1: LIMITS

Calculus means "a method of calculation or reasoning." When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17^{th} century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then "distance = rate $\times$ time." But what if the rate is not constant – can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was "a" calculus. However, as the power and importance of their discovery took hold, it became known to many as "the" calculus. Today, we generally shorten this to discuss "calculus."

The foundation of "the calculus" is the *limit.* It is a tool to describe a particular behavior of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make "finding limits" tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

1.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \frac{\sin x}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes "near the value

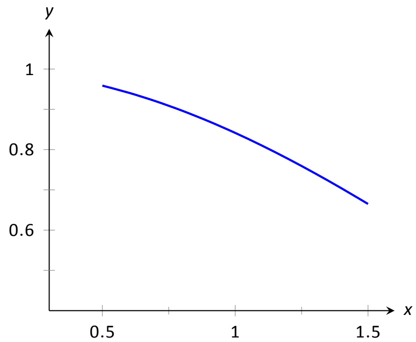

Figure 1.1: $\sin(x)/x$ near $x = 1$.

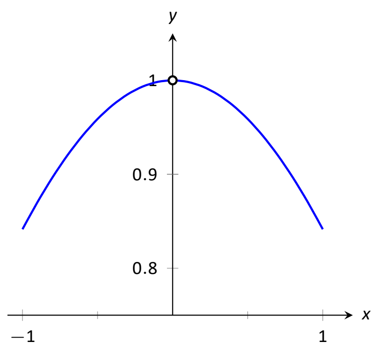

Figure 1.2: $\sin(x)/x$ near $x = 0$.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

Figure 1.3: Values of $\sin(x)/x$ with x near 1.

1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure 1.1, where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

Consider this again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 1.2, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \rightarrow \frac{\sin 0}{0} \rightarrow \text{"}\frac{0}{0}\text{"}.$$

The expression "0/0" has no value; it is *indeterminate.* Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition (that will come in the next section); this is a pseudo-definition that will allow us to explore the idea of a limit.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin x}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} \approx 1.$$

(We *approximated* these limits, hence used the "$\approx$" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Figure 1.3.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is in bold to highlight the fact that when considering limits, we are *not* concerned

Notes:

with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

Now approximate $\lim_{x\to 0}\sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure 1.2. The table in Figure 1.4 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row in bold again to stress that we are not concerned with the value of our function at $x = 0$, only on the behavior of the function *near* 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0}\sin(x)/x$; that is,

$$\lim_{x\to 0}\sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 1.4: Values of $\sin(x)/x$ with x near 0.

Example 1 **Approximating the value of a limit**

Use graphical and numerical methods to approximate

$$\lim_{x\to 3}\frac{x^2-x-6}{6x^2-19x+3}.$$

SOLUTION To graphically approximate the limit, graph

$$y = (x^2-x-6)/(6x^2-19x+3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figures 1.5 and 1.6, respectively.

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3}\frac{x^2-x-6}{6x^2-19x+3} \approx 0.294.$$

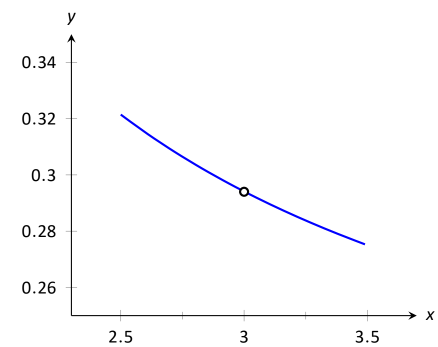

Figure 1.5: Graphically approximating a limit in Example 1.

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	**not defined**
3.001	0.294073
3.01	0.293669
3.1	0.289773

Figure 1.6: Numerically approximating a limit in Example 1.

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?

2. How many values of x in a table are “enough?” In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

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Graphs are useful since they give a visual understanding concerning the behavior of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically. Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example 1, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

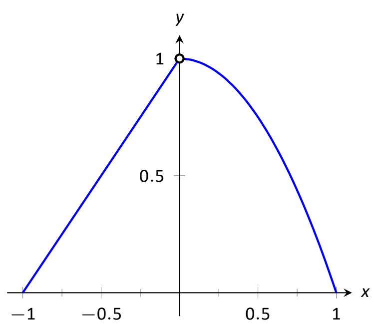

Figure 1.7: Graphically approximating a limit in Example 2.

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Figure 1.8: Numerically approximating a limit in Example 2.

Example 2 Approximating the value of a limit

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x+1 & x<0 \\ -x^2+1 & x>0 \end{cases}.$$

Solution Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 1.7 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

The table shown in Figure 1.8 shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ for some numbers L for all values of c, for there may not be a number that $f(x)$ is approaching. There are three ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.
2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c.

Notes:

We'll explore each of these in turn.

Example 3 **Different Values Approached From Left and Right**

Explore why $\lim_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \leq 1 \\ x & x > 1 \end{cases}.$$

SOLUTION A graph of $f(x)$ around $x = 1$ and a table are given in Figures 1.9 and 1.10, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behavior is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is not approaching one value as x approaches 1.

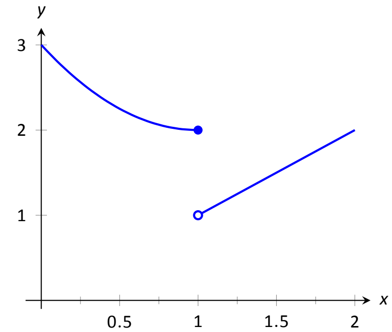

Figure 1.9: Observing no limit as $x \to 1$ in Example 3.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Figure 1.10: Values of $f(x)$ near $x = 1$ in Example 3.

Example 4 **The Function Grows Without Bound**

Explore why $\lim_{x\to 1} 1/(x-1)^2$ does not exist.

SOLUTION A graph and table of $f(x) = 1/(x-1)^2$ are given in Figures 1.11 and 1.12, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number}.$$

Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

does not exist.

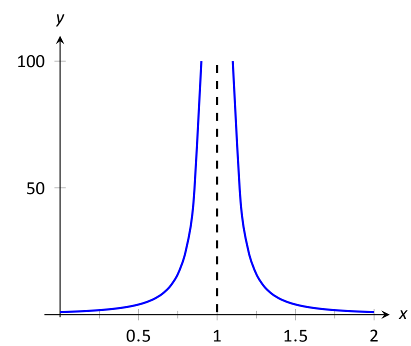

Figure 1.11: Observing no limit as $x \to 1$ in Example 4.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Figure 1.12: Values of $f(x)$ near $x = 1$ in Example 4.

Example 5 **The Function Oscillates**

Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

SOLUTION Two graphs of $f(x) = \sin(1/x)$ are given in Figures 1.13. Figure 1.13(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate

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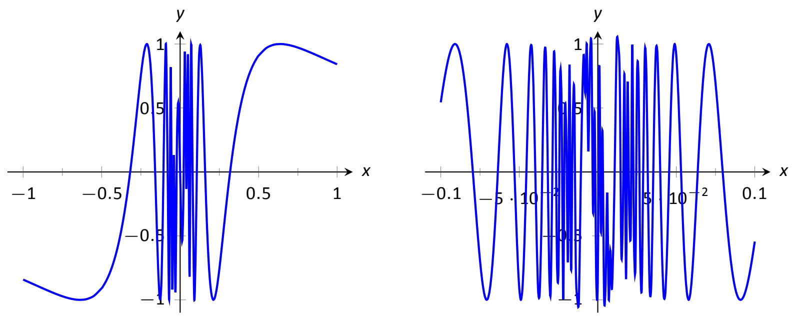
near $x = 0$. One might think that despite the oscillation, as x approaches 0, $f(x)$ approaches 0. However, Figure 1.13(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in the table in Figure 1.13(c), we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinite times! Because of this oscillation,

$\lim\limits_{x\to 0} \sin(1/x)$ does not exist.



(a) (b)

x	$\sin(1/x)$
0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	-0.349994
$1. \times 10^{-7}$	0.420548

(c)

Figure 1.13: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 5.

Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \quad \text{and} \quad f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's *average velocity* was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a *difference quotient*:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{10}{4} = 2.5\text{ft/s}.$$

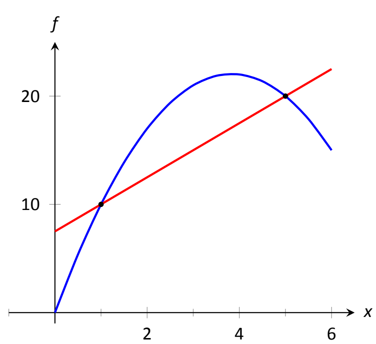


Figure 1.14: Interpreting a difference quotient as the slope of a secant line.

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This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing: given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 1.14.

Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure 1.15. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \ ?$$

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 1.16. The table gives us reason to assume the value of the limit is about 8.5.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

In the next section we give the formal definition of the limit and begin our study of finding limits analytically. In the following exercises, we continue our introduction and approximate the value of limits.

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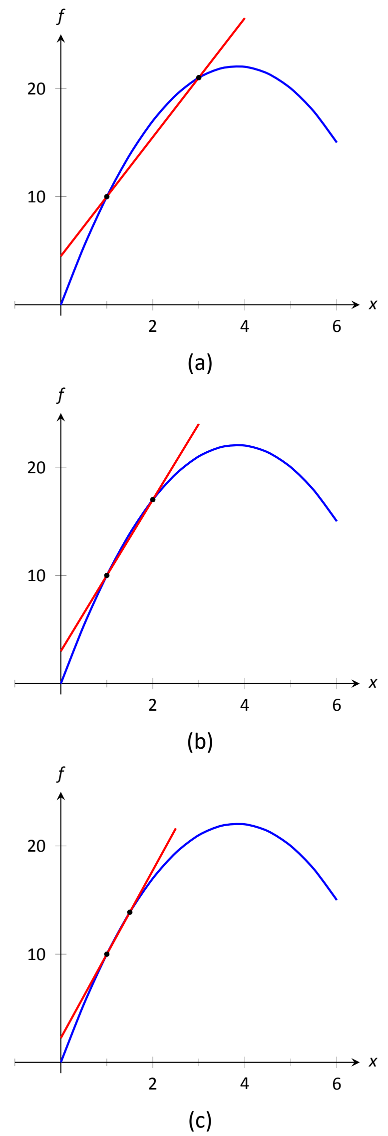


Figure 1.15: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Figure 1.16: The difference quotient evaluated at values of h near 0.

Exercises 1.1

Terms and Concepts

01 01 ex 19 1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

01 01 ex 20 2. An expression of the form $\frac{0}{0}$ is called ____.

01 01 ex 21 3. T/F: The limit of $f(x)$ as x approaches 5 is $f(5)$.

01 01 ex 22 4. Describe three situations where $\lim\limits_{x\to c} f(x)$ does not exist.

01 01 ex 23 5. In your own words, what is a difference quotient?

Problems

01 01 exset 01 **In Exercises 6 – 16, approximate the given limits both numerically and graphically.**

01 01 ex 01 6. $\lim\limits_{x\to 1} x^2 + 3x - 5$

01 01 ex 02 7. $\lim\limits_{x\to 0} x^3 - 3x^2 + x - 5$

01 01 ex 03 8. $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+3x}$

01 01 ex 04 9. $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

01 01 ex 05 10. $\lim\limits_{x\to -1} \dfrac{x^2+8x+7}{x^2+6x+5}$

01 01 ex 06 11. $\lim\limits_{x\to 2} \dfrac{x^2+7x+10}{x^2-4x+4}$

01 01 ex 07 12. $\lim\limits_{x\to 2} f(x)$, where

$$f(x) = \begin{cases} x+2 & x \leq 2 \\ 3x-5 & x > 2 \end{cases}.$$

01 01 ex 08 13. $\lim\limits_{x\to 3} f(x)$, where

$$f(x) = \begin{cases} x^2-x+1 & x \leq 3 \\ 2x+1 & x > 3 \end{cases}.$$

01 01 ex 09 14. $\lim\limits_{x\to 0} f(x)$, where

$$f(x) = \begin{cases} \cos x & x \leq 0 \\ x^2+3x+1 & x > 0 \end{cases}.$$

01 01 ex 10 15. $\lim\limits_{x\to \pi/2} f(x)$, where

$$f(x) = \begin{cases} \sin x & x \leq \pi/2 \\ \cos x & x > \pi/2 \end{cases}.$$

01 01 exset 02 **In Exercises 16 – 23, a function f and a value a are given. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.**

01 01 ex 11 16. $f(x) = -7x + 2, \quad a = 3$

01 01 ex 12 17. $f(x) = 9x + 0.06, \quad a = -1$

01 01 ex 13 18. $f(x) = x^2 + 3x - 7, \quad a = 1$

01 01 ex 14 19. $f(x) = \dfrac{1}{x+1}, \quad a = 2$

01 01 ex 15 20. $f(x) = -4x^2 + 5x - 1, \quad a = -3$

01 01 ex 16 21. $f(x) = \ln x, \quad a = 5$

01 01 ex 17 22. $f(x) = \sin x, \quad a = \pi$

01 01 ex 18 23. $f(x) = \cos x, \quad a = \pi$

1.2 Epsilon-Delta Definition of a Limit

This section introduces the formal definition of a limit. Many refer to this as "the epsilon-delta," definition, referring to the letters ε and δ of the Greek alphabet.

Note: the common phrase "the ε-δ definition" is read aloud as "the epsilon delta definition." The hyphen between ε and δ is not a minus sign.

Before we give the actual definition, let's consider a few informal ways of describing a limit. Given a function $y = f(x)$ and an x-value, c, we say that "the limit of the function f, as x approaches c, is a value L":

1. if "y tends to L" as "x tends to c."

2. if "y approaches L" as "x approaches c."

3. if "y is near L" whenever "x is near c."

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c? How near do x and y have to be to c and L, respectively?

The definition we describe in this section comes from formalizing **3**. A quick restatement gets us closer to what we want:

3′. If x is within a certain *tolerance level* of c, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L.

The traditional notation for the x-tolerance is the lowercase Greek letter delta, or δ, and the y-tolerance is denoted by lowercase epsilon, or ε. One more rephrasing of **3′** nearly gets us to the actual definition:

3″. If x is within δ units of c, then the corresponding value of y is within ε units of L.

We can write "x is within δ units of c" mathematically as

$$|x - c| < \delta, \qquad \text{which is equivalent to} \qquad c - \delta < x < c + \delta.$$

Letting the symbol "$\longrightarrow$" represent the word "implies," we can rewrite **3″** as

$$|x - c| < \delta \longrightarrow |y - L| < \varepsilon \qquad \text{or} \qquad c - \delta < x < c + \delta \longrightarrow L - \varepsilon < y < L + \varepsilon.$$

The point is that δ and ε, being tolerances, can be any positive (but typically small) values. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Notes:

Definition 1 The Limit of a Function f

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The **limit of** $f(x)$**, as** x **approaches** c**, is** L, denoted by

$$\lim_{x\to c} f(x) = L,$$

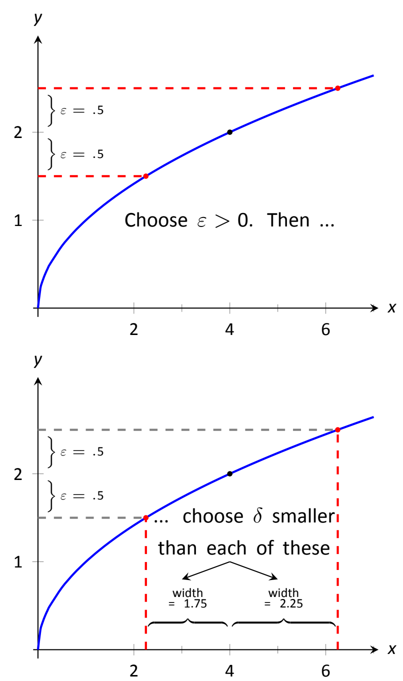
means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \neq c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

(Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x\to c} f(x) = L \iff \forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \longrightarrow |f(x) - L| < \varepsilon.)$$

Note the order in which ε and δ are given. In the definition, the y-tolerance ε is given *first* and then the limit will exist **if** we can find an x-tolerance δ that works.

An example will help us understand this definition. Note that the explanation is long, but it will take one through all steps necessary to understand the ideas.



With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Figure 1.17: Illustrating the $\varepsilon - \delta$ process.

Example 6 Evaluating a limit using the definition

Show that $\lim_{x\to 4} \sqrt{x} = 2$.

Solution Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or $\varepsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2, i.e., $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$\begin{aligned} 1.5 <\ & y < 2.5 \\ 1.5 <\ & \sqrt{x} < 2.5 \\ 1.5^2 <\ & x < 2.5^2 \\ 2.25 <\ & x < 6.25. \end{aligned}$$

So, what is the desired x tolerance? Remember, we want to find a symmetric interval of x values, namely $4 - \delta < x < 4 + \delta$. The lower bound of 2.25 is 1.75 units from 4; the upper bound of 6.25 is 2.25 units from 4. We need the smaller of these two distances; we must have $\delta \leq 1.75$. See Figure 1.17.

Notes:

Given the y tolerance $\varepsilon = 0.5$, we have found an x tolerance, $\delta \leq 1.75$, such that whenever x is within δ units of 4, then y is within ε units of 2. That's what we were trying to find.

Let's try another value of ε.

What if the y tolerance is 0.01, i.e., $\varepsilon = 0.01$? How close to 4 does x have to be in order for y to be within 0.01 units of 2 (or $1.99 < y < 2.01$)? Again, we just square these values to get $1.99^2 < x < 2.01^2$, or

$$3.9601 < x < 4.0401.$$

What is the desired x tolerance? In this case we must have $\delta \leq 0.0399$, which is the minimum distance from 4 of the two bounds given above.

What we have so far: if $\varepsilon = 0.5$, then $\delta \leq 1.75$ and if $\varepsilon = 0.01$, then $\delta \leq 0.0399$. A pattern is not easy to see, so we switch to general ε try to determine δ symbolically. We start by assuming $y = \sqrt{x}$ is within ε units of 2:

$$\begin{aligned}
|y-2| &< \varepsilon \\
-\varepsilon < y-2 &< \varepsilon && \text{(Definition of absolute value)} \\
-\varepsilon < \sqrt{x}-2 &< \varepsilon && (y=\sqrt{x}) \\
2-\varepsilon < \sqrt{x} &< 2+\varepsilon && \text{(Add 2)} \\
(2-\varepsilon)^2 < x &< (2+\varepsilon)^2 && \text{(Square all)} \\
4-4\varepsilon+\varepsilon^2 < x &< 4+4\varepsilon+\varepsilon^2 && \text{(Expand)} \\
4-(4\varepsilon-\varepsilon^2) < x &< 4+(4\varepsilon+\varepsilon^2). && \text{(Rewrite in the desired form)}
\end{aligned}$$

The "desired form" in the last step is "$4 -$ *something* $< x < 4 +$ *something*." Since we want this last interval to describe an x tolerance around 4, we have that either $\delta \leq 4\varepsilon - \varepsilon^2$ or $\delta \leq 4\varepsilon + \varepsilon^2$, whichever is smaller:

$$\delta \leq \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\}.$$

Since $\varepsilon > 0$, the minimum is $\delta \leq 4\varepsilon - \varepsilon^2$. That's the formula: given an ε, set $\delta \leq 4\varepsilon - \varepsilon^2$.

We can check this for our previous values. If $\varepsilon = 0.5$, the formula gives $\delta \leq 4(0.5) - (0.5)^2 = 1.75$ and when $\varepsilon = 0.01$, the formula gives $\delta \leq 4(0.01) - (0.01)^2 = 0.399$.

So given any $\varepsilon > 0$, set $\delta \leq 4\varepsilon - \varepsilon^2$. Then if $|x-4| < \delta$ (and $x \neq 4$), then $|f(x) - 2| < \varepsilon$, satisfying the definition of the limit. We have shown formally (and finally!) that $\lim\limits_{x\to 4} \sqrt{x} = 2$.

Notes:

The previous example was a little long in that we sampled a few specific cases of ε before handling the general case. Normally this is not done. The previous example is also a bit unsatisfying in that $\sqrt{4} = 2$; why work so hard to prove something so obvious? Many ε-δ proofs are long and difficult to do. In this section, we will focus on examples where the answer is, frankly, obvious, because the non–obvious examples are even harder. In the next section we will learn some theorems that allow us to evaluate limits *analytically*, that is, without using the ε-δ definition.

Example 7 **Evaluating a limit using the definition**

Show that $\lim\limits_{x\to 2} x^2 = 4$.

Solution Let's do this example symbolically from the start. Let $\varepsilon > 0$ be given; we want $|y-4| < \varepsilon$, i.e., $|x^2-4| < \varepsilon$. How do we find δ such that when $|x-2| < \delta$, we are guaranteed that $|x^2-4| < \varepsilon$?

This is a bit trickier than the previous example, but let's start by noticing that $|x^2-4| = |x-2|\cdot|x+2|$. Consider:

$$|x^2-4| < \varepsilon \longrightarrow |x-2|\cdot|x+2| < \varepsilon \longrightarrow |x-2| < \frac{\varepsilon}{|x+2|}. \tag{1.1}$$

Could we not set $\delta = \dfrac{\varepsilon}{|x+2|}$?

We are close to an answer, but the catch is that δ must be a *constant* value (so it can't contain x). There is a way to work around this, but we do have to make an assumption. Remember that ε is supposed to be a small number, which implies that δ will also be a small value. In particular, we can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$, giving $1 < x < 3$.

Now, back to the fraction $\dfrac{\varepsilon}{|x+2|}$. If $1 < x < 3$, then $3 < x+2 < 5$ (add 2 to all terms in the inequality). Taking reciprocals, we have

$$\begin{aligned} \frac{1}{5} &< \frac{1}{|x+2|} < \frac{1}{3} && \text{which implies} \\ \frac{1}{5} &< \frac{1}{|x+2|} && \text{which implies} \\ \frac{\varepsilon}{5} &< \frac{\varepsilon}{|x+2|}. && \end{aligned} \tag{1.2}$$

This suggests that we set $\delta \leq \dfrac{\varepsilon}{5}$. To see why, let consider what follows when we assume $|x-2| < \delta$:

Notes:

$$\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\varepsilon}{5} && \text{(Our choice of } \delta\text{)} \\
|x-2|\cdot|x+2| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Multiply by } |x+2|\text{)} \\
|x^2-4| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Combine left side)} \\
|x^2-4| &< |x+2|\cdot\frac{\varepsilon}{5} < |x+2|\cdot\frac{\varepsilon}{|x+2|} = \varepsilon && \text{(Using (1.2) as long as } \delta < 1\text{)}
\end{aligned}$$

We have arrived at $|x^2-4| < \varepsilon$ as desired. Note again, in order to make this happen we needed δ to first be less than 1. That is a safe assumption; we want ε to be arbitrarily small, forcing δ to also be small.

We have also picked δ to be smaller than "necessary." We could get by with a slightly larger δ, as shown in Figure 1.18. The dashed outer lines show the boundaries defined by our choice of ε. The dotted inner lines show the boundaries defined by setting $\delta = \varepsilon/5$. Note how these dotted lines are within the dashed lines. That is perfectly fine; by choosing x within the dotted lines we are guaranteed that $f(x)$ will be within ε of 4.

In summary, given $\varepsilon > 0$, set $\delta = \varepsilon/5$. Then $|x-2| < \delta$ implies $|x^2-4| < \varepsilon$ (i.e. $|y-4| < \varepsilon$) as desired. This shows that $\lim\limits_{x\to 2} x^2 = 4$. Figure 1.18 gives a visualization of this; by restricting x to values within $\delta = \varepsilon/5$ of 2, we see that $f(x)$ is within ε of 4.

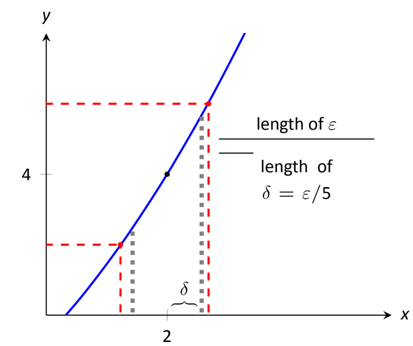


Figure 1.18: Choosing $\delta = \varepsilon/5$ in Example 7.

Make note of the general pattern exhibited in these last two examples. In some sense, each starts out "backwards." That is, while we want to

1. start with $|x-c| < \delta$ and conclude that
2. $|f(x)-L| < \varepsilon$,

we actually start by assuming

1. $|f(x)-L| < \varepsilon$, then perform some algebraic manipulations to give an inequality of the form
2. $|x-c| <$ *something*.

When we have properly done this, the *something* on the "greater than" side of the inequality becomes our δ. We can refer to this as the "scratch–work" phase of our proof. Once we have δ, we can formally start with $|x-c| < \delta$ and use algebraic manipulations to conclude that $|f(x)-L| < \varepsilon$, usually by using the same steps of our "scratch–work" in reverse order.

Notes:

We highlight this process in the following example.

Example 8 **Evaluating a limit using the definition**
Prove that $\lim\limits_{x\to 1} x^3 - 2x = -1$.

Solution We start our scratch–work by considering $|f(x) - (-1)| < \varepsilon$:

$$\begin{aligned} |f(x) - (-1)| &< \varepsilon \\ |x^3 - 2x + 1| &< \varepsilon && \text{(Now factor)} \\ |(x-1)(x^2+x-1)| &< \varepsilon \\ |x-1| &< \frac{\varepsilon}{|x^2+x-1|}. && (1.3) \end{aligned}$$

We are at the phase of saying that $|x-1| <$ *something*, where *something*$=$ $\varepsilon/|x^2+x-1|$. We want to turn that *something* into δ.

Since x is approaching 1, we are safe to assume that x is between 0 and 2. So

$$\begin{aligned} &0 < x < 2 \\ &0 < x^2 < 4. && \text{(squared each term)} \end{aligned}$$

Since $0 < x < 2$, we can add 0, x and 2, respectively, to each part of the inequality and maintain the inequality.

$$\begin{aligned} 0 < x^2 &+ x < 6 \\ -1 < x^2 &+ x - 1 < 5. && \text{(subtracted 1 from each part)} \end{aligned}$$

In Equation (1.3), we wanted $|x-1| < \varepsilon/|x^2+x-1|$. The above shows that given any x in $[0, 2]$, we know that

$$\begin{aligned} x^2 + x - 1 &< 5 && \text{which implies that} \\ \frac{1}{5} &< \frac{1}{x^2+x-1} && \text{which implies that} \\ \frac{\varepsilon}{5} &< \frac{\varepsilon}{x^2+x-1}. && (1.4) \end{aligned}$$

So we set $\delta \leq \varepsilon/5$. This ends our scratch–work, and we begin the formal proof (which also helps us understand why this was a good choice of δ).

Given ε, let $\delta \leq \varepsilon/5$. We want to show that when $|x-1| < \delta$, then $|(x^3 -$

Notes:

$2x) - (-1)| < \varepsilon$. We start with $|x-1| < \delta$:

$$
\begin{aligned}
|x-1| &< \delta \\
|x-1| &< \frac{\varepsilon}{5} \\
|x-1| &< \frac{\varepsilon}{5} < \frac{\varepsilon}{|x^2+x-1|} \quad \text{(for } x \text{ near 1, from Equation (1.4))} \\
|x-1| \cdot |x^2+x-1| &< \varepsilon \\
|x^3-2x+1| &< \varepsilon \\
|(x^3-2x)-(-1)| &< \varepsilon,
\end{aligned}
$$

which is what we wanted to show. Thus $\lim\limits_{x\to1} x^3-2x = -1$.

We illustrate evaluating limits once more.

Example 9 **Evaluating a limit using the definition**
Prove that $\lim\limits_{x\to0} e^x = 1$.

Solution Symbolically, we want to take the equation $|e^x-1| < \varepsilon$ and unravel it to the form $|x-0| < \delta$. Here is our scratch–work:

$$
\begin{aligned}
|e^x-1| &< \varepsilon \\
-\varepsilon < e^x - 1 &< \varepsilon && \text{(Definition of absolute value)} \\
1-\varepsilon < e^x &< 1+\varepsilon && \text{(Add 1)} \\
\ln(1-\varepsilon) < x &< \ln(1+\varepsilon) && \text{(Take natural logs)}
\end{aligned}
$$

Note: Recall $\ln 1 = 0$ and $\ln x < 0$ when $0 < x < 1$. So $\ln(1-\varepsilon) < 0$, hence we consider its absolute value.

Making the safe assumption that $\varepsilon < 1$ ensures the last inequality is valid (i.e., so that $\ln(1-\varepsilon)$ is defined). We can then set δ to be the minimum of $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$; i.e.,

$$\delta = \min\{|\ln(1-\varepsilon)|, \ln(1+\varepsilon)\} = \ln(1+\varepsilon).$$

Now, we work through the actual the proof:

$$
\begin{aligned}
|x-0| &< \delta \\
-\delta < x &< \delta && \text{(Definition of absolute value)} \\
-\ln(1+\varepsilon) < x &< \ln(1+\varepsilon). \\
\ln(1-\varepsilon) < x &< \ln(1+\varepsilon). && \text{(since } \ln(1-\varepsilon) < -\ln(1+\varepsilon))
\end{aligned}
$$

Notes:

The above line is true by our choice of δ and by the fact that since $|\ln(1-\varepsilon)| > \ln(1+\varepsilon)$ and $\ln(1-\varepsilon) < 0$, we know $\ln(1-\varepsilon) < -\ln(1+\varepsilon)$.

$$1-\varepsilon < e^x < 1+\varepsilon \qquad \text{(Exponentiate)}$$
$$-\varepsilon < e^x - 1 < \varepsilon \qquad \text{(Subtract 1)}$$

In summary, given $\varepsilon > 0$, let $\delta = \ln(1+\varepsilon)$. Then $|x-0| < \delta$ implies $|e^x - 1| < \varepsilon$ as desired. We have shown that $\lim\limits_{x\to 0} e^x = 1$.

We note that we could actually show that $\lim_{x\to c} e^x = e^c$ for any constant c. We do this by factoring out e^c from both sides, leaving us to show $\lim_{x\to c} e^{x-c} = 1$ instead. By using the substitution $u = x-c$, this reduces to showing $\lim_{u\to 0} e^u = 1$ which we just did in the last example. As an added benefit, this shows that in fact the function $f(x) = e^x$ is *continuous* at all values of x, an important concept we will define in Section 1.5.

This formal definition of the limit is not an easy concept grasp. Our examples are actually "easy" examples, using "simple" functions like polynomials, square–roots and exponentials. It is very difficult to prove, using the techniques given above, that $\lim\limits_{x\to 0}(\sin x)/x = 1$, as we approximated in the previous section.

There is hope. The next section shows how one can evaluate complicated limits using certain basic limits as building blocks. While limits are an incredibly important part of calculus (and hence much of higher mathematics), rarely are limits evaluated using the definition. Rather, the techniques of the following section are employed.

Notes:

Exercises 1.2

Terms and Concepts

01 02 ex 01 1. What is wrong with the following "definition" of a limit?

> "The limit of $f(x)$, as x approaches a, is K" means that given any $\delta > 0$ there exists $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$, we have $|x - a| < \delta$.

01 02 ex 02 2. Which is given first in establishing a limit, the x–tolerance or the y–tolerance?

01 02 ex 09 3. T/F: ε must always be positive.

01 02 ex 10 4. T/F: δ must always be positive.

Problems

01 02 exset 02 **In Exercises 5 – 11, prove the given limit using an $\varepsilon - \delta$ proof.**

01 02 ex 04 5. $\lim\limits_{x\to 5} 3 - x = -2$

01 02 ex 05 6. $\lim\limits_{x\to 3} x^2 - 3 = 6$

01 02 ex 11 7. $\lim\limits_{x\to 4} x^2 + x - 5 = 15$

01 02 ex 06 8. $\lim\limits_{x\to 2} x^3 - 1 = 7$

01 02 ex 03 9. $\lim\limits_{x\to 2} 5 = 5$

01 02 ex 07 10. $\lim\limits_{x\to 0} e^{2x} - 1 = 0$

01 02 ex 08 11. $\lim\limits_{x\to 0} \sin x = 0$ (Hint: use the fact that $|\sin x| \leq |x|$, with equality only when $x = 0$.)

1.3 Finding Limits Analytically

In Section 1.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. In the previous section we gave the definition of the limit and demonstrated how to use it to verify our approximations were correct. Thus far, our method of finding a limit is 1) make a really good approximation either graphically or numerically, and 2) verify our approximation is correct using a ε-δ proof.

Recognizing that ε-δ proofs are cumbersome, this section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x) + g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 1 Basic Limit Properties

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim\limits_{x\to c} b = b$
2. Identity $\lim\limits_{x\to c} x = c$
3. Sums/Differences: $\lim\limits_{x\to c}(f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim\limits_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim\limits_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim\limits_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Powers: $\lim\limits_{x\to c} f(x)^n = L^n$
8. Roots: $\lim\limits_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
9. Compositions: Adjust our previously given limit situation to:

   $$\lim_{x\to c} f(x) = L,\ \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$

   Then $\lim\limits_{x\to c} g(f(x)) = K$.

Notes:

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

We apply the theorem to an example.

Example 10 Using basic limit properties

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim_{x\to 2} p(x)$

Solution

1. Using the Sum/Difference rule, we know that $\lim_{x\to 2} \big(f(x)+g(x)\big) = 2+3 = 5$.
2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim_{x\to 2} \big(5f(x)+g(x)^2\big) = 5\cdot 2 + 3^2 = 19$.
3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2 - 5x + 7)\\ &= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7\\ &= 3\cdot 2^2 - 5\cdot 2 + 7\\ &= 9\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 1. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Notes:

Theorem 2 Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim\limits_{x\to c} p(x) = p(c)$
2. $\lim\limits_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 11 Finding a limit of a rational function

Using Theorem 2, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution Using Theorem 2, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

It was likely frustrating in Section 1.2 to do a lot of work to prove that

$$\lim_{x\to 2} x^2 = 4$$

as it seemed fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 11.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 3 Special Limits

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim\limits_{x\to c} \sin x = \sin c$
2. $\lim\limits_{x\to c} \cos x = \cos c$
3. $\lim\limits_{x\to c} \tan x = \tan c$
4. $\lim\limits_{x\to c} \csc x = \csc c$
5. $\lim\limits_{x\to c} \sec x = \sec c$
6. $\lim\limits_{x\to c} \cot x = \cot c$
7. $\lim\limits_{x\to c} a^x = a^c$ $(a > 0)$
8. $\lim\limits_{x\to c} \ln x = \ln c$
9. $\lim\limits_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Notes:

Example 12 **Evaluating limits analytically**

Evaluate the following limits.

1. $\displaystyle\lim_{x\to\pi} \cos x$
2. $\displaystyle\lim_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\displaystyle\lim_{x\to\pi/2} \cos x \sin x$
4. $\displaystyle\lim_{x\to 1} e^{\ln x}$
5. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$

Solution

1. This is a straightforward application of Theorem 3. $\displaystyle\lim_{x\to\pi} \cos x = \cos\pi = -1$.

2. We can approach this in at least two ways. First, by directly applying Theorem 3, we have:

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \sec^2 3 - \tan^2 3.$$

   Using the Pythagorean Theorem, this last expression is 1; therefore

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = 1.$$

   We can also use the Pythagorean Theorem from the start.

$$\lim_{x\to 3} (\sec^2 x - \tan^2 x) = \lim_{x\to 3} 1 = 1,$$

   using the Constant limit rule. Either way, we find the limit is 1.

3. Applying the Product limit rule of Theorem 1 and Theorem 3 gives

$$\lim_{x\to\pi/2} \cos x \sin x = \cos(\pi/2)\sin(\pi/2) = 0 \cdot 1 = 0.$$

4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln x} = x$ and evaluate $\displaystyle\lim_{x\to 1} e^{\ln x} = \lim_{x\to 1} x = 1$.

   We can also use the limit Composition Rule of Theorem 1. Using Theorem 3, we have $\displaystyle\lim_{x\to 1} \ln x = \ln 1 = 0$ and $\displaystyle\lim_{x\to 0} e^x = e^0 = 1$, satisfying the conditions of the Composition Rule. Applying this rule,

$$\lim_{x\to 1} e^{\ln x} = \lim_{x\to 0} e^x = e^0 = 1.$$

   Both approaches are valid, giving the same result.

Notes:

5. We encountered this limit in Section 1.1. Applying our theorems, we attempt to find the limit as

$$\lim_{x\to 0}\frac{\sin x}{x} \to \frac{\sin 0}{0} \to \text{“}\frac{0}{0}\text{”}.$$

This, of course, violates a condition of Theorem 1, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

The section could have been titled “Using Known Limits to Find Unknown Limits.” By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always “squeezed” between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem 4 Squeeze Theorem

Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to “squeeze” a given function. However, that is generally the only place where work is necessary; the theorem makes the “evaluating the limit part” very simple.

We use the Squeeze Theorem in the following example to finally prove that $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$.

Notes:

Example 13 Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Solution We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 1.19. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \le \theta \le \pi/2$. Later we will show that we can also consider $\theta \le 0$.)

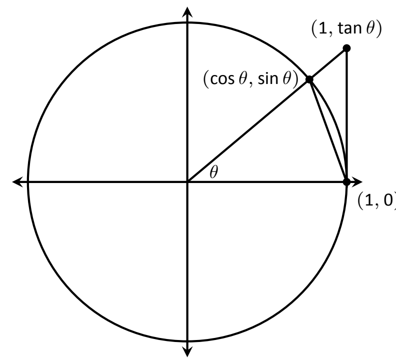
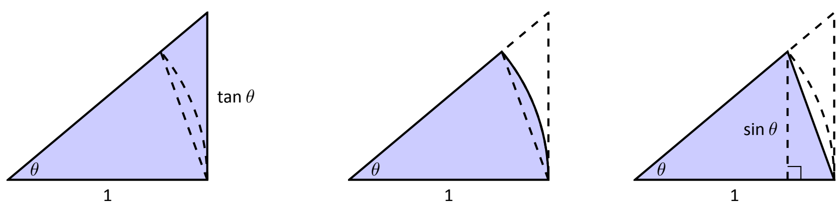


Figure 1.19: The unit circle and related triangles.

Figure 1.19 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



$$\frac{\tan\theta}{2} \quad\ge\quad \frac{\theta}{2} \quad\ge\quad \frac{\sin\theta}{2}$$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \ge \frac{\theta}{\sin\theta} \ge 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \le \frac{\sin\theta}{\theta} \le 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0} \cos\theta \le \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \le \lim_{\theta\to 0} 1$$

Notes:

$$\cos 0 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \leq 1$$

Clearly this means that $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio* of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 5 Special Limits

1. $\lim_{x \to 0} \frac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$
3. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
4. $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Notes:

Our final theorem for this section will be motivated by the following example.

Example 14 **Using algebra to evaluate a limit**
Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution We begin by attempting to apply Theorem 2 and substituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \text{“}\frac{0}{0}\text{”},$$

an indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 1.20, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x=1$, but for all other x,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1.$$

Clearly $\lim\limits_{x\to 1} x+1 = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x=1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

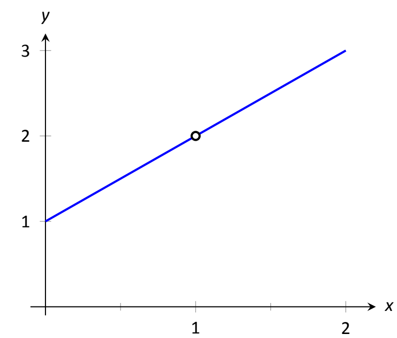


Figure 1.20: Graphing f in Example 14 to understand a limit.

The key to the above example is that the functions $y=(x^2-1)/(x-1)$ and $y=x+1$ are identical except at $x=1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Notes:

Theorem 6 Limits of Functions Equal At All But One Point

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim\limits_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim\limits_{x\to c} \dfrac{g(x)}{f(x)}$ returns "0/0", then $(x-c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 6. We demonstrate this once more.

Example 15 Evaluating a limit using Theorem 6

Evaluate $\lim\limits_{x\to 3} \dfrac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}$.

Solution We attempt to apply Theorem 2 by substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}.$$

We can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem 6 we conclude:

$$\begin{aligned}\lim_{x\to 3} \frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} &= \lim_{x\to 3} \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}\\ &= \lim_{x\to 3} \frac{(x^2+x-2)}{(2x^2+9x-5)}\\ &= \frac{10}{40} = \frac{1}{4}.\end{aligned}$$

We end this section by revisiting a limit first seen in Section 1.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Notes:

Example 16 **Evaluating the limit of a difference quotient**

Let $f(x) = -1.5x^2 + 11.5x$; find $\displaystyle\lim_{h\to 0} \frac{f(1+h)-f(1)}{h}$.

Solution Since f is a polynomial, our first attempt should be to employ Theorem 2 and substitute 0 for h. However, we see that this gives us "$0/0$." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0} \frac{-1.5(1+h)^2 + 11.5(1+h) - \big(-1.5(1)^2 + 11.5(1)\big)}{h} \\
&= \lim_{h\to 0} \frac{-1.5(1+2h+h^2) + 11.5 + 11.5h - 10}{h} \\
&= \lim_{h\to 0} \frac{-1.5h^2 + 8.5h}{h} \\
&= \lim_{h\to 0} \frac{h(-1.5h + 8.5)}{h} \\
&= \lim_{h\to 0} (-1.5h + 8.5) \quad \big(\text{using Theorem 6, as } h \neq 0\big) \\
&= 8.5 \quad \big(\text{using Theorem 3}\big)
\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 3; it states that many functions that we use regularly behave in a very nice, predictable way. In the next section we give a name to this nice behavior; we label such functions as *continuous.* Defining that term will require us to look again at what a limit is and what causes limits to not exist.

Notes:

Exercises 1.3

Terms and Concepts

01 03 ex 01 1. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} b = b$.

01 03 ex 02 2. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} x = c$.

01 03 ex 03 3. What does the text mean when it says that certain functions' "behavior is 'nice' in terms of limits"? What, in particular, is "nice"?

01 03 ex 04 4. Sketch a graph that visually demonstrates the Squeeze Theorem.

01 03 ex 05 5. You are given the following information:

(a) $\lim\limits_{x\to 1} f(x) = 0$

(b) $\lim\limits_{x\to 1} g(x) = 0$

(c) $\lim\limits_{x\to 1} f(x)/g(x) = 2$

What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

Problems

01 03 exset 02 **In Exercises 6 – 13, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.**

- $\lim\limits_{x\to 9} f(x) = 6, \quad \lim\limits_{x\to 6} f(x) = 9, \quad f(9) = 6$
- $\lim\limits_{x\to 9} g(x) = 3, \quad \lim\limits_{x\to 6} g(x) = 3, \quad g(6) = 9$

01 03 ex 06 6. $\lim\limits_{x\to 9}(f(x) + g(x))$

01 03 ex 07 7. $\lim\limits_{x\to 9}(3f(x)/g(x))$

01 03 ex 08 8. $\lim\limits_{x\to 9}\left(\dfrac{f(x) - 2g(x)}{g(x)}\right)$

01 03 ex 09 9. $\lim\limits_{x\to 6}\left(\dfrac{f(x)}{3 - g(x)}\right)$

01 03 ex 10 10. $\lim\limits_{x\to 9} g\big(f(x)\big)$

01 03 ex 11 11. $\lim\limits_{x\to 6} f\big(g(x)\big)$

01 03 ex 12 12. $\lim\limits_{x\to 6} g\big(f(f(x))\big)$

01 03 ex 13 13. $\lim\limits_{x\to 6} f(x)g(x) - f^2(x) + g^2(x)$

01 03 exset 03 **In Exercises 14 – 17, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.**

- $\lim\limits_{x\to 1} f(x) = 2, \quad \lim\limits_{x\to 10} f(x) = 1, \quad f(1) = 1/5$
- $\lim\limits_{x\to 1} g(x) = 0, \quad \lim\limits_{x\to 10} g(x) = \pi, \quad g(10) = \pi$

01 03 ex 14 14. $\lim\limits_{x\to 1} f(x)^{g(x)}$

01 03 ex 15 15. $\lim\limits_{x\to 10} \cos\big(g(x)\big)$

01 03 ex 16 16. $\lim\limits_{x\to 1} f(x)g(x)$

01 03 ex 17 17. $\lim\limits_{x\to 1} g\big(5f(x)\big)$

01 03 exset 04 **In Exercises 18 – 32, evaluate the given limit.**

01 03 ex 18 18. $\lim\limits_{x\to 3} x^2 - 3x + 7$

01 03 ex 21 19. $\lim\limits_{x\to \pi}\left(\dfrac{x-3}{x-5}\right)^7$

01 03 ex 22 20. $\lim\limits_{x\to \pi/4} \cos x \sin x$

01 03 ex 23 21. $\lim\limits_{x\to 0} \ln x$

01 03 ex 24 22. $\lim\limits_{x\to 3} 4^{x^3 - 8x}$

01 03 ex 25 23. $\lim\limits_{x\to \pi/6} \csc x$

01 03 ex 26 24. $\lim\limits_{x\to 0} \ln(1 + x)$

01 03 ex 20 25. $\lim\limits_{x\to \pi} \dfrac{x^2 + 3x + 5}{5x^2 - 2x - 3}$

01 03 ex 19 26. $\lim\limits_{x\to \pi} \dfrac{3x + 1}{1 - x}$

01 03 ex 27 27. $\lim\limits_{x\to 6} \dfrac{x^2 - 4x - 12}{x^2 - 13x + 42}$

01 03 ex 28 28. $\lim\limits_{x\to 0} \dfrac{x^2 + 2x}{x^2 - 2x}$

01 03 ex 29 29. $\lim\limits_{x\to 2} \dfrac{x^2 + 6x - 16}{x^2 - 3x + 2}$

01 03 ex 30 30. $\lim\limits_{x\to 2} \dfrac{x^2 - 10x + 16}{x^2 - x - 2}$

01 03 ex 31 31. $\lim\limits_{x\to -2} \dfrac{x^2 - 5x - 14}{x^2 + 10x + 16}$

01 03 ex 32 32. $\lim\limits_{x\to -1} \dfrac{x^2 + 9x + 8}{x^2 - 6x - 7}$

01 03 exset 06 **Use the Squeeze Theorem in Exercises 33 – 36, where appropriate, to evaluate the given limit.**

01 03 ex 38 33. $\lim\limits_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

01 03 ex 40 34. $\lim\limits_{x\to 0} \sin x \cos\left(\dfrac{1}{x^2}\right)$

01 03 ex 42 35. $\lim\limits_{x\to 1} f(x)$, where $3x-2 \leq f(x) \leq x^3$.

01 03 ex 41 36. $\lim\limits_{x\to 3} f(x)$, where $6x-9 \leq f(x) \leq x^2$.

01 03 exset 05 **Exercises 37 – 41 challenge your understanding of limits but can be evaluated using the knowledge gained in this section.**

01 03 ex 33 37. $\lim\limits_{x\to 0} \dfrac{\sin 3x}{x}$

01 03 ex 34 38. $\lim\limits_{x\to 0} \dfrac{\sin 5x}{8x}$

01 03 ex 35 39. $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x}$

01 03 ex 36 40. $\lim\limits_{x\to 0} \dfrac{\sin x}{x}$, where x is measured in degrees, not radians.

01 03 ex 43 41. Let $f(x) = 0$ and $g(x) = \dfrac{x}{x}$.

(a) Show why $\lim\limits_{x\to 2} f(x) = 0$.

(b) Show why $\lim\limits_{x\to 0} g(x) = 1$.

(c) Show why $\lim\limits_{x\to 2} g\big(f(x)\big)$ does not exist.

(d) Show why the answer to part (c) does not violate the Composition Rule of Theorem 1.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

01 01 ex 19 1. Answers will vary.

01 01 ex 20 2. An indeterminate form.

01 01 ex 21 3. F

01 01 ex 22 4. The function may approach different values from the left and right, the function may grow without bound, or the function might oscillate.

01 01 ex 23 5. Answers will vary.

01 01 ex 01 6. -1

01 01 ex 02 7. -5

01 01 ex 03 8. Limit does not exist

01 01 ex 04 9. 2

01 01 ex 05 10. 1.5

01 01 ex 06 11. Limit does not exist.

01 01 ex 07 12. Limit does not exist.

01 01 ex 08 13. 7

01 01 ex 09 14. 1

01 01 ex 10 15. Limit does not exist.

01 01 ex 11 16.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-7
-0.01	-7
0.01	-7
0.1	-7

The limit seems to be exactly 7.

01 01 ex 12 17.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	9
-0.01	9
0.01	9
0.1	9

The limit seems to be exactly 9.

01 01 ex 13 18.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	4.9
-0.01	4.99
0.01	5.01
0.1	5.1

The limit is approx. 5.

01 01 ex 14 19.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.114943
-0.01	-0.111483
0.01	-0.110742
0.1	-0.107527

The limit is approx. -0.11.

01 01 ex 15 20.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	29.4
-0.01	29.04
0.01	28.96
0.1	28.6

The limit is approx. 29.

01 01 ex 16 21.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.202027
-0.01	0.2002
0.01	0.1998
0.1	0.198026

The limit is approx. 0.2.

01 01 ex 17 22.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.998334
-0.01	-0.999983
0.01	-0.999983
0.1	-0.998334

The limit is approx. -1.

01 01 ex 18 23.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.0499583
-0.01	-0.00499996
0.01	0.00499996
0.1	0.0499583

The limit is approx. 0.005.

Section 1.2

01 02 ex 01 1. ε should be given first, and the restriction $|x-a|<\delta$ implies $|f(x)-K|<\varepsilon$, not the other way around.

01 02 ex 02 2. The y–tolerance.

01 02 ex 09 3. T

01 02 ex 10 4. T

01 02 ex 04 5. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-5|<\delta$, $|f(x)-(-2)|<\varepsilon$.

Consider $|f(x)-(-2)|<\varepsilon$:

$$\begin{gathered}|f(x)+2|<\varepsilon\\ |(3-x)+2|<\varepsilon\\ |5-x|<\varepsilon\\ -\varepsilon<5-x<\varepsilon\\ -\varepsilon<x-5<\varepsilon.\end{gathered}$$

This implies we can let $\delta=\varepsilon$. Then:

$$\begin{gathered}|x-5|<\delta\\ -\delta<x-5<\delta\\ -\varepsilon<x-5<\varepsilon\\ -\varepsilon<(x-3)-2<\varepsilon\\ -\varepsilon<(-x+3)-(-2)<\varepsilon\\ |3-x-(-2)|<\varepsilon,\end{gathered}$$

which is what we wanted to prove.

01 02 ex 05 6. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-3|<\delta$, $|f(x)-6|<\varepsilon$.

Consider $|f(x)-6|<\varepsilon$, keeping in mind we want to make a statement about $|x-3|$:

$$
\begin{aligned}
|f(x)-6| &< \varepsilon \\
|x^2-3-6| &< \varepsilon \\
|x^2-9| &< \varepsilon \\
|x-3|\cdot|x+3| &< \varepsilon \\
|x-3| &< \varepsilon/|x+3|
\end{aligned}
$$

Since x is near 3, we can safely assume that, for instance, $2<x<4$. Thus

$$
\begin{gathered}
2+3<x+3<4+3 \\
5<x+3<7 \\
\frac{1}{7}<\frac{1}{x+3}<\frac{1}{5} \\
\frac{\varepsilon}{7}<\frac{\varepsilon}{x+3}<\frac{\varepsilon}{5}
\end{gathered}
$$

Let $\delta=\frac{\varepsilon}{7}$. Then:

$$
\begin{aligned}
|x-3| &< \delta \\
|x-3| &< \frac{\varepsilon}{7} \\
|x-3| &< \frac{\varepsilon}{x+3} \\
|x-3|\cdot|x+3| &< \frac{\varepsilon}{x+3}\cdot|x+3|
\end{aligned}
$$

Assuming x is near 3, $x+3$ is positive and we can drop the absolute value signs on the right.

$$
\begin{aligned}
|x-3|\cdot|x+3| &< \frac{\varepsilon}{x+3}\cdot(x+3) \\
|x^2-9| &< \varepsilon \\
|(x^2-3)-6| &< \varepsilon,
\end{aligned}
$$

which is what we wanted to prove.

01 02 ex 11

7. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-4|<\delta$, $|f(x)-15|<\varepsilon$.
Consider $|f(x)-15|<\varepsilon$, keeping in mind we want to make a statement about $|x-4|$:

$$
\begin{aligned}
|f(x)-15| &< \varepsilon \\
|x^2+x-5-15| &< \varepsilon \\
|x^2+x-20| &< \varepsilon \\
|x-4|\cdot|x+5| &< \varepsilon \\
|x-4| &< \varepsilon/|x+5|
\end{aligned}
$$

Since x is near 4, we can safely assume that, for instance, $3<x<5$. Thus

$$
\begin{gathered}
3+5<x+5<5+5 \\
8<x+5<10 \\
\frac{1}{10}<\frac{1}{x+5}<\frac{1}{8} \\
\frac{\varepsilon}{10}<\frac{\varepsilon}{x+5}<\frac{\varepsilon}{8}
\end{gathered}
$$

Let $\delta=\frac{\varepsilon}{10}$. Then:

$$
\begin{aligned}
|x-4| &< \delta \\
|x-4| &< \frac{\varepsilon}{10} \\
|x-4| &< \frac{\varepsilon}{x+5} \\
|x-4|\cdot|x+5| &< \frac{\varepsilon}{x+5}\cdot|x+5|
\end{aligned}
$$

Assuming x is near 4, $x+5$ is positive and we can drop the absolute value signs on the right.

$$
\begin{aligned}
|x-4|\cdot|x+5| &< \frac{\varepsilon}{x+5}\cdot(x+5) \\
|x^2+x-20| &< \varepsilon \\
|(x^2+x-5)-15| &< \varepsilon,
\end{aligned}
$$

which is what we wanted to prove.

01 02 ex 06

8. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-2|<\delta$, $|f(x)-7|<\varepsilon$.
Consider $|f(x)-7|<\varepsilon$, keeping in mind we want to make a statement about $|x-2|$:

$$
\begin{aligned}
|f(x)-7| &< \varepsilon \\
|x^3-1-7| &< \varepsilon \\
|x^3-8| &< \varepsilon \\
|x-2|\cdot|x^2+2x+4| &< \varepsilon \\
|x-3| &< \varepsilon/|x^2+2x+4|
\end{aligned}
$$

Since x is near 2, we can safely assume that, for instance, $1<x<3$. Thus

$$
\begin{gathered}
1^2+2\cdot 1+4<x^2+2x+4<3^2+2\cdot 3+4 \\
7<x^2+2x+4<19 \\
\frac{1}{19}<\frac{1}{x^2+2x+4}<\frac{1}{7} \\
\frac{\varepsilon}{19}<\frac{\varepsilon}{x^2+2x+4}<\frac{\varepsilon}{7}
\end{gathered}
$$

Let $\delta=\frac{\varepsilon}{19}$. Then:

$$
\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\varepsilon}{19} \\
|x-2| &< \frac{\varepsilon}{x^2+2x+4} \\
|x-2|\cdot|x^2+2x+4| &< \frac{\varepsilon}{x^2+2x+4}\cdot|x^2+2x+4|
\end{aligned}
$$

Assuming x is near 2, x^2+2x+4 is positive and we can drop the absolute value signs on the right.

$$
\begin{aligned}
|x-2|\cdot|x^2+2x+4| &< \frac{\varepsilon}{x^2+2x+4}\cdot(x^2+2x+4) \\
|x^3-8| &< \varepsilon \\
|(x^3-1)-7| &< \varepsilon,
\end{aligned}
$$

which is what we wanted to prove.

01 02 ex 03 9. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-5| < \varepsilon$. However, since $f(x) = 5$, a constant function, the latter inequality is simply $|5-5| < \varepsilon$, which is always true. Thus we can choose any δ we like; we arbitrarily choose $\delta = \varepsilon$.

01 02 ex 07 10. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0| < \delta$, $|f(x)-0| < \varepsilon$.

Consider $|f(x)-0| < \varepsilon$, keeping in mind we want to make a statement about $|x-0|$ (i.e., $|x|$):

$$\begin{aligned}
|f(x)-0| &< \varepsilon\\
|e^{2x}-1| &< \varepsilon\\
-\varepsilon < e^{2x}-1 &< \varepsilon\\
1-\varepsilon < e^{2x} &< 1+\varepsilon\\
\ln(1-\varepsilon) < 2x &< \ln(1+\varepsilon)\\
\frac{\ln(1-\varepsilon)}{2} < x &< \frac{\ln(1+\varepsilon)}{2}
\end{aligned}$$

Let $\delta = \min\left\{\left|\frac{\ln(1-\varepsilon)}{2}\right|, \frac{\ln(1+\varepsilon)}{2}\right\} = \frac{\ln(1+\varepsilon)}{2}$.

Thus:

$$\begin{aligned}
|x| &< \delta\\
|x| < \frac{\ln(1+\varepsilon)}{2} &< \left|\frac{\ln(1-\varepsilon)}{2}\right|\\
\frac{\ln(1-\varepsilon)}{2} < x &< \frac{\ln(1+\varepsilon)}{2}\\
\ln(1-\varepsilon) < 2x &< \ln(1+\varepsilon)\\
1-\varepsilon < e^{2x} &< 1+\varepsilon\\
-\varepsilon < e^{2x}-1 &< \varepsilon\\
|e^{2x}-1-(0)| &< \varepsilon,
\end{aligned}$$

which is what we wanted to prove.

01 02 ex 08 11. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0| < \delta$, $|f(x)-0| < \varepsilon$. In simpler terms, we want to show that when $|x| < \delta$, $|\sin x| < \varepsilon$.

Set $\delta = \varepsilon$. We start with assuming that $|x| < \delta$. Using the hint, we have that $|\sin x| < |x| < \delta = \varepsilon$. Hence if $|x| < \delta$, we know immediately that $|\sin x| < \varepsilon$.

Section 1.3

01 03 ex 01 1. Answers will vary.

01 03 ex 02 2. Answers will vary.

01 03 ex 03 3. Answers will vary.

01 03 ex 04 4. Answers will vary.

01 03 ex 05 5. As x is near 1, both f and g are near 0, but f is approximately twice the size of g. (I.e., $f(x) \approx 2g(x)$.)

01 03 ex 06 6. 9

01 03 ex 07 7. 6

01 03 ex 08 8. 0

01 03 ex 09 9. Limit does not exist.

01 03 ex 10 10. 3

01 03 ex 11 11. Not possible to know; as x approaches 6, $g(x)$ approaches 3, but we know nothing of the behavior of $f(x)$ when x is near 3.

01 03 ex 12 12. 3

01 03 ex 13 13. -45

01 03 ex 14 14. 1

01 03 ex 15 15. -1

01 03 ex 16 16. 0

01 03 ex 17 17. π

01 03 ex 18 18. 7

01 03 ex 21 19. $-0.000000015 \approx 0$

01 03 ex 22 20. $1/2$

01 03 ex 23 21. Limit does not exist

01 03 ex 24 22. 64

01 03 ex 25 23. 2

01 03 ex 26 24. 0

01 03 ex 20 25. $\frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064$

01 03 ex 19 26. $\frac{3\pi+1}{1-\pi}$

01 03 ex 27 27. -8

01 03 ex 28 28. -1

01 03 ex 29 29. 10

01 03 ex 30 30. -2

01 03 ex 31 31. $-3/2$

01 03 ex 32 32. $-7/8$

01 03 ex 38 33. 0

01 03 ex 40 34. 0

01 03 ex 42 35. 1

01 03 ex 41 36. 9

01 03 ex 3[illegible] 37. 3

01 03 ex 34 38. $5/8$

01 03 ex 35 39. 1

01 03 ex 36 40. $\pi/180$

01 03 ex 43 41.

(a) Apply Part 1 of Theorem 1.

(b) Apply Theorem 6; $g(x) = \frac{x}{x}$ is the same as $g(x) = 1$ everywhere except at $x = 0$. Thus $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} 1 = 1$.

(c) The function $f(x)$ is always 0, so $g\big(f(x)\big)$ is never defined as $g(x)$ is not defined at $x = 0$. Therefore the limit does not exist.

(d) The Composition Rule requires that $\lim\limits_{x\to 0} g(x)$ be equal to $g(0)$. They are not equal, so the conditions of the Composition Rule are not satisfied, and hence the rule is not violated.