyle\int_0^3 \int_0^{1-x/3} \int_0^{2-2x/3-2y} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_0^1 \int_0^{3-3y} \int_0^{2-2x/3-2y} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_0^3 \int_0^{2-2x/3} \int_0^{1-x/3-z/2} dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^2 \int_0^{3-3z/2} \int_0^{1-x/3-z/2} dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^1 \int_0^{2-2y} \int_0^{3-3y-3z/2} dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^2 \int_0^{1-z/2} \int_0^{3-3y-3z/2} dx\,dy\,dz$

$\displaystyle V = \int_0^3 \int_0^{1-x/3} \int_0^{2-2x/3-2y} dz\,dy\,dx = 1.$

13 06 ex 08 10. $dz\,dy\,dx$: $\displaystyle\int_1^3 \int_0^2 \int_0^{(3-x)/2} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_0^2 \int_1^3 \int_0^{(3-x)/2} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_1^3 \int_0^{(3-x)/2} \int_0^2 dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^1 \int_1^{3-2z} \int_0^2 dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^2 \int_0^1 \int_1^{3-2z} dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^1 \int_0^2 \int_1^{3-2z} dx\,dy\,dz$

$\displaystyle V = \int_0^1 \int_0^2 \int_1^{3-2z} dx\,dy\,dz = 2.$

13 06 ex 09 11. $dz\,dy\,dx$: $\displaystyle\int_0^2 \int_{-2}^0 \int_{y^2/2}^{-y} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_{-2}^0 \int_0^2 \int_{y^2/2}^{-y} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_{-2}^0 \int_{y^2/2}^{-y} \int_0^2 dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^2 \int_{-\sqrt{2z}}^{-z} \int_0^2 dx\,dy\,dz$

$\displaystyle V = \int_0^2 \int_0^2 \int_{-\sqrt{2z}}^{-z} dy\,dz\,dx = 4/3.$

13 06 ex 10 12. $dz\,dy\,dx$: $\displaystyle\int_0^3 \int_{3x}^9 \int_0^{\sqrt{y^2-9x^2}} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_0^9 \int_0^{y/3} \int_0^{\sqrt{y^2-9x^2}} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_0^3 \int_0^{\sqrt{81-9x^2}} \int_{\sqrt{z^2+9x^2}}^9 dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^9 \int_0^{\sqrt{9-z^2/9}} \int_{\sqrt{z^2+9x^2}}^9 dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^9 \int_0^y \int_0^{\frac{1}{3}\sqrt{y^2-z^2}} dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^9 \int_z^9 \int_0^{\frac{1}{3}\sqrt{y^2-z^2}} dx\,dy\,dz$

13 06 ex 11 13. $dz\,dy\,dx$: $\displaystyle\int_0^2 \int_{1-x/2}^1 \int_0^{2x+4y-4} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_0^1 \int_{2-2y}^2 \int_0^{2x+4y-4} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_0^2 \int_0^{2x} \int_{z/4-x/2+1}^1 dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^4 \int_{z/2}^2 \int_{z/4-x/2+1}^1 dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^1 \int_0^{4y} \int_{z/2-2y+2}^2 dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^4 \int_{z/4}^1 \int_{z/2-2y+2}^2 dx\,dy\,dz$

$\displaystyle V = \int_0^4 \int_{z/4}^1 \int_{z/2-2y+2}^2 dx\,dy\,dz = 4/3.$

13 06 ex 12 14. $dz\,dy\,dx$: $\displaystyle\int_{-2}^2 \int_0^{4-x^2} \int_0^{2y} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_0^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} \int_0^{2x+4y-4} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_{-2}^2 \int_0^{8-2x^2} \int_{z/2}^{4-x^2} dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^8 \int_{-\sqrt{4-z/2}}^{\sqrt{4-z/2}} \int_{z/2}^{4-x^2} dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^4 \int_0^{2y} \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^8 \int_{z/2}^4 \int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy\,dz$

$\displaystyle V = \int_{-2}^2 \int_0^{4-x^2} \int_0^{2y} dz\,dy\,dx = 512/15.$

13 06 ex 13 15. $dz\,dy\,dx$: $\displaystyle\int_0^1 \int_0^{1-x^2} \int_0^{\sqrt{1-y}} dz\,dy\,dx$

$dz\,dx\,dy$: $\displaystyle\int_0^1 \int_0^{\sqrt{1-y}} \int_0^{\sqrt{1-y}} dz\,dx\,dy$

$dy\,dz\,dx$: $\displaystyle\int_0^1 \int_0^x \int_0^{1-x^2} dy\,dz\,dx + \int_0^1 \int_x^1 \int_0^{1-z^2} dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^1 \int_0^z \int_0^{1-z^2} dy\,dx\,dz + \int_0^1 \int_z^1 \int_0^{1-x^2} dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^1 \int_0^{\sqrt{1-y}} \int_0^{\sqrt{1-y}} dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^1 \int_0^{1-z^2} \int_0^{\sqrt{1-y}} dx\,dy\,dz$

Answers will vary. Neither order is particularly "hard." The order *dz dy dx* requires integrating a square root, so powers can be messy; the order *dy dz dx* requires two triple integrals, but each uses only polynomials.

13 06 ex 14 16. $dz\,dy\,dx$:
$\displaystyle\int_0^1 \int_0^{3x} \int_0^{1-x} dz\,dy\,dx + \int_0^1 \int_{3x}^3 \int_0^{1-y/3} dz\,dy\,dx$

$dz\,dx\,dy$:
$$\int_0^3 \int_0^{y/3} \int_0^{1-y/3} dz\,dy\,dx + \int_0^3 \int_{y/3}^1 \int_0^{1-x} dz\,dx\,dy$$

$dy\,dz\,dx$: $\displaystyle\int_0^1 \int_0^{1-x} \int_0^{3-3z} dy\,dz\,dx$

$dy\,dx\,dz$: $\displaystyle\int_0^1 \int_0^{1-z} \int_0^{3-3z} dy\,dx\,dz$

$dx\,dz\,dy$: $\displaystyle\int_0^3 \int_0^{1-y/3} \int_0^{1-z} dx\,dz\,dy$

$dx\,dy\,dz$: $\displaystyle\int_0^1 \int_0^{3-3z} \int_0^{1-z} dx\,dy\,dz$

$$V = \int_0^1 \int_0^{3-3z} \int_0^{1-z} dx\,dy\,dz = 1.$$

13 06 ex 19 17. 8

13 06 ex 20 18. $7/8$

13 06 ex 22 19. π

13 06 ex 21 20. 0

13 06 ex 15 21. $M = 10$, $M_{yz} = 15/2$, $M_{xz} = 5/2$, $M_{xy} = 5$; $(\bar{x}, \bar{y}, \bar{z}) = (3/4, 1/4, 1/2)$

13 06 ex 16 22. $M = 4$, $M_{yz} = 20/3$, $M_{xz} = 4$, $M_{xy} = 4/3$; $(\bar{x}, \bar{y}, \bar{z}) = (5/3, 1, 1/3)$

13 06 ex 17 23. $M = 16/5$, $M_{yz} = 16/3$, $M_{xz} = 104/45$, $M_{xy} = 32/9$; $(\bar{x}, \bar{y}, \bar{z}) = (5/3, 13/18, 10/9) \approx (1.67, 0.72, 1.11)$

13 06 ex 18 24. $M = \frac{65,536}{15} \approx 208.05$, $M_{yz} = 0$, $M_{xz} = \frac{2,097,152}{3465} \approx 605.24$, $M_{xy} = \frac{2,097,152}{3465} \approx 605.24$; $(\bar{x}, \bar{y}, \bar{z}) = (0, 32/11, 32/11) \approx (0, 2.91, 2.91)$