12.1 Triple Integration with Cylindrical and Spherical Coordinates

Just as polar coordinates gave us a new way of describing curves in the plane, in this section we will see how *cylindrical* and *spherical* coordinates give us new ways of desribing surfaces and regions in space.

Cylindrical Coordinates

In short, cylindrical coordinates can be thought of as a combination of the polar and rectangular coordinate systems. One can identify a point (x_0, y_0, z_0), given in rectangular coordinates, with the point (r_0, θ_0, z_0), given in cylindrical coordinates, where the z-value in both systems is the same, and the point (x_0, y_0) in the x-y plane is identified with the polar point $P(r_0, \theta_0)$; see Figure 12.1.1. So that each point in space that does not lie on the z-axis is defined uniquely, we will restrict $r \geq 0$ and $0 \leq \theta \leq 2\pi$.

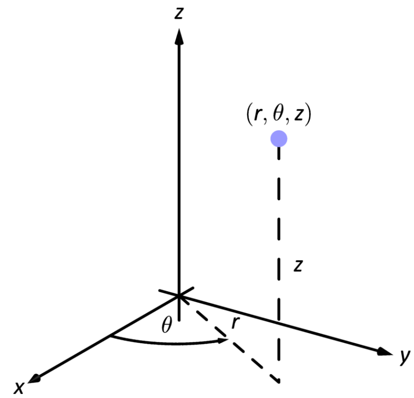


Figure 12.1.1: Illustrating the principles behind cylindrical coordinates.

We use the identity $z = z$ along with the identities found in Key Idea 9.4.1 to convert between the rectangular coordinate (x, y, z) and the cylindrical coordinate (r, θ, z), namely

From rectangular to cylindrical: $r = \sqrt{x^2 + y^2}, \quad \tan\theta = y/x \quad \text{and} \quad z = z;$

From cylindrical to rectangular: $x = r\cos\theta \quad y = r\sin\theta \quad \text{and} \quad z = z.$

Note: Our rectangular to polar conversion formulas used $r^2 = x^2 + y^2$, allowing for negative r values. Since we now restrict $r \geq 0$, we can use $r = \sqrt{x^2 + y^2}$.

These identities, along with conversions related to spherical coordinates, are given later in Key Idea 12.1.1.

Example 12.1.1 Converting between rectangular and cylindrical coordinates

Convert the rectangular point $(2, -2, 1)$ to cylindrical coordinates, and convert the cylindrical point $(4, 3\pi/4, 5)$ to rectangular.

Solution Following the identities given above (and, later in Key Idea 12.1.1), we have $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$. Using $\tan\theta = y/x$, we find $\theta = \tan^{-1}(-2/2) = -\pi/4$. As we restrict θ to being between 0 and 2π, we set $\theta = 7\pi/4$. Finally, $z = 1$, giving the cylindrical point $(2\sqrt{2}, 7\pi/4, 1)$.

In converting the cylindrical point $(4, 3\pi/4, 5)$ to rectangular, we have $x = 4\cos\big(3\pi/4\big) = -2\sqrt{2}$, $y = 4\sin\big(3\pi/4\big) = 2\sqrt{2}$ and $z = 5$, giving the rectangular point $(-2\sqrt{2}, 2\sqrt{2}, 5)$.

Setting each of r, θ and z equal to a constant defines a surface in space, as illustrated in the following example.

Notes:

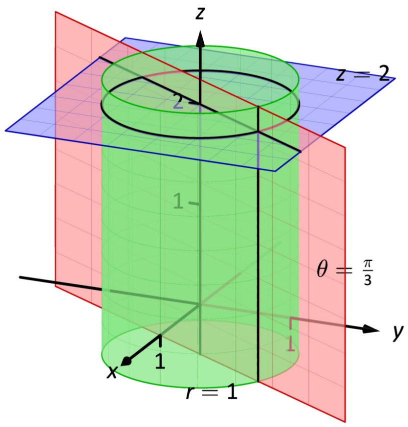


Figure 12.1.2: Graphing the canoncial surfaces in cylindrical coordinates from Example 12.1.2.

Example 12.1.2 Canonical surfaces in cylindrical coordinates

Describe the surfaces $r = 1$, $\theta = \pi/3$ and $z = 2$, given in cylindrical coordinates.

Solution The equation $r = 1$ describes all points in space that are 1 unit away from the z-axis. This surface is a "tube" or "cylinder" of radius 1, centered on the z-axis, as graphed in Figure 10.1.8 (which describes the cylinder $x^2 + y^2 = 1$ in space).

The equation $\theta = \pi/3$ describes the plane formed by extending the line $\theta = \pi/3$, as given by polar coordinates in the x-y plane, parallel to the z-axis.

The equation $z = 2$ describes the plane of all points in space that are 2 units above the x-y plane. This plane is the same as the plane described by $z = 2$ in rectangular coordinates.

All three surfaces are graphed in Figure 12.1.2. Note how their intersection uniquely defines the point $P = (1, \pi/3, 2)$.

Cylindrical coordinates are useful when describing certain domains in space, allowing us to evaluate triple integrals over these domains more easily than if we used rectangular coordinates.

Theorem 13.6.4 shows how to evaluate $\iiint_D h(x,y,z)\, dV$ using rectangular coordinates. In that evaluation, we use $dV = dz\,dy\,dx$ (or one of the other five orders of integration). Recall how, in this order of integration, the bounds on y are "curve to curve" and the bounds on x are "point to point": these bounds describe a region R in the x-y plane. We could describe R using polar coordinates as done in Section 13.3. In that section, we saw how we used $dA = r\,dr\,d\theta$ instead of $dA = dy\,dx$.

Considering the above thoughts, we have $dV = dz\big(r\,dr\,d\theta\big) = r\,dz\,dr\,d\theta$. We set bounds on z as "surface to surface" as done in the previous section, and then use "curve to curve" and "point to point" bounds on r and θ, respectively. Finally, using the identities given above, we replace the integrand $h(x,y,z)$ to $h(r,\theta,z)$.

This process should sound plausible; the following theorem states it is truly a way of evaluating a triple integral.

Theorem 12.1.1 Triple Integration in Cylindrical Coordinates

Let $w = h(r,\theta,z)$ be a continuous function on a closed, bounded region D in space, bounded in cylindrical coordinates by $\alpha \leq \theta \leq \beta$, $g_1(\theta) \leq r \leq g_2(\theta)$ and $f_1(r,\theta) \leq z \leq f_2(r,\theta)$. Then

$$\iiint_D h(r,\theta,z)\, dV = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r,\theta)}^{f_2(r,\theta)} h(r,\theta,z) r\, dz\, dr\, d\theta.$$

Notes:

Example 12.1.3 Evaluating a triple integral with cylindrical coordinates

Find the mass of the solid represented by the region in space bounded by $z = 0$, $z = \sqrt{4 - x^2 - y^2} + 3$ and the cylinder $x^2 + y^2 = 4$ (as shown in Figure 12.1.3), with density function $\delta(x, y, z) = x^2 + y^2 + z + 1$, using a triple integral in cylindrical coordinates. Distances are measured in centimeters and density is measured in grams/cm^3.

Solution We begin by describing this region of space with cylindrical coordinates. The plane $z = 0$ is left unchanged; with the identity $r = \sqrt{x^2 + y^2}$, we convert the hemisphere of radius 2 to the equation $z = \sqrt{4 - r^2}$; the cylinder $x^2 + y^2 = 4$ is converted to $r^2 = 4$, or, more simply, $r = 2$. We also convert the density function: $\delta(r, \theta, z) = r^2 + z + 1$.

To describe this solid with the bounds of a triple integral, we bound z with $0 \le z \le \sqrt{4 - r^2} + 3$; we bound r with $0 \le r \le 2$; we bound θ with $0 \le \theta \le 2\pi$.

Using Definition 13.6.4 and Theorem 12.1.1, we have the mass of the solid is

$$
\begin{aligned}
M = \iiint_D \delta(x, y, z)\, dV &= \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}+3} \big(r^2 + z + 1\big) r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 \Big((r^3 + 4r)\sqrt{4 - r^2} + \frac{5}{2}r^3 + \frac{19}{2}r\Big)\, dr\, d\theta \\
&= \frac{1318\pi}{15} \approx 276.04 \text{ gm},
\end{aligned}
$$

where we leave the details of the remaining double integral to the reader.

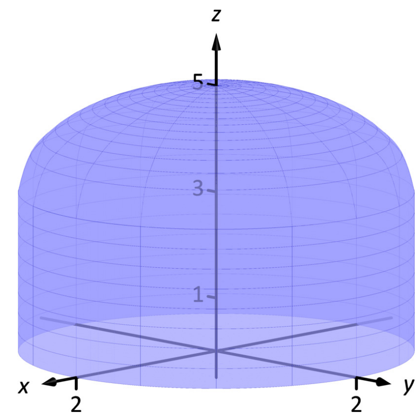


Figure 12.1.3: Visualizing the solid used in Example 12.1.3.

Example 12.1.4 Finding the center of mass using cylindrical coordinates

Find the center of mass of the solid with constant density whose base can be described by the polar curve $r = \cos(3\theta)$ and whose top is defined by the plane $z = 1 - x + 0.1y$, where distances are measured in feet, as seen in Figure 12.1.4. (The volume of this solid was found in Example 13.3.5.)

Solution We convert the equation of the plane to use cylindrical coordinates: $z = 1 - r\cos\theta + 0.1r\sin\theta$. Thus the region is space is bounded by $0 \le z \le 1 - r\cos\theta + 0.1r\sin\theta$, $0 \le r \le \cos(3\theta)$, $0 \le \theta \le \pi$ (recall that the rose curve $r = \cos(3\theta)$ is traced out once on $[0, \pi]$.

Since density is constant, we set $\delta = 1$ and finding the mass is equivalent to finding the volume of the solid. We set up the triple integral to compute this but do not evaluate it; we leave it to the reader to confirm it evaluates to the same

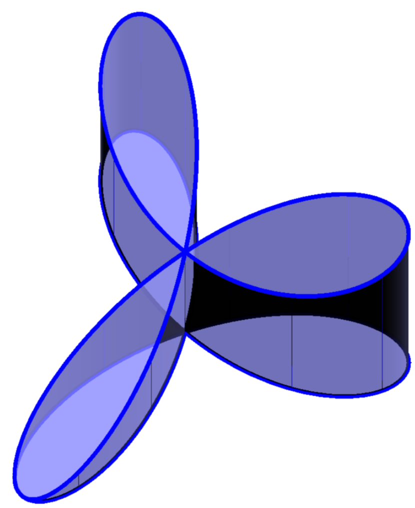


Figure 12.1.4: Visualizing the solid used in Example 12.1.4.

Notes:

result found in Example 13.3.5.

$$M = \iiint_D \delta \, dV = \int_0^{\pi} \int_0^{\cos(3\theta)} \int_0^{1 - r\cos\theta + 0.1r\sin\theta} r \, dz \, dr \, d\theta \approx 0.785.$$

From Definition 13.6.4 we set up the triple integrals to compute the moments about the three coordinate planes. The computation of each is left to the reader (using technology is recommended):

$$\begin{aligned} M_{yz} = \iiint_D x \, dV &= \int_0^{\pi} \int_0^{\cos(3\theta)} \int_0^{1 - r\cos\theta + 0.1r\sin\theta} (r\cos\theta) r \, dz \, dr \, d\theta \\ &= -0.147. \end{aligned}$$

$$\begin{aligned} M_{xz} = \iiint_D y \, dV &= \int_0^{\pi} \int_0^{\cos(3\theta)} \int_0^{1 - r\cos\theta + 0.1r\sin\theta} (r\sin\theta) r \, dz \, dr \, d\theta \\ &= 0.015. \\ M_{xy} = \iiint_D z \, dV &= \int_0^{\pi} \int_0^{\cos(3\theta)} \int_0^{1 - r\cos\theta + 0.1r\sin\theta} (z) r \, dz \, dr \, d\theta \\ &= 0.467. \end{aligned}$$

The center of mass, in rectangular coordinates, is located at $(-0.147, 0.015, 0.467)$, which lies outside the bounds of the solid.

Spherical Coordinates

In short, spherical coordinates can be thought of as a "double application" of the polar coordinate system. In spherical coordinates, a point *P* is identified with (ρ, θ, φ), where ρ is the distance from the origin to *P*, θ is the same angle as would be used to describe *P* in the cylindrical coordinate system, and φ is the angle between the positive *z*-axis and the ray from the origin to *P*; see Figure 12.1.5. So that each point in space that does not lie on the *z*-axis is defined uniquely, we will restrict $\rho \geq 0$, $0 \leq \theta \leq 2\pi$ and $0 \leq \varphi \leq \pi$.

The following Key Idea gives conversions to/from our three spatial coordinate systems.

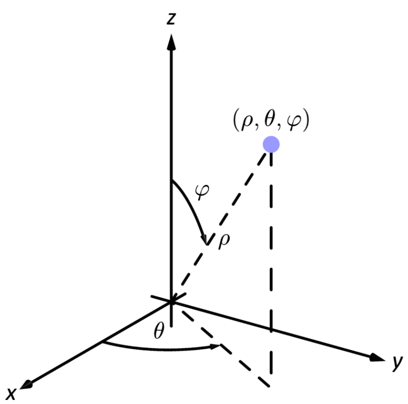


Figure 12.1.5: Illustrating the principles behind spherical coordinates.

Note: The symbol ρ is the Greek letter "rho." Traditionally it is used in the spherical coordinate system, while *r* is used in the polar and cylindrical coordinate systems.

Notes:

Key Idea 12.1.1 Converting Between Rectangular, Cylindrical and Spherical Coordinates

Rectangular and Cylindrical

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x, \quad z = z$$
$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

Rectangular and Spherical

$$\rho = \sqrt{x^2+y^2+z^2}, \quad \tan\theta = y/x, \quad \cos\varphi = z/\sqrt{x^2+y^2+z^2}$$
$$x = \rho\sin\varphi\cos\theta, \quad y = \rho\sin\varphi\sin\theta, \quad z = \rho\cos\varphi$$

Cylindrical and Spherical

$$\rho = \sqrt{r^2+z^2}, \quad \theta = \theta, \quad \tan\varphi = r/z$$
$$r = \rho\sin\varphi, \quad \theta = \theta, \quad z = \rho\cos\varphi$$

Example 12.1.5 Converting between rectangular and spherical coordinates

Convert the rectangular point $(2, -2, 1)$ to spherical coordinates, and convert the spherical point $(6, \pi/3, \pi/2)$ to rectangular and cylindrical coordinates.

Solution This rectangular point is the same as used in Example 12.1.1. Using Key Idea 12.1.1, we find $\rho = \sqrt{2^2 + (-1)^2 + 1^2} = 3$. Using the same logic as in Example 12.1.1, we find $\theta = 7\pi/4$. Finally, $\cos\varphi = 1/3$, giving $\varphi = \cos^{-1}(1/3) \approx 1.23$, or about $70.53°$. Thus the spherical coordinates are approximately $(3, 7\pi/4, 1.23)$.

Converting the spherical point $(6, \pi/3, \pi/2)$ to rectangular, we have $x = 6\sin(\pi/2)\cos(\pi/3) = 3$, $y = 6\sin(\pi/2)\sin(\pi/3) = 3\sqrt{3}$ and $z = 6\cos(\pi/2) = 0$. Thus the rectangular coordinates are $(3, 3\sqrt{3}, 0)$.

To convert this spherical point to cylindrical, we have $r = 6\sin(\pi/2) = 6$, $\theta = \pi/3$ and $z = 6\cos(\pi/2) = 0$, giving the cylindrical point $(6, \pi/3, 0)$.

Example 12.1.6 Canonical surfaces in spherical coordinates

Describe the surfaces $\rho = 1$, $\theta = \pi/3$ and $\varphi = \pi/6$, given in spherical coordinates.

Solution The equation $\rho = 1$ describes all points in space that are 1 unit away from the origin: this is the sphere of radius 1, centered at the origin.

The equation $\theta = \pi/3$ describes the same surface in spherical coordinates as it does in cylindrical coordinates: beginning with the line $\theta = \pi/3$ in the x-y plane as given by polar coordinates, extend the line parallel to the z-axis, forming

Notes:

a plane.

The equation $\varphi = \pi/6$ describes all points P in space where the ray from the origin to P makes an angle of $\pi/6$ with the positive z-axis. This describes a cone, with the positive z-axis its axis of symmetry, with point at the origin.

All three surfaces are graphed in Figure 12.1.6. Note how their intersection uniquely defines the point $P = (1, \pi/3, \pi/6)$.

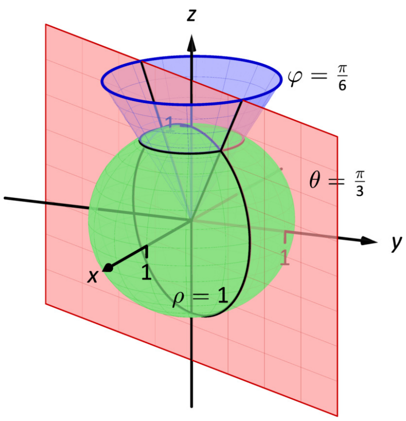


Figure 12.1.6: Graphing the canonical surfaces in spherical coordinates from Example 12.1.6.

Spherical coordinates are useful when describing certain domains in space, allowing us to evaluate triple integrals over these domains more easily than if we used rectangular coordinates or cylindrical coordinates. The crux of setting up a triple integral in spherical coordinates is appropriately describing the "small amount of volume," dV, used in the integral.

Considering Figure **??**, we can make a small "spherical wedge" by varying ρ, θ and φ each a small amount, $\Delta\rho$, $\Delta\theta$ and $\Delta\varphi$, respectively. This wedge is approximately a rectangular solid when the change in each coordinate is small, giving a volume of about

$$\Delta V \approx \Delta\rho \ \times \ \rho\Delta\varphi \ \times \ \rho\sin(\varphi)\Delta\theta.$$

Given a region D in space, we can approximate the volume of D with many such wedges. As the size of each of $\Delta\rho$, $\Delta\theta$ and $\Delta\varphi$ goes to zero, the number of wedges increases to infinity and the volume of D is more accurately approximated, giving

$$dV = d\rho \ \times \ \rho\, d\varphi \ \times \ \rho\sin(\varphi) d\theta = \rho^2\sin(\varphi)\, d\rho\, d\theta\, d\varphi.$$

Again, this development of dV should sound reasonable, and the following theorem states it is the appropriate manner by which triple integrals are to be evaluated in spherical coordinates.

Note: It is generally most intuitive to evaluate the triple integral in Theorem 12.1.2 by integrating with respect to ρ first; it often does not matter whether we next integrate with respect to θ or φ. Different texts present different standard orders, some preferring $d\varphi\, d\theta$ instead of $d\theta\, d\varphi$. As the bounds for these variables are usually constants in practice, it generally is a matter of preference.

Theorem 12.1.2 Triple Integration in Spherical Coordinates

Let $w = h(\rho, \theta, \varphi)$ be a continuous function on a closed, bounded region D in space, bounded in spherical coordinates by $\alpha_1 \leq \varphi \leq \alpha_2$, $\beta_1 \leq \theta \leq \beta_2$ and $f_1(\theta, \varphi) \leq \rho \leq f_2(\theta, \varphi)$. Then

$$\iiint_D h(\rho, \theta, \varphi)\, dV = \int_{\alpha_1}^{\alpha_2}\int_{\beta_1}^{\beta_2}\int_{f_1(\theta,\varphi)}^{f_2(\theta,\varphi)} h(\rho, \theta, \varphi)\rho^2\sin(\varphi)\, d\rho\, d\theta\, d\varphi.$$

Example 12.1.7 Establishing the volume of a sphere

Let D be the region in space bounded by the sphere, centered at the origin, of radius r. Use a triple integral in spherical coordinates to find the volume V of D.

Notes:

Solution The sphere of radius r, centered at the origin, has equation $\rho = r$. To obtain the full sphere, the bounds on θ and φ are $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi$. This leads us to:

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^{r} \big(\rho^2 \sin(\varphi)\big)\, d\rho\, d\theta\, d\varphi \\ &= \int_0^{\pi}\int_0^{2\pi} \left(\frac{1}{3}\rho^3 \sin(\varphi)\Big|_0^r\right) d\theta\, d\varphi \\ &= \int_0^{\pi}\int_0^{2\pi} \left(\frac{1}{3}r^3 \sin(\varphi)\right) d\theta\, d\varphi \\ &= \int_0^{\pi} \left(\frac{2\pi}{3}r^3 \sin(\varphi)\right) d\varphi \\ &= \left(-\frac{2\pi}{3}r^3 \cos(\varphi)\right)\Big|_0^{\pi} \\ &= \frac{4\pi}{3}r^3, \end{aligned}$$

the familiar formula for the volume of a sphere. Note how the integration steps were easy, not using square–roots nor integration steps such as Substitution.

Example 12.1.8 Finding the center of mass using spherical coordinates
Find the center of mass of the solid with constant density whose surface can be represented by $\rho = 4$, $\varphi = \pi/6$, where θ ranges from 0 to 2π, as illustrated in Figure 12.1.7.

Solution We will set up the four triple integrals needed to find the center of mass (i.e., to compute M, M_{yz}, M_{xz} and M_{xy}) and leave it to the reader to evaluate each integral. Because of symmetry, we expect the x and y coordinates of the center of mass to be 0.

While the surfaces describing the solid are given in the statement of the problem, to describe the full solid D, we use the following bounds: $0 \le \rho \le 4$, $0 \le \theta \le 2\pi$ and $0 \le \varphi \le \pi/6$. Since density δ is constant, we assume $\delta = 1$.

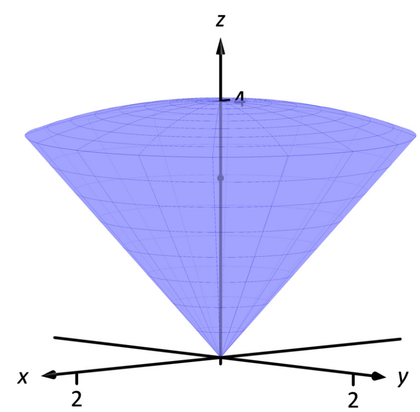


Figure 12.1.7: Graphing the solid, and its center of mass, from Example 12.1.8.

Notes:

The mass of the solid:

$$\begin{aligned} M &= \iiint_D dm = \iiint_D dV \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \left(\rho^2 \sin(\varphi)\right) d\rho\, d\theta\, d\varphi \\ &= \frac{64}{3}\left(2 - \sqrt{3}\right)\pi \approx 17.958. \end{aligned}$$

To compute M_{yz}, the integrand is x; using Key Idea 12.1.1, we have $x = \rho \sin\varphi \cos\theta$. This gives:

$$\begin{aligned} M_{yz} &= \iiint_D x\, dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \left((\rho \sin(\varphi) \cos(\theta))\rho^2 \sin(\varphi)\right) d\rho\, d\theta\, d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \left(\rho^3 \sin^2(\varphi) \cos(\theta)\right) d\rho\, d\theta\, d\varphi \\ &= 0, \end{aligned}$$

which we expected as we expect $\bar{x} = 0$.

To compute M_{xz}, the integrand is y; using Key Idea 12.1.1, we have $y = \rho \sin\varphi \sin\theta$. This gives:

$$\begin{aligned} M_{xz} &= \iiint_D y\, dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \left((\rho \sin(\varphi) \sin(\theta))\rho^2 \sin(\varphi)\right) d\rho\, d\theta\, d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 \left(\rho^3 \sin^2(\varphi) \sin(\theta)\right) d\rho\, d\theta\, d\varphi \\ &= 0, \end{aligned}$$

which we also expected as we expect $\bar{y} = 0$.

To compute M_{xy}, the integrand is z; using Key Idea 12.1.1, we have $z =$

Notes:

$\rho\cos\varphi$. This gives:

$$\begin{aligned} M_{xy} &= \iiint_D z\, dm \\ &= \int_0^{\pi/6}\int_0^{2\pi}\int_0^4 \big((\rho\cos(\varphi))\rho^2\sin(\varphi)\big)\, d\rho\, d\theta\, d\varphi \\ &= \int_0^{\pi/6}\int_0^{2\pi}\int_0^4 \big(\rho^3\cos(\varphi)\sin(\varphi)\big)\, d\rho\, d\theta\, d\varphi \\ &= 16\pi \approx 50.266. \end{aligned}$$

Thus the center of mass is $(0, 0, M_{xy}/M) \approx (0, 0, 2.799)$, as indicated in Figure 12.1.7.

Notes:

Exercises 12.1

Terms and Concepts

1. The strategy for establishing bounds for triple integrals is "________ to ________, ________ to ________ and ________ to ________."

2. Give an informal interpretation of what "$\iiint_D dV$" means.

3. Give two uses of triple integration.

4. If an object has a constant density δ and a volume V, what is its mass?

Problems

In Exercises 5 – 8, two surfaces $f_1(x,y)$ and $f_2(x,y)$ and a region R in the x,y plane are given. Set up and evaluate the double integral that finds the volume between these surfaces over R.

5. $f_1(x,y) = 8 - x^2 - y^2$, $f_2(x,y) = 2x + y$;
   R is the square with corners $(-1,-1)$ and $(1,1)$.

6. $f_1(x,y) = x^2 + y^2$, $f_2(x,y) = -x^2 - y^2$;
   R is the square with corners $(0,0)$ and $(2,3)$.

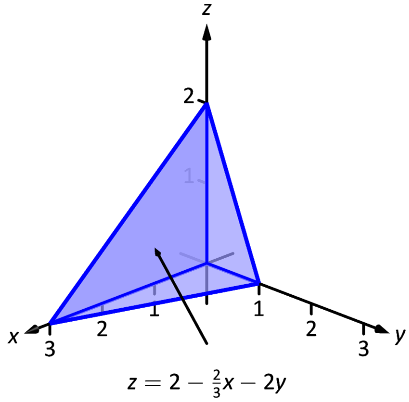
7. $f_1(x,y) = \sin x \cos y$, $f_2(x,y) = \cos x \sin y + 2$;
   R is the triangle with corners $(0,0)$, $(\pi,0)$ and (π,π).

8. $f_1(x,y) = 2x^2 + 2y^2 + 3$, $f_2(x,y) = 6 - x^2 - y^2$;
   R is the disk bounded by $x^2 + y^2 = 1$.

In Exercises 9 – 16, a domain D is described by its bounding surfaces, along with a graph. Set up the triple integrals that give the volume of D in all 6 orders of integration, and find the volume of D by evaluating the indicated triple integral.

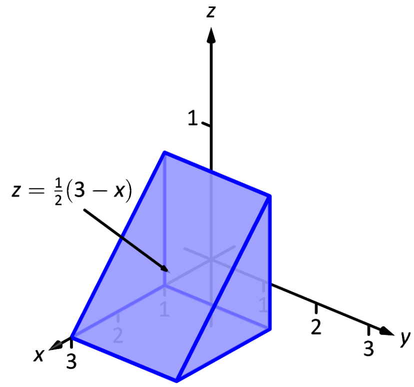
9. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$.

   Evaluate the triple integral with order $dz\,dy\,dx$.



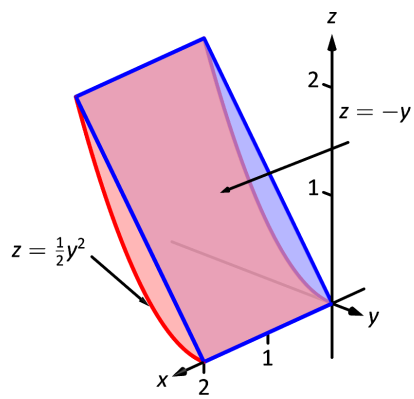
10. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$.

    Evaluate the triple integral with order $dx\,dy\,dz$.



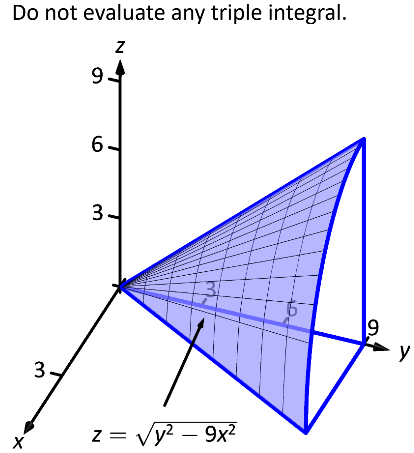
11. D is bounded by the planes $x = 0$, $x = 2$, $z = -y$ and by $z = y^2/2$.

    Evaluate the triple integral with the order $dy\,dz\,dx$.



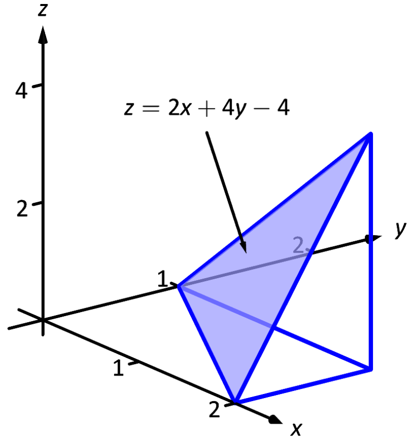
12. D is bounded by the planes $z = 0$, $y = 9$, $x = 0$ and by $z = \sqrt{y^2 - 9x^2}$.

    Do not evaluate any triple integral.



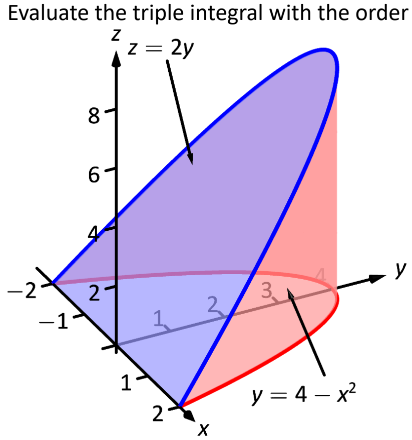
13. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$.

Evaluate the triple integral with the order $dx\,dy\,dz$.

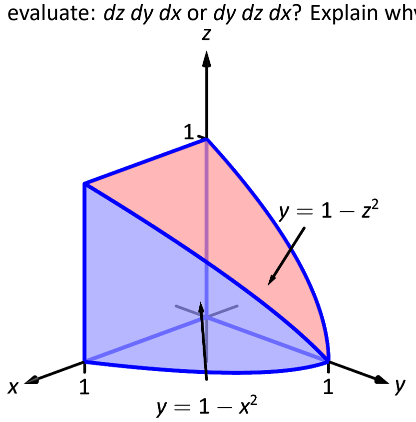


14. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$.

Evaluate the triple integral with the order $dz\,dy\,dx$.

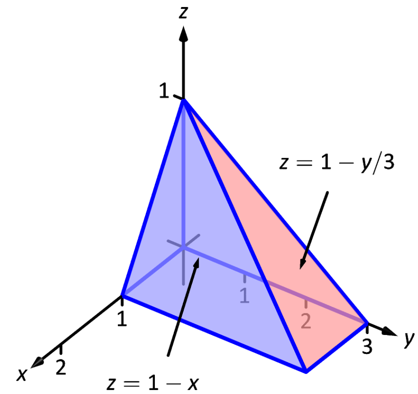


15. D is bounded by the coordinate planes and by $y = 1 - x^2$ and $y = 1 - z^2$.
Do not evaluate any triple integral. Which order is easier to evaluate: $dz\,dy\,dx$ or $dy\,dz\,dx$? Explain why.



16. D is bounded by the coordinate planes and by $z = 1 - y/3$ and $z = 1 - x$.

Evaluate the triple integral with order $dx\,dy\,dz$.



In Exercises 17 – 20, evaluate the triple integral.

17. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^{\pi}\big(\cos x\sin y\sin z\big)\,dz\,dy\,dx$

18. $\displaystyle\int_0^1\int_0^x\int_0^{x+y}\big(x+y+z\big)\,dz\,dy\,dx$

19. $\displaystyle\int_0^{\pi}\int_0^1\int_0^z\big(\sin(yz)\big)\,dx\,dy\,dz$

20. $\displaystyle\int_{\pi}^{\pi^2}\int_x^{x^3}\int_{-y^2}^{y^2}\left(z\frac{x^2y+y^2x}{e^{x^2+y^2}}\right)\,dz\,dy\,dx$

In Exercises 21 – 24, find the center of mass of the solid represented by the indicated space region D with density function $\delta(x,y,z)$.

21. D is bounded by the coordinate planes and $z = 2 - 2x/3 - 2y$; $\quad\delta(x,y,z) = 10\text{gm/cm}^3$.
(Note: this is the same region as used in Exercise 9.)

22. D is bounded by the planes $y = 0$, $y = 2$, $x = 1$, $z = 0$ and $z = (3 - x)/2$; $\quad\delta(x,y,z) = 2\text{gm/cm}^3$.
(Note: this is the same region as used in Exercise 10.)

23. D is bounded by the planes $x = 2$, $y = 1$, $z = 0$ and $z = 2x + 4y - 4$; $\quad\delta(x,y,z) = x^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 13.)

24. D is bounded by the plane $z = 2y$ and by $y = 4 - x^2$. $\delta(x,y,z) = y^2\text{lb/in}^3$.
(Note: this is the same region as used in Exercise 14.)

A: Solutions To Selected Problems

Chapter 12

Section 12.1

1. surface to surface, curve to curve and point to point
2. One possible answer is "sum up lots of little volumes over D."
3. Answers can vary. From this section we used triple integration to find the volume of a solid region, the mass of a solid, and the center of mass of a solid.
4. δV.
5. $V = \int_{-1}^{1}\int_{-1}^{1}\left(8 - x^2 - y^2 - (2x+y)\right)\,dx\,dy = 88/3$
6. $V = \int_{0}^{2}\int_{0}^{3}\left(x^2 + y^2 - (-x^2 - y^2)\right)\,dy\,dx = 52$
7. $V = \int_{0}^{\pi}\int_{0}^{x}\left(\cos x \sin y + 2 - \sin x \cos y\right)\,dy\,dx = \pi^2 - \pi \approx 6.728$
8. $V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\left(6 - x^2 - y^2 - (2x^2 + 2y^2 + 3)\right)\,dy\,dx.$
   Integrating in polar is easier, giving
   $V = \int_{0}^{2\pi}\int_{0}^{1}\left(3 - 3r^2\right)r\,dr\,d\theta = 3\pi/2.$
9. dz dy dx: $\int_0^3\int_0^{1-x/3}\int_0^{2-2x/3-2y} dz\,dy\,dx$
   dz dx dy: $\int_0^1\int_0^{3-3y}\int_0^{2-2x/3-2y} dz\,dx\,dy$
   dy dz dx: $\int_0^3\int_0^{2-2x/3}\int_0^{1-x/3-z/2} dy\,dz\,dx$
   dy dx dz: $\int_0^2\int_0^{3-3z/2}\int_0^{1-x/3-z/2} dy\,dx\,dz$
   dx dz dy: $\int_0^1\int_0^{2-2y}\int_0^{3-3y-3z/2} dx\,dz\,dy$
   dx dy dz: $\int_0^2\int_0^{1-z/2}\int_0^{3-3y-3z/2} dx\,dy\,dz$
   $V = \int_0^3\int_0^{1-x/3}\int_0^{2-2x/3-2y} dz\,dy\,dx = 1.$
10. dz dy dx: $\int_1^3\int_0^2\int_0^{(3-x)/2} dz\,dy\,dx$
    dz dx dy: $\int_0^2\int_1^3\int_0^{(3-x)/2} dz\,dx\,dy$
    dy dz dx: $\int_1^3\int_0^{(3-x)/2}\int_0^2 dy\,dz\,dx$
    dy dx dz: $\int_0^1\int_1^{3-2z}\int_0^2 dy\,dx\,dz$
    dx dz dy: $\int_0^2\int_0^1\int_1^{3-2z} dx\,dz\,dy$
    dx dy dz: $\int_0^1\int_0^2\int_1^{3-2z} dx\,dy\,dz$
    $V = \int_0^1\int_0^2\int_1^{3-2z} dx\,dy\,dz = 2.$
11. dz dy dx: $\int_0^2\int_{-2}^0\int_{y^2/2}^{-y} dz\,dy\,dx$
    dz dx dy: $\int_{-2}^0\int_0^2\int_{y^2/2}^{-y} dz\,dx\,dy$
    dy dz dx: $\int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\,dz\,dx$
    dy dx dz: $\int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\,dx\,dz$
    dx dz dy: $\int_{-2}^0\int_{y^2/2}^{-y}\int_0^2 dx\,dz\,dy$
    dx dy dz: $\int_0^2\int_{-\sqrt{2z}}^{-z}\int_0^2 dx\,dy\,dz$
    $V = \int_0^2\int_0^2\int_{-\sqrt{2z}}^{-z} dy\,dz\,dx = 4/3.$
12. dz dy dx: $\int_0^3\int_{3x}^9\int_0^{\sqrt{y^2-9x^2}} dz\,dy\,dx$
    dz dx dy: $\int_0^9\int_0^{y/3}\int_0^{\sqrt{y^2-9x^2}} dz\,dx\,dy$
    dy dz dx: $\int_0^3\int_0^{\sqrt{81-9x^2}}\int_{\sqrt{z^2+9x^2}}^9 dy\,dz\,dx$
    dy dx dz: $\int_0^9\int_0^{\sqrt{9-z^2/9}}\int_{\sqrt{z^2+9x^2}}^9 dy\,dx\,dz$
    dx dz dy: $\int_0^9\int_0^y\int_0^{\frac{1}{3}\sqrt{y^2-z^2}} dx\,dz\,dy$
    dx dy dz: $\int_0^9\int_z^9\int_0^{\frac{1}{3}\sqrt{y^2-z^2}} dx\,dy\,dz$
13. dz dy dx: $\int_0^2\int_{1-x/2}^1\int_0^{2x+4y-4} dz\,dy\,dx$
    dz dx dy: $\int_0^1\int_{2-2y}^2\int_0^{2x+4y-4} dz\,dx\,dy$
    dy dz dx: $\int_0^2\int_0^{2x}\int_{z/4-x/2+1}^1 dy\,dz\,dx$
    dy dx dz: $\int_0^4\int_{z/2}^2\int_{z/4-x/2+1}^1 dy\,dx\,dz$
    dx dz dy: $\int_0^1\int_0^{4y}\int_{z/2-2y+2}^2 dx\,dz\,dy$
    dx dy dz: $\int_0^4\int_{z/4}^1\int_{z/2-2y+2}^2 dx\,dy\,dz$
    $V = \int_0^4\int_{z/4}^1\int_{z/2-2y+2}^2 dx\,dy\,dz = 4/3.$
14. dz dy dx: $\int_{-2}^2\int_0^{4-x^2}\int_0^{2y} dz\,dy\,dx$
    dz dx dy: $\int_0^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}}\int_0^{2x+4y-4} dz\,dx\,dy$
    dy dz dx: $\int_{-2}^2\int_0^{8-2x^2}\int_{z/2}^{4-x^2} dy\,dz\,dx$
    dy dx dz: $\int_0^8\int_{-\sqrt{4-z/2}}^{\sqrt{4-z/2}}\int_{z/2}^{4-x^2} dy\,dx\,dz$
    dx dz dy: $\int_0^4\int_0^{2y}\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dz\,dy$
    dx dy dz: $\int_0^8\int_{z/2}^4\int_{-\sqrt{4-y}}^{\sqrt{4-y}} dx\,dy\,dz$
    $V = \int_{-2}^2\int_0^{4-x^2}\int_0^{2y} dz\,dy\,dx = 512/15.$
15. dz dy dx: $\int_0^1\int_0^{1-x^2}\int_0^{\sqrt{1-y}} dz\,dy\,dx$
    dz dx dy: $\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} dz\,dx\,dy$
    dy dz dx: $\int_0^1\int_0^x\int_0^{1-x^2} dy\,dz\,dx + \int_0^1\int_x^1\int_0^{1-z^2} dy\,dz\,dx$
    dy dx dz: $\int_0^1\int_0^z\int_0^{1-z^2} dy\,dx\,dz + \int_0^1\int_z^1\int_0^{1-x^2} dy\,dx\,dz$
    dx dz dy: $\int_0^1\int_0^{\sqrt{1-y}}\int_0^{\sqrt{1-y}} dx\,dz\,dy$

dx dy dz: $\int_0^1 \int_0^{1-z^2} \int_0^{\sqrt{1-y}} dx\, dy\, dz$

Answers will vary. Neither order is particularly "hard." The order *dz dy dx* requires integrating a square root, so powers can be messy; the order *dy dz dx* requires two triple integrals, but each uses only polynomials.

16. *dz dy dx*: $\int_0^1 \int_0^{3x} \int_0^{1-x} dz\, dy\, dx + \int_0^1 \int_{3x}^3 \int_0^{1-y/3} dz\, dy\, dx$

dz dx dy: $\int_0^3 \int_0^{y/3} \int_0^{1-y/3} dz\, dy\, dx + \int_0^3 \int_{y/3}^1 \int_0^{1-x} dz\, dx\, dy$

dy dz dx: $\int_0^1 \int_0^{1-x} \int_0^{3-3z} dy\, dz\, dx$

dy dx dz: $\int_0^1 \int_0^{1-z} \int_0^{3-3z} dy\, dx\, dz$

dx dz dy: $\int_0^3 \int_0^{1-y/3} \int_0^{1-z} dx\, dz\, dy$

dx dy dz: $\int_0^1 \int_0^{3-3z} \int_0^{1-z} dx\, dy\, dz$

$$V = \int_0^1 \int_0^{3-3z} \int_0^{1-z} dx\, dy\, dz = 1.$$

17. 8

18. $7/8$

19. π

20. 0

21. $M = 10$, $M_{yz} = 15/2$, $M_{xz} = 5/2$, $M_{xy} = 5$; $(\overline{x}, \overline{y}, \overline{z}) = (3/4, 1/4, 1/2)$

22. $M = 4$, $M_{yz} = 20/3$, $M_{xz} = 4$, $M_{xy} = 4/3$; $(\overline{x}, \overline{y}, \overline{z}) = (5/3, 1, 1/3)$

23. $M = 16/5$, $M_{yz} = 16/3$, $M_{xz} = 104/45$, $M_{xy} = 32/9$; $(\overline{x}, \overline{y}, \overline{z}) = (5/3, 13/18, 10/9) \approx (1.67, 0.72, 1.11)$

24. $M = \frac{65,536}{15} \approx 208.05$, $M_{yz} = 0$, $M_{xz} = \frac{2,097,152}{3465} \approx 605.24$, $M_{xy} = \frac{2,097,152}{3465} \approx 605.24$; $(\overline{x}, \overline{y}, \overline{z}) = (0, 32/11, 32/11) \approx (0, 2.91, 2.91)$