1: LIMITS

1.1 Limits Involving Infinity

In Definition 1 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 1.1. Note how, as x approaches 0, $f(x)$ grows very, very large. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

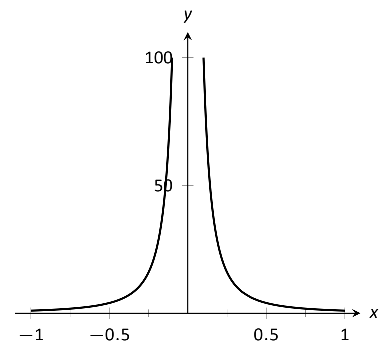

Figure 1.1: Graphing $f(x) = 1/x^2$ for values of x near 0.

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to \infty} \frac{1}{x^2} = 0.$$

We explore both types of use of ∞ in turn.

> **Definition 1   Limit of Infinity, ∞**
>
> We say $\lim_{x\to c} f(x) = \infty$ if for every $M > 0$ there exists $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$, then $f(x) \geq M$.

This is just like the ε–δ definition from Section 1.2. In that definition, given any (small) value ε, if we let x get close enough to c (within δ units of c) then $f(x)$ is guaranteed to be within ε of $f(c)$. Here, given any (large) value M, if we let x get close enough to c (within δ units of c), then $f(x)$ will be at least as large as M. In other words, if we get close enough to c, then we can make $f(x)$ as large as we want. We can define limits equal to $-\infty$ in a similar way.

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist.* A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive.

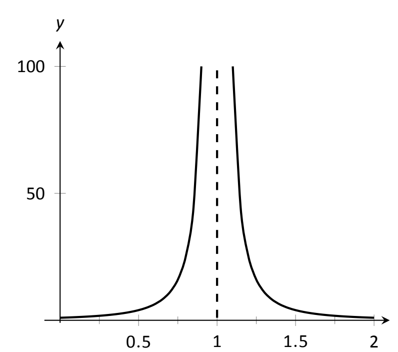

Figure 1.2: Observing infinite limit as $x \to 1$ in Example 1.

Example 1   Evaluating limits involving infinity

Find $\lim_{x\to 1} \dfrac{1}{(x-1)^2}$ as shown in Figure 1.2.

Solution In Example 4 of Section 1.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. In general, let a "large" value M be given. Let $\delta = 1/\sqrt{M}$. If x is within δ of 1, i.e., if $|x-1| < 1/\sqrt{M}$, then:

$$\begin{aligned} |x-1| &< \frac{1}{\sqrt{M}} \\ (x-1)^2 &< \frac{1}{M} \\ \frac{1}{(x-1)^2} &> M, \end{aligned}$$

which is what we wanted to show. So we may say $\lim\limits_{x\to 1} 1/(x-1)^2 = \infty$.

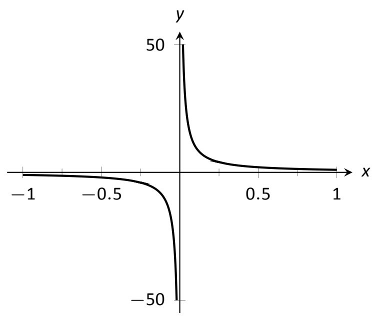

Figure 1.3: Evaluating $\lim\limits_{x\to 0} \frac{1}{x}$.

Example 2 **Evaluating limits involving infinity**

Find $\lim\limits_{x\to 0} \frac{1}{x}$, as shown in Figure 1.3.

Solution It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behavior is not consistent, we cannot say that $\lim\limits_{x\to 0} \frac{1}{x} = \infty$. However, we can make a statement about one–sided limits. We can state that $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

Vertical asymptotes

If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, we say the function has a **vertical asymptote** at c.

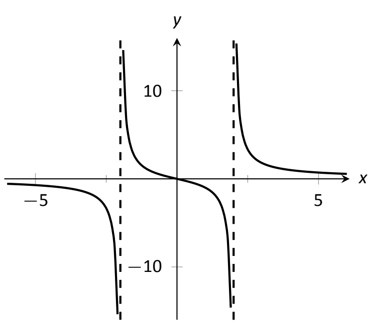

Figure 1.4: Graphing $f(x) = \frac{3x}{x^2-4}$.

Example 3 **Finding vertical asymptotes**

Find the vertical asymptotes of $f(x) = \frac{3x}{x^2-4}$.

Solution Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 1.4. Thus the vertical asymptotes are at $x = \pm 2$.

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When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 1.5. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote, rather a hole exists in the graph at $x = 1$.

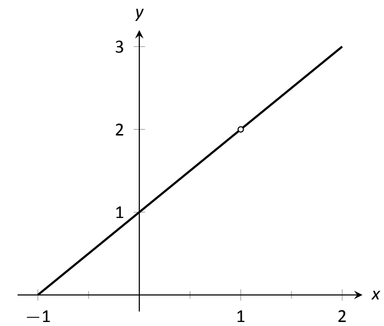

Figure 1.5: Graphically showing that $f(x) = \dfrac{x^2 - 1}{x - 1}$ does not have an asymptote at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin x)/x$. We have considered this function several times in the previous sections. We found that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 1.3 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

We have seen how the limits

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2 - 1}{x - 1}$$

each return the indeterminate form "0/0" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up with $0/0$ expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called *indeterminate*.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that

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the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling) or it may require a tool such as the Squeeze Theorem. In a later section we will learn a technique called l'Hôspital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 2 Limits at Infinity and Horizontal Asymptote

1. We say $\lim\limits_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.

2. We say $\lim\limits_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.

3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say that $y = L$ is a **horizontal asymptote** of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition

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with Definition 1.

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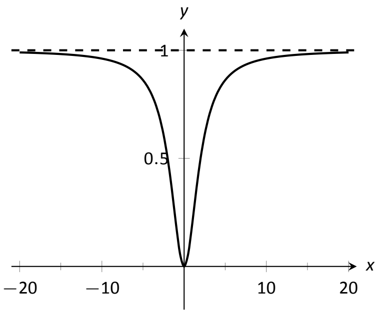


(a)

x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
−10	0.9615
−100	0.9996
−10000	0.999996

(b)

Figure 1.6: Using a graph and a table to approximate a horizontal asymptote in Example 4.

Example 4 Approximating horizontal asymptotes

Approximate the horizontal asymptote(s) of $f(x) = \dfrac{x^2}{x^2+4}$.

Solution We will approximate the horizontal asymptotes by approximating the limits

$$\lim_{x\to-\infty} \frac{x^2}{x^2+4} \quad \text{and} \quad \lim_{x\to\infty} \frac{x^2}{x^2+4}.$$

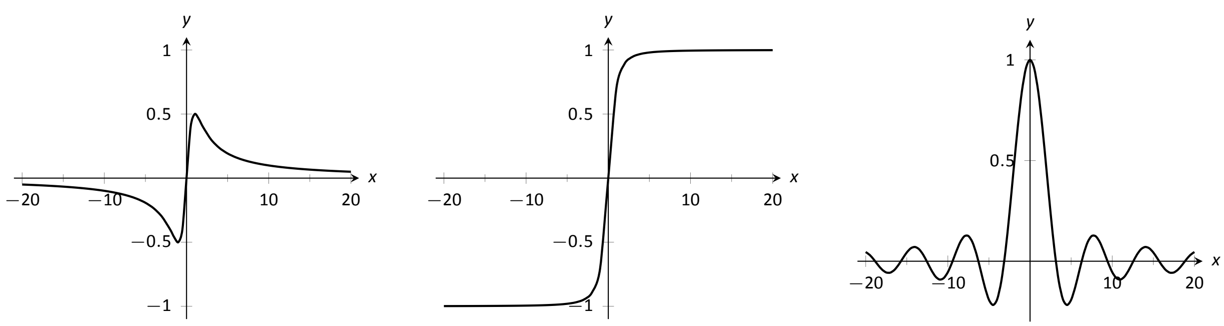
Figure 1.6(a) shows a sketch of f, and part (b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.

Later, we will show how to determine this analytically.

Horizontal asymptotes can take on a variety of forms. Figure 1.7(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 1.7(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 1.7(c) shows that $f(x) = (\sin x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.



(a) (b) (c)

Figure 1.7: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim\limits_{x\to\infty} 1/x$. As x gets larger and larger, the $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large

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enough value of x. Given ε, we can make $1/x < \varepsilon$ by choosing $x > 1/\varepsilon$. Thus we have $\lim_{x\to\infty} 1/x = 0$.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence dividing by 1), which is the largest power of x to appear in the function. Doing this, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} \\
&= \lim_{x\to\infty} \frac{x^3/x^3 + 2x/x^3 + 1/x^3}{4x^3/x^3 - 2x^2/x^3 + 9/x^3} \\
&= \lim_{x\to\infty} \frac{1 + 2/x^2 + 1/x^3}{4 - 2/x + 9/x^3}.
\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0} = \frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Notes:

Theorem 1 Limits of Rational Functions at Infinity

Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0},$$

where any of the coefficients may be 0 except for a_n and b_m.

1. If $n = m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{a_n}{b_m}$.
2. If $n < m$, then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = 0$.
3. If $n > m$, then $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ are both infinite.

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n = m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for $a_n x^n/x^n$ and $b_m x^m/x^n$. Since $n = m$, this will leave us with the limit a_n/b_m. If $n < m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n > m$, and we try dividing through by x^n, we end up with all the terms in the denominator tending toward 0, while the x^n term in the numerator does not approach 0. This is indicative of some sort of infinite limit.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to $a_n x^n$, and likewise all the terms in the denominator are small compared to $b_n x^m$. If $n = m$, looking only at these two important terms, we have $(a_n x^n)/(b_n x^m)$. This reduces to a_n/b_m. If $n < m$, the function behaves like $a_n/(b_m x^{m-n})$, which tends toward 0. If $n > m$, the function behaves like $a_n x^{n-m}/b_m$, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the largest powers of x. For instance, consider again $\lim\limits_{x\to\pm\infty} \dfrac{x}{\sqrt{x^2+1}}$, graphed in Figure 1.7(b). When x is very large, $x^2 + 1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x|, \quad \text{and} \quad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y = 1$ and $y = -1$, respectively.

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Example 5 **Finding a limit of a rational function**

Confirm analytically that $y = 1$ is the horizontal asymptote of $f(x) = \dfrac{x^2}{x^2+4}$, as approximated in Example 4.

Solution Before using Theorem 1, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{x^2+4} &= \lim_{x\to\infty}\frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

We can also use Theorem 1 directly; in this case $n = m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., 1/1 = 1.

Example 6 **Finding limits of rational functions**

Use Theorem 1 to evaluate each of the following limits.

1. $\displaystyle\lim_{x\to-\infty}\frac{x^2+2x-1}{x^3+1}$
2. $\displaystyle\lim_{x\to\infty}\frac{x^2+2x-1}{1-x-3x^2}$
3. $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{3-x}$

Solution

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 1.8(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 1.8(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 1.8(c).

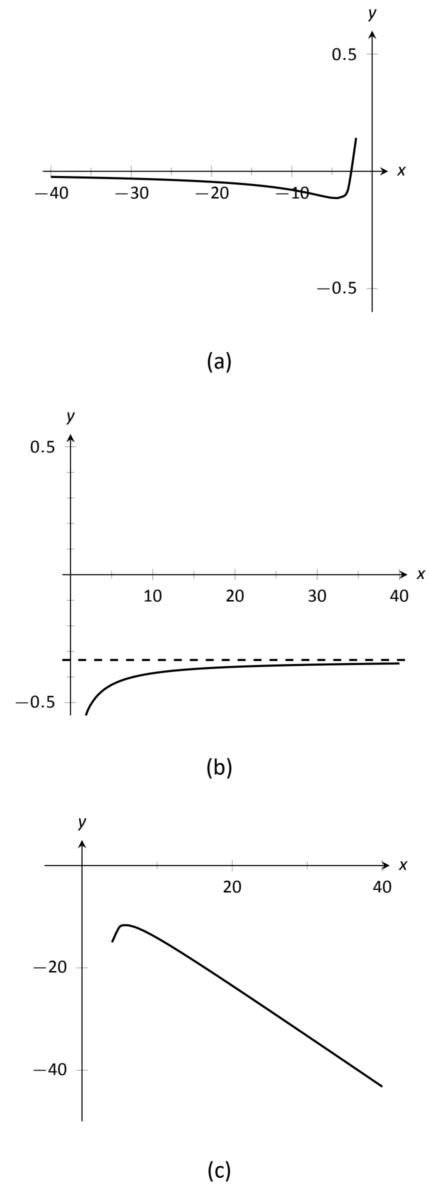

Figure 1.8: Visualizing the functions in Example 6.

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Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not–so–easy method of proving the value of a limit (ε-δ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Notes:

Exercises 1.1

Terms and Concepts

01 06 ex 01 1. T/F: If $\lim\limits_{x\to 5} f(x) = \infty$, then we are implicitly stating that the limit exists.

01 06 ex 02 2. T/F: If $\lim\limits_{x\to \infty} f(x) = 5$, then we are implicitly stating that the limit exists.

01 06 ex 03 3. T/F: If $\lim\limits_{x\to 1^-} f(x) = -\infty$, then $\lim\limits_{x\to 1^+} f(x) = \infty$

01 06 ex 04 4. T/F: If $\lim\limits_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$.

01 06 ex 05 5. T/F: $\infty/0$ is not an indeterminate form.

01 06 ex 06 6. List 5 indeterminate forms.

01 06 ex 07 7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

01 06 ex 08 8. Let $\lim\limits_{x\to 7} f(x) = \infty$. Explain how we know that f is/is not continuous at $x = 7$.

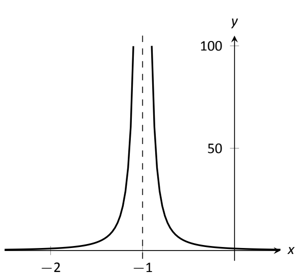
Problems

01 06 exset 01 **In Exercises 9 – 14, evaluate the given limits using the graph of the function.**

01 06 ex 09 9. $f(x) = \dfrac{1}{(x+1)^2}$

(a) $\lim\limits_{x\to -1^-} f(x)$

(b) $\lim\limits_{x\to -1^+} f(x)$



01 06 ex 10 10. $f(x) = \dfrac{1}{(x-3)(x-5)^2}$.

(a) $\lim\limits_{x\to 3^-} f(x)$

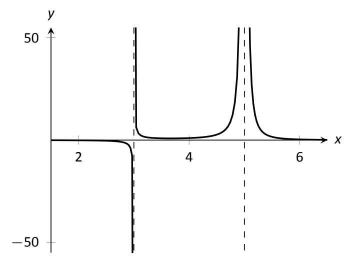
(b) $\lim\limits_{x\to 3^+} f(x)$

(c) $\lim\limits_{x\to 3} f(x)$

(d) $\lim\limits_{x\to 5^-} f(x)$

(e) $\lim\limits_{x\to 5^+} f(x)$

(f) $\lim\limits_{x\to 5} f(x)$



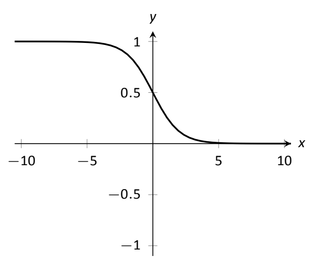
01 06 ex 11 11. $f(x) = \dfrac{1}{e^x + 1}$

(a) $\lim\limits_{x\to -\infty} f(x)$

(b) $\lim\limits_{x\to \infty} f(x)$

(c) $\lim\limits_{x\to 0^-} f(x)$

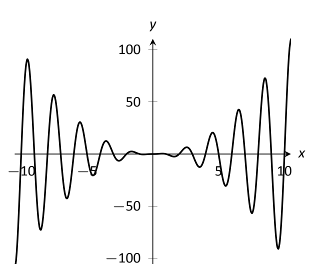
(d) $\lim\limits_{x\to 0^+} f(x)$



01 06 ex 12 12. $f(x) = x^2 \sin(\pi x)$

(a) $\lim\limits_{x\to -\infty} f(x)$

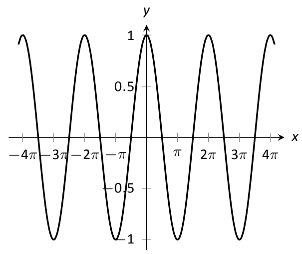
(b) $\lim\limits_{x\to \infty} f(x)$



01 06 ex 13 13. $f(x) = \cos(x)$

(a) $\lim\limits_{x\to-\infty} f(x)$

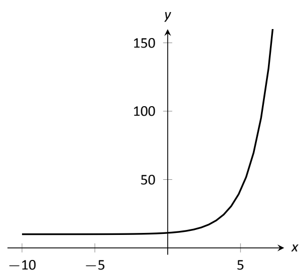
(b) $\lim\limits_{x\to\infty} f(x)$



01 06 ex 40 14. $f(x) = 2^x + 10$

(a) $\lim\limits_{x\to-\infty} f(x)$

(b) $\lim\limits_{x\to\infty} f(x)$



01 06 exset 02 **In Exercises 15 – 18, numerically approximate the following limits:**

(a) $\lim\limits_{x\to3^-} f(x)$

(b) $\lim\limits_{x\to3^+} f(x)$

(c) $\lim\limits_{x\to3} f(x)$

01 06 ex 14 15. $f(x) = \dfrac{x^2-1}{x^2-x-6}$

01 06 ex 15 16. $f(x) = \dfrac{x^2+5x-36}{x^3-5x^2+3x+9}$

01 06 ex 16 17. $f(x) = \dfrac{x^2-11x+30}{x^3-4x^2-3x+18}$

01 06 ex 17 18. $f(x) = \dfrac{x^2-9x+18}{x^2-x-6}$

01 06 exset 03 **In Exercises 19 – 24, identify the horizontal and vertical asymptotes, if any, of the given function.**

01 06 ex 18 19. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-20}$

01 06 ex 19 20. $f(x) = \dfrac{-3x^2-9x-6}{5x^2-10x-15}$

01 06 ex 20 21. $f(x) = \dfrac{x^2+x-12}{7x^3-14x^2-21x}$

01 06 ex 21 22. $f(x) = \dfrac{x^2-9}{9x-9}$

01 06 ex 22 23. $f(x) = \dfrac{x^2-9}{9x+27}$

01 06 ex 41 24. $f(x) = \dfrac{x^2-1}{-x^2-1}$

01 06 exset 04 **In Exercises 25 – 28, evaluate the given limit.**

01 06 ex 23 25. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{x-5}$

01 06 ex 24 26. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{5-x}$

01 06 ex 25 27. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{x^2-5}$

01 06 ex 26 28. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{5-x^2}$

Review

01 06 ex 27 29. Use an $\varepsilon - \delta$ proof to show that $\lim\limits_{x\to1} 5x-2 = 3$.

01 06 ex 28 30. Let $\lim\limits_{x\to2} f(x) = 3$ and $\lim\limits_{x\to2} g(x) = -1$. Evaluate the following limits.

(a) $\lim\limits_{x\to2} (f+g)(x)$

(b) $\lim\limits_{x\to2} (fg)(x)$

(c) $\lim\limits_{x\to2} (f/g)(x)$

(d) $\lim\limits_{x\to2} f(x)^{g(x)}$

01 06 ex 29 31. Let $f(x) = \begin{cases} x^2-1 & x<3 \\ x+5 & x\geq3 \end{cases}$.
Is f continuous everywhere?

01 06 ex 30 32. Evaluate the limit: $\lim\limits_{x\to e} \ln x$.

1.2 Differentials

In Section 2.2 we explored the meaning and use of the derivative. This section starts by revisiting some of those ideas.

Recall that the derivative of a function f can be used to find the slopes of lines tangent to the graph of f. At $x = c$, the tangent line to the graph of f has equation

$$y = f'(c)(x - c) + f(c).$$

The tangent line can be used to find good approximations of $f(x)$ for values of x near c.

For instance, we can approximate $\sin 1.1$ using the tangent line to the graph of $f(x) = \sin x$ at $x = \pi/3 \approx 1.05$. Recall that $\sin(\pi/3) = \sqrt{3}/2 \approx 0.866$, and $\cos(\pi/3) = 1/2$. Thus the tangent line to $f(x) = \sin x$ at $x = \pi/3$ is:

$$\ell(x) = \frac{1}{2}(x - \pi/3) + 0.866.$$

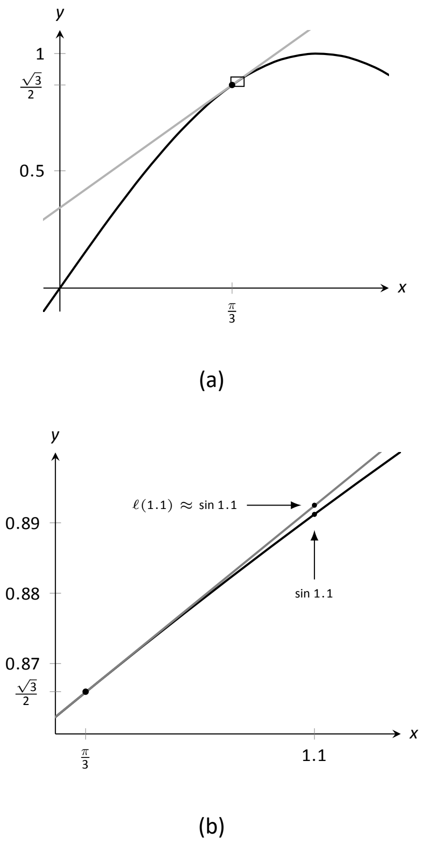


Figure 1.9: Graphing $f(x) = \sin x$ and its tangent line at $x = \pi/3$ in order to estimate $\sin 1.1$.

In Figure 1.9(a), we see a graph of $f(x) = \sin x$ graphed along with its tangent line at $x = \pi/3$. The small rectangle shows the region that is displayed in Figure 1.9(b). In this figure, we see how we are approximating $\sin 1.1$ with the tangent line, evaluated at 1.1. Together, the two figures show how close these values are.

Using this line to approximate $\sin 1.1$, we have:

$$\begin{aligned} \ell(1.1) &= \frac{1}{2}(1.1 - \pi/3) + 0.866 \\ &= \frac{1}{2}(0.053) + 0.866 = 0.8925. \end{aligned}$$

(We leave it to the reader to see how good of an approximation this is.)

We now generalize this concept. Given $f(x)$ and an x–value c, the tangent line is $\ell(x) = f'(c)(x-c) + f(c)$. Clearly, $f(c) = \ell(c)$. Let Δx be a small number, representing a small change in x value. We assert that:

$$f(c + \Delta x) \approx \ell(c + \Delta x),$$

since the tangent line to a function approximates well the values of that function near $x = c$.

As the x value changes from c to $c + \Delta x$, the y value of f changes from $f(c)$ to $f(c + \Delta x)$. We call this change of y value Δy. That is:

$$\Delta y = f(c + \Delta x) - f(c).$$

Notes:

Replacing $f(c + \Delta x)$ with its tangent line approximation, we have

$$\begin{aligned} \Delta y &\approx \ell(c + \Delta x) - f(c) \\ &= f'(c)\big((c + \Delta x) - c\big) + f(c) - f(c) \\ &= f'(c)\Delta x \end{aligned} \tag{1.1}$$

This final equation is important; we'll come back to it in Key Idea 1.

We introduce two new variables, dx and dy in the context of a formal definition.

Definition 3 Differentials of x and y.

Let $y = f(x)$ be differentiable. The **differential of** x, denoted dx, is any nonzero real number (usually taken to be a small number). The **differential of** y, denoted dy, is

$$dy = f'(x)dx.$$

We can solve for $f'(x)$ in the above equation: $f'(x) = dy/dx$. This states that the derivative of f with respect to x is the differential of y divided by the differential of x; this is **not** the alternate notation for the derivative, $\frac{dy}{dx}$. This latter notation was chosen because of the fraction–like qualities of the derivative, but again, it is one symbol and not a fraction.

It is helpful to organize our new concepts and notations in one place.

Key Idea 1 Differential Notation

Let $y = f(x)$ be a differentiable function.

1. Δx represents a small, nonzero change in x value.
2. dx represents a small, nonzero change in x value (i.e., $\Delta x = dx$).
3. Δy is the change in y value as x changes by Δx; hence

$$\Delta y = f(x + \Delta x) - f(x).$$

4. $dy = f'(x)dx$ which, by Equation (1.1), is an *approximation* of the change in y value as x changes by Δx; $dy \approx \Delta y$.

Notes:

What is the value of differentials? Like many mathematical concepts, differentials provide both practical and theoretical benefits. We explore both here.

Example 7 **Finding and using differentials**
Consider $f(x) = x^2$. Knowing $f(3) = 9$, approximate $f(3.1)$.

Solution The x value is changing from $x = 3$ to $x = 3.1$; therefore, we see that $dx = 0.1$. If we know how much the y value changes from $f(3)$ to $f(3.1)$ (i.e., if we know Δy), we will know exactly what $f(3.1)$ is (since we already know $f(3)$). We can approximate Δy with dy.

$$\begin{aligned}\Delta y &\approx dy \\ &= f'(3)dx \\ &= 2\cdot 3\cdot 0.1 = 0.6.\end{aligned}$$

We expect the y value to change by about 0.6, so we approximate $f(3.1) \approx 9.6$.

We leave it to the reader to verify this, but the preceding discussion links the differential to the tangent line of $f(x)$ at $x = 3$. One can verify that the tangent line, evaluated at $x = 3.1$, also gives $y = 9.6$.

Of course, it is easy to compute the actual answer (by hand or with a calculator): $3.1^2 = 9.61$. (Before we get too cynical and say “Then why bother?”, note our approximation is *really* good!)

So why bother?

In “most” real life situations, we do not know the function that describes a particular behavior. Instead, we can only take measurements of how things change – measurements of the derivative.

Imagine water flowing down a winding channel. It is easy to measure the speed and direction (i.e., the *velocity*) of water at any location. It is very hard to create a function that describes the overall flow, hence it is hard to predict where a floating object placed at the beginning of the channel will end up. However, we can *approximate* the path of an object using differentials. Over small intervals, the path taken by a floating object is essentially linear. Differentials allow us to approximate the true path by piecing together lots of short, linear paths. This technique is called Euler’s Method, studied in introductory Differential Equations courses.

We use differentials once more to approximate the value of a function. Even though calculators are very accessible, it is neat to see how these techniques can sometimes be used to easily compute something that looks rather hard.

Notes:

Example 8 **Using differentials to approximate a function value**

Approximate $\sqrt{4.5}$.

Solution We expect $\sqrt{4.5} \approx 2$, yet we can do better. Let $f(x) = \sqrt{x}$, and let $c = 4$. Thus $f(4) = 2$. We can compute $f'(x) = 1/(2\sqrt{x})$, so $f'(4) = 1/4$.

We approximate the difference between $f(4.5)$ and $f(4)$ using differentials, with $dx = 0.5$:

$$f(4.5) - f(4) = \Delta y \approx dy = f'(4) \cdot dx = 1/4 \cdot 1/2 = 1/8 = 0.125.$$

The approximate change in f from $x = 4$ to $x = 4.5$ is 0.125, so we approximate $\sqrt{4.5} \approx 2.125$.

Differentials are important when we discuss *integration*. When we study that topic, we will use notation such as

$$\int f(x)\, dx$$

quite often. While we don't discuss here what all of that notation means, note the existence of the differential dx. Proper handling of *integrals* comes with proper handling of differentials.

In light of that, we practice finding differentials in general.

Example 9 **Finding differentials**

In each of the following, find the differential dy.

1. $y = \sin x$ 2. $y = e^x(x^2 + 2)$ 3. $y = \sqrt{x^2 + 3x - 1}$

Solution

1. $y = \sin x$: As $f(x) = \sin x$, $f'(x) = \cos x$. Thus

$$dy = \cos(x)dx.$$

2. $y = e^x(x^2 + 2)$: Let $f(x) = e^x(x^2 + 2)$. We need $f'(x)$, requiring the Product Rule.

   We have $f'(x) = e^x(x^2 + 2) + 2xe^x$, so

$$dy = \big(e^x(x^2 + 2) + 2xe^x\big)dx.$$

Notes:

3. $y = \sqrt{x^2+3x-1}$: Let $f(x) = \sqrt{x^2+3x-1}$; we need $f'(x)$, requiring the Chain Rule.

   We have $f'(x) = \dfrac{1}{2}(x^2+3x-1)^{-\frac{1}{2}}(2x+3) = \dfrac{2x+3}{2\sqrt{x^2+3x-1}}$. Thus

$$dy = \frac{(2x+3)dx}{2\sqrt{x^2+3x-1}}.$$

Finding the differential dy of $y = f(x)$ is really no harder than finding the derivative of f; we just *multiply* $f'(x)$ by dx. It is important to remember that we are not simply adding the symbol "dx" at the end.

We have seen a practical use of differentials as they offer a good method of making certain approximations. Another use is *error propagation.* Suppose a length is measured to be x, although the actual value is $x+\Delta x$ (where we hope Δx is small). This measurement of x may be used to compute some other value; we can think of this as $f(x)$ for some function f. As the true length is $x+\Delta x$, one really should have computed $f(x+\Delta x)$. The difference between $f(x)$ and $f(x+\Delta x)$ is the propagated error.

How close are $f(x)$ and $f(x+\Delta x)$? This is a difference in "y" values;

$$f(x+\Delta x) - f(x) = \Delta y \approx dy.$$

We can approximate the propagated error using differentials.

Example 10 Using differentials to approximate propagated error

A steel ball bearing is to be manufactured with a diameter of 2cm. The manufacturing process has a tolerance of ± 0.1mm in the diameter. Given that the density of steel is about 7.85g/cm^3, estimate the propagated error in the mass of the ball bearing.

Solution The mass of a ball bearing is found using the equation "mass = volume $\times$ density." In this situation the mass function is a product of the radius of the ball bearing, hence it is $m = 7.85\frac{4}{3}\pi r^3$. The differential of the mass is

$$dm = 31.4\pi r^2 dr.$$

The radius is to be 1cm; the manufacturing tolerance in the radius is ± 0.05mm, or ± 0.005cm. The propagated error is approximately:

$$\begin{aligned}\Delta m &\approx dm \\ &= 31.4\pi(1)^2(\pm 0.005) \\ &= \pm 0.493\text{g}\end{aligned}$$

Notes:

Is this error significant? It certainly depends on the application, but we can get an idea by computing the *relative error*. The ratio between amount of error to the total mass is

$$\begin{aligned} \frac{dm}{m} &= \pm\frac{0.493}{7.85\frac{4}{3}\pi} \\ &= \pm\frac{0.493}{32.88} \\ &= \pm 0.015, \end{aligned}$$

or $\pm 1.5\%$.

We leave it to the reader to confirm this, but if the diameter of the ball was supposed to be 10cm, the same manufacturing tolerance would give a propagated error in mass of ± 12.33g, which corresponds to a *percent error* of $\pm 0.188\%$. While the amount of error is much greater ($12.33 > 0.493$), the percent error is much lower.

We first learned of the derivative in the context of instantaneous rates of change and slopes of tangent lines. We furthered our understanding of the power of the derivative by studying how it relates to the graph of a function (leading to ideas of increasing/decreasing and concavity). This chapter has put the derivative to yet more uses:

- Equation solving (Newton's Method)
- Related Rates (furthering our use of the derivative to find instantaneous rates of change)
- Optimization (applied extreme values), and
- Differentials (useful for various approximations and for something called integration).

In the next chapters, we will consider the "reverse" problem to computing the derivative: given a function *f*, can we find a function whose derivative is *f*? Being able to do so opens up an incredible world of mathematics and applications.

Notes:

Exercises 1.2

Terms and Concepts

1. T/F: Given a differentiable function $y = f(x)$, we are generally free to choose a value for dx, which then determines the value of dy.

2. T/F: The symbols "dx" and "Δx" represent the same concept.

3. T/F: The symbols "dy" and "Δy" represent the same concept.

4. T/F: Differentials are important in the study of integration.

5. How are differentials and tangent lines related?

Problems

In Exercises 6 – 17, use differentials to approximate the given value by hand.

6. 2.05^2

7. 5.93^2

8. 5.1^3

9. 6.8^3

10. $\sqrt{16.5}$

11. $\sqrt{24}$

12. $\sqrt[3]{63}$

13. $\sqrt[3]{8.5}$

14. $\sin 3$

15. $\cos 1.5$

16. $e^{0.1}$

In Exercises 17 – 29, compute the differential dy.

17. $y = x^2 + 3x - 5$

18. $y = x^7 - x^5$

19. $y = \dfrac{1}{4x^2}$

20. $y = (2x + \sin x)^2$

21. $y = x^2 e^{3x}$

22. $y = \dfrac{4}{x^4}$

23. $y = \dfrac{2x}{\tan x + 1}$

24. $y = \ln(5x)$

25. $y = e^x \sin x$

26. $y = \cos(\sin x)$

27. $y = \dfrac{x+1}{x+2}$

28. $y = 3^x \ln x$

29. $y = x \ln x - x$

30. A set of plastic spheres are to be made with a diameter of 1cm. If the manufacturing process is accurate to 1mm, what is the propagated error in volume of the spheres?

31. The distance, in feet, a stone drops in t seconds is given by $d(t) = 16t^2$. The depth of a hole is to be approximated by dropping a rock and listening for it to hit the bottom. What is the propagated error if the time measurement is accurate to $2/10^{\text{ths}}$ of a second and the measured time is:

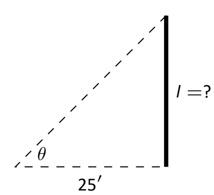
    (a) 2 seconds?
    (b) 5 seconds?

32. What is the propagated error in the measurement of the cross sectional area of a circular log if the diameter is measured at $15''$, accurate to $1/4''$?

33. A wall is to be painted that is $8'$ high and is measured to be $10',\ 7''$ long. Find the propagated error in the measurement of the wall's surface area if the measurement is accurate to $1/2''$.

Exercises 34 – 38 explore some issues related to surveying in which distances are approximated using other measured distances and measured angles. ***(Hint: Convert all angles to radians before computing.)***

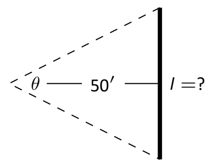
34. The length l of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85.2°, accurate to 1°. Assume that the triangle formed is a right triangle.



    (a) What is the measured length l of the wall?
    (b) What is the propagated error?
    (c) What is the percent error?

04 04 ex 36 35. Answer the questions of Exercise 34, but with a measured angle of 71.5°, accurate to 1°, measured from a point $100'$ from the wall.

04 04 ex 34 36. The length l of a long wall is to be calculated by measuring the angle θ shown in the diagram (not to scale). Assume the formed triangle is an isosceles triangle. The measured angle is 143°, accurate to 1°.



(a) What is the measured length of the wall?

(b) What is the propagated error?

(c) What is the percent error?

04 04 ex 37 37. The length of the walls in Exercises 34 – 36 are essentially the same. Which setup gives the most accurate result?

04 04 ex 38 38. Consider the setup in Exercise 36. This time, assume the angle measurement of 143° is exact but the measured $50'$ from the wall is accurate to $6''$. What is the approximate percent error?

1.3 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

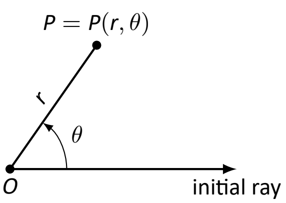


Figure 1.10: Illustrating polar coordinates.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 9.36

Practice will make this process more clear.

Example 11 Plotting Polar Coordinates
Plot the following polar coordinates:

$$A = P(1, \pi/4) \quad B = P(1.5, \pi) \quad C = P(2, -\pi/3) \quad D = P(-1, \pi/4)$$

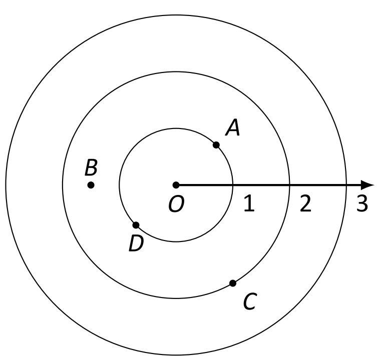


Figure 1.11: Plotting polar points in Example 298.

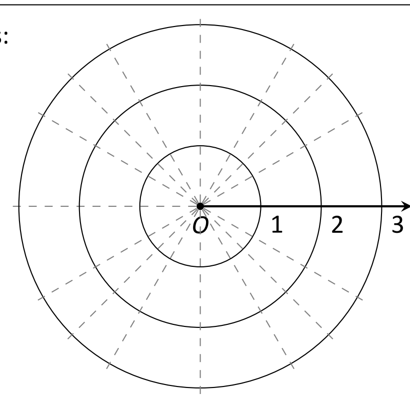
Solution To aid in the drawing, a polar grid is provided at the bottom of this page. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).

To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units – in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 9.37.

Notes:



Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 9.38 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

Key Idea 2 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

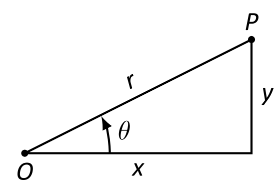


Figure 1.12: Converting between rectangular and polar coordinates.

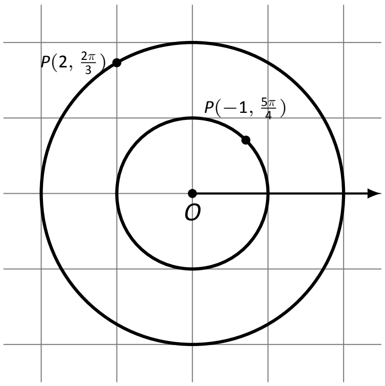
Example 12 Converting Between Polar and Rectangular Coordinates

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.

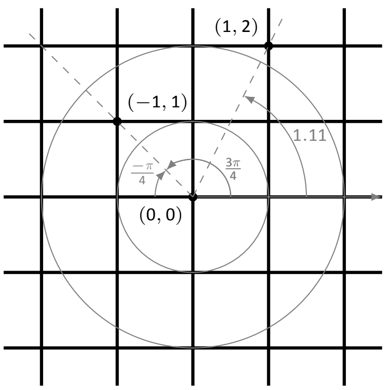
2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

Solution

Notes:



(a)



(b)

Figure 1.13: Plotting rectangular and polar points in Example 299.

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 40, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 9.39 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $P(\sqrt{5}, 1.11)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4 = -45^\circ.$$

This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1st and 4th quadrants. To find locations in the 2nd and 3rd quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P(\sqrt{2}, 3\pi/4)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $P(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 9.39 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.

Notes:

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 13 **Introduction to Graphing Polar Functions**
Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centered at $(0, 0)$ with radius 1.5. This is sketched in Figure 9.40.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45°$ with the initial ray.

   We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

   $$\tan\pi/4 = y/x \quad \Rightarrow x\tan\pi/4 = y \quad \Rightarrow y = x.$$

   This graph is also plotted in Figure 9.40.

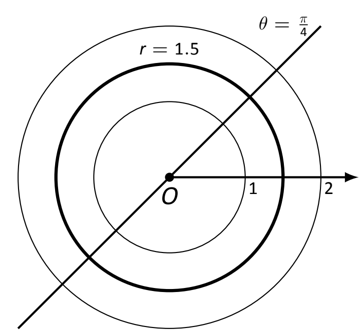


Figure 1.14: Plotting standard polar plots.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

Notes:

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 14 **Sketching Polar Functions**
Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

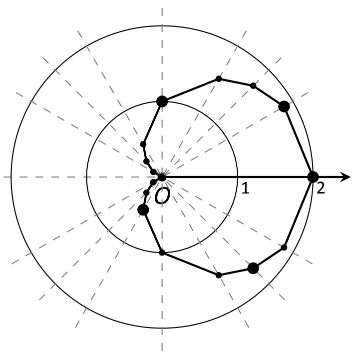


Figure 1.15: Graphing a polar function in Example 301 by plotting points.

SOLUTION A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values – integers, then add more if needed. When plotting polar equations, start with the "common" angles – multiples of $\pi/6$ and $\pi/4$. Figure 9.41 gives a table of just a few values of θ in $[0, \pi]$.

Consider the point $P(2, 0)$ determined by the first line of the table. The angle is 0 radians – we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is $1 + \sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.

The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the *x*-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called $\theta_{\mathtt{min}}$ and $\theta_{\mathtt{max}}$) as well as the $\theta_{\mathtt{step}}$ – that is, how far apart the θ values are spaced. The smaller the $\theta_{\mathtt{step}}$ value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1 + \cos\theta$ from Example 301 in Figure 9.42.

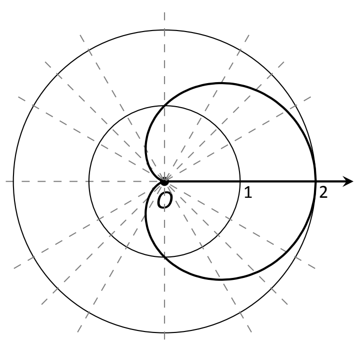


Figure 1.16: Using technology to graph a polar function.

Example 15 **Sketching Polar Functions**
Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

SOLUTION We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 9.43. These points are then plotted in Figure 9.44 (a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.

Notes:

Pt.	θ	$\cos(2\theta)$	Pt.	θ	$\cos(2\theta)$
1	0	1.	10	$7\pi/6$	0.5
2	$\pi/6$	0.5	11	$5\pi/4$	0.
3	$\pi/4$	0.	12	$4\pi/3$	-0.5
4	$\pi/3$	-0.5	13	$3\pi/2$	$-1.$
5	$\pi/2$	$-1.$	14	$5\pi/3$	-0.5
6	$2\pi/3$	-0.5	15	$7\pi/4$	0.
7	$3\pi/4$	0.	16	$11\pi/6$	0.5
8	$5\pi/6$	0.5	17	2π	1.
9	π	1.			

Figure 1.17: Tables of points for plotting a polar curve.

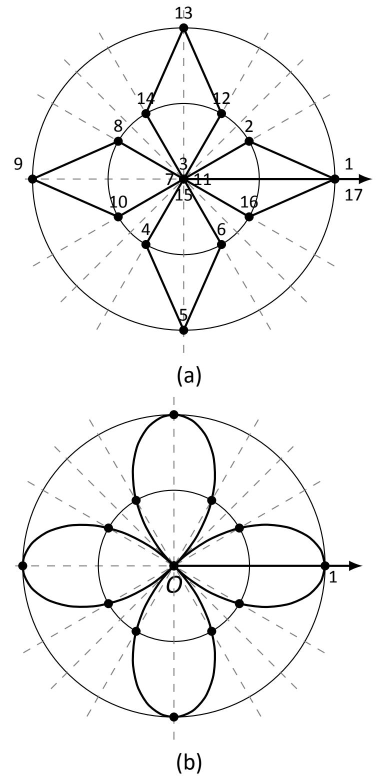


Figure 1.18: Polar plots from Example 302.

Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 9.44 (b). This plot is an example of a *rose curve*.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 40.

Example 16 **Converting between rectangular and polar equations.**

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

Solution

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

Notes:

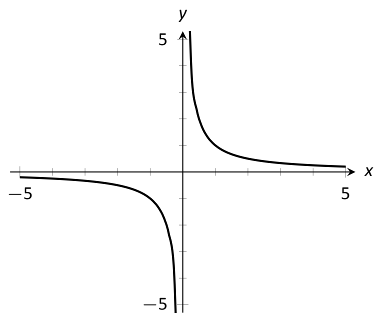


Figure 1.19: Graphing $xy = 1$ from Example 303.

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta \cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

   This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

   We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 9.45; note how it only exists in the first and third quadrants.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

   The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

Notes:

We recognize this as a circle; by completing the square we can find its radius and center.

$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1.$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

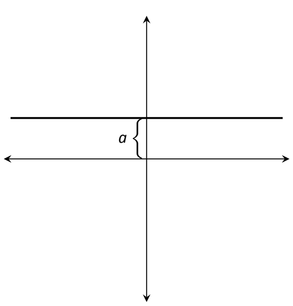
Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardiod* (a shape important the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

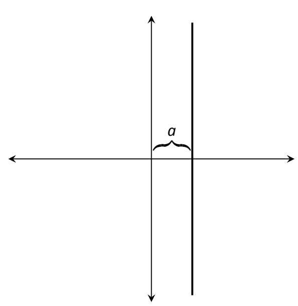
Horizontal line:

$r = a\csc\theta$



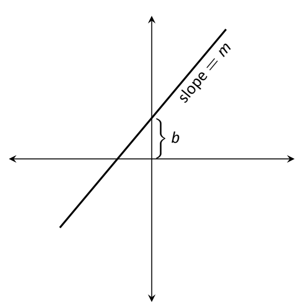
Vertical line:

$r = a\sec\theta$



Not through origin:

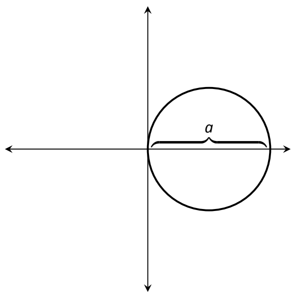
$$r = \frac{b}{\sin\theta - m\cos\theta}$$



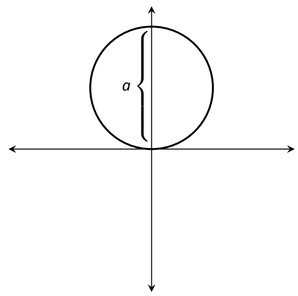
Notes:

Circles

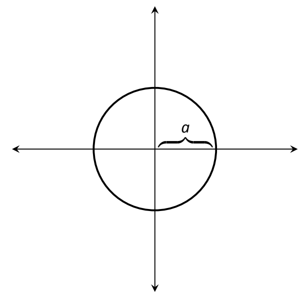
Centered on x-axis:
$r = a\cos\theta$



Centered on y-axis:
$r = a\sin\theta$

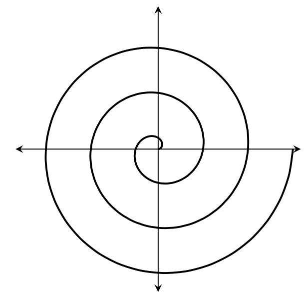


Centered on origin:
$r = a$



Spiral

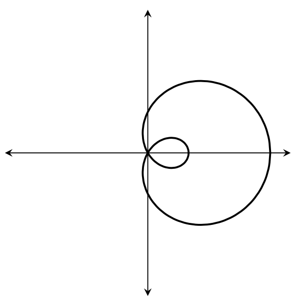
Archimedean spiral
$r = \theta$



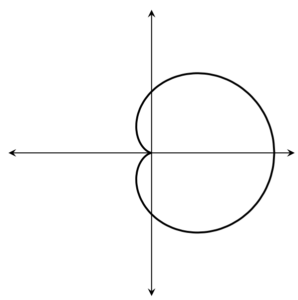
Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$;   Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

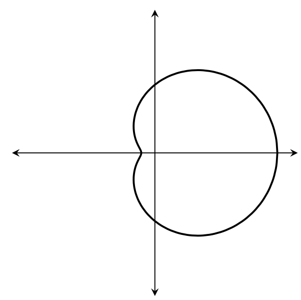
With inner loop:
$\frac{a}{b} < 1$



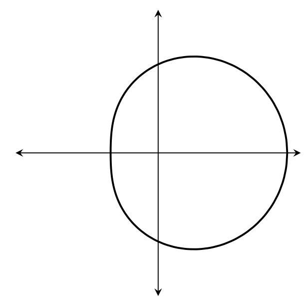
Cardiod:
$\frac{a}{b} = 1$



Dimpled:
$1 < \frac{a}{b} < 2$



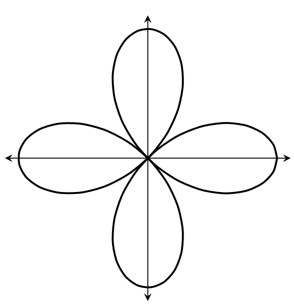
Convex:
$\frac{a}{b} > 2$



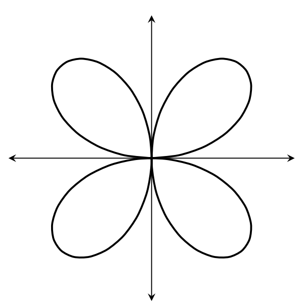
Rose Curves

Symmetric about x-axis: $r = a\cos(n\theta)$;   Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

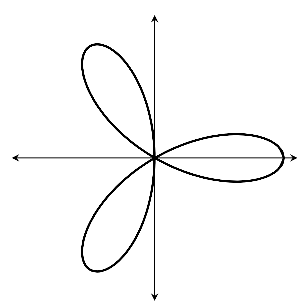
$r = a\cos(2\theta)$



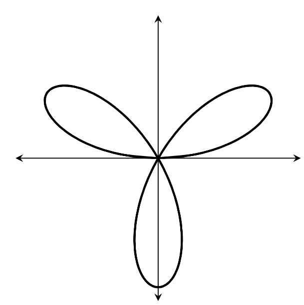
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



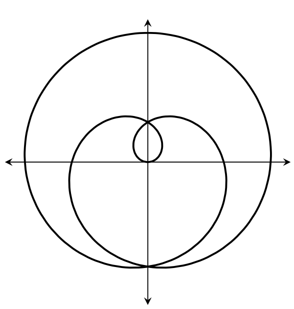
$r = a\sin(3\theta)$



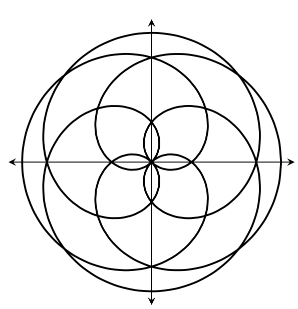
Special Curves

Rose curves

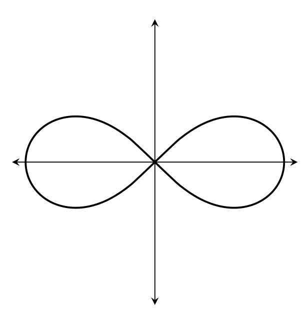
$r = a\sin(\theta/5)$



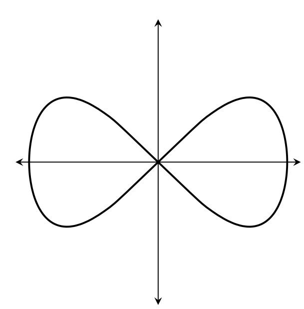
$r = a\sin(2\theta/5)$



Lemniscate:
$r^2 = a^2\cos(2\theta)$



Eight Curve:
$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 17 Finding points of intersection with polar curves

Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.

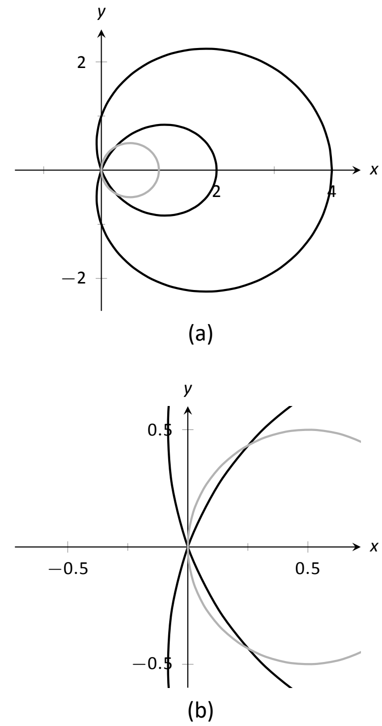


Figure 1.20: Graphs to help determine the points of intersection of the polar functions given in Example 304.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 9.46(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned}1+3\cos\theta &= \cos\theta\\ 2\cos\theta &= -1\\ \cos\theta &= -\frac{1}{2}\\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}.\end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4$^{\text{th}}$ quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 2$^{\text{nd}}$ quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly

Notes:

traced). The limaçon intersects the pole when $1+3\cos\theta=0$; this occurs when $\cos\theta=-1/3$, or for $\theta=\cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta=1.9106=109.47^\circ$. The limaçon intersects the pole twice in $[0,2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta=4.3726=250.53^\circ$.

If all one is concerned with is the (x,y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0,0)$; we might not care at what θ value. Likewise, using $\theta=2\pi/3$ and $\theta=4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Notes:

Exercises 1.3

Terms and Concepts

09 04 ex 01 1. In your own words, describe how to plot the polar point $P(r, \theta)$.

09 04 ex 02 2. T/F: When plotting a point with polar coordinate $P(r, \theta)$, r must be positive.

09 04 ex 03 3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.

09 04 ex 04 4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

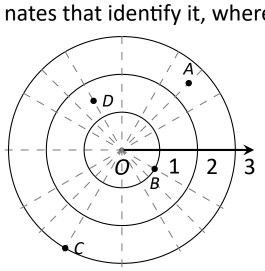
09 04 ex 05 5. Plot the points with the given polar coordinates.

(a) $A = P(2, 0)$
(b) $B = P(1, \pi)$
(c) $C = P(-2, \pi/2)$
(d) $D = P(1, \pi/4)$

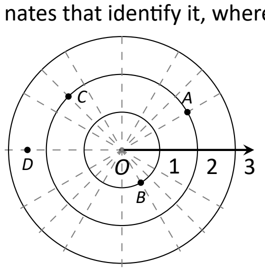
09 04 ex 06 6. Plot the points with the given polar coordinates.

(a) $A = P(2, 3\pi)$
(b) $B = P(1, -\pi)$
(c) $C = P(1, 2)$
(d) $D = P(1/2, 5\pi/6)$

09 04 ex 07 7. For each of the given points give two sets of polar coordinates that identify it, where $0 \leq \theta \leq 2\pi$.



09 04 ex 08 8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \leq \theta \leq \pi$.



09 04 ex 09 9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

(a) $A = P(2, \pi/4)$
(b) $B = P(2, -\pi/4)$
(c) $C = (2, -1)$
(d) $D = (-2, 1)$

09 04 ex 10 10. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

(a) $A = P(3, \pi)$
(b) $B = P(1, 2\pi/3)$
(c) $C = (0, 4)$
(d) $D = (1, -\sqrt{3})$

09 04 exset 01 **In Exercises 11 – 29, graph the polar function on the given interval.**

09 04 ex 11 11. $r = 2, \quad 0 \leq \theta \leq \pi/2$

09 04 ex 12 12. $\theta = \pi/6, \quad -1 \leq r \leq 2$

09 04 ex 13 13. $r = 1 - \cos\theta, \quad [0, 2\pi]$

09 04 ex 14 14. $r = 2 + \sin\theta, \quad [0, 2\pi]$

09 04 ex 15 15. $r = 2 - \sin\theta, \quad [0, 2\pi]$

09 04 ex 16 16. $r = 1 - 2\sin\theta, \quad [0, 2\pi]$

09 04 ex 17 17. $r = 1 + 2\sin\theta, \quad [0, 2\pi]$

09 04 ex 18 18. $r = \cos(2\theta), \quad [0, 2\pi]$

09 04 ex 19 19. $r = \sin(3\theta), \quad [0, \pi]$

09 04 ex 20 20. $r = \cos(\theta/3), \quad [0, 3\pi]$

09 04 ex 21 21. $r = \cos(2\theta/3), \quad [0, 6\pi]$

09 04 ex 22 22. $r = \theta/2, \quad [0, 4\pi]$

09 04 ex 23 23. $r = 3\sin(\theta), \quad [0, \pi]$

09 04 ex 24 24. $r = \cos\theta\sin\theta, \quad [0, 2\pi]$

09 04 ex 25 25. $r = \theta^2 - (\pi/2)^2, \quad [-\pi, \pi]$

09 04 ex 26 26. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0, 2\pi]$

09 04 ex 27 27. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0, 2\pi]$

09 04 ex 28 28. $r = 3\sec\theta, \quad (-\pi/2, \pi/2)$

09 04 ex 29 29. $r = 3\csc\theta, \quad (0, \pi)$

09 04 exset 02 **In Exercises 30 – 38, convert the polar equation to a rectangular equation.**

09 04 ex 30 30. $r = 2\cos\theta$

09 04 ex 31 31. $r = -4\sin\theta$

09 04 ex 32 32. $r = \cos\theta + \sin\theta$

09 04 ex 33 33. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$

09 04 ex 34 34. $r = \dfrac{3}{\cos\theta}$

09 04 ex 35 35. $r = \dfrac{4}{\sin\theta}$

09 04 ex 36 36. $r = \tan\theta$

09 04 ex 37 37. $r = 2$

09 04 ex 38 38. $\theta = \pi/6$

09 04 exset 03 **In Exercises 39 – 46, convert the rectangular equation to a polar equation.**

09 04 ex 39 39. $y = x$

09 04 ex 40 40. $y = 4x + 7$

09 04 ex 41 41. $x = 5$

09 04 ex 42 42. $y = 5$

09 04 ex 43 43. $x = y^2$

09 04 ex 44 44. $x^2y = 1$

09 04 ex 45 45. $x^2 + y^2 = 7$

09 04 ex 46 46. $(x+1)^2 + y^2 = 1$

09 04 exset 04 **In Exercises 47 – 54, find the points of intersection of the polar graphs.**

09 04 ex 47 47. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

09 04 ex 48 48. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

09 04 ex 49 49. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

09 04 ex 50 50. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

09 04 ex 51 51. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

09 04 ex 52 52. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

09 04 ex 53 53. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

09 04 ex 54 54. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

09 04 ex 55 55. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\dfrac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

09 04 ex 56 56. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

01 06 ex 01 1. F

01 06 ex 02 2. T

01 06 ex 03 3. F

01 06 ex 04 4. T

01 06 ex 05 5. T

01 06 ex 06 6. Answers will vary.

01 06 ex 07 7. Answers will vary.

01 06 ex 08 8. The limit of f as x approaches 7 does not exist, hence f is not continuous. (Note: f could be defined at 7!)

01 06 ex 09 9.

(a) ∞

(b) ∞

01 06 ex 10 10.

(a) $-\infty$

(b) ∞

(c) Limit does not exist

(d) ∞

(e) ∞

(f) ∞

01 06 ex 11 11.

(a) 1

(b) 0

(c) $1/2$

(d) $1/2$

01 06 ex 12 12.

(a) Limit does not exist

(b) Limit does not exist

01 06 ex 13 13.

(a) Limit does not exist

(b) Limit does not exist

01 06 ex 40 14.

(a) 10

(b) ∞

01 06 ex 14 15. Tables will vary.

(a)

x	$f(x)$
2.9	-15.1224
2.99	-159.12
2.999	-1599.12

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	16.8824
3.01	160.88
3.001	1600.88

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x)$ does not exist.

01 06 ex 15 16. Tables will vary.

(a)

x	$f(x)$
2.9	-335.64
2.99	-30350.6

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	-265.61
3.01	-29650.6

It seems $\lim_{x\to 3^+} f(x) = -\infty$.

(c) It seems $\lim_{x\to 3} f(x) = -\infty$.

01 06 ex 16 17. Tables will vary.

(a)

x	$f(x)$
2.9	132.857
2.99	12124.4

It seems $\lim_{x\to 3^-} f(x) = \infty$.

(b)

x	$f(x)$
3.1	108.039
3.01	11876.4

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x) = \infty$.

01 06 ex 17 18. Tables will vary.

(a)

x	$f(x)$
2.9	-0.632
2.99	-0.6032
2.999	-0.60032

It seems $\lim_{x\to 3^-} f(x) = -0.6$.

(b)

x	$f(x)$
3.1	-0.5686
3.01	-0.5968
3.001	-0.59968

It seems $\lim_{x\to 3^+} f(x) = -0.6$.

(c) It seems $\lim_{x\to 3} f(x) = -0.6$.

01 06 ex 18 19. Horizontal asymptote at $y = 2$; vertical asymptotes at $x = -5, 4$.

01 06 ex 19 20. Horizontal asymptote at $y = -3/5$; vertical asymptote at $x = 3$.

01 06 ex 20 21. Horizontal asymptote at $y = 0$; vertical asymptotes at $x = -1, 0$.

01 06 ex 21 22. No horizontal asymptote; vertical asymptote at $x = 1$.

01 06 ex 22 23. No horizontal or vertical asymptotes.

01 06 ex 41 24. Horizontal asymptote at $y = -1$; no vertical asymptotes

01 06 ex 23 25. ∞

01 06 ex 24 26. $-\infty$

01 06 ex 25 27. $-\infty$

01 06 ex 26 28. ∞

01 06 ex 27 29. Solution omitted.

01 06 ex 28 30.

(a) 2

(b) -3

(c) -3

(d) $1/3$

01 06 ex 29 31. Yes. The only "questionable" place is at $x = 3$, but the left and right limits agree.

01 06 ex 30 32. 1

Section 1.2

04 04 ex 01 1. T

04 04 ex 02 2. T

04 04 ex 03 3. F

04 04 ex 04 4. T

04 04 ex 05 5. Answers will vary.

04 04 ex 06 6. Use $y = x^2$; $dy = 2x \cdot dx$ with $x = 2$ and $dx = 0.05$. Thus $dy = .2$; knowing $2^2 = 4$, we have $2.05^2 \approx 4.2$.

04 04 ex 07 7. Use $y = x^2$; $dy = 2x \cdot dx$ with $x = 6$ and $dx = -0.07$. Thus $dy = -0.84$; knowing $6^2 = 36$, we have $5.93^2 \approx 35.16$.

04 04 ex 08 8. Use $y = x^3$; $dy = 3x^2 \cdot dx$ with $x = 5$ and $dx = 0.1$. Thus $dy = 7.5$; knowing $5^3 = 125$, we have $5.1^3 \approx 132.5$.

04 04 ex 09 9. Use $y = x^3$; $dy = 3x^2 \cdot dx$ with $x = 7$ and $dx = -0.2$. Thus $dy = -29.4$; knowing $7^3 = 343$, we have $6.8^3 \approx 313.6$.

04 04 ex 10 10. Use $y = \sqrt{x}$; $dy = 1/(2\sqrt{x}) \cdot dx$ with $x = 16$ and $dx = 0.5$. Thus $dy = .0625$; knowing $\sqrt{16} = 4$, we have $\sqrt{16.5} \approx 4.0625$.

04 04 ex 11 11. Use $y = \sqrt{x}$; $dy = 1/(2\sqrt{x}) \cdot dx$ with $x = 25$ and $dx = -1$. Thus $dy = -0.1$; knowing $\sqrt{25} = 5$, we have $\sqrt{24} \approx 4.9$.

04 04 ex 12 12. Use $y = \sqrt[3]{x}$; $dy = 1/(3\sqrt[3]{x^2}) \cdot dx$ with $x = 64$ and $dx = -1$. Thus $dy = -1/48 \approx 0.0208$; we could use $-1/48 \approx -1/50 = -0.02$; knowing $\sqrt[3]{64} = 4$, we have $\sqrt[3]{63} \approx 3.98$.

04 04 ex 13 13. Use $y = \sqrt[3]{x}$; $dy = 1/(3\sqrt[3]{x^2}) \cdot dx$ with $x = 8$ and $dx = 0.5$. Thus $dy = 1/24 \approx 1/25 = 0.04$; knowing $\sqrt[3]{8} = 2$, we have $\sqrt[3]{8.5} \approx 2.04$.

04 04 ex 14 14. Use $y = \sin x$; $dy = \cos x \cdot dx$ with $x = \pi$ and $dx \approx -0.14$. Thus $dy = 0.14$; knowing $\sin \pi = 0$, we have $\sin 3 \approx 0.14$.

04 04 ex 15 15. Use $y = \cos x$; $dy = -\sin x \cdot dx$ with $x = \pi/2 \approx 1.57$ and $dx \approx -0.07$. Thus $dy = 0.07$; knowing $\cos \pi/2 = 0$, we have $\cos 1.5 \approx 0.07$.

04 04 ex 16 16. Use $y = e^x$; $dy = e^x \cdot dx$ with $x = 0$ and $dx = 0.1$. Thus $dy = 0.1$; knowing $e^0 = 1$, we have $e^{0.1} \approx 1.1$.

04 04 ex 17 17. $dy = (2x + 3)dx$

04 04 ex 18 18. $dy = (7x^6 - 5x^4)dx$

04 04 ex 19 19. $dy = \frac{-2}{4x^3}dx$

04 04 ex 20 20. $dy = 2(2x + \sin x)(2 + \cos x)dx$

04 04 ex 21 21. $dy = \left(2xe^{3x} + 3x^2e^{3x}\right)dx$

04 04 ex 22 22. $dy = \frac{-16}{x^5}dx$

04 04 ex 23 23. $dy = \frac{2(\tan x + 1) - 2x \sec^2 x}{(\tan x + 1)^2}dx$

04 04 ex 24 24. $dy = \frac{1}{x}dx$

04 04 ex 25 25. $dy = (e^x \sin x + e^x \cos x)dx$

04 04 ex 26 26. $dy = (-\sin(\sin x) \cos x)dx$

04 04 ex 27 27. $dy = \frac{1}{(x+2)^2}dx$

04 04 ex 28 28. $dy = ((\ln 3)3^x \ln x + \frac{3^x}{x})dx$

04 04 ex 29 29. $dy = (\ln x)dx$

04 04 ex 30 30. $dV = \pm 0.157$

04 04 ex 31 31.

(a) ± 12.8 feet

(b) ± 32 feet

04 04 ex 32 32. $\pm 15\pi/8 \approx \pm 5.89\text{in}^2$

04 04 ex 33 33. $\pm 48\text{in}^2$, or $1/3\text{ft}^2$

04 04 ex 35 34.

(a) 297.8 feet

(b) ± 62.3 ft

(c) $\pm 20.9\%$

04 04 ex 36 35.

(a) 298.8 feet

(b) ± 17.3 ft

(c) $\pm 5.8\%$

04 04 ex 34 36.

(a) 298.9 feet

(b) ± 8.67 ft

(c) $\pm 2.9\%$

04 04 ex 37 37. The isosceles triangle setup works the best with the smallest percent error.

04 04 ex 38 38. 1%

Section 1.3

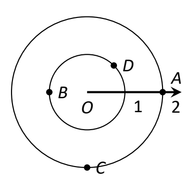
09 04 ex 01 1. Answers will vary.

09 04 ex 02 2. F

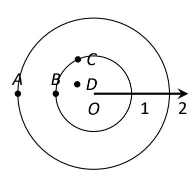
09 04 ex 03 3. T

09 04 ex 04 4. F

09 04 ex 05 5.



09 04 ex 06 6.



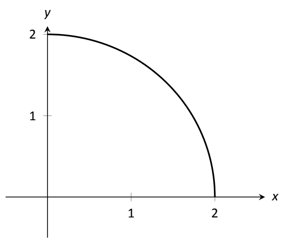
09 04 ex 07 7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
$B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
$C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
$D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

09 04 ex 08 8. $A = P(2, \pi/6)$ and $P(-2, -5\pi/6)$;
$B = P(1, -\pi/3)$ and $P(-1, 2\pi/3)$;
$C = P(2, 3\pi/4)$ and $P(-2, -\pi/4)$;
$D = P(2.5, \pi)$ and $P(2.5, -\pi)$;

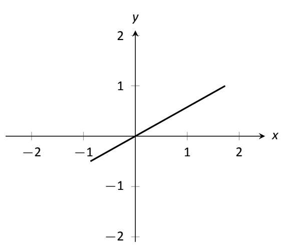
09 04 ex 09 9. $A = (\sqrt{2}, \sqrt{2})$
$B = (\sqrt{2}, -\sqrt{2})$
$C = P(\sqrt{5}, -0.46)$
$D = P(\sqrt{5}, 2.68)$

09 04 ex 10 10. $A = (-3, 0)$
$B = (-1/2, \sqrt{3}/2)$
$C = P(4, \pi/2)$
$D = P(2, -\pi/3)$

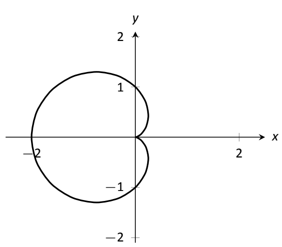
09 04 ex 11 11.



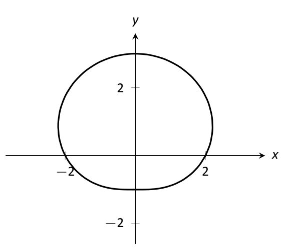
09 04 ex 12 12.



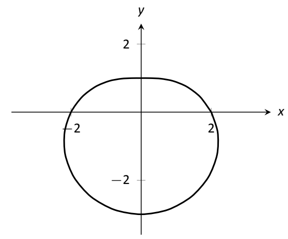
09 04 ex 13 13.



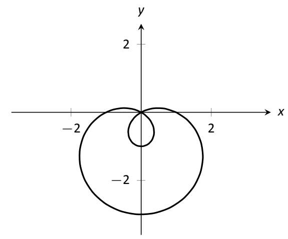
09 04 ex 14 14.



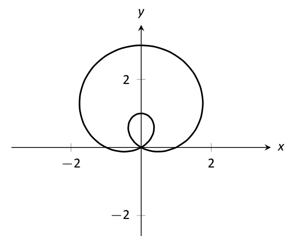
09 04 ex 15 15.



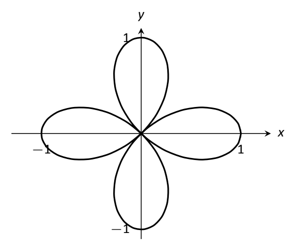
09 04 ex 16 16.



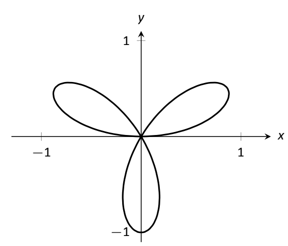
09 04 ex 17 17.



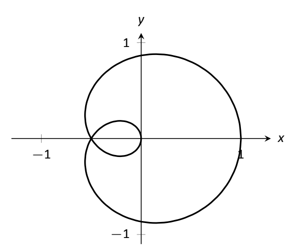
09 04 ex 18 18.



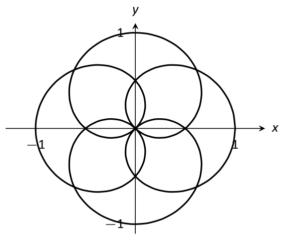
09 04 ex 19 19.



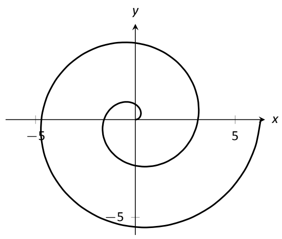
09 04 ex 20 20.



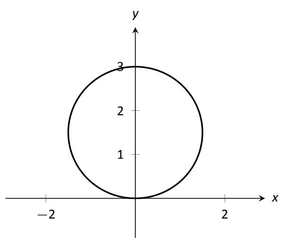
09 04 ex 21 21.



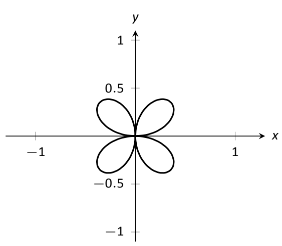
09 04 ex 22 22.



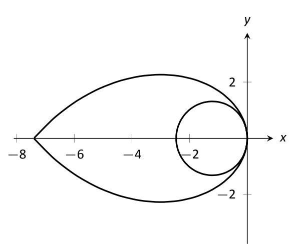
09 04 ex 23 23.



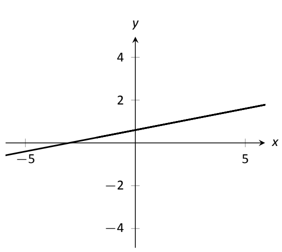
09 04 ex 24 24.



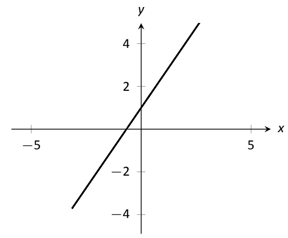
09 04 ex 25 25.



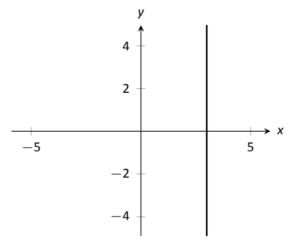
09 04 ex 26 26.



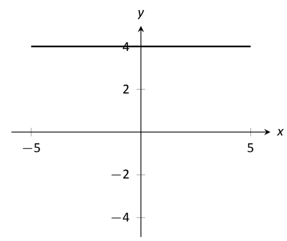
09 04 ex 27 27.



09 04 ex 28 28.



09 04 ex 29 29.



09 04 ex 30 30. $(x-1)^2 + y^2 = 1$

09 04 ex 31 31. $x^2 + (y+2)^2 = 4$

09 04 ex 32 32. $(x-1/2)^2 + (y-1/2)^2 = 1/2$

09 04 ex 33 33. $y = 2/5x + 7/5$

09 04 ex 34 34. $x = 3$

09 04 ex 35 35. $y = 4$

09 04 ex 36 36. $x^4 + x^2y^2x^2 - y^2 = 0$

09 04 ex 37 37. $x^2 + y^2 = 4$

09 04 ex 38 38. $y = x/\sqrt{3}$

09 04 ex 39 39. $\theta = \pi/4$

09 04 ex 40 40. $r = 7/(\sin\theta - 4\cos\theta)$

09 04 ex 41 41. $r = 5\sec\theta$

09 04 ex 42 42. $r = 5\csc\theta$

09 04 ex 43 43. $r = \cos\theta/\sin^2\theta$

09 04 ex 44 44. $r = 1/\sqrt[3]{\cos^2\theta\sin\theta}$

09 04 ex 45 45. $r = \sqrt{7}$

09 04 ex 46 46. $r = -2\cos\theta$

09 04 ex 47 47. $P(\sqrt{3}/2, \pi/6)$, $P(0, \pi/2)$, $P(-\sqrt{3}/2, 5\pi/6)$

09 04 ex 48 48. $P(1, 0)$, $P(0, \pi/2) = P(0, \pi/4)$, $P(-1/2, \pi/3)$

09 04 ex 49 49. $P(0, 0) = P(0, \pi/2)$, $P(\sqrt{2}, \pi/4)$

09 04 ex 50 50. $P(\sqrt{3}/2, \pi/3) = P(-\sqrt{3}/2, 4\pi/3)$,
$P(\sqrt{3}/2, 2\pi/3) = P(-\sqrt{3}/2, 5\pi/3)$, $P(0, \pi/2)$

09 04 ex 51 51. $P(\sqrt{2}/2, \pi/12)$, $P(-\sqrt{2}/2, 5\pi/12)$, $P(\sqrt{2}/2, 3\pi/4)$

09 04 ex 52 52. $P(3/2, \pi/3)$, $P(3/2, -\pi/3)$

09 04 ex 53 53. For all points, $r = 1$; $\theta =$
$\pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12,\ 13\pi/12,\ 17\pi/12,\ 19\pi/12,\ 23\pi/12.$

09 04 ex 54 54. $P(0,0) = P(0, 3\pi/2)$, $P(1+\sqrt{2}/2, 3\pi/4)$, $P(1-\sqrt{2}/2, 7\pi/4)$

09 04 ex 55 55. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

09 04 ex 56 56. Answers will vary.