1: Curves in the Plane

We have explored functions of the form $y = f(x)$ closely throughout this text. We have explored their limits, their derivatives and their antiderivatives; we have learned to identify key features of their graphs, such as relative maxima and minima, inflection points and asymptotes; we have found equations of their tangent lines, the areas between portions of their graphs and the x-axis, and the volumes of solids generated by revolving portions of their graphs about a horizontal or vertical axis.

Despite all this, the graphs created by functions of the form $y = f(x)$ are limited. Since each x-value can correspond to only 1 y-value, common shapes like circles cannot be fully described by a function in this form. Fittingly, the "vertical line test" excludes vertical lines from being functions of x, even though these lines are important in mathematics.

In this chapter we'll explore new ways of drawing curves in the plane. We'll still work within the framework of functions, as an input will still only correspond to one output. However, our new techniques of drawing curves will render the vertical line test pointless, and allow us to create important – and beautiful – new curves. Once these curves are defined, we'll apply the concepts of calculus to them, continuing to find equations of tangent lines and the areas of enclosed regions.

1.1 Conic Sections

The ancient Greeks recognized that interesting shapes can be formed by intersecting a plane with a *double napped* cone (i.e., two identical cones placed tip–to–tip as shown in the following figures). As these shapes are formed as sections of conics, they have earned the official name "conic sections."

The three "most interesting" conic sections are given in the top row of Figure 1.1.1. They are the parabola, the ellipse (which includes circles) and the hyperbola. In each of these cases, the plane does not intersect the tips of the cones (usually taken to be the origin).

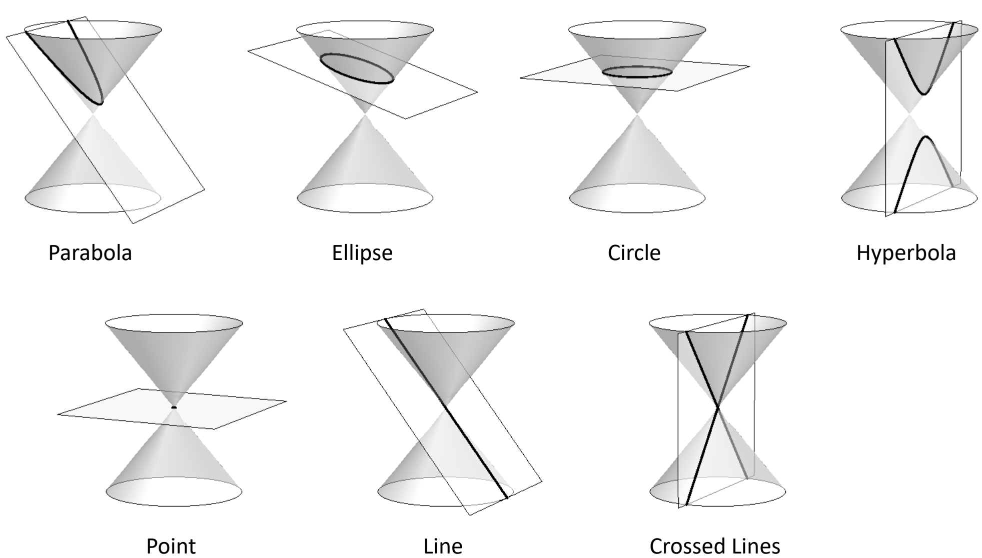


Figure 1.1.1: Conic Sections

When the plane does contain the origin, three **degenerate** cones can be formed as shown the bottom row of Figure 1.1.1: a point, a line, and crossed lines. We focus here on the nondegenerate cases.

While the above geometric constructs define the conics in an intuitive, visual way, these constructs are not very helpful when trying to analyze the shapes algebraically or consider them as the graph of a function. It can be shown that all conics can be defined by the general second–degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

While this algebraic definition has its uses, most find another geometric perspective of the conics more beneficial.

Each nondegenerate conic can be defined as the **locus**, or set, of points that satisfy a certain distance property. These distance properties can be used to generate an algebraic formula, allowing us to study each conic as the graph of a function.

Parabolas

Definition 1.1.1 Parabola

A **parabola** is the locus of all points equidistant from a point (called a **focus**) and a line (called the **directrix**) that does not contain the focus.

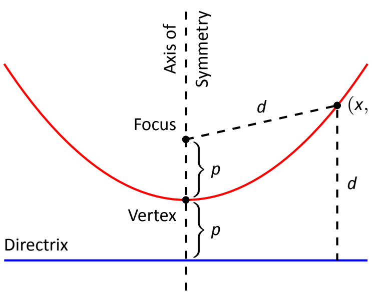


Figure 1.1.2: Illustrating the definition of the parabola and establishing an algebraic formula.

Figure 1.1.2 illustrates this definition. The point halfway between the focus and the directrix is the **vertex**. The line through the focus, perpendicular to the directrix, is the **axis of symmetry**, as the portion of the parabola on one side of this line is the mirror–image of the portion on the opposite side.

The definition leads us to an algebraic formula for the parabola. Let $P = (x, y)$ be a point on a parabola whose focus is at $F = (0, p)$ and whose directrix is at $y = -p$. (We'll assume for now that the focus lies on the y-axis; by placing the focus p units above the x-axis and the directrix p units below this axis, the vertex will be at $(0, 0)$.)

We use the Distance Formula to find the distance d_1 between F and P:

$$d_1 = \sqrt{(x-0)^2 + (y-p)^2}.$$

The distance d_2 from P to the directrix is more straightforward:

$$d_2 = y - (-p) = y + p.$$

Notes:

These two distances are equal. Setting $d_1 = d_2$, we can solve for y in terms of x:

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{x^2 + (y-p)^2} &= y + p \end{aligned}$$

Now square both sides.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2yp + p^2 &= y^2 + 2yp + p^2 \\ x^2 &= 4yp \\ y &= \frac{1}{4p}x^2. \end{aligned}$$

The geometric definition of the parabola has led us to the familiar quadratic function whose graph is a parabola with vertex at the origin. When we allow the vertex to not be at $(0,0)$, we get the following standard form of the parabola.

Key Idea 1.1.1 General Equation of a Parabola

1. **Vertical Axis of Symmetry:** The equation of the parabola with vertex at (h,k) and directrix $y = k - p$ in standard form is

$$y = \frac{1}{4p}(x-h)^2 + k.$$

The focus is at $(h, k+p)$.

2. **Horizontal Axis of Symmetry:** The equation of the parabola with vertex at (h,k) and directrix $x = h - p$ in standard form is

$$x = \frac{1}{4p}(y-k)^2 + h.$$

The focus is at $(h+p, k)$.

Note: p is not necessarily a positive number.

Example 1.1.1 Finding the equation of a parabola

Give the equation of the parabola with focus at $(1,2)$ and directrix at $y = 3$.

Solution The vertex is located halfway between the focus and directrix, so $(h,k) = (1, 2.5)$. This gives $p = -0.5$. Using Key Idea 1.1.1 we have the

Notes:

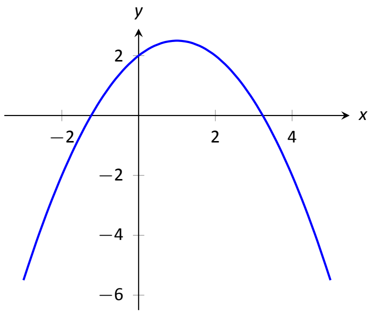


Figure 1.1.3: The parabola described in Example 1.1.1.

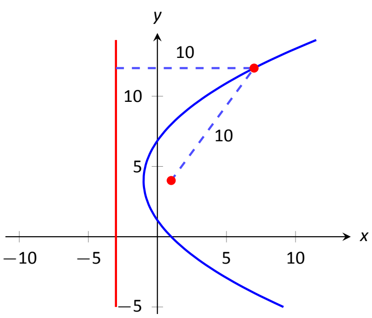


Figure 1.1.4: The parabola described in Example 1.1.2. The distances from a point on the parabola to the focus and directrix is given.

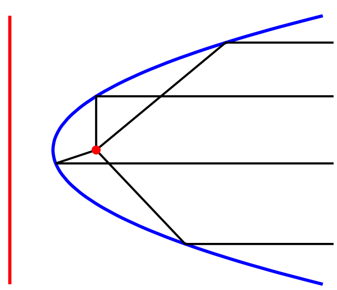


Figure 1.1.5: Illustrating the parabola's reflective property.

equation of the parabola as

$$y = \frac{1}{4(-0.5)}(x-1)^2 + 2.5 = -\frac{1}{2}(x-1)^2 + 2.5.$$

The parabola is sketched in Figure 1.1.3.

Example 1.1.2 Finding the focus and directrix of a parabola

Find the focus and directrix of the parabola $x = \frac{1}{8}y^2 - y + 1$. The point $(7, 12)$ lies on the graph of this parabola; verify that it is equidistant from the focus and directrix.

SOLUTION We need to put the equation of the parabola in its general form. This requires us to complete the square:

$$\begin{aligned} x &= \frac{1}{8}y^2 - y + 1 \\ &= \frac{1}{8}\left(y^2 - 8y + 8\right) \\ &= \frac{1}{8}\left(y^2 - 8y + 16 - 16 + 8\right) \\ &= \frac{1}{8}\left((y-4)^2 - 8\right) \\ &= \frac{1}{8}(y-4)^2 - 1. \end{aligned}$$

Hence the vertex is located at $(-1, 4)$. We have $\frac{1}{8} = \frac{1}{4p}$, so $p = 2$. We conclude that the focus is located at $(1, 4)$ and the directrix is $x = -3$. The parabola is graphed in Figure 1.1.4, along with its focus and directrix.

The point $(7, 12)$ lies on the graph and is $7 - (-3) = 10$ units from the directrix. The distance from $(7, 12)$ to the focus is:

$$\sqrt{(7-1)^2 + (12-4)^2} = \sqrt{100} = 10.$$

Indeed, the point on the parabola is equidistant from the focus and directrix.

Reflective Property

One of the fascinating things about the nondegenerate conic sections is their reflective properties. Parabolas have the following reflective property:

> Any ray emanating from the focus that intersects the parabola reflects off along a line perpendicular to the directrix.

This is illustrated in Figure 1.1.5. The following theorem states this more rigorously.

Notes:

Theorem 1.1.1 Reflective Property of the Parabola

Let P be a point on a parabola. The tangent line to the parabola at P makes equal angles with the following two lines:

1. The line containing P and the focus F, and
2. The line perpendicular to the directrix through P.

Because of this reflective property, paraboloids (the 3D analogue of parabolas) make for useful flashlight reflectors as the light from the bulb, ideally located at the focus, is reflected along parallel rays. Satellite dishes also have paraboloid shapes. Signals coming from satellites effectively approach the dish along parallel rays. The dish then *focuses* these rays at the focus, where the sensor is located.

Ellipses

Definition 1.1.2 Ellipse

An **ellipse** is the locus of all points whose sum of distances from two fixed points, each a **focus** of the ellipse, is constant.

An easy way to visualize this construction of an ellipse is to pin both ends of a string to a board. The pins become the foci. Holding a pencil tight against the string places the pencil on the ellipse; the sum of distances from the pencil to the pins is constant: the length of the string. See Figure 1.1.6.

We can again find an algebraic equation for an ellipse using this geometric definition. Let the foci be located along the x-axis, c units from the origin. Let these foci be labeled as $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P = (x, y)$ be a point on the ellipse. The sum of distances from F_1 to P (d_1) and from F_2 to P (d_2) is a constant d. That is, $d_1 + d_2 = d$. Using the Distance Formula, we have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = d.$$

Using a fair amount of algebra can produce the following equation of an ellipse (note that the equation is an implicitly defined function; it has to be, as an ellipse fails the Vertical Line Test):

$$\frac{x^2}{\left(\frac{d}{2}\right)^2} + \frac{y^2}{\left(\frac{d}{2}\right)^2 - c^2} = 1.$$

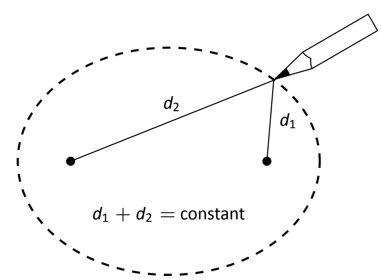


Figure 1.1.6: Illustrating the construction of an ellipse with pins, pencil and string.

Notes:

This is not particularly illuminating, but by making the substitution $a = d/2$ and $b = \sqrt{a^2 - c^2}$, we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

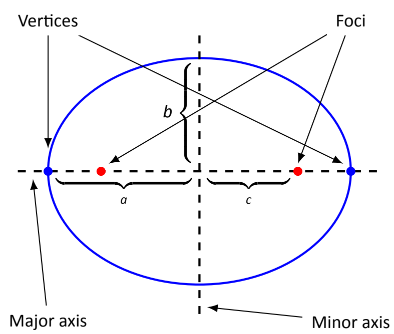


Figure 1.1.7: Labeling the significant features of an ellipse.

This choice of a and b is not without reason; as shown in Figure 1.1.7, the values of a and b have geometric meaning in the graph of the ellipse.

In general, the two foci of an ellipse lie on the **major axis** of the ellipse, and the midpoint of the segment joining the two foci is the **center**. The major axis intersects the ellipse at two points, each of which is a **vertex**. The line segment through the center and perpendicular to the major axis is the **minor axis**. The "constant sum of distances" that defines the ellipse is the length of the major axis, i.e., $2a$.

Allowing for the shifting of the ellipse gives the following standard equations.

Key Idea 1.1.2 Standard Equation of the Ellipse

The equation of an ellipse centered at (h, k) with major axis of length $2a$ and minor axis of length $2b$ in standard form is:

1. **Horizontal major axis:** $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1.$

2. **Vertical major axis:** $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1.$

The foci lie along the major axis, c units from the center, where $c^2 = a^2 - b^2$.

Example 1.1.3 Finding the equation of an ellipse

Find the general equation of the ellipse graphed in Figure 1.1.8.

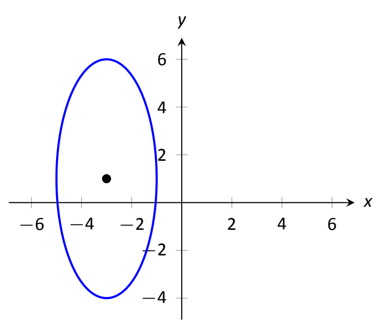


Figure 1.1.8: The ellipse used in Example 1.1.3.

Solution The center is located at $(-3, 1)$. The distance from the center to a vertex is 5 units, hence $a = 5$. The minor axis seems to have length 4, so $b = 2$. Thus the equation of the ellipse is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{25} = 1.$$

Example 1.1.4 Graphing an ellipse

Graph the ellipse defined by $4x^2 + 9y^2 - 8x - 36y = -4$.

Notes:

Solution It is simple to graph an ellipse once it is in standard form. In order to put the given equation in standard form, we must complete the square with both the x and y terms. We first rewrite the equation by regrouping:

$$4x^2+9y^2-8x-36y=-4 \quad\Rightarrow\quad (4x^2-8x)+(9y^2-36y)=-4.$$

Now we complete the squares.

$$\begin{aligned}
(4x^2-8x)+(9y^2-36y)&=-4\\
4(x^2-2x)+9(y^2-4y)&=-4\\
4(x^2-2x+1-1)+9(y^2-4y+4-4)&=-4\\
4\big((x-1)^2-1\big)+9\big((y-2)^2-4\big)&=-4\\
4(x-1)^2-4+9(y-2)^2-36&=-4\\
4(x-1)^2+9(y-2)^2&=36\\
\frac{(x-1)^2}{9}+\frac{(y-2)^2}{4}&=1.
\end{aligned}$$

We see the center of the ellipse is at $(1,2)$. We have $a=3$ and $b=2$; the major axis is horizontal, so the vertices are located at $(-2,2)$ and $(4,2)$. We find $c=\sqrt{9-4}=\sqrt{5}\approx 2.24$. The foci are located along the major axis, approximately 2.24 units from the center, at $(1\pm 2.24, 2)$. This is all graphed in Figure 1.1.9 .

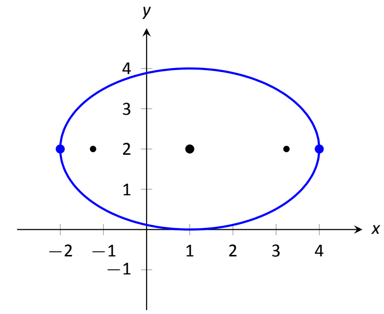


Figure 1.1.9: Graphing the ellipse in Example 1.1.4.

Eccentricity

When $a=b$, we have a circle. The general equation becomes

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=1 \quad\Rightarrow (x-h)^2+(y-k)^2=a^2,$$

the familiar equation of the circle centered at (h,k) with radius a. Since $a=b$, $c=\sqrt{a^2-b^2}=0$. The circle has "two" foci, but they lie on the same point, the center of the circle.

Consider Figure 1.1.10, where several ellipses are graphed with $a=1$. In (a), we have $c=0$ and the ellipse is a circle. As c grows, the resulting ellipses look less and less circular. A measure of this "noncircularness" is *eccentricity*.

> **Definition 1.1.3 Eccentricity of an Ellipse**
>
> The eccentricity e of an ellipse is $e=\dfrac{c}{a}$.

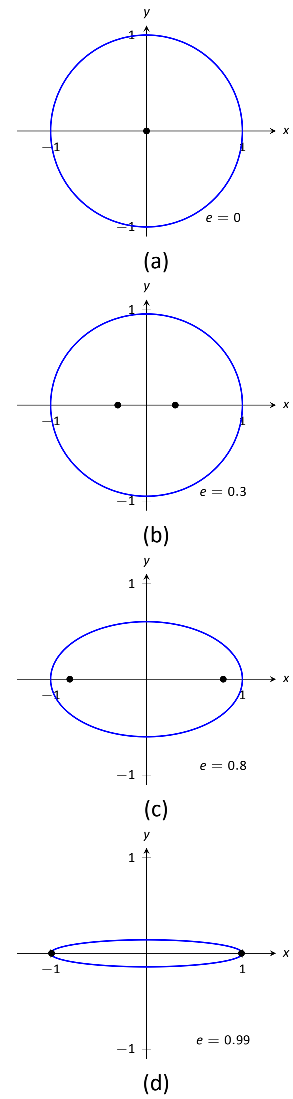


Figure 1.1.10: Understanding the eccentricity of an ellipse.

Notes:

The eccentricity of a circle is 0; that is, a circle has no "noncircularness." As c approaches a, e approaches 1, giving rise to a very noncircular ellipse, as seen in Figure 1.1.10 (d).

It was long assumed that planets had circular orbits. This is known to be incorrect; the orbits are elliptical. Earth has an eccentricity of 0.0167 – it has a nearly circular orbit. Mercury's orbit is the most eccentric, with $e = 0.2056$. (Pluto's eccentricity is greater, at $e = 0.248$, the greatest of all the currently known dwarf planets.) The planet with the most circular orbit is Venus, with $e = 0.0068$. The Earth's moon has an eccentricity of $e = 0.0549$, also very circular.

Reflective Property

The ellipse also possesses an interesting reflective property. Any ray emanating from one focus of an ellipse reflects off the ellipse along a line through the other focus, as illustrated in Figure 1.1.11. This property is given formally in the following theorem.

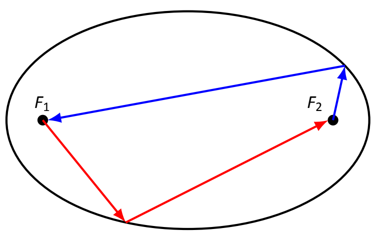


Figure 1.1.11: Illustrating the reflective property of an ellipse.

Theorem 1.1.2 Reflective Property of an Ellipse

Let P be a point on a ellipse with foci F_1 and F_2. The tangent line to the ellipse at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

This reflective property is useful in optics and is the basis of the phenomena experienced in whispering halls.

Hyperbolas

The definition of a hyperbola is very similar to the definition of an ellipse; we essentially just change the word "sum" to "difference."

Definition 1.1.4 Hyperbola

A **hyperbola** is the locus of all points where the absolute value of difference of distances from two fixed points, each a focus of the hyperbola, is constant.

Notes:

We do not have a convenient way of visualizing the construction of a hyperbola as we did for the ellipse. The geometric definition does allow us to find an algebraic expression that describes it. It will be useful to define some terms first.

The two foci lie on the **transverse axis** of the hyperbola; the midpoint of the line segment joining the foci is the **center** of the hyperbola. The transverse axis intersects the hyperbola at two points, each a **vertex** of the hyperbola. The line through the center and perpendicular to the transverse axis is the **conjugate axis.** This is illustrated in Figure 1.1.12. It is easy to show that the constant difference of distances used in the definition of the hyperbola is the distance between the vertices, i.e., $2a$.

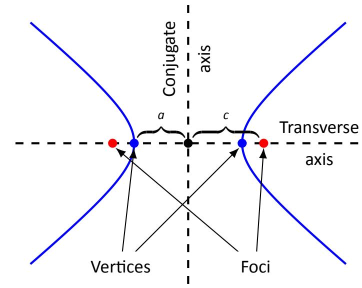


Figure 1.1.12: Labeling the significant features of a hyperbola.

Key Idea 1.1.3 Standard Equation of a Hyperbola

The equation of a hyperbola centered at (h,k) in standard form is:

1. **Horizontal Transverse Axis:** $$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1.$$
2. **Vertical Transverse Axis:** $$\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1.$$

The vertices are located a units from the center and the foci are located c units from the center, where $c^2=a^2+b^2$.

Graphing Hyperbolas

Consider the hyperbola $\frac{x^2}{9}-\frac{y^2}{1}=1$. Solving for y, we find $y=\pm\sqrt{x^2/9-1}$. As x grows large, the "-1" part of the equation for y becomes less significant and $y\approx\pm\sqrt{x^2/9}=\pm x/3$. That is, as x gets large, the graph of the hyperbola looks very much like the lines $y=\pm x/3$. These lines are asymptotes of the hyperbola, as shown in Figure 1.1.13.

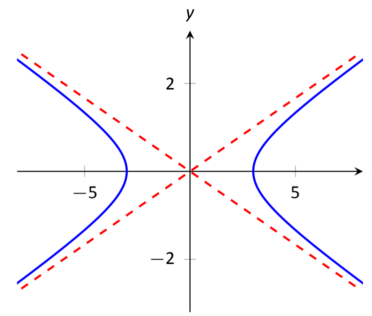


Figure 1.1.13: Graphing the hyperbola $\frac{x^2}{9}-\frac{y^2}{1}=1$ along with its asymptotes, $y=\pm x/3$.

This is a valuable tool in sketching. Given the equation of a hyperbola in general form, draw a rectangle centered at (h,k) with sides of length $2a$ parallel to the transverse axis and sides of length $2b$ parallel to the conjugate axis. (See Figure 1.1.14 for an example with a horizontal transverse axis.) The diagonals of the rectangle lie on the asymptotes.

These lines pass through (h,k). When the transverse axis is horizontal, the slopes are $\pm b/a$; when the transverse axis is vertical, their slopes are $\pm a/b$. This gives equations:

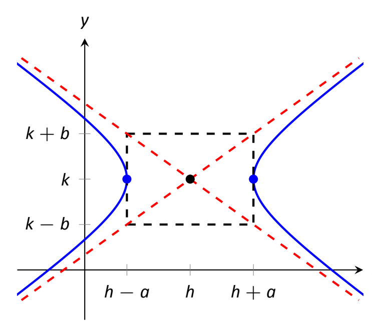


Figure 1.1.14: Using the asymptotes of a hyperbola as a graphing aid.

Notes:

Horizontal Transverse Axis	Vertical Transverse Axis
$y = \pm\frac{b}{a}(x-h)+k$	$y = \pm\frac{a}{b}(x-h)+k.$

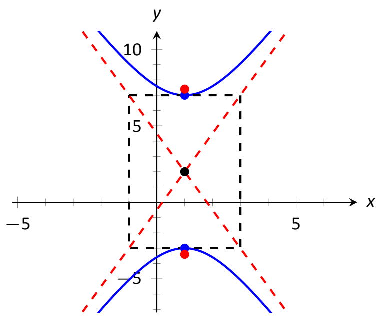


Figure 1.1.15: Graphing the hyperbola in Example 1.1.5.

Example 1.1.5 Graphing a hyperbola

Sketch the hyperbola given by $\frac{(y-2)^2}{25} - \frac{(x-1)^2}{4} = 1.$

Solution The hyperbola is centered at $(1,2)$; $a=5$ and $b=2$. In Figure 1.1.15 we draw the prescribed rectangle centered at $(1,2)$ along with the asymptotes defined by its diagonals. The hyperbola has a vertical transverse axis, so the vertices are located at $(1,7)$ and $(1,-3)$. This is enough to make a good sketch.

We also find the location of the foci: as $c^2 = a^2+b^2$, we have $c=\sqrt{29}\approx 5.4$. Thus the foci are located at $(1, 2\pm 5.4)$ as shown in the figure.

Example 1.1.6 Graphing a hyperbola

Sketch the hyperbola given by $9x^2 - y^2 + 2y = 10$.

Solution We must complete the square to put the equation in general form. (We recognize this as a hyperbola since it is a general quadratic equation and the x^2 and y^2 terms have opposite signs.)

$$\begin{aligned} 9x^2 - y^2 + 2y &= 10 \\ 9x^2 - (y^2 - 2y) &= 10 \\ 9x^2 - (y^2 - 2y + 1 - 1) &= 10 \\ 9x^2 - \left((y-1)^2 - 1\right) &= 10 \\ 9x^2 - (y-1)^2 &= 9 \\ x^2 - \frac{(y-1)^2}{9} &= 1 \end{aligned}$$

We see the hyperbola is centered at $(0,1)$, with a horizontal transverse axis, where $a=1$ and $b=3$. The appropriate rectangle is sketched in Figure 1.1.16 along with the asymptotes of the hyperbola. The vertices are located at $(\pm 1, 1)$. We have $c=\sqrt{10}\approx 3.2$, so the foci are located at $(\pm 3.2, 1)$ as shown in the figure.

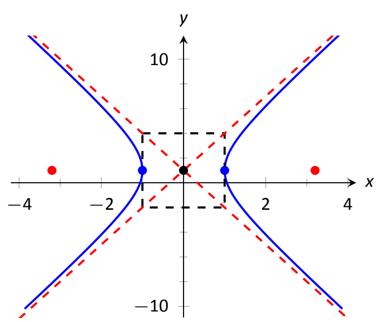


Figure 1.1.16: Graphing the hyperbola in Example 1.1.6.

Notes:

Eccentricity

Definition 1.1.5 Eccentricity of a Hyperbola

The eccentricity of a hyperbola is $e = \dfrac{c}{a}$.

Note that this is the definition of eccentricity as used for the ellipse. When c is close in value to a (i.e., $e \approx 1$), the hyperbola is very narrow (looking almost like crossed lines). Figure 1.1.17 shows hyperbolas centered at the origin with $a = 1$. The graph in (a) has $c = 1.05$, giving an eccentricity of $e = 1.05$, which is close to 1. As c grows larger, the hyperbola widens and begins to look like parallel lines, as shown in part (d) of the figure.

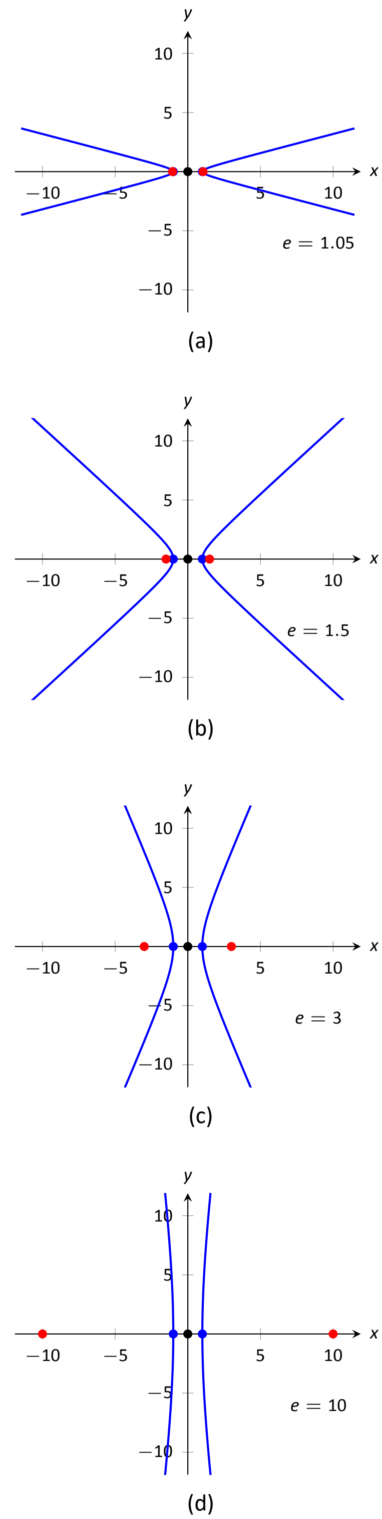


Figure 1.1.17: Understanding the eccentricity of a hyperbola.

Reflective Property

Hyperbolas share a similar reflective property with ellipses. However, in the case of a hyperbola, a ray emanating from a focus that intersects the hyperbola reflects along a line containing the other focus, but moving *away* from that focus. This is illustrated in Figure 1.1.19 (on the next page). Hyperbolic mirrors are commonly used in telescopes because of this reflective property. It is stated formally in the following theorem.

Theorem 1.1.3 Reflective Property of Hyperbolas

Let P be a point on a hyperbola with foci F_1 and F_2. The tangent line to the hyperbola at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

Location Determination

Determining the location of a known event has many practical uses (locating the epicenter of an earthquake, an airplane crash site, the position of the person speaking in a large room, etc.).

To determine the location of an earthquake's epicenter, seismologists use *trilateration* (not to be confused with *triangulation*). A seismograph allows one

Notes:

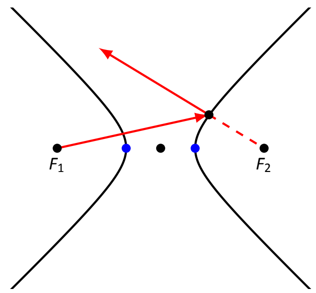


Figure 1.1.19: Illustrating the reflective property of a hyperbola.

to determine how far away the epicenter was; using three separate readings, the location of the epicenter can be approximated.

A key to this method is knowing distances. What if this information is not available? Consider three microphones at positions A, B and C which all record a noise (a person's voice, an explosion, etc.) created at unknown location D. The microphone does not "know" when the sound was *created*, only when the sound was *detected*. How can the location be determined in such a situation?

If each location has a clock set to the same time, hyperbolas can be used to determine the location. Suppose the microphone at position A records the sound at exactly 12:00, location B records the time exactly 1 second later, and location C records the noise exactly 2 seconds after that. We are interested in the *difference* of times. Since the speed of sound is approximately 340 m/s, we can conclude quickly that the sound was created 340 meters closer to position A than position B. If A and B are a known distance apart (as shown in Figure 1.1.18 (a)), then we can determine a hyperbola on which D must lie.

The "difference of distances" is 340; this is also the distance between vertices of the hyperbola. So we know $2a = 340$. Positions A and B lie on the foci, so $2c = 1000$. From this we can find $b \approx 470$ and can sketch the hyperbola, given in part (b) of the figure. We only care about the side closest to A. (Why?)

We can also find the hyperbola defined by positions B and C. In this case, $2a = 680$ as the sound traveled an extra 2 seconds to get to C. We still have $2c = 1000$, centering this hyperbola at $(-500, 500)$. We find $b \approx 367$. This hyperbola is sketched in part (c) of the figure. The intersection point of the two graphs is the location of the sound, at approximately $(188, -222.5)$.

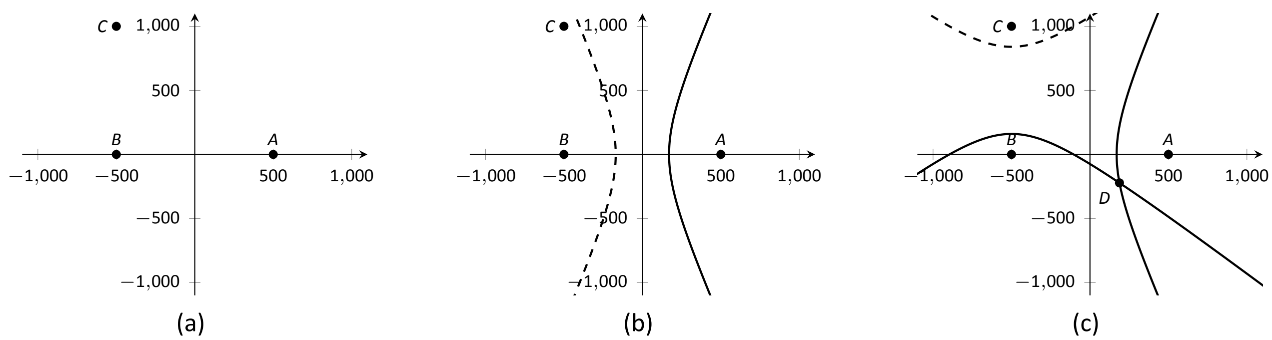


Figure 1.1.18: Using hyperbolas in location detection.

This chapter explores curves in the plane, in particular curves that cannot be described by functions of the form $y = f(x)$. In this section, we learned of ellipses and hyperbolas that are defined implicitly, not explicitly. In the following sections, we will learn completely new ways of describing curves in the plane, using *parametric equations* and *polar coordinates*, then study these curves using calculus techniques.

Notes:

Exercises 1.1

Terms and Concepts

09 01 ex 01 1. What is the difference between degenerate and nondegenerate conics?

09 01 ex 03 2. Use your own words to explain what the eccentricity of an ellipse measures.

09 01 ex 02 3. What has the largest eccentricity: an ellipse or a hyperbola?

09 01 ex 44 4. Explain why the following is true: "If the coefficient of the x^2 term in the equation of an ellipse in standard form is smaller than the coefficient of the y^2 term, then the ellipse has a horizontal major axis."

09 01 ex 45 5. Explain how one can quickly look at the equation of a hyperbola in standard form and determine whether the transverse axis is horizontal or vertical.

09 01 ex 46 6. Fill in the blank: It can be said that ellipses and hyperbolas share the *same* reflective property: "A ray emanating from one focus will reflect off the conic along a ________ that contains the other focus."

Problems

09 01 exset 01 **In Exercises 7 – 14, find the equation of the parabola defined by the given information. Sketch the parabola.**

09 01 ex 04 7. Focus: $(3,2)$; directrix: $y=1$

09 01 ex 05 8. Focus: $(-1,-4)$; directrix: $y=2$

09 01 ex 06 9. Focus: $(1,5)$; directrix: $x=3$

09 01 ex 07 10. Focus: $(1/4,0)$; directrix: $x=-1/4$

09 01 ex 08 11. Focus: $(1,1)$; vertex: $(1,2)$

09 01 ex 09 12. Focus: $(-3,0)$; vertex: $(0,0)$

09 01 ex 10 13. Vertex: $(0,0)$; directrix: $y=-1/16$

09 01 ex 11 14. Vertex: $(2,3)$; directrix: $x=4$

09 01 exset 02 **In Exercises 15 – 16, the equation of a parabola and a point on its graph are given. Find the focus and directrix of the parabola, and verify that the given point is equidistant from the focus and directrix.**

09 01 ex 12 15. $y=\frac{1}{4}x^2$, $P=(2,1)$

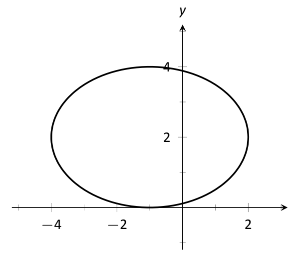
09 01 ex 13 16. $x=\frac{1}{8}(y-2)^2+3$, $P=(11,10)$

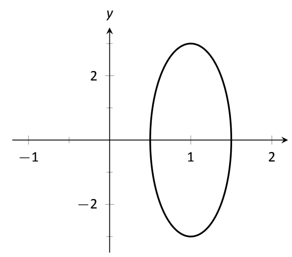
09 01 exset 03 **In Exercises 17 – 18, sketch the ellipse defined by the given equation. Label the center, foci and vertices.**

09 01 ex 18 17. $\dfrac{(x-1)^2}{3}+\dfrac{(y-2)^2}{5}=1$

09 01 ex 19 18. $\dfrac{1}{25}x^2+\dfrac{1}{9}(y+3)^2=1$

09 01 exset 05 **In Exercises 19 – 20, find the equation of the ellipse shown in the graph. Give the location of the foci and the eccentricity of the ellipse.**

09 01 ex 20 19. 

09 01 ex 21 20. 

09 01 exset 04 **In Exercises 21 – 24, find the equation of the ellipse defined by the given information. Sketch the elllipse.**

09 01 ex 14 21. Foci: $(\pm 2,0)$; vertices: $(\pm 3,0)$

09 01 ex 15 22. Foci: $(-1,3)$ and $(5,3)$; vertices: $(-3,3)$ and $(7,3)$

09 01 ex 16 23. Foci: $(2,\pm 2)$; vertices: $(2,\pm 7)$

09 01 ex 17 24. Focus: $(-1,5)$; vertex: $(-1,-4)$; center: $(-1,1)$

09 01 exset 06 **In Exercises 25 – 28, write the equation of the given ellipse in standard form.**

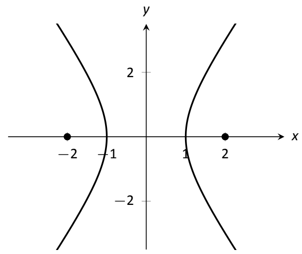
09 01 ex 22 25. $x^2-2x+2y^2-8y=-7$

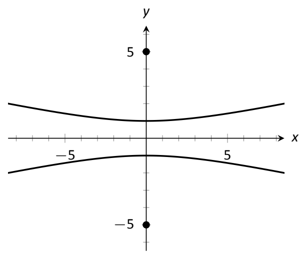
09 01 ex 23 26. $5x^2+3y^2=15$

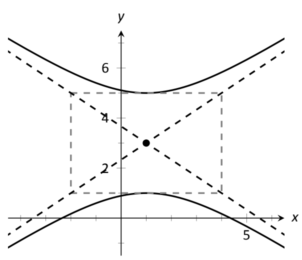
09 01 ex 24 27. $3x^2+2y^2-12y+6=0$

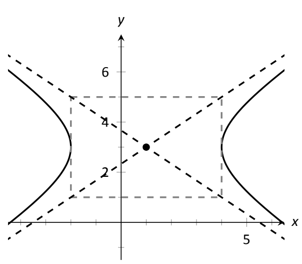
09 01 ex 25 28. $x^2+y^2-4x-4y+4=0$

09 01 exset 07 **In Exercises 29 – 32, find the equation of the hyperbola shown in the graph.**

09 01 ex 27 29. 

09 01 ex 28 30. 

09 01 ex 29 31. 

09 01 ex 30 32. 

09 01 exset 08 **In Exercises 33 – 34, sketch the hyperbola defined by the given equation. Label the center and foci.**

09 01 ex 31 33. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$

09 01 ex 32 34. $(y-4)^2 - \dfrac{(x+1)^2}{25} = 1$

09 01 exset 09 **In Exercises 35 – 38, find the equation of the hyperbola defined by the given information. Sketch the hyperbola.**

09 01 ex 33 35. Foci: $(\pm 3, 0)$; vertices: $(\pm 2, 0)$

09 01 ex 34 36. Foci: $(0, \pm 3)$; vertices: $(0, \pm 2)$

09 01 ex 35 37. Foci: $(-2, 3)$ and $(8, 3)$; vertices: $(-1, 3)$ and $(7, 3)$

09 01 ex 36 38. Foci: $(3, -2)$ and $(3, 8)$; vertices: $(3, 0)$ and $(3, 6)$

09 01 exset 10 **In Exercises 39 – 42, write the equation of the hyperbola in standard form.**

09 01 ex 37 39. $3x^2 - 4y^2 = 12$

09 01 ex 38 40. $3x^2 - y^2 + 2y = 10$

09 01 ex 39 41. $x^2 - 10y^2 + 40y = 30$

09 01 ex 40 42. $(4y - x)(4y + x) = 4$

09 01 ex 26 43. Consider the ellipse given by $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{12} = 1$.

(a) Verify that the foci are located at $(1, 3 \pm 2\sqrt{2})$.

(b) The points $P_1 = (2, 6)$ and $P_2 = (1+\sqrt{2}, 3+\sqrt{6}) \approx (2.414, 5.449)$ lie on the ellipse. Verify that the sum of distances from each point to the foci is the same.

09 01 ex 42 44. Johannes Kepler discovered that the planets of our solar system have elliptical orbits with the Sun at one focus. The Earth's elliptical orbit is used as a standard unit of distance; the distance from the center of Earth's elliptical orbit to one vertex is 1 Astronomical Unit, or A.U.

The following table gives information about the orbits of three planets.

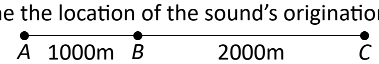
	Distance from center to vertex	eccentricity
Mercury	0.387 A.U.	0.2056
Earth	1 A.U.	0.0167
Mars	1.524 A.U.	0.0934

(a) In an ellipse, knowing $c^2 = a^2 - b^2$ and $e = c/a$ allows us to find b in terms of a and e. Show $b = a\sqrt{1-e^2}$.

(b) For each planet, find equations of their elliptical orbit of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (This places the center at $(0, 0)$, but the Sun is in a different location for each planet.)

(c) Shift the equations so that the Sun lies at the origin. Plot the three elliptical orbits.

09 01 ex 43 45. A loud sound is recorded at three stations that lie on a line as shown in the figure below. Station A recorded the sound 1 second after Station B, and Station C recorded the sound 3 seconds after B. Using the speed of sound as 340m/s, determine the location of the sound's origination.



1.2 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

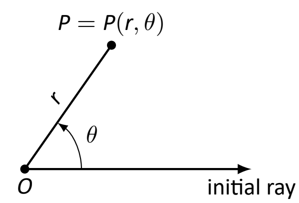


Figure 1.2.1: Illustrating polar coordinates.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 1.2.1

Practice will make this process more clear.

Example 1.2.1 Plotting Polar Coordinates
Plot the following polar coordinates:

$$A = P(1, \pi/4) \quad B = P(1.5, \pi) \quad C = P(2, -\pi/3) \quad D = P(-1, \pi/4)$$

Solution To aid in the drawing, a polar grid is provided at the bottom of this page. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).

To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units – in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 1.2.2.

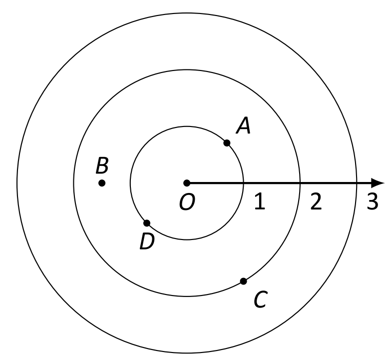
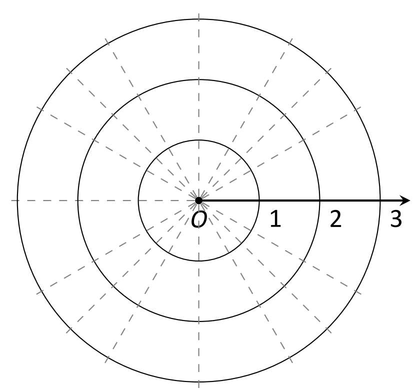


Figure 1.2.2: Plotting polar points in Example 1.2.1.

Notes:



Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 1.2.3 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

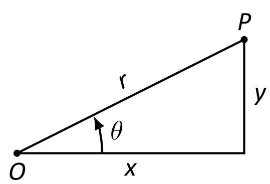


Figure 1.2.3: Converting between rectangular and polar coordinates.

Key Idea 1.2.1 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 1.2.2 Converting Between Polar and Rectangular Coordinates

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.
2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

Solution

Notes:

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 1.2.1, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 1.2.4 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $P(\sqrt{5}, 1.11)$.

(b) To convert $(-1, 1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

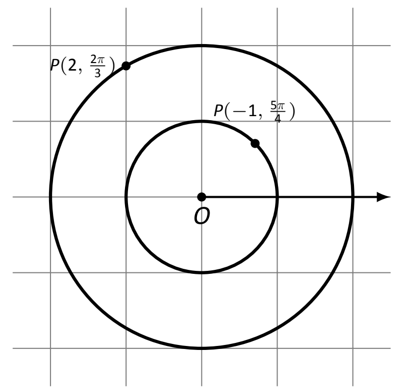
Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4 = -45^\circ.$$

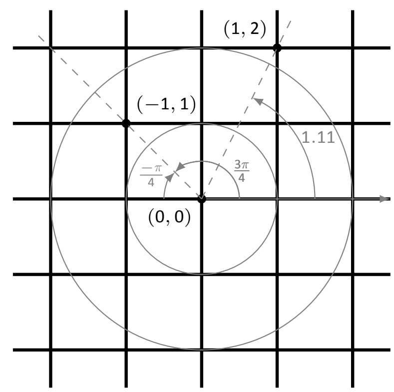
This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1^{st} and 4^{th} quadrants. To find locations in the 2^{nd} and 3^{rd} quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P(\sqrt{2}, 3\pi/4)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1, 1)$ as $P(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 1.2.4 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.



(a)



(b)

Figure 1.2.4: Plotting rectangular and polar points in Example 1.2.2.

Notes:

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 1.2.3 Introduction to Graphing Polar Functions

Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centered at $(0, 0)$ with radius 1.5. This is sketched in Figure 1.2.5.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45^\circ$ with the initial ray.

   We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

$$\tan \pi/4 = y/x \quad \Rightarrow x \tan \pi/4 = y \quad \Rightarrow y = x.$$

   This graph is also plotted in Figure 1.2.5.

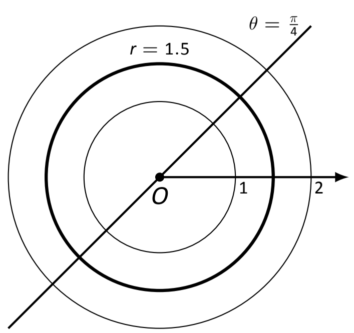


Figure 1.2.5: Plotting standard polar plots.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

Notes:

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

Example 1.2.4 Sketching Polar Functions
Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

Solution A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values – integers, then add more if needed. When plotting polar equations, start with the "common" angles – multiples of $\pi/6$ and $\pi/4$. Figure 1.2.6 gives a table of just a few values of θ in $[0, \pi]$.

Consider the point $P(2, 0)$ determined by the first line of the table. The angle is 0 radians – we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is $1 + \sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.

The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

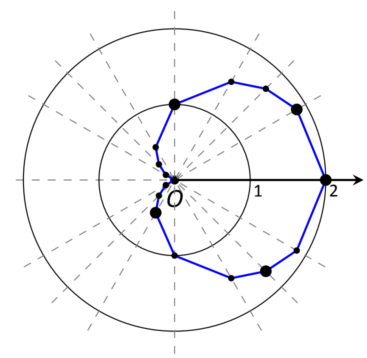


Figure 1.2.6: Graphing a polar function in Example 1.2.4 by plotting points.

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the *x*-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called $\theta_{\mathtt{min}}$ and $\theta_{\mathtt{max}}$) as well as the $\theta_{\mathtt{step}}$ – that is, how far apart the θ values are spaced. The smaller the $\theta_{\mathtt{step}}$ value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1 + \cos\theta$ from Example 1.2.4 in Figure 1.2.7.

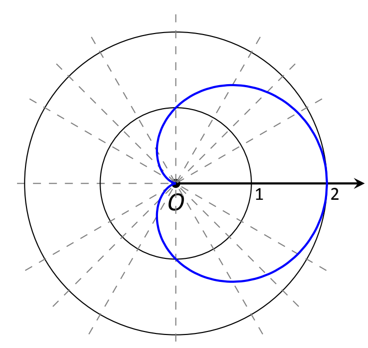
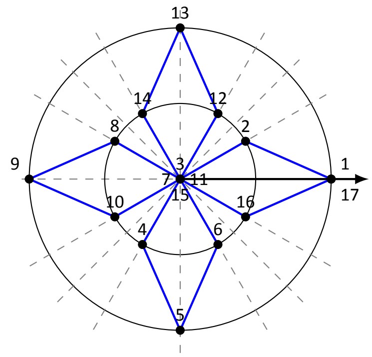


Figure 1.2.7: Using technology to graph a polar function.

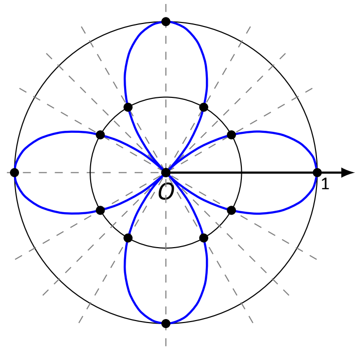
Example 1.2.5 Sketching Polar Functions
Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 1.2.8. These points are then plotted in Figure 1.2.9 (a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.

Notes:



(a)



(b)

Figure 1.2.9: Polar plots from Example 1.2.5.

Pt.	θ	$\cos(2\theta)$
1	0	1.
2	$\pi/6$	0.5
3	$\pi/4$	0.
4	$\pi/3$	-0.5
5	$\pi/2$	$-1.$
6	$2\pi/3$	-0.5
7	$3\pi/4$	0.
8	$5\pi/6$	0.5
9	π	1.

Pt.	θ	$\cos(2\theta)$
10	$7\pi/6$	0.5
11	$5\pi/4$	0.
12	$4\pi/3$	-0.5
13	$3\pi/2$	$-1.$
14	$5\pi/3$	-0.5
15	$7\pi/4$	0.
16	$11\pi/6$	0.5
17	2π	1.

Figure 1.2.8: Tables of points for plotting a polar curve.

Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 1.2.9 (b). This plot is an example of a *rose curve*.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 1.2.1.

Example 1.2.6 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

SOLUTION

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

Notes:

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta \cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 1.2.10; note how it only exists in the first and third quadrants.

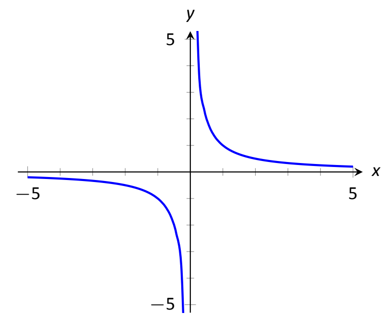


Figure 1.2.10: Graphing $xy = 1$ from Example 1.2.6.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

Notes:

We recognize this as a circle; by completing the square we can find its radius and center.

$$x^2 - 2x + y^2 = 0$$
$$(x-1)^2 + y^2 = 1.$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardioid* (a shape important to the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

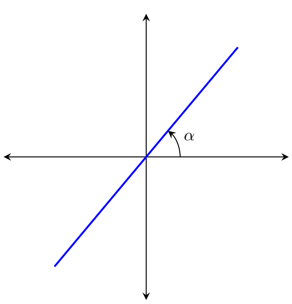
Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

Lines

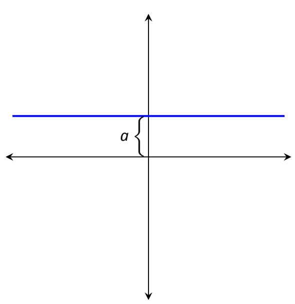
Through the origin:

$\theta = \alpha$



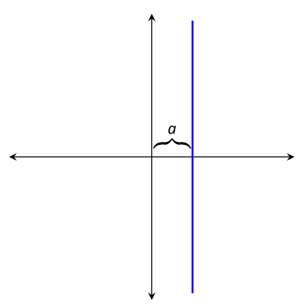
Horizontal line:

$r = a\csc\theta$



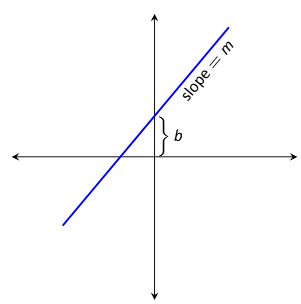
Vertical line:

$r = a\sec\theta$



Not through origin:

$$r = \frac{b}{\sin\theta - m\cos\theta}$$

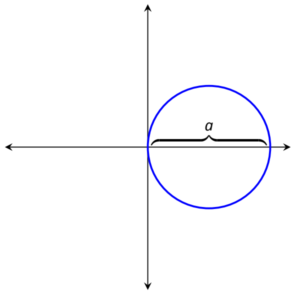


Notes:

Circles

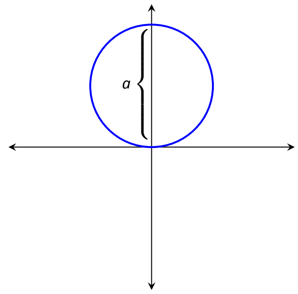
Centered on x-axis:

$r = a\cos\theta$



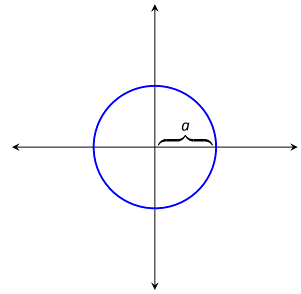
Centered on y-axis:

$r = a\sin\theta$



Centered on origin:

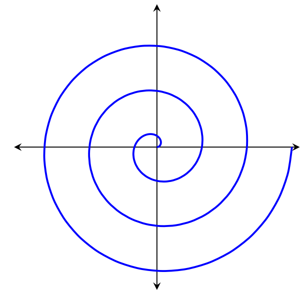
$r = a$



Spiral

Archimedean spiral

$r = \theta$

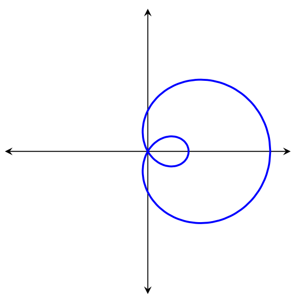


Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$; Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

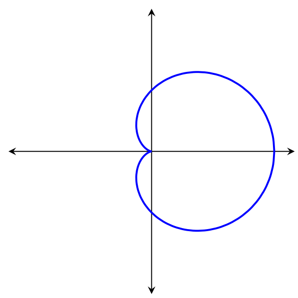
With inner loop:

$\frac{a}{b} < 1$



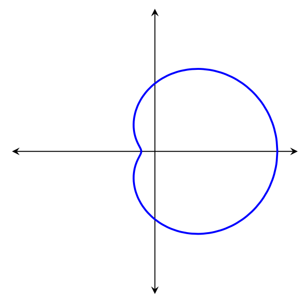
Cardioid:

$\frac{a}{b} = 1$



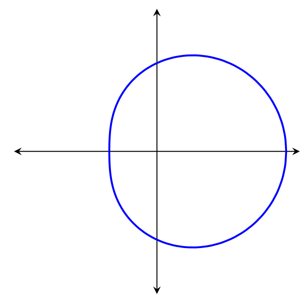
Dimpled:

$1 < \frac{a}{b} < 2$



Convex:

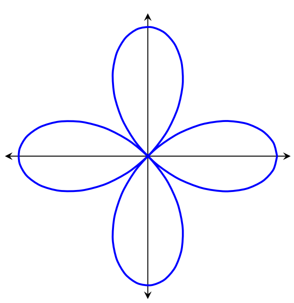
$\frac{a}{b} > 2$



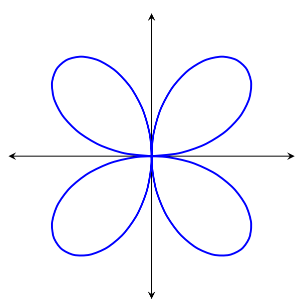
Rose Curves

Symmetric about x-axis: $r = a\cos(n\theta)$; Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

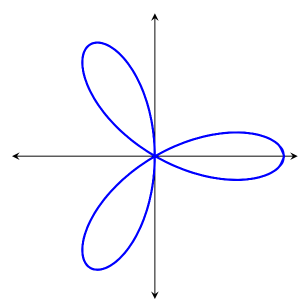
$r = a\cos(2\theta)$



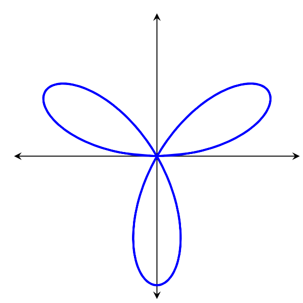
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



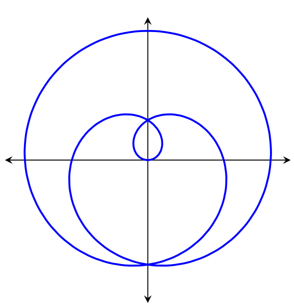
$r = a\sin(3\theta)$



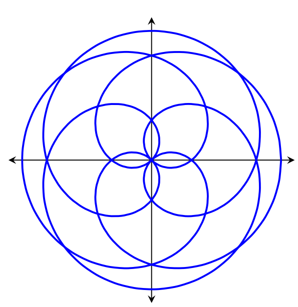
Special Curves

Rose curves

$r = a\sin(\theta/5)$

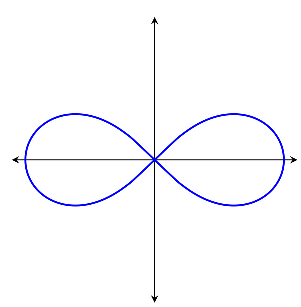


$r = a\sin(2\theta/5)$



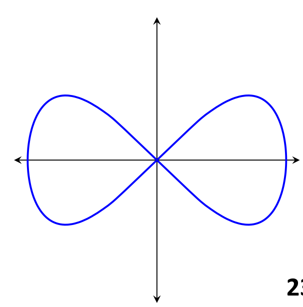
Lemniscate:

$r^2 = a^2\cos(2\theta)$



Eight Curve:

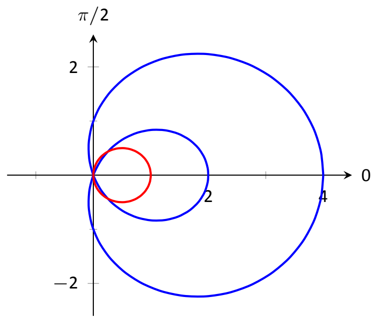
$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 1.2.7 Finding points of intersection with polar curves

Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.



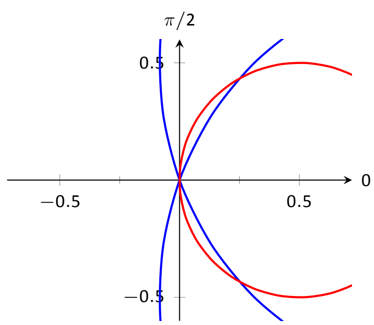


Figure 1.2.11: Graphs to help determine the points of intersection of the polar functions given in Example 1.2.7.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 1.2.11(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1+3\cos\theta &= \cos\theta \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4$^{\text{th}}$ quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 2$^{\text{nd}}$ quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly

Notes:

traced). The limaçon intersects the pole when $1 + 3\cos\theta = 0$; this occurs when $\cos\theta = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106 = 109.47^\circ$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta = 4.3726 = 250.53^\circ$.

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Notes:

Exercises 1.2

Terms and Concepts

09 04 ex 01 1. In your own words, describe how to plot the polar point $P(r,\theta)$.

09 04 ex 02 2. T/F: When plotting a point with polar coordinate $P(r,\theta)$, r must be positive.

09 04 ex 03 3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.

09 04 ex 04 4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

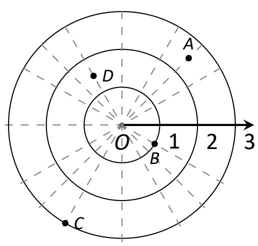
09 04 ex 05 5. Plot the points with the given polar coordinates.

(a) $A = P(2,0)$ (b) $B = P(1,\pi)$ (c) $C = P(-2,\pi/2)$ (d) $D = P(1,\pi/4)$

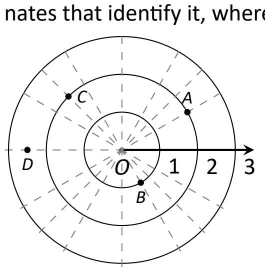
09 04 ex 06 6. Plot the points with the given polar coordinates.

(a) $A = P(2,3\pi)$ (b) $B = P(1,-\pi)$ (c) $C = P(1,2)$ (d) $D = P(1/2,5\pi/6)$

09 04 ex 07 7. For each of the given points give two sets of polar coordinates that identify it, where $0 \le \theta \le 2\pi$.



09 04 ex 08 8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \le \theta \le \pi$.



09 04 ex 09 9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

(a) $A = P(2,\pi/4)$ (b) $B = P(2,-\pi/4)$ (c) $C = (2,-1)$ (d) $D = (-2,1)$

09 04 ex 10 10. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

(a) $A = P(3,\pi)$ (b) $B = P(1,2\pi/3)$ (c) $C = (0,4)$ (d) $D = (1,-\sqrt{3})$

09 04 exset 01 **In Exercises 11 – 30, graph the polar function on the given interval.**

09 04 ex 11 11. $r = 2, \quad 0 \le \theta \le \pi/2$

09 04 ex 12 12. $\theta = \pi/6, \quad -1 \le r \le 2$

09 04 ex 13 13. $r = 1 - \cos\theta, \quad [0,2\pi]$

09 04 ex 14 14. $r = 2 + \sin\theta, \quad [0,2\pi]$

09 04 ex 15 15. $r = 2 - \sin\theta, \quad [0,2\pi]$

09 04 ex 16 16. $r = 1 - 2\sin\theta, \quad [0,2\pi]$

09 04 ex 17 17. $r = 1 + 2\sin\theta, \quad [0,2\pi]$

09 04 ex 18 18. $r = \cos(2\theta), \quad [0,2\pi]$

09 04 ex 19 19. $r = \sin(3\theta), \quad [0,\pi]$

09 04 ex 20 20. $r = \cos(\theta/3), \quad [0,3\pi]$

09 04 ex 21 21. $r = \cos(2\theta/3), \quad [0,6\pi]$

09 04 ex 22 22. $r = \theta/2, \quad [0,4\pi]$

09 04 ex 23 23. $r = 3\sin(\theta), \quad [0,\pi]$

09 04 ex 57 24. $r = 2\cos(\theta), \quad [0,\pi/2]$

09 04 ex 24 25. $r = \cos\theta\sin\theta, \quad [0,2\pi]$

09 04 ex 25 26. $r = \theta^2 - (\pi/2)^2, \quad [-\pi,\pi]$

09 04 ex 26 27. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0,2\pi]$

09 04 ex 27 28. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0,2\pi]$

09 04 ex 28 29. $r = 3\sec\theta, \quad (-\pi/2,\pi/2)$

09 04 ex 29 30. $r = 3\csc\theta, \quad (0,\pi)$

09 04 exset 02 **In Exercises 31 – 40, convert the polar equation to a rectangular equation.**

09 04 ex 30 31. $r = 6\cos\theta$

09 04 ex 31 32. $r = -4\sin\theta$

09 04 ex 32 33. $r = \cos\theta + \sin\theta$

09 04 ex 33 34. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$

09 04 ex 34 35. $r = \dfrac{3}{\cos\theta}$

09 04 ex 35 36. $r = \dfrac{4}{\sin\theta}$

09 04 ex 36 37. $r = \tan\theta$

09 04 ex 58 38. $r = \cot\theta$

09 04 ex 37 39. $r = 2$

09 04 ex 38 40. $\theta = \pi/6$

09 04 exset 03 **In Exercises 41 – 48, convert the rectangular equation to a polar equation.**

09 04 ex 39 41. $y = x$

09 04 ex 40 42. $y = 4x + 7$

09 04 ex 41 43. $x = 5$

09 04 ex 42 44. $y = 5$

09 04 ex 43 45. $x = y^2$

09 04 ex 44 46. $x^2 y = 1$

09 04 ex 45 47. $x^2 + y^2 = 7$

09 04 ex 46 48. $(x+1)^2 + y^2 = 1$

09 04 exset 04 **In Exercises 49 – 56, find the points of intersection of the polar graphs.**

09 04 ex 47 49. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

09 04 ex 48 50. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

09 04 ex 49 51. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

09 04 ex 50 52. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

09 04 ex 51 53. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

09 04 ex 52 54. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

09 04 ex 53 55. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

09 04 ex 54 56. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

09 04 ex 55 57. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\dfrac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

09 04 ex 56 58. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

09 01 ex 01 1. When defining the conics as the intersections of a plane and a double napped cone, degenerate conics are created when the plane intersects the tips of the cones (usually taken as the origin). Nondegenerate conics are formed when this plane does not contain the origin.

09 01 ex 03 2. Answers will vary.

09 01 ex 02 3. Hyperbola

09 01 ex 44 4. With the equation $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$, the ellipse has a horizontal major axis if $a>b$. But the coefficient of the x^2 term is $1/a^2$ (not a^2), so if $1/a^2<1/b^2$, then $a>b$ and the major axis is horizontal.

09 01 ex 45 5. With a horizontal transverse axis, the x^2 term has a positive coefficient; with a vertical transverse axis, the y^2 term has a positive coefficient.

09 01 ex 46 6. line

09 01 ex 04 7. $y=\frac{1}{2}(x-3)^2+\frac{3}{2}$

09 01 ex 05 8. $y=\frac{-1}{12}(x+1)^2-1$

09 01 ex 06 9. $x=-\frac{1}{4}(y-5)^2+2$

09 01 ex 07 10. $x=y^2$

09 01 ex 08 11. $y=-\frac{1}{4}(x-1)^2+2$

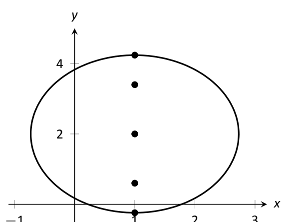
09 01 ex 09 12. $x=-\frac{1}{12}y^2$

09 01 ex 10 13. $y=4x^2$

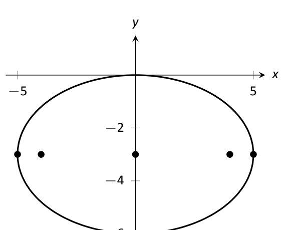
09 01 ex 11 14. $x=-\frac{1}{8}(y-3)^2+2$

09 01 ex 12 15. focus: $(0,1)$; directrix: $y=-1$. The point P is 2 units from each.

09 01 ex 13 16. focus: $(5,2)$; directrix: $x=1$. The point P is 10 units from each.



09 01 ex 18 17.



09 01 ex 19 18.

09 01 ex 20 19. $\frac{(x+1)^2}{9}+\frac{(y-2)^2}{4}=1$; foci at $(-1\pm\sqrt{5},2)$; $e=\sqrt{5}/3$

09 01 ex 21 20. $\frac{(x-1)^2}{1/4}+\frac{y^2}{9}=1$; foci at $(1,\pm\sqrt{8.75})$; $e=\sqrt{8.75}/3\approx 0.99$

09 01 ex 14 21. $\frac{x^2}{9}+\frac{y^2}{5}=1$

09 01 ex 15 22. $\frac{(x-2)^2}{25}+\frac{(y-3)^2}{16}=1$

09 01 ex 16 23. $\frac{(x-2)^2}{45}+\frac{y^2}{49}=1$

09 01 ex 17 24. $\frac{(x+1)^2}{9}+\frac{(y-1)^2}{25}=1$

09 01 ex 22 25. $\frac{(x-1)^2}{2}+(y-2)^2=1$

09 01 ex 23 26. $\frac{x^2}{3}+\frac{y^2}{5}=1$

09 01 ex 24 27. $\frac{x^2}{4}+\frac{(y-3)^2}{6}=1$

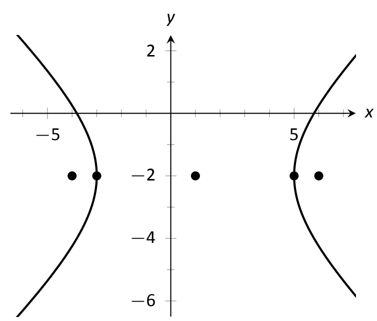
09 01 ex 25 28. $\frac{(x-2)^2}{4}+\frac{(y-2)^2}{4}=1$

09 01 ex 27 29. $x^2-\frac{y^2}{3}=1$

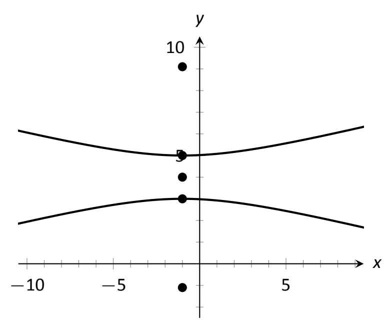
09 01 ex 28 30. $y^2-\frac{x^2}{24}=1$

09 01 ex 29 31. $\frac{(y-3)^2}{4}-\frac{(x-1)^2}{9}=1$

09 01 ex 30 32. $\frac{(x-1)^2}{9}-\frac{(y-3)^2}{4}=1$



09 01 ex 31 33.



09 01 ex 32 34.

09 01 ex 33 35. $\frac{x^2}{4}-\frac{y^2}{5}=1$

09 01 ex 34 36. $\frac{y^2}{4}-\frac{x^2}{5}=1$

09 01 ex 35 37. $\frac{(x-3)^2}{16}-\frac{(y-3)^2}{9}=1$

09 01 ex 36 38. $\frac{(y-3)^2}{9}-\frac{(x-3)^2}{16}=1$

09 01 ex 37 39. $\frac{x^2}{4}-\frac{y^2}{3}=1$

09 01 ex 38 40. $\frac{x^2}{3}-\frac{(y-1)^2}{9}=1$

09 01 ex 39 41. $(y-2)^2-\frac{x^2}{10}=1$

09 01 ex 40 42. $4y^2-\frac{x^2}{4}=1$

09 01 ex 26 43.

(a) $c = \sqrt{12 - 4} = 2\sqrt{2}$.

(b) The sum of distances for each point is $2\sqrt{12} \approx 6.9282$.

09 01 ex 42 44.

(a) Solve for c in $e = c/a$: $c = ae$. Thus $a^2e^2 = a^2 - b^2$, and $b^2 = a^2 - a^2e^2$. The result follows.

(b) Mercury: $x^2/(0.387)^2 + y^2/(0.3787)^2 = 1$
Earth: $x^2 + y^2/(0.99986)^2 = 1$
Mars: $x^2/(1.524)^2 + y^2/(1.517)^2 = 1$

(c) Mercury: $(x - 0.08)^2/(0.387)^2 + y^2/(0.3787)^2 = 1$
Earth: $(x - 0.0167)^2 + y^2/(0.99986)^2 = 1$
Mars: $(x - 0.1423)^2/(1.524)^2 + y^2/(1.517)^2 = 1$

09 01 ex 43 45. The sound originated from a point approximately 31m to the left of B and 1340m above it.

Section 1.2

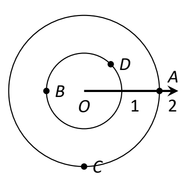
09 04 ex 01 1. Answers will vary.

09 04 ex 02 2. F

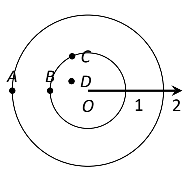
09 04 ex 03 3. T

09 04 ex 04 4. F

09 04 ex 05 5.



09 04 ex 06 6.



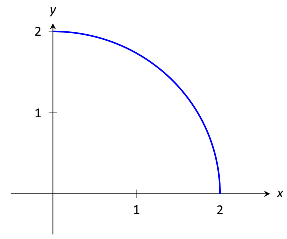
09 04 ex 07 7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
$B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
$C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
$D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

09 04 ex 08 8. $A = P(2, \pi/6)$ and $P(-2, -5\pi/6)$;
$B = P(1, -\pi/3)$ and $P(-1, 2\pi/3)$;
$C = P(2, 3\pi/4)$ and $P(-2, -\pi/4)$;
$D = P(2.5, \pi)$ and $P(2.5, -\pi)$;

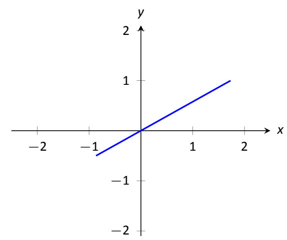
09 04 ex 09 9. $A = (\sqrt{2}, \sqrt{2})$
$B = (\sqrt{2}, -\sqrt{2})$
$C = P(\sqrt{5}, -0.46)$
$D = P(\sqrt{5}, 2.68)$

09 04 ex 10 10. $A = (-3, 0)$
$B = (-1/2, \sqrt{3}/2)$
$C = P(4, \pi/2)$
$D = P(2, -\pi/3)$

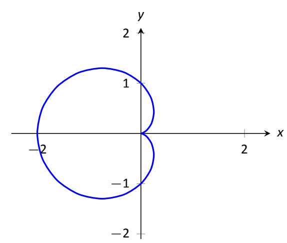
09 04 ex 11 11.



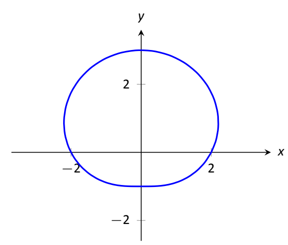
09 04 ex 12 12.



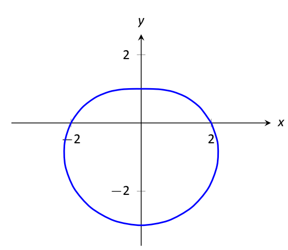
09 04 ex 13 13.



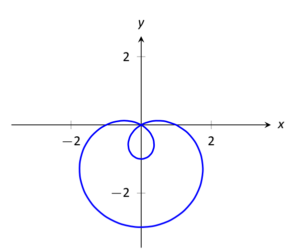
09 04 ex 14 14.



09 04 ex 15 15.



09 04 ex 16 16.



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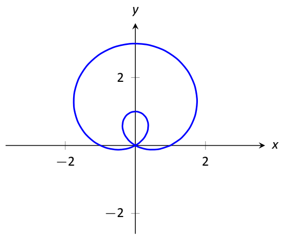
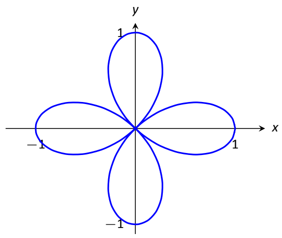
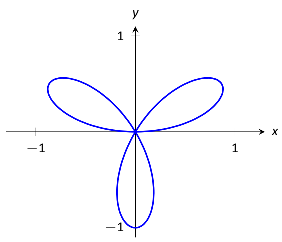


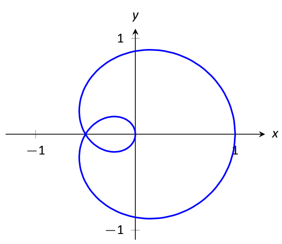
09 04 ex 18 18.



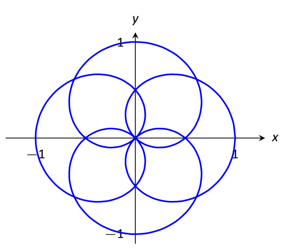
09 04 ex 19 19.



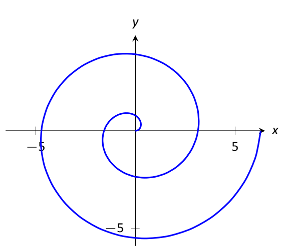
09 04 ex 20 20.



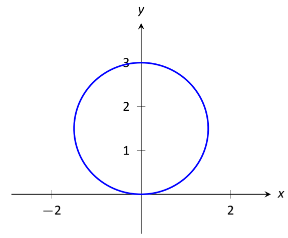
09 04 ex 21 21.



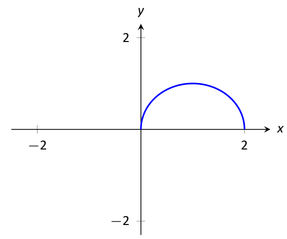
09 04 ex 22 22.



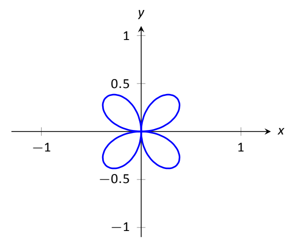
09 04 ex 23 23.



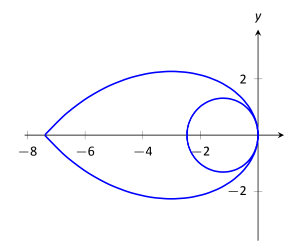
09 04 ex 57 24.



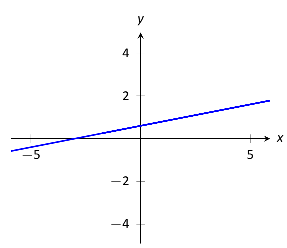
09 04 ex 24 25.



09 04 ex 25 26.



09 04 ex 26 27.



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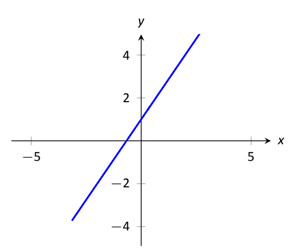

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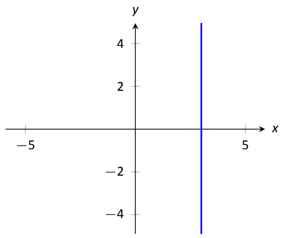
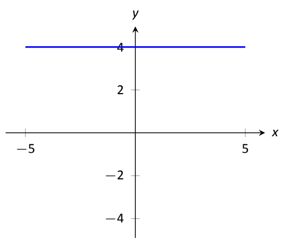


09 04 ex 29 30.



09 04 ex 30 31. $(x-3)^2+y^2=3$

09 04 ex 31 32. $x^2+(y+2)^2=4$

09 04 ex 32 33. $(x-1/2)^2+(y-1/2)^2=1/2$

09 04 ex 33 34. $y=2/5x+7/5$

09 04 ex 34 35. $x=3$

09 04 ex 35 36. $y=4$

09 04 ex 36 37. $x^4+x^2y^2-y^2=0$

09 04 ex 58 38. $y^4+x^2y^2-x^2=0$

09 04 ex 37 39. $x^2+y^2=4$

09 04 ex 38 40. $y=x/\sqrt{3}$

09 04 ex 39 41. $\theta=\pi/4$

09 04 ex 40 42. $r=7/(\sin\theta-4\cos\theta)$

09 04 ex 41 43. $r=5\sec\theta$

09 04 ex 42 44. $r=5\csc\theta$

09 04 ex 43 45. $r=\cos\theta/\sin^2\theta$

09 04 ex 44 46. $r=1/\sqrt[3]{\cos^2\theta\sin\theta}$

09 04 ex 45 47. $r=\sqrt{7}$

09 04 ex 46 48. $r=-2\cos\theta$

09 04 ex 47 49. $P(\sqrt{3}/2,\pi/6)$, $P(0,\pi/2)$, $P(-\sqrt{3}/2,5\pi/6)$

09 04 ex 48 50. $P(1,0)$, $P(0,\pi/2)=P(0,\pi/4)$, $P(-1/2,\pi/3)$

09 04 ex 49 51. $P(0,0)=P(0,\pi/2)$, $P(\sqrt{2},\pi/4)$

09 04 ex 50 52. $P(\sqrt{3}/2,\pi/3)=P(-\sqrt{3}/2,4\pi/3)$,
$P(\sqrt{3}/2,2\pi/3)=P(-\sqrt{3}/2,5\pi/3)$, $P(0,\pi/2)$

09 04 ex 51 53. $P(\sqrt{2}/2,\pi/12)$, $P(-\sqrt{2}/2,5\pi/12)$, $P(\sqrt{2}/2,3\pi/4)$

09 04 ex 52 54. $P(3/2,\pi/3)$, $P(3/2,-\pi/3)$

09 04 ex 53 55. For all points, $r=1$; $\theta=$
$\pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12,\ 13\pi/12,\ 17\pi/12,\ 19\pi/12,\ 23\pi/12.$

09 04 ex 54 56. $P(0,0)=P(0,3\pi/2)$, $P(1+\sqrt{2}/2,3\pi/4)$,
$P(1-\sqrt{2}/2,7\pi/4)$

09 04 ex 55 57. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

09 04 ex 56 58. Answers will vary.