12: Functions of Several Variables

A function of the form $y = f(x)$ is a function of a single variable; given a value of x, we can find a value y. Even the vector–valued functions of Chapter 11 are single–variable functions; the input is a single variable though the output is a vector.

There are many situations where a desired quantity is a function of two or more variables. For instance, wind chill is measured by knowing the temperature and wind speed; the volume of a gas can be computed knowing the pressure and temperature of the gas; to compute a baseball player's batting average, one needs to know the number of hits and the number of at–bats.

This chapter studies **multivariable** functions, that is, functions with more than one input.

12.1 Introduction to Multivariable Functions

Definition 12.1.1 Function of Two Variables

Let D be a subset of $\mathbb{R}^2$. A **function f of two variables** is a rule that assigns each pair (x, y) in D a value $z = f(x, y)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Example 12.1.1 Understanding a function of two variables

Let $z = f(x, y) = x^2 - y$. Evaluate $f(1, 2)$, $f(2, 1)$, and $f(-2, 4)$; find the domain and range of f.

Solution Using the definition $f(x, y) = x^2 - y$, we have:

$$\begin{aligned} f(1,2) &= 1^2 - 2 = -1 \\ f(2,1) &= 2^2 - 1 = 3 \\ f(-2,4) &= (-2)^2 - 4 = 0 \end{aligned}$$

The domain is not specified, so we take it to be all possible pairs in $\mathbb{R}^2$ for which f is defined. In this example, f is defined for *all* pairs (x, y), so the domain D of f is $\mathbb{R}^2$.

The output of f can be made as large or small as possible; any real number r can be the output. (In fact, given any real number r, $f(0, -r) = r$.) So the range R of f is $\mathbb{R}$.

Example 12.1.2 Understanding a function of two variables

Let $f(x,y)=\sqrt{1-\frac{x^2}{9}-\frac{y^2}{4}}$. Find the domain and range of f.

SOLUTION The domain is all pairs (x,y) allowable as input in f. Because of the square–root, we need (x,y) such that $0\leq 1-\frac{x^2}{9}-\frac{y^2}{4}$:

$$0\leq 1-\frac{x^2}{9}-\frac{y^2}{4}$$
$$\frac{x^2}{9}+\frac{y^2}{4}\leq 1$$

The above equation describes an ellipse and its interior as shown in Figure 12.1.1. We can represent the domain D graphically with the figure; in set notation, we can write $D=\{(x,y)|\ \frac{x^2}{9}+\frac{y^2}{4}\leq 1\}$.

The range is the set of all possible output values. The square–root ensures that all output is ≥ 0. Since the x and y terms are squared, then subtracted, inside the square–root, the largest output value comes at $x=0$, $y=0$: $f(0,0)=1$. Thus the range R is the interval $[0,1]$.

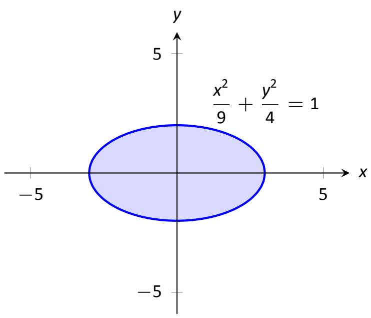

Figure 12.1.1: Illustrating the domain of $f(x,y)$ in Example 12.1.2.

Graphing Functions of Two Variables

The **graph** of a function f of two variables is the set of all points $\big(x,y,f(x,y)\big)$ where (x,y) is in the domain of f. This creates a **surface** in space.

One can begin sketching a graph by plotting points, but this has limitations. Consider Figure 12.1.2(a) where 25 points have been plotted of $f(x,y)=\dfrac{1}{x^2+y^2+1}$. More points have been plotted than one would reasonably want to do by hand, yet it is not clear at all what the graph of the function looks like. Technology allows us to plot lots of points, connect adjacent points with lines and add shading to create a graph like Figure 12.1.2b which does a far better job of illustrating the behavior of f.

While technology is readily available to help us graph functions of two variables, there is still a paper–and–pencil approach that is useful to understand and master as it, combined with high–quality graphics, gives one great insight into the behavior of a function. This technique is known as sketching **level curves**.

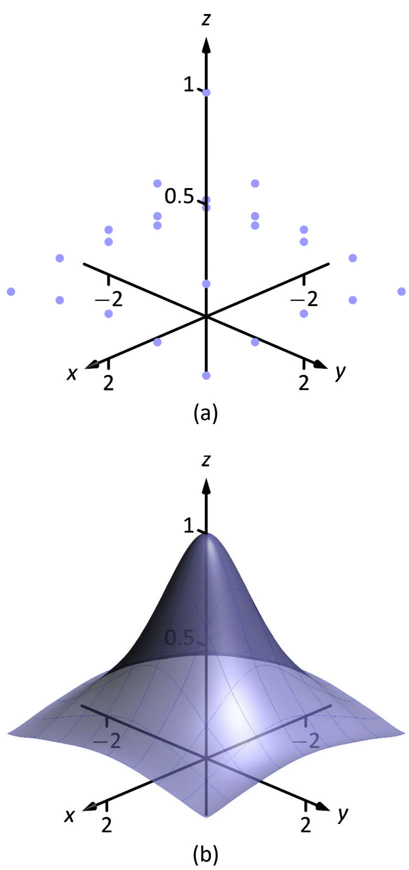

Figure 12.1.2: Graphing a function of two variables.

Level Curves

It may be surprising to find that the problem of representing a three dimensional surface on paper is familiar to most people (they just don't realize it). Topographical maps, like the one shown in Figure 12.1.3, represent the surface of Earth by indicating points with the same elevation with **contour lines**. The

Notes:

elevations marked are equally spaced; in this example, each thin line indicates an elevation change in 50ft increments and each thick line indicates a change of 200ft. When lines are drawn close together, elevation changes rapidly (as one does not have to travel far to rise 50ft). When lines are far apart, such as near "Aspen Campground," elevation changes more gradually as one has to walk farther to rise 50ft.

Given a function $z = f(x, y)$, we can draw a "topographical map" of f by drawing **level curves** (or, contour lines). A level curve at $z = c$ is a curve in the x-y plane such that for all points (x, y) on the curve, $f(x, y) = c$.

When drawing level curves, it is important that the c values are spaced equally apart as that gives the best insight to how quickly the "elevation" is changing. Examples will help one understand this concept.

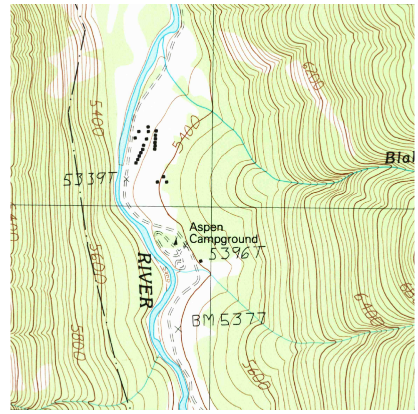

Figure 12.1.3: A topographical map displays elevation by drawing contour lines, along with the elevation is constant.
Sample taken from the public domain USGS Digital Raster Graphics, http://topmaps.usgs.gove/drg/.

Example 12.1.3 Drawing Level Curves

Let $f(x, y) = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Find the level curves of f for $c = 0$, 0.2, 0.4, 0.6, 0.8 and 1.

Solution Consider first $c = 0$. The level curve for $c = 0$ is the set of all points (x, y) such that $0 = \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}}$. Squaring both sides gives us

$$\frac{x^2}{9} + \frac{y^2}{4} = 1,$$

an ellipse centered at $(0, 0)$ with horizontal major axis of length 6 and minor axis of length 4. Thus for any point (x, y) on this curve, $f(x, y) = 0$.

Now consider the level curve for $c = 0.2$

$$\begin{aligned} 0.2 &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ 0.04 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 0.96 \\ \frac{x^2}{8.64} + \frac{y^2}{3.84} &= 1. \end{aligned}$$

This is also an ellipse, where $a = \sqrt{8.64} \approx 2.94$ and $b = \sqrt{3.84} \approx 1.96$.

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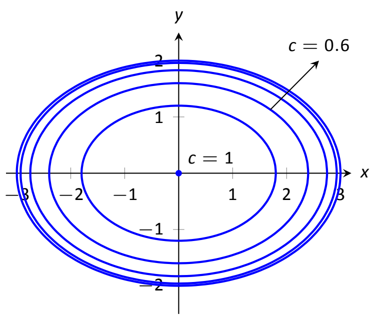
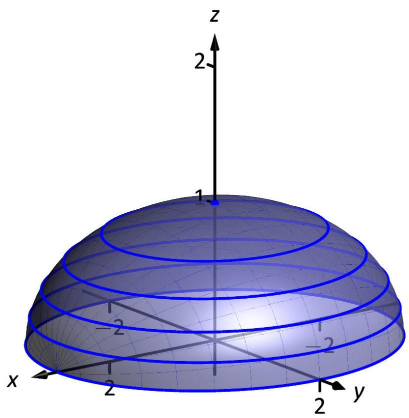


(a)



(b)

Figure 12.1.4: Graphing the level curves in Example 12.1.3.

In general, for $z = c$, the level curve is:

$$\begin{aligned} c &= \sqrt{1 - \frac{x^2}{9} - \frac{y^2}{4}} \\ c^2 &= 1 - \frac{x^2}{9} - \frac{y^2}{4} \\ \frac{x^2}{9} + \frac{y^2}{4} &= 1 - c^2 \\ \frac{x^2}{9(1-c^2)} + \frac{y^2}{4(1-c^2)} &= 1, \end{aligned}$$

ellipses that are decreasing in size as c increases. A special case is when $c = 1$; there the ellipse is just the point $(0, 0)$.

The level curves are shown in Figure 12.1.4(a). Note how the level curves for $c = 0$ and $c = 0.2$ are very, very close together: this indicates that f is growing rapidly along those curves.

In Figure 12.1.4(b), the curves are drawn on a graph of f in space. Note how the elevations are evenly spaced. Near the level curves of $c = 0$ and $c = 0.2$ we can see that f indeed is growing quickly.

Example 12.1.4 Analyzing Level Curves

Let $f(x, y) = \dfrac{x + y}{x^2 + y^2 + 1}$. Find the level curves for $z = c$.

Solution We begin by setting $f(x, y) = c$ for an arbitrary c and seeing if algebraic manipulation of the equation reveals anything significant.

$$\begin{aligned} \frac{x+y}{x^2+y^2+1} &= c \\ x + y &= c(x^2 + y^2 + 1). \end{aligned}$$

We recognize this as a circle, though the center and radius are not yet clear. By completing the square, we can obtain:

$$\left(x - \frac{1}{2c}\right)^2 + \left(y - \frac{1}{2c}\right)^2 = \frac{1}{2c^2} - 1,$$

a circle centered at $\big(1/(2c), 1/(2c)\big)$ with radius $\sqrt{1/(2c^2) - 1}$, where $|c| < 1/\sqrt{2}$. The level curves for $c = \pm 0.2$, ± 0.4 and ± 0.6 are sketched in Figure 12.1.5(a). To help illustrate "elevation," we use thicker lines for c values near 0, and dashed lines indicate where $c < 0$.

There is one special level curve, when $c = 0$. The level curve in this situation is $x + y = 0$, the line $y = -x$.

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In Figure 12.1.5(b) we see a graph of the surface. Note how the y-axis is pointing away from the viewer to more closely resemble the orientation of the level curves in (a).

Seeing the level curves helps us understand the graph. For instance, the graph does not make it clear that one can "walk" along the line $y = -x$ without elevation change, though the level curve does.

Functions of Three Variables

We extend our study of multivariable functions to functions of three variables. (One can make a function of as many variables as one likes; we limit our study to three variables.)

Definition 12.1.2 Function of Three Variables

Let D be a subset of $\mathbb{R}^3$. A **function f of three variables** is a rule that assigns each triple (x, y, z) in D a value $w = f(x, y, z)$ in $\mathbb{R}$. D is the **domain** of f; the set of all outputs of f is the **range**.

Note how this definition closely resembles that of Definition 12.1.1.

Example 12.1.5 Understanding a function of three variables

Let $f(x, y, z) = \dfrac{x^2 + z + 3\sin y}{x + 2y - z}$. Evaluate f at the point $(3, 0, 2)$ and find the domain and range of f.

Solution
$$f(3, 0, 2) = \frac{3^2 + 2 + 3\sin 0}{3 + 2(0) - 2} = 11.$$

As the domain of f is not specified, we take it to be the set of all triples (x, y, z) for which $f(x, y, z)$ is defined. As we cannot divide by 0, we find the domain D is

$$D = \{(x, y, z) \mid x + 2y - z \neq 0\}.$$

We recognize that the set of all points in $\mathbb{R}^3$ that *are not* in D form a plane in space that passes through the origin (with normal vector $\langle 1, 2, -1 \rangle$).

We determine the range R is $\mathbb{R}$; that is, all real numbers are possible outputs of f. There is no set way of establishing this. Rather, to get numbers near 0 we can let $y = 0$ and choose $z \approx -x^2$. To get numbers of arbitrarily large magnitude, we can let $z \approx x + 2y$.

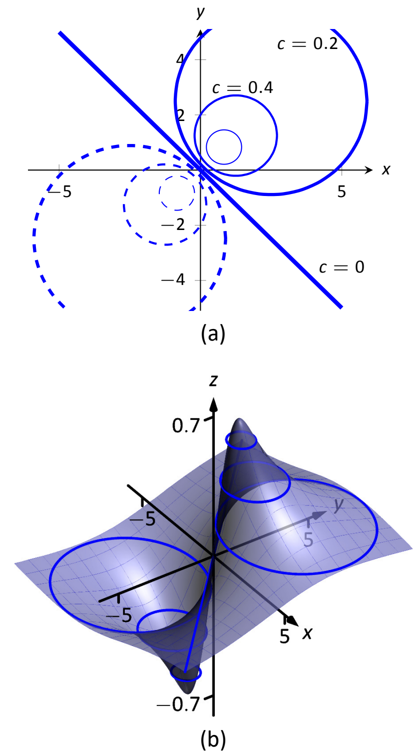

Figure 12.1.5: Graphing the level curves in Example 12.1.4.

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Level Surfaces

It is very difficult to produce a meaningful graph of a function of three variables. A function of *one* variable is a *curve* drawn in *2* dimensions; a function of *two* variables is a *surface* drawn in *3* dimensions; a function of *three* variables is a *hypersurface* drawn in *4* dimensions.

There are a few techniques one can employ to try to "picture" a graph of three variables. One is an analogue of level curves: **level surfaces**. Given $w = f(x, y, z)$, the level surface at $w = c$ is the surface in space formed by all points (x, y, z) where $f(x, y, z) = c$.

Example 12.1.6 Finding level surfaces

If a point source S is radiating energy, the intensity I at a given point P in space is inversely proportional to the square of the distance between S and P. That is, when $S = (0, 0, 0)$, $I(x, y, z) = \dfrac{k}{x^2 + y^2 + z^2}$ for some constant k.

Let $k = 1$; find the level surfaces of I.

Solution We can (mostly) answer this question using "common sense." If energy (say, in the form of light) is emanating from the origin, its intensity will be the same at all points equidistant from the origin. That is, at any point on the surface of a sphere centered at the origin, the intensity should be the same. Therefore, the level surfaces are spheres.

We now find this mathematically. The level surface at $I = c$ is defined by

$$c = \frac{1}{x^2 + y^2 + z^2}.$$

A small amount of algebra reveals

$$x^2 + y^2 + z^2 = \frac{1}{c}.$$

Given an intensity c, the level surface $I = c$ is a sphere of radius $1/\sqrt{c}$, centered at the origin.

Figure 12.1.6 gives a table of the radii of the spheres for given c values. Normally one would use equally spaced c values, but these values have been chosen purposefully. At a distance of 0.25 from the point source, the intensity is 16; to move to a point of half that intensity, one just moves out 0.1 to 0.35 – not much at all. To again halve the intensity, one moves 0.15, a little more than before.

Note how each time the intensity if halved, the distance required to move away grows. We conclude that the closer one is to the source, the more rapidly the intensity changes.

c	r
16.	0.25
8.	0.35
4.	0.5
2.	0.71
1.	1.
0.5	1.41
0.25	2.
0.125	2.83
0.0625	4.

Figure 12.1.6: A table of c values and the corresponding radius r of the spheres of constant value in Example 12.1.6.

In the next section we apply the concepts of limits to functions of two or more variables.

Notes:

Exercises 12.1

Terms and Concepts

12 01 ex 01 1. Give two examples (other than those given in the text) of "real world" functions that require more than one input.

12 01 ex 02 2. The graph of a function of two variables is a ________.

12 01 ex 03 3. Most people are familiar with the concept of level curves in the context of ________ maps.

12 01 ex 04 4. T/F: Along a level curve, the output of a function does not change.

12 01 ex 05 5. The analogue of a level curve for functions of three variables is a level ________.

12 01 ex 06 6. What does it mean when level curves are close together? Far apart?

Problems

12 01 exset 01 **In Exercises 7 – 14, give the domain and range of the multivariable function.**

12 01 ex 07 7. $f(x,y) = x^2 + y^2 + 2$

12 01 ex 08 8. $f(x,y) = x + 2y$

12 01 ex 09 9. $f(x,y) = x - 2y$

12 01 ex 10 10. $f(x,y) = \dfrac{1}{x+2y}$

12 01 ex 11 11. $f(x,y) = \dfrac{1}{x^2+y^2+1}$

12 01 ex 12 12. $f(x,y) = \sin x \cos y$

12 01 ex 13 13. $f(x,y) = \sqrt{9-x^2-y^2}$

12 01 ex 14 14. $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2-9}}$

12 01 exset 02 **In Exercises 15 – 22, describe in words and sketch the level curves for the function and given c values.**

12 01 ex 15 15. $f(x,y) = 3x - 2y$; $c = -2, 0, 2$

12 01 ex 16 16. $f(x,y) = x^2 - y^2$; $c = -1, 0, 1$

12 01 ex 17 17. $f(x,y) = x - y^2$; $c = -2, 0, 2$

12 01 ex 18 18. $f(x,y) = \dfrac{1-x^2-y^2}{2y-2x}$; $c = -2, 0, 2$

12 01 ex 19 19. $f(x,y) = \dfrac{2x-2y}{x^2+y^2+1}$; $c = -1, 0, 1$

12 01 ex 20 20. $f(x,y) = \dfrac{y-x^3-1}{x}$; $c = -3, -1, 0, 1, 3$

12 01 ex 21 21. $f(x,y) = \sqrt{x^2+4y^2}$; $c = 1, 2, 3, 4$

12 01 ex 22 22. $f(x,y) = x^2 + 4y^2$; $c = 1, 2, 3, 4$

12 01 exset 03 **In Exercises 23 – 26, give the domain and range of the functions of three variables.**

12 01 ex 24 23. $f(x,y,z) = \dfrac{x}{x+2y-4z}$

12 01 ex 25 24. $f(x,y,z) = \dfrac{1}{1-x^2-y^2-z^2}$

12 01 ex 26 25. $f(x,y,z) = \sqrt{z-x^2+y^2}$

12 01 ex 27 26. $f(x,y,z) = z^2 \sin x \cos y$

12 01 exset 04 **In Exercises 27 – 30, describe the level surfaces of the given functions of three variables.**

12 01 ex 28 27. $f(x,y,z) = x^2 + y^2 + z^2$

12 01 ex 29 28. $f(x,y,z) = z - x^2 + y^2$

12 01 ex 30 29. $f(x,y,z) = \dfrac{x^2+y^2}{z}$

12 01 ex 31 30. $f(x,y,z) = \dfrac{z}{x-y}$

12 01 ex 32 31. Compare the level curves of Exercises 21 and 22. How are they similar, and how are they different? Each surface is a quadric surface; describe how the level curves are consistent with what we know about each surface.

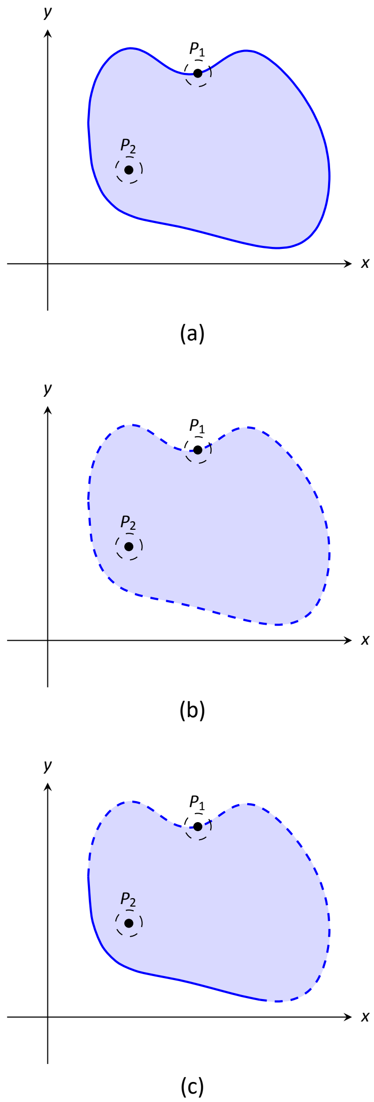

Figure 12.2.1: Illustrating open and closed sets in the x-y plane.

12.2 Limits and Continuity of Multivariable Functions

We continue with the pattern we have established in this text: after defining a new kind of function, we apply calculus ideas to it. The previous section defined functions of two and three variables; this section investigates what it means for these functions to be "continuous."

We begin with a series of definitions. We are used to "open intervals" such as $(1, 3)$, which represents the set of all x such that $1 < x < 3$, and "closed intervals" such as $[1, 3]$, which represents the set of all x such that $1 \leq x \leq 3$. We need analogous definitions for open and closed sets in the x-y plane.

Definition 12.2.1 Open Disk, Boundary and Interior Points, Open and Closed Sets, Bounded Sets

An **open disk** B in $\mathbb{R}^2$ centered at (x_0, y_0) with radius r is the set of all points (x, y) such that $\sqrt{(x-x_0)^2+(y-y_0)^2} < r$.

Let S be a set of points in $\mathbb{R}^2$. A point P in $\mathbb{R}^2$ is a **boundary point** of S if all open disks centered at P contain both points in S and points not in S.

A point P in S is an **interior point** of S if there is an open disk centered at P that contains only points in S.

A set S is **open** if every point in S is an interior point.

A set S is **closed** if it contains all of its boundary points.

A set S is **bounded** if there is an $M > 0$ such that the open disk, centered at the origin with radius M, contains S. A set that is not bounded is **unbounded**.

Figure 12.2.1 shows several sets in the x-y plane. In each set, point P_1 lies on the boundary of the set as all open disks centered there contain both points in, and not in, the set. In contrast, point P_2 is an interior point for there is an open disk centered there that lies entirely within the set.

The set depicted in Figure 12.2.1(a) is a closed set as it contains all of its boundary points. The set in (b) is open, for all of its points are interior points (or, equivalently, it does not contain any of its boundary points). The set in (c) is neither open nor closed as it contains some of its boundary points.

Notes:

Example 12.2.1 Determining open/closed, bounded/unbounded

Determine if the domain of the function $f(x,y) = \sqrt{1 - x^2/9 - y^2/4}$ is open, closed, or neither, and if it is bounded.

SOLUTION This domain of this function was found in Example 12.1.2 to be $D = \{(x,y) \mid \frac{x^2}{9} + \frac{y^2}{4} \leq 1\}$, the region *bounded* by the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$. Since the region includes the boundary (indicated by the use of "$\leq$"), the set contains all of its boundary points and hence is closed. The region is bounded as a disk of radius 4, centered at the origin, contains D.

Example 12.2.2 Determining open/closed, bounded/unbounded

Determine if the domain of $f(x,y) = \frac{1}{x-y}$ is open, closed, or neither.

SOLUTION As we cannot divide by 0, we find the domain to be $D = \{(x,y) \mid x - y \neq 0\}$. In other words, the domain is the set of all points (x,y) *not* on the line $y = x$.

The domain is sketched in Figure 12.2.2. Note how we can draw an open disk around any point in the domain that lies entirely inside the domain, and also note how the only boundary points of the domain are the points on the line $y = x$. We conclude the domain is an open set. The set is unbounded.

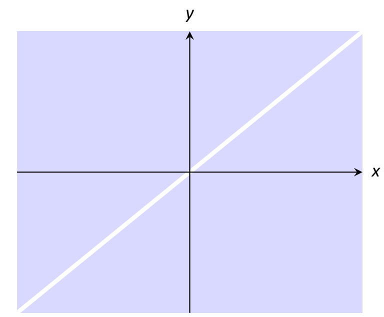

Figure 12.2.2: Sketching the domain of the function in Example 12.2.2.

Limits

Recall a pseudo–definition of the limit of a function of one variable: "$\lim_{x\to c} f(x) = L$" means that if x is "really close" to c, then $f(x)$ is "really close" to L. A similar pseudo–definition holds for functions of two variables. We'll say that

$$\text{“}\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\text{”}$$

means "if the point (x,y) is really close to the point (x_0,y_0), then $f(x,y)$ is really close to L." The formal definition is given below.

Definition 12.2.2 Limit of a Function of Two Variables

Let S be a set containing $P = (x_0,y_0)$ where every open disk centered at P contains points in S other than P, let f be a function of two variables defined on S, except possibly at P, and let L be a real number. The **limit of $f(x,y)$ as (x,y) approaches (x_0,y_0) is L**, denoted

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L,$$

means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all (x,y) in S, where $(x,y) \neq (x_0,y_0)$, if (x,y) is in the open disk centered at (x_0,y_0) with radius δ, then $|f(x,y) - L| < \varepsilon$.

Note: While our first limit definition was defined over an open interval, we now define limits over a set S in the plane (where S does not have to be open). As planar sets can be far more complicated than intervals, our definition adds the restriction ". . . where every open disk centered at P contains points in S other than P." In this text, all sets we'll consider will satisfy this condition and we won't bother to check; it is included in the definition for completeness.

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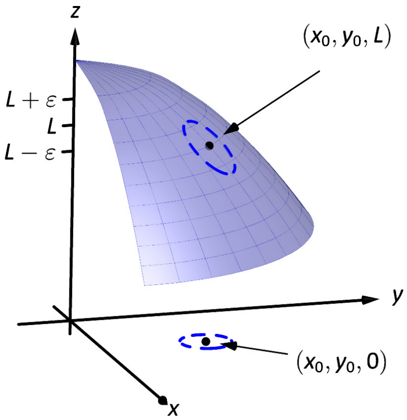


Figure 12.2.3: **Illustrating the definition of a limit.** The open disk in the x-y plane has radius δ. Let (x, y) be any point in this disk; $f(x, y)$ is within ε of L.

The concept behind Definition 12.2.2 is sketched in Figure 12.2.3. Given $\varepsilon > 0$, find $\delta > 0$ such that if (x, y) is any point in the open disk centered at (x_0, y_0) in the x-y plane with radius δ, then $f(x, y)$ should be within ε of L.

Computing limits using this definition is rather cumbersome. The following theorem allows us to evaluate limits much more easily.

Theorem 12.2.1 Basic Limit Properties of Functions of Two Variables

Let b, x_0, y_0, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x, y) = K.$$

The following limits hold.

1. Constants: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b = b$
2. Identity $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0$; $\displaystyle\lim_{(x,y)\to(x_0,y_0)} y = y_0$
3. Sums/Differences: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big(f(x, y) \pm g(x, y)\big) = L \pm K$
4. Scalar Multiples: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} b \cdot f(x, y) = bL$
5. Products: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y) \cdot g(x, y) = LK$
6. Quotients: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)/g(x, y) = L/K$, $(K \neq 0)$
7. Powers: $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)^n = L^n$

This theorem, combined with Theorems 1.3.2 and 1.3.3 of Section 1.3, allows us to evaluate many limits.

Example 12.2.3 Evaluating a limit

Evaluate the following limits:

$$1. \lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) \qquad 2. \lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$$

Notes:

Solution

1. The aforementioned theorems allow us to simply evaluate $y/x + \cos(xy)$ when $x = 1$ and $y = \pi$. If an indeterminate form is returned, we must do more work to evaluate the limit; otherwise, the result is the limit. Therefore

$$\begin{aligned}\lim_{(x,y)\to(1,\pi)} \frac{y}{x} + \cos(xy) &= \frac{\pi}{1} + \cos\pi \\ &= \pi - 1.\end{aligned}$$

2. We attempt to evaluate the limit by substituting 0 in for x and y, but the result is the indeterminate form "0/0." To evaluate this limit, we must "do more work," but we have not yet learned what "kind" of work to do. Therefore we cannot yet evaluate this limit.

When dealing with functions of a single variable we also considered one–sided limits and stated

$$\lim_{x\to c} f(x) = L \quad \text{if, and only if,} \quad \lim_{x\to c^+} f(x) = L \quad \textbf{and} \quad \lim_{x\to c^-} f(x) = L.$$

That is, the limit is L if and only if $f(x)$ approaches L when x approaches c from **either** direction, the left or the right.

In the plane, there are infinite directions from which (x, y) might approach (x_0, y_0). In fact, we do not have to restrict ourselves to approaching (x_0, y_0) from a particular direction, but rather we can approach that point along a path that is not a straight line. It is possible to arrive at different limiting values by approaching (x_0, y_0) along different paths. If this happens, we say that $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist (this is analogous to the left and right hand limits of single variable functions not being equal).

Our theorems tell us that we can evaluate most limits quite simply, without worrying about paths. When indeterminate forms arise, the limit may or may not exist. If it does exist, it can be difficult to prove this as we need to show the same limiting value is obtained regardless of the path chosen. The case where the limit does not exist is often easier to deal with, for we can often pick two paths along which the limit is different.

Example 12.2.4 Showing limits do not exist

1. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2 + y^2}$ does not exist by finding the limits along the lines $y = mx$.

Notes:

2. Show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist by finding the limit along the path $y = -\sin x$.

Solution

1. Evaluating $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy}{x^2+y^2}$ along the lines $y = mx$ means replace all y's with mx and evaluating the resulting limit:

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{3x(mx)}{x^2+(mx)^2} &= \lim_{x\to 0} \frac{3mx^2}{x^2(m^2+1)}\\ &= \lim_{x\to 0} \frac{3m}{m^2+1}\\ &= \frac{3m}{m^2+1}.\end{aligned}$$

While the limit exists for each choice of m, we get a *different* limit for each choice of m. That is, along different lines we get differing limiting values, meaning *the* limit does not exist.

2. Let $f(x,y) = \frac{\sin(xy)}{x+y}$. We are to show that $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist by finding the limit along the path $y = -\sin x$. First, however, consider the limits found along the lines $y = mx$ as done above.

$$\begin{aligned}\lim_{(x,mx)\to(0,0)} \frac{\sin\big(x(mx)\big)}{x+mx} &= \lim_{x\to 0} \frac{\sin(mx^2)}{x(m+1)}\\ &= \lim_{x\to 0} \frac{\sin(mx^2)}{x}\cdot\frac{1}{m+1}.\end{aligned}$$

By applying L'Hôpital's Rule, we can show this limit is 0 *except* when $m = -1$, that is, along the line $y = -x$. This line is not in the domain of f, so we have found the following fact: along every line $y = mx$ in the domain of f, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Now consider the limit along the path $y = -\sin x$:

$$\lim_{(x,-\sin x)\to(0,0)} \frac{\sin\big(-x\sin x\big)}{x-\sin x} = \lim_{x\to 0} \frac{\sin\big(-x\sin x\big)}{x-\sin x}$$

Now apply L'Hôpital's Rule twice:

$$\begin{aligned}&= \lim_{x\to 0} \frac{\cos\big(-x\sin x\big)(-\sin x - x\cos x)}{1-\cos x} \qquad (\text{“} = 0/0\text{”})\\ &= \lim_{x\to 0} \frac{-\sin\big(-x\sin x\big)(-\sin x - x\cos x)^2 + \cos\big(-x\sin x\big)(-2\cos x + x\sin x)}{\sin x}\\ &= \text{“}-2/0\text{”} \Rightarrow \text{the limit does not exist.}\end{aligned}$$

Notes:

Step back and consider what we have just discovered. Along any line $y = mx$ in the domain of the $f(x, y)$, the limit is 0. However, along the path $y = -\sin x$, which lies in the domain of $f(x, y)$ for all $x \neq 0$, the limit does not exist. Since the limit is not the same along every path to $(0, 0)$, we say $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{x+y}$ does not exist.

Example 12.2.5 Finding a limit

Let $f(x, y) = \dfrac{5x^2y^2}{x^2+y^2}$. Find $\displaystyle\lim_{(x,y)\to(0,0)} f(x, y)$.

Solution It is relatively easy to show that along any line $y = mx$, the limit is 0. This is not enough to prove that the limit exists, as demonstrated in the previous example, but it tells us that if the limit does exist then it must be 0.

To prove the limit is 0, we apply Definition 12.2.2. Let $\varepsilon > 0$ be given. We want to find $\delta > 0$ such that if $\sqrt{(x-0)^2+(y-0)^2} < \delta$, then $|f(x, y) - 0| < \varepsilon$.

Set $\delta < \sqrt{\varepsilon/5}$. Note that $\left|\dfrac{5y^2}{x^2+y^2}\right| < 5$ for all $(x, y) \neq (0, 0)$, and that if $\sqrt{x^2+y^2} < \delta$, then $x^2 < \delta^2$.

Let $\sqrt{(x-0)^2+(y-0)^2} = \sqrt{x^2+y^2} < \delta$. Consider $|f(x, y) - 0|$:

$$\begin{aligned} |f(x, y) - 0| &= \left|\frac{5x^2y^2}{x^2+y^2} - 0\right| \\ &= \left|x^2 \cdot \frac{5y^2}{x^2+y^2}\right| \\ &< \delta^2 \cdot 5 \\ &< \frac{\varepsilon}{5} \cdot 5 \\ &= \varepsilon. \end{aligned}$$

Thus if $\sqrt{(x-0)^2+(y-0)^2} < \delta$ then $|f(x, y) - 0| < \varepsilon$, which is what we wanted to show. Thus $\displaystyle\lim_{(x,y)\to(0,0)} \frac{5x^2y^2}{x^2+y^2} = 0$.

Continuity

Definition 1.5.1 defines what it means for a function of one variable to be continuous. In brief, it meant that the graph of the function did not have breaks, holes, jumps, etc. We define continuity for functions of two variables in a similar way as we did for functions of one variable.

Notes:

Definition 12.2.3 Continuous

Let a function $f(x,y)$ be defined on a set S containing the point (x_0,y_0).

1. f is **continuous** at (x_0,y_0) if $\displaystyle\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$.

2. f is **continuous on** S if f is continuous at all points in S. If f is continuous at all points in $\mathbb{R}^2$, we say that f is **continuous everywhere**.

Example 12.2.6 Continuity of a function of two variables

Let $f(x,y) = \begin{cases} \frac{\cos y \sin x}{x} & x \neq 0 \\ \cos y & x = 0 \end{cases}$. Is f continuous at $(0,0)$? Is f continuous everywhere?

Solution To determine if f is continuous at $(0,0)$, we need to compare $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ to $f(0,0)$.

Applying the definition of f, we see that $f(0,0) = \cos 0 = 1$.

We now consider the limit $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$. Substituting 0 for x and y in $(\cos y \sin x)/x$ returns the indeterminate form "0/0", so we need to do more work to evaluate this limit.

Consider two related limits: $\displaystyle\lim_{(x,y)\to(0,0)} \cos y$ and $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}$. The first limit does not contain x, and since $\cos y$ is continuous,

$$\lim_{(x,y)\to(0,0)} \cos y = \lim_{y\to 0} \cos y = \cos 0 = 1.$$

The second limit does not contain y. By Theorem 1.3.5 we can say

$$\lim_{(x,y)\to(0,0)} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Finally, Theorem 12.2.1 of this section states that we can combine these two limits as follows:

$$\begin{aligned} \lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} &= \lim_{(x,y)\to(0,0)} (\cos y)\left(\frac{\sin x}{x}\right) \\ &= \left(\lim_{(x,y)\to(0,0)} \cos y\right)\left(\lim_{(x,y)\to(0,0)} \frac{\sin x}{x}\right) \\ &= (1)(1) \\ &= 1. \end{aligned}$$

Notes:

We have found that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos y \sin x}{x} = f(0,0)$, so f is continuous at $(0,0)$.

A similar analysis shows that f is continuous at all points in $\mathbb{R}^2$. As long as $x \neq 0$, we can evaluate the limit directly; when $x = 0$, a similar analysis shows that the limit is $\cos y$. Thus we can say that f is continuous everywhere. A graph of f is given in Figure 12.2.4. Notice how it has no breaks, jumps, etc.

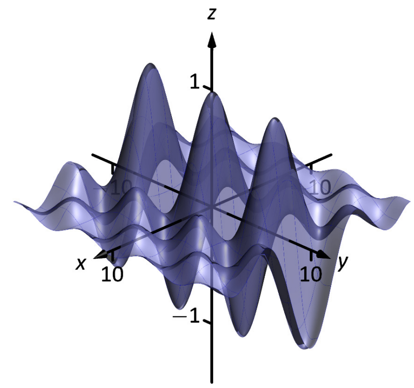


Figure 12.2.4: A graph of $f(x,y)$ in Example 12.2.6.

The following theorem is very similar to Theorem 1.5.1, giving us ways to combine continuous functions to create other continuous functions.

Theorem 12.2.2 Properties of Continuous Functions

Let f and g be continuous on a set S, let c be a real number, and let n be a positive integer. The following functions are continuous on S.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as longs as $g \neq 0$ on S)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (if n is even then $f \geq 0$ on S; if n is odd, then true for all values of f on S.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on S, where the range of f on S is J, and let g be a single variable function that is continuous on J. Then $g \circ f$, i.e., $g(f(x,y))$, is continuous on S.

Example 12.2.7 Establishing continuity of a function

Let $f(x,y) = \sin(x^2 \cos y)$. Show f is continuous everywhere.

Solution We will apply both Theorems 1.5.1 and 12.2.2. Let $f_1(x,y) = x^2$. Since y is not actually used in the function, and polynomials are continuous (by Theorem 1.5.1), we conclude f_1 is continuous everywhere. A similar statement can be made about $f_2(x,y) = \cos y$. Part 3 of Theorem 12.2.2 states that $f_3 = f_1 \cdot f_2$ is continuous everywhere, and Part 7 of the theorem states the composition of sine with f_3 is continuous: that is, $\sin(f_3) = \sin(x^2 \cos y)$ is continuous everywhere.

Notes:

Functions of Three Variables

The definitions and theorems given in this section can be extended in a natural way to definitions and theorems about functions of three (or more) variables. We cover the key concepts here; some terms from Definitions 12.2.1 and 12.2.3 are not redefined but their analogous meanings should be clear to the reader.

Definition 12.2.4 Open Balls, Limit, Continuous

1. An **open ball** in $\mathbb{R}^3$ centered at (x_0, y_0, z_0) with radius r is the set of all points (x, y, z) such that $\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}=r$.

2. Let D be a set in $\mathbb{R}^3$ containing (x_0, y_0, z_0) where every open ball centered at (x_0, y_0, z_0) contains points of D other than (x_0, y_0, z_0), and let $f(x, y, z)$ be a function of three variables defined on D, except possibly at (x_0, y_0, z_0). The **limit** of $f(x, y, z)$ as (x, y, z) approaches (x_0, y_0, z_0) is L, denoted
$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = L,$$
means that given any $\varepsilon > 0$, there is a $\delta > 0$ such that for all (x, y, z) in D, $(x, y, z) \neq (x_0, y_0, z_0)$, if (x, y, z) is in the open ball centered at (x_0, y_0, z_0) with radius δ, then $|f(x, y, z) - L| < \varepsilon$.

3. Let $f(x, y, z)$ be defined on a set D containing (x_0, y_0, z_0). f is **continuous** at (x_0, y_0, z_0) if $\displaystyle\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = f(x_0,y_0,z_0)$; if f is continuous at all points in D, we say f is **continuous on** D.

These definitions can also be extended naturally to apply to functions of four or more variables. Theorem 12.2.2 also applies to function of three or more variables, allowing us to say that the function
$$f(x,y,z) = \frac{e^{x^2+y}\sqrt{y^2+z^2+3}}{\sin(xyz)+5}$$
is continuous everywhere.

When considering single variable functions, we studied limits, then continuity, then the derivative. In our current study of multivariable functions, we have studied limits and continuity. In the next section we study derivation, which takes on a slight twist as we are in a multivarible context.

Notes:

Exercises 12.2

Terms and Concepts

12 02 ex 01 1. Describe in your own words the difference between boundary and interior points of a set.

12 02 ex 02 2. Use your own words to describe (informally) what $\lim\limits_{(x,y)\to(1,2)} f(x,y) = 17$ means.

12 02 ex 17 3. Give an example of a closed, bounded set.

12 02 ex 18 4. Give an example of a closed, unbounded set.

12 02 ex 19 5. Give an example of a open, bounded set.

12 02 ex 20 6. Give an example of a open, unbounded set.

Problems

12 02 exset 01 **In Exercises 7 – 10, a set S is given.**

(a) Give one boundary point and one interior point, when possible, of S.

(b) State whether S is open, closed, or neither.

(c) State whether S is bounded or unbounded.

12 02 ex 03 7. $S = \left\{ (x,y) \;\middle|\; \dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{9} \leq 1 \right\}$

12 02 ex 04 8. $S = \left\{ (x,y) \mid y \neq x^2 \right\}$

12 02 ex 05 9. $S = \left\{ (x,y) \mid x^2 + y^2 = 1 \right\}$

12 02 ex 06 10. $S = \{(x,y) | y > \sin x\}$

12 02 exset 02 **In Exercises 11 – 14:**

(a) Find the domain D of the given function.

(b) State whether D is an open or closed set.

(c) State whether D is bounded or unbounded.

12 02 ex 09 11. $f(x,y) = \sqrt{9 - x^2 - y^2}$

12 02 ex 07 12. $f(x,y) = \sqrt{y - x^2}$

12 02 ex 08 13. $f(x,y) = \dfrac{1}{\sqrt{y - x^2}}$

12 02 ex 10 14. $f(x,y) = \dfrac{x^2 - y^2}{x^2 + y^2}$

12 02 exset 03 **In Exercises 15 – 20, a limit is given. Evaluate the limit along the paths given, then state why these results show the given limit does not exist.**

12 02 ex 12 15. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 - y^2}{x^2 + y^2}$

(a) Along the path $y = 0$.

(b) Along the path $x = 0$.

12 02 ex 11 16. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x + y}{x - y}$

(a) Along the path $y = mx$.

12 02 ex 13 17. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy - y^2}{y^2 + x}$

(a) Along the path $y = mx$.

(b) Along the path $x = 0$.

12 02 ex 14 18. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2)}{y}$

(a) Along the path $y = mx$.

(b) Along the path $y = x^2$.

12 02 ex 15 19. $\lim\limits_{(x,y)\to(1,2)} \dfrac{x + y - 3}{x^2 - 1}$

(a) Along the path $y = 2$.

(b) Along the path $y = x + 1$.

12 02 ex 16 20. $\lim\limits_{(x,y)\to(\pi,\pi/2)} \dfrac{\sin x}{\cos y}$

(a) Along the path $x = \pi$.

(b) Along the path $y = x - \pi/2$.

12.3 Partial Derivatives

Let y be a function of x. We have studied in great detail the derivative of y with respect to x, that is, $\frac{dy}{dx}$, which measures the rate at which y changes with respect to x. Consider now $z = f(x, y)$. It makes sense to want to know how z changes with respect to x and/or y. This section begins our investigation into these rates of change.

Consider the function $z = f(x, y) = x^2 + 2y^2$, as graphed in Figure 12.3.1(a). By fixing $y = 2$, we focus our attention to all points on the surface where the y-value is 2, shown in both parts (a) and (b) of the figure. These points form a curve in space: $z = f(x, 2) = x^2 + 8$ which is a function of just one variable. We can take the derivative of z with respect to x along this curve and find equations of tangent lines, etc.

The key notion to extract from this example is: by treating y as constant (it does not vary) we can consider how z changes with respect to x. In a similar fashion, we can hold x constant and consider how z changes with respect to y. This is the underlying principle of **partial derivatives**. We state the formal, limit–based definition first, then show how to compute these partial derivatives without directly taking limits.

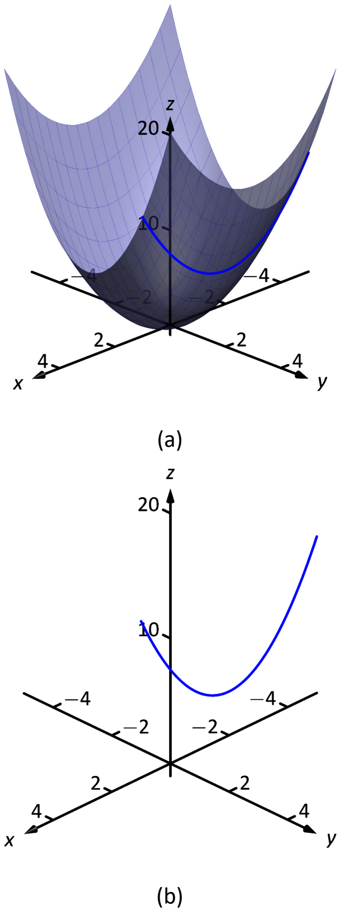


Figure 12.3.1: By fixing $y = 2$, the surface $f(x, y) = x^2 + 2y^2$ is a curve in space.

Alternate notations for $f_x(x, y)$ include:

$$\frac{\partial}{\partial x} f(x, y), \quad \frac{\partial f}{\partial x}, \quad \frac{\partial z}{\partial x}, \quad \text{and } z_x,$$

with similar notations for $f_y(x, y)$. For ease of notation, $f_x(x, y)$ is often abbreviated f_x.

Definition 12.3.1 Partial Derivative

Let $z = f(x, y)$ be a continuous function on a set S in $\mathbb{R}^2$.

1. The **partial derivative of f with respect to x** is:

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}.$$

2. The **partial derivative of f with respect to y** is:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}.$$

Example 12.3.1 Computing partial derivatives with the limit definition

Let $f(x, y) = x^2y + 2x + y^3$. Find $f_x(x, y)$ using the limit definition.

Notes:

Solution Using Definition 12.3.1, we have:

$$\begin{aligned}
f_x(x,y) &= \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}\\
&= \lim_{h\to 0}\frac{(x+h)^2y+2(x+h)+y^3-(x^2y+2x+y^3)}{h}\\
&= \lim_{h\to 0}\frac{x^2y+2xhy+h^2y+2x+2h+y^3-(x^2y+2x+y^3)}{h}\\
&= \lim_{h\to 0}\frac{2xhy+h^2y+2h}{h}\\
&= \lim_{h\to 0} 2xy+hy+2\\
&= 2xy+2.
\end{aligned}$$

We have found $f_x(x,y) = 2xy+2$.

Example 12.3.1 found a partial derivative using the formal, limit–based definition. Using limits is not necessary, though, as we can rely on our previous knowledge of derivatives to compute partial derivatives easily. When computing $f_x(x,y)$, we hold y fixed – it does not vary. Therefore we can compute the derivative with respect to x by treating y as a constant or coefficient.

Just as $\frac{d}{dx}\big(5x^2\big) = 10x$, we compute $\frac{\partial}{\partial x}\big(x^2y\big) = 2xy$. Here we are treating y as a coefficient.

Just as $\frac{d}{dx}\big(5^3\big) = 0$, we compute $\frac{\partial}{\partial x}\big(y^3\big) = 0$. Here we are treating y as a constant. More examples will help make this clear.

Example 12.3.2 Finding partial derivatives

Find $f_x(x,y)$ and $f_y(x,y)$ in each of the following.

1. $f(x,y) = x^3y^2+5y^2-x+7$
2. $f(x,y) = \cos(xy^2)+\sin x$
3. $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$

Solution

1. We have $f(x,y) = x^3y^2+5y^2-x+7$.
   Begin with $f_x(x,y)$. Keep y fixed, treating it as a constant or coefficient, as appropriate:
   $$f_x(x,y) = 3x^2y^2-1.$$
   Note how the $5y^2$ and 7 terms go to zero.

Notes:

To compute $f_y(x,y)$, we hold x fixed:

$$f_y(x,y) = 2x^3y + 10y.$$

Note how the $-x$ and 7 terms go to zero.

2. We have $f(x,y) = \cos(xy^2) + \sin x$.
Begin with $f_x(x,y)$. We need to apply the Chain Rule with the cosine term; y^2 is the coefficient of the x-term inside the cosine function.

$$f_x(x,y) = -\sin(xy^2)(y^2) + \cos x = -y^2\sin(xy^2) + \cos x.$$

To find $f_y(x,y)$, note that x is the coefficient of the y^2 term inside of the cosine term; also note that since x is fixed, $\sin x$ is also fixed, and we treat it as a constant.

$$f_y(x,y) = -\sin(xy^2)(2xy) = -2xy\sin(xy^2).$$

3. We have $f(x,y) = e^{x^2y^3}\sqrt{x^2+1}$.
Beginning with $f_x(x,y)$, note how we need to apply the Product Rule.

$$\begin{aligned} f_x(x,y) &= e^{x^2y^3}(2xy^3)\sqrt{x^2+1} + e^{x^2y^3}\frac{1}{2}\left(x^2+1\right)^{-1/2}(2x) \\ &= 2xy^3e^{x^2y^3}\sqrt{x^2+1} + \frac{xe^{x^2y^3}}{\sqrt{x^2+1}}. \end{aligned}$$

Note that when finding $f_y(x,y)$ we do not have to apply the Product Rule; since $\sqrt{x^2+1}$ does not contain y, we treat it as fixed and hence becomes a coefficient of the $e^{x^2y^3}$ term.

$$f_y(x,y) = e^{x^2y^3}(3x^2y^2)\sqrt{x^2+1} = 3x^2y^2e^{x^2y^3}\sqrt{x^2+1}.$$

We have shown *how* to compute a partial derivative, but it may still not be clear what a partial derivative *means*. Given $z = f(x,y)$, $f_x(x,y)$ measures the rate at which z changes as only x varies: y is held constant.

Imagine standing in a rolling meadow, then beginning to walk due east. Depending on your location, you might walk up, sharply down, or perhaps not change elevation at all. This is similar to measuring z_x: you are moving only east (in the "x"-direction) and not north/south at all. Going back to your original location, imagine now walking due north (in the "y"-direction). Perhaps walking due north does not change your elevation at all. This is analogous to $z_y = 0$: z does not change with respect to y. We can see that z_x and z_y do not have to be the same, or even similar, as it is easy to imagine circumstances where walking east means you walk downhill, though walking north makes you walk uphill.

Notes:

The following example helps us visualize this more.

Example 12.3.3 Evaluating partial derivatives
Let $z = f(x, y) = -x^2 - \frac{1}{2}y^2 + xy + 10$. Find $f_x(2, 1)$ and $f_y(2, 1)$ and interpret their meaning.

Solution We begin by computing $f_x(x, y) = -2x + y$ and $f_y(x, y) = -y + x$. Thus

$$f_x(2, 1) = -3 \quad \text{and} \quad f_y(2, 1) = 1.$$

It is also useful to note that $f(2, 1) = 7.5$. What does each of these numbers mean?

Consider $f_x(2, 1) = -3$, along with Figure 12.3.2(a). If one "stands" on the surface at the point $(2, 1, 7.5)$ and moves parallel to the x-axis (i.e., only the x-value changes, not the y-value), then the instantaneous rate of change is -3. Increasing the x-value will decrease the z-value; decreasing the x-value will increase the z-value.

Now consider $f_y(2, 1) = 1$, illustrated in Figure 12.3.2(b). Moving along the curve drawn on the surface, i.e., parallel to the y-axis and not changing the x-values, increases the z-value instantaneously at a rate of 1. Increasing the y-value by 1 would increase the z-value by approximately 1.

Since the magnitude of f_x is greater than the magnitude of f_y at $(2, 1)$, it is "steeper" in the x-direction than in the y-direction.

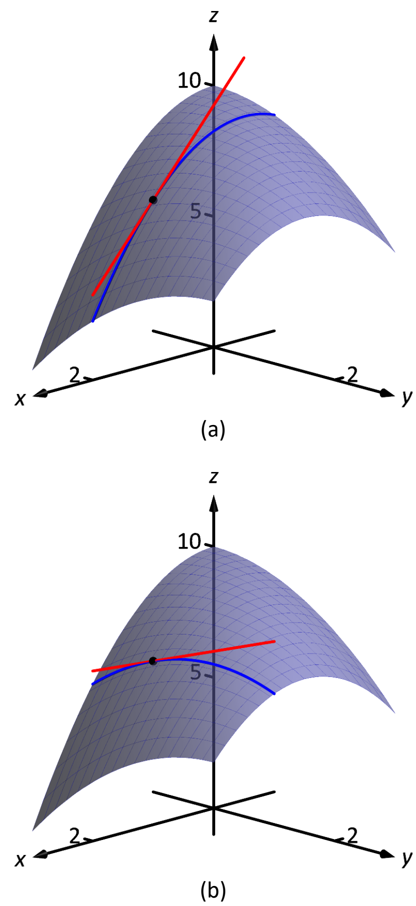


Figure 12.3.2: Illustrating the meaning of partial derivatives.

Second Partial Derivatives

Let $z = f(x, y)$. We have learned to find the partial derivatives $f_x(x, y)$ and $f_y(x, y)$, which are each functions of x and y. Therefore we can take partial derivatives of them, each with respect to x and y. We define these "second partials" along with the notation, give examples, then discuss their meaning.

Notes:

Note: The terms in Definition 12.3.2 all depend on limits, so each definition comes with the caveat "where the limit exists."

Definition 12.3.2 Second Partial Derivative, Mixed Partial Derivative

Let $z = f(x, y)$ be continuous on a set S.

1. The **second partial derivative of f with respect to x then x** is

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \left(f_x\right)_x = f_{xx}$$

2. The **second partial derivative of f with respect to x then y** is

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \left(f_x\right)_y = f_{xy}$$

Similar definitions hold for $\dfrac{\partial^2 f}{\partial y^2} = f_{yy}$ and $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx}$.

The second partial derivatives f_{xy} and f_{yx} are **mixed partial derivatives**.

The notation of second partial derivatives gives some insight into the notation of the second derivative of a function of a single variable. If $y = f(x)$, then $f''(x) = \dfrac{d^2y}{dx^2}$. The "$d^2y$" portion means "take the derivative of y twice," while "dx^2" means "with respect to x both times." When we only know of functions of a single variable, this latter phrase seems silly: there is only one variable to take the derivative with respect to. Now that we understand functions of multiple variables, we see the importance of specifying which variables we are referring to.

Example 12.3.4 Second partial derivatives

For each of the following, find all six first and second partial derivatives. That is, find

$$f_x, \quad f_y, \quad f_{xx}, \quad f_{yy}, \quad f_{xy} \quad \text{and} \quad f_{yx}\,.$$

1. $f(x, y) = x^3y^2 + 2xy^3 + \cos x$

2. $f(x, y) = \dfrac{x^3}{y^2}$

3. $f(x, y) = e^x \sin(x^2y)$

Notes:

SOLUTION In each, we give f_x and f_y immediately and then spend time deriving the second partial derivatives.

1. $f(x,y) = x^3y^2 + 2xy^3 + \cos x$

   $f_x(x,y) = 3x^2y^2 + 2y^3 - \sin x$

   $f_y(x,y) = 2x^3y + 6xy^2$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6xy^2 - \cos x$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\big(f_y\big) = \dfrac{\partial}{\partial y}\big(2x^3y + 6xy^2\big) = 2x^3 + 12xy$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\big(f_x\big) = \dfrac{\partial}{\partial y}\big(3x^2y^2 + 2y^3 - \sin x\big) = 6x^2y + 6y^2$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\big(f_y\big) = \dfrac{\partial}{\partial x}\big(2x^3y + 6xy^2\big) = 6x^2y + 6y^2$

2. $f(x,y) = \dfrac{x^3}{y^2} = x^3y^{-2}$

   $f_x(x,y) = \dfrac{3x^2}{y^2}$

   $f_y(x,y) = -\dfrac{2x^3}{y^3}$

   $f_{xx}(x,y) = \dfrac{\partial}{\partial x}\big(f_x\big) = \dfrac{\partial}{\partial x}\big(\dfrac{3x^2}{y^2}\big) = \dfrac{6x}{y^2}$

   $f_{yy}(x,y) = \dfrac{\partial}{\partial y}\big(f_y\big) = \dfrac{\partial}{\partial y}\big(-\dfrac{2x^3}{y^3}\big) = \dfrac{6x^3}{y^4}$

   $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\big(f_x\big) = \dfrac{\partial}{\partial y}\big(\dfrac{3x^2}{y^2}\big) = -\dfrac{6x^2}{y^3}$

   $f_{yx}(x,y) = \dfrac{\partial}{\partial x}\big(f_y\big) = \dfrac{\partial}{\partial x}\big(-\dfrac{2x^3}{y^3}\big) = -\dfrac{6x^2}{y^3}$

3. $f(x,y) = e^x\sin(x^2y)$

   Because the following partial derivatives get rather long, we omit the extra notation and just give the results. In several cases, multiple applications of the Product and Chain Rules will be necessary, followed by some basic combination of like terms.

   $f_x(x,y) = e^x\sin(x^2y) + 2xye^x\cos(x^2y)$

   $f_y(x,y) = x^2e^x\cos(x^2y)$

   $f_{xx}(x,y) = e^x\sin(x^2y) + 4xye^x\cos(x^2y) + 2ye^x\cos(x^2y) - 4x^2y^2e^x\sin(x^2y)$

   $f_{yy}(x,y) = -x^4e^x\sin(x^2y)$

   $f_{xy}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

   $f_{yx}(x,y) = x^2e^x\cos(x^2y) + 2xe^x\cos(x^2y) - 2x^3ye^x\sin(x^2y)$

Notes:

Notice how in each of the three functions in Example 12.3.4, $f_{xy} = f_{yx}$. Due to the complexity of the examples, this likely is not a coincidence. The following theorem states that it is not.

Theorem 12.3.1 Mixed Partial Derivatives

Let f be defined such that f_{xy} and f_{yx} are continuous on a set S. Then for each point (x, y) in S, $f_{xy}(x, y) = f_{yx}(x, y)$.

Finding f_{xy} and f_{yx} independently and comparing the results provides a convenient way of checking our work.

Understanding Second Partial Derivatives

Now that we know *how* to find second partials, we investigate *what* they tell us.

Again we refer back to a function $y = f(x)$ of a single variable. The second derivative of f is "the derivative of the derivative," or "the rate of change of the rate of change." The second derivative measures how much the derivative is changing. If $f''(x) < 0$, then the derivative is getting smaller (so the graph of f is concave down); if $f''(x) > 0$, then the derivative is growing, making the graph of f concave up.

Now consider $z = f(x, y)$. Similar statements can be made about f_{xx} and f_{yy} as could be made about $f''(x)$ above. When taking derivatives with respect to x twice, we measure how much f_x changes with respect to x. If $f_{xx}(x, y) < 0$, it means that as x increases, f_x decreases, and the graph of f will be concave down *in the x-direction*. Using the analogy of standing in the rolling meadow used earlier in this section, f_{xx} measures whether one's path is concave up/down when walking due east.

Similarly, f_{yy} measures the concavity in the y-direction. If $f_{yy}(x, y) > 0$, then f_y is increasing with respect to y and the graph of f will be concave up in the y-direction. Appealing to the rolling meadow analogy again, f_{yy} measures whether one's path is concave up/down when walking due north.

We now consider the mixed partials f_{xy} and f_{yx}. The mixed partial f_{xy} measures how much f_x changes with respect to y. Once again using the rolling meadow analogy, f_x measures the slope if one walks due east. Looking east, begin walking *north* (side–stepping). Is the path towards the east getting steeper? If so, $f_{xy} > 0$. Is the path towards the east not changing in steepness? If so, then $f_{xy} = 0$. A similar thing can be said about f_{yx}: consider the steepness of paths heading north while side–stepping to the east.

The following example examines these ideas with concrete numbers and

Notes:

graphs.

Example 12.3.5 Understanding second partial derivatives

Let $z = x^2 - y^2 + xy$. Evaluate the 6 first and second partial derivatives at $(-1/2, 1/2)$ and interpret what each of these numbers mean.

Solution We find that:

$f_x(x, y) = 2x + y$, $f_y(x, y) = -2y + x$, $f_{xx}(x, y) = 2$, $f_{yy}(x, y) = -2$ and $f_{xy}(x, y) = f_{yx}(x, y) = 1$. Thus at $(-1/2, 1/2)$ we have

$$f_x(-1/2, 1/2) = -1/2, \qquad f_y(-1/2, 1/2) = -3/2.$$

The slope of the tangent line at $(-1/2, 1/2, -1/4)$ in the direction of x is $-1/2$: if one moves from that point parallel to the x-axis, the instantaneous rate of change will be $-1/2$. The slope of the tangent line at this point in the direction of y is $-3/2$: if one moves from this point parallel to the y-axis, the instantaneous rate of change will be $-3/2$. These tangents lines are graphed in Figure 12.3.3(a) and (b), respectively, where the tangent lines are drawn in a solid line.

Now consider only Figure 12.3.3(a). Three directed tangent lines are drawn (two are dashed), each in the direction of x; that is, each has a slope determined by f_x. Note how as y increases, the slope of these lines get closer to 0. Since the slopes are all negative, getting closer to 0 means the *slopes are increasing.* The slopes given by f_x are increasing as y increases, meaning f_{xy} must be positive.

Since $f_{xy} = f_{yx}$, we also expect f_y to increase as x increases. Consider Figure 12.3.3(b) where again three directed tangent lines are drawn, this time each in the direction of y with slopes determined by f_y. As x increases, the slopes become less steep (closer to 0). Since these are negative slopes, this means the slopes are increasing.

Thus far we have a visual understanding of f_x, f_y, and $f_{xy} = f_{yx}$. We now interpret f_{xx} and f_{yy}. In Figure 12.3.3(a), we see a curve drawn where x is held constant at $x = -1/2$: only y varies. This curve is clearly concave down, corresponding to the fact that $f_{yy} < 0$. In part (b) of the figure, we see a similar curve where y is constant and only x varies. This curve is concave up, corresponding to the fact that $f_{xx} > 0$.

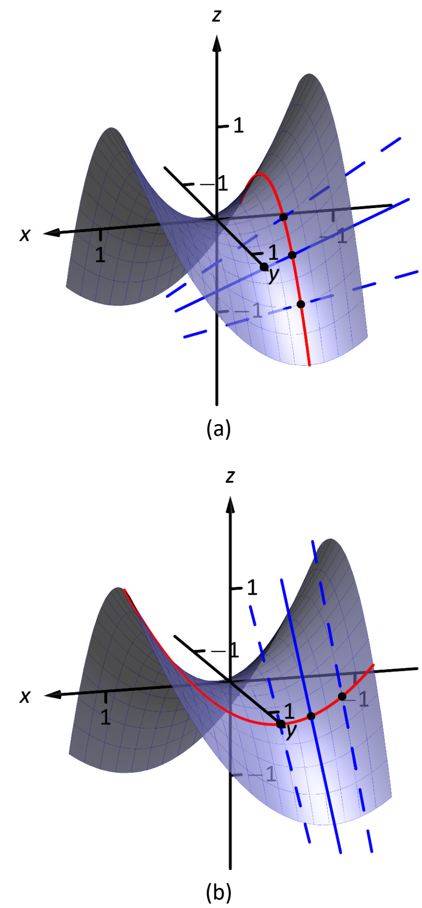


Figure 12.3.3: Understanding the second partial derivatives in Example 12.3.5.

Partial Derivatives and Functions of Three Variables

The concepts underlying partial derivatives can be easily extend to more than two variables. We give some definitions and examples in the case of three variables and trust the reader can extend these definitions to more variables if needed.

Notes:

Definition 12.3.3 Partial Derivatives with Three Variables

Let $w = f(x, y, z)$ be a continuous function on a set D in $\mathbb{R}^3$.
The **partial derivative of f with respect to x** is:

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}.$$

Similar definitions hold for $f_y(x, y, z)$ and $f_z(x, y, z)$.

By taking partial derivatives of partial derivatives, we can find second partial derivatives of f with respect to z then y, for instance, just as before.

Example 12.3.6 Partial derivatives of functions of three variables

For each of the following, find f_x, f_y, f_z, f_{xz}, f_{yz}, and f_{zz}.

1. $f(x, y, z) = x^2y^3z^4 + x^2y^2 + x^3z^3 + y^4z^4$
2. $f(x, y, z) = x\sin(yz)$

Solution

1. $f_x = 2xy^3z^4 + 2xy^2 + 3x^2z^3$; $f_y = 3x^2y^2z^4 + 2x^2y + 4y^3z^4$;
   $f_z = 4x^2y^3z^3 + 3x^3z^2 + 4y^4z^3$; $f_{xz} = 8xy^3z^3 + 9x^2z^2$;
   $f_{yz} = 12x^2y^2z^3 + 16y^3z^3$; $f_{zz} = 12x^2y^3z^2 + 6x^3z + 12y^4z^2$
2. $f_x = \sin(yz)$; $f_y = xz\cos(yz)$; $f_z = xy\cos(yz)$;
   $f_{xz} = y\cos(yz)$; $f_{yz} = x\cos(yz) - xyz\sin(yz)$; $f_{zz} = -xy^2\sin(xy)$

Higher Order Partial Derivatives

We can continue taking partial derivatives of partial derivatives of partial derivatives of ...; we do not have to stop with second partial derivatives. These higher order partial derivatives do not have a tidy graphical interpretation; nevertheless they are not hard to compute and worthy of some practice.

We do not formally define each higher order derivative, but rather give just a few examples of the notation.

$$f_{xyx}(x, y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right) \quad \text{and}$$

$$f_{xyz}(x, y, z) = \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\right).$$

Notes:

Example 12.3.7 Higher order partial derivatives

1. Let $f(x,y) = x^2y^2 + \sin(xy)$. Find f_{xxy} and f_{yxx}.
2. Let $f(x,y,z) = x^3e^{xy} + \cos(z)$. Find f_{xyz}.

Solution

1. To find f_{xxy}, we first find f_x, then f_{xx}, then f_{xxy}:

$$\begin{aligned} f_x &= 2xy^2 + y\cos(xy) \qquad f_{xx} = 2y^2 - y^2\sin(xy) \\ f_{xxy} &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   To find f_{yxx}, we first find f_y, then f_{yx}, then f_{yxx}:

$$\begin{aligned} f_y &= 2x^2y + x\cos(xy) \qquad f_{yx} = 4xy + \cos(xy) - xy\sin(xy) \\ f_{yxx} &= 4y - y\sin(xy) - \big(y\sin(xy) + xy^2\cos(xy)\big) \\ &= 4y - 2y\sin(xy) - xy^2\cos(xy). \end{aligned}$$

   Note how $f_{xxy} = f_{yxx}$.

2. To find f_{xyz}, we find f_x, then f_{xy}, then f_{xyz}:

$$\begin{aligned} f_x &= 3x^2e^{xy} + x^3ye^{xy} \qquad f_{xy} = 3x^3e^{xy} + x^3e^{xy} + x^4ye^{xy} = 4x^3e^{xy} + x^4ye^{xy} \\ f_{xyz} &= 0. \end{aligned}$$

In the previous example we saw that $f_{xxy} = f_{yxx}$; this is not a coincidence. While we do not state this as a formal theorem, as long as each partial derivative is continuous, it does not matter the order in which the partial derivatives are taken. For instance, $f_{xxy} = f_{xyx} = f_{yxx}$.

This can be useful at times. Had we known this, the second part of Example 12.3.7 would have been much simpler to compute. Instead of computing f_{xyz} in the x, y then z orders, we could have applied the z, then x then y order (as $f_{xyz} = f_{zxy}$). It is easy to see that $f_z = -\sin z$; then f_{zx} and f_{zxy} are clearly 0 as f_z does not contain an x or y.

Notes:

A brief review of this section: partial derivatives measure the instantaneous rate of change of a multivariable function with respect to one variable. With $z = f(x, y)$, the partial derivatives f_x and f_y measure the instantaneous rate of change of z when moving parallel to the x- and y-axes, respectively. How do we measure the rate of change at a point when we do not move parallel to one of these axes? What if we move in the direction given by the vector $\langle 2, 1\rangle$? Can we measure that rate of change? The answer is, of course, yes, we can. This is the topic of Section 12.6. First, we need to define what it means for a function of two variables to be *differentiable.*

Notes:

Exercises 12.3

Terms and Concepts

12 03 ex 01 1. What is the difference between a constant and a coefficient?

12 03 ex 02 2. Given a function $z = f(x, y)$, explain in your own words how to compute f_x.

12 03 ex 03 3. In the mixed partial fraction f_{xy}, which is computed first, f_x or f_y?

12 03 ex 04 4. In the mixed partial fraction $\dfrac{\partial^2 f}{\partial x \partial y}$, which is computed first, f_x or f_y?

Problems

12 03 exset 01 **In Exercises 5 – 8, evaluate $f_x(x, y)$ and $f_y(x, y)$ at the indicated point.**

12 03 ex 05 5. $f(x, y) = x^2y - x + 2y + 3$ at $(1, 2)$

12 03 ex 06 6. $f(x, y) = x^3 - 3x + y^2 - 6y$ at $(-1, 3)$

12 03 ex 07 7. $f(x, y) = \sin y \cos x$ at $(\pi/3, \pi/3)$

12 03 ex 08 8. $f(x, y) = \ln(xy)$ at $(-2, -3)$

12 03 exset 02 **In Exercises 9 – 26, find f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx}.**

12 03 ex 09 9. $f(x, y) = x^2y + 3x^2 + 4y - 5$

12 03 ex 10 10. $f(x, y) = y^3 + 3xy^2 + 3x^2y + x^3$

12 03 ex 13 11. $f(x, y) = \dfrac{x}{y}$

12 03 ex 14 12. $f(x, y) = \dfrac{4}{xy}$

12 03 ex 15 13. $f(x, y) = e^{x^2+y^2}$

12 03 ex 16 14. $f(x, y) = e^{x+2y}$

12 03 ex 17 15. $f(x, y) = \sin x \cos y$

12 03 ex 18 16. $f(x, y) = (x + y)^3$

12 03 ex 11 17. $f(x, y) = \cos(5xy^3)$

12 03 ex 12 18. $f(x, y) = \sin(5x^2 + 2y^3)$

12 03 ex 19 19. $f(x, y) = \sqrt{4xy^2 + 1}$

12 03 ex 20 20. $f(x, y) = (2x + 5y)\sqrt{y}$

12 03 ex 21 21. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

12 03 ex 22 22. $f(x, y) = 5x - 17y$

12 03 ex 23 23. $f(x, y) = 3x^2 + 1$

12 03 ex 24 24. $f(x, y) = \ln(x^2 + y)$

12 03 ex 25 25. $f(x, y) = \dfrac{\ln x}{4y}$

12 03 ex 26 26. $f(x, y) = 5e^x \sin y + 9$

12 03 exset 03 **In Exercises 27 – 30, form a function $z = f(x, y)$ such that f_x and f_y match those given.**

12 03 ex 27 27. $f_x = \sin y + 1, \quad f_y = x \cos y$

12 03 ex 28 28. $f_x = x + y, \quad f_y = x + y$

12 03 ex 29 29. $f_x = 6xy - 4y^2, \quad f_y = 3x^2 - 8xy + 2$

12 03 ex 30 30. $f_x = \dfrac{2x}{x^2 + y^2}, \quad f_y = \dfrac{2y}{x^2 + y^2}$

12 03 exset 04 **In Exercises 31 – 34, find f_x, f_y, f_z, f_{yz} and f_{zy}.**

12 03 ex 31 31. $f(x, y, z) = x^2e^{2y-3z}$

12 03 ex 32 32. $f(x, y, z) = x^3y^2 + x^3z + y^2z$

12 03 ex 33 33. $f(x, y, z) = \dfrac{3x}{7y^2z}$

12 03 ex 34 34. $f(x, y, z) = \ln(xyz)$

12.4 Differentiability and the Total Differential

We studied **differentials** in Section 4.4, where Definition 4.4.1 states that if $y = f(x)$ and f is differentiable, then $dy = f'(x)dx$. One important use of this differential is in Integration by Substitution. Another important application is approximation. Let $\Delta x = dx$ represent a change in x. When dx is small, $dy \approx \Delta y$, the change in y resulting from the change in x. Fundamental in this understanding is this: as dx gets small, the difference between Δy and dy goes to 0. Another way of stating this: as dx goes to 0, the *error* in approximating Δy with dy goes to 0.

We extend this idea to functions of two variables. Let $z = f(x, y)$, and let $\Delta x = dx$ and $\Delta y = dy$ represent changes in x and y, respectively. Let $\Delta z = f(x + dx, y + dy) - f(x, y)$ be the change in z over the change in x and y. Recalling that f_x and f_y give the instantaneous rates of z-change in the x- and y-directions, respectively, we can approximate Δz with $dz = f_x dx + f_y dy$; in words, the total change in z is approximately the change caused by changing x plus the change caused by changing y. In a moment we give an indication of whether or not this approximation is any good. First we give a name to dz.

Note: From Definition 12.4.1, we can write

$$dz = \langle f_x, f_y\rangle \cdot \langle dx, dy\rangle.$$

While not explored in this section, the vector $\langle f_x, f_y\rangle$ is seen again in the next section and fully defined in Section 12.6.

Definition 12.4.1 Total Differential

Let $z = f(x, y)$ be continuous on an open set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of** z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Example 12.4.1 Finding the total differential
Let $z = x^4e^{3y}$. Find dz.

Solution We compute the partial derivatives: $f_x = 4x^3e^{3y}$ and $f_y = 3x^4e^{3y}$. Following Definition 12.4.1, we have

$$dz = 4x^3e^{3y}dx + 3x^4e^{3y}dy.$$

We *can* approximate Δz with dz, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point (x_0, y_0), let E_x and E_y be functions of dx and dy such that $E_xdx + E_ydy$ describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_xdx + E_ydy\\ &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + E_xdx + E_ydy.\end{aligned}$$

Notes:

If the approximation of Δz by dz is good, then as dx and dy get small, so does $E_x dx + E_y dy$. The approximation of Δz by dz is even better if, as dx and dy go to 0, so do E_x and E_y. This leads us to our definition of differentiability.

Definition 12.4.2 Multivariable Differentiability

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_x and E_y be functions of dx and dy such that

$$\Delta z = dz + E_x dx + E_y dy.$$

1. f is **differentiable at** (x_0, y_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy \rangle || < \delta$, then $|| \langle E_x, E_y \rangle || < \varepsilon$. That is, as dx and dy go to 0, so do E_x and E_y.
2. f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Example 12.4.2 Showing a function is differentiable

Show $f(x, y) = xy + 3y^2$ is differentiable using Definition 12.4.2.

Solution We begin by finding $f(x + dx, y + dy)$, Δz, f_x and f_y.

$$\begin{aligned} f(x + dx, y + dy) &= (x + dx)(y + dy) + 3(y + dy)^2 \\ &= xy + xdy + ydx + dxdy + 3y^2 + 6ydy + 3dy^2. \end{aligned}$$

$\Delta z = f(x + dx, y + dy) - f(x, y)$, so

$$\Delta z = xdy + ydx + dxdy + 6ydy + 3dy^2.$$

It is straightforward to compute $f_x = y$ and $f_y = x + 6y$. Consider once more Δz:

$$\begin{aligned} \Delta z &= xdy + ydx + dxdy + 6ydy + 3dy^2 \qquad \text{(now reorder)} \\ &= ydx + xdy + 6ydy + dxdy + 3dy^2 \\ &= \underbrace{(y)}_{f_x} dx + \underbrace{(x + 6y)}_{f_y} dy + \underbrace{(dy)}_{E_x} dx + \underbrace{(3dy)}_{E_y} dy \\ &= f_x dx + f_y dy + E_x dx + E_y dy. \end{aligned}$$

With $E_x = dy$ and $E_y = 3dy$, it is clear that as dx and dy go to 0, E_x and E_y also go to 0. Since this did not depend on a specific point (x_0, y_0), we can say that $f(x, y)$

Notes:

is differentiable for all pairs (x, y) in $\mathbb{R}^2$, or, equivalently, that f is differentiable everywhere.

Our intuitive understanding of differentiability of functions $y = f(x)$ of one variable was that the graph of f was "smooth." A similar intuitive understanding of functions $z = f(x, y)$ of two variables is that the surface defined by f is also "smooth," not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

Theorem 12.4.1 Continuity and Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem 12.4.2 Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f_x and f_y are both continuous on S, then f is differentiable on S.

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 12.4.2 and Theorem 12.4.2, though: it is possible for a function f to be differentiable yet f_x and/or f_y is *not* continuous. Such strange behavior of functions is a source of delight for many mathematicians.

When f_x and f_y exist at a point but are not continuous at that point, we need to use other methods to determine whether or not f is differentiable at that point.

For instance, consider the function

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Notes:

We can find $f_x(0,0)$ and $f_y(0,0)$ using Definition 12.3.1:

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0; \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0. \end{aligned}$$

Both f_x and f_y *exist* at $(0,0)$, but they are not continuous at $(0,0)$, as

$$f_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \quad \text{and} \quad f_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

are not continuous at $(0,0)$. (Take the limit of f_x as $(x,y) \to (0,0)$ along the x- and y-axes; they give different results.) So even though f_x and f_y *exist* at every point in the x-y plane, they are not continuous. Therefore it is possible, by Theorem 12.4.2, for f to not be differentiable.

Indeed, it is not. One can show that f is not continuous at $(0,0)$ (see Example 12.2.4), and by Theorem 12.4.1, this means f is not differentiable at $(0,0)$.

Approximating with the Total Differential

By the definition, when f is differentiable dz is a good approximation for Δz when dx and dy are small. We give some simple examples of how this is used here.

Example 12.4.3 Approximating with the total differential
Let $z = \sqrt{x}\sin y$. Approximate $f(4.1, 0.8)$.

Solution Recognizing that $\pi/4 \approx 0.785 \approx 0.8$, we can approximate $f(4.1, 0.8)$ using $f(4, \pi/4)$. We can easily compute $f(4,\pi/4) = \sqrt{4}\sin(\pi/4) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.414$. Without calculus, this is the best approximation we could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let $\Delta z = f(4.1, 0.8) - f(4, \pi/4)$. The total differential dz is approximately equal to Δz, so

$$f(4.1, 0.8) - f(4,\pi/4) \approx dz \quad \Rightarrow \quad f(4.1, 0.8) \approx dz + f(4, \pi/4). \tag{12.1}$$

To find dz, we need f_x and f_y.

Notes:

$$f_x(x,y) = \frac{\sin y}{2\sqrt{x}} \quad \Rightarrow \quad f_x(4,\pi/4) = \frac{\sin \pi/4}{2\sqrt{4}} = \frac{\sqrt{2}/2}{4} = \sqrt{2}/8.$$

$$f_y(x,y) = \sqrt{x}\cos y \quad \Rightarrow \quad f_y(4,\pi/4) = \sqrt{4}\frac{\sqrt{2}}{2} = \sqrt{2}.$$

Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.8 with $\pi/4$ gives $dy \approx 0.015$. Thus

$$\begin{aligned} dz(4,\pi/4) &= f_x(4,\pi/4)(0.1) + f_y(4,\pi/4)(0.015) \\ &= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \\ &\approx 0.039. \end{aligned}$$

Returning to Equation (12.1), we have

$$f(4.1,0.8) \approx 0.039 + 1.414 = 1.4531.$$

We, of course, can compute the actual value of $f(4.1,0.8)$ with a calculator; the actual value, accurate to 5 places after the decimal, is 1.45254. Obviously our approximation is quite good.

The point of the previous example was *not* to develop an approximation method for known functions. After all, we can very easily compute $f(4.1,0.8)$ using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

"New position = old position + amount of change," so
"New position $\approx$ old position + approximate amount of change."

In the previous example, we could easily compute $f(4,\pi/4)$ and could approximate the amount of z-change when computing $f(4.1,0.8)$, letting us approximate the new z-value.

It may be surprising to learn that it is not uncommon to know the values of f, f_x and f_y at a particular point without actually knowing the function f. The total differential gives a good method of approximating f at nearby points.

Example 12.4.4 Approximating an unknown function
Given that $f(2,-3) = 6$, $f_x(2,-3) = 1.3$ and $f_y(2,-3) = -0.6$, approximate $f(2.1,-3.03)$.

Notes:

Solution The total differential approximates how much f changes from the point $(2,-3)$ to the point $(2.1,-3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have

$$\begin{aligned} dz &= f_x(2,-3)dx + f_y(2,-3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148. \end{aligned}$$

The change in z is approximately 0.148, so we approximate $f(2.1,-3.03) \approx 6.148$.

Error/Sensitivity Analysis

The total differential gives an approximation of the change in z given small changes in x and y. We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

Example 12.4.5 Sensitivity analysis

A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Solution A cylindrical solid with height h and radius r has volume $V = \pi r^2 h$. We can view V as a function of two variables, r and h. We can compute partial derivatives of V:

$$\frac{\partial V}{\partial r} = V_r(r,h) = 2\pi rh \qquad \text{and} \qquad \frac{\partial V}{\partial h} = V_h(r,h) = \pi r^2.$$

The total differential is $dV = (2\pi rh)dr + (\pi r^2)dh$. When $h = 10$ and $r = 2$, we have $dV = 40\pi dr + 4\pi dh$. Note that the coefficient of dr is $40\pi \approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

Notes:

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

Definition 12.4.3 Total Differential

Let $w = f(x, y, z)$ be continuous on an open set S. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the **total differential of** w is

$$dw = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

This differential can be a good approximation of the change in w when $w = f(x, y, z)$ is **differentiable**.

Definition 12.4.4 Multivariable Differentiability

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0), let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions of dx, dy and dz such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. f is **differentiable at** (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy, dz\rangle || < \delta$, then $|| \langle E_x, E_y, E_z\rangle || < \varepsilon$.

2. f is **differentiable on** B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is **differentiable everywhere**.

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 12.4.2.

Notes:

Theorem 12.4.3 Continuity and Differentiability of Functions of Three Variables

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).
2. If f_x, f_y and f_z are continuous on B, then f is differentiable on B.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we encounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a functions to be "differentiable," along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Notes:

Exercises 12.4

Terms and Concepts

12 04 ex 01 1. T/F: If $f(x,y)$ is differentiable on S, the f is continuous on S.

12 04 ex 02 2. T/F: If f_x and f_y are continuous on S, then f is differentiable on S.

12 04 ex 03 3. T/F: If $z=f(x,y)$ is differentiable, then the change in z over small changes dx and dy in x and y is approximately dz.

12 04 ex 04 4. Finish the sentence: "The new z-value is approximately the old z-value plus the approximate ________."

Problems

12 04 exset 01 **In Exercises 5 – 8, find the total differential dz.**

12 04 ex 05 5. $z = x\sin y + x^2$

12 04 ex 06 6. $z = (2x^2+3y)^2$

12 04 ex 07 7. $z = 5x - 7y$

12 04 ex 08 8. $z = xe^{x+y}$

12 04 exset 02 **In Exercises 9 – 12, a function $z=f(x,y)$ is given. Give the indicated approximation using the total differential.**

12 04 ex 09 9. $f(x,y) = \sqrt{x^2+y}$. Approximate $f(2.95, 7.1)$ knowing $f(3,7)=4$.

12 04 ex 10 10. $f(x,y) = \sin x\cos y$. Approximate $f(0.1,-0.1)$ knowing $f(0,0)=0$.

12 04 ex 11 11. $f(x,y) = x^2y - xy^2$. Approximate $f(2.04, 3.06)$ knowing $f(2,3)=-6$.

12 04 ex 12 12. $f(x,y) = \ln(x-y)$. Approximate $f(5.1, 3.98)$ knowing $f(5,4)=0$.

12 04 exset 04 **Exercises 13 – 16 ask a variety of questions dealing with approximating error and sensitivity analysis.**

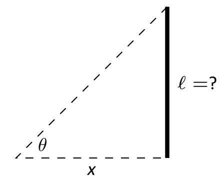
12 04 ex 15 13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

12 04 ex 16 14. **Projectile Motion:** The x-value of an object moving under the principles of projectile motion is $x(\theta, v_0, t) = (v_0\cos\theta)t$. A particular projectile is fired with an initial velocity of $v_0 = 250$ft/s and an angle of elevation of $\theta = 60^\circ$. It travels a distance of 375ft in 3 seconds.

Is the projectile more sensitive to errors in initial speed or angle of elevation?

12 04 ex 17 15. The length ℓ of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85°, and the distance x is measured to be 30'. Assume that the triangle formed is a right triangle.

Is the measurement of the length of ℓ more sensitive to errors in the measurement of x or in θ?



12 04 ex 18 16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance D can be viewed as the product of the length ℓ of a measuring tape times the number n of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (I.e., $30 = 12\times 2.5$.) Thus $D = n\ell$.

Suppose each time a measurement is taken with the tape, the recorded distance is within 1/16" of the actual distance. (I.e., $d\ell = 1/16'' \approx 0.005$ft). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

12 04 exset 05 **In Exercises 17 – 18, find the total differential dw.**

12 04 ex 19 17. $w = x^2yz^3$

12 04 ex 20 18. $w = e^x\sin y\ln z$

12 04 exset 03 **In Exercises 19 – 22, use the information provided and the total differential to make the given approximation.**

12 04 ex 13 19. $f(3,1) = 7$, $f_x(3,1) = 9$, $f_y(3,1) = -2$. Approximate $f(3.05, 0.9)$.

12 04 ex 14 20. $f(-4,2) = 13$, $f_x(-4,2) = 2.6$, $f_y(-4,2) = 5.1$. Approximate $f(-4.12, 2.07)$.

12 04 ex 21 21. $f(2,4,5) = -1$, $f_x(2,4,5) = 2$, $f_y(2,4,5) = -3$, $f_z(2,4,5) = 3.7$. Approximate $f(2.5, 4.1, 4.8)$.

12 04 ex 22 22. $f(3,3,3) = 5$, $f_x(3,3,3) = 2$, $f_y(3,3,3) = 0$, $f_z(3,3,3) = -2$. Approximate $f(3.1, 3.1, 3.1)$.

A: SOLUTIONS TO SELECTED PROBLEMS

Chapter 12

Section 12.1

12 01 ex 01 1. Answers will vary.

12 01 ex 02 2. surface

12 01 ex 03 3. topographical

12 01 ex 04 4. T

12 01 ex 05 5. surface

12 01 ex 06 6. When level curves are close together, it means the function is changing z-values rapidly. When far apart, it changes z-values slowly.

12 01 ex 07 7. domain: $\mathbb{R}^2$
range: $z \geq 2$

12 01 ex 08 8. domain: $\mathbb{R}^2$
range: $\mathbb{R}$

12 01 ex 09 9. domain: $\mathbb{R}^2$
range: $\mathbb{R}$

12 01 ex 10 10. domain: $x \neq 2y$; in set notation, $\{(x, y) \mid x \neq 2y\}$
range: $z \neq 0$

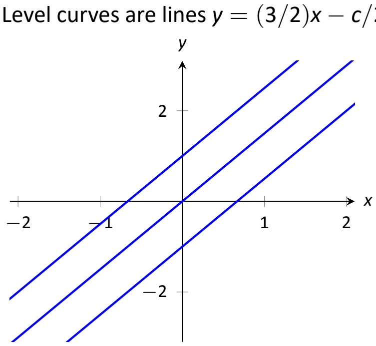
12 01 ex 11 11. domain: $\mathbb{R}^2$
range: $0 < z \leq 1$

12 01 ex 12 12. domain: $\mathbb{R}^2$
range: $-1 \leq z \leq 1$

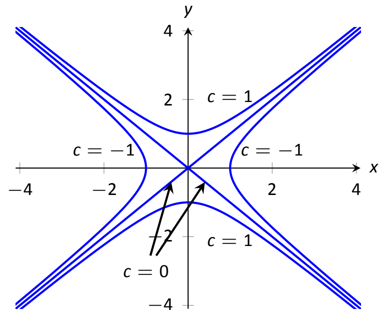
12 01 ex 13 13. domain: $\{(x, y) \mid x^2 + y^2 \leq 9\}$, i.e., the domain is the circle and interior of a circle centered at the origin with radius 3.
range: $0 \leq z \leq 3$

12 01 ex 14 14. domain: $\{(x, y) \mid x^2 + y^2 \geq 9\}$, i.e., the domain is the exterior of the circle (not including the circle itself) centered at the origin with radius 3.
range: $0 < z < \infty$, or $(0, \infty)$

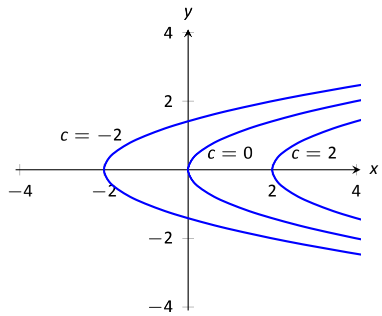
12 01 ex 15 15. Level curves are lines $y = (3/2)x - c/2$.



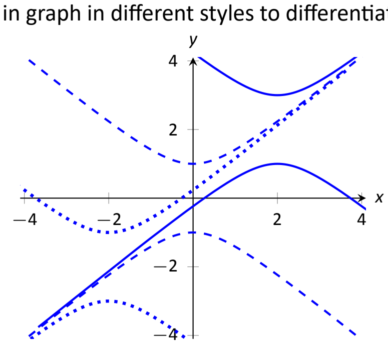
12 01 ex 16 16. Level curves are hyperbolas $\frac{x^2}{c} - \frac{y^2}{c} = 1$, except for $c = 0$, where the level curve is the pair of lines $y = x$, $y = -x$.



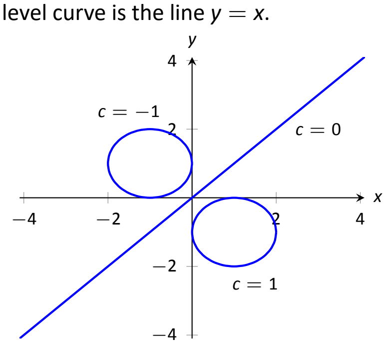
12 01 ex 17 17. Level curves are parabolas $x = y^2 + c$.



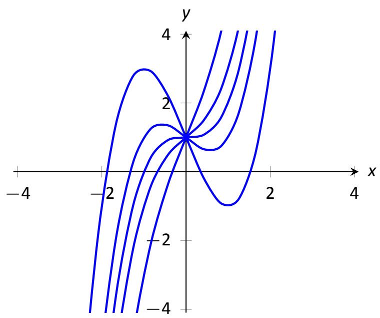
12 01 ex 18 18. Level curves are hyperbolas $(x-c)^2 - (y-c)^2 = 1$, drawn in graph in different styles to differentiate the curves.



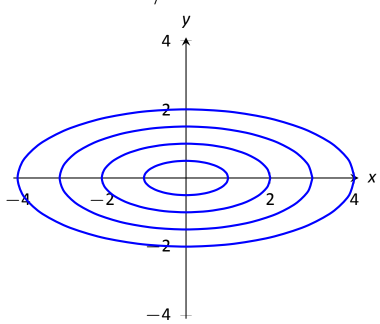
12 01 ex 19 19. When $c \neq 0$, the level curves are circles, centered at $(1/c, -1/c)$ with radius $\sqrt{2/c^2 - 1}$. When $c = 0$, the level curve is the line $y = x$.



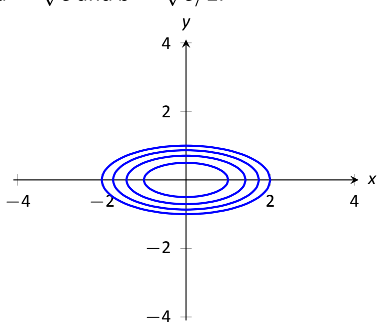
12 01 ex 20 20. Level curves are cubics of the form $y = x^3 + cx + 1$. Note how each curve passes through $(0, 1)$ and that the function is not defined at $x = 0$.



12 01 ex 21 21. Level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$.



12 01 ex 22 22. Level curves are ellipses of the form $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$, i.e., $a = \sqrt{c}$ and $b = \sqrt{c}/2$.



12 01 ex 24 23. domain: $x + 2y - 4z \neq 0$; the set of points in $\mathbb{R}^3$ NOT in the domain form a plane through the origin.
range: $\mathbb{R}$

12 01 ex 25 24. domain: $x^2 + y^2 + z^2 \neq 1$; the set of points in $\mathbb{R}^3$ NOT in the domain form a sphere of radius 1.
range: $(-\infty, 0) \cup [1, \infty)$

12 01 ex 26 25. domain: $z \geq x^2 - y^2$; the set of points in $\mathbb{R}^3$ above (and including) the hyperbolic paraboloid $z = x^2 - y^2$.
range: $[0, \infty)$

12 01 ex 27 26. domain: $\mathbb{R}^3$
range: $\mathbb{R}$

12 01 ex 28 27. The level surfaces are spheres, centered at the origin, with radius $\sqrt{c}$.

12 01 ex 29 28. The level surfaces are hyperbolic paraboloids of the form $z = x^2 - y^2 + c$; each is shifted up/down by c.

12 01 ex 30 29. The level surfaces are paraboloids of the form $z = \frac{x^2}{c} + \frac{y^2}{c}$; the larger c, the "wider" the paraboloid.

12 01 ex 31 30. The level surfaces are planes through the origin of the form $cx - cy - z = 0$, that is, planes through the origin with normal vector $\langle c, -c, -1\rangle$.

12 01 ex 23 31. The level curves for each surface are similar; for $z = \sqrt{x^2 + 4y^2}$ the level curves are ellipses of the form $\frac{x^2}{c^2} + \frac{y^2}{c^2/4} = 1$, i.e., $a = c$ and $b = c/2$; whereas for $z = x^2 + 4y^2$ the level curves are ellipses of the form $\frac{x^2}{c} + \frac{y^2}{c/4} = 1$, i.e., $a = \sqrt{c}$ and $b = \sqrt{c}/2$. The first set of ellipses are spaced evenly apart, meaning the function grows at a constant rate; the second set of ellipses are more closely spaced together as c grows, meaning the function grows faster and faster as c increases.
The function $z = \sqrt{x^2 + 4y^2}$ can be rewritten as $z^2 = x^2 + 4y^2$, an elliptic cone; the function $z = x^2 + 4y^2$ is a paraboloid, each matching the description above.

Section 12.2

12 02 ex 01 1. Answers will vary.

12 02 ex 02 2. Answers will vary. One answer is "As (x, y) gets close to $(1, 2)$, $f(x, y)$ gets close to 17."

12 02 ex 17 3. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 \leq 1\}$

12 02 ex 18 4. Answers will vary.
One possible answer: $\{(x, y) | y \geq x^2\}$

12 02 ex 19 5. Answers will vary.
One possible answer: $\{(x, y) | x^2 + y^2 < 1\}$

12 02 ex 20 6. Answers will vary.
One possible answer: $\{(x, y) | y > x^2\}$

12 02 ex 03 7.
   (a) Answers will vary.
   interior point: $(1, 3)$
   boundary point: $(3, 3)$
   (b) S is a closed set
   (c) S is bounded

12 02 ex 04 8.
   (a) Answers will vary.
   interior point: $(-5, 28)$
   boundary point: $(3, 9)$
   (b) S is an open set
   (c) S is unbounded

12 02 ex 05 9.
   (a) Answers will vary.
   interior point: none
   boundary point: $(0, -1)$
   (b) S is a closed set, consisting only of boundary points
   (c) S is bounded

12 02 ex 06 10.
   (a) Answers will vary.
   Interior point: $(0, 1)$
   Boundary point: $(0, 0)$
   (b) S is a closed set, containing all of its boundary points.
   (c) S is unbounded.

12 02 ex 09 11.

(a) $D = \{(x,y) \mid 9 - x^2 - y^2 \geq 0\}$.

(b) D is a closed set.

(c) D is bounded.

12 02 ex 07 12.

(a) $D = \{(x,y) \mid y \geq x^2\}$.

(b) D is a closed set.

(c) D is unbounded.

12 02 ex 08 13.

(a) $D = \{(x,y) \mid y > x^2\}$.

(b) D is an open set.

(c) D is unbounded.

12 02 ex 10 14.

(a) $D = \{(x,y) \mid (x,y) \neq (0,0)\}$.

(b) D is an open set.

(c) D is unbounded.

12 02 ex 12 15.

(a) Along $y = 0$, the limit is 1.

(b) Along $x = 0$, the limit is -1.

Since the above limits are not equal, the limit does not exist.

12 02 ex 11 16.

(a) Along $y = mx$, the limit is $\frac{m+1}{m-1}$.

Since the above limit varies according to what m is used, each limit is different, meaning the overall limit does not exist.

12 02 ex 13 17.

(a) Along $y = mx$, the limit is $\dfrac{mx(1-m)}{m^2x+1} = 0$ for all m.

(b) Along $x = 0$, the limit is -1.

Since the above limits are not equal, the limit does not exist.

12 02 ex 14 18.

(a) Along $y = mx$, the limit is:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2)}{y} = \lim_{x\to 0} \frac{\sin(x^2)}{mx}$$

apply L'Hôpital's Rule

$$= \lim_{x\to 0} \frac{2x\cos(x^2)}{m} = 0.$$

(b) Along $x = 0$, the limit is:

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2)}{y} = \lim_{x\to 0} \frac{\sin(x^2)}{x^2}.$$

This can be evaluated with L'Hôpital's Rule or from known limits; it is 1.

Since the limits along the lines $y = mx$ are not the same as the limit along $y = x^2$, the overall limit does not exist.

12 02 ex 15 19.

(a) Along $y = 2$, the limit is:

$$\lim_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1} = \lim_{x\to 1} \frac{x-1}{x^2-1} = \lim_{x\to 1} \frac{1}{x+1} = 1/2.$$

(b) Along $y = x + 1$, the limit is:

$$\lim_{(x,y)\to(1,2)} \frac{x+y-3}{x^2-1} = \lim_{x\to 1} \frac{2(x-1)}{x^2-1} = \lim_{x\to 1} \frac{2}{x+1} = 1.$$

Since the limits along the lines $y = 2$ and $y = x + 1$ differ, the overall limit does not exist.

12 02 ex 16 20.

(a) Along $x = \pi$, the limit is:

$$\lim_{(x,y)\to(\pi,\pi/2)} \frac{\sin x}{\cos y} = \lim_{y\to\pi/2} \frac{0}{\cos y} = 0.$$

(b) Along $y = x - \pi/2$, the limit is:

$$\lim_{(x,y)\to(\pi,\pi/2)} \frac{\sin x}{\cos y} = \lim_{x\to\pi} \frac{\sin x}{\cos(x-\pi/2)}$$

Apply L'Hôpital's Rule:

$$= \lim_{x\to\pi} \frac{\cos x}{\sin(x-\pi/2)} = -1.$$

Since the limits along the lines $x = \pi$ and $y = x - \pi$ differ, the overall limit does not exist.

Section 12.3

12 03 ex 01 1. A constant is a number that is added or subtracted in an expression; a coefficient is a number that is being multiplied by a nonconstant function.

12 03 ex 02 2. Answers will vary; each should include something about treating y as a constant or a coefficient.

12 03 ex 03 3. f_x

12 03 ex 04 4. f_y

12 03 ex 05 5. $f_x = 2xy - 1, f_y = x^2 + 2$
$f_x(1,2) = 3, f_y(1,2) = 3$

12 03 ex 06 6. $f_x = 3x^2 - 3, f_y = 2y - 6$
$f_x(-1,3) = 0, f_y(-1,3) = 0$

12 03 ex 07 7. $f_x = -\sin x \sin y, f_y = \cos x \cos y$
$f_x(\pi/3,\pi/3) = -3/4, f_y(\pi/3,\pi/3) = 1/4$

12 03 ex 08 8. $f_x = 1/x, f_y = 1/y$
$f_x(-2,-3) = -1/2, f_y(-2,-3) = -1/3$

12 03 ex 09 9. $f_x = 2xy + 6x, f_y = x^2 + 4$
$f_{xx} = 2y + 6, f_{yy} = 0$
$f_{xy} = 2x, f_{yx} = 2x$

12 03 ex 10 10. $f_x = 3x^2 + 6xy + 3y^2, f_y = 3x^2 + 6xy + 3y^2$
$f_{xx} = 6x + 6y, f_{yy} = 6x + 6y$
$f_{xy} = 6x + 6y, f_{yx} = 6x + 6y$

12 03 ex 13 11. $f_x = 1/y, f_y = -x/y^2$
$f_{xx} = 0, f_{yy} = 2x/y^3$
$f_{xy} = -1/y^2, f_{yx} = -1/y^2$

12 03 ex 14 12. $f_x = -4/(x^2y), f_y = -4/(xy^2)$
$f_{xx} = 8/(x^3y), f_{yy} = 8/(xy^3)$
$f_{xy} = 4/(x^2y^2), f_{yx} = 4/(x^2y^2)$

12 03 ex 15 13. $f_x = 2xe^{x^2+y^2}, f_y = 2ye^{x^2+y^2}$
$f_{xx} = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2}, f_{yy} = 2e^{x^2+y^2} + 4y^2e^{x^2+y^2}$
$f_{xy} = 4xye^{x^2+y^2}, f_{yx} = 4xye^{x^2+y^2}$

12 03 ex 16 14. $f_x = e^{x+2y}, f_y = 2e^{x+2y}$
$f_{xx} = e^{x+2y}, f_{yy} = 4e^{x+2y}$
$f_{xy} = 2e^{x+2y}, f_{yx} = 2e^{x+2y}$

12 03 ex 17 15. $f_x = \cos x \cos y, f_y = -\sin x \sin y$
$f_{xx} = -\sin x \cos y, f_{yy} = -\sin x \cos y$
$f_{xy} = -\sin y \cos x, f_{yx} = -\sin y \cos x$

12 03 ex 18 16. $f_x = 3(x+y)^2, f_y = 3(x+y)^2$
$f_{xx} = 6(x+y), f_{yy} = 6(x+y)$
$f_{xy} = 6(x+y), f_{yx} = 6(x+y)$

12 03 ex 11 17. $f_x = -5y^3 \sin(5xy^3), f_y = -15xy^2 \sin(5xy^3)$
$f_{xx} = -25y^6 \cos(5xy^3),$
$f_{yy} = -225x^2y^4 \cos(5xy^3) - 30xy \sin(5xy^3)$
$f_{xy} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3),$
$f_{yx} = -75xy^5 \cos(5xy^3) - 15y^2 \sin(5xy^3)$

12 03 ex 12 18. $f_x = 10x \cos(5x^2 + 2y^3), f_y = 6y^2 \cos(5x^2 + 2y^3)$
$f_{xx} = 10 \cos(5x^2 + 2y^3) - 100x^2 \sin(5x^2 + 2y^3),$
$f_{yy} = 12y \cos(5x^2 + 2y^3) - 36y^4 \sin(5x^2 + 2y^3)$
$f_{xy} = -60xy^2 \sin(5x^2 + 2y^3), f_{yx} = -60xy^2 \sin(5x^2 + 2y^3)$

12 03 ex 19 19. $f_x = \frac{2y^2}{\sqrt{4xy^2+1}}, f_y = \frac{4xy}{\sqrt{4xy^2+1}}$
$f_{xx} = -\frac{4y^4}{\sqrt{4xy^2+1}^3}, f_{yy} = -\frac{16x^2y^2}{\sqrt{4xy^2+1}^3} + \frac{4x}{\sqrt{4xy^2+1}}$
$f_{xy} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}, f_{yx} = -\frac{8xy^3}{\sqrt{4xy^2+1}^3} + \frac{4y}{\sqrt{4xy^2+1}}$

12 03 ex 20 20. $f_x = 2\sqrt{y}, f_y = 5\sqrt{y} + \frac{2x+5y}{2\sqrt{y}}$
$f_{xx} = 0, f_{yy} = \frac{5}{\sqrt{y}} - \frac{2x+5y}{4y^{3/2}}$
$f_{xy} = \frac{1}{\sqrt{y}}, f_{yx} = \frac{1}{\sqrt{y}}$

12 03 ex 21 21. $f_x = -\frac{2x}{(x^2+y^2+1)^2}, f_y = -\frac{2y}{(x^2+y^2+1)^2}$
$f_{xx} = \frac{8x^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}, f_{yy} = \frac{8y^2}{(x^2+y^2+1)^3} - \frac{2}{(x^2+y^2+1)^2}$
$f_{xy} = \frac{8xy}{(x^2+y^2+1)^3}, f_{yx} = \frac{8xy}{(x^2+y^2+1)^3}$

12 03 ex 22 22. $f_x = 5, f_y = -17$
$f_{xx} = 0, f_{yy} = 0$
$f_{xy} = 0, f_{yx} = 0$

12 03 ex 23 23. $f_x = 6x, f_y = 0$
$f_{xx} = 6, f_{yy} = 0$
$f_{xy} = 0, f_{yx} = 0$

12 03 ex 24 24. $f_x = \frac{2x}{(x^2+y)}, f_y = \frac{1}{(x^2+y)}$
$f_{xx} = -\frac{4x^2}{(x^2+y)^2} + \frac{2}{(x^2+y)}, f_{yy} = -\frac{1}{(x^2+y)^2}$
$f_{xy} = -\frac{2x}{(x^2+y)^2}, f_{yx} = -\frac{2x}{(x^2+y)^2}$

12 03 ex 25 25. $f_x = \frac{1}{4xy}, f_y = -\frac{\ln x}{4y^2}$
$f_{xx} = -\frac{1}{4x^2y}, f_{yy} = \frac{\ln x}{2y^3}$
$f_{xy} = -\frac{1}{4xy^2}, f_{yx} = -\frac{1}{4xy^2}$

12 03 ex 26 26. $f_x = 5e^x \sin y, f_y = 5e^x \cos y$
$f_{xx} = 5e^x \sin y, f_{yy} = -5e^x \sin y$
$f_{xy} = 5e^x \cos y, f_{yx} = 5e^x \cos y$

12 03 ex 27 27. $f(x,y) = x \sin y + x + C$, where C is any constant.

12 03 ex 28 28. $f(x,y) = \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 + C$, where C is any constant.

12 03 ex 29 29. $f(x,y) = 3x^2y - 4xy^2 + 2y + C$, where C is any constant.

12 03 ex 30 30. $f(x,y) = \ln(x^2 + y^2) + C$, where C is any constant.

12 03 ex 31 31. $f_x = 2xe^{2y-3z}, f_y = 2x^2e^{2y-3z}, f_z = -3x^2e^{2y-3z}$
$f_{yz} = -6x^2e^{2y-3z}, f_{zy} = -6x^2e^{2y-3z}$

12 03 ex 32 32. $f_x = 3x^2y^2 + 3x^2z, f_y = 2x^3y + 2yz, f_z = x^3 + y^2$
$f_{yz} = 2y, f_{zy} = 2y$

12 03 ex 33 33. $f_x = \frac{3}{7y^2z}, f_y = -\frac{6x}{7y^3z}, f_z = -\frac{3x}{7y^2z^2}$
$f_{yz} = \frac{6x}{7y^3z^2}, f_{zy} = \frac{6x}{7y^3z^2}$

12 03 ex 34 34. $f_x = \frac{1}{x}, f_y = \frac{1}{y}, f_z = \frac{1}{z}$
$f_{yz} = 0, f_{zy} = 0$

Section 12.4

12 04 ex 01 1. T

12 04 ex 02 2. T

12 04 ex 03 3. T

12 04 ex 04 4. amount of change

12 04 ex 05 5. $dz = (\sin y + 2x)dx + (x \cos y)dy$

12 04 ex 06 6. $dz = 8x(2x^2 + 3y)dx + 6(2x^2 + 3y)dy$

12 04 ex 07 7. $dz = 5dx - 7dy$

12 04 ex 08 8. $dz = (e^{x+y} + xe^{x+y})dx + xe^{x+y}dy$

12 04 ex 09 9. $dz = \frac{x}{\sqrt{x^2+y}}dx + \frac{1}{2\sqrt{x^2+y}}dy$, with $dx = -0.05$ and $dy = .1$. At $(3,7)$, $dz = 3/4(-0.05) + 1/8(.1) = -0.025$, so $f(2.95, 7.1) \approx -0.025 + 4 = 3.975$.

12 04 ex 10 10. $dz = (\cos x \cos y)dx - (\sin x \sin y)dy$, with $dx = 0.1$ and $dy = -0.1$. At $(0,0)$, $dz = 1(.1) - (0)(-0.1) = 0.1$, so $f(0.1, -0.1) \approx 0.1 + 0 = 0.1$.

12 04 ex 11 11. $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$, with $dx = 0.04$ and $dy = 0.06$. At $(2,3)$,
$dz = 3(0.04) + (-8)(0.06) = -0.36$, so
$f(2.04, 3.06) \approx -0.36 - 6 = -6.36$.

12 04 ex 12 12. $dz = \frac{1}{x-y}dx - \frac{1}{x-y}dy$, with $dx = 0.1$ and $dy = -0.02$. At $(5,4)$, $dz = 1(0.1) + (-1)(-0.02) = 0.12$, so $f(5.1, 3.98) \approx 0.12 + 0 = 0.12$.

12 04 ex 15 13. The total differential of volume is $dV = 4\pi dr + \pi dh$. The coefficient of dr is greater than the coefficient of dh, so the volume is more sensitive to changes in the radius.

12 04 ex 16 14. Distance of the projectile is a function of two variables (leaving $t = 3$): $D(v_0, \theta) = 3v_0 \cos\theta$. The total differential of D is $dD = 3\cos\theta dv_0 - 3v_0 \sin\theta d\theta$. The coefficient of $d\theta$ has a much greater magnitude than the coefficient of dv_0, so a small change in the angle of elevation has a much greater effect on distance traveled than a small change in initial velocity.

12 04 ex 17

15. Using trigonometry, $\ell = x\tan\theta$, so $d\ell = \tan\theta dx + x\sec^2\theta d\theta$. With $\theta = 85^\circ$ and $x = 30$, we have $d\ell = 11.43dx + 3949.38d\theta$. The measured length of the wall is much more sensitive to errors in θ than in x. While it can be difficult to compare sensitivities between measuring feet and measuring degrees (it is somewhat like "comparing apples to oranges"), here the coefficients are so different that the result is clear: a small error in degree has a much greater impact than a small error in distance.

12 04 ex 18

16. With $D = n\ell$, the total differential is $dD = \ell\, dn + n\, d\ell$. If one measures with a short tape, n must be large and hence $n\, d\ell$ is going to be greater than when a large tape is used (wherein n will be small).

12 04 ex 19

17. $dw = 2xyz^3\, dx + x^2z^3\, dy + 3x^2yz^2\, dz$

12 04 ex 20

18. $dw = e^x \sin y \ln z\, dx + e^x \cos y \ln z\, dy + e^x \sin y \frac{1}{z}\, dz$

12 04 ex 13

19. $dx = 0.05$, $dy = -0.1$.
$dz = 9(.05) + (-2)(-0.1) = 0.65$. So
$f(3.05, 0.9) \approx 7 + 0.65 = 7.65$.

12 04 ex 14

20. $dx = -0.12$, $dy = 0.07$.
$dz = 2.6(-.12) + (5.1)(0.07) = 0.045$. So
$f(-4.12, 2.07) \approx 13 + 0.045 = 13.045$.

12 04 ex 21

21. $dx = 0.5$, $dy = 0.1$, $dz = -0.2$.
$dw = 2(0.5) + (-3)(0.1) + 3.7(-0.2) = -0.04$, so
$f(2.5, 4.1, 4.8) \approx -1 - 0.04 = -1.04$.

12 04 ex 22

22. $dx = 0.1$, $dy = 0.1$, $dz = 0.1$.
$dw = 2(0.1) + (0)(0.1) + (-2)(.1) = 0$, so
$f(3.1, 3.1, 3.1) \approx 5 + 0 = 5$.

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