1: Vectors

1.1 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction ("... with winds from the southeast gusting up to 30 mph ..."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One "bare–bones" definition of a vector is based on what we wrote above: "a vector is a mathematical object with magnitude and direction parameters." This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

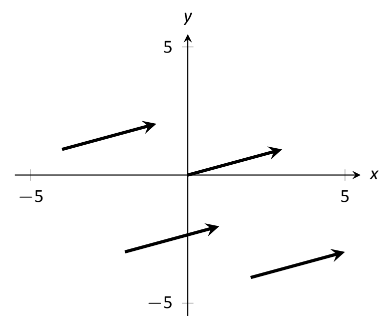


Figure 1.1.1: Drawing the same vector with different initial points.

Definition 1.1.1 Vector

A **vector** is a directed line segment.

Given points P and Q (either in the plane or in space), we denote with $\overrightarrow{PQ}$ the vector *from P to Q*. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.

The **magnitude**, **length** or **norm** of $\overrightarrow{PQ}$ is the length of the line segment $\overline{PQ}$: $||\overrightarrow{PQ}|| = ||\overline{PQ}||$.

Two vectors are **equal** if they have the same magnitude and direction.

Figure 10.2.1 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use $\mathbb{R}^2$ (pronounced "r two") to represent all the vectors in the plane, and use $\mathbb{R}^3$ (pronounced "r three") to represent all the vectors in space.

Consider the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ as shown in Figure 10.2.2. The vectors look to

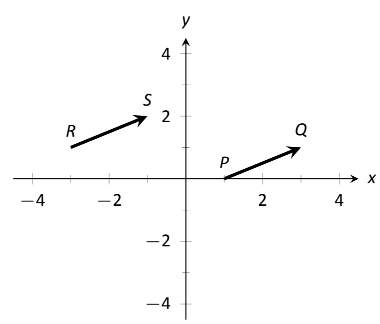


Figure 1.1.2: Illustrating how equal vectors have the same displacement.

be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through P and Q or R and S). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the x, y and possibly z directions the terminal point is from the initial point. Both the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in Figure 10.2.2 have an x-displacement of 2 and a y-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose x-displacement is a and whose y-displacement is b will have terminal point (a, b) when the initial point is the origin, $(0, 0)$. This leads us to a definition of a standard and concise way of referring to vectors.

Definition 1.1.2 Component Form of a Vector

1. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a, b) when its initial point is $(0, 0)$, is $\langle a, b \rangle$.
2. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a, b, c) when its initial point is $(0, 0, 0)$, is $\langle a, b, c \rangle$.

The numbers a, b (and c, respectively) are the **components** of $\vec{v}$.

It follows from the definition that the component form of the vector $\overrightarrow{PQ}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1 \rangle ;$$

in space, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, the component form of $\overrightarrow{PQ}$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle .$$

We practice using this notation in the following example.

Example 1.1.1 Using component form notation for vectors

1. Sketch the vector $\vec{v} = \langle 2, -1 \rangle$ starting at $P = (3, 2)$ and find its magnitude.
2. Find the component form of the vector $\vec{w}$ whose initial point is $R = (-3, -2)$ and whose terminal point is $S = (-1, 2)$.

Notes:

3. Sketch the vector $\vec{u} = \langle 2, -1, 3\rangle$ starting at the point $Q = (1, 1, 1)$ and find its magnitude.

SOLUTION

1. Using P as the initial point, we move 2 units in the positive x-direction and -1 units in the positive y-direction to arrive at the terminal point $P' = (5, 1)$, as drawn in Figure 10.2.3(a).

   The magnitude of $\vec{v}$ is determined directly from the component form:

   $$||\,\vec{v}\,|| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

2. Using the note following Definition 10.2.2, we have

   $$\overrightarrow{RS} = \langle -1 - (-3), 2 - (-2)\rangle = \langle 2, 4\rangle\,.$$

   One can readily see from Figure 10.2.3(a) that the x- and y-displacement of $\overrightarrow{RS}$ is 2 and 4, respectively, as the component form suggests.

3. Using Q as the initial point, we move 2 units in the positive x-direction, -1 unit in the positive y-direction, and 3 units in the positive z-direction to arrive at the terminal point $Q' = (3, 0, 4)$, illustrated in Figure 10.2.3(b).

   The magnitude of $\vec{u}$ is:

   $$||\,\vec{u}\,|| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

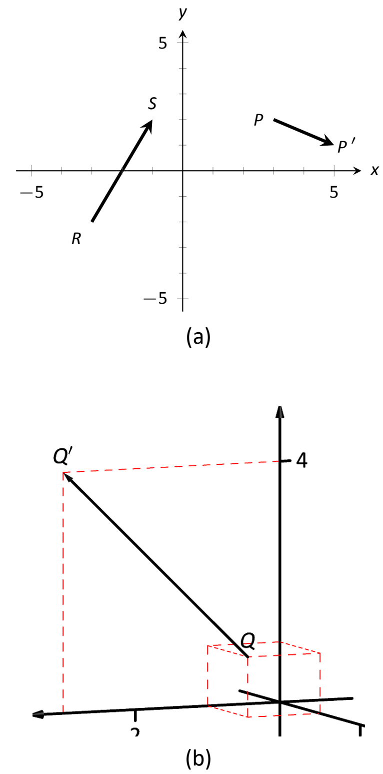


Figure 1.1.3: Graphing vectors in Example 10.2.1.

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

Notes:

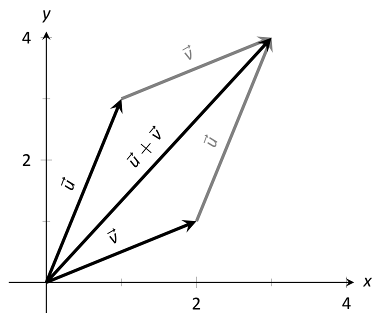


Figure 1.1.5: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

Definition 1.1.3 Vector Algebra

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ be vectors in $\mathbb{R}^2$, and let c be a scalar.
   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector
   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle .$$
   (b) The scalar product of c and $\vec{v}$ is the vector
   $$c\vec{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle .$$
2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$, and let c be a scalar.
   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector
   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$
   (b) The scalar product of c and $\vec{v}$ is the vector
   $$c\vec{v} = c\langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle .$$

In short, we say addition and scalar multiplication are computed "component–wise."

Example 1.1.2 Adding vectors

Sketch the vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 1 \rangle$ and $\vec{u} + \vec{v}$ all with initial point at the origin.

Solution We first compute $\vec{u} + \vec{v}$.

$$\begin{aligned}\vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle .\end{aligned}$$

These are all sketched in Figure 10.2.4.

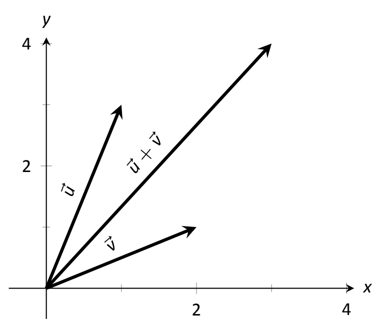


Figure 1.1.4: Graphing the sum of vectors in Example 10.2.2.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding $\vec{u} + \vec{v}$ suggests the following idea:

"Starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$."

Notes:

This idea is sketched in Figure 10.2.5, where the initial point of $\vec{v}$ is the terminal point of $\vec{u}$. This is known as the "Head to Tail Rule" of adding vectors. Vector addition is very important. For instance, if the vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u}+\vec{v}$ gives the resulting force. Because of various physical applications of vector addition, the sum $\vec{u}+\vec{v}$ is often referred to as the **resultant vector**, or just the "resultant."

Analytically, it is easy to see that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. Figure 10.2.5 also gives a graphical representation of this, using gray vectors. Note that the vectors $\vec{u}$ and $\vec{v}$, when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector $\vec{u}+\vec{v}$ is defined by forming the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$; the initial point of $\vec{u}+\vec{v}$ is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in $\mathbb{R}^3$ as well.

It follows from the properties of the real numbers and Definition 10.2.3 that

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

Example 1.1.3 Vector Subtraction
Let $\vec{u}=\langle 3,1\rangle$ and $\vec{v}=\langle 1,2\rangle$. Compute and sketch $\vec{u}-\vec{v}$.

Solution The computation of $\vec{u}-\vec{v}$ is straightforward, and we show all steps below. Usually the formal step of multiplying by (-1) is omitted and we "just subtract."

$$\begin{aligned}\vec{u}-\vec{v}&=\vec{u}+(-1)\vec{v}\\&=\langle 3,1\rangle+\langle -1,-2\rangle\\&=\langle 2,-1\rangle .\end{aligned}$$

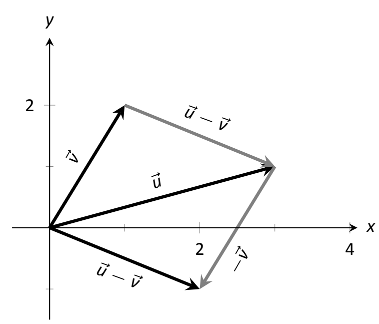


Figure 1.1.6: Illustrating how to subtract vectors graphically.

Figure 10.2.6 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum $\vec{u}+(-\vec{v})$. The figure also illustrates how $\vec{u}-\vec{v}$ can be obtained by looking only at the terminal points of $\vec{u}$ and $\vec{v}$ (when their initial points are the same).

Example 1.1.4 Scaling vectors

1. Sketch the vectors $\vec{v}=\langle 2,1\rangle$ and $2\vec{v}$ with initial point at the origin.
2. Compute the magnitudes of $\vec{v}$ and $2\vec{v}$.

Notes:

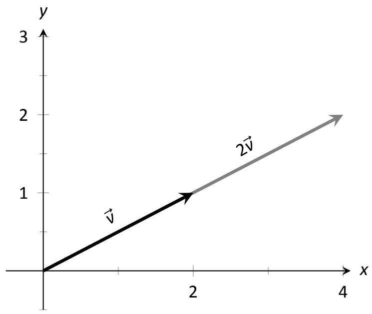


Figure 1.1.7: Graphing vectors $\vec{v}$ and $2\vec{v}$ in Example 10.2.4.

Solution

1. We compute $2\vec{v}$:

$$\begin{aligned} 2\vec{v} &= 2\left\langle 2,1\right\rangle \\ &= \left\langle 4,2\right\rangle . \end{aligned}$$

 Both $\vec{v}$ and $2\vec{v}$ are sketched in Figure 10.2.7. Make note that $2\vec{v}$ does not start at the terminal point of $\vec{v}$; rather, its initial point is also the origin.

2. The figure suggests that $2\vec{v}$ is twice as long as $\vec{v}$. We compute their magnitudes to confirm this.

$$\begin{aligned} ||\,\vec{v}\,|| &= \sqrt{2^2+1^2} \\ &= \sqrt{5}. \\ ||\,2\vec{v}\,|| &= \sqrt{4^2+2^2} \\ &= \sqrt{20} \\ &= \sqrt{4\cdot 5} = 2\sqrt{5}. \end{aligned}$$

 As we suspected, $2\vec{v}$ is twice as long as $\vec{v}$.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\vec{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0,0\rangle$; in $\mathbb{R}^3$, it is $\langle 0,0,0\rangle$. Usually the context makes is clear whether $\vec{0}$ is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

Notes:

Theorem 1.1.1 Properties of Vector Operations

The following are true for all scalars c and d, and for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^2$ or where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^3$:

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ Commutative Property
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ Associative Property
3. $\vec{v}+\vec{0}=\vec{v}$ Additive Identity
4. $(cd)\vec{v}=c(d\vec{v})$
5. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ Distributive Property
6. $(c+d)\vec{v}=c\vec{v}+d\vec{v}$ Distributive Property
7. $0\vec{v}=\vec{0}$
8. $||\, c\vec{v}\, ||=|c|\cdot||\, \vec{v}\, ||$
9. $||\, \vec{u}\, ||=0$ if, and only if, $\vec{u}=\vec{0}$.

As stated before, each vector $\vec{v}$ conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as $||\, \vec{v}\, ||$. *Unit vectors* are a way of extracting just the direction information from a vector.

Definition 1.1.4 Unit Vector

A **unit vector** is a vector $\vec{v}$ with a magnitude of 1; that is,

$$||\, \vec{v}\, ||=1.$$

Consider this scenario: you are given a vector $\vec{v}$ and are told to create a vector of length 10 in the direction of $\vec{v}$. How does one do that? If we knew that $\vec{u}$ was the unit vector in the direction of $\vec{v}$, the answer would be easy: $10\vec{u}$. So how do we find $\vec{u}$?

Property 8 of Theorem 10.2.1 holds the key. If we divide $\vec{v}$ by its magnitude, it becomes a vector of length 1. Consider:

$$\begin{aligned} \left|\left| \frac{1}{||\, \vec{v}\, ||}\vec{v} \right|\right| &= \frac{1}{||\, \vec{v}\, ||}||\, \vec{v}\, || \qquad \text{(we can pull out } \tfrac{1}{||\, \vec{v}\, ||} \text{ as it is a scalar)} \\ &= 1. \end{aligned}$$

Notes:

So the vector of length 10 in the direction of $\vec{v}$ is $10\dfrac{1}{\|\vec{v}\|}\vec{v}$. An example will make this more clear.

Example 1.1.5 Using Unit Vectors
Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

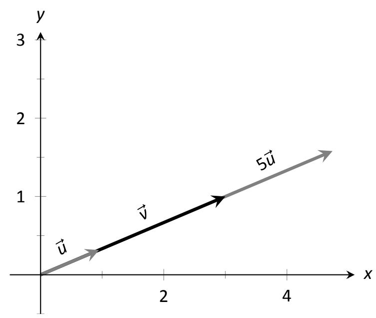


Figure 1.1.8: Graphing vectors in Example 10.2.5. All vectors shown have their initial point at the origin.

Solution

1. We find $\|\vec{v}\| = \sqrt{10}$. So the unit vector $\vec{u}$ in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{\sqrt{10}}\vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle.$$
2. We find $\|\vec{w}\| = 3$, so the unit vector $\vec{z}$ in the direction of $\vec{w}$ is
$$\vec{u} = \frac{1}{3}\vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle.$$
3. To create a vector with magnitude 5 in the direction of $\vec{v}$, we multiply the unit vector $\vec{u}$ by 5. Thus $5\vec{u} = \langle 15/\sqrt{10}, 5/\sqrt{10}\rangle$ is the vector we seek. This is sketched in Figure 10.2.8.

The basic formation of the unit vector $\vec{u}$ in the direction of a vector $\vec{v}$ leads to a interesting equation. It is:
$$\vec{v} = \|\vec{v}\|\frac{1}{\|\vec{v}\|}\vec{v}.$$
We rewrite the equation with parentheses to make a point:
$$\vec{v} = \underbrace{\|\vec{v}\|}_{\text{magnitude}} \cdot \underbrace{\left(\frac{1}{\|\vec{v}\|}\vec{v}\right)}_{\text{direction}}.$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

Notes:

Definition 1.1.5 Parallel Vectors

1. Unit vectors $\vec{u}_1$ and $\vec{u}_2$ are **parallel** if $\vec{u}_1 = \pm\vec{u}_2$.
2. Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are **parallel** if their respective unit vectors are parallel.

Note: $\vec{0}$ is directionless; because $|| \, \vec{0} \, || = 0$, there is no unit vector in the "direction" of $\vec{0}$.
Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, $\vec{0}$ is parallel to all vectors as $\vec{0} = 0\vec{v}$ for all $\vec{v}$.
We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that $\vec{0}$ is parallel to all vectors if they desire. (See also the marginal note on page 604.)

It is equivalent to say that vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$ (see marginal note).

If one graphed all unit vectors in $\mathbb{R}^2$ with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.

A similar construction in $\mathbb{R}^3$ shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in $\mathbb{R}^2$. Important concepts about unit vectors are given in the following Key Idea.

Key Idea 1.1.1 Unit Vectors

1. The unit vector in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{|| \, \vec{v} \, ||}\vec{v}.$$
2. A vector $\vec{u}$ in $\mathbb{R}^2$ is a unit vector if, and only if, its component form is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.
3. A vector $\vec{u}$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$ for some angles θ and φ.

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 1.1.6 Finding Component Forces
Consider a weight of 50lb hanging from two chains, as shown in Figure 10.2.9. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

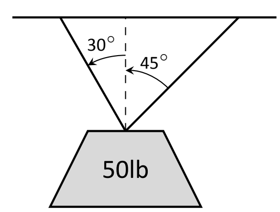


Figure 1.1.9: A diagram of a weight hanging from 2 chains in Example 10.2.6.

Solution Knowing that gravity is pulling the 50lb weight straight down,

Notes:

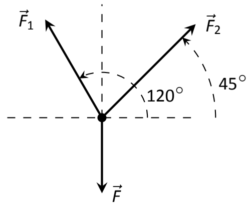


Figure 1.1.10: A diagram of the force vectors from Example 10.2.6.

we can create a vector $\vec{F}$ to represent this force.

$$\vec{F} = 50\langle 0, -1\rangle = \langle 0, -50\rangle.$$

We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let $\vec{F}_1$ represent the force from the chain making an angle of 30° with the vertical, and let $\vec{F}_2$ represent the force form the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 10.2.10), and apply Key Idea 10.2.1. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use m_1 and m_2 to represent them.

$$\vec{F}_1 = m_1\langle \cos 120^\circ, \sin 120^\circ\rangle$$

$$\vec{F}_2 = m_2\langle \cos 45^\circ, \sin 45^\circ\rangle$$

As the weight is not moving, we know the sum of the forces is $\vec{0}$. This gives:

$$\begin{aligned} \vec{F} + \vec{F}_1 + \vec{F}_2 &= \vec{0} \\ \langle 0, -50\rangle + m_1\langle \cos 120^\circ, \sin 120^\circ\rangle + m_2\langle \cos 45^\circ, \sin 45^\circ\rangle &= \vec{0} \end{aligned}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$\begin{aligned} m_1 \cos 120^\circ + m_2 \cos 45^\circ &= 0 \\ m_1 \sin 120^\circ + m_2 \sin 45^\circ &= 50 \end{aligned}$$

This is a simple 2-equation, 2-unkown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \qquad m_2 = \frac{50\sqrt{2}}{1+\sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an "additional" horizontal force while holding the weight in place.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component–wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

Notes:

Definition 1.1.6 Standard Unit Vectors

1. In $\mathbb{R}^2$, the standard unit vectors are
$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle .$$
2. In $\mathbb{R}^3$, the standard unit vectors are
$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle .$$

Example 1.1.7 Using standard unit vectors

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.
2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

Solution

1. $$\begin{aligned} \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2\langle 1, 0 \rangle - 3\langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \end{aligned}$$
2. $$\begin{aligned} \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

Example 1.1.8 Finding Component Force

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 10.2.11. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

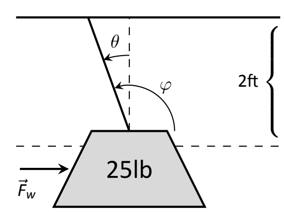


Figure 1.1.11: A figure of a weight being pushed by the wind in Example 10.2.8.

Notes:

Solution The force of the wind is represented by the vector $\vec{F}_w = 5\vec{i}$. The force of gravity on the weight is represented by $\vec{F}_g = -25\vec{j}$. The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m\,\langle \cos\varphi, \sin\varphi\rangle = m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j}$$

for some magnitude m and some angle with the horizontal φ. (Note: θ is the angle the chain makes with the *vertical*; φ is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is $\vec{0}$:

$$\begin{aligned} \vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0} \end{aligned}$$

Thus the sum of the $\vec{i}$ and $\vec{j}$ components are 0, leading us to the following system of equations:

$$\begin{aligned} 5 + m\cos\varphi &= 0 \\ -25 + m\sin\varphi &= 0 \end{aligned} \tag{1.1}$$

This is enough to determine $\vec{F}_c$ already, as we know $m\cos\varphi = -5$ and $m\sin\varphi = 25$. Thus $F_c = \langle -5, 25\rangle$. We can use this to find the magnitude m:

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (10.1) to solve for φ. We choose the first equality as using arccosine will return an angle in the 2$^{\text{nd}}$ quadrant:

$$5 + 5\sqrt{26}\cos\varphi = 0 \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting 90° from this angle gives us an angle of 11.31° with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of 11.31°. The length of the adjacent side (in the diagram, the dashed vertical line) is $2\cos 11.31^\circ \approx 1.96$ft. Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

Notes:

Exercises 1.1

Terms and Concepts

10 02 ex 01 1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.

10 02 ex 02 2. What is the difference between $(1, 2)$ and $\langle 1, 2\rangle$?

10 02 ex 03 3. What is a unit vector?

10 02 ex 36 4. Unit vectors can be thought of as conveying what type of information?

10 02 ex 04 5. What does it mean for two vectors to be parallel?

10 02 ex 05 6. What effect does multiplying a vector by -2 have?

Problems

10 02 exset 01 **In Exercises 7 – 10, points P and Q are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.**

10 02 ex 06 7. $P = (2, -1), \quad Q = (3, 5)$

10 02 ex 07 8. $P = (3, 2), \quad Q = (7, -2)$

10 02 ex 08 9. $P = (0, 3, -1), \quad Q = (6, 2, 5)$

10 02 ex 09 10. $P = (2, 1, 2), \quad Q = (4, 3, 2)$

10 02 ex 10 11. Let $\vec{u} = \langle 1, -2\rangle$ and $\vec{v} = \langle 1, 1\rangle$.

(a) Find $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $2\vec{u} - 3\vec{v}$.

(b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.

(c) Find $\vec{x}$ where $\vec{u} + \vec{x} = 2\vec{v} - \vec{x}$.

10 02 ex 11 12. Let $\vec{u} = \langle 1, 1, -1\rangle$ and $\vec{v} = \langle 2, 1, 2\rangle$.

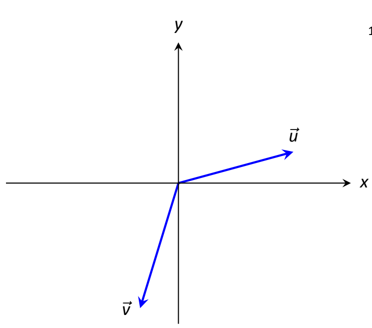
(a) Find $\vec{u} + \vec{v}$, $\vec{u} - \vec{v}$, $\pi\vec{u} - \sqrt{2}\vec{v}$.

(b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.

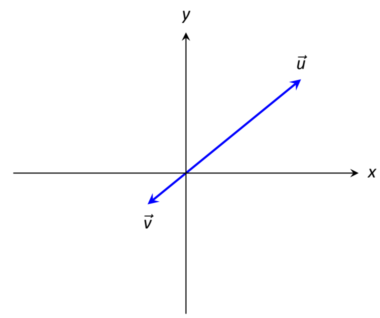
(c) Find $\vec{x}$ where $\vec{u} + \vec{x} = \vec{v} + 2\vec{x}$.

10 02 exset 02 **In Exercises 13 – 16, sketch $\vec{u}$, $\vec{v}$, $\vec{u} + \vec{v}$ and $\vec{u} - \vec{v}$ on the same axes.**

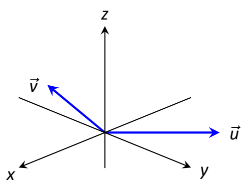
10 02 ex 12 13.



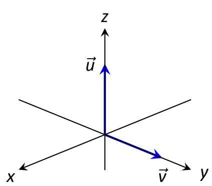
10 02 ex 13 14.



10 02 ex 14 15.



10 02 ex 15 16.



10 02 exset 03 **In Exercises 17 – 20, find $||\,\vec{u}\,||$, $||\,\vec{v}\,||$, $||\,\vec{u} + \vec{v}\,||$ and $||\,\vec{u} - \vec{v}\,||$.**

10 02 ex 16 17. $\vec{u} = \langle 2, 1\rangle, \quad \vec{v} = \langle 3, -2\rangle$

10 02 ex 17 18. $\vec{u} = \langle -3, 2, 2\rangle, \quad \vec{v} = \langle 1, -1, 1\rangle$

10 02 ex 18 19. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle -3, -6\rangle$

10 02 ex 19 20. $\vec{u} = \langle 2, -3, 6\rangle, \quad \vec{v} = \langle 10, -15, 30\rangle$

10 02 ex 20 21. Under what conditions is $||\,\vec{u}\,|| + ||\,\vec{v}\,|| = ||\,\vec{u} + \vec{v}\,||$?

10 02 exset 04 **In Exercises 22 – 25, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.**

10 02 ex 21 22. $\vec{v} = \langle 3, 7\rangle$

10 02 ex 22 23. $\vec{v} = \langle 6, 8\rangle$

10 02 ex 23 24. $\vec{v} = \langle 1, -2, 2\rangle$

10 02 ex 24 25. $\vec{v} = \langle 2, -2, 2\rangle$

10 02 ex 25 26. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the *x*-axis.

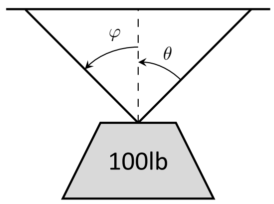
10 02 ex 26 27. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the *y*-axis.

10 02 ex 27 28. Verify, from Key Idea 10.2.1, that

$$\vec{u} = \langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$$

is a unit vector for all angles θ and φ.

10 02 exset 05 **A weight of 100lb is suspended from two chains, making angles with the vertical of θ and φ as shown in the figure below.**



In Exercises 29 – 32, angles θ and φ are given. Find the force applied to each chain.

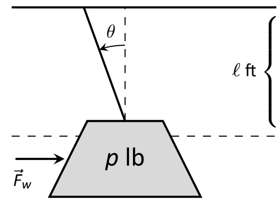
10 02 ex 28 29. $\theta = 30^\circ, \quad \varphi = 30^\circ$

10 02 ex 29 30. $\theta = 60^\circ, \quad \varphi = 60^\circ$

10 02 ex 30 31. $\theta = 20^\circ, \quad \varphi = 15^\circ$

10 02 ex 31 32. $\theta = 0^\circ, \quad \varphi = 0^\circ$

10 02 exset 06 **A weight of $p$lb is suspended from a chain of length ℓ while a constant force of $\vec{F}_w$ pushes the weight to the right, making an angle of θ with the vertical, as shown in the figure below.**



In Exercises 33 – 36, a force $\vec{F}_w$ and length ℓ are given. Find the angle θ and the height the weight is lifted as it moves to the right.

10 02 ex 32 33. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 1\text{lb}$

10 02 ex 33 34. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 10\text{lb}$

10 02 ex 34 35. $\vec{F}_w = 1\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

10 02 ex 35 36. $\vec{F}_w = 10\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

1.2 The Dot Product

The previous section introduced vectors and described how to add them together and how to multiply them by scalars. This section introduces *a* multiplication on vectors called the **dot product**.

Definition 1.2.1 Dot Product

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ in $\mathbb{R}^2$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2.$$
2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ in $\mathbb{R}^3$. The **dot product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \cdot \vec{v}$, is
$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Note how this product of vectors returns a *scalar*, not another vector. We practice evaluating a dot product in the following example, then we will discuss why this product is useful.

Example 1.2.1 Evaluating dot products

1. Let $\vec{u} = \langle 1, 2 \rangle$, $\vec{v} = \langle 3, -1 \rangle$ in $\mathbb{R}^2$. Find $\vec{u} \cdot \vec{v}$.
2. Let $\vec{x} = \langle 2, -2, 5 \rangle$ and $\vec{y} = \langle -1, 0, 3 \rangle$ in $\mathbb{R}^3$. Find $\vec{x} \cdot \vec{y}$.

Solution

1. Using Definition 1.1.1, we have
$$\vec{u} \cdot \vec{v} = 1(3) + 2(-1) = 1.$$
2. Using the definition, we have
$$\vec{x} \cdot \vec{y} = 2(-1) - 2(0) + 5(3) = 13.$$

The dot product, as shown by the preceding example, is very simple to evaluate. It is only the sum of products. While the definition gives no hint as to why

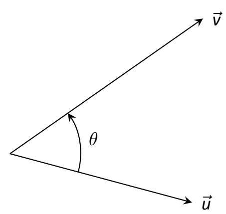
Notes:

we would care about this operation, there is an amazing connection between the dot product and angles formed by the vectors. Before stating this connection, we give a theorem stating some of the properties of the dot product.

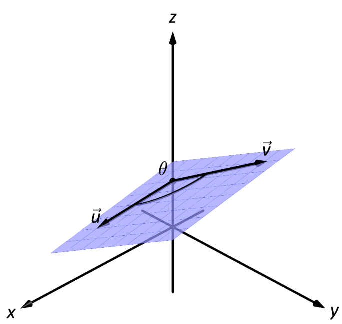
Theorem 1.2.1 Properties of the Dot Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ and let c be a scalar.

1. $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$ Commutative Property
2. $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$ Distributive Property
3. $c(\vec{u}\cdot\vec{v}) = (c\vec{u})\cdot\vec{v} = \vec{u}\cdot(c\vec{v})$
4. $\vec{0}\cdot\vec{v} = 0$
5. $\vec{v}\cdot\vec{v} = ||\,\vec{v}\,||^2$



(a)



(b)

Figure 1.2.1: Illustrating the angle formed by two vectors with the same initial point.

The last statement of the theorem makes a handy connection between the magnitude of a vector and the dot product with itself. Our definition and theorem give properties of the dot product, but we are still likely wondering "What does the dot product *mean*?" It is helpful to understand that the dot product of a vector with itself is connected to its magnitude.

The next theorem extends this understanding by connecting the dot product to magnitudes and angles. Given vectors $\vec{u}$ and $\vec{v}$ in the plane, an angle θ is clearly formed when $\vec{u}$ and $\vec{v}$ are drawn with the same initial point as illustrated in Figure 1.1.1(a). (We always take θ to be the angle in $[0,\pi]$ as two angles are actually created.)

The same is also true of 2 vectors in space: given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, there is a plane that contains both $\vec{u}$ and $\vec{v}$. (When $\vec{u}$ and $\vec{v}$ are colinear, there are infinite planes that contain both vectors.) In that plane, we can again find an angle θ between them (and again, $0\leq\theta\leq\pi$). This is illustrated in Figure 1.1.1(b).

The following theorem connects this angle θ to the dot product of $\vec{u}$ and $\vec{v}$.

Notes:

Theorem 1.2.2 The Dot Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$. Then

$$\vec{u}\cdot\vec{v} = ||\,\vec{u}\,||\,||\,\vec{v}\,||\cos\theta,$$

where θ, $0 \le \theta \le \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Using Theorem 1.1.1, we can rewrite this theorem as

$$\frac{\vec{u}}{||\,\vec{u}\,||}\cdot\frac{\vec{v}}{||\,\vec{v}\,||} = \cos\theta.$$

Note how on the left hand side of the equation, we are computing the dot product of two unit vectors. Recalling that unit vectors essentially only provide direction information, we can informally restate Theorem 1.1.2 as saying "The dot product of two directions gives the cosine of the angle between them."

When θ is an acute angle (i.e., $0 \le \theta < \pi/2$), $\cos\theta$ is positive; when $\theta = \pi/2$, $\cos\theta = 0$; when θ is an obtuse angle ($\pi/2 < \theta \le \pi$), $\cos\theta$ is negative. Thus the sign of the dot product gives a general indication of the angle between the vectors, illustrated in Figure 1.1.2.

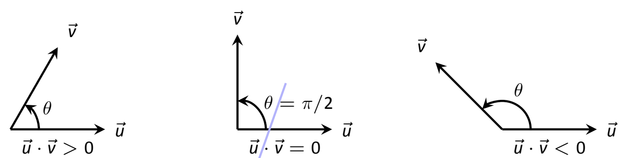


Figure 1.2.2: Illustrating the relationship between the angle between vectors and the sign of their dot product.

We *can* use Theorem 1.1.2 to compute the dot product, but generally this theorem is used to find the angle between known vectors (since the dot product is generally easy to compute). To this end, we rewrite the theorem's equation as

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||} \quad\Leftrightarrow\quad \theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right).$$

We practice using this theorem in the following example.

Example 1.2.2 Using the dot product to find angles

Let $\vec{u} = \langle 3, 1\rangle$, $\vec{v} = \langle -2, 6\rangle$ and $\vec{w} = \langle -4, 3\rangle$, as shown in Figure 1.1.3. Find the angles α, β and θ.

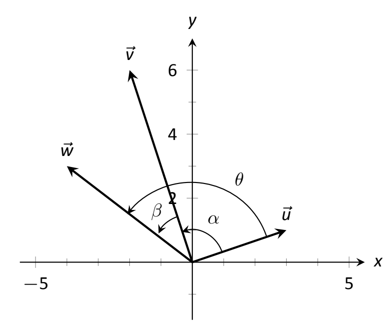


Figure 1.2.3: Vectors used in Example 1.1.2.

Notes:

Solution We start by computing the magnitude of each vector.

$$|| \vec{u} || = \sqrt{10}; \quad || \vec{v} || = 2\sqrt{10}; \quad || \vec{w} || = 5.$$

We now apply Theorem 1.1.2 to find the angles.

$$\begin{aligned} \alpha &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{(\sqrt{10})(2\sqrt{10})}\right) \\ &= \cos^{-1}(0) = \frac{\pi}{2} = 90^\circ. \end{aligned}$$

$$\begin{aligned} \beta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{(2\sqrt{10})(5)}\right) \\ &= \cos^{-1}\left(\frac{26}{10\sqrt{10}}\right) \\ &\approx 0.6055 \approx 34.7^\circ. \end{aligned}$$

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{(\sqrt{10})(5)}\right) \\ &= \cos^{-1}\left(\frac{-9}{5\sqrt{10}}\right) \\ &\approx 2.1763 \approx 124.7^\circ \end{aligned}$$

We see from our computation that $\alpha + \beta = \theta$, as indicated by Figure 1.1.3. While we knew this should be the case, it is nice to see that this non-intuitive formula indeed returns the results we expected.

We do a similar example next in the context of vectors in space.

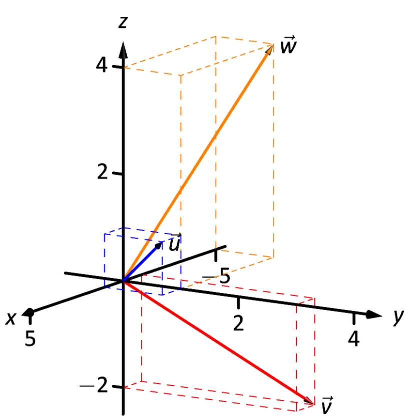


Figure 1.2.4: Vectors used in Example 1.1.3.

Example 1.2.3 **Using the dot product to find angles**

Let $\vec{u} = \langle 1, 1, 1\rangle$, $\vec{v} = \langle -1, 3, -2\rangle$ and $\vec{w} = \langle -5, 1, 4\rangle$, as illustrated in Figure 1.1.4. Find the angle between each pair of vectors.

Solution

1. Between $\vec{u}$ and $\vec{v}$:

$$\begin{aligned} \theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|| \vec{u} |||| \vec{v} ||}\right) \\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{14}}\right) \\ &= \frac{\pi}{2}. \end{aligned}$$

Notes:

2. Between $\vec{u}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{w}}{||\,\vec{u}\,||||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{3}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

3. Between $\vec{v}$ and $\vec{w}$:

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{v}\cdot\vec{w}}{||\,\vec{v}\,||||\,\vec{w}\,||}\right)\\ &= \cos^{-1}\left(\frac{0}{\sqrt{14}\sqrt{42}}\right)\\ &= \frac{\pi}{2}.\end{aligned}$$

While our work shows that each angle is $\pi/2$, i.e., 90°, none of these angles looks to be a right angle in Figure 1.1.4. Such is the case when drawing three–dimensional objects on the page.

All three angles between these vectors was $\pi/2$, or 90°. We know from geometry and everyday life that 90° angles are "nice" for a variety of reasons, so it should seem significant that these angles are all $\pi/2$. Notice the common feature in each calculation (and also the calculation of α in Example 1.1.2): the dot products of each pair of angles was 0. We use this as a basis for a definition of the term **orthogonal**, which is essentially synonymous to *perpendicular*.

Note: The term *perpendicular* originally referred to lines. As mathematics progressed, the concept of "being at right angles to" was applied to other objects, such as vectors and planes, and the term *orthogonal* was introduced. It is especially used when discussing objects that are hard, or impossible, to visualize: two vectors in 5-dimensional space are orthogonal if their dot product is 0. It is not wrong to say they are *perpendicular*, but common convention gives preference to the word *orthogonal*.

Definition 1.2.2 Orthogonal

Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if their dot product is 0.

Example 1.2.4 Finding orthogonal vectors
Let $\vec{u} = \langle 3, 5\rangle$ and $\vec{v} = \langle 1, 2, 3\rangle$.

1. Find two vectors in $\mathbb{R}^2$ that are orthogonal to $\vec{u}$.

2. Find two non–parallel vectors in $\mathbb{R}^3$ that are orthogonal to $\vec{v}$.

Solution

Notes:

1. Recall that a line perpendicular to a line with slope m has slope $-1/m$, the "opposite reciprocal slope." We can think of the slope of $\vec{u}$ as $5/3$, its "rise over run." A vector orthogonal to $\vec{u}$ will have slope $-3/5$. There are many such choices, though all parallel:

$$\langle -5, 3\rangle \quad \text{or} \quad \langle 5, -3\rangle \quad \text{or} \quad \langle -10, 6\rangle \quad \text{or} \quad \langle 15, -9\rangle \text{, etc.}$$

2. There are infinite directions in space orthogonal to any given direction, so there are an infinite number of non–parallel vectors orthogonal to $\vec{v}$. Since there are so many, we have great leeway in finding some.

   One way is to arbitrarily pick values for the first two components, leaving the third unknown. For instance, let $\vec{v}_1 = \langle 2, 7, z\rangle$. If $\vec{v}_1$ is to be orthogonal to $\vec{v}$, then $\vec{v}_1 \cdot \vec{v} = 0$, so

$$2 + 14 + 3z = 0 \quad \Rightarrow z = \frac{-16}{3}.$$

   So $\vec{v}_1 = \langle 2, 7, -16/3\rangle$ is orthogonal to $\vec{v}$. We can apply a similar technique by leaving the first or second component unknown.

   Another method of finding a vector orthogonal to $\vec{v}$ mirrors what we did in part 1. Let $\vec{v}_2 = \langle -2, 1, 0\rangle$. Here we switched the first two components of $\vec{v}$, changing the sign of one of them (similar to the "opposite reciprocal" concept before). Letting the third component be 0 effectively ignores the third component of $\vec{v}$, and it is easy to see that

$$\vec{v}_2 \cdot \vec{v} = \langle -2, 1, 0\rangle \cdot \langle 1, 2, 3\rangle = 0.$$

   Clearly $\vec{v}_1$ and $\vec{v}_2$ are not parallel.

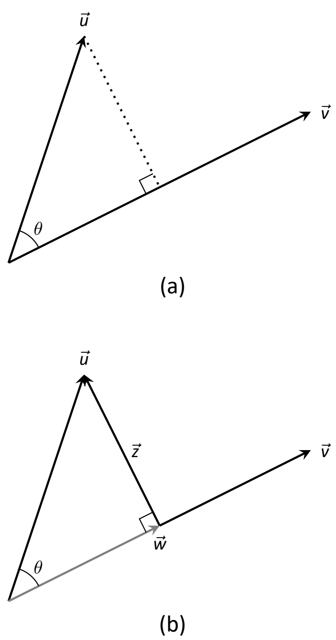


Figure 1.2.5: Developing the construction of the *orthogonal projection*.

An important construction is illustrated in Figure 1.1.5, where vectors $\vec{u}$ and $\vec{v}$ are sketched. In part (a), a dotted line is drawn from the tip of $\vec{u}$ to the line containing $\vec{v}$, where the dotted line is orthogonal to $\vec{v}$. In part (b), the dotted line is replaced with the vector $\vec{z}$ and $\vec{w}$ is formed, parallel to $\vec{v}$. It is clear by the diagram that $\vec{u} = \vec{w} + \vec{z}$. What is important about this construction is this: $\vec{u}$ is *decomposed* as the sum of two vectors, one of which is parallel to $\vec{v}$ and one that is perpendicular to $\vec{v}$. It is hard to overstate the importance of this construction (as we'll see in upcoming examples).

The vectors $\vec{w}$, $\vec{z}$ and $\vec{u}$ as shown in Figure 1.1.5 (b) form a right triangle, where the angle between $\vec{v}$ and $\vec{u}$ is labeled θ. We can find $\vec{w}$ in terms of $\vec{v}$ and $\vec{u}$.

Using trigonometry, we can state that

$$||\,\vec{w}\,|| = ||\,\vec{u}\,|| \cos\theta. \tag{1.2}$$

Notes:

We also know that $\vec{w}$ is parallel to to $\vec{v}$; that is, the direction of $\vec{w}$ is the direction of $\vec{v}$, described by the unit vector $\vec{v}/||\,\vec{v}\,||$. The vector $\vec{w}$ is the vector in the direction $\vec{v}/||\,\vec{v}\,||$ with magnitude $||\,\vec{u}\,||\cos\theta$:

$$\vec{w} = \Big(||\,\vec{u}\,||\cos\theta\Big)\frac{1}{||\,\vec{v}\,||}\vec{v}.$$

Replace $\cos\theta$ using Theorem 1.1.2:

$$\begin{aligned} &= \left(||\,\vec{u}\,||\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||||\,\vec{v}\,||}\right)\frac{1}{||\,\vec{v}\,||}\vec{v} \\ &= \frac{\vec{u}\cdot\vec{v}}{||\,\vec{v}\,||^2}\vec{v}. \end{aligned}$$

Now apply Theorem 1.1.1.

$$= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Since this construction is so important, it is given a special name.

Definition 1.2.3 Orthogonal Projection

Let $\vec{u}$ and $\vec{v}\neq\vec{0}$ be given. The **orthogonal projection of $\vec{u}$ onto $\vec{v}$**, denoted $\text{proj}_{\,\vec{v}}\,\vec{u}$, is

$$\text{proj}_{\,\vec{v}}\,\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}.$$

Example 1.2.5 Computing the orthogonal projection

1. Let $\vec{u}=\langle -2,1\rangle$ and $\vec{v}=\langle 3,1\rangle$. Find $\text{proj}_{\,\vec{v}}\,\vec{u}$, and sketch all three vectors with initial points at the origin.

2. Let $\vec{w}=\langle 2,1,3\rangle$ and $\vec{x}=\langle 1,1,1\rangle$. Find $\text{proj}_{\,\vec{x}}\,\vec{w}$, and sketch all three vectors with initial points at the origin.

Solution

Notes:

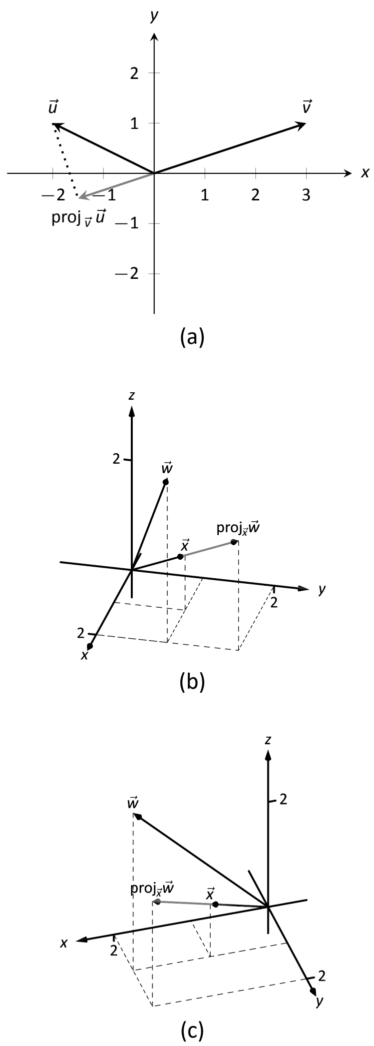


Figure 1.2.6: Graphing the vectors used in Example 1.1.5.

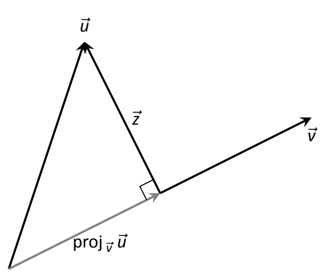


Figure 1.2.7: Illustrating the orthogonal projection.

1. Applying Definition 1.1.3, we have

$$\begin{aligned}\operatorname{proj}_{\vec{v}}\vec{u} &= \frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v}\\ &= \frac{-5}{10}\left\langle 3, 1\right\rangle\\ &= \left\langle -\frac{3}{2}, -\frac{1}{2}\right\rangle.\end{aligned}$$

   Vectors $\vec{u}$, $\vec{v}$ and $\operatorname{proj}_{\vec{v}}\vec{u}$ are sketched in Figure 1.1.6(a). Note how the projection is parallel to $\vec{v}$; that is, it lies on the same line through the origin as $\vec{v}$, although it points in the opposite direction. That is because the angle between $\vec{u}$ and $\vec{v}$ is obtuse (i.e., greater than 90°).

2. Apply the definition:

$$\begin{aligned}\operatorname{proj}_{\vec{x}}\vec{w} &= \frac{\vec{w}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\vec{x}\\ &= \frac{6}{3}\left\langle 1, 1, 1\right\rangle\\ &= \left\langle 2, 2, 2\right\rangle.\end{aligned}$$

   These vectors are sketched in Figure 1.1.6(b), and again in part (c) from a different perspective. Because of the nature of graphing these vectors, the sketch in part (b) makes it difficult to recognize that the drawn projection has the geometric properties it should. The graph shown in part (c) illustrates these properties better.

A special case of orthogonal projection occurs when $\vec{v}$ is a unit vector. In this situation, the formula for the orthogonal projection of a vector $\vec{u}$ onto $\vec{v}$ reduces to just $\operatorname{proj}_{\vec{v}}\vec{u} = (\vec{u}\cdot\vec{v})\vec{v}$, as $\vec{v}\cdot\vec{v} = 1$.

This gives us a new understanding of the dot product. When $\vec{v}$ is a unit vector, essentially providing only direction information, the dot product of $\vec{u}$ and $\vec{v}$ gives "how much of $\vec{u}$ is in the direction of $\vec{v}$." This use of the dot product will be very useful in future sections.

Now consider Figure 1.1.7 where the concept of the orthogonal projection is again illustrated. It is clear that

$$\vec{u} = \operatorname{proj}_{\vec{v}}\vec{u} + \vec{z}. \tag{1.3}$$

As we know what $\vec{u}$ and $\operatorname{proj}_{\vec{v}}\vec{u}$ are, we can solve for $\vec{z}$ and state that

$$\vec{z} = \vec{u} - \operatorname{proj}_{\vec{v}}\vec{u}.$$

Notes:

This leads us to rewrite Equation (1.2) in a seemingly silly way:

$$\vec{u} = \text{proj}_{\vec{v}}\,\vec{u} + (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}).$$

This is not nonsense, as pointed out in the following Key Idea. (Notation note: the expression "$\parallel \vec{y}$" means "is parallel to $\vec{y}$." We can use this notation to state "$\vec{x} \parallel \vec{y}$" which means "$\vec{x}$ is parallel to $\vec{y}$." The expression "$\perp \vec{y}$" means "is orthogonal to $\vec{y}$," and is used similarly.)

Key Idea 1.2.1 Orthogonal Decomposition of Vectors

Let $\vec{u}$ and $\vec{v}$ be given. Then $\vec{u}$ can be written as the sum of two vectors, one of which is parallel to $\vec{v}$, and one of which is orthogonal to $\vec{v}$:

$$\vec{u} = \underbrace{\text{proj}_{\vec{v}}\,\vec{u}}_{\parallel \vec{v}} + (\underbrace{\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}}_{\perp \vec{v}}).$$

We illustrate the use of this equality in the following example.

Example 1.2.6 Orthogonal decomposition of vectors

1. Let $\vec{u} = \langle -2, 1\rangle$ and $\vec{v} = \langle 3, 1\rangle$ as in Example 1.1.5. Decompose $\vec{u}$ as the sum of a vector parallel to $\vec{v}$ and a vector orthogonal to $\vec{v}$.
2. Let $\vec{w} = \langle 2, 1, 3\rangle$ and $\vec{x} = \langle 1, 1, 1\rangle$ as in Example 1.1.5. Decompose $\vec{w}$ as the sum of a vector parallel to $\vec{x}$ and a vector orthogonal to $\vec{x}$.

Solution

1. In Example 1.1.5, we found that $\text{proj}_{\vec{v}}\,\vec{u} = \langle -1.5, -0.5\rangle$. Let

   $$\vec{z} = \vec{u} - \text{proj}_{\vec{v}}\,\vec{u} = \langle -2, 1\rangle - \langle -1.5, -0.5\rangle = \langle -0.5, 1.5\rangle\,.$$

   Is $\vec{z}$ orthogonal to $\vec{v}$? (I.e, is $\vec{z} \perp \vec{v}$?) We check for orthogonality with the dot product:

   $$\vec{z} \cdot \vec{v} = \langle -0.5, 1.5\rangle \cdot \langle 3, 1\rangle = 0.$$

   Since the dot product is 0, we know $\vec{z} \perp \vec{v}$. Thus:

   $$\begin{aligned} \vec{u} &= \text{proj}_{\vec{v}}\,\vec{u} \;+\; (\vec{u} - \text{proj}_{\vec{v}}\,\vec{u}) \\ \langle -2, 1\rangle &= \underbrace{\langle -1.5, -0.5\rangle}_{\parallel \vec{v}} \;+\; \underbrace{\langle -0.5, 1.5\rangle}_{\perp \vec{v}}\,. \end{aligned}$$

Notes:

2. We found in Example 1.1.5 that $\text{proj}_{\vec{x}}\,\vec{w} = \langle 2,2,2\rangle$. Applying the Key Idea, we have:

$$\vec{z} = \vec{w} - \text{proj}_{\vec{x}}\,\vec{w} = \langle 2,1,3\rangle - \langle 2,2,2\rangle = \langle 0,-1,1\rangle\,.$$

We check to see if $\vec{z} \perp \vec{x}$:

$$\vec{z}\cdot\vec{x} = \langle 0,-1,1\rangle \cdot \langle 1,1,1\rangle = 0.$$

Since the dot product is 0, we know the two vectors are orthogonal. We now write $\vec{w}$ as the sum of two vectors, one parallel and one orthogonal to $\vec{x}$:

$$\begin{aligned}\vec{w} &= \text{proj}_{\vec{x}}\,\vec{w} \;+\; (\vec{w} - \text{proj}_{\vec{x}}\,\vec{w})\\ \langle 2,1,3\rangle &= \underbrace{\langle 2,2,2\rangle}_{\parallel\,\vec{x}} \;+\; \underbrace{\langle 0,-1,1\rangle}_{\perp\,\vec{x}}\end{aligned}$$

We give an example of where this decomposition is useful.

Example 1.2.7 Orthogonally decomposing a force vector

Consider Figure 1.1.8(a), showing a box weighing 50lb on a ramp that rises 5ft over a span of 20ft. Find the components of force, and their magnitudes, acting on the box (as sketched in part (b) of the figure):

1. in the direction of the ramp, and
2. orthogonal to the ramp.

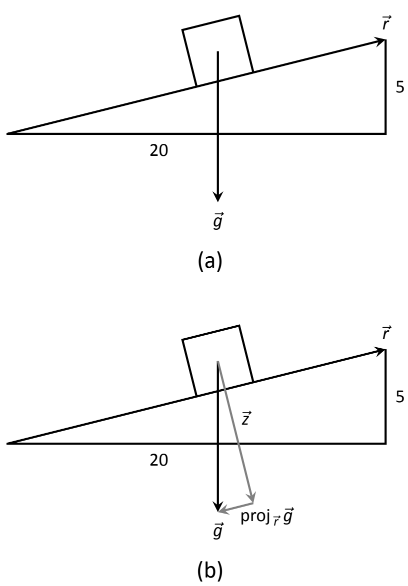


Figure 1.2.8: Sketching the ramp and box in Example 1.1.7. Note: *The vectors are not drawn to scale.*

SOLUTION As the ramp rises 5ft over a horizontal distance of 20ft, we can represent the direction of the ramp with the vector $\vec{r} = \langle 20,5\rangle$. Gravity pulls down with a force of 50lb, which we represent with $\vec{g} = \langle 0,-50\rangle$.

1. To find the force of gravity in the direction of the ramp, we compute $\text{proj}_{\vec{r}}\,\vec{g}$:

$$\begin{aligned}\text{proj}_{\vec{r}}\,\vec{g} &= \frac{\vec{g}\cdot\vec{r}}{\vec{r}\cdot\vec{r}}\vec{r}\\ &= \frac{-250}{425}\langle 20,5\rangle\\ &= \left\langle -\frac{200}{17}, -\frac{50}{17}\right\rangle \approx \langle -11.76, -2.94\rangle\,.\end{aligned}$$

The magnitude of $\text{proj}_{\vec{r}}\,\vec{g}$ is $||\,\text{proj}_{\vec{r}}\,\vec{g}\,|| = 50/\sqrt{17} \approx 12.13$lb. Though the box weighs 50lb, a force of about 12lb is enough to keep the box from sliding down the ramp.

Notes:

2. To find the component $\vec{z}$ of gravity orthogonal to the ramp, we use Key Idea 1.1.1.

$$\begin{aligned}\vec{z} &= \vec{g} - \text{proj}_{\vec{r}}\,\vec{g}\\ &= \left\langle \frac{200}{17}, -\frac{800}{17} \right\rangle \approx \langle 11.76, -47.06\rangle .\end{aligned}$$

The magnitude of this force is $\|\,\vec{z}\,\| \approx 48.51$lb. In physics and engineering, knowing this force is important when computing things like static frictional force. (For instance, we could easily compute if the static frictional force alone was enough to keep the box from sliding down the ramp.)

Application to Work

In physics, the application of a force F to move an object in a straight line a distance d produces *work*; the amount of work W is $W = Fd$, (where F is in the direction of travel). The orthogonal projection allows us to compute work when the force is not in the direction of travel.

Consider Figure 1.1.9, where a force $\vec{F}$ is being applied to an object moving in the direction of $\vec{d}$. (The distance the object travels is the magnitude of $\vec{d}$.) The work done is the amount of force in the direction of $\vec{d}$, $\|\,\text{proj}_{\vec{d}}\,\vec{F}\,\|$, times $\|\,\vec{d}\,\|$:

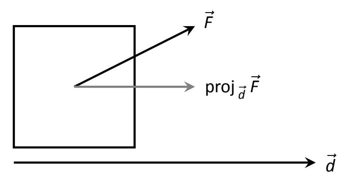


Figure 1.2.9: Finding work when the force and direction of travel are given as vectors.

$$\begin{aligned}\|\,\text{proj}_{\vec{d}}\,\vec{F}\,\| \cdot \|\,\vec{d}\,\| &= \left\|\, \frac{\vec{F}\cdot\vec{d}}{\vec{d}\cdot\vec{d}}\vec{d} \,\right\| \cdot \|\,\vec{d}\,\|\\ &= \left| \frac{\vec{F}\cdot\vec{d}}{\|\,\vec{d}\,\|^2} \right| \cdot \|\,\vec{d}\,\| \cdot \|\,\vec{d}\,\|\\ &= \frac{\left|\vec{F}\cdot\vec{d}\right|}{\|\,\vec{d}\,\|^2} \|\,\vec{d}\,\|^2\\ &= \left|\vec{F}\cdot\vec{d}\right| .\end{aligned}$$

The expression $\vec{F}\cdot\vec{d}$ will be positive if the angle between $\vec{F}$ and $\vec{d}$ is acute; when the angle is obtuse (hence $\vec{F}\cdot\vec{d}$ is negative), the force is causing motion in the opposite direction of $\vec{d}$, resulting in "negative work." We want to capture this sign, so we drop the absolute value and find that $W = \vec{F}\cdot\vec{d}$.

Definition 1.2.4 Work

Let $\vec{F}$ be a constant force that moves an object in a straight line from point P to point Q. Let $\vec{d} = \overrightarrow{PQ}$. The **work** W done by $\vec{F}$ along $\vec{d}$ is $W = \vec{F}\cdot\vec{d}$.

Notes:

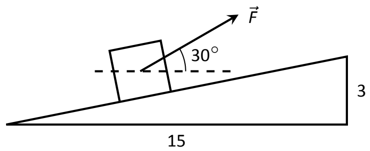


Figure 1.2.10: Computing work when sliding a box up a ramp in Example 1.1.8.

Example 1.2.8 Computing work

A man slides a box along a ramp that rises 3ft over a distance of 15ft by applying 50lb of force as shown in Figure 1.1.10. Compute the work done.

Solution The figure indicates that the force applied makes a 30° angle with the horizontal, so $\vec{F} = 50\left\langle \cos 30^\circ, \sin 30^\circ\right\rangle \approx \left\langle 43.3, 25\right\rangle$. The ramp is represented by $\vec{d} = \langle 15, 3\rangle$. The work done is simply

$$\vec{F}\cdot\vec{d} = 50\left\langle \cos 30^\circ, \sin 30^\circ\right\rangle \cdot \langle 15, 3\rangle \approx 724.5\text{ft–lb}.$$

Note how we did not actually compute the distance the object traveled, nor the magnitude of the force in the direction of travel; this is all inherently computed by the dot product!

The dot product is a powerful way of evaluating computations that depend on angles without actually using angles. The next section explores another "product" on vectors, the *cross product.* Once again, angles play an important role, though in a much different way.

Notes:

Exercises 1.2

Terms and Concepts

1. The dot product of two vectors is a ________, not a vector.

2. How are the concepts of the dot product and vector magnitude related?

3. How can one quickly tell if the angle between two vectors is acute or obtuse?

4. Give a synonym for “orthogonal.”

Problems

In Exercises 5 – 10, find the dot product of the given vectors.

5. $\vec{u} = \langle 2, -4\rangle$, $\vec{v} = \langle 3, 7\rangle$

6. $\vec{u} = \langle 5, 3\rangle$, $\vec{v} = \langle 6, 1\rangle$

7. $\vec{u} = \langle 1, -1, 2\rangle$, $\vec{v} = \langle 2, 5, 3\rangle$

8. $\vec{u} = \langle 3, 5, -1\rangle$, $\vec{v} = \langle 4, -1, 7\rangle$

9. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

10. $\vec{u} = \langle 1, 2, 3\rangle$, $\vec{v} = \langle 0, 0, 0\rangle$

11. Create your own vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^2$ and show that $\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$.

12. Create your own vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ and scalar c and show that $c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$.

In Exercises 13 – 16, find the measure of the angle between the two vectors in both radians and degrees.

13. $\vec{u} = \langle 1, 1\rangle$, $\vec{v} = \langle 1, 2\rangle$

14. $\vec{u} = \langle -2, 1\rangle$, $\vec{v} = \langle 3, 5\rangle$

15. $\vec{u} = \langle 8, 1, -4\rangle$, $\vec{v} = \langle 2, 2, 0\rangle$

16. $\vec{u} = \langle 1, 7, 2\rangle$, $\vec{v} = \langle 4, -2, 5\rangle$

In Exercises 17 – 20, a vector $\vec{v}$ is given. Give two vectors that are orthogonal to $\vec{v}$.

17. $\vec{v} = \langle 4, 7\rangle$

18. $\vec{v} = \langle -3, 5\rangle$

19. $\vec{v} = \langle 1, 1, 1\rangle$

20. $\vec{v} = \langle 1, -2, 3\rangle$

In Exercises 21 – 26, vectors $\vec{u}$ and $\vec{v}$ are given. Find $\text{proj}_{\vec{v}}\,\vec{u}$, the orthogonal projection of $\vec{u}$ onto $\vec{v}$, and sketch all three vectors with the same initial point.

21. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

22. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

23. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

24. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

25. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

26. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

In Exercises 27 – 32, vectors $\vec{u}$ and $\vec{v}$ are given. Write $\vec{u}$ as the sum of two vectors, one of which is parallel to $\vec{v}$ and one of which is perpendicular to $\vec{v}$. Note: these are the same pairs of vectors as found in Exercises 21 – 26.

27. $\vec{u} = \langle 1, 2\rangle$, $\vec{v} = \langle -1, 3\rangle$

28. $\vec{u} = \langle 5, 5\rangle$, $\vec{v} = \langle 1, 3\rangle$

29. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 1, 1\rangle$

30. $\vec{u} = \langle -3, 2\rangle$, $\vec{v} = \langle 2, 3\rangle$

31. $\vec{u} = \langle 1, 5, 1\rangle$, $\vec{v} = \langle 1, 2, 3\rangle$

32. $\vec{u} = \langle 3, -1, 2\rangle$, $\vec{v} = \langle 2, 2, 1\rangle$

33. A 10lb box sits on a ramp that rises 4ft over a distance of 20ft. How much force is required to keep the box from sliding down the ramp?

34. A 10lb box sits on a 15ft ramp that makes a 30° angle with the horizontal. How much force is required to keep the box from sliding down the ramp?

35. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 45° to the horizontal?

36. How much work is performed in moving a box horizontally 10ft with a force of 20lb applied at an angle of 10° to the horizontal?

37. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied horizontally?

38. How much work is performed in moving a box up the length of a ramp that rises 2ft over a distance of 10ft, with a force of 50lb applied at an angle of 45° to the horizontal?

39. How much work is performed in moving a box up the length of a 10ft ramp that makes a 5° angle with the horizontal, with 50lb of force applied in the direction of the ramp?

1.3 The Cross Product

"Orthogonality" is immensely important. A quick scan of your current environment will undoubtedly reveal numerous surfaces and edges that are perpendicular to each other (including the edges of this page). The dot product provides a quick test for orthogonality: vectors $\vec{u}$ and $\vec{v}$ are perpendicular if, and only if, $\vec{u}\cdot\vec{v}=0$.

Given two non–parallel, nonzero vectors $\vec{u}$ and $\vec{v}$ in space, it is very useful to find a vector $\vec{w}$ that is perpendicular to both $\vec{u}$ and $\vec{v}$. There is a operation, called the **cross product**, that creates such a vector. This section defines the cross product, then explores its properties and applications.

Definition 1.3.1 Cross Product

Let $\vec{u}=\langle u_1,u_2,u_3\rangle$ and $\vec{v}=\langle v_1,v_2,v_3\rangle$ be vectors in $\mathbb{R}^3$. The **cross product of $\vec{u}$ and $\vec{v}$**, denoted $\vec{u}\times\vec{v}$, is the vector

$$\vec{u}\times\vec{v}=\langle u_2v_3-u_3v_2,-(u_1v_3-u_3v_1),u_1v_2-u_2v_1\rangle\,.$$

This definition can be a bit cumbersome to remember. After an example we will give a convenient method for computing the cross product. For now, careful examination of the products and differences given in the definition should reveal a pattern that is not too difficult to remember. (For instance, in the first component only 2 and 3 appear as subscripts; in the second component, only 1 and 3 appear as subscripts. Further study reveals the order in which they appear.)

Let's practice using this definition by computing a cross product.

Example 1.3.1 Computing a cross product

Let $\vec{u}=\langle 2,-1,4\rangle$ and $\vec{v}=\langle 3,2,5\rangle$. Find $\vec{u}\times\vec{v}$, and verify that it is orthogonal to both $\vec{u}$ and $\vec{v}$.

Solution Using Definition 10.4.1, we have

$$\vec{u}\times\vec{v}=\big\langle(-1)5-(4)2,-\big((2)5-(4)3\big),(2)2-(-1)3\big\rangle=\langle -13,2,7\rangle\,.$$

(We encourage the reader to compute this product on their own, then verify their result.)

We test whether or not $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$ using the dot product:

$$\big(\vec{u}\times\vec{v}\big)\cdot\vec{u}=\langle -13,2,7\rangle\cdot\langle 2,-1,4\rangle=0,$$

$$\big(\vec{u}\times\vec{v}\big)\cdot\vec{v}=\langle -13,2,7\rangle\cdot\langle 3,2,5\rangle=0.$$

Since both dot products are zero, $\vec{u}\times\vec{v}$ is indeed orthogonal to both $\vec{u}$ and $\vec{v}$.

Notes:

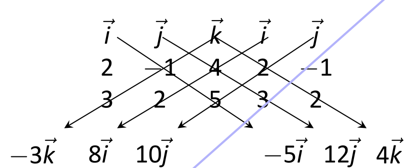
A convenient method of computing the cross product starts with forming a particular 3×3 *matrix*, or rectangular array. The first row comprises the standard unit vectors $\vec{i}$, $\vec{j}$, and $\vec{k}$. The second and third rows are the vectors $\vec{u}$ and $\vec{v}$, respectively. Using $\vec{u}$ and $\vec{v}$ from Example 10.4.1, we begin with:

$$\begin{array}{ccc} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 4 \\ 3 & 2 & 5 \end{array}$$

Now repeat the first two columns after the original three:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 2 & -1 & 4 & 2 & -1 \\ 3 & 2 & 5 & 3 & 2 \end{array}$$

This gives three full "upper left to lower right" diagonals, and three full "upper right to lower left" diagonals, as shown. Compute the products along each diagonal, then add the products on the right and subtract the products on the left:



$$\vec{u} \times \vec{v} = \big(-5\vec{i} + 12\vec{j} + 4\vec{k}\big) - \big(-3\vec{k} + 8\vec{i} + 10\vec{j}\big) = -13\vec{i} + 2\vec{j} + 7\vec{k} = \langle -13, 2, 7 \rangle .$$

We practice using this method.

Example 1.3.2 Computing a cross product

Let $\vec{u} = \langle 1, 3, 6 \rangle$ and $\vec{v} = \langle -1, 2, 1 \rangle$. Compute both $\vec{u} \times \vec{v}$ and $\vec{v} \times \vec{u}$.

Solution To compute $\vec{u} \times \vec{v}$, we form the matrix as prescribed above, complete with repeated first columns:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ 1 & 3 & 6 & 1 & 3 \\ -1 & 2 & 1 & -1 & 2 \end{array}$$

We let the reader compute the products of the diagonals; we give the result:

$$\vec{u} \times \vec{v} = \big(3\vec{i} - 6\vec{j} + 2\vec{k}\big) - \big(-3\vec{k} + 12\vec{i} + \vec{j}\big) = \langle -9, -7, 5 \rangle .$$

Notes:

To compute $\vec{v} \times \vec{u}$, we switch the second and third rows of the above matrix, then multiply along diagonals and subtract:

$$\begin{array}{ccccc} \vec{i} & \vec{j} & \vec{k} & \vec{i} & \vec{j} \\ -1 & 2 & 1 & -1 & 2 \\ 1 & 3 & 6 & 1 & 3 \end{array}$$

Note how with the rows being switched, the products that once appeared on the right now appear on the left, and vice–versa. Thus the result is:

$$\vec{v} \times \vec{u} = \left(12\vec{i} + \vec{j} - 3\vec{k}\right) - \left(2\vec{k} + 3\vec{i} - 6\vec{j}\right) = \langle 9, 7, -5 \rangle \,,$$

which is the opposite of $\vec{u} \times \vec{v}$. We leave it to the reader to verify that each of these vectors is orthogonal to $\vec{u}$ and $\vec{v}$.

Properties of the Cross Product

It is not coincidence that $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ in the preceding example; one can show using Definition 10.4.1 that this will always be the case. The following theorem states several useful properties of the cross product, each of which can be verified by referring to the definition.

Theorem 1.3.1 Properties of the Cross Product

Let $\vec{u}$, $\vec{v}$ and $\vec{w}$ be vectors in $\mathbb{R}^3$ and let c be a scalar. The following identities hold:

1. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ Anticommutative Property
2. (a) $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$ Distributive Properties
   (b) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
3. $c(\vec{u} \times \vec{v}) = (c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v})$
4. (a) $(\vec{u} \times \vec{v}) \cdot \vec{u} = 0$ Orthogonality Properties
   (b) $(\vec{u} \times \vec{v}) \cdot \vec{v} = 0$
5. $\vec{u} \times \vec{u} = \vec{0}$
6. $\vec{u} \times \vec{0} = \vec{0}$
7. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$ Triple Scalar Product

Notes:

We introduced the cross product as a way to find a vector orthogonal to two given vectors, but we did not give a proof that the construction given in Definition 10.4.1 satisfies this property. Theorem 10.4.1 asserts this property holds; we leave it as a problem in the Exercise section to verify this.

Property 5 from the theorem is also left to the reader to prove in the Exercise section, but it reveals something more interesting than "the cross product of a vector with itself is $\vec{0}$." Let $\vec{u}$ and $\vec{v}$ be parallel vectors; that is, let there be a scalar c such that $\vec{v} = c\vec{u}$. Consider their cross product:

$$\begin{aligned} \vec{u} \times \vec{v} &= \vec{u} \times (c\vec{u}) \\ &= c(\vec{u} \times \vec{u}) && \text{(by Property 3 of Theorem 10.4.1)} \\ &= \vec{0}. && \text{(by Property 5 of Theorem 10.4.1)} \end{aligned}$$

We have just shown that the cross product of parallel vectors is $\vec{0}$. This hints at something deeper. Theorem 1.1.2 related the angle between two vectors and their dot product; there is a similar relationship relating the cross product of two vectors and the angle between them, given by the following theorem.

Theorem 1.3.2 The Cross Product and Angles

Let $\vec{u}$ and $\vec{v}$ be vectors in $\mathbb{R}^3$. Then

$$||\,\vec{u} \times \vec{v}\,|| = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta,$$

where θ, $0 \leq \theta \leq \pi$, is the angle between $\vec{u}$ and $\vec{v}$.

Note: Definition 1.1.2 (through Theorem 1.1.2) defines $\vec{u}$ and $\vec{v}$ to be orthogonal if $\vec{u}\cdot\vec{v} = \vec{0}$. We could use Theorem 10.4.2 to define $\vec{u}$ and $\vec{v}$ are parallel if $\vec{u} \times \vec{v} = \vec{0}$. By such a definition, $\vec{0}$ would be both orthogonal and parallel to every vector. Apparent paradoxes such as this are not uncommon in mathematics and can be very useful. (See also the marginal note on page 582.)

Note that this theorem makes a statement about the *magnitude* of the cross product. When the angle between $\vec{u}$ and $\vec{v}$ is 0 or π (i.e., the vectors are parallel), the magnitude of the cross product is 0. The only vector with a magnitude of 0 is $\vec{0}$ (see Property 9 of Theorem 10.2.1), hence the cross product of parallel vectors is $\vec{0}$.

We demonstrate the truth of this theorem in the following example.

Example 1.3.3 The cross product and angles

Let $\vec{u} = \langle 1, 3, 6\rangle$ and $\vec{v} = \langle -1, 2, 1\rangle$ as in Example 10.4.2. Verify Theorem 10.4.2 by finding θ, the angle between $\vec{u}$ and $\vec{v}$, and the magnitude of $\vec{u} \times \vec{v}$.

Notes:

Solution We use Theorem 1.1.2 to find the angle between $\vec{u}$ and $\vec{v}$.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{||\,\vec{u}\,||\,||\,\vec{v}\,||}\right)\\ &= \cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\\ &\approx 0.8471 = 48.54^\circ.\end{aligned}$$

Our work in Example 10.4.2 showed that $\vec{u}\times\vec{v} = \langle -9, -7, 5\rangle$, hence $||\,\vec{u}\times\vec{v}\,|| = \sqrt{155}$. Is $||\,\vec{u}\times\vec{v}\,|| = ||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta$? Using numerical approximations, we find:

$$\begin{aligned}||\,\vec{u}\times\vec{v}\,|| &= \sqrt{155}\\ &\approx 12.45.\end{aligned} \qquad \begin{aligned}||\,\vec{u}\,||\,||\,\vec{v}\,||\sin\theta &= \sqrt{46}\sqrt{6}\sin 0.8471\\ &\approx 12.45.\end{aligned}$$

Numerically, they seem equal. Using a right triangle, one can show that

$$\sin\left(\cos^{-1}\left(\frac{11}{\sqrt{46}\sqrt{6}}\right)\right) = \frac{\sqrt{155}}{\sqrt{46}\sqrt{6}},$$

which allows us to verify the theorem exactly.

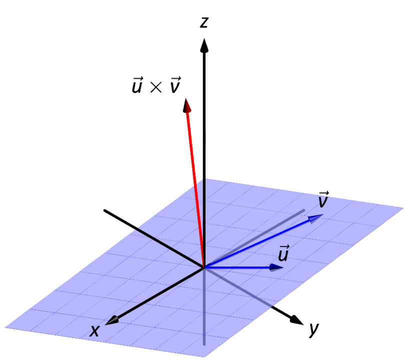


Figure 1.3.1: Illustrating the Right Hand Rule of the cross product.

Right Hand Rule

The anticommutative property of the cross product demonstrates that $\vec{u}\times\vec{v}$ and $\vec{v}\times\vec{u}$ differ only by a sign – these vectors have the same magnitude but point in the opposite direction. When seeking a vector perpendicular to $\vec{u}$ and $\vec{v}$, we essentially have two directions to choose from, one in the direction of $\vec{u}\times\vec{v}$ and one in the direction of $\vec{v}\times\vec{u}$. Does it matter which we choose? How can we tell which one we will get without graphing, etc.?

Another wonderful property of the cross product, as defined, is that it follows the **right hand rule.** Given $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ with the same initial point, point the index finger of your right hand in the direction of $\vec{u}$ and let your middle finger point in the direction of $\vec{v}$ (much as we did when establishing the right hand rule for the 3-dimensional coordinate system). Your thumb will naturally extend in the direction of $\vec{u}\times\vec{v}$. One can "practice" this using Figure 10.4.1. If you switch, and point the index finder in the direction of $\vec{v}$ and the middle finger in the direction of $\vec{u}$, your thumb will now point in the opposite direction, allowing you to "visualize" the anticommutative property of the cross product.

Applications of the Cross Product

Notes:

There are a number of ways in which the cross product is useful in mathematics, physics and other areas of science beyond "just" finding a vector perpendicular to two others. We highlight a few here.

Area of a Parallelogram

It is a standard geometry fact that the area of a parallelogram is $A = bh$, where b is the length of the base and h is the height of the parallelogram, as illustrated in Figure 10.4.2(a). As shown when defining the Parallelogram Law of vector addition, two vectors $\vec{u}$ and $\vec{v}$ define a parallelogram when drawn from the same initial point, as illustrated in Figure 10.4.2(b). Trigonometry tells us that $h = ||\,\vec{u}\,||\sin\theta$, hence the area of the parallelogram is

$$A = ||\,\vec{u}\,||\;||\,\vec{v}\,||\sin\theta = ||\,\vec{u}\times\vec{v}\,||, \tag{1.4}$$

where the second equality comes from Theorem 10.4.2. We illustrate using Equation (10.4) in the following example.

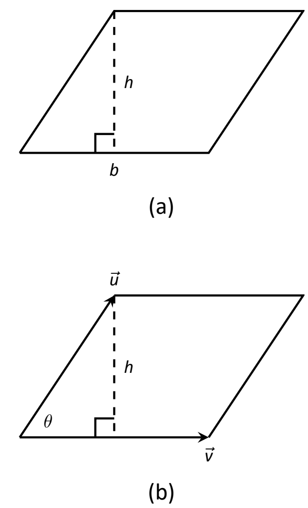


Figure 1.3.2: Using the cross product to find the area of a parallelogram.

Example 1.3.4 Finding the area of a parallelogram

1. Find the area of the parallelogram defined by the vectors $\vec{u} = \langle 2,1\rangle$ and $\vec{v} = \langle 1,3\rangle$.
2. Verify that the points $A = (1,1,1)$, $B = (2,3,2)$, $C = (4,5,3)$ and $D = (3,3,2)$ are the vertices of a parallelogram. Find the area of the parallelogram.

Solution

1. Figure 10.4.3(a) sketches the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$. We have a slight problem in that our vectors exist in $\mathbb{R}^2$, not $\mathbb{R}^3$, and the cross product is only defined on vectors in $\mathbb{R}^3$. We skirt this issue by viewing $\vec{u}$ and $\vec{v}$ as vectors in the $x-y$ plane of $\mathbb{R}^3$, and rewrite them as $\vec{u} = \langle 2,1,0\rangle$ and $\vec{v} = \langle 1,3,0\rangle$. We can now compute the cross product. It is easy to show that $\vec{u}\times\vec{v} = \langle 0,0,5\rangle$; therefore the area of the parallelogram is $A = ||\,\vec{u}\times\vec{v}\,|| = 5$.
2. To show that the quadrilateral $ABCD$ is a parallelogram (shown in Figure 10.4.3(b)), we need to show that the opposite sides are parallel. We can quickly show that $\overrightarrow{AB} = \overrightarrow{DC} = \langle 1,2,1\rangle$ and $\overrightarrow{BC} = \overrightarrow{AD} = \langle 2,2,1\rangle$. We find the area by computing the magnitude of the cross product of $\overrightarrow{AB}$ and $\overrightarrow{BC}$:
$$\overrightarrow{AB}\times\overrightarrow{BC} = \langle 0,1,-2\rangle \quad\Rightarrow\quad ||\,\overrightarrow{AB}\times\overrightarrow{BC}\,|| = \sqrt{5}\approx 2.236.$$

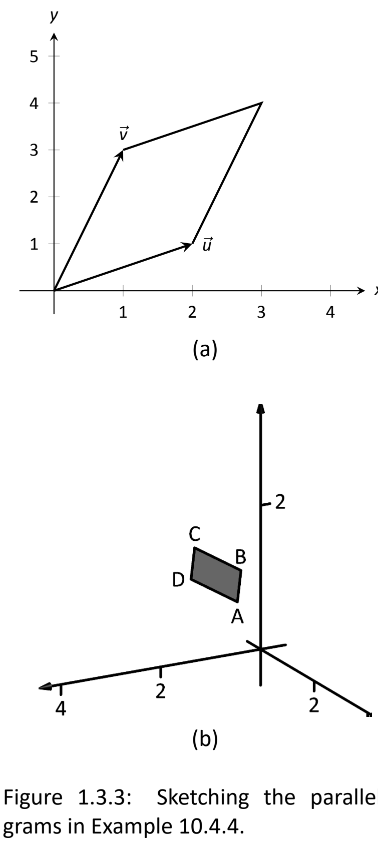


Figure 1.3.3: Sketching the parallelograms in Example 10.4.4.

Notes:

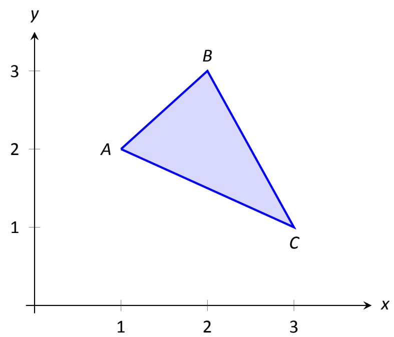


Figure 1.3.4: Finding the area of a triangle in Example 10.4.5.

This application is perhaps more useful in finding the area of a triangle (in short, triangles are used more often than parallelograms). We illustrate this in the following example.

Example 1.3.5 Area of a triangle
Find the area of the triangle with vertices $A = (1, 2)$, $B = (2, 3)$ and $C = (3, 1)$, as pictured in Figure 10.4.4.

Solution We found the area of this triangle in Example 7.1.4 to be 1.5 using integration. There we discussed the fact that finding the area of a triangle can be inconvenient using the "$\frac{1}{2}bh$" formula as one has to compute the height, which generally involves finding angles, etc. Using a cross product is much more direct.

We can choose any two sides of the triangle to use to form vectors; we choose $\overrightarrow{AB} = \langle 1, 1\rangle$ and $\overrightarrow{AC} = \langle 2, -1\rangle$. As in the previous example, we will rewrite these vectors with a third component of 0 so that we can apply the cross product. The area of the triangle is

$$\frac{1}{2}||\,\overrightarrow{AB}\times\overrightarrow{AC}\,|| = \frac{1}{2}||\ \langle 1,1,0\rangle\times\langle 2,-1,0\rangle\ || = \frac{1}{2}||\ \langle 0,0,-3\rangle\ || = \frac{3}{2}.$$

We arrive at the same answer as before with less work.

Volume of a Parallelepiped

Note: The word "parallelepiped" is pronounced "parallel–eh–pipe–ed."

The three dimensional analogue to the parallelogram is the **parallelepiped**. Each face is parallel to the opposite face, as illustrated in Figure 10.4.5. By crossing $\vec v$ and $\vec w$, one gets a vector whose magnitude is the area of the base. Dotting this vector with $\vec u$ computes the volume of parallelepiped! (Up to a sign; take the absolute value.)

Thus the volume of a parallelepiped defined by vectors $\vec u$, $\vec v$ and $\vec w$ is

$$V = |\vec u\cdot(\vec v\times\vec w)|. \tag{1.5}$$

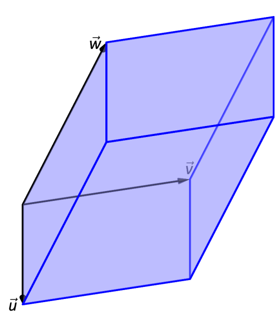


Figure 1.3.5: A parallelepiped is the three dimensional analogue to the parallelogram.

Note how this is the Triple Scalar Product, first seen in Theorem 10.4.1. Applying the identities given in the theorem shows that we can apply the Triple Scalar Product in any "order" we choose to find the volume. That is,

$$V = |\vec u\cdot(\vec v\times\vec w)| = |\vec u\cdot(\vec w\times\vec v)| = |(\vec u\times\vec v)\cdot\vec w|,\quad \text{etc.}$$

Example 1.3.6 Finding the volume of parallelepiped
Find the volume of the parallepiped defined by the vectors $\vec u = \langle 1, 1, 0\rangle$, $\vec v = \langle -1, 1, 0\rangle$ and $\vec w = \langle 0, 1, 1\rangle$.

Notes:

Solution We apply Equation (10.5). We first find $\vec{v} \times \vec{w} = \langle 1, 1, -1 \rangle$. Then

$$|\vec{u} \cdot (\vec{v} \times \vec{w})| = |\langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle| = 2.$$

So the volume of the parallelepiped is 2 cubic units.

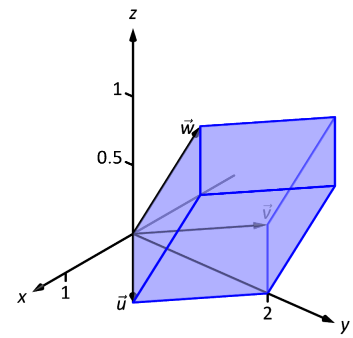


Figure 1.3.6: A parallelepiped in Example 10.4.6.

While this application of the Triple Scalar Product is interesting, it is not used all that often: parallelepipeds are not a common shape in physics and engineering. The last application of the cross product is very applicable in engineering.

Torque

Torque is a measure of the turning force applied to an object. A classic scenario involving torque is the application of a wrench to a bolt. When a force is applied to the wrench, the bolt turns. When we represent the force and wrench with vectors $\vec{F}$ and $\vec{\ell}$, we see that the bolt moves (because of the threads) in a direction orthogonal to $\vec{F}$ and $\vec{\ell}$. Torque is usually represented by the Greek letter τ, or tau, and has units of N·m, a Newton–meter, or ft·lb, a foot–pound.

While a full understanding of torque is beyond the purposes of this book, when a force $\vec{F}$ is applied to a lever arm $\vec{\ell}$, the resulting torque is

$$\vec{\tau} = \vec{\ell} \times \vec{F}. \tag{1.6}$$

Example 1.3.7 Computing torque

A lever of length 2ft makes an angle with the horizontal of 45°. Find the resulting torque when a force of 10lb is applied to the end of the level where:

1. the force is perpendicular to the lever, and
2. the force makes an angle of 60° with the lever, as shown in Figure 10.4.7.

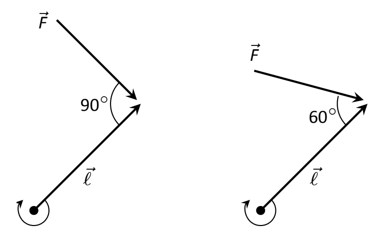


Figure 1.3.7: Showing a force being applied to a lever in Example 10.4.7.

Solution

1. We start by determining vectors for the force and lever arm. Since the lever arm makes a 45° angle with the horizontal and is 2ft long, we can state that $\vec{\ell} = 2\langle \cos 45°, \sin 45° \rangle = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$.

   Since the force vector is perpendicular to the lever arm (as seen in the left hand side of Figure 10.4.7), we can conclude it is making an angle of $-45°$ with the horizontal. As it has a magnitude of 10lb, we can state $\vec{F} = 10\langle \cos(-45°), \sin(-45°) \rangle = \left\langle 5\sqrt{2}, -5\sqrt{2} \right\rangle$.

   Using Equation (10.6) to find the torque requires a cross product. We again let the third component of each vector be 0 and compute the cross

Notes:

product:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \times \left\langle 5\sqrt{2}, -5\sqrt{2}, 0 \right\rangle \\ &= \langle 0, 0, -20 \rangle\end{aligned}$$

This clearly has a magnitude of 20 ft-lb.

We can view the force and lever arm vectors as lying "on the page"; our computation of $\vec{\tau}$ shows that the torque goes "into the page." This follows the Right Hand Rule of the cross product, and it also matches well with the example of the wrench turning the bolt. Turning a bolt clockwise moves it in.

2. Our lever arm can still be represented by $\vec{\ell} = \left\langle \sqrt{2}, \sqrt{2} \right\rangle$. As our force vector makes a 60° angle with $\vec{\ell}$, we can see (referencing the right hand side of the figure) that $\vec{F}$ makes a -15° angle with the horizontal. Thus

$$\begin{aligned}\vec{F} = 10 \langle \cos -15^\circ, \sin -15^\circ \rangle &= \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, \frac{5(-1+\sqrt{3})}{\sqrt{2}} \right\rangle \\ &\approx \langle 9.659, -2.588 \rangle .\end{aligned}$$

We again make the third component 0 and take the cross product to find the torque:

$$\begin{aligned}\vec{\tau} &= \vec{\ell} \times \vec{F} \\ &= \left\langle \sqrt{2}, \sqrt{2}, 0 \right\rangle \times \left\langle \frac{5(1+\sqrt{3})}{\sqrt{2}}, \frac{5(-1+\sqrt{3})}{\sqrt{2}}, 0 \right\rangle \\ &= \left\langle 0, 0, -10\sqrt{3} \right\rangle \\ &\approx \langle 0, 0, -17.321 \rangle .\end{aligned}$$

As one might expect, when the force and lever arm vectors *are* orthogonal, the magnitude of force is greater than when the vectors *are not* orthogonal.

While the cross product has a variety of applications (as noted in this chapter), its fundamental use is finding a vector perpendicular to two others. Knowing a vector is orthogonal to two others is of incredible importance, as it allows us to find the equations of lines and planes in a variety of contexts. The importance of the cross product, in some sense, relies on the importance of lines and planes, which see widespread use throughout engineering, physics and mathematics. We study lines and planes in the next two sections.

Notes:

Exercises 1.3

Terms and Concepts

1. The cross product of two vectors is a ________, not a scalar.

2. One can visualize the direction of $\vec{u} \times \vec{v}$ using the ________ ________ ________.

3. Give a synonym for "orthogonal."

4. T/F: A fundamental principle of the cross product is that $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

5. ________ is a measure of the turning force applied to an object.

6. T/F: If $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

Problems

In Exercises 7 – 16, vectors $\vec{u}$ and $\vec{v}$ are given. Compute $\vec{u} \times \vec{v}$ and show this is orthogonal to both $\vec{u}$ and $\vec{v}$.

7. $\vec{u} = \langle 3, 2, -2\rangle, \quad \vec{v} = \langle 0, 1, 5\rangle$

8. $\vec{u} = \langle 5, -4, 3\rangle, \quad \vec{v} = \langle 2, -5, 1\rangle$

9. $\vec{u} = \langle 4, -5, -5\rangle, \quad \vec{v} = \langle 3, 3, 4\rangle$

10. $\vec{u} = \langle -4, 7, -10\rangle, \quad \vec{v} = \langle 4, 4, 1\rangle$

11. $\vec{u} = \langle 1, 0, 1\rangle, \quad \vec{v} = \langle 5, 0, 7\rangle$

12. $\vec{u} = \langle 1, 5, -4\rangle, \quad \vec{v} = \langle -2, -10, 8\rangle$

13. $\vec{u} = \langle a, b, 0\rangle, \quad \vec{v} = \langle c, d, 0\rangle$

14. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{j}$

15. $\vec{u} = \vec{i}, \quad \vec{v} = \vec{k}$

16. $\vec{u} = \vec{j}, \quad \vec{v} = \vec{k}$

17. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

18. Pick any vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^3$ and show that $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$.

In Exercises 19 – 22, the magnitudes of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$ are given, along with the angle θ between them. Use this information to find the magnitude of $\vec{u} \times \vec{v}$.

19. $||\,\vec{u}\,|| = 2, \quad ||\,\vec{v}\,|| = 5, \quad \theta = 30^\circ$

20. $||\,\vec{u}\,|| = 3, \quad ||\,\vec{v}\,|| = 7, \quad \theta = \pi/2$

21. $||\,\vec{u}\,|| = 3, \quad ||\,\vec{v}\,|| = 4, \quad \theta = \pi$

22. $||\,\vec{u}\,|| = 2, \quad ||\,\vec{v}\,|| = 5, \quad \theta = 5\pi/6$

In Exercises 23 – 26, find the area of the parallelogram defined by the given vectors.

23. $\vec{u} = \langle 1, 1, 2\rangle, \quad \vec{v} = \langle 2, 0, 3\rangle$

24. $\vec{u} = \langle -2, 1, 5\rangle, \quad \vec{v} = \langle -1, 3, 1\rangle$

25. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle 2, 1\rangle$

26. $\vec{u} = \langle 2, 0\rangle, \quad \vec{v} = \langle 0, 3\rangle$

In Exercises 27 – 30, find the area of the triangle with the given vertices.

27. Vertices: $(0, 0, 0)$, $(1, 3, -1)$ and $(2, 1, 1)$.

28. Vertices: $(5, 2, -1)$, $(3, 6, 2)$ and $(1, 0, 4)$.

29. Vertices: $(1, 1)$, $(1, 3)$ and $(2, 2)$.

30. Vertices: $(3, 1)$, $(1, 2)$ and $(4, 3)$.

In Exercises 31 – 32, find the area of the quadrilateral with the given vertices. (Hint: break the quadrilateral into 2 triangles.)

31. Vertices: $(0, 0)$, $(1, 2)$, $(3, 0)$ and $(4, 3)$.

32. Vertices: $(0, 0, 0)$, $(2, 1, 1)$, $(-1, 2, -8)$ and $(1, -1, 5)$.

In Exercises 33 – 34, find the volume of the parallelepiped defined by the given vectors.

33. $\vec{u} = \langle 1, 1, 1\rangle, \quad \vec{v} = \langle 1, 2, 3\rangle, \quad \vec{w} = \langle 1, 0, 1\rangle$

34. $\vec{u} = \langle -1, 2, 1\rangle, \quad \vec{v} = \langle 2, 2, 1\rangle, \quad \vec{w} = \langle 3, 1, 3\rangle$

In Exercises 35 – 38, find a unit vector orthogonal to both $\vec{u}$ and $\vec{v}$.

35. $\vec{u} = \langle 1, 1, 1\rangle, \quad \vec{v} = \langle 2, 0, 1\rangle$

36. $\vec{u} = \langle 1, -2, 1\rangle, \quad \vec{v} = \langle 3, 2, 1\rangle$

37. $\vec{u} = \langle 5, 0, 2\rangle, \quad \vec{v} = \langle -3, 0, 7\rangle$

38. $\vec{u} = \langle 1, -2, 1\rangle, \quad \vec{v} = \langle -2, 4, -2\rangle$

39. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in horizontally from the crankshaft. Find the magnitude of the torque applied to the crankshaft.

10 04 ex 37 40. A bicycle rider applies 150lb of force, straight down, onto a pedal that extends 7in from the crankshaft, making a 30° angle with the horizontal. Find the magnitude of the torque applied to the crankshaft.

10 04 ex 38 41. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench. What is the maximum amount of torque that can be applied to the bolt?

10 04 ex 39 42. To turn a stubborn bolt, 80lb of force is applied to a 10in wrench in a confined space, where the direction of applied force makes a 10° angle with the wrench. How much torque is subsequently applied to the wrench?

10 04 ex 40 43. Show, using the definition of the Cross Product, that $\vec{u}\cdot(\vec{u}\times\vec{v}) = 0$; that is, that $\vec{u}$ is orthogonal to the cross product of $\vec{u}$ and $\vec{v}$.

10 04 ex 41 44. Show, using the definition of the Cross Product, that $\vec{u}\times\vec{u} = \vec{0}$.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

10 02 ex 01 1. Answers will vary.

10 02 ex 02 2. $(1,2)$ is a point; $\langle 1,2\rangle$ is a vector that describes a displacement of 1 unit in the x-direction and 2 units in the y-direction.

10 02 ex 03 3. A vector with magnitude 1.

10 02 ex 36 4. Direction

10 02 ex 04 5. Their respective unit vectors are parallel; unit vectors $\vec{u}_1$ and $\vec{u}_2$ are parallel if $\vec{u}_1 = \pm\vec{u}_2$.

10 02 ex 05 6. It stretches the vector by a factor of 2, and points it in the opposite direction.

10 02 ex 06 7. $\overrightarrow{PQ} = \langle 1,6\rangle = 1\vec{i} + 6\vec{j}$

10 02 ex 07 8. $\overrightarrow{PQ} = \langle 4,-4\rangle = 4\vec{i} - 4\vec{j}$

10 02 ex 08 9. $\overrightarrow{PQ} = \langle 6,-1,6\rangle = 6\vec{i} - \vec{j} + 6\vec{k}$

10 02 ex 09 10. $\overrightarrow{PQ} = \langle 2,2,0\rangle = 2\vec{i} + 2\vec{j}$

10 02 ex 10 11.

(a) $\vec{u}+\vec{v} = \langle 2,-1\rangle$; $\vec{u}-\vec{v} = \langle 0,-3\rangle$; $2\vec{u}-3\vec{v} = \langle -1,-7\rangle$.

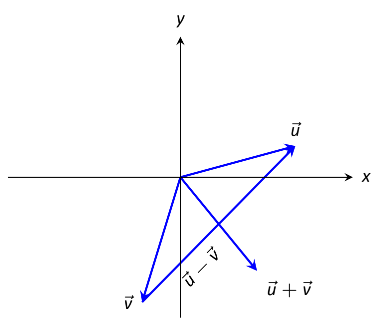
(c) $\vec{x} = \langle 1/2, 2\rangle$.

10 02 ex 11 12.

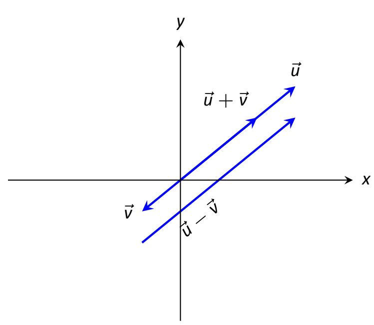
(a) $\vec{u}+\vec{v} = \langle 3,2,1\rangle$; $\vec{u}-\vec{v} = \langle -1,0,-3\rangle$; $\pi\vec{u} - \sqrt{2}\vec{v} = \left\langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2}\right\rangle$.

(c) $\vec{x} = \langle -1,0,-3\rangle$.

10 02 ex 12 13.

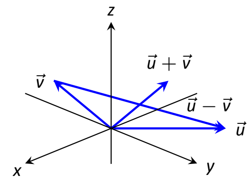


10 02 ex 13 14.

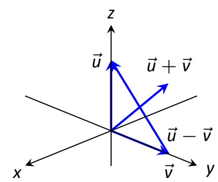


Sketch of $\vec{u}-\vec{v}$ shifted for clarity.

10 02 ex 14 15.



10 02 ex 15 16.



10 02 ex 16 17. $||\,\vec{u}\,|| = \sqrt{5}$, $||\,\vec{v}\,|| = \sqrt{13}$, $||\,\vec{u}+\vec{v}\,|| = \sqrt{26}$, $||\,\vec{u}-\vec{v}\,|| = \sqrt{10}$

10 02 ex 17 18. $||\,\vec{u}\,|| = \sqrt{17}$, $||\,\vec{v}\,|| = \sqrt{3}$, $||\,\vec{u}+\vec{v}\,|| = \sqrt{14}$, $||\,\vec{u}-\vec{v}\,|| = \sqrt{26}$

10 02 ex 18 19. $||\,\vec{u}\,|| = \sqrt{5}$, $||\,\vec{v}\,|| = 3\sqrt{5}$, $||\,\vec{u}+\vec{v}\,|| = 2\sqrt{5}$, $||\,\vec{u}-\vec{v}\,|| = 4\sqrt{5}$

10 02 ex 19 20. $||\,\vec{u}\,|| = 7$, $||\,\vec{v}\,|| = 35$, $||\,\vec{u}+\vec{v}\,|| = 42$, $||\,\vec{u}-\vec{v}\,|| = 28$

10 02 ex 20 21. When $\vec{u}$ and $\vec{v}$ have the same direction. (Note: parallel is not enough.)

10 02 ex 21 22. $\vec{u} = \left\langle 3/\sqrt{58}, 7/\sqrt{58}\right\rangle$

10 02 ex 22 23. $\vec{u} = \langle 0.6, 0.8\rangle$

10 02 ex 23 24. $\vec{u} = \langle 1/3, -2/3, 2/3\rangle$

10 02 ex 24 25. $\vec{u} = \left\langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\right\rangle$

10 02 ex 25 26. $\vec{u} = \langle \cos 50^\circ, \sin 50^\circ\rangle \approx \langle 0.643, 0.766\rangle$.

10 02 ex 26 27. $\vec{u} = \langle \cos 120^\circ, \sin 120^\circ\rangle = \left\langle -1/2, \sqrt{3}/2\right\rangle$.

10 02 ex 27 28.

$$\begin{aligned}
||\,\vec{u}\,|| &= \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} \\
&= \sqrt{\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta} \\
&= \sqrt{\sin^2\theta + \cos^2\theta} \\
&= 1.
\end{aligned}$$

10 02 ex 28 29. The force on each chain is $100/\sqrt{3} \approx 57.735$lb.

10 02 ex 29 30. The force on each chain is 100lb.

10 02 ex 30 31. The force on the chain with angle θ is approx. 45.124lb; the force on the chain with angle φ is approx. 59.629lb.

10 02 ex 31 32. The force on each chain is 50lb.

10 02 ex 32 33. $\theta = 45^\circ$; the weight is lifted 0.29 ft (about 3.5in).

10 02 ex 33 34. $\theta = 5.71^\circ$; the weight is lifted 0.005 ft (about 1/16th of an inch).

10 02 ex 34 35. $\theta = 45^\circ$; the weight is lifted 2.93 ft.

10 02 ex 35 36. $\theta = 84.29^\circ$; the weight is lifted 9 ft.

Section 1.2

10 03 ex 01 1. Scalar

10 03 ex 02 2. The magnitude of a vectors is the square root of the dot product of a vector with itself; that is, $||\,\vec{v}\,|| = \sqrt{\vec{v}\cdot\vec{v}}$.

10 03 ex 03 3. By considering the sign of the dot product of the two vectors. If the dot product is positive, the angle is acute; if the dot product is negative, the angle is obtuse.

10 03 ex 04 4. “Perpendicular” is one answer.

10 03 ex 05 5. -22

10 03 ex 06 6. 33

10 03 ex 07 7. 3

10 03 ex 08 8. 0

10 03 ex 09 9. not defined

10 03 ex 10 10. 0

10 03 ex 11 11. Answers will vary.

10 03 ex 12 12. Answers will vary.

10 03 ex 13 13. $\theta = 0.3218 \approx 18.43^\circ$

10 03 ex 14 14. $\theta = 1.6476 \approx 94.4^\circ$

10 03 ex 15 15. $\theta = \pi/4 = 45^\circ$

10 03 ex 16 16. $\theta = \pi/2 = 90^\circ$

10 03 ex 17 17. Answers will vary; two possible answers are $\langle -7, 4\rangle$ and $\langle 14, -8\rangle$.

10 03 ex 18 18. Answers will vary; two possible answers are $\langle 5, 3\rangle$ and $\langle -15, -9\rangle$.

10 03 ex 19 19. Answers will vary; two possible answers are $\langle 1, 0, -1\rangle$ and $\langle 4, 5, -9\rangle$.

10 03 ex 20 20. Answers will vary; two possible answers are $\langle 2, 1, 0\rangle$ and $\langle 1, 1, 1/3\rangle$.

10 03 ex 21 21. $\text{proj}_{\,\vec{v}}\,\vec{u} = \langle -1/2, 3/2\rangle$.

10 03 ex 22 22. $\text{proj}_{\,\vec{v}}\,\vec{u} = \langle 2, 6\rangle$.

10 03 ex 25 23. $\text{proj}_{\,\vec{v}}\,\vec{u} = \langle -1/2, -1/2\rangle$.

10 03 ex 26 24. $\text{proj}_{\,\vec{v}}\,\vec{u} = \langle 0, 0\rangle$.

10 03 ex 23 25. $\text{proj}_{\,\vec{v}}\,\vec{u} = \langle 1, 2, 3\rangle$.

10 03 ex 24 26. $\text{proj}_{\,\vec{v}}\,\vec{u} = \langle 4/3, 4/3, 2/3\rangle$.

10 03 ex 27 27. $\vec{u} = \langle -1/2, 3/2\rangle + \langle 3/2, 1/2\rangle$.

10 03 ex 28 28. $\vec{u} = \langle 2, 6\rangle + \langle 3, -1\rangle$.

10 03 ex 29 29. $\vec{u} = \langle -1/2, -1/2\rangle + \langle -5/2, 5/2\rangle$.

10 03 ex 30 30. $\vec{u} = \langle 0, 0\rangle + \langle -3, 2\rangle$.

10 03 ex 31 31. $\vec{u} = \langle 1, 2, 3\rangle + \langle 0, 3, -2\rangle$.

10 03 ex 32 32. $\vec{u} = \langle 4/3, 4/3, 2/3\rangle + \langle 5/3, -7/3, 4/3\rangle$.

10 03 ex 33 33. 1.96lb

10 03 ex 34 34. 5lb

10 03 ex 35 35. 141.42ft–lb

10 03 ex 36 36. 196.96ft–lb

10 03 ex 37 37. 500ft–lb

10 03 ex 38 38. 424.26ft–lb

10 03 ex 39 39. 500ft–lb

Section 1.3

10 04 ex 01 1. vector

10 04 ex 02 2. right hand rule

10 03 ex 04 3. “Perpendicular” is one answer.

10 04 ex 03 4. T

10 04 ex 04 5. Torque

10 04 ex 42 6. T

10 04 ex 07 7. $\vec{u}\times\vec{v} = \langle 12, -15, 3\rangle$

10 04 ex 08 8. $\vec{u}\times\vec{v} = \langle 11, 1, -17\rangle$

10 04 ex 05 9. $\vec{u}\times\vec{v} = \langle -5, -31, 27\rangle$

10 04 ex 06 10. $\vec{u}\times\vec{v} = \langle 47, -36, -44\rangle$

10 04 ex 09 11. $\vec{u}\times\vec{v} = \langle 0, -2, 0\rangle$

10 04 ex 10 12. $\vec{u}\times\vec{v} = \langle 0, 0, 0\rangle$

10 04 ex 43 13. $\vec{u}\times\vec{v} = \langle 0, 0, ad - bc\rangle$

10 04 ex 11 14. $\vec{i}\times\vec{j} = \vec{k}$

10 04 ex 12 15. $\vec{i}\times\vec{k} = -\vec{j}$

10 04 ex 13 16. $\vec{j}\times\vec{k} = \vec{i}$

10 04 ex 14 17. Answers will vary.

10 04 ex 15 18. Answers will vary.

10 04 ex 16 19. 5

10 04 ex 17 20. 21

10 04 ex 18 21. 0

10 04 ex 19 22. 5

10 04 ex 20 23. $\sqrt{14}$

10 04 ex 21 24. $\sqrt{230}$

10 04 ex 22 25. 3

10 04 ex 23 26. 6

10 04 ex 24 27. $5\sqrt{2}/2$

10 04 ex 25 28. $3\sqrt{30}$

10 04 ex 26 29. 1

10 04 ex 27 30. $5/2$

10 04 ex 28 31. 7

10 04 ex 29 32. $8\sqrt{7/2}$

10 04 ex 30 33. 2

10 04 ex 31 34. 15

10 04 ex 35 35. $\pm\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$

10 04 ex 32 36. $\pm\frac{1}{\sqrt{21}}\langle -2, 1, 4\rangle$

10 04 ex 33 37. $\langle 0, \pm 1, 0 \rangle$

10 04 ex 34 38. Since $\vec{u}$ and $\vec{v}$ are parallel, any unit vector orthogonal to $\vec{u}$ works (such as $\frac{1}{\sqrt{2}} \langle 1, 0, -1 \rangle$).

10 04 ex 36 39. 87.5ft–lb

10 04 ex 37 40. $43.75\sqrt{3} \approx 75.78$ft–lb

10 04 ex 38 41. $200/3 \approx 66.67$ft–lb

10 04 ex 39 42. 11.58ft–lb

10 04 ex 40 43. With $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, we have

$$\begin{aligned}\vec{u} \cdot (\vec{u} \times \vec{v}) &= \langle u_1, u_2, u_3 \rangle \cdot (\langle u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \rangle) \\ &= u_1(u_2 v_3 - u_3 v_2) - u2(u_1 v_3 - u_3 v_1) + u3(u_1 v_2 - u_2 v_1) \\ &= 0.\end{aligned}$$

10 04 ex 41 44. With $\vec{u} = \langle u_1, u_2, u_3 \rangle$, we have

$$\begin{aligned}\vec{u} \times \vec{u} &= \langle u_2 u_3 - u_3 u_2, -(u_1 u_3 - u_3 u_1), u_1 u_2 - u_2 u_1 \rangle \\ &= \langle 0, 0, 0 \rangle \\ &= \vec{0}.\end{aligned}$$