1: Sequences and Series

This chapter introduces **sequences** and **series**, important mathematical constructions that are useful when solving a large variety of mathematical problems. The content of this chapter is considerably different from the content of the chapters before it. While the material we learn here definitely falls under the scope of "calculus," we will make very little use of derivatives or integrals. Limits are extremely important, though, especially limits that involve infinity.

One of the problems addressed by this chapter is this: suppose we know information about a function and its derivatives at a point, such as $f(1) = 3$, $f'(1) = 1$, $f''(1) = -2$, $f'''(1) = 7$, and so on. What can I say about $f(x)$ itself? Is there any reasonable approximation of the value of $f(2)$? The topic of Taylor Series addresses this problem, and allows us to make excellent approximations of functions when limited knowledge of the function is available.

Notation: We use $\mathbb{N}$ to describe the set of natural numbers, that is, the integers 1, 2, 3, ...

Factorial: The expression $3!$ refers to the number $3 \cdot 2 \cdot 1 = 6$.

In general, $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$, where n is a natural number.

We define $0! = 1$. While this does not immediately make sense, it makes many mathematical formulas work properly.

1.1 Sequences

We commonly refer to a set of events that occur one after the other as a *sequence* of events. In mathematics, we use the word *sequence* to refer to an ordered set of numbers, i.e., a set of numbers that "occur one after the other."

For instance, the numbers 2, 4, 6, 8, ..., form a sequence. The order is important; the first number is 2, the second is 4, etc. It seems natural to seek a formula that describes a given sequence, and often this can be done. For instance, the sequence above could be described by the function $a(n) = 2n$, for the values of $n = 1, 2, \ldots$ To find the 10^{th} term in the sequence, we would compute $a(10)$. This leads us to the following, formal definition of a sequence.

Definition 1 Sequence

A **sequence** is a function $a(n)$ whose domain is $\mathbb{N}$. The **range** of a sequence is the set of all distinct values of $a(n)$.

The **terms** of a sequence are the values $a(1)$, $a(2)$, ..., which are usually denoted with subscripts as a_1, a_2, ...

A sequence $a(n)$ is often denoted as $\{a_n\}$.

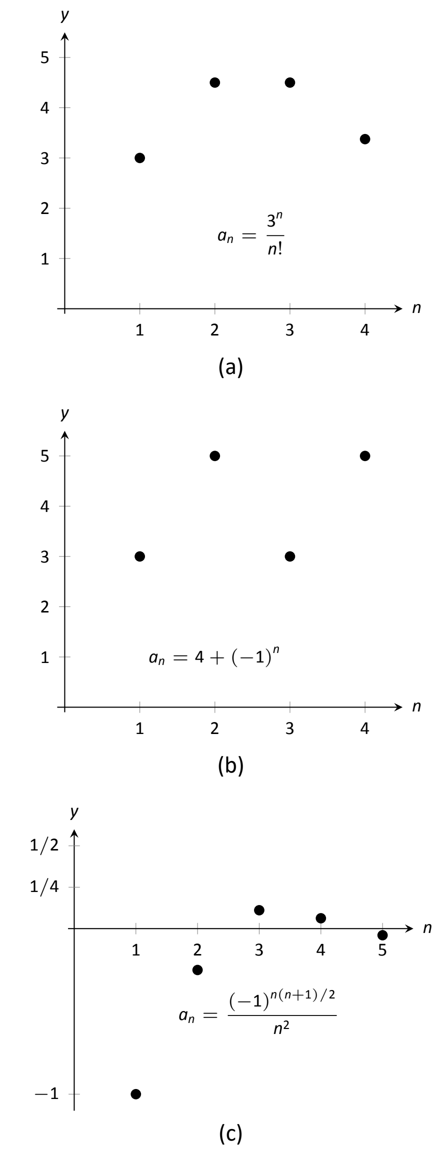


Figure 1.1: Plotting sequences in Example 1.

Example 1 Listing terms of a sequence

List the first four terms of the following sequences.

1. $\{a_n\} = \left\{ \dfrac{3^n}{n!} \right\}$ 2. $\{a_n\} = \{4 + (-1)^n\}$ 3. $\{a_n\} = \left\{ \dfrac{(-1)^{n(n+1)/2}}{n^2} \right\}$

Solution

1. $a_1 = \dfrac{3^1}{1!} = 3;\quad a_2 = \dfrac{3^2}{2!} = \dfrac{9}{2};\quad a_3 = \dfrac{3^3}{3!} = \dfrac{9}{2};\quad a_4 = \dfrac{3^4}{4!} = \dfrac{27}{8}$

   We can plot the terms of a sequence with a scatter plot. The "x"-axis is used for the values of n, and the values of the terms are plotted on the y-axis. To visualize this sequence, see Figure 1.1(a).

2. $a_1 = 4 + (-1)^1 = 3;\quad a_2 = 4 + (-1)^2 = 5;$

   $a_3 = 4 + (-1)^3 = 3;\quad a_4 = 4 + (-1)^4 = 5$. Note that the range of this sequence is finite, consisting of only the values 3 and 5. This sequence is plotted in Figure 1.1(b).

3. $a_1 = \dfrac{(-1)^{1(2)/2}}{1^2} = -1;\quad a_2 = \dfrac{(-1)^{2(3)/2}}{2^2} = -\dfrac{1}{4}$

   $a_3 = \dfrac{(-1)^{3(4)/2}}{3^2} = \dfrac{1}{9}\quad a_4 = \dfrac{(-1)^{4(5)/2}}{4^2} = \dfrac{1}{16};$

   $a_5 = \dfrac{(-1)^{5(6)/2}}{5^2} = -\dfrac{1}{25}.$

   We gave one extra term to begin to show the pattern of signs is "$-, -, +, +, -, -, \ldots$, due to the fact that the exponent of -1 is a special quadratic. This sequence is plotted in Figure 1.1(c).

Example 2 Determining a formula for a sequence

Find the n^{th} term of the following sequences, i.e., find a function that describes each of the given sequences.

1. $2, 5, 8, 11, 14, \ldots$
2. $2, -5, 10, -17, 26, -37, \ldots$
3. $1, 1, 2, 6, 24, 120, 720, \ldots$
4. $\dfrac{5}{2}, \dfrac{5}{2}, \dfrac{15}{8}, \dfrac{5}{4}, \dfrac{25}{32}, \ldots$

Notes:

Solution We should first note that there is never exactly one function that describes a finite set of numbers as a sequence. There are many sequences that start with 2, then 5, as our first example does. We are looking for a simple formula that describes the terms given, knowing there is possibly more than one answer.

1. Note how each term is 3 more than the previous one. This implies a linear function would be appropriate: $a(n) = a_n = 3n + b$ for some appropriate value of b. As we want $a_1 = 2$, we set $b = -1$. Thus $a_n = 3n - 1$.

2. First notice how the sign changes from term to term. This is most commonly accomplished by multiplying the terms by either $(-1)^n$ or $(-1)^{n+1}$. Using $(-1)^n$ multiplies the odd terms by (-1); using $(-1)^{n+1}$ multiplies the even terms by (-1). As this sequence has negative even terms, we will multiply by $(-1)^{n+1}$.

   After this, we might feel a bit stuck as to how to proceed. At this point, we are just looking for a pattern of some sort: what do the numbers 2, 5, 10, 17, etc., have in common? There are many correct answers, but the one that we'll use here is that each is one more than a perfect square. That is, $2 = 1^2 + 1$, $5 = 2^2 + 1$, $10 = 3^2 + 1$, etc. Thus our formula is $a_n = (-1)^{n+1}(n^2 + 1)$.

3. One who is familiar with the factorial function will readily recognize these numbers. They are $0!, 1!, 2!, 3!$, etc. Since our sequences start with $n = 1$, we cannot write $a_n = n!$, for this misses the $0!$ term. Instead, we shift by 1, and write $a_n = (n-1)!$.

4. This one may appear difficult, especially as the first two terms are the same, but a little "sleuthing" will help. Notice how the terms in the numerator are always multiples of 5, and the terms in the denominator are always powers of 2. Does something as simple as $a_n = \frac{5n}{2^n}$ work?

   When $n = 1$, we see that we indeed get $5/2$ as desired. When $n = 2$, we get $10/4 = 5/2$. Further checking shows that this formula indeed matches the other terms of the sequence.

A common mathematical endeavor is to create a new mathematical object (for instance, a sequence) and then apply previously known mathematics to the new object. We do so here. The fundamental concept of calculus is the limit, so we will investigate what it means to find the limit of a sequence.

Notes:

Definition 2 Limit of a Sequence, Convergent, Divergent

Let $\{a_n\}$ be a sequence and let L be a real number. Given any $\varepsilon > 0$, if an m can be found such that $|a_n - L| < \varepsilon$ for all $n > m$, then we say the **limit of $\{a_n\}$, as n approaches infinity, is L**, denoted

$$\lim_{n\to\infty} a_n = L.$$

If $\lim_{n\to\infty} a_n$ exists, we say the sequence **converges**; otherwise, the sequence **diverges**.

This definition states, informally, that if the limit of a sequence is L, then if you go far enough out along the sequence, all subsequent terms will be *really close* to L. Of course, the terms "far enough" and "really close" are subjective terms, but hopefully the intent is clear.

This definition is reminiscent of the ε–δ proofs of Chapter 1. In that chapter we developed other tools to evaluate limits apart from the formal definition; we do so here as well.

Theorem 1 Limit of a Sequence

Let $\{a_n\}$ be a sequence and let $f(x)$ be a function whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$.

If $\lim_{x\to\infty} f(x) = L$, then $\lim_{n\to\infty} a_n = L$.

Theorem 1 allows us, in certain cases, to apply the tools developed in Chapter 1 to limits of sequences. Note two things *not* stated by the theorem:

1. If $\lim_{x\to\infty} f(x)$ does not exist, we cannot conclude that $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist. For instance, we can define a sequence $\{a_n\} = \{\cos(2\pi n)\}$. Let $f(x) = \cos(2\pi x)$. Since the cosine function oscillates over the real numbers, the limit $\lim_{x\to\infty} f(x)$ does not exist.

   However, for every positive integer n, $\cos(2\pi n) = 1$, so $\lim_{n\to\infty} a_n = 1$.

2. If we cannot find a function $f(x)$ whose domain contains the positive real numbers where $f(n) = a_n$ for all n in $\mathbb{N}$, we cannot conclude $\lim_{n\to\infty} a_n$ does not exist. It may, or may not, exist.

Notes:

Example 3 **Determining convergence/divergence of a sequence**

Determine the convergence or divergence of the following sequences.

1. $\{a_n\} = \left\{\dfrac{3n^2 - 2n + 1}{n^2 - 1000}\right\}$ 2. $\{a_n\} = \{\cos n\}$ 3. $\{a_n\} = \left\{\dfrac{(-1)^n}{n}\right\}$

SOLUTION

1. Using Theorem 11, we can state that $\lim\limits_{x\to\infty} \dfrac{3x^2 - 2x + 1}{x^2 - 1000} = 3$. (We could have also directly applied l'Hôpital's Rule.) Thus the sequence $\{a_n\}$ converges, and its limit is 3. A scatter plot of every 5 values of a_n is given in Figure 1.2 (a). The values of a_n vary widely near $n = 30$, ranging from about -73 to 125, but as n grows, the values approach 3.

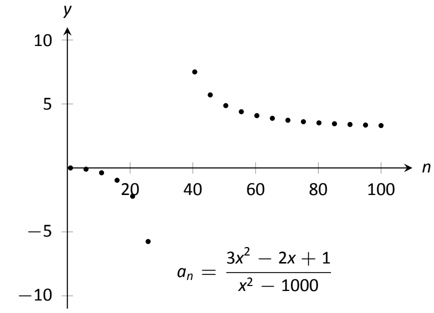
2. The limit $\lim\limits_{x\to\infty} \cos x$ does not exist as $\cos x$ oscillates (and takes on every value in $[-1, 1]$ infinitely many times). Thus we cannot apply Theorem 1.

   The fact that the cosine function oscillates strongly hints that $\cos n$, when n is restricted to $\mathbb{N}$, will also oscillate. Figure 1.2 (b), where the sequence is plotted, implies that this is true. Because only discrete values of cosine are plotted, it does not bear strong resemblance to the familiar cosine wave. The proof of the following statement is beyond the scope of this text, but it is true: there are infinitely many integers n that are arbitrarily (i.e., *very*) close to an even multiple of π, so that $\cos n \approx 1$. Similarly, there are infinitely many integers m that are arbitrarily close to an odd multiple of π, so that $\cos m \approx -1$. As the sequence takes on values near 1 and -1 infinitely many times, we conclude that $\lim\limits_{n\to\infty} a_n$ does not exist.

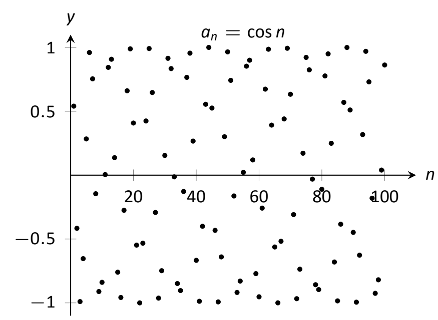
3. We cannot actually apply Theorem 1 here, as the function $f(x) = (-1)^x/x$ is not well defined. (What does $(-1)^{\sqrt{2}}$ mean? In actuality, there is an answer, but it involves *complex analysis*, beyond the scope of this text.)

   Instead, we invoke the definition of the limit of a sequence. By looking at the plot in Figure 1.2 (c), we would like to conclude that the sequence converges to $L = 0$. Let $\varepsilon > 0$ be given. We can find a natural number m such that $1/m < \varepsilon$. Let $n > m$, and consider $|a_n - L|$:

$$\begin{aligned}|a_n - L| &= \left|\frac{(-1)^n}{n} - 0\right| \\ &= \frac{1}{n} \\ &< \frac{1}{m} \quad \text{(since } n > m\text{)} \\ &< \varepsilon.\end{aligned}$$



(a)



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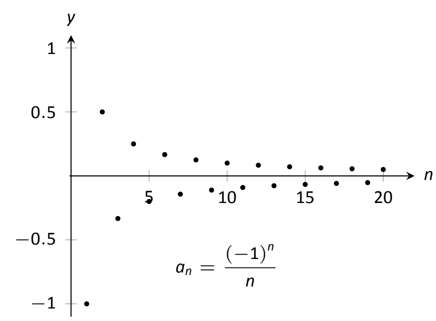

(c)

Figure 1.2: Scatter plots of the sequences in Example 3.

Notes:

We have shown that by picking m large enough, we can ensure that a_n is arbitrarily close to our limit, $L = 0$, hence by the definition of the limit of a sequence, we can say $\lim_{n\to\infty} a_n = 0$.

In the previous example we used the definition of the limit of a sequence to determine the convergence of a sequence as we could not apply Theorem 1. In general, we like to avoid invoking the definition of a limit, and the following theorem gives us tool that we could use in that example instead.

Theorem 2 Absolute Value Theorem

Let $\{a_n\}$ be a sequence. If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$

Example 4 Determining the convergence/divergence of a sequence

Determine the convergence or divergence of the following sequences.

$$1.\ \{a_n\} = \left\{\frac{(-1)^n}{n}\right\} \qquad 2.\ \{a_n\} = \left\{\frac{(-1)^n(n+1)}{n}\right\}$$

Solution

1. This appeared in Example 3. We want to apply Theorem 2, so consider the limit of $\{|a_n|\}$:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty}\left|\frac{(-1)^n}{n}\right| \\ &= \lim_{n\to\infty}\frac{1}{n} \\ &= 0.\end{aligned}$$

Since this limit is 0, we can apply Theorem 2 and state that $\lim_{n\to\infty} a_n = 0$.

2. Because of the alternating nature of this sequence (i.e., every other term is multiplied by -1), we cannot simply look at the limit $\lim_{x\to\infty}\frac{(-1)^x(x+1)}{x}$. We can try to apply the techniques of Theorem 2:

$$\begin{aligned}\lim_{n\to\infty} |a_n| &= \lim_{n\to\infty}\left|\frac{(-1)^n(n+1)}{n}\right| \\ &= \lim_{n\to\infty}\frac{n+1}{n} \\ &= 1.\end{aligned}$$

We have concluded that when we ignore the alternating sign, the sequence approaches 1. This means we cannot apply Theorem 2; it states the the limit must be 0 in order to conclude anything.

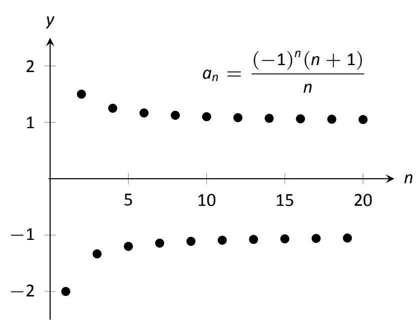

Figure 1.3: A plot of a sequence in Example 4, part 2.

Notes:

Since we know that the signs of the terms alternate *and* we know that the limit of $|a_n|$ is 1, we know that as n approaches infinity, the terms will alternate between values close to 1 and -1, meaning the sequence diverges. A plot of this sequence is given in Figure 1.3.

We continue our study of the limits of sequences by considering some of the properties of these limits.

Theorem 3 Properties of the Limits of Sequences

Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$, $\lim\limits_{n\to\infty} b_n = K$, and let c be a real number.

1. $\lim\limits_{n\to\infty} (a_n \pm b_n) = L \pm K$
2. $\lim\limits_{n\to\infty} (a_n \cdot b_n) = L \cdot K$
3. $\lim\limits_{n\to\infty} (a_n / b_n) = L/K$, $K \neq 0$
4. $\lim\limits_{n\to\infty} c \cdot a_n = c \cdot L$

Example 5 Applying properties of limits of sequences

Let the following sequences, and their limits, be given:

- $\{a_n\} = \left\{\dfrac{n+1}{n^2}\right\}$, and $\lim\limits_{n\to\infty} a_n = 0$;
- $\{b_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$, and $\lim\limits_{n\to\infty} b_n = e$; and
- $\{c_n\} = \{n \cdot \sin(5/n)\}$, and $\lim\limits_{n\to\infty} c_n = 5$.

Evaluate the following limits.

1. $\lim\limits_{n\to\infty} (a_n + b_n)$
2. $\lim\limits_{n\to\infty} (b_n \cdot c_n)$
3. $\lim\limits_{n\to\infty} (1000 \cdot a_n)$

Solution We will use Theorem 3 to answer each of these.

1. Since $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} b_n = e$, we conclude that $\lim\limits_{n\to\infty} (a_n + b_n) = 0 + e = e$. So even though we are adding something to each term of the sequence b_n, we are adding something so small that the final limit is the same as before.
2. Since $\lim\limits_{n\to\infty} b_n = e$ and $\lim\limits_{n\to\infty} c_n = 5$, we conclude that $\lim\limits_{n\to\infty} (b_n \cdot c_n) = e \cdot 5 = 5e$.

Notes:

3. Since $\lim_{n\to\infty} a_n = 0$, we have $\lim_{n\to\infty} 1000a_n = 1000 \cdot 0 = 0$. It does not matter that we multiply each term by 1000; the sequence still approaches 0. (It just takes longer to get close to 0.)

There is more to learn about sequences than just their limits. We will also study their range and the relationships terms have with the terms that follow. We start with some definitions describing properties of the range.

Definition 3 Bounded and Unbounded Sequences

A sequence $\{a_n\}$ is said to be **bounded** if there exists real numbers m and M such that $m < a_n < M$ for all n in $\mathbb{N}$.

A sequence $\{a_n\}$ is said to be **unbounded** if it is not bounded.

A sequence $\{a_n\}$ is said to be **bounded above** if there exists an M such that $a_n < M$ for all n in $\mathbb{N}$; it is **bounded below** if there exists an m such that $m < a_n$ for all n in $\mathbb{N}$.

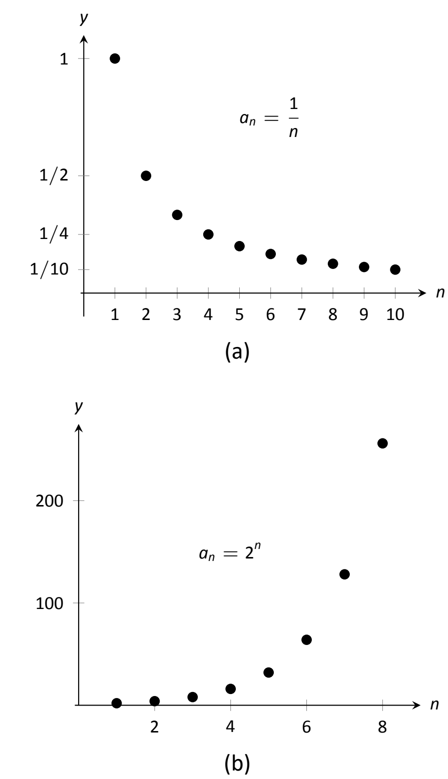

Figure 1.4: A plot of $\{a_n\} = \{1/n\}$ and $\{a_n\} = \{2^n\}$ from Example 6.

It follows from this definition that an unbounded sequence may be bounded above or bounded below; a sequence that is both bounded above and below is simply a bounded sequence.

Example 6 Determining boundedness of sequences

Determine the boundedness of the following sequences.

1. $\{a_n\} = \left\{\frac{1}{n}\right\}$ 2. $\{a_n\} = \{2^n\}$

Solution

1. The terms of this sequence are always positive but are decreasing, so we have $0 < a_n < 2$ for all n. Thus this sequence is bounded. Figure 1.4(a) illustrates this.
2. The terms of this sequence obviously grow without bound. However, it is also true that these terms are all positive, meaning $0 < a_n$. Thus we can say the sequence is unbounded, but also bounded below. Figure 1.4(b) illustrates this.

The previous example produces some interesting concepts. First, we can recognize that the sequence $\{1/n\}$ converges to 0. This says, informally, that "most" of the terms of the sequence are "really close" to 0. This implies that the sequence is bounded, using the following logic. First, "most" terms are near

Notes:

0, so we could find some sort of bound on these terms (using Definition 2, the bound is ε). That leaves a "few" terms that are not near 0 (i.e., a *finite* number of terms). A finite list of numbers is always bounded.

This logic implies that if a sequence converges, it must be bounded. This is indeed true, as stated by the following theorem.

Theorem 4 Convergent Sequences are Bounded

Let $\{a_n\}$ be a convergent sequence. Then $\{a_n\}$ is bounded.

In Example 5 we saw the sequence $\{b_n\} = \{(1+1/n)^n\}$, where it was stated that $\lim\limits_{n\to\infty} b_n = e$. (Note that this is simply restating part of Theorem 5.) Even though it may be difficult to intuitively grasp the behavior of this sequence, we know immediately that it is bounded.

Another interesting concept to come out of Example 6 again involves the sequence $\{1/n\}$. We stated, without proof, that the terms of the sequence were decreasing. That is, that $a_{n+1} < a_n$ for all n. (This is easy to show. Clearly $n < n+1$. Taking reciprocals flips the inequality: $1/n > 1/(n+1)$. This is the same as $a_n > a_{n+1}$.) Sequences that either steadily increase or decrease are important, so we give this property a name.

Definition 4 Monotonic Sequences

1. A sequence $\{a_n\}$ is **monotonically increasing** if $a_n \leq a_{n+1}$ for all n, i.e.,
$$a_1 \leq a_2 \leq a_3 \leq \cdots a_n \leq a_{n+1} \cdots$$
2. A sequence $\{a_n\}$ is **monotonically decreasing** if $a_n \geq a_{n+1}$ for all n, i.e.,
$$a_1 \geq a_2 \geq a_3 \geq \cdots a_n \geq a_{n+1} \cdots$$
3. A sequence is **monotonic** if it is monotonically increasing or monotonically decreasing.

Example 7 Determining monotonicity

Determine the monotonicity of the following sequences.

Notes:

Note: Keep in mind what Theorem 4 does *not* say. It does not say that bounded sequences must converge, nor does it say that if a sequence does not converge, it is not bounded.

Note: It is sometimes useful to call a monotonically increasing sequence *strictly increasing* if $a_n < a_{n+1}$ for all n; i.e, we remove the possibility that subsequent terms are equal.
A similar statement holds for *strictly decreasing.*

1. $\{a_n\} = \left\{\dfrac{n+1}{n}\right\}$
2. $\{a_n\} = \left\{\dfrac{n^2+1}{n+1}\right\}$
3. $\{a_n\} = \left\{\dfrac{n^2-9}{n^2-10n+26}\right\}$
4. $\{a_n\} = \left\{\dfrac{n^2}{n!}\right\}$

Solution In each of the following, we will examine $a_{n+1}-a_n$. If $a_{n+1}-a_n > 0$, we conclude that $a_n < a_{n+1}$ and hence the sequence is increasing. If $a_{n+1}-a_n < 0$, we conclude that $a_n > a_{n+1}$ and the sequence is decreasing. Of course, a sequence need not be monotonic and perhaps neither of the above will apply.

We also give a scatter plot of each sequence. These are useful as they suggest a pattern of monotonicity, but analytic work should be done to confirm a graphical trend.

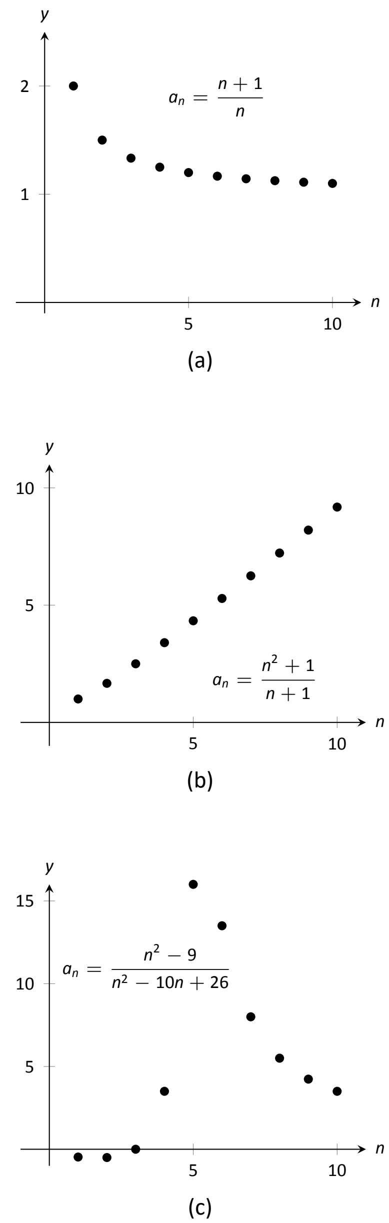

Figure 1.5: Plots of sequences in Example 7.

1. $$\begin{aligned} a_{n+1}-a_n &= \frac{n+2}{n+1} - \frac{n+1}{n} \\ &= \frac{(n+2)(n)-(n+1)^2}{(n+1)n} \\ &= \frac{-1}{n(n+1)} \\ &< 0 \quad \text{for all } n. \end{aligned}$$

   Since $a_{n+1}-a_n < 0$ for all n, we conclude that the sequence is decreasing.

2. $$\begin{aligned} a_{n+1}-a_n &= \frac{(n+1)^2+1}{n+2} - \frac{n^2+1}{n+1} \\ &= \frac{\big((n+1)^2+1\big)(n+1)-(n^2+1)(n+2)}{(n+1)(n+2)} \\ &= \frac{n^2+4n+1}{(n+1)(n+2)} \\ &> 0 \quad \text{for all } n. \end{aligned}$$

   Since $a_{n+1}-a_n > 0$ for all n, we conclude the sequence is increasing.

3. We can clearly see in Figure 1.5 (c), where the sequence is plotted, that it is not monotonic. However, it does seem that after the first 4 terms it is decreasing. To understand why, perform the same analysis as done before:

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$$
\begin{aligned}
a_{n+1} - a_n &= \frac{(n+1)^2 - 9}{(n+1)^2 - 10(n+1) + 26} - \frac{n^2 - 9}{n^2 - 10n + 26} \\
&= \frac{n^2 + 2n - 8}{n^2 - 8n + 17} - \frac{n^2 - 9}{n^2 - 10n + 26} \\
&= \frac{(n^2 + 2n - 8)(n^2 - 10n + 26) - (n^2 - 9)(n^2 - 8n + 17)}{(n^2 - 8n + 17)(n^2 - 10n + 26)} \\
&= \frac{-10n^2 + 60n - 55}{(n^2 - 8n + 17)(n^2 - 10n + 26)}.
\end{aligned}
$$

We want to know when this is greater than, or less than, 0. The denominator is always positive, therefore we are only concerned with the numerator. For small values of n, the numerator is positive. As n grows large, the numerator is dominated by $-10n^2$, meaning the entire fraction will be negative; i.e., for large enough n, $a_{n+1} - a_n < 0$. Using the quadratic formula we can determine that the numerator is negative for $n \geq 5$.

In short, the sequence is simply not monotonic, though it is useful to note that for $n \geq 5$, the sequence is monotonically decreasing.

4. Again, the plot in Figure 1.6 shows that the sequence is not monotonic, but it suggests that it is monotonically decreasing after the first term. We perform the usual analysis to confirm this.

$$
\begin{aligned}
a_{n+1} - a_n &= \frac{(n+1)^2}{(n+1)!} - \frac{n^2}{n!} \\
&= \frac{(n+1)^2 - n^2(n+1)}{(n+1)!} \\
&= \frac{-n^3 + 2n + 1}{(n+1)!}
\end{aligned}
$$

When $n = 1$, the above expression is > 0; for $n \geq 2$, the above expression is < 0. Thus this sequence is not monotonic, but it is monotonically decreasing after the first term.

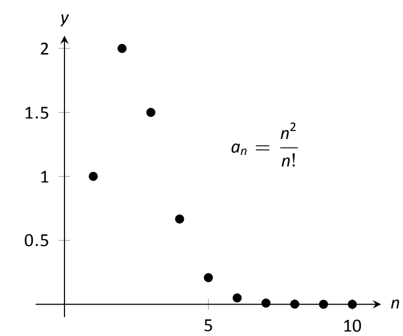


Figure 1.6: A plot of $\{a_n\} = \{n^2/n!\}$ in Example 7.

Knowing that a sequence is monotonic can be useful. Consider, for example, a sequence that is monotonically decreasing and is bounded below. We know the sequence is always getting smaller, but that there is a bound to how small it can become. This is enough to prove that the sequence will converge, as stated in the following theorem.

Notes:

Theorem 5 Bounded Monotonic Sequences are Convergent

1. Let $\{a_n\}$ be a monotonically increasing sequence that is bounded above. Then $\{a_n\}$ converges.
2. Let $\{a_n\}$ be a monotonically decreasing sequence that is bounded below. Then $\{a_n\}$ converges.

Consider once again the sequence $\{a_n\} = \{1/n\}$. It is easy to show it is monotonically decreasing and that it is always positive (i.e., bounded below by 0). Therefore we can conclude by Theorem 5 that the sequence converges. We already knew this by other means, but in the following section this theorem will become very useful.

We can replace Theorem 5 with the statement "Let $\{a_n\}$ be a bounded, monotonic sequence. Then $\{a_n\}$ converges; i.e., $\lim\limits_{n\to\infty} a_n$ exists." We leave it to the reader in the exercises to show the theorem and the above statement are equivalent.

Sequences are a great source of mathematical inquiry. The On-Line Encyclopedia of Integer Sequences (`http://oeis.org`) contains thousands of sequences and their formulae. (As of this writing, there are 257,537 sequences in the database.) Perusing this database quickly demonstrates that a single sequence can represent several different "real life" phenomena.

Interesting as this is, our interest actually lies elsewhere. We are more interested in the *sum* of a sequence. That is, given a sequence $\{a_n\}$, we are very interested in $a_1 + a_2 + a_3 + \cdots$. Of course, one might immediately counter with "Doesn't this just add up to 'infinity'?" Many times, yes, but there are many important cases where the answer is no. This is the topic of *series*, which we begin to investigate in the next section.

Notes:

Exercises 1.1

Terms and Concepts

1. Use your own words to define a *sequence.*

2. The domain of a sequence is the ________ numbers.

3. Use your own words to describe the *range* of a sequence.

4. Describe what it means for a sequence to be *bounded*.

Problems

In Exercises 5 – 8, give the first five terms of the given sequence.

5. $\{a_n\} = \left\{\dfrac{4^n}{(n+1)!}\right\}$

6. $\{b_n\} = \left\{\left(-\dfrac{3}{2}\right)^n\right\}$

7. $\{c_n\} = \left\{-\dfrac{n^{n+1}}{n+2}\right\}$

8. $\{d_n\} = \left\{\dfrac{1}{\sqrt{5}}\left(\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right)\right\}$

In Exercises 9 – 12, determine the n^{th} term of the given sequence.

9. 4, 7, 10, 13, 16, ...

10. $3,\ -\dfrac{3}{2},\ \dfrac{3}{4},\ -\dfrac{3}{8},\ \ldots$

11. $10,\ 20,\ 40,\ 80,\ 160,\ \ldots$

12. $1, 1,\ \dfrac{1}{2},\ \dfrac{1}{6},\ \dfrac{1}{24},\ \dfrac{1}{120},\ \ldots$

In Exercises 13 – 16, use the following information to determine the limit of the given sequences.

- $\{a_n\} = \left\{\dfrac{2^n - 20}{2^n}\right\}$; $\quad \lim\limits_{n\to\infty} a_n = 1$
- $\{b_n\} = \left\{\left(1+\dfrac{2}{n}\right)^n\right\}$; $\quad \lim\limits_{n\to\infty} b_n = e^2$
- $\{c_n\} = \{\sin(3/n)\}$; $\quad \lim\limits_{n\to\infty} c_n = 0$

13. $\{a_n\} = \left\{\dfrac{2^n - 20}{7\cdot 2^n}\right\}$

14. $\{a_n\} = \{3b_n - a_n\}$

15. $\{a_n\} = \left\{\sin(3/n)\left(1+\dfrac{2}{n}\right)^n\right\}$

16. $\{a_n\} = \left\{\left(1+\dfrac{2}{n}\right)^{2n}\right\}$

In Exercises 17 – 28, determine whether the sequence converges or diverges. If convergent, give the limit of the sequence.

17. $\{a_n\} = \left\{(-1)^n\dfrac{n}{n+1}\right\}$

18. $\{a_n\} = \left\{\dfrac{4n^2 - n + 5}{3n^2+1}\right\}$

19. $\{a_n\} = \left\{\dfrac{4^n}{5^n}\right\}$

20. $\{a_n\} = \left\{\dfrac{n-1}{n} - \dfrac{n}{n-1}\right\}, n \geq 2$

21. $\{a_n\} = \{\ln(n)\}$

22. $\{a_n\} = \left\{\dfrac{3n}{\sqrt{n^2+1}}\right\}$

23. $\{a_n\} = \left\{\left(1+\dfrac{1}{n}\right)^n\right\}$

24. $\{a_n\} = \left\{5 - \dfrac{1}{n}\right\}$

25. $\{a_n\} = \left\{\dfrac{(-1)^{n+1}}{n}\right\}$

26. $\{a_n\} = \left\{\dfrac{1.1^n}{n}\right\}$

27. $\{a_n\} = \left\{\dfrac{2n}{n+1}\right\}$

28. $\{a_n\} = \left\{(-1)^n\dfrac{n^2}{2^n - 1}\right\}$

In Exercises 29 – 34, determine whether the sequence is bounded, bounded above, bounded below, or none of the above.

29. $\{a_n\} = \{\sin n\}$

30. $\{a_n\} = \{\tan n\}$

31. $\{a_n\} = \left\{(-1)^n\dfrac{3n-1}{n}\right\}$

32. $\{a_n\} = \left\{\dfrac{3n^2-1}{n}\right\}$

33. $\{a_n\} = \{n\cos n\}$

34. $\{a_n\} = \{2^n - n!\}$

In Exercises 35 – 38, determine whether the sequence is monotonically increasing or decreasing. If it is not, determine if there is an m such that it is monotonic for all $n \geq m$.

35. $\{a_n\} = \left\{\dfrac{n}{n+2}\right\}$

36. $\{a_n\} = \left\{\dfrac{n^2 - 6n + 9}{n}\right\}$

37. $\{a_n\} = \left\{(-1)^n \dfrac{1}{n^3}\right\}$

38. $\{a_n\} = \left\{\dfrac{n^2}{2^n}\right\}$

39. Prove Theorem 2; that is, use the definition of the limit of a sequence to show that if $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

40. Let $\{a_n\}$ and $\{b_n\}$ be sequences such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = K$.

    (a) Show that if $a_n < b_n$ for all n, then $L \leq K$.

    (b) Give an example where $L = K$.

41. Prove the Squeeze Theorem for sequences: Let $\{a_n\}$ and $\{b_n\}$ be such that $\lim\limits_{n\to\infty} a_n = L$ and $\lim\limits_{n\to\infty} b_n = L$, and let $\{c_n\}$ be such that $a_n \leq c_n \leq b_n$ for all n. Then $\lim\limits_{n\to\infty} c_n = L$

A: Solutions To Selected Problems

Chapter 1

Section 1.1

1. Answers will vary.
2. natural
3. Answers will vary.
4. Answers will vary.
5. $2, \frac{8}{3}, \frac{8}{3}, \frac{32}{15}, \frac{64}{45}$
6. $-\frac{3}{2}, \frac{9}{4}, -\frac{27}{8}, \frac{81}{16}, -\frac{243}{32}$
7. $\frac{1}{3}, 2, \frac{81}{5}, \frac{512}{3}, \frac{15625}{7}$
8. $1, 1, 2, 3, 5$
9. $a_n = 3n + 1$
10. $a_n = (-1)^{n+1}\frac{3}{2^{n-1}}$
11. $a_n = 10 \cdot 2^{n-1}$
12. $a_n = 1/(n-1)!$
13. $1/7$
14. $3e^2 - 1$
15. 0
16. e^4
17. diverges
18. converges to $4/3$
19. converges to 0
20. converges to 0
21. diverges
22. converges to 3
23. converges to e
24. converges to 5
25. converges to 0
26. diverges
27. converges to 2
28. converges to 0
29. bounded
30. neither bounded above or below
31. bounded
32. bounded below
33. neither bounded above or below
34. bounded above
35. monotonically increasing
36. monotonically increasing for $n \geq 3$
37. never monotonic
38. monotonically decreasing for $n \geq 3$
39. Let $\{a_n\}$ be given such that $\lim\limits_{n\to\infty} |a_n| = 0$. By the definition of the limit of a sequence, given any $\varepsilon > 0$, there is a m such that for all $n > m$, $|\,|a_n| - 0| < \varepsilon$. Since $|\,|a_n| - 0| = |a_n - 0|$, this directly implies that for all $n > m$, $|a_n - 0| < \varepsilon$, meaning that $\lim\limits_{n\to\infty} a_n = 0$.
40. (a) Left to reader

    (b) $a_n = 1/3^n$ and $b_n = 1/2^n$
41. Left to reader

Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$

   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln |x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

22. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

23. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

24. $\displaystyle\int \cosh x\,dx = \sinh x + C$

25. $\displaystyle\int \sinh x\,dx = \cosh x + C$

26. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$

27. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

28. $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$

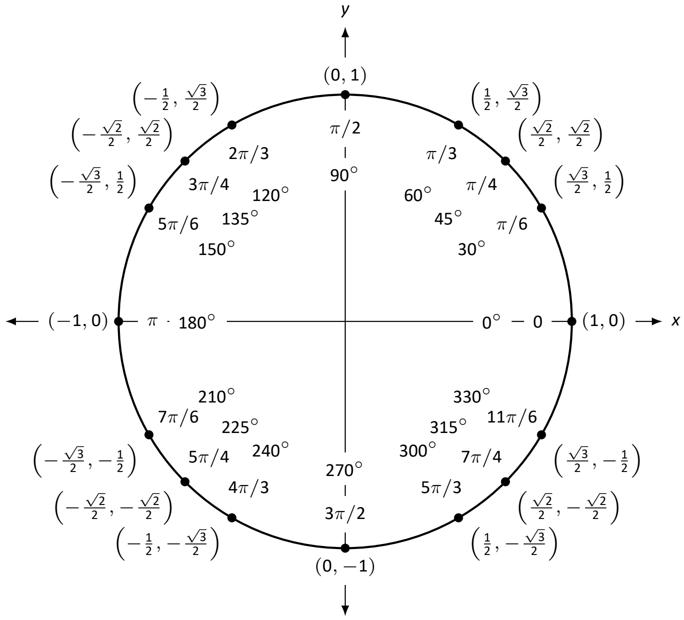
29. $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$

30. $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$

31. $\displaystyle\int \frac{1}{x\sqrt{a^2 - x^2}}\,dx = \frac{1}{a}\ln\left(\frac{x}{a + \sqrt{a^2 - x^2}}\right) + C$

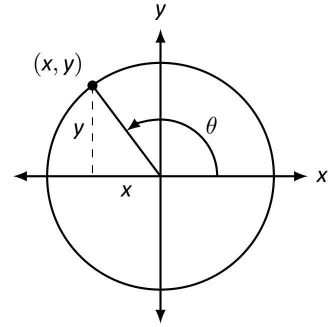
32. $\displaystyle\int \frac{1}{x\sqrt{x^2 + a^2}}\,dx = \frac{1}{a}\ln\left|\frac{x}{a + \sqrt{x^2 + a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

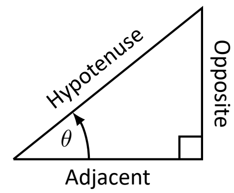


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition



$\sin\theta = \dfrac{\text{O}}{\text{H}} \qquad \csc\theta = \dfrac{\text{H}}{\text{O}}$

$\cos\theta = \dfrac{\text{A}}{\text{H}} \qquad \sec\theta = \dfrac{\text{H}}{\text{A}}$

$\tan\theta = \dfrac{\text{O}}{\text{A}} \qquad \cot\theta = \dfrac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$$\sin x \sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x \cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x \cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

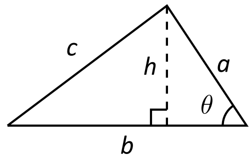
Areas and Volumes

Triangles

$h = a \sin\theta$

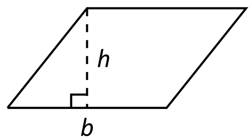
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



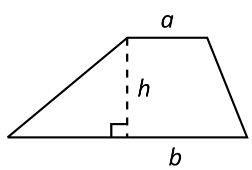
Parallelograms

Area = bh



Trapezoids

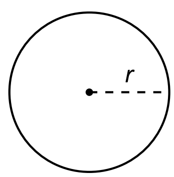
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

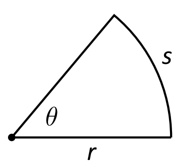


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

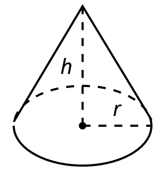
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

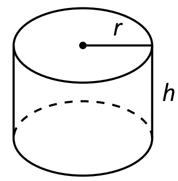
Surface Area =
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

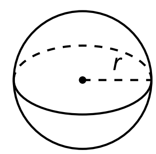
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

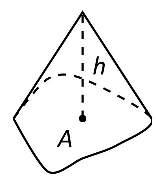
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

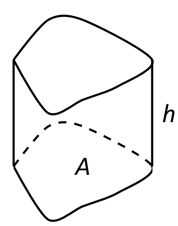
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x-a)(x+a)$ $\qquad$ $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x+a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x-y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x+y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x-y)^2 = x^2 - 2xy + y^2$

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$ab + ac = a(b+c)$ $\qquad$ $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ $\qquad$ $\dfrac{a+b}{c} = \dfrac{a}{c} + \dfrac{b}{c}$

$\dfrac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\dfrac{a}{b}\right)\left(\dfrac{d}{c}\right) = \dfrac{ad}{bc}$ $\qquad$ $\dfrac{\left(\frac{a}{b}\right)}{c} = \dfrac{a}{bc}$ $\qquad$ $\dfrac{a}{\left(\frac{b}{c}\right)} = \dfrac{ac}{b}$

$a\left(\dfrac{b}{c}\right) = \dfrac{ab}{c}$ $\qquad$ $\dfrac{a-b}{c-d} = \dfrac{b-a}{d-c}$ $\qquad$ $\dfrac{ab + ac}{a} = b + c$

Exponents and Radicals

$a^0 = 1, \quad a \neq 0$ $\qquad$ $(ab)^x = a^x b^x$ $\qquad$ $a^x a^y = a^{x+y}$ $\qquad$ $\sqrt{a} = a^{1/2}$ $\qquad$ $\dfrac{a^x}{a^y} = a^{x-y}$ $\qquad$ $\sqrt[n]{a} = a^{1/n}$

$\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ $\qquad$ $\sqrt[n]{a^m} = a^{m/n}$ $\qquad$ $a^{-x} = \dfrac{1}{a^x}$ $\qquad$ $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$ $\qquad$ $(a^x)^y = a^{xy}$ $\qquad$ $\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ \ dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\ \ dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\ \ dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\ \ dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\ \ dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$