1: DERIVATIVES

The previous chapter introduced the most fundamental of calculus topics: the limit. This chapter introduces the second most fundamental of calculus topics: the derivative. Limits describe *where* a function is going; derivatives describe *how fast* the function is going.

1.1 Instantaneous Rates of Change: The Derivative

A common amusement park ride lifts riders to a height then allows them to freefall a certain distance before safely stopping them. Suppose such a ride drops riders from a height of 150 feet. Students of physics may recall that the height (in feet) of the riders, t seconds after freefall (and ignoring air resistance, etc.) can be accurately modeled by $f(t) = -16t^2 + 150$.

Using this formula, it is easy to verify that, without intervention, the riders will hit the ground at $t = 2.5\sqrt{1.5} \approx 3.06$ seconds. Suppose the designers of the ride decide to begin slowing the riders' fall after 2 seconds (corresponding to a height of 86 ft.). How fast will the riders be traveling at that time?

We have been given a *position* function, but what we want to compute is a velocity at a specific point in time, i.e., we want an *instantaneous velocity*. We do not currently know how to calculate this.

However, we do know from common experience how to calculate an *average velocity*. (If we travel 60 miles in 2 hours, we know we had an average velocity of 30 mph.) We looked at this concept in Section 1.1 when we introduced the difference quotient. We have

$$\frac{\text{change in distance}}{\text{change in time}} = \frac{\text{“ rise ”}}{\text{run}} = \text{average velocity.}$$

We can approximate the instantaneous velocity at $t = 2$ by considering the average velocity over some time period containing $t = 2$. If we make the time interval small, we will get a good approximation. (This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of the velocity.)

Consider the interval from $t = 2$ to $t = 3$ (just before the riders hit the ground). On that interval, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{f(3) - f(2)}{1} = -80 \text{ ft/s},$$

where the minus sign indicates that the riders are moving *down*. By narrowing the interval we consider, we will likely get a better approximation of the instantaneous velocity. On $[2, 2.5]$ we have

$$\frac{f(2.5)-f(2)}{2.5-2} = \frac{f(2.5)-f(2)}{0.5} = -72 \text{ ft/s}.$$

We can do this for smaller and smaller intervals of time. For instance, over a time span of $1/10^{\text{th}}$ of a second, i.e., on $[2, 2.1]$, we have

$$\frac{f(2.1)-f(2)}{2.1-2} = \frac{f(2.1)-f(2)}{0.1} = -65.6 \text{ ft/s}.$$

Over a time span of $1/100^{\text{th}}$ of a second, on $[2, 2.01]$, the average velocity is

$$\frac{f(2.01)-f(2)}{2.01-2} = \frac{f(2.01)-f(2)}{0.01} = -64.16 \text{ ft/s}.$$

What we are really computing is the average velocity on the interval $[2, 2+h]$ for small values of h. That is, we are computing

$$\frac{f(2+h)-f(2)}{h}$$

where h is small.

We really want to use $h = 0$, but this, of course, returns the familiar "0/0" indeterminate form. So we employ a limit, as we did in Section 1.1.

We can approximate the value of this limit numerically with small values of h as seen in Figure 1.1.1. It looks as though the velocity is approaching -64 ft/s. Computing the limit directly gives

$$\begin{aligned}
\lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}\frac{-16(2+h)^2+150-(-16(2)^2+150)}{h}\\
&= \lim_{h\to 0}\frac{-64h-16h^2}{h}\\
&= \lim_{h\to 0} -64-16h\\
&= -64.
\end{aligned}$$

h	Average Velocity ft/s
1	-80
0.5	-72
0.1	-65.6
0.01	-64.16
0.001	-64.016

Figure 1.1.1: Approximating the instantaneous velocity with average velocities over a small time period h.

Graphically, we can view the average velocities we computed numerically as the slopes of secant lines on the graph of f going through the points $(2, f(2))$ and $(2+h, f(2+h))$. In Figure 1.1.2, the secant line corresponding to $h = 1$ is shown in three contexts. Figure 1.1.2(a) shows a "zoomed out" version of f with its secant line. In (b), we zoom in around the points of intersection between f and the secant line. Notice how well this secant line approximates f between

Notes:

those two points – it is a common practice to approximate functions with straight lines.

As $h \to 0$, these secant lines approach the *tangent line*, a line that goes through the point $(2, f(2))$ with the special slope of -64. In parts (c) and (d) of Figure 1.1.2, we zoom in around the point $(2, 86)$. In (c) we see the secant line, which approximates f well, but not as well the tangent line shown in (d).

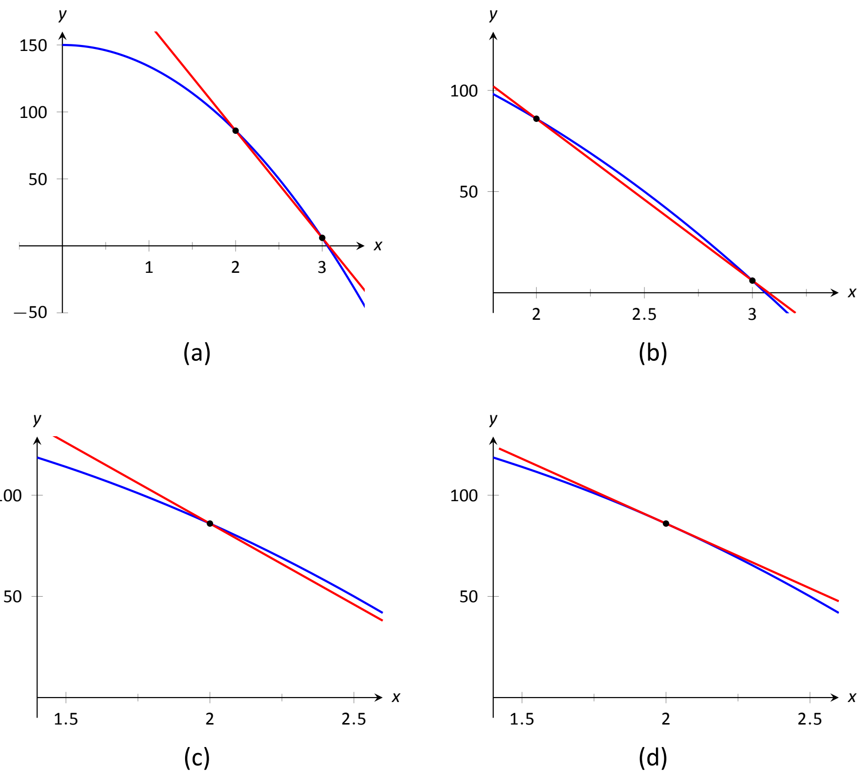


Figure 1.1.2: Parts (a), (b) and (c) show the secant line to $f(x)$ with $h = 1$, zoomed in different amounts. Part (d) shows the tangent line to f at $x = 2$.

We have just introduced a number of important concepts that we will flesh out more within this section. First, we formally define two of them.

Notes:

Definition 1.1.1 Derivative at a Point

Let f be a continuous function on an open interval I and let c be in I. The **derivative of f at c**, denoted $f'(c)$, is

$$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h},$$

provided the limit exists. If the limit exists, we say that f **is differentiable at** c; if the limit does not exist, then f **is not differentiable at** c. If f is differentiable at every point in I, then f **is differentiable on** I.

Definition 1.1.2 Tangent Line

Let f be continuous on an open interval I and differentiable at c, for some c in I. The line with equation $\ell(x) = f'(c)(x-c)+f(c)$ is the **tangent line** to the graph of f at c; that is, it is the line through $(c, f(c))$ whose slope is the derivative of f at c.

Some examples will help us understand these definitions.

Example 1.1.1 Finding derivatives and tangent lines

Let $f(x) = 3x^2+5x-7$. Find:

1. $f'(1)$
2. The equation of the tangent line to the graph of f at $x=1$.
3. $f'(3)$
4. The equation of the tangent line to the graph f at $x=3$.

Solution

1. We compute this directly using Definition 1.1.1.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h)^2+5(1+h)-7-(3(1)^2+5(1)-7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2+11h}{h} \\ &= \lim_{h\to 0} 3h+11 = 11. \end{aligned}$$

Notes:

2. The tangent line at $x = 1$ has slope $f'(1)$ and goes through the point $(1, f(1)) = (1, 1)$. Thus the tangent line has equation, in point-slope form, $y = 11(x-1)+1$. In slope-intercept form we have $y = 11x - 10$.

3. Again, using the definition,

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{3(3+h)^2+5(3+h)-7-(3(3)^2+5(3)-7)}{h} \\ &= \lim_{h\to 0} \frac{3h^2+23h}{h} \\ &= \lim_{h\to 0} 3h+23 \\ &= 23. \end{aligned}$$

4. The tangent line at $x = 3$ has slope 23 and goes through the point $(3, f(3)) = (3, 35)$. Thus the tangent line has equation $y = 23(x-3)+35 = 23x-34$.

A graph of f is given in Figure 1.1.3 along with the tangent lines at $x = 1$ and $x = 3$.

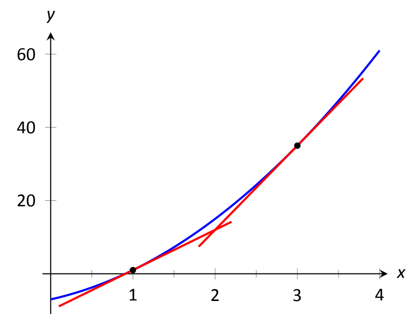


Figure 1.1.3: A graph of $f(x) = 3x^2+5x-7$ and its tangent lines at $x = 1$ and $x = 3$.

Another important line that can be created using information from the derivative is the **normal line.** It is perpendicular to the tangent line, hence its slope is the opposite–reciprocal of the tangent line's slope.

Definition 1.1.3 Normal Line

Let f be continuous on an open interval I and differentiable at c, for some c in I. The **normal line** to the graph of f at c is the line with equation

$$n(x) = \frac{-1}{f'(c)}(x-c)+f(c),$$

where $f'(c) \neq 0$. When $f'(c) = 0$, the normal line is the vertical line through $\big(c, f(c)\big)$; that is, $x = c$.

Example 1.1.2 Finding equations of normal lines

Let $f(x) = 3x^2+5x-7$, as in Example 1.1.1. Find the equations of the normal lines to the graph of f at $x = 1$ and $x = 3$.

SOLUTION In Example 1.1.1, we found that $f'(1) = 11$. Hence at $x = 1$,

Notes:

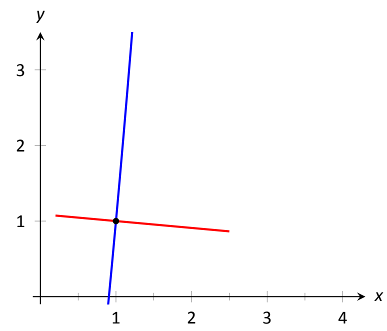


Figure 1.1.4: A graph of $f(x) = 3x^2 + 5x - 7$, along with its normal line at $x = 1$.

the normal line will have slope $-1/11$. An equation for the normal line is

$$n(x) = \frac{-1}{11}(x-1) + 1.$$

The normal line is plotted with $y = f(x)$ in Figure 1.1.4. Note how the line looks perpendicular to f. (A key word here is "looks." Mathematically, we say that the normal line *is* perpendicular to f at $x = 1$ as the slope of the normal line is the opposite–reciprocal of the slope of the tangent line. However, normal lines may not always *look* perpendicular. The aspect ratio of the picture of the graph plays a big role in this.)

We also found that $f'(3) = 23$, so the normal line to the graph of f at $x = 3$ will have slope $-1/23$. An equation for the normal line is

$$n(x) = \frac{-1}{23}(x-3) + 35.$$

Linear functions are easy to work with; many functions that arise in the course of solving real problems are not easy to work with. A common practice in mathematical problem solving is to approximate difficult functions with not–so–difficult functions. Lines are a common choice. It turns out that at any given point on the graph of a differentiable function f, the best linear approximation to f is its tangent line. That is one reason we'll spend considerable time finding tangent lines to functions.

One type of function that does not benefit from a tangent–line approximation is a line; it is rather simple to recognize that the tangent line to a line is the line itself. We look at this in the following example.

Example 1.1.3 Finding the Derivative of a Line

Consider $f(x) = 3x + 5$. Find the equation of the tangent line to f at $x = 1$ and $x = 7$.

Solution We find the slope of the tangent line by using Definition 1.1.1.

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h\to 0} \frac{3(1+h) + 5 - (3+5)}{h} \\ &= \lim_{h\to 0} \frac{3h}{h} \\ &= \lim_{h\to 0} 3 \\ &= 3. \end{aligned}$$

Notes:

We just found that $f'(1) = 3$. That is, we found the *instantaneous rate of change* of $f(x) = 3x + 5$ is 3. This is not surprising; lines are characterized by being the *only* functions with a *constant rate of change.* That rate of change is called the *slope* of the line. Since their rates of change are constant, their *instantaneous* rates of change are always the same; they are all the slope.

So given a line $f(x) = ax + b$, the derivative at any point x will be a; that is, $f'(x) = a$.

It is now easy to see that the tangent line to the graph of f at $x = 1$ is just f, with the same being true for $x = 7$.

We often desire to find the tangent line to the graph of a function without knowing the actual derivative of the function. In these cases, the best we may be able to do is approximate the tangent line. We demonstrate this in the next example.

Example 1.1.4 Numerical Approximation of the Tangent Line

Approximate the equation of the tangent line to the graph of $f(x) = \sin x$ at $x = 0$.

Solution In order to find the equation of the tangent line, we need a slope and a point. The point is given to us: $(0, \sin 0) = (0, 0)$. To compute the slope, we need the derivative. This is where we will make an approximation. Recall that

$$f'(0) \approx \frac{\sin(0 + h) - \sin 0}{h}$$

for a small value of h. We choose (somewhat arbitrarily) to let $h = 0.1$. Thus

$$f'(0) \approx \frac{\sin(0.1) - \sin 0}{0.1} \approx 0.9983.$$

Thus our approximation of the equation of the tangent line is $y = 0.9983(x - 0) + 0 = 0.9983x$; it is graphed in Figure 1.1.5. The graph seems to imply the approximation is rather good.

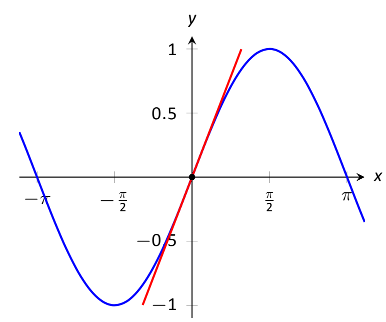


Figure 1.1.5: $f(x) = \sin x$ graphed with an approximation to its tangent line at $x = 0$.

Recall from Section 1.3 that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, meaning for values of x near 0, $\sin x \approx x$. Since the slope of the line $y = x$ is 1 at $x = 0$, it should seem reasonable that "the slope of $f(x) = \sin x$" is near 1 at $x = 0$. In fact, since we *approximated* the value of the slope to be 0.9983, we might guess the *actual value* is 1. We'll come back to this later.

Consider again Example 1.1.1. To find the derivative of f at $x = 1$, we needed to evaluate a limit. To find the derivative of f at $x = 3$, we needed to again evaluate a limit. We have this process:

Notes:

input specific number c $\longrightarrow$ $\boxed{\text{do something to } f \text{ and } c}$ $\longrightarrow$ return number $f'(c)$

This process describes a *function*; given one input (the value of c), we return exactly one output (the value of $f'(c)$). The "do something" box is where the tedious work (taking limits) of this function occurs.

Instead of applying this function repeatedly for different values of c, let us apply it just once to the variable x. We then take a limit just once. The process now looks like:

input variable x $\longrightarrow$ $\boxed{\text{do something to } f \text{ and } x}$ $\longrightarrow$ return function $f'(x)$

The output is the "derivative function," $f'(x)$. The $f'(x)$ function will take a number c as input and return the derivative of f at c. This calls for a definition.

Definition 1.1.4 Derivative Function

Let f be a differentiable function on an open interval I. The function

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

is **the derivative of** f.

Notation:
Let $y = f(x)$. The following notations all represent the derivative of f:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f) = \frac{d}{dx}(y).$$

Important: The notation $\dfrac{dy}{dx}$ is one symbol; it is **not** the fraction "dy/dx". The notation, while somewhat confusing at first, was chosen with care. A fraction–looking symbol was chosen because the derivative has many fraction–like properties. Among other places, we see these properties at work when we talk about the units of the derivative, when we discuss the Chain Rule, and when we learn about integration (topics that appear in later sections and chapters).

Examples will help us understand this definition.

Example 1.1.5 Finding the derivative of a function
Let $f(x) = 3x^2 + 5x - 7$ as in Example 1.1.1. Find $f'(x)$.

Notes:

Solution We apply Definition 1.1.4.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{3(x+h)^2+5(x+h)-7-(3x^2+5x-7)}{h}\\ &= \lim_{h\to 0}\frac{3h^2+6xh+5h}{h}\\ &= \lim_{h\to 0}3h+6x+5\\ &= 6x+5 \end{aligned}$$

So $f'(x) = 6x+5$. Recall earlier we found that $f'(1) = 11$ and $f'(3) = 23$. Note our new computation of $f'(x)$ affirm these facts.

Example 1.1.6 **Finding the derivative of a function**

Let $f(x) = \dfrac{1}{x+1}$. Find $f'(x)$.

Solution We apply Definition 1.1.4.

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{x+h+1}-\frac{1}{x+1}}{h} \end{aligned}$$

Now find common denominator then subtract; pull $1/h$ out front to facilitate reading.

$$\begin{aligned} &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x+1}{(x+1)(x+h+1)}-\frac{x+h+1}{(x+1)(x+h+1)}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}\right)\\ &= \lim_{h\to 0}\frac{1}{h}\cdot\left(\frac{-h}{(x+1)(x+h+1)}\right)\\ &= \lim_{h\to 0}\frac{-1}{(x+1)(x+h+1)}\\ &= \frac{-1}{(x+1)(x+1)}\\ &= \frac{-1}{(x+1)^2}. \end{aligned}$$

Notes:

So $f'(x) = \dfrac{-1}{(x+1)^2}$. To practice using our notation, we could also state

$$\frac{d}{dx}\left(\frac{1}{x+1}\right) = \frac{-1}{(x+1)^2}.$$

Example 1.1.7 Finding the derivative of a function

Find the derivative of $f(x) = \sin x$.

SOLUTION Before applying Definition 1.1.4, note that once this is found, we can find the actual tangent line to $f(x) = \sin x$ at $x = 0$, whereas we settled for an approximation in Example 1.1.4.

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} && \left(\begin{array}{c}\text{Use trig identity}\\ \sin(x+h) = \sin x\cos h + \cos x \sin h\end{array}\right)\\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x\sin h - \sin x}{h} && \text{(regroup)}\\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x\sin h}{h} && \text{(split into two fractions)}\\
&= \lim_{h\to 0} \left(\frac{\sin x(\cos h - 1)}{h} + \frac{\cos x\sin h}{h}\right) && \left(\text{use } \lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \text{ and } \lim_{h\to 0}\frac{\sin h}{h} = 1\right)\\
&= \sin x \cdot 0 + \cos x \cdot 1\\
&= \cos x\,!
\end{aligned}$$

We have found that when $f(x) = \sin x$, $f'(x) = \cos x$. This should be somewhat surprising; the result of a tedious limit process and the sine function is a nice function. Then again, perhaps this is not entirely surprising. The sine function is periodic – it repeats itself on regular intervals. Therefore its rate of change also repeats itself on the same regular intervals. We should have known the derivative would be periodic; we now know exactly which periodic function it is.

Thinking back to Example 1.1.4, we can find the slope of the tangent line to $f(x) = \sin x$ at $x = 0$ using our derivative. We approximated the slope as 0.9983; we now know the slope is *exactly* $\cos 0 = 1$.

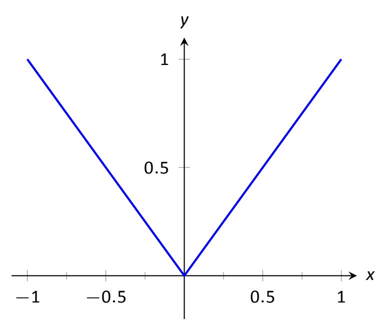


Figure 1.1.6: The absolute value function, $f(x) = |x|$. Notice how the slope of the lines (and hence the tangent lines) abruptly changes at $x = 0$.

Example 1.1.8 Finding the derivative of a piecewise defined function

Find the derivative of the absolute value function,

$$f(x) = |x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}.$$

See Figure 1.1.6.

SOLUTION We need to evaluate $\lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$. As f is piecewise–defined, we need to consider separately the limits when $x < 0$ and when $x > 0$.

Notes:

When $x < 0$:

$$\begin{aligned}\frac{d}{dx}\big(-x\big) &= \lim_{h\to 0}\frac{-(x+h)-(-x)}{h}\\ &= \lim_{h\to 0}\frac{-h}{h}\\ &= \lim_{h\to 0} -1\\ &= -1.\end{aligned}$$

When $x > 0$, a similar computation shows that $\frac{d}{dx}\big(x\big) = 1$.

We need to also find the derivative at $x = 0$. By the definition of the derivative at a point, we have

$$f'(0) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}.$$

Since $x = 0$ is the point where our function's definition switches from one piece to the other, we need to consider left and right-hand limits. Consider the following, where we compute the left and right hand limits side by side.

$$\begin{array}{r|r} \displaystyle\lim_{h\to 0^-}\frac{f(0+h)-f(0)}{h} = & \displaystyle\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h} = \\ \displaystyle\lim_{h\to 0^-}\frac{-h-0}{h} = & \displaystyle\lim_{h\to 0^+}\frac{h-0}{h} = \\ \displaystyle\lim_{h\to 0^-} -1 = -1 & \displaystyle\lim_{h\to 0^+} 1 = 1 \end{array}$$

The last lines of each column tell the story: the left and right hand limits are not equal. Therefore the limit does not exist at 0, and f is not differentiable at 0. So we have

$$f'(x) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}.$$

At $x = 0$, $f'(x)$ does not exist; there is a jump discontinuity at 0; see Figure 1.1.7. So $f(x) = |x|$ is differentiable everywhere except at 0.

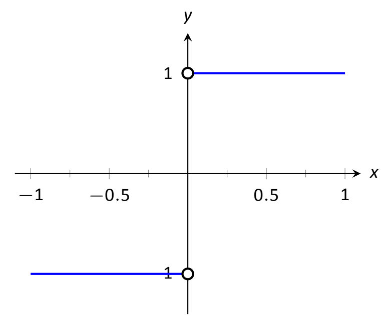


Figure 1.1.7: A graph of the derivative of $f(x) = |x|$.

The point of non-differentiability came where the piecewise defined function switched from one piece to the other. Our next example shows that this does not always cause trouble.

Example 1.1.9 Finding the derivative of a piecewise defined function

Find the derivative of $f(x)$, where $f(x) = \begin{cases} \sin x & x \leq \pi/2 \\ 1 & x > \pi/2 \end{cases}$. See Figure 1.1.8.

Notes:

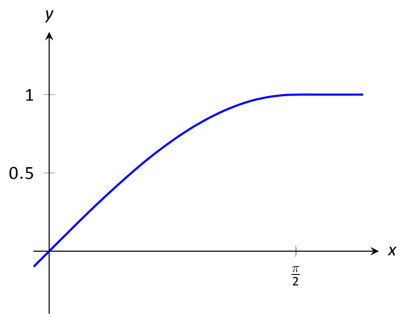


Figure 1.1.8: A graph of $f(x)$ as defined in Example 1.1.9.

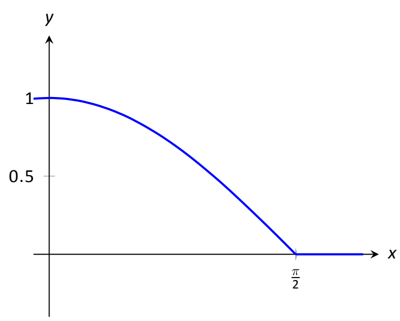


Figure 1.1.9: A graph of $f'(x)$ in Example 1.1.9.

Solution Using Example 1.1.7, we know that when $x < \pi/2$, $f'(x) = \cos x$. It is easy to verify that when $x > \pi/2$, $f'(x) = 0$; consider:

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{1-1}{h} = \lim_{h\to 0} 0 = 0.$$

So far we have

$$f'(x) = \begin{cases} \cos x & x < \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

We still need to find $f'(\pi/2)$. Notice at $x = \pi/2$ that both pieces of f' are 0, meaning we can state that $f'(\pi/2) = 0$.

Being more rigorous, we can again evaluate the difference quotient limit at $x = \pi/2$, utilizing again left and right–hand limits:

$$\begin{aligned} \lim_{h\to 0^-} \frac{f(\pi/2+h) - f(\pi/2)}{h} &= \\ \lim_{h\to 0^-} \frac{\sin(\pi/2+h) - \sin(\pi/2)}{h} &= \\ \lim_{h\to 0^-} \frac{\sin(\frac{\pi}{2})\cos(h) + \sin(h)\cos(\frac{\pi}{2}) - \sin(\frac{\pi}{2})}{h} &= \\ \lim_{h\to 0^-} \frac{1\cdot\cos(h) + \sin(h)\cdot 0 - 1}{h} &= \\ 0. & \end{aligned}$$

$$\begin{aligned} \lim_{h\to 0^+} \frac{f(\pi/2+h) - f(\pi/2)}{h} &= \\ \lim_{h\to 0^+} \frac{1-1}{h} &= \\ \lim_{h\to 0^+} \frac{0}{h} &= \\ 0. & \end{aligned}$$

Since both the left and right hand limits are 0 at $x = \pi/2$, the limit exists and $f'(\pi/2)$ exists (and is 0). Therefore we can fully write f' as

$$f'(x) = \begin{cases} \cos x & x \leq \pi/2 \\ 0 & x > \pi/2 \end{cases}.$$

See Figure 1.1.9 for a graph of this function.

Recall we pseudo–defined a continuous function as one in which we could sketch its graph without lifting our pencil. We can give a pseudo–definition for differentiability as well: it is a continuous function that does not have any "sharp corners." One such sharp corner is shown in Figure 1.1.6. Even though the function f in Example 1.1.9 is piecewise–defined, the transition is "smooth" hence it is differentiable. Note how in the graph of f in Figure 1.1.8 it is difficult to tell when f switches from one piece to the other; there is no "corner."

This section defined the derivative; in some sense, it answers the question of "What *is* the derivative?" The next section addresses the question "What does the derivative *mean*?"

Notes:

Exercises 1.1

Terms and Concepts

02 01 ex 01 1. T/F: Let f be a position function. The average rate of change on $[a, b]$ is the slope of the line through the points $(a, f(a))$ and $(b, f(b))$.

02 01 ex 02 2. T/F: The definition of the derivative of a function at a point involves taking a limit.

02 01 ex 03 3. In your own words, explain the difference between the average rate of change and instantaneous rate of change.

02 01 ex 04 4. In your own words, explain the difference between Definitions 1.1.1 and 1.1.4.

02 01 ex 05 5. Let $y = f(x)$. Give three different notations equivalent to "$f'(x)$."

02 01 ex 35 6. If two lines are perpendicular, what is true of their slopes?

Problems

02 01 exset 02 **In Exercises 7 – 14, use the definition of the derivative to compute the derivative of the given function.**

02 01 ex 10 7. $f(x) = 6$

02 01 ex 11 8. $f(x) = 2x$

02 01 ex 12 9. $f(t) = 4 - 3t$

02 01 ex 13 10. $g(x) = x^2$

02 01 ex 36 11. $h(x) = x^3$

02 01 ex 14 12. $f(x) = 3x^2 - x + 4$

02 01 ex 15 13. $r(x) = \dfrac{1}{x}$

02 01 ex 16 14. $r(s) = \dfrac{1}{s-2}$

02 01 exset 03 **In Exercises 15 – 22, a function and an x–value c are given. (Note: these functions are the same as those given in Exercises 7 through 14.)**

(a) Give the equation of the tangent line at $x = c$.

(b) Give the equation of the normal line at $x = c$.

02 01 ex 17 15. $f(x) = 6$, at $x = -2$.

02 01 ex 18 16. $f(x) = 2x$, at $x = 3$.

02 01 ex 19 17. $f(x) = 4 - 3x$, at $x = 7$.

02 01 ex 20 18. $g(x) = x^2$, at $x = 2$.

02 01 ex 37 19. $h(x) = x^3$, at $x = 4$.

02 01 ex 21 20. $f(x) = 3x^2 - x + 4$, at $x = -1$.

02 01 ex 22 21. $r(x) = \dfrac{1}{x}$, at $x = -2$.

02 01 ex 23 22. $r(x) = \dfrac{1}{x-2}$, at $x = 3$.

02 01 exset 01 **In Exercises 23 – 26, a function f and an x–value a are given. Approximate the equation of the tangent line to the graph of f at $x = a$ by numerically approximating $f'(a)$, using $h = 0.1$.**

02 01 ex 06 23. $f(x) = x^2 + 2x + 1$, $x = 3$

02 01 ex 07 24. $f(x) = \dfrac{10}{x+1}$, $x = 9$

02 01 ex 08 25. $f(x) = e^x$, $x = 2$

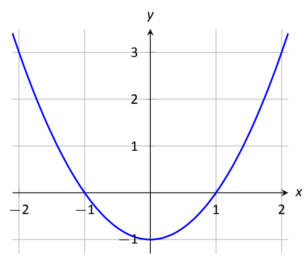
02 01 ex 09 26. $f(x) = \cos x$, $x = 0$

02 01 ex 24 27. The graph of $f(x) = x^2 - 1$ is shown.

(a) Use the graph to approximate the slope of the tangent line to f at the following points: $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

(b) Using the definition, find $f'(x)$.

(c) Find the slope of the tangent line at the points $(-1, 0)$, $(0, -1)$ and $(2, 3)$.

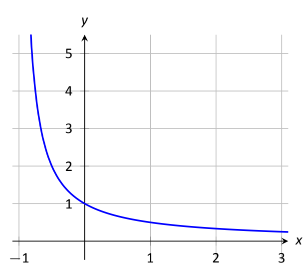


02 01 ex 25 28. The graph of $f(x) = \dfrac{1}{x+1}$ is shown.

(a) Use the graph to approximate the slope of the tangent line to f at the following points: $(0, 1)$ and $(1, 0.5)$.

(b) Using the definition, find $f'(x)$.

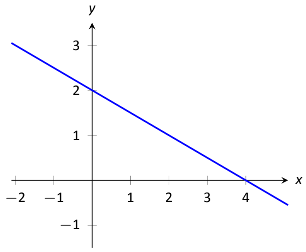
(c) Find the slope of the tangent line at the points $(0, 1)$ and $(1, 0.5)$.



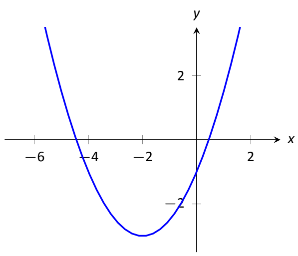
02 01 exset 04

In Exercises 29 – 32, a graph of a function $f(x)$ is given. Using the graph, sketch $f'(x)$.

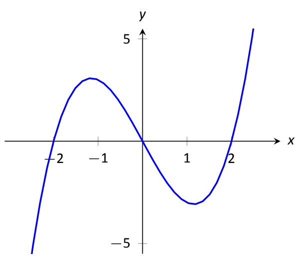
02 01 ex 26

29. 

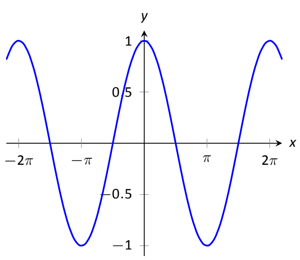
02 01 ex 27

30. 

02 01 ex 28

31. 

02 01 ex 29

32. 

02 01 ex 30

33. Using the graph of $g(x)$ below, answer the following questions.

(a) Where is $g(x) > 0$?

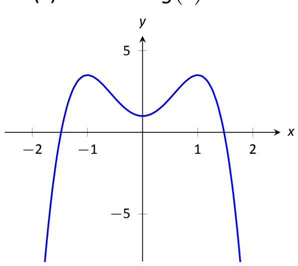
(b) Where is $g(x) < 0$?

(c) Where is $g(x) = 0$?

(d) Where is $g'(x) < 0$?

(e) Where is $g'(x) > 0$?

(f) Where is $g'(x) = 0$?



Review

02 01 ex 31

34. Approximate $\lim\limits_{x\to 5} \dfrac{x^2+2x-35}{x^2-10.5x+27.5}$.

02 01 ex 32

35. Use the Bisection Method to approximate, accurate to two decimal places, the root of $g(x) = x^3 + x^2 + x - 1$ on $[0.5, 0.6]$.

02 01 ex 33

36. Give intervals on which each of the following functions are continuous.

(a) $\dfrac{1}{e^x+1}$

(b) $\dfrac{1}{x^2-1}$

(c) $\sqrt{5-x}$

(d) $\sqrt{5-x^2}$

02 01 ex 34

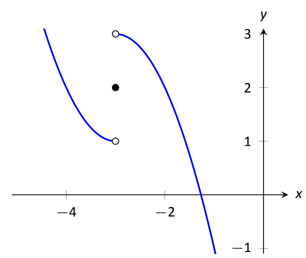
37. Use the graph of $f(x)$ provided to answer the following.

(a) $\lim\limits_{x\to -3^-} f(x) = ?$

(b) $\lim\limits_{x\to -3^+} f(x) = ?$

(c) $\lim\limits_{x\to -3} f(x) = ?$

(d) Where is f continuous?



1.2 Interpretations of the Derivative

The previous section defined the derivative of a function and gave examples of how to compute it using its definition (i.e., using limits). The section also started with a brief motivation for this definition, that is, finding the instantaneous velocity of a falling object given its position function. The next section will give us more accessible tools for computing the derivative, tools that are easier to use than repeated use of limits.

This section falls in between the "What is the definition of the derivative?" and "How do I compute the derivative?" sections. Here we are concerned with "What does the derivative mean?", or perhaps, when read with the right emphasis, "What *is* the derivative?" We offer two interconnected interpretations of the derivative, hopefully explaining why we care about it and why it is worthy of study.

Interpretation of the Derivative #1: Instantaneous Rate of Change

The previous section started with an example of using the position of an object (in this case, a falling amusement–park rider) to find the object's velocity. This type of example is often used when introducing the derivative because we tend to readily recognize that velocity is the *instantaneous rate of change of position.* In general, if f is a function of x, then $f'(x)$ measures the instantaneous rate of change of f with respect to x. Put another way, the derivative answers "When x changes, at what rate does f change?" Thinking back to the amusement–park ride, we asked "When time changed, at what rate did the height change?" and found the answer to be "By -64 feet per second."

Now imagine driving a car and looking at the speedometer, which reads "60 mph." Five minutes later, you wonder how far you have traveled. Certainly, lots of things could have happened in those 5 minutes; you could have intentionally sped up significantly, you might have come to a complete stop, you might have slowed to 20 mph as you passed through construction. But suppose that you know, as the driver, none of these things happened. You know you maintained a fairly consistent speed over those 5 minutes. What is a good approximation of the distance traveled?

One could argue the *only* good approximation, given the information provided, would be based on "distance = rate $\times$ time." In this case, we assume a constant rate of 60 mph with a time of $5/60$ hours. Hence we would approximate the distance traveled as 5 miles.

Referring back to the falling amusement–park ride, knowing that at $t = 2$ the velocity was -64 ft/s, we could reasonably assume that 1 second later the rid-

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Notes:

ers' height would have dropped by about 64 feet. Knowing that the riders were *accelerating* as they fell would inform us that this is an *under–approximation*. If all we knew was that $f(2) = 86$ and $f'(2) = -64$, we'd know that we'd have to stop the riders quickly otherwise they would hit the ground.

Units of the Derivative

It is useful to recognize the *units* of the derivative function. If y is a function of x, i.e., $y = f(x)$ for some function f, and y is measured in feet and x in seconds, then the units of $y' = f'$ are "feet per second," commonly written as "ft/s." In general, if y is measured in units P and x is measured in units Q, then y' will be measured in units "P per Q", or "P/Q." Here we see the fraction–like behavior of the derivative in the notation:

$$\text{the units of } \quad \frac{dy}{dx} \quad \text{are} \quad \frac{\text{units of } y}{\text{units of } x}.$$

Example 1.2.1 The meaning of the derivative: World Population

Let $P(t)$ represent the world population t minutes after 12:00 a.m., January 1, 2012. It is fairly accurate to say that $P(0) = 7,028,734,178$ (`www.prb.org`). It is also fairly accurate to state that $P'(0) = 156$; that is, at midnight on January 1, 2012, the population of the world was growing by about 156 *people per minute* (note the units). Twenty days later (or, 28,800 minutes later) we could reasonably assume the population grew by about $28,800 \cdot 156 = 4,492,800$ people.

Example 1.2.2 The meaning of the derivative: Manufacturing

The term *widget* is an economic term for a generic unit of manufacturing output. Suppose a company produces widgets and knows that the market supports a price of \$10 per widget. Let $P(n)$ give the profit, in dollars, earned by manufacturing and selling n widgets. The company likely cannot make a (positive) profit making just one widget; the start–up costs will likely exceed \$10. Mathematically, we would write this as $P(1) < 0$.

What do $P(1000) = 500$ and $P'(1000) = 0.25$ mean? Approximate $P(1100)$.

Solution The equation $P(1000) = 500$ means that selling 1,000 widgets returns a profit of \$500. We interpret $P'(1000) = 0.25$ as meaning that the profit is increasing at rate of \$0.25 per widget (the units are "dollars per widget.") Since we have no other information to use, our best approximation for $P(1100)$ is:

$$P(1100) \approx P(1000) + P'(1000) \times 100 = \$500 + 100 \cdot 0.25 = \$525.$$

We approximate that selling 1,100 widgets returns a profit of \$525.

Notes:

The previous examples made use of an important approximation tool that we first used in our previous "driving a car at 60 mph" example at the beginning of this section. Five minutes after looking at the speedometer, our best approximation for distance traveled assumed the rate of change was constant. In Examples 1.2.1 and 1.2.2 we made similar approximations. We were given rate of change information which we used to approximate total change. Notationally, we would say that

$$f(c+h) \approx f(c) + f'(c) \cdot h.$$

This approximation is best when h is "small." "Small" is a relative term; when dealing with the world population, $h = 22$ days = 28,800 minutes is small in comparison to years. When manufacturing widgets, 100 widgets is small when one plans to manufacture thousands.

The Derivative and Motion

One of the most fundamental applications of the derivative is the study of motion. Let $s(t)$ be a position function, where t is time and $s(t)$ is distance. For instance, s could measure the height of a projectile or the distance an object has traveled.

Let's let $s(t)$ measure the distance traveled, in feet, of an object after t seconds of travel. Then $s'(t)$ has units "feet per second," and $s'(t)$ measures the *instantaneous rate of distance change* – it measures **velocity**.

Now consider $v(t)$, a velocity function. That is, at time t, $v(t)$ gives the velocity of an object. The derivative of v, $v'(t)$, gives the *instantaneous rate of velocity change* – **acceleration**. (We often think of acceleration in terms of cars: a car may "go from 0 to 60 in 4.8 seconds." This is an *average* acceleration, a measurement of how quickly the velocity changed.) If velocity is measured in feet per second, and time is measured in seconds, then the units of acceleration (i.e., the units of $v'(t)$) are "feet per second per second," or (ft/s)/s. We often shorten this to "feet per second squared," or ft/s^2, but this tends to obscure the meaning of the units.

Perhaps the most well known acceleration is that of gravity. In this text, we use $g = 32\text{ft/s}^2$ or $g = 9.8\text{m/s}^2$. What do these numbers mean?

A constant acceleration of 32(ft/s)/s means that the velocity changes by 32ft/s each second. For instance, let $v(t)$ measures the velocity of a ball thrown straight up into the air, where v has units ft/s and t is measured in seconds. The ball will have a positive velocity while traveling upwards and a negative velocity while falling down. The acceleration is thus -32ft/s^2. If $v(1) = 20\text{ft/s}$, then when $t = 2$, the velocity will have decreased by 32ft/s; that is, $v(2) = -12\text{ft/s}$. We can continue: $v(3) = -44\text{ft/s}$, and we can also figure that $v(0) = 52\text{ft/s}$.

These ideas are so important we write them out as a Key Idea.

Notes:

Key Idea 1.2.1 The Derivative and Motion

1. Let $s(t)$ be the position function of an object. Then $s'(t)$ is the velocity function of the object.
2. Let $v(t)$ be the velocity function of an object. Then $v'(t)$ is the acceleration function of the object.

We now consider the second interpretation of the derivative given in this section. This interpretation is not independent from the first by any means; many of the same concepts will be stressed, just from a slightly different perspective.

Interpretation of the Derivative #2: The Slope of the Tangent Line

Given a function $y = f(x)$, the difference quotient $\dfrac{f(c+h)-f(c)}{h}$ gives a change in y values divided by a change in x values; i.e., it is a measure of the "rise over run," or "slope," of the line that goes through two points on the graph of f: $\big(c, f(c)\big)$ and $\big(c+h, f(c+h)\big)$. As h shrinks to 0, these two points come close together; in the limit we find $f'(c)$, the slope of a special line called the tangent line that intersects f only once near $x = c$.

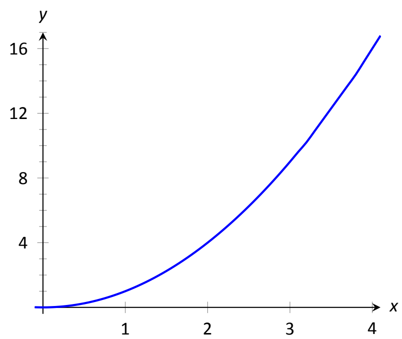


Figure 1.2.1: A graph of $f(x) = x^2$.

Lines have a constant rate of change, their slope. Nonlinear functions do not have a constant rate of change, but we can measure their *instantaneous rate of change* at a given x value c by computing $f'(c)$. We can get an idea of how f is behaving by looking at the slopes of its tangent lines. We explore this idea in the following example.

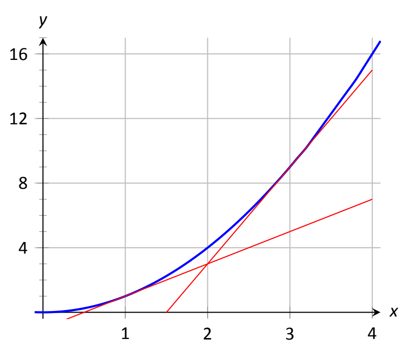


Figure 1.2.2: A graph of $f(x) = x^2$ and tangent lines.

Example 1.2.3 Understanding the derivative: the rate of change

Consider $f(x) = x^2$ as shown in Figure 1.2.1. It is clear that at $x = 3$ the function is growing faster than at $x = 1$, as it is steeper at $x = 3$. How much faster is it growing?

Solution We can answer this directly after the following section, where we learn to quickly compute derivatives. For now, we will answer graphically, by considering the slopes of the respective tangent lines.

With practice, one can fairly effectively sketch tangent lines to a curve at a particular point. In Figure 1.2.2, we have sketched the tangent lines to f at $x = 1$ and $x = 3$, along with a grid to help us measure the slopes of these lines. At

Notes:

$x = 1$, the slope is 2; at $x = 3$, the slope is 6. Thus we can say not only is f growing faster at $x = 3$ than at $x = 1$, it is growing *three times as fast*.

Example 1.2.4 Understanding the graph of the derivative

Consider the graph of $f(x)$ and its derivative, $f'(x)$, in Figure 1.2.3(a). Use these graphs to find the slopes of the tangent lines to the graph of f at $x = 1$, $x = 2$, and $x = 3$.

Solution To find the appropriate slopes of tangent lines to the graph of f, we need to look at the corresponding values of f'.

The slope of the tangent line to f at $x = 1$ is $f'(1)$; this looks to be about -1.

The slope of the tangent line to f at $x = 2$ is $f'(2)$; this looks to be about 4.

The slope of the tangent line to f at $x = 3$ is $f'(3)$; this looks to be about 3.

Using these slopes, the tangent lines to f are sketched in Figure 1.2.3(b). Included on the graph of f' in this figure are filled circles where $x = 1$, $x = 2$ and $x = 3$ to help better visualize the y value of f' at those points.

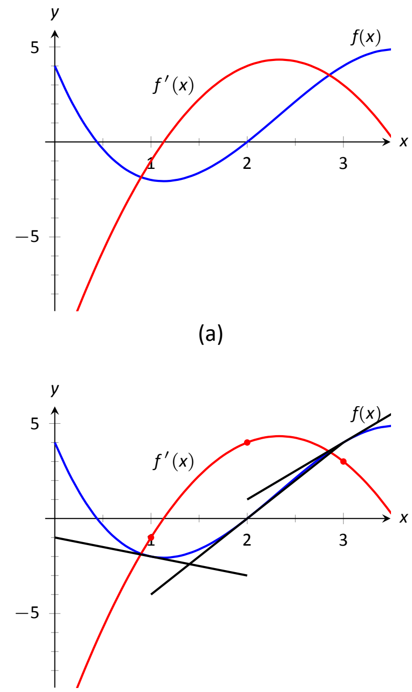


Figure 1.2.3: Graphs of f and f' in Example 1.2.4, along with tangent lines in (b).

Example 1.2.5 Approximation with the derivative

Consider again the graph of $f(x)$ and its derivative $f'(x)$ in Example 1.2.4. Use the tangent line to f at $x = 3$ to approximate the value of $f(3.1)$.

Solution Figure 1.2.4 shows the graph of f along with its tangent line, zoomed in at $x = 3$. Notice that near $x = 3$, the tangent line makes an excellent approximation of f. Since lines are easy to deal with, often it works well to approximate a function with its tangent line. (This is especially true when you don't actually know much about the function at hand, as we don't in this example.)

While the tangent line to f was drawn in Example 1.2.4, it was not explicitly computed. Recall that the tangent line to f at $x = c$ is $y = f'(c)(x - c) + f(c)$. While f is not explicitly given, by the graph it looks like $f(3) = 4$. Recalling that $f'(3) = 3$, we can compute the tangent line to be approximately $y = 3(x-3)+4$. It is often useful to leave the tangent line in point–slope form.

To use the tangent line to approximate $f(3.1)$, we simply evaluate y at 3.1 instead of f.

$$f(3.1) \approx y(3.1) = 3(3.1 - 3) + 4 = .1 * 3 + 4 = 4.3.$$

We approximate $f(3.1) \approx 4.3$.

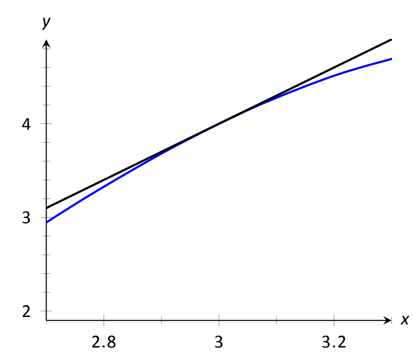


Figure 1.2.4: Zooming in on f and its tangent line at $x = 3$ for the function given in Examples 1.2.4 and 1.2.5.

To demonstrate the accuracy of the tangent line approximation, we now state that in Example 1.2.5, $f(x) = -x^3 + 7x^2 - 12x + 4$. We can evaluate $f(3.1) = 4.279$. Had we known f all along, certainly we could have just made this computation. In reality, we often only know two things:

Notes:

1. what $f(c)$ is, for some value of c, and
2. what $f'(c)$ is.

For instance, we can easily observe the location of an object and its instantaneous velocity at a particular point in time. We do not have a "function f" for the location, just an observation. This is enough to create an approximating function for f.

This last example has a direct connection to our approximation method explained above after Example 1.2.2. We stated there that

$$f(c+h) \approx f(c) + f'(c)\cdot h.$$

If we know $f(c)$ and $f'(c)$ for some value $x = c$, then computing the tangent line at $(c, f(c))$ is easy: $y(x) = f'(c)(x-c) + f(c)$. In Example 1.2.5, we used the tangent line to approximate a value of f. Let's use the tangent line at $x = c$ to approximate a value of f near $x = c$; i.e., compute $y(c+h)$ to approximate $f(c+h)$, assuming again that h is "small." Note:

$$y(c+h) = f'(c)\big((c+h)-c\big) + f(c) = f'(c)\cdot h + f(c).$$

This is the exact same approximation method used above! Not only does it make intuitive sense, as explained above, it makes analytical sense, as this approximation method is simply using a tangent line to approximate a function's value.

The importance of understanding the derivative cannot be understated. When f is a function of x, $f'(x)$ measures the instantaneous rate of change of f with respect to x and gives the slope of the tangent line to f at x.

Notes:

Exercises 1.2

Terms and Concepts

02 02 exset 01

02 02 ex 01 1. What is the instantaneous rate of change of position called?

02 02 ex 02 2. Given a function $y = f(x)$, in your own words describe how to find the units of $f'(x)$.

02 02 ex 03 3. What functions have a constant rate of change?

Problems

02 02 ex 04 4. Given $f(5) = 10$ and $f'(5) = 2$, approximate $f(6)$.

02 02 ex 05 5. Given $P(100) = -67$ and $P'(100) = 5$, approximate $P(110)$.

02 02 ex 06 6. Given $z(25) = 187$ and $z'(25) = 17$, approximate $z(20)$.

02 02 ex 07 7. Knowing $f(10) = 25$ and $f'(10) = 5$ and the methods described in this section, which approximation is likely to be most accurate: $f(10.1)$, $f(11)$, or $f(20)$? Explain your reasoning.

02 02 ex 08 8. Given $f(7) = 26$ and $f(8) = 22$, approximate $f'(7)$.

02 02 ex 09 9. Given $H(0) = 17$ and $H(2) = 29$, approximate $H'(2)$.

02 02 ex 10 10. Let $V(x)$ measure the volume, in decibels, measured inside a restaurant with x customers. What are the units of $V'(x)$?

02 02 ex 11 11. Let $v(t)$ measure the velocity, in ft/s, of a car moving in a straight line t seconds after starting. What are the units of $v'(t)$?

02 02 ex 12 12. The height H, in feet, of a river is recorded t hours after midnight, April 1. What are the units of $H'(t)$?

02 02 ex 13 13. P is the profit, in thousands of dollars, of producing and selling c cars.

(a) What are the units of $P'(c)$?

(b) What is likely true of $P(0)$?

02 02 ex 14 14. T is the temperature in degrees Fahrenheit, h hours after midnight on July 4 in Sidney, NE.

(a) What are the units of $T'(h)$?

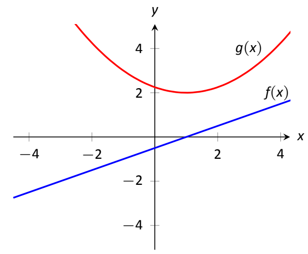
(b) Is $T'(8)$ likely greater than or less than 0? Why?

(c) Is $T(8)$ likely greater than or less than 0? Why?

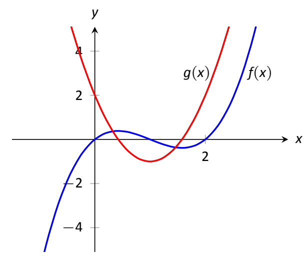
02 02 exset 02

In Exercises 15 – 18, graphs of functions $f(x)$ and $g(x)$ are given. Identify which function is the derivative of the other.

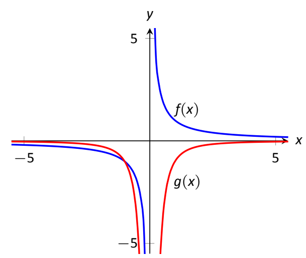
02 02 ex 15

15. 

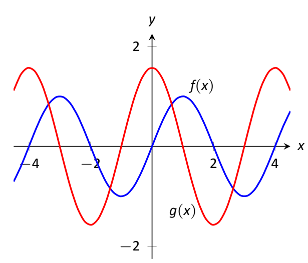
02 02 ex 16

16. 

02 02 ex 17

17. 

02 02 ex 18

18. 

Review

02 02 exset 03

In Exercises 19 – 20, use the definition to compute the derivatives of the following functions.

02 02 ex 19 19. $f(x) = 5x^2$

02 02 ex 20 20. $f(x) = (x-2)^3$

In Exercises 21 – 22, numerically approximate the value of $f'(x)$ at the indicated x value.

02 02 ex 21 21. $f(x) = \cos x$ at $x = \pi$.

02 02 ex 22 22. $f(x) = \sqrt{x}$ at $x = 9$.

1.3 Basic Differentiation Rules

The derivative is a powerful tool but is admittedly awkward given its reliance on limits. Fortunately, one thing mathematicians are good at is *abstraction.* For instance, instead of continually finding derivatives at a point, we abstracted and found the derivative function.

Let's practice abstraction on linear functions, $y = mx + b$. What is y'? Without limits, recognize that linear function are characterized by being functions with a constant rate of change (the slope). The derivative, y', gives the instantaneous rate of change; with a linear function, this is constant, m. Thus $y' = m$.

Let's abstract once more. Let's find the derivative of the general quadratic function, $f(x) = ax^2 + bx + c$. Using the definition of the derivative, we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{a(x+h)^2 + b(x+h) + c - (ax^2 + bx + c)}{h} \\ &= \lim_{h\to 0} \frac{ah^2 + 2ahx + bh}{h} \\ &= \lim_{h\to 0} ah + 2ax + b \\ &= 2ax + b. \end{aligned}$$

So if $y = 6x^2 + 11x - 13$, we can immediately compute $y' = 12x + 11$.

In this section (and in some sections to follow) we will learn some of what mathematicians have already discovered about the derivatives of certain functions and how derivatives interact with arithmetic operations. We start with a theorem.

Theorem 1.3.1 Derivatives of Common Functions

1. **Constant Rule:**
   $\frac{d}{dx}\big(c\big) = 0$, where c is a constant.
2. **Power Rule:**
   $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$, where n is an integer, $n > 0$.
5. $\frac{d}{dx}(\sin x) = \cos x$
6. $\frac{d}{dx}(\cos x) = -\sin x$
7. $\frac{d}{dx}(e^x) = e^x$
8. $\frac{d}{dx}(\ln x) = \frac{1}{x}$

Notes:

This theorem starts by stating an intuitive fact: constant functions have no rate of change as they are *constant*. Therefore their derivative is 0 (they change at the rate of 0). The theorem then states some fairly amazing things. The Power Rule states that the derivatives of Power Functions (of the form $y = x^n$) are very straightforward: multiply by the power, then subtract 1 from the power. We see something incredible about the function $y = e^x$: it is its own derivative. We also see a new connection between the sine and cosine functions.

One special case of the Power Rule is when $n = 1$, i.e., when $f(x) = x$. What is $f'(x)$? According to the Power Rule,

$$f'(x) = \frac{d}{dx}\big(x\big) = \frac{d}{dx}\big(x^1\big) = 1 \cdot x^0 = 1.$$

In words, we are asking "At what rate does f change with respect to x?" Since f *is* x, we are asking "At what rate does x change with respect to x?" The answer is: 1. They change at the same rate.

Let's practice using this theorem.

Example 1.3.1 Using Theorem 1.3.1 to find, and use, derivatives

Let $f(x) = x^3$.

1. Find $f'(x)$.
2. Find the equation of the line tangent to the graph of f at $x = -1$.
3. Use the tangent line to approximate $(-1.1)^3$.
4. Sketch f, f' and the found tangent line on the same axis.

Solution

1. The Power Rule states that if $f(x) = x^3$, then $f'(x) = 3x^2$.
2. To find the equation of the line tangent to the graph of f at $x = -1$, we need a point and the slope. The point is $(-1, f(-1)) = (-1, -1)$. The slope is $f'(-1) = 3$. Thus the tangent line has equation $y = 3(x-(-1)) + (-1) = 3x + 2$.
3. We can use the tangent line to approximate $(-1.1)^3$ as -1.1 is close to -1. We have
$$(-1.1)^3 \approx 3(-1.1) + 2 = -1.3.$$
We can easily find the actual answer; $(-1.1)^3 = -1.331$.
4. See Figure 1.3.1.

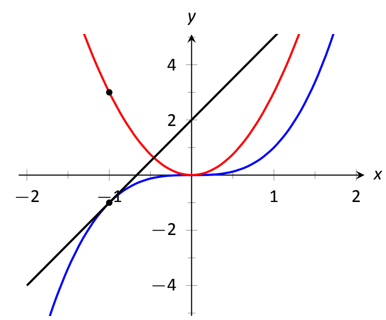


Figure 1.3.1: A graph of $f(x) = x^3$, along with its derivative $f'(x) = 3x^2$ and its tangent line at $x = -1$.

Notes:

Theorem 1.3.1 gives useful information, but we will need much more. For instance, using the theorem, we can easily find the derivative of $y = x^3$, but it does not tell how to compute the derivative of $y = 2x^3$, $y = x^3 + \sin x$ nor $y = x^3 \sin x$. The following theorem helps with the first two of these examples (the third is answered in the next section).

Theorem 1.3.2 Properties of the Derivative

Let f and g be differentiable on an open interval I and let c be a real number. Then:

1. **Sum/Difference Rule:**
$$\frac{d}{dx}\Big(f(x) \pm g(x)\Big) = \frac{d}{dx}\Big(f(x)\Big) \pm \frac{d}{dx}\Big(g(x)\Big) = f'(x) \pm g'(x)$$
2. **Constant Multiple Rule:**
$$\frac{d}{dx}\Big(c \cdot f(x)\Big) = c \cdot \frac{d}{dx}\Big(f(x)\Big) = c \cdot f'(x).$$

Theorem 1.3.2 allows us to find the derivatives of a wide variety of functions. It can be used in conjunction with the Power Rule to find the derivatives of any polynomial. Recall in Example 1.1.5 that we found, using the limit definition, the derivative of $f(x) = 3x^2 + 5x - 7$. We can now find its derivative without expressly using limits:

$$\begin{aligned}\frac{d}{dx}\Big(3x^2 + 5x + 7\Big) &= 3\frac{d}{dx}\Big(x^2\Big) + 5\frac{d}{dx}\Big(x\Big) + \frac{d}{dx}\Big(7\Big)\\ &= 3 \cdot 2x + 5 \cdot 1 + 0\\ &= 6x + 5.\end{aligned}$$

We were a bit pedantic here, showing every step. Normally we would do all the arithmetic and steps in our head and readily find $\frac{d}{dx}\Big(3x^2+5x+7\Big) = 6x+5$.

Example 1.3.2 Using the tangent line to approximate a function value

Let $f(x) = \sin x + 2x + 1$. Approximate $f(3)$ using an appropriate tangent line.

SOLUTION This problem is intentionally ambiguous; we are to *approximate* using an *appropriate* tangent line. How good of an approximation are we seeking? What does appropriate mean?

In the "real world," people solving problems deal with these issues all time. One must make a judgment using whatever seems reasonable. In this example, the actual answer is $f(3) = \sin 3 + 7$, where the real problem spot is $\sin 3$. What is $\sin 3$?

Notes:

Since 3 is close to π, we can assume $\sin 3 \approx \sin\pi = 0$. Thus one guess is $f(3) \approx 7$. Can we do better? Let's use a tangent line as instructed and examine the results; it seems best to find the tangent line at $x = \pi$.

Using Theorem 1.3.1 we find $f'(x) = \cos x + 2$. The slope of the tangent line is thus $f'(\pi) = \cos\pi + 2 = 1$. Also, $f(\pi) = 2\pi + 1 \approx 7.28$. So the tangent line to the graph of f at $x = \pi$ is $y = 1(x - \pi) + 2\pi + 1 = x + \pi + 1 \approx x + 4.14$. Evaluated at $x = 3$, our tangent line gives $y = 3 + 4.14 = 7.14$. Using the tangent line, our final approximation is that $f(3) \approx 7.14$.

Using a calculator, we get an answer accurate to 4 places after the decimal: $f(3) = 7.1411$. Our initial guess was 7; our tangent line approximation was more accurate, at 7.14.

The point is *not* "Here's a cool way to do some math without a calculator." Sure, that might be handy sometime, but your phone could probably give you the answer. Rather, the point is to say that tangent lines are a good way of approximating, and many scientists, engineers and mathematicians often face problems too hard to solve directly. So they approximate.

Higher Order Derivatives

The derivative of a function f is itself a function, therefore we can take its derivative. The following definition gives a name to this concept and introduces its notation.

Definition 1.3.1 Higher Order Derivatives

Let $y = f(x)$ be a differentiable function on I. The following are defined, provided the corresponding limits exist.

1. The **second derivative of** f is:
$$f''(x) = \frac{d}{dx}\Big(f'(x)\Big) = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y''.$$
2. The **third derivative of** f is:
$$f'''(x) = \frac{d}{dx}\Big(f''(x)\Big) = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d^3y}{dx^3} = y'''.$$
3. The **n$^{\text{th}}$ derivative of** f is:
$$f^{(n)}(x) = \frac{d}{dx}\left(f^{(n-1)}(x)\right) = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right) = \frac{d^ny}{dx^n} = y^{(n)}.$$

Note: The second derivative notation could be written as
$$\frac{d^2y}{dx^2} = \frac{d^2y}{(dx)^2} = \frac{d^2}{(dx)^2}(y).$$
That is, we take the derivative of y twice (hence d^2), both times with respect to x (hence $(dx)^2 = dx^2$).

In general, when finding the fourth derivative and on, we resort to the $f^{(4)}(x)$

Notes:

notation, not $f''''(x)$; after a while, too many ticks is confusing.

Let's practice using this new concept.

Example 1.3.3 Finding higher order derivatives

Find the first four derivatives of the following functions:

1. $f(x) = 4x^2$
2. $f(x) = \sin x$
3. $f(x) = 5e^x$

Solution

1. Using the Power and Constant Multiple Rules, we have: $f'(x) = 8x$. Continuing on, we have

$$f''(x) = \frac{d}{dx}\big(8x\big) = 8; \qquad f'''(x) = 0; \qquad f^{(4)}(x) = 0.$$

Notice how all successive derivatives will also be 0.

2. We employ Theorem 1.3.1 repeatedly.

$$f'(x) = \cos x; \qquad f''(x) = -\sin x; \qquad f'''(x) = -\cos x; \qquad f^{(4)}(x) = \sin x.$$

Note how we have come right back to $f(x)$ again. (Can you quickly figure what $f^{(23)}(x)$ is?)

3. Employing Theorem 1.3.1 and the Constant Multiple Rule, we can see that

$$f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = 5e^x.$$

Interpreting Higher Order Derivatives

What do higher order derivatives *mean*? What is the practical interpretation?

Our first answer is a bit wordy, but is technically correct and beneficial to understand. That is,

> The second derivative of a function f is the rate of change of the rate of change of f.

One way to grasp this concept is to let f describe a position function. Then, as stated in Key Idea 1.2.1, f' describes the rate of position change: velocity. We now consider f'', which describes the rate of velocity change. Sports car

Notes:

enthusiasts talk of how fast a car can go from 0 to 60 mph; they are bragging about the *acceleration* of the car.

We started this chapter with amusement–park riders free–falling with position function $f(t) = -16t^2 + 150$. It is easy to compute $f'(t) = -32t$ ft/s and $f''(t) = -32$ (ft/s)/s. We may recognize this latter constant; it is the acceleration due to gravity. In keeping with the unit notation introduced in the previous section, we say the units are "feet per second per second." This is usually shortened to "feet per second squared," written as "ft/s^2."

It can be difficult to consider the meaning of the third, and higher order, derivatives. The third derivative is "the rate of change of the rate of change of the rate of change of f." That is essentially meaningless to the uninitiated. In the context of our position/velocity/acceleration example, the third derivative is the "rate of change of acceleration," commonly referred to as "jerk."

Make no mistake: higher order derivatives have great importance even if their practical interpretations are hard (or "impossible") to understand. The mathematical topic of *series* makes extensive use of higher order derivatives.

Notes:

Exercises 1.3

Terms and Concepts

02 03 ex 01 1. What is the name of the rule which states that $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, where $n > 0$ is an integer?

02 03 ex 02 2. What is $\frac{d}{dx}\big(\ln x\big)$?

02 03 ex 03 3. Give an example of a function $f(x)$ where $f'(x) = f(x)$.

02 03 ex 04 4. Give an example of a function $f(x)$ where $f'(x) = 0$.

02 03 ex 05 5. The derivative rules introduced in this section explain how to compute the derivative of which of the following functions?

- $f(x) = \frac{3}{x^2}$
- $g(x) = 3x^2 - x + 17$
- $h(x) = 5\ln x$
- $j(x) = \sin x \cos x$
- $k(x) = e^{x^2}$
- $m(x) = \sqrt{x}$

02 03 ex 06 6. Explain in your own words how to find the third derivative of a function $f(x)$.

02 03 ex 07 7. Give an example of a function where $f'(x) \neq 0$ and $f''(x) = 0$.

02 03 ex 08 8. Explain in your own words what the second derivative "means."

02 03 ex 09 9. If $f(x)$ describes a position function, then $f'(x)$ describes what kind of function? What kind of function is $f''(x)$?

02 03 ex 10 10. Let $f(x)$ be a function measured in pounds, where x is measured in feet. What are the units of $f''(x)$?

Problems

02 03 exset 01 **In Exercises 11 – 26, compute the derivative of the given function.**

02 03 ex 11 11. $f(x) = 7x^2 - 5x + 7$

02 03 ex 12 12. $g(x) = 14x^3 + 7x^2 + 11x - 29$

02 03 ex 13 13. $m(t) = 9t^5 - \frac{1}{8}t^3 + 3t - 8$

02 03 ex 14 14. $f(\theta) = 9\sin\theta + 10\cos\theta$

02 03 ex 15 15. $f(r) = 6e^r$

02 03 ex 16 16. $g(t) = 10t^4 - \cos t + 7\sin t$

02 03 ex 17 17. $f(x) = 2\ln x - x$

02 03 ex 18 18. $p(s) = \frac{1}{4}s^4 + \frac{1}{3}s^3 + \frac{1}{2}s^2 + s + 1$

02 03 ex 19 19. $h(t) = e^t - \sin t - \cos t$

02 03 ex 20 20. $f(x) = \ln(5x^2)$

02 03 ex 21 21. $f(t) = \ln(17) + e^2 + \sin\pi/2$

02 03 ex 22 22. $g(t) = (1 + 3t)^2$

02 03 ex 23 23. $g(x) = (2x - 5)^3$

02 03 ex 24 24. $f(x) = (1 - x)^3$

02 03 ex 25 25. $f(x) = (2 - 3x)^2$

02 03 ex 50 26. A property of logarithms is that $\log_a x = \frac{\log_b x}{\log_b a}$, for all bases $a, b > 0, \neq 1$.

   (a) Rewrite this identity when $b = e$, i.e., using $\log_e x = \ln x$, with $a = 10$.

   (b) Use part (a) to find the derivative of $y = \log_{10} x$.

   (c) Use part (b) to find the derivative of $y = \log_a x$, for any $a > 0, \neq 1$.

02 03 exset 02 **In Exercises 27 – 32, compute the first four derivatives of the given function.**

02 03 ex 26 27. $f(x) = x^6$

02 03 ex 27 28. $g(x) = 2\cos x$

02 03 ex 28 29. $h(t) = t^2 - e^t$

02 03 ex 29 30. $p(\theta) = \theta^4 - \theta^3$

02 03 ex 30 31. $f(\theta) = \sin\theta - \cos\theta$

02 03 ex 31 32. $f(x) = 1,100$

02 03 exset 03 **In Exercises 33 – 38, find the equations of the tangent and normal lines to the graph of the function at the given point.**

02 03 ex 32 33. $f(x) = x^3 - x$ at $x = 1$

02 03 ex 33 34. $f(t) = e^t + 3$ at $t = 0$

02 03 ex 34 35. $g(x) = \ln x$ at $x = 1$

02 03 ex 35 36. $f(x) = 4\sin x$ at $x = \pi/2$

02 03 ex 36 37. $f(x) = -2\cos x$ at $x = \pi/4$

02 03 ex 37 38. $f(x) = 2x + 3$ at $x = 5$

Review

02 03 ex 39 39. Given that $e^0 = 1$, approximate the value of $e^{0.1}$ using the tangent line to $f(x) = e^x$ at $x = 0$.

1.4 The Product and Quotient Rules

The previous section showed that, in some ways, derivatives behave nicely. The Constant Multiple and Sum/Difference Rules established that the derivative of $f(x) = 5x^2 + \sin x$ was not complicated. We neglected computing the derivative of things like $g(x) = 5x^2 \sin x$ and $h(x) = \frac{5x^2}{\sin x}$ on purpose; their derivatives are *not* as straightforward. (If you had to guess what their respective derivatives are, you would probably guess wrong.) For these, we need the Product and Quotient Rules, respectively, which are defined in this section.

We begin with the Product Rule.

Theorem 1.4.1 Product Rule

Let f and g be differentiable functions on an open interval I. Then fg is a differentiable function on I, and

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = f(x)g'(x) + f'(x)g(x).$$

Important: $\frac{d}{dx}\Big(f(x)g(x)\Big) \neq f'(x)g'(x)$! While this answer is simpler than the Product Rule, it is wrong.

We practice using this new rule in an example, followed by an example that demonstrates why this theorem is true.

Example 1.4.1 Using the Product Rule

Use the Product Rule to compute the derivative of $y = 5x^2 \sin x$. Evaluate the derivative at $x = \pi/2$.

Solution To make our use of the Product Rule explicit, let's set $f(x) = 5x^2$ and $g(x) = \sin x$. We easily compute/recall that $f'(x) = 10x$ and $g'(x) = \cos x$. Employing the rule, we have

$$\frac{d}{dx}\Big(5x^2 \sin x\Big) = 5x^2 \cos x + 10x \sin x.$$

At $x = \pi/2$, we have

$$y'(\pi/2) = 5\left(\frac{\pi}{2}\right)^2 \cos\left(\frac{\pi}{2}\right) + 10\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right) = 5\pi.$$

We graph y and its tangent line at $x = \pi/2$, which has a slope of 5π, in Figure 1.4.1. While this does not *prove* that the Product Rule is the correct way to handle derivatives of products, it helps validate its truth.

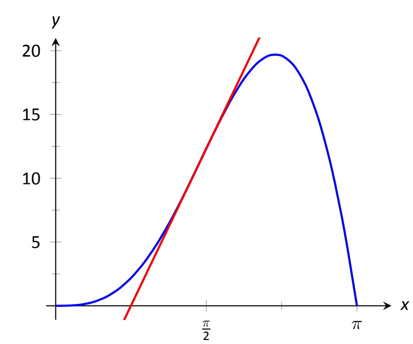


Figure 1.4.1: A graph of $y = 5x^2 \sin x$ and its tangent line at $x = \pi/2$.

Notes:

We now investigate why the Product Rule is true.

Example 1.4.2 A proof of the Product Rule

Use the definition of the derivative to prove Theorem 1.4.1.

Solution By the limit definition, we have

$$\frac{d}{dx}\Big(f(x)g(x)\Big) = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h}.$$

We now do something a bit unexpected; add 0 to the numerator (so that nothing is changed) in the form of $-f(x+h)g(x)+f(x+h)g(x)$, then do some regrouping as shown.

$$\begin{aligned}
\frac{d}{dx}\Big(f(x)g(x)\Big) &= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \quad \text{(now add 0 to the numerator)}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h} \quad \text{(regroup)}\\
&= \lim_{h\to 0}\frac{\Big(f(x+h)g(x+h)-f(x+h)g(x)\Big)+\Big(f(x+h)g(x)-f(x)g(x)\Big)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x+h)g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)g(x)-f(x)g(x)}{h} \quad \text{(factor)}\\
&= \lim_{h\to 0}f(x+h)\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x) \quad \text{(apply limits)}\\
&= f(x)g'(x)+f'(x)g(x).
\end{aligned}$$

It is often true that we can recognize that a theorem is true through its proof yet somehow doubt its applicability to real problems. In the following example, we compute the derivative of a product of functions in two ways to verify that the Product Rule is indeed "right."

Example 1.4.3 Exploring alternate derivative methods

Let $y = (x^2+3x+1)(2x^2-3x+1)$. Find y' two ways: first, by expanding the given product and then taking the derivative, and second, by applying the Product Rule. Verify that both methods give the same answer.

Solution We first expand the expression for y; a little algebra shows that $y = 2x^4+3x^3-6x^2+1$. It is easy to compute y':

$$y' = 8x^3+9x^2-12x.$$

Notes:

Now apply the Product Rule.

$$\begin{aligned} y' &= (x^2+3x+1)(4x-3)+(2x+3)(2x^2-3x+1) \\ &= \big(4x^3+9x^2-5x-3\big)+\big(4x^3-7x+3\big) \\ &= 8x^3+9x^2-12x. \end{aligned}$$

The uninformed usually assume that "the derivative of the product is the product of the derivatives." Thus we are tempted to say that $y' = (2x+3)(4x-3) = 8x^2+6x-9$. Obviously this is not correct.

Example 1.4.4 Using the Product Rule with a product of three functions

Let $y = x^3 \ln x \cos x$. Find y'.

Solution We have a product of three functions while the Product Rule only specifies how to handle a product of two functions. Our method of handling this problem is to simply group the latter two functions together, and consider $y = x^3\big(\ln x \cos x\big)$. Following the Product Rule, we have

$$y' = (x^3)\big(\ln x \cos x\big)' + 3x^2\big(\ln x \cos x\big)$$

To evaluate $\big(\ln x \cos x\big)'$, we apply the Product Rule again:

$$\begin{aligned} &= (x^3)\Big(\ln x(-\sin x)+\frac{1}{x}\cos x\Big)+3x^2\big(\ln x\cos x\big) \\ &= x^3\ln x(-\sin x)+x^3\frac{1}{x}\cos x+3x^2\ln x\cos x \end{aligned}$$

Recognize the pattern in our answer above: when applying the Product Rule to a product of three functions, there are three terms added together in the final derivative. Each term contains only one derivative of one of the original functions, and each function's derivative shows up in only one term. It is straightforward to extend this pattern to finding the derivative of a product of 4 or more functions.

We consider one more example before discussing another derivative rule.

Example 1.4.5 Using the Product Rule

Find the derivatives of the following functions.

1. $f(x) = x \ln x$
2. $g(x) = x \ln x - x$.

Notes:

Solution Recalling that the derivative of $\ln x$ is $1/x$, we use the Product Rule to find our answers.

1. $\dfrac{d}{dx}\Big(x\ln x\Big) = x\cdot 1/x + 1\cdot \ln x = 1+\ln x.$

2. Using the result from above, we compute

$$\frac{d}{dx}\Big(x\ln x - x\Big) = 1+\ln x - 1 = \ln x.$$

This seems significant; if the natural log function $\ln x$ is an important function (it is), it seems worthwhile to know a function whose derivative is $\ln x$. We have found one. (We leave it to the reader to find another; a correct answer will be *very* similar to this one.)

We have learned how to compute the derivatives of sums, differences, and products of functions. We now learn how to find the derivative of a quotient of functions.

Theorem 1.4.2 Quotient Rule

Let f and g be differentiable functions defined on an open interval I, where $g(x)\neq 0$ on I. Then f/g is differentiable on I, and

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}.$$

The Quotient Rule is not hard to use, although it might be a bit tricky to remember. A useful mnemonic works as follows. Consider a fraction's numerator and denominator as "HI" and "LO", respectively. Then

$$\frac{d}{dx}\left(\frac{\text{HI}}{\text{LO}}\right) = \frac{\text{LO}\cdot\,\text{dHI} - \text{HI}\cdot\,\text{dLO}}{\text{LOLO}},$$

read "low dee high minus high dee low, over low low." Said fast, that phrase can roll off the tongue, making it easy to memorize. The "dee high" and "dee low" parts refer to the derivatives of the numerator and denominator, respectively.

Let's practice using the Quotient Rule.

Example 1.4.6 Using the Quotient Rule

Let $f(x) = \dfrac{5x^2}{\sin x}$. Find $f'(x)$.

Notes:

Solution Directly applying the Quotient Rule gives:

$$\frac{d}{dx}\left(\frac{5x^2}{\sin x}\right) = \frac{\sin x \cdot 10x - 5x^2 \cdot \cos x}{\sin^2 x} = \frac{10x\sin x - 5x^2\cos x}{\sin^2 x}.$$

The Quotient Rule allows us to fill in holes in our understanding of derivatives of the common trigonometric functions. We start with finding the derivative of the tangent function.

Example 1.4.7 Using the Quotient Rule to find $\frac{d}{dx}\big(\tan x\big)$.

Find the derivative of $y = \tan x$.

Solution At first, one might feel unequipped to answer this question. But recall that $\tan x = \sin x/\cos x$, so we can apply the Quotient Rule.

$$\begin{aligned}\frac{d}{dx}\Big(\tan x\Big) &= \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)\\ &= \frac{\cos x\cos x - \sin x(-\sin x)}{\cos^2 x}\\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x.\end{aligned}$$

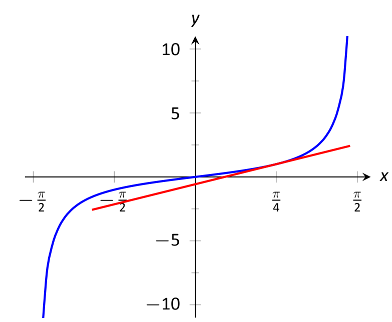


Figure 1.4.2: A graph of $y = \tan x$ along with its tangent line at $x = \pi/4$.

This is a beautiful result. To confirm its truth, we can find the equation of the tangent line to $y = \tan x$ at $x = \pi/4$. The slope is $\sec^2(\pi/4) = 2$; $y = \tan x$, along with its tangent line, is graphed in Figure 1.4.2.

We include this result in the following theorem about the derivatives of the trigonometric functions. Recall we found the derivative of $y = \sin x$ in Example 1.1.7 and stated the derivative of the cosine function in Theorem 1.3.1. The derivatives of the cotangent, cosecant and secant functions can all be computed directly using Theorem 1.3.1 and the Quotient Rule.

Notes:

Theorem 1.4.3 Derivatives of Trigonometric Functions

1. $\dfrac{d}{dx}\big(\sin x\big) = \cos x$
2. $\dfrac{d}{dx}\big(\cos x\big) = -\sin x$
3. $\dfrac{d}{dx}\big(\tan x\big) = \sec^2 x$
4. $\dfrac{d}{dx}\big(\cot x\big) = -\csc^2 x$
5. $\dfrac{d}{dx}\big(\sec x\big) = \sec x \tan x$
6. $\dfrac{d}{dx}\big(\csc x\big) = -\csc x \cot x$

To remember the above, it may be helpful to keep in mind that the derivatives of the trigonometric functions that start with "c" have a minus sign in them.

Example 1.4.8 Exploring alternate derivative methods

In Example 1.4.6 the derivative of $f(x) = \dfrac{5x^2}{\sin x}$ was found using the Quotient Rule. Rewriting f as $f(x) = 5x^2 \csc x$, find f' using Theorem 1.4.3 and verify the two answers are the same.

SOLUTION We found in Example 1.4.6 that the $f'(x) = \dfrac{10x\sin x - 5x^2\cos x}{\sin^2 x}$. We now find f' using the Product Rule, considering f as $f(x) = 5x^2 \csc x$.

$$
\begin{aligned}
f'(x) &= \frac{d}{dx}\Big(5x^2\csc x\Big) \\
&= 5x^2(-\csc x\cot x) + 10x\csc x && \text{(now rewrite trig functions)} \\
&= 5x^2\cdot\frac{-1}{\sin x}\cdot\frac{\cos x}{\sin x} + \frac{10x}{\sin x} \\
&= \frac{-5x^2\cos x}{\sin^2 x} + \frac{10x}{\sin x} && \text{(get common denominator)} \\
&= \frac{10x\sin x - 5x^2\cos x}{\sin^2 x}
\end{aligned}
$$

Finding f' using either method returned the same result. At first, the answers looked different, but some algebra verified they are the same. In general, there is not one final form that we seek; the immediate result from the Product Rule is fine. Work to "simplify" your results into a form that is most readable and useful to you.

The Quotient Rule gives other useful results, as show in the next example.

Notes:

Example 1.4.9 Using the Quotient Rule to expand the Power Rule
Find the derivatives of the following functions.

1. $f(x) = \dfrac{1}{x}$

2. $f(x) = \dfrac{1}{x^n}$, where $n > 0$ is an integer.

SOLUTION We employ the Quotient Rule.

1. $f'(x) = \dfrac{x \cdot 0 - 1 \cdot 1}{x^2} = -\dfrac{1}{x^2}$.

2. $f'(x) = \dfrac{x^n \cdot 0 - 1 \cdot nx^{n-1}}{(x^n)^2} = -\dfrac{nx^{n-1}}{x^{2n}} = -\dfrac{n}{x^{n+1}}$.

The derivative of $y = \dfrac{1}{x^n}$ turned out to be rather nice. It gets better. Consider:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x^n}\right) &= \frac{d}{dx}\left(x^{-n}\right) && \text{(apply result from Example 1.4.9)}\\
&= -\frac{n}{x^{n+1}} && \text{(rewrite algebraically)}\\
&= -nx^{-(n+1)}\\
&= -nx^{-n-1}.
\end{aligned}$$

This is reminiscent of the Power Rule: multiply by the power, then subtract 1 from the power. We now add to our previous Power Rule, which had the restriction of $n > 0$.

> **Theorem 1.4.4 Power Rule with Integer Exponents**
> Let $f(x) = x^n$, where $n \neq 0$ is an integer. Then
> $$f'(x) = n \cdot x^{n-1}.$$

Taking the derivative of many functions is relatively straightforward. It is clear (with practice) what rules apply and in what order they should be applied. Other functions present multiple paths; different rules may be applied depending on how the function is treated. One of the beautiful things about calculus is that there is not "the" right way; each path, when applied correctly, leads to

Notes:

the same result, the derivative. We demonstrate this concept in an example.

Example 1.4.10 Exploring alternate derivative methods

Let $f(x) = \dfrac{x^2 - 3x + 1}{x}$. Find $f'(x)$ in each of the following ways:

1. By applying the Quotient Rule,
2. by viewing f as $f(x) = \big(x^2 - 3x + 1\big) \cdot x^{-1}$ and applying the Product and Power Rules, and
3. by "simplifying" first through division.

Verify that all three methods give the same result.

Solution

1. Applying the Quotient Rule gives:

$$f'(x) = \frac{x \cdot \big(2x - 3\big) - \big(x^2 - 3x + 1\big) \cdot 1}{x^2} = \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}.$$

2. By rewriting f, we can apply the Product and Power Rules as follows:

$$\begin{aligned} f'(x) &= \big(x^2 - 3x + 1\big) \cdot (-1)x^{-2} + \big(2x - 3\big) \cdot x^{-1} \\ &= -\frac{x^2 - 3x + 1}{x^2} + \frac{2x - 3}{x} \\ &= -\frac{x^2 - 3x + 1}{x^2} + \frac{2x^2 - 3x}{x^2} \\ &= \frac{x^2 - 1}{x^2} = 1 - \frac{1}{x^2}, \end{aligned}$$

   the same result as above.

3. As $x \neq 0$, we can divide through by x first, giving $f(x) = x - 3 + \dfrac{1}{x}$. Now apply the Power Rule.

$$f'(x) = 1 - \frac{1}{x^2},$$

   the same result as before.

Example 1.4.10 demonstrates three methods of finding f'. One is hard pressed to argue for a "best method" as all three gave the same result without too much difficulty, although it is clear that using the Product Rule required more steps. Ultimately, the important principle to take away from this is: reduce the answer

Notes:

to a form that seems "simple" and easy to interpret. In that example, we saw different expressions for f', including:

$$1-\frac{1}{x^2}=\frac{x\cdot\big(2x-3\big)-\big(x^2-3x+1\big)\cdot 1}{x^2}=\big(x^2-3x+1\big)\cdot(-1)x^{-2}+\big(2x-3\big)\cdot x^{-1}.$$

They are equal; they are all correct; only the first is "clear." Work to make answers clear.

In the next section we continue to learn rules that allow us to more easily compute derivatives than using the limit definition directly. We have to memorize the derivatives of a certain set of functions, such as "the derivative of $\sin x$ is $\cos x$." The Sum/Difference, Constant Multiple, Power, Product and Quotient Rules show us how to find the derivatives of certain combinations of these functions. The next section shows how to find the derivatives when we *compose* these functions together.

Notes:

Exercises 1.4

Terms and Concepts

02 04 ex 01 1. T/F: The Product Rule states that $\frac{d}{dx}\left(x^2 \sin x\right) = 2x\cos x$.

02 04 ex 02 2. T/F: The Quotient Rule states that $\frac{d}{dx}\left(\frac{x^2}{\sin x}\right) = \frac{\cos x}{2x}$.

02 04 ex 03 3. T/F: The derivatives of the trigonometric functions that start with “c” have minus signs in them.

02 04 ex 04 4. What derivative rule is used to extend the Power Rule to include negative integer exponents?

02 04 ex 05 5. T/F: Regardless of the function, there is always exactly one right way of computing its derivative.

02 04 ex 06 6. In your own words, explain what it means to make your answers “clear.”

Problems

02 04 exset 01 **In Exercises 7 – 10:**

(a) Use the Product Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Product Rule.

(c) Show that the answers from (a) and (b) are equivalent.

02 04 ex 07 7. $f(x) = x(x^2 + 3x)$

02 04 ex 08 8. $g(x) = 2x^2(5x^3)$

02 04 ex 09 9. $h(s) = (2s - 1)(s + 4)$

02 04 ex 10 10. $f(x) = (x^2 + 5)(3 - x^3)$

02 04 exset 02 **In Exercises 11 – 14:**

(a) Use the Quotient Rule to differentiate the function.

(b) Manipulate the function algebraically and differentiate without the Quotient Rule.

(c) Show that the answers from (a) and (b) are equivalent.

02 04 ex 11 11. $f(x) = \frac{x^2 + 3}{x}$

02 04 ex 12 12. $g(x) = \frac{x^3 - 2x^2}{2x^2}$

02 04 ex 13 13. $h(s) = \frac{3}{4s^3}$

02 04 ex 14 14. $f(t) = \frac{t^2 - 1}{t + 1}$

02 04 exset 03 **In Exercises 15 – 36, compute the derivative of the given function.**

02 04 ex 16 15. $f(x) = x\sin x$

02 04 ex 48 16. $f(x) = x^2\cos x$

02 04 ex 49 17. $f(x) = e^x \ln x$

02 04 ex 17 18. $f(t) = \frac{1}{t^2}(\csc t - 4)$

02 04 ex 18 19. $g(x) = \frac{x + 7}{x - 5}$

02 04 ex 19 20. $g(t) = \frac{t^5}{\cos t - 2t^2}$

02 04 ex 20 21. $h(x) = \cot x - e^x$

02 04 ex 50 22. $f(x) = \big(\tan x\big)\ln x$

02 04 ex 21 23. $h(t) = 7t^2 + 6t - 2$

02 04 ex 15 24. $f(x) = \frac{x^4 + 2x^3}{x + 2}$

02 04 ex 51 25. $f(x) = \big(3x^2 + 8x + 7\big)e^x$

02 04 ex 52 26. $g(t) = \frac{t^5 - t^3}{e^t}$

02 04 ex 22 27. $f(x) = (16x^3 + 24x^2 + 3x)\frac{7x - 1}{16x^3 + 24x^2 + 3x}$

02 04 ex 23 28. $f(t) = t^5(\sec t + e^t)$

02 04 ex 46 29. $f(x) = \frac{\sin x}{\cos x + 3}$

02 04 ex 53 30. $f(\theta) = \theta^3\sin\theta + \frac{\sin\theta}{\theta^3}$

02 04 ex 54 31. $f(x) = \frac{\cos x}{x} + \frac{x}{\tan x}$

02 04 ex 24 32. $g(x) = e^2\big(\sin(\pi/4) - 1\big)$

02 04 ex 25 33. $g(t) = 4t^3e^t - \sin t\cos t$

02 04 ex 47 34. $h(t) = \frac{t^2\sin t + 3}{t^2\cos t + 2}$

02 04 ex 26 35. $f(x) = x^2e^x\tan x$

02 04 ex 27 36. $g(x) = 2x\sin x\sec x$

02 04 exset 04

In Exercises 37 – 40, find the equations of the tangent and normal lines to the graph of g at the indicated point.

02 04 ex 28

37. $g(s) = e^s(s^2 + 2)$ at $(0, 2)$.

02 04 ex 29

38. $g(t) = t \sin t$ at $\left(\frac{3\pi}{2}, -\frac{3\pi}{2}\right)$

02 04 ex 30

39. $g(x) = \dfrac{x^2}{x - 1}$ at $(2, 4)$

02 04 ex 31

40. $g(\theta) = \dfrac{\cos\theta - 8\theta}{\theta + 1}$ at $(0, 1)$

02 04 exset 05

In Exercises 41 – 44, find the x–values where the graph of the function has a horizontal tangent line.

02 04 ex 32

41. $f(x) = 6x^2 - 18x - 24$

02 04 ex 33

42. $f(x) = x \sin x$ on $[-1, 1]$

02 04 ex 34

43. $f(x) = \dfrac{x}{x + 1}$

02 04 ex 35

44. $f(x) = \dfrac{x^2}{x + 1}$

02 04 exset 06

In Exercises 45 – 48, find the requested derivative.

02 04 ex 36

45. $f(x) = x \sin x$; find $f''(x)$.

02 04 ex 37

46. $f(x) = x \sin x$; find $f^{(4)}(x)$.

02 04 ex 38

47. $f(x) = \csc x$; find $f''(x)$.

02 04 ex 39

48. $f(x) = (x^3 - 5x + 2)(x^2 + x - 7)$; find $f^{(8)}(x)$.

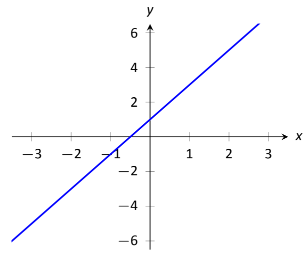
Review

02 04 exset 07

In Exercises 49 – 52, use the graph of $f(x)$ to sketch $f'(x)$.

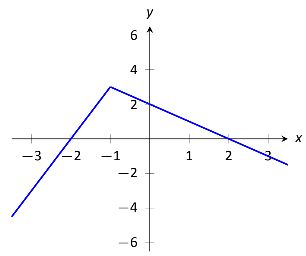
02 04 ex 42

49.



02 04 ex 43

50.



02 04 ex 44

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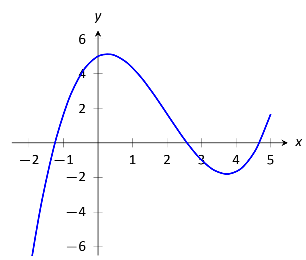
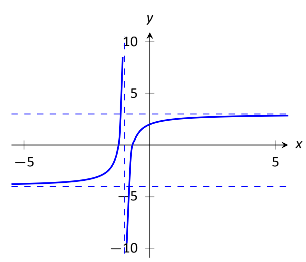


02 04 ex 45

52.



1.5 The Chain Rule

We have covered almost all of the derivative rules that deal with combinations of two (or more) functions. The operations of addition, subtraction, multiplication (including by a constant) and division led to the Sum and Difference rules, the Constant Multiple Rule, the Power Rule, the Product Rule and the Quotient Rule. To complete the list of differentiation rules, we look at the last way two (or more) functions can be combined: the process of composition (i.e. one function "inside" another).

One example of a composition of functions is $f(x) = \cos(x^2)$. We currently do not know how to compute this derivative. If forced to guess, one would likely guess $f'(x) = -\sin(2x)$, where we recognize $-\sin x$ as the derivative of $\cos x$ and $2x$ as the derivative of x^2. However, this is not the case; $f'(x) \neq -\sin(2x)$. In Example 1.5.4 we'll see the correct answer, which employs the new rule this section introduces, the **Chain Rule**.

Before we define this new rule, recall the notation for composition of functions. We write $(f \circ g)(x)$ or $f(g(x))$, read as "f of g of x," to denote composing f with g. In shorthand, we simply write $f \circ g$ or $f(g)$ and read it as "f of g." Before giving the corresponding differentiation rule, we note that the rule extends to multiple compositions like $f(g(h(x)))$ or $f(g(h(j(x))))$, etc.

To motivate the rule, let's look at three derivatives we can already compute.

Example 1.5.1 Exploring similar derivatives

Find the derivatives of $F_1(x) = (1-x)^2$, $F_2(x) = (1-x)^3$, and $F_3(x) = (1-x)^4$. (We'll see later why we are using subscripts for different functions and an uppercase F.)

Solution In order to use the rules we already have, we must first expand each function as $F_1(x) = 1 - 2x + x^2$, $F_2(x) = 1 - 3x + 3x^2 - x^3$ and $F_3(x) = 1 - 4x + 6x^2 - 4x^3 + x^4$.

It is not hard to see that:

$$F_1'(x) = -2 + 2x,$$
$$F_2'(x) = -3 + 6x - 3x^2 \text{ and}$$
$$F_3'(x) = -4 + 12x - 12x^2 + 4x^3.$$

An interesting fact is that these can be rewritten as

$$F_1'(x) = -2(1-x), \quad F_2'(x) = -3(1-x)^2 \text{ and } F_3'(x) = -4(1-x)^3.$$

A pattern might jump out at you; note how the we end up multiplying by the old power and the new power is reduced by 1. We also always multiply by (-1).

Notes:

Recognize that each of these functions is a composition, letting $g(x) = 1-x$:

$$\begin{aligned} F_1(x) &= f_1(g(x)), \quad \text{where } f_1(x) = x^2, \\ F_2(x) &= f_2(g(x)), \quad \text{where } f_2(x) = x^3, \\ F_3(x) &= f_3(g(x)), \quad \text{where } f_3(x) = x^4. \end{aligned}$$

We'll come back to this example after giving the formal statements of the Chain Rule; for now, we are just illustrating a pattern.

When composing functions, we need to make sure that the new function is actually defined. For instance, consider $f(x) = \sqrt{x}$ and $g(x) = -x^2 - 1$. The domain of f excludes all negative numbers, but the range of g is only negative numbers. Therefore the composition $f\big(g(x)\big) = \sqrt{-x^2-1}$ is not defined for any x, and hence is not differentiable.

The following definition takes care to ensure this problem does not arise. We'll focus more on the derivative result than on the domain/range conditions.

Theorem 1.5.1 The Chain Rule

Let g be a differentiable function on an interval I, let the range of g be a subset of the interval J, and let f be a differentiable function on J. Then $y = f(g(x))$ is a differentiable function on I, and

$$y' = f'(g(x)) \cdot g'(x).$$

To help understand the Chain Rule, we return to Example 1.5.1.

Example 1.5.2 Using the Chain Rule

Use the Chain Rule to find the derivatives of the following functions, as given in Example 1.5.1.

Solution Example 1.5.1 ended with the recognition that each of the given functions was actually a composition of functions. To avoid confusion, we ignore most of the subscripts here.

$F_1(x) = (1-x)^2$**:**

We found that

$$y = (1-x)^2 = f(g(x)), \text{ where } f(x) = x^2 \text{ and } g(x) = 1-x.$$

To find y', we apply the Chain Rule. We need $f'(x) = 2x$ and $g'(x) = -1$.

Notes:

Part of the Chain Rule uses $f'(g(x))$. This means substitute $g(x)$ for x in the equation for $f'(x)$. That is, $f'(x) = 2(1-x)$. Finishing out the Chain Rule we have

$$y' = f'(g(x)) \cdot g'(x) = 2(1-x) \cdot (-1) = -2(1-x) = 2x - 2.$$

$F_2(x) = (1-x)^3$:

Let $y = (1-x)^3 = f(g(x))$, where $f(x) = x^3$ and $g(x) = (1-x)$. We have $f'(x) = 3x^2$, so $f'(g(x)) = 3(1-x)^2$. The Chain Rule then states

$$y' = f'(g(x)) \cdot g'(x) = 3(1-x)^2 \cdot (-1) = -3(1-x)^2.$$

$F_3(x) = (1-x)^4$:

Finally, when $y = (1-x)^4$, we have $f(x) = x^4$ and $g(x) = (1-x)$. Thus $f'(x) = 4x^3$ and $f'(g(x)) = 4(1-x)^3$. Thus

$$y' = f'(g(x)) \cdot g'(x) = 4(1-x)^3 \cdot (-1) = -4(1-x)^3.$$

Example 1.5.2 demonstrated a particular pattern: when $f(x) = x^n$, then $y' = n \cdot (g(x))^{n-1} \cdot g'(x)$. This is called the Generalized Power Rule.

Theorem 1.5.2 Generalized Power Rule

Let $g(x)$ be a differentiable function and let $n \neq 0$ be an integer. Then

$$\frac{d}{dx}\Big(g(x)^n\Big) = n \cdot \big(g(x)\big)^{n-1} \cdot g'(x).$$

This allows us to quickly find the derivative of functions like $y = (3x^2 - 5x + 7 + \sin x)^{20}$. While it may look intimidating, the Generalized Power Rule states that

$$y' = 20(3x^2 - 5x + 7 + \sin x)^{19} \cdot (6x - 5 + \cos x).$$

Treat the derivative–taking process step–by–step. In the example just given, first multiply by 20, then rewrite the inside of the parentheses, raising it all to the 19$^{\text{th}}$ power. Then think about the derivative of the expression inside the parentheses, and multiply by that.

We now consider more examples that employ the Chain Rule.

Notes:

Example 1.5.3 Using the Chain Rule

Find the derivatives of the following functions:

1. $y = \sin 2x$
2. $y = \ln(4x^3 - 2x^2)$
3. $y = e^{-x^2}$

SOLUTION

1. Consider $y = \sin 2x$. Recognize that this is a composition of functions, where $f(x) = \sin x$ and $g(x) = 2x$. Thus

$$y' = f'(g(x)) \cdot g'(x) = \cos(2x) \cdot 2 = 2\cos 2x.$$

2. Recognize that $y = \ln(4x^3 - 2x^2)$ is the composition of $f(x) = \ln x$ and $g(x) = 4x^3 - 2x^2$. Also, recall that

$$\frac{d}{dx}\Big(\ln x\Big) = \frac{1}{x}.$$

This leads us to:

$$y' = \frac{1}{4x^3 - 2x^2} \cdot (12x^2 - 4x) = \frac{12x^2 - 4x}{4x^3 - 2x^2} = \frac{4x(3x-1)}{2x(2x^2 - x)} = \frac{2(3x-1)}{2x^2 - x}.$$

3. Recognize that $y = e^{-x^2}$ is the composition of $f(x) = e^x$ and $g(x) = -x^2$. Remembering that $f'(x) = e^x$, we have

$$y' = e^{-x^2} \cdot (-2x) = (-2x)e^{-x^2}.$$

Example 1.5.4 Using the Chain Rule to find a tangent line

Let $f(x) = \cos x^2$. Find the equation of the line tangent to the graph of f at $x = 1$.

SOLUTION The tangent line goes through the point $(1, f(1)) \approx (1, 0.54)$ with slope $f'(1)$. To find f', we need the Chain Rule.

$f'(x) = -\sin(x^2) \cdot (2x) = -2x\sin x^2$. Evaluated at $x = 1$, we have $f'(1) = -2\sin 1 \approx -1.68$. Thus the equation of the tangent line is

$$y = -1.68(x-1) + 0.54.$$

The tangent line is sketched along with f in Figure 1.5.1.

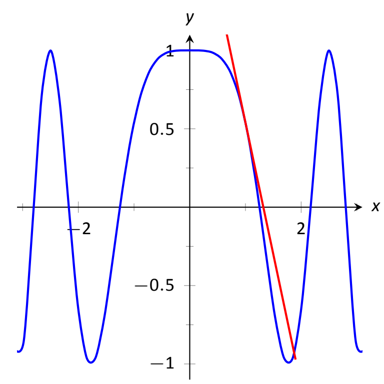


Figure 1.5.1: $f(x) = \cos x^2$ sketched along with its tangent line at $x = 1$.

The Chain Rule is used often in taking derivatives. Because of this, one can become familiar with the basic process and learn patterns that facilitate finding derivatives quickly. For instance,

$$\frac{d}{dx}\Big(\ln(\text{anything})\Big) = \frac{1}{\text{anything}} \cdot (\text{anything})' = \frac{(\text{anything})'}{\text{anything}}.$$

Notes:

A concrete example of this is

$$\frac{d}{dx}\Big(\ln(3x^{15} - \cos x + e^x)\Big) = \frac{45x^{14} + \sin x + e^x}{3x^{15} - \cos x + e^x}.$$

While the derivative may look intimidating at first, look for the pattern. The denominator is the same as what was inside the natural log function; the numerator is simply its derivative.

This pattern recognition process can be applied to lots of functions. In general, instead of writing "anything", we use u as a generic function of x. We then say

$$\frac{d}{dx}\Big(\ln u\Big) = \frac{u'}{u}.$$

The following is a short list of how the Chain Rule can be quickly applied to familiar functions.

1. $\frac{d}{dx}\Big(u^n\Big) = n \cdot u^{n-1} \cdot u'$.
2. $\frac{d}{dx}\Big(e^u\Big) = u' \cdot e^u$.
3. $\frac{d}{dx}\Big(\sin u\Big) = u' \cdot \cos u$.
4. $\frac{d}{dx}\Big(\cos u\Big) = -u' \cdot \sin u$.
5. $\frac{d}{dx}\Big(\tan u\Big) = u' \cdot \sec^2 u$.

Of course, the Chain Rule can be applied in conjunction with any of the other rules we have already learned. We practice this next.

Example 1.5.5 Using the Product, Quotient and Chain Rules

Find the derivatives of the following functions.

1. $f(x) = x^5 \sin 2x^3$ 2. $f(x) = \dfrac{5x^3}{e^{-x^2}}$.

Solution

1. We must use the Product and Chain Rules. Do not think that you must be able to "see" the whole answer immediately; rather, just proceed step–by–step.

$$f'(x) = x^5\big(6x^2 \cos 2x^3\big) + 5x^4\big(\sin 2x^3\big) = 6x^7 \cos 2x^3 + 5x^4 \sin 2x^3.$$

2. We must employ the Quotient Rule along with the Chain Rule. Again, pro-

Notes:

ceed step–by–step.

$$f'(x) = \frac{e^{-x^2}\big(15x^2\big) - 5x^3\big((-2x)e^{-x^2}\big)}{\big(e^{-x^2}\big)^2} = \frac{e^{-x^2}\big(10x^4 + 15x^2\big)}{e^{-2x^2}}$$
$$= e^{x^2}\big(10x^4 + 15x^2\big).$$

A key to correctly working these problems is to break the problem down into smaller, more manageable pieces. For instance, when using the Product and Chain Rules together, just consider the first part of the Product Rule at first: $f(x)g'(x)$. Just rewrite $f(x)$, then find $g'(x)$. Then move on to the $f'(x)g(x)$ part. Don't attempt to figure out both parts at once.

Likewise, using the Quotient Rule, approach the numerator in two steps and handle the denominator after completing that. Only simplify afterward.

We can also employ the Chain Rule itself several times, as shown in the next example.

Example 1.5.6 Using the Chain Rule multiple times

Find the derivative of $y = \tan^5(6x^3 - 7x)$.

Solution Recognize that we have the $g(x) = \tan(6x^3 - 7x)$ function "inside" the $f(x) = x^5$ function; that is, we have $y = \big(\tan(6x^3 - 7x)\big)^5$. We begin using the Generalized Power Rule; in this first step, we do not fully compute the derivative. Rather, we are approaching this step–by–step.

$$y' = 5\big(\tan(6x^3 - 7x)\big)^4 \cdot g'(x).$$

We now find $g'(x)$. We again need the Chain Rule;

$$g'(x) = \sec^2(6x^3 - 7x) \cdot (18x^2 - 7).$$

Combine this with what we found above to give

$$\begin{aligned} y' &= 5\big(\tan(6x^3 - 7x)\big)^4 \cdot \sec^2(6x^3 - 7x) \cdot (18x^2 - 7) \\ &= (90x^2 - 35)\sec^2(6x^3 - 7x)\tan^4(6x^3 - 7x). \end{aligned}$$

This function is frankly a ridiculous function, possessing no real practical value. It is very difficult to graph, as the tangent function has many vertical asymptotes and $6x^3 - 7x$ grows so very fast. The important thing to learn from this is that the derivative can be found. In fact, it is not "hard;" one can take several simple steps and should be careful to keep track of how to apply each of these steps.

Notes:

It is a traditional mathematical exercise to find the derivatives of arbitrarily complicated functions just to demonstrate that it *can be done*. Just break everything down into smaller pieces.

Example 1.5.7 Using the Product, Quotient and Chain Rules

Find the derivative of $f(x) = \dfrac{x\cos(x^{-2}) - \sin^2(e^{4x})}{\ln(x^2 + 5x^4)}$.

Solution This function likely has no practical use outside of demonstrating derivative skills. The answer is given below without simplification. It employs the Quotient Rule, the Product Rule, and the Chain Rule three times.

$$f'(x) = \frac{\left(\begin{array}{l}\ln(x^2+5x^4)\cdot\Big[\big(x\cdot(-\sin(x^{-2}))\cdot(-2x^{-3})+1\cdot\cos(x^{-2})\big)-2\sin(e^{4x})\cdot\cos(e^{4x})\cdot(4e^{4x})\Big]\\ \qquad -\Big(x\cos(x^{-2})-\sin^2(e^{4x})\Big)\cdot\frac{2x+20x^3}{x^2+5x^4}\end{array}\right)}{\big(\ln(x^2+5x^4)\big)^2}.$$

The reader is highly encouraged to look at each term and recognize why it is there. (I.e., the Quotient Rule is used; in the numerator, identify the "LOdHI" term, etc.) This example demonstrates that derivatives can be computed systematically, no matter how arbitrarily complicated the function is.

The Chain Rule also has theoretic value. That is, it can be used to find the derivatives of functions that we have not yet learned as we do in the following example.

Example 1.5.8 The Chain Rule and exponential functions

Use the Chain Rule to find the derivative of $y = 2^x$.

Solution We only know how to find the derivative of one exponential function, $y = e^x$. We can accomplish our goal by rewriting 2 in terms of e. Recalling that e^x and $\ln x$ are inverse functions, we can write

$$2 = e^{\ln 2} \quad \text{and so} \quad y = 2^x = \left(e^{\ln 2}\right)^x = e^{x(\ln 2)}.$$

The function is now the composition $y = f(g(x))$, with $f(x) = e^x$ and $g(x) = x(\ln 2)$. Since $f'(x) = e^x$ and $g'(x) = \ln 2$, the Chain Rule gives

$$y' = e^{x(\ln 2)} \cdot \ln 2.$$

Recall that the $e^{x(\ln 2)}$ term on the right hand side is just 2^x, our original function. Thus, the derivative contains the original function itself. We have

$$y' = y \cdot \ln 2 = 2^x \cdot \ln 2.$$

Notes:

We can extend this process to use any base a, where $a > 0$ and $a \neq 1$. All we need to do is replace each "2" in our work with "a." The Chain Rule, coupled with the derivative rule of e^x, allows us to find the derivatives of all exponential functions.

The comment at the end of previous example is important and is restated formally as a theorem.

Theorem 1.5.3 Derivatives of Exponential Functions

Let $f(x) = a^x$, for $a > 0, a \neq 1$. Then f is differentiable for all real numbers (i.e., differentiable everywhere) and

$$f'(x) = \ln a \cdot a^x.$$

Alternate Chain Rule Notation

It is instructive to understand what the Chain Rule "looks like" using "$\frac{dy}{dx}$" notation instead of y' notation. Suppose that $y = f(u)$ is a function of u, where $u = g(x)$ is a function of x, as stated in Theorem 1.5.1. Then, through the composition $f \circ g$, we can think of y as a function of x, as $y = f(g(x))$. Thus the derivative of y with respect to x makes sense; we can talk about $\frac{dy}{dx}$. This leads to an interesting progression of notation:

$$\begin{aligned}
y' &= f'(g(x)) \cdot g'(x) \\
\frac{dy}{dx} &= y'(u) \cdot u'(x) && \text{(since } y = f(u) \text{ and } u = g(x)\text{)} \\
\frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} && \text{(using "fractional" notation for the derivative)}
\end{aligned}$$

Here the "fractional" aspect of the derivative notation stands out. On the right hand side, it seems as though the "du" terms cancel out, leaving

$$\frac{dy}{dx} = \frac{dy}{dx}.$$

It is important to realize that we *are not* canceling these terms; the derivative notation of $\frac{dy}{du}$ is one symbol. It is equally important to realize that this notation was chosen precisely because of this behavior. It makes applying the Chain Rule easy with multiple variables. For instance,

Notes:

$$\frac{dy}{dt} = \frac{dy}{d\bigcirc} \cdot \frac{d\bigcirc}{d\triangle} \cdot \frac{d\triangle}{dt}.$$

where $\bigcirc$ and $\triangle$ are any variables you'd like to use.

One of the most common ways of "visualizing" the Chain Rule is to consider a set of gears, as shown in Figure 1.5.2. The gears have 36, 18, and 6 teeth, respectively. That means for every revolution of the *x* gear, the *u* gear revolves twice. That is, the rate at which the *u* gear makes a revolution is twice as fast as the rate at which the *x* gear makes a revolution. Using the terminology of calculus, the rate of *u*-change, with respect to *x*, is $\frac{du}{dx} = 2$.

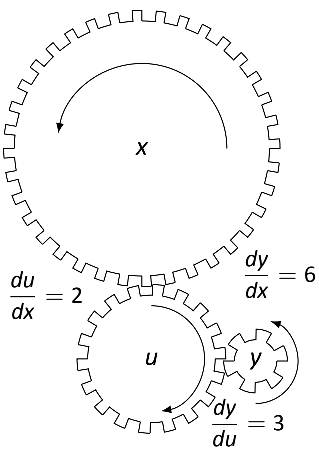


Figure 1.5.2: A series of gears to demonstrate the Chain Rule. Note how $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Likewise, every revolution of *u* causes 3 revolutions of *y*: $\frac{dy}{du} = 3$. How does *y* change with respect to *x*? For each revolution of *x*, *y* revolves 6 times; that is,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2 \cdot 3 = 6.$$

We can then extend the Chain Rule with more variables by adding more gears to the picture.

It is difficult to overstate the importance of the Chain Rule. So often the functions that we deal with are compositions of two or more functions, requiring us to use this rule to compute derivatives. It is also often used in real life when actual functions are unknown. Through measurement, we can calculate (or, approximate) $\frac{dy}{du}$ and $\frac{du}{dx}$. With our knowledge of the Chain Rule, we can find $\frac{dy}{dx}$.

In the next section, we use the Chain Rule to justify another differentiation technique. There are many curves that we can draw in the plane that fail the "vertical line test." For instance, consider $x^2 + y^2 = 1$, which describes the unit circle. We may still be interested in finding slopes of tangent lines to the circle at various points. The next section shows how we can find $\frac{dy}{dx}$ without first "solving for *y*." While we can in this instance, in many other instances solving for *y* is impossible. In these situations, *implicit differentiation* is indispensable.

Notes:

Exercises 1.5

Terms and Concepts

02 05 ex 05 1. T/F: The Chain Rule describes how to evaluate the derivative of a composition of functions.

02 05 ex 01 2. T/F: The Generalized Power Rule states that $\frac{d}{dx}\Big(g(x)^n\Big) = n\big(g(x)\big)^{n-1}$.

02 05 ex 02 3. T/F: $\frac{d}{dx}\big(\ln(x^2)\big) = \frac{1}{x^2}$.

02 05 ex 03 4. T/F: $\frac{d}{dx}\big(3^x\big) \approx 1.1 \cdot 3^x$.

02 05 ex 04 5. T/F: $\frac{dx}{dy} = \frac{dx}{dt} \cdot \frac{dt}{dy}$

02 05 ex 53 6. $f(x) = \big(\ln x + x^2\big)^3$

Problems

02 05 exset 01 **In Exercises 7 – 36, compute the derivative of the given function.**

02 05 ex 06 7. $f(x) = (4x^3 - x)^{10}$

02 05 ex 07 8. $f(t) = (3t - 2)^5$

02 05 ex 08 9. $g(\theta) = (\sin\theta + \cos\theta)^3$

02 05 ex 09 10. $h(t) = e^{3t^2+t-1}$

02 05 ex 53 11. $f(x) = \big(\ln x + x^2\big)^3$

02 05 ex 54 12. $f(x) = 2^{x^3+3x}$

02 05 ex 10 13. $f(x) = \big(x + \frac{1}{x}\big)^4$

02 05 ex 11 14. $f(x) = \cos(3x)$

02 05 ex 12 15. $g(x) = \tan(5x)$

02 05 ex 55 16. $h(\theta) = \tan\big(\theta^2 + 4\theta\big)$

02 05 ex 56 17. $g(t) = \sin\big(t^5 + \frac{1}{t}\big)$

02 05 ex 13 18. $h(t) = \sin^4(2t)$

02 05 ex 14 19. $p(t) = \cos^3(t^2 + 3t + 1)$

02 05 ex 15 20. $f(x) = \ln(\cos x)$

02 05 ex 16 21. $f(x) = \ln(x^2)$

02 05 ex 17 22. $f(x) = 2\ln(x)$

02 05 ex 18 23. $g(r) = 4^r$

02 05 ex 19 24. $g(t) = 5^{\cos t}$

02 05 ex 20 25. $g(t) = 15^2$

02 05 ex 21 26. $m(w) = \frac{3^w}{2^w}$

02 05 ex 42 27. $h(t) = \frac{2^t + 3}{3^t + 2}$

02 05 ex 22 28. $m(w) = \frac{3^w + 1}{2^w}$

02 05 ex 23 29. $f(x) = \frac{3^{x^2} + x}{2^{x^2}}$

02 05 ex 24 30. $f(x) = x^2\sin(5x)$

02 05 ex 58 31. $f(x) = (x^2 + x)^5(3x^4 + 2x)^3$

02 05 ex 25 32. $g(t) = \cos(t^2 + 3t)\sin(5t - 7)$

02 05 ex 57 33. $f(x) = \sin(3x + 4)\cos(5 - 2x)$

02 05 ex 26 34. $g(t) = \cos(\frac{1}{t})e^{5t^2}$

02 05 ex 59 35. $f(x) = \frac{\sin\big(4x + 1\big)}{(5x - 9)^3}$

02 05 ex 60 36. $f(x) = \frac{(4x + 1)^2}{\tan(5x)}$

02 05 exset 02 **In Exercises 37 – 40, find the equations of tangent and normal lines to the graph of the function at the given point. Note: the functions here are the same as in Exercises 7 through 10.**

02 05 ex 27 37. $f(x) = (4x^3 - x)^{10}$ at $x = 0$

02 05 ex 28 38. $f(t) = (3t - 2)^5$ at $t = 1$

02 05 ex 29 39. $g(\theta) = (\sin\theta + \cos\theta)^3$ at $\theta = \pi/2$

02 05 ex 30 40. $h(t) = e^{3t^2+t-1}$ at $t = -1$

02 05 ex 31 41. Compute $\frac{d}{dx}\big(\ln(kx)\big)$ two ways:

(a) Using the Chain Rule, and

(b) by first using the logarithm rule $\ln(ab) = \ln a + \ln b$, then taking the derivative.

02 05 ex 32 42. Compute $\frac{d}{dx}\big(\ln(x^k)\big)$ two ways:

(a) Using the Chain Rule, and

(b) by first using the logarithm rule $\ln(a^p) = p\ln a$, then taking the derivative.

Review

02 05 ex 33 43. The "wind chill factor" is a measurement of how cold it "feels" during cold, windy weather. Let $W(w)$ be the wind chill factor, in degrees Fahrenheit, when it is 25°F outside with a wind of w mph.

(a) What are the units of $W'(w)$?

(b) What would you expect the sign of $W'(10)$ to be?

02 05 ex 34 44. Find the derivatives of the following functions.

(a) $f(x) = x^2e^x\cot x$

(b) $g(x) = 2^x3^x4^x$

1.6 Implicit Differentiation

In the previous sections we learned to find the derivative, $\frac{dy}{dx}$, or y', when y is given *explicitly* as a function of x. That is, if we know $y = f(x)$ for some function f, we can find y'. For example, given $y = 3x^2 - 7$, we can easily find $y' = 6x$. (Here we explicitly state how x and y are related. Knowing x, we can directly find y.)

Sometimes the relationship between y and x is not explicit; rather, it is *implicit*. For instance, we might know that $x^2 - y = 4$. This equality defines a relationship between x and y; if we know x, we could figure out y. Can we still find y'? In this case, sure; we solve for y to get $y = x^2 - 4$ (hence we now know y explicitly) and then differentiate to get $y' = 2x$.

Sometimes the *implicit* relationship between x and y is complicated. Suppose we are given $\sin(y) + y^3 = 6 - x^3$. A graph of this implicit function is given in Figure 1.6.1. In this case there is absolutely no way to solve for y in terms of elementary functions. The surprising thing is, however, that we can still find y' via a process known as **implicit differentiation**.

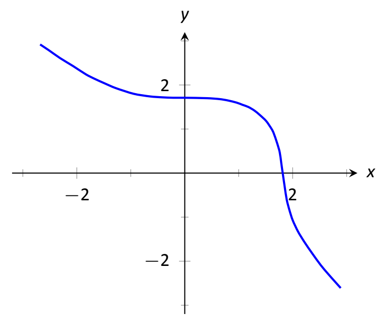


Figure 1.6.1: A graph of the implicit function $\sin(y) + y^3 = 6 - x^3$.

Implicit differentiation is a technique based on the Chain Rule that is used to find a derivative when the relationship between the variables is given implicitly rather than explicitly (solved for one variable in terms of the other).

We begin by reviewing the Chain Rule. Let f and g be functions of x. Then

$$\frac{d}{dx}\Big(f(g(x))\Big) = f'(g(x)) \cdot g'(x).$$

Suppose now that $y = g(x)$. We can rewrite the above as

$$\frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot y', \quad \text{or} \quad \frac{d}{dx}\Big(f(y)\Big) = f'(y) \cdot \frac{dy}{dx}. \tag{1.1}$$

These equations look strange; the key concept to learn here is that we can find y' even if we don't exactly know how y and x relate.

We demonstrate this process in the following example.

Example 1.6.1 Using Implicit Differentiation

Find y' given that $\sin(y) + y^3 = 6 - x^3$.

Solution We start by taking the derivative of both sides (thus maintaining the equality.) We have :

$$\frac{d}{dx}\Big(\sin(y) + y^3\Big) = \frac{d}{dx}\Big(6 - x^3\Big).$$

Notes:

The right hand side is easy; it returns $-3x^2$.

The left hand side requires more consideration. We take the derivative term–by–term. Using the technique derived from Equation 1.1 above, we can see that

$$\frac{d}{dx}\Big(\sin y\Big) = \cos y \cdot y'.$$

We apply the same process to the y^3 term.

$$\frac{d}{dx}\Big(y^3\Big) = \frac{d}{dx}\Big((y)^3\Big) = 3(y)^2 \cdot y'.$$

Putting this together with the right hand side, we have

$$\cos(y)y' + 3y^2y' = -3x^2.$$

Now solve for y'.

$$\begin{aligned} \cos(y)y' + 3y^2y' &= -3x^2. \\ \big(\cos y + 3y^2\big)y' &= -3x^2 \\ y' &= \frac{-3x^2}{\cos y + 3y^2} \end{aligned}$$

This equation for y' probably seems unusual for it contains both x and y terms. How is it to be used? We'll address that next.

Implicit functions are generally harder to deal with than explicit functions. With an explicit function, given an x value, we have an explicit formula for computing the corresponding y value. With an implicit function, one often has to find x and y values *at the same time* that satisfy the equation. It is much easier to demonstrate that a given point satisfies the equation than to actually find such a point.

For instance, we can affirm easily that the point $(\sqrt[3]{6}, 0)$ lies on the graph of the implicit function $\sin y + y^3 = 6 - x^3$. Plugging in 0 for y, we see the left hand side is 0. Setting $x = \sqrt[3]{6}$, we see the right hand side is also 0; the equation is satisfied. The following example finds the equation of the tangent line to this function at this point.

Example 1.6.2 Using Implicit Differentiation to find a tangent line

Find the equation of the line tangent to the curve of the implicitly defined function $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$.

Solution In Example 1.6.1 we found that

$$y' = \frac{-3x^2}{\cos y + 3y^2}.$$

Notes:

We find the slope of the tangent line at the point $(\sqrt[3]{6}, 0)$ by substituting $\sqrt[3]{6}$ for x and 0 for y. Thus at the point $(\sqrt[3]{6}, 0)$, we have the slope as

$$y' = \frac{-3(\sqrt[3]{6})^2}{\cos 0 + 3 \cdot 0^2} = \frac{-3\sqrt[3]{36}}{1} \approx -9.91.$$

Therefore the equation of the tangent line to the implicitly defined function $\sin y + y^3 = 6 - x^3$ at the point $(\sqrt[3]{6}, 0)$ is

$$y = -3\sqrt[3]{36}(x - \sqrt[3]{6}) + 0 \approx -9.91x + 18.$$

The curve and this tangent line are shown in Figure 1.6.2.

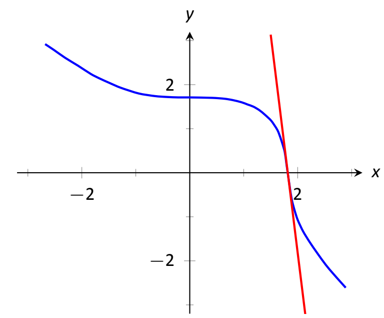


Figure 1.6.2: The function $\sin y + y^3 = 6 - x^3$ and its tangent line at the point $(\sqrt[3]{6}, 0)$.

This suggests a general method for implicit differentiation. For the steps below assume y is a function of x.

1. Take the derivative of each term in the equation. Treat the x terms like normal. When taking the derivatives of y terms, the usual rules apply except that, because of the Chain Rule, we need to multiply each term by y'.

2. Get all the y' terms on one side of the equal sign and put the remaining terms on the other side.

3. Factor out y'; solve for y' by dividing.

Practical Note: When working by hand, it may be beneficial to use the symbol $\frac{dy}{dx}$ instead of y', as the latter can be easily confused for y or y^1.

Example 1.6.3 Using Implicit Differentiation

Given the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$, find y'.

Solution We will take the implicit derivatives term by term. The derivative of y^3 is $3y^2y'$.

The second term, x^2y^4, is a little tricky. It requires the Product Rule as it is the product of two functions of x: x^2 and y^4. Its derivative is $x^2(4y^3y') + 2xy^4$. The first part of this expression requires a y' because we are taking the derivative of a y term. The second part does not require it because we are taking the derivative of x^2.

The derivative of the right hand side is easily found to be 2. In all, we get:

$$3y^2y' + 4x^2y^3y' + 2xy^4 = 2.$$

Move terms around so that the left side consists only of the y' terms and the right side consists of all the other terms:

$$3y^2y' + 4x^2y^3y' = 2 - 2xy^4.$$

Notes:

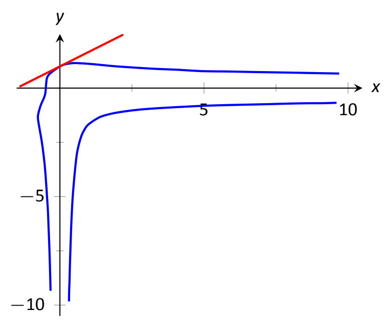


Figure 1.6.3: A graph of the implicitly defined function $y^3 + x^2y^4 = 1 + 2x$ along with its tangent line at the point $(0, 1)$.

Factor out y' from the left side and solve to get

$$y' = \frac{2 - 2xy^4}{3y^2 + 4x^2y^3}.$$

To confirm the validity of our work, let's find the equation of a tangent line to this function at a point. It is easy to confirm that the point $(0, 1)$ lies on the graph of this function. At this point, $y' = 2/3$. So the equation of the tangent line is $y = 2/3(x-0)+1$. The function and its tangent line are graphed in Figure 1.6.3.

Notice how our function looks much different than other functions we have seen. For one, it fails the vertical line test. Such functions are important in many areas of mathematics, so developing tools to deal with them is also important.

Example 1.6.4 Using Implicit Differentiation

Given the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$, find y'.

Solution Differentiating term by term, we find the most difficulty in the first term. It requires both the Chain and Product Rules.

$$\begin{aligned}\frac{d}{dx}\Big(\sin(x^2y^2)\Big) &= \cos(x^2y^2)\cdot\frac{d}{dx}\Big(x^2y^2\Big)\\ &= \cos(x^2y^2)\cdot\big(x^2(2yy') + 2xy^2\big)\\ &= 2(x^2yy' + xy^2)\cos(x^2y^2).\end{aligned}$$

We leave the derivatives of the other terms to the reader. After taking the derivatives of both sides, we have

$$2(x^2yy' + xy^2)\cos(x^2y^2) + 3y^2y' = 1 + y'.$$

We now have to be careful to properly solve for y', particularly because of the product on the left. It is best to multiply out the product. Doing this, we get

$$2x^2y\cos(x^2y^2)y' + 2xy^2\cos(x^2y^2) + 3y^2y' = 1 + y'.$$

From here we can safely move around terms to get the following:

$$2x^2y\cos(x^2y^2)y' + 3y^2y' - y' = 1 - 2xy^2\cos(x^2y^2).$$

Then we can solve for y' to get

$$y' = \frac{1 - 2xy^2\cos(x^2y^2)}{2x^2y\cos(x^2y^2) + 3y^2 - 1}.$$

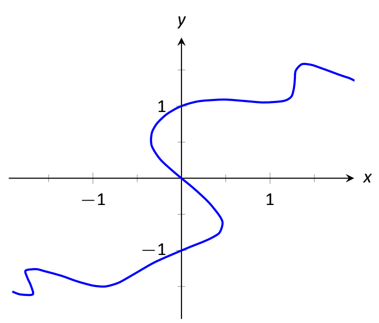


Figure 1.6.4: A graph of the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$.

Notes:

A graph of this implicit function is given in Figure 1.6.4. It is easy to verify that the points $(0,0)$, $(0,1)$ and $(0,-1)$ all lie on the graph. We can find the slopes of the tangent lines at each of these points using our formula for y'.

At $(0,0)$, the slope is -1.

At $(0,1)$, the slope is $1/2$.

At $(0,-1)$, the slope is also $1/2$.

The tangent lines have been added to the graph of the function in Figure 1.6.5.

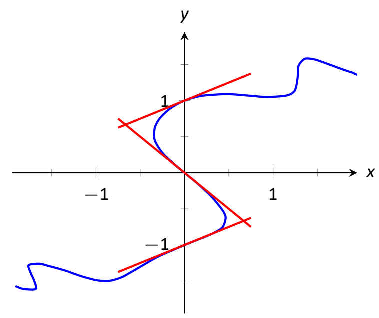


Figure 1.6.5: A graph of the implicitly defined function $\sin(x^2y^2)+y^3=x+y$ and certain tangent lines.

Quite a few "famous" curves have equations that are given implicitly. We can use implicit differentiation to find the slope at various points on those curves. We investigate two such curves in the next examples.

Example 1.6.5 Finding slopes of tangent lines to a circle

Find the slope of the tangent line to the circle $x^2+y^2=1$ at the point $(1/2, \sqrt{3}/2)$.

SOLUTION Taking derivatives, we get $2x+2yy'=0$. Solving for y' gives:

$$y' = \frac{-x}{y}.$$

This is a clever formula. Recall that the slope of the line through the origin and the point (x,y) on the circle will be y/x. We have found that the slope of the tangent line to the circle at that point is the opposite reciprocal of y/x, namely, $-x/y$. Hence these two lines are always perpendicular.

At the point $(1/2, \sqrt{3}/2)$, we have the tangent line's slope as

$$y' = \frac{-1/2}{\sqrt{3}/2} = \frac{-1}{\sqrt{3}} \approx -0.577.$$

A graph of the circle and its tangent line at $(1/2, \sqrt{3}/2)$ is given in Figure 1.6.6, along with a thin dashed line from the origin that is perpendicular to the tangent line. (It turns out that all normal lines to a circle pass through the center of the circle.)

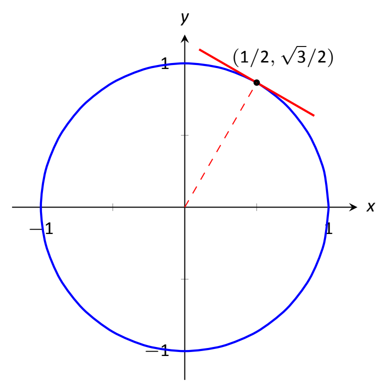


Figure 1.6.6: The unit circle with its tangent line at $(1/2, \sqrt{3}/2)$.

This section has shown how to find the derivatives of implicitly defined functions, whose graphs include a wide variety of interesting and unusual shapes. Implicit differentiation can also be used to further our understanding of "regular" differentiation.

One hole in our current understanding of derivatives is this: what is the derivative of the square root function? That is,

$$\frac{d}{dx}\left(\sqrt{x}\right) = \frac{d}{dx}\left(x^{1/2}\right) = ?$$

Notes:

We allude to a possible solution, as we can write the square root function as a power function with a rational (or, fractional) power. We are then tempted to apply the Power Rule and obtain

$$\frac{d}{dx}\big(x^{1/2}\big) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}.$$

The trouble with this is that the Power Rule was initially defined only for positive integer powers, $n > 0$. While we did not justify this at the time, generally the Power Rule is proved using something called the Binomial Theorem, which deals only with positive integers. The Quotient Rule allowed us to extend the Power Rule to negative integer powers. Implicit Differentiation allows us to extend the Power Rule to rational powers, as shown below.

Let $y = x^{m/n}$, where m and n are integers with no common factors (so $m = 2$ and $n = 5$ is fine, but $m = 2$ and $n = 4$ is not). We can rewrite this explicit function implicitly as $y^n = x^m$. Now apply implicit differentiation.

$$\begin{aligned}
y &= x^{m/n}\\
y^n &= x^m\\
\frac{d}{dx}\big(y^n\big) &= \frac{d}{dx}\big(x^m\big)\\
n\cdot y^{n-1}\cdot y' &= m\cdot x^{m-1}\\
y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \qquad \text{(now substitute } x^{m/n} \text{ for } y\text{)}\\
&= \frac{m}{n}\frac{x^{m-1}}{\big(x^{m/n}\big)^{n-1}} \qquad \text{(apply lots of algebra)}\\
&= \frac{m}{n}x^{(m-n)/n}\\
&= \frac{m}{n}x^{m/n-1}.
\end{aligned}$$

The above derivation is the key to the proof extending the Power Rule to rational powers. Using limits, we can extend this once more to include *all* powers, including irrational (even transcendental!) powers, giving the following theorem.

Theorem 1.6.1 Power Rule for Differentiation

Let $f(x) = x^n$, where $n \neq 0$ is a real number. Then f is differentiable on its domain, except possibly at $x = 0$, and $f'(x) = n\cdot x^{n-1}$.

Notes:

This theorem allows us to say the derivative of x^{π} is $\pi x^{\pi-1}$.

We now apply this final version of the Power Rule in the next example, the second investigation of a "famous" curve.

Example 1.6.6 Using the Power Rule

Find the slope of $x^{2/3} + y^{2/3} = 8$ at the point $(8, 8)$.

SOLUTION This is a particularly interesting curve called an *astroid*. It is the shape traced out by a point on the edge of a circle that is rolling around inside of a larger circle, as shown in Figure 1.6.7.

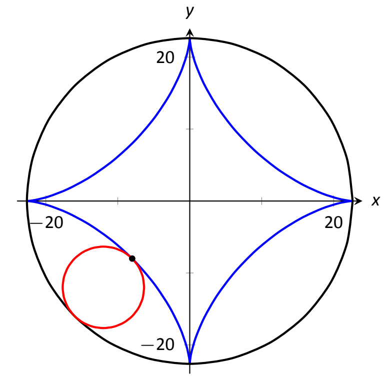


Figure 1.6.7: An astroid, traced out by a point on the smaller circle as it rolls inside the larger circle.

To find the slope of the astroid at the point $(8, 8)$, we take the derivative implicitly.

$$\begin{aligned}
\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}y' &= 0 \\
\frac{2}{3}y^{-1/3}y' &= -\frac{2}{3}x^{-1/3} \\
y' &= -\frac{x^{-1/3}}{y^{-1/3}} \\
y' &= -\frac{y^{1/3}}{x^{1/3}} = -\sqrt[3]{\frac{y}{x}}.
\end{aligned}$$

Plugging in $x = 8$ and $y = 8$, we get a slope of -1. The astroid, with its tangent line at $(8, 8)$, is shown in Figure 1.6.8.

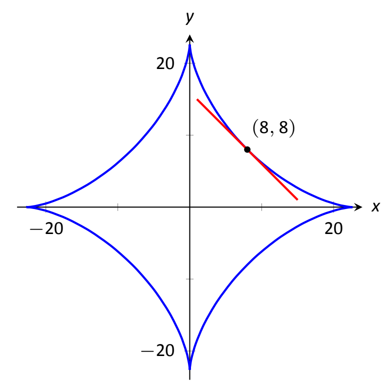


Figure 1.6.8: An astroid with a tangent line.

Implicit Differentiation and the Second Derivative

We can use implicit differentiation to find higher order derivatives. In theory, this is simple: first find $\frac{dy}{dx}$, then take its derivative with respect to x. In practice, it is not hard, but it often requires a bit of algebra. We demonstrate this in an example.

Example 1.6.7 Finding the second derivative

Given $x^2 + y^2 = 1$, find $\dfrac{d^2y}{dx^2} = y''$.

SOLUTION We found that $y' = \frac{dy}{dx} = -x/y$ in Example 1.6.5. To find y'',

Notes:

we apply implicit differentiation to y'.

$$\begin{aligned} y'' &= \frac{d}{dx}\left(y'\right) \\ &= \frac{d}{dx}\left(-\frac{x}{y}\right) \qquad \text{(Now use the Quotient Rule.)} \\ &= -\frac{y(1) - x(y')}{y^2} \end{aligned}$$

replace y' with $-x/y$:

$$\begin{aligned} &= -\frac{y - x(-x/y)}{y^2} \\ &= -\frac{y + x^2/y}{y^2}. \end{aligned}$$

While this is not a particularly simple expression, it is usable. We can see that $y'' > 0$ when $y < 0$ and $y'' < 0$ when $y > 0$. In Section 3.4, we will see how this relates to the shape of the graph.

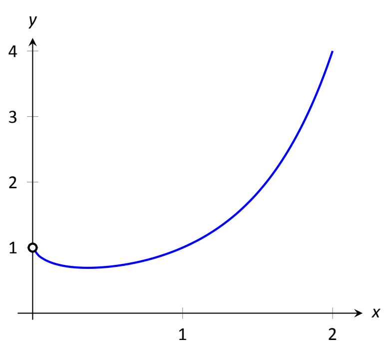


Figure 1.6.9: A plot of $y = x^x$.

Logarithmic Differentiation

Consider the function $y = x^x$; it is graphed in Figure 1.6.9. It is well–defined for $x > 0$ and we might be interested in finding equations of lines tangent and normal to its graph. How do we take its derivative?

The function is not a power function: it has a "power" of x, not a constant. It is not an exponential function: it has a "base" of x, not a constant.

A differentiation technique known as *logarithmic differentiation* becomes useful here. The basic principle is this: take the natural log of both sides of an equation $y = f(x)$, then use implicit differentiation to find y'. We demonstrate this in the following example.

Example 1.6.8 Using Logarithmic Differentiation

Given $y = x^x$, use logarithmic differentiation to find y'.

SOLUTION As suggested above, we start by taking the natural log of

Notes:

both sides then applying implicit differentiation.

$$\begin{aligned}
y &= x^x \\
\ln(y) &= \ln(x^x) && \text{(apply logarithm rule)} \\
\ln(y) &= x\ln x && \text{(now use implicit differentiation)} \\
\frac{d}{dx}\Big(\ln(y)\Big) &= \frac{d}{dx}\Big(x\ln x\Big) \\
\frac{y'}{y} &= \ln x + x\cdot\frac{1}{x} \\
\frac{y'}{y} &= \ln x + 1 \\
y' &= y\big(\ln x + 1\big) && \text{(substitute } y = x^x\text{)} \\
y' &= x^x\big(\ln x + 1\big).
\end{aligned}$$

To "test" our answer, let's use it to find the equation of the tangent line at $x = 1.5$. The point on the graph our tangent line must pass through is $(1.5, 1.5^{1.5}) \approx (1.5, 1.837)$. Using the equation for y', we find the slope as

$$y' = 1.5^{1.5}\big(\ln 1.5 + 1\big) \approx 1.837(1.405) \approx 2.582.$$

Thus the equation of the tangent line is $y = 1.6833(x - 1.5) + 1.837$. Figure 1.6.10 graphs $y = x^x$ along with this tangent line.

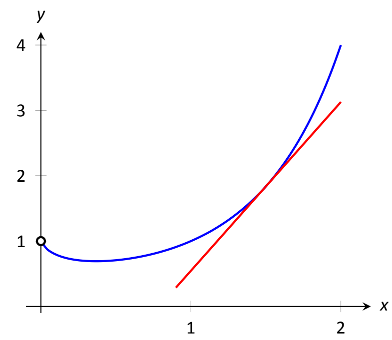


Figure 1.6.10: A graph of $y = x^x$ and its tangent line at $x = 1.5$.

Implicit differentiation proves to be useful as it allows us to find the instantaneous rates of change of a variety of functions. In particular, it extended the Power Rule to rational exponents, which we then extended to all real numbers. In the next section, implicit differentiation will be used to find the derivatives of *inverse* functions, such as $y = \sin^{-1} x$.

Notes:

Exercises 1.6

Terms and Concepts

02 06 ex 01 1. In your own words, explain the difference between implicit functions and explicit functions.

02 06 ex 02 2. Implicit differentiation is based on what other differentiation rule?

02 06 ex 03 3. T/F: Implicit differentiation can be used to find the derivative of $y = \sqrt{x}$.

02 06 ex 04 4. T/F: Implicit differentiation can be used to find the derivative of $y = x^{3/4}$.

Problems

02 06 exset 01 **In Exercises 5 – 12, compute the derivative of the given function.**

02 05 ex 50 5. $f(x) = \sqrt{x} + \dfrac{1}{\sqrt{x}}$

02 05 ex 51 6. $f(x) = \sqrt[3]{x} + x^{2/3}$

02 06 ex 06 7. $f(t) = \sqrt{1-t^2}$

02 06 ex 07 8. $g(t) = \sqrt{t}\sin t$

02 06 ex 08 9. $h(x) = x^{1.5}$

02 05 ex 52 10. $f(x) = x^{\pi} + x^{1.9} + \pi^{1.9}$

02 05 ex 40 11. $g(x) = \dfrac{x+7}{\sqrt{x}}$

02 05 ex 41 12. $f(t) = \sqrt[5]{t}(\sec t + e^t)$

02 06 exset 02 **In Exercises 13 – 25, find $\dfrac{dy}{dx}$ using implicit differentiation.**

02 06 ex 09 13. $x^4 + y^2 + y = 7$

02 06 ex 10 14. $x^{2/5} + y^{2/5} = 1$

02 06 ex 11 15. $\cos(x) + \sin(y) = 1$

02 06 ex 12 16. $\dfrac{x}{y} = 10$

02 06 ex 13 17. $\dfrac{y}{x} = 10$

02 06 ex 14 18. $x^2e^2 + 2^y = 5$

02 06 ex 15 19. $x^2\tan y = 50$

02 06 ex 16 20. $(3x^2 + 2y^3)^4 = 2$

02 06 ex 17 21. $(y^2 + 2y - x)^2 = 200$

02 06 ex 18 22. $\dfrac{x^2+y}{x+y^2} = 17$

02 06 ex 19 23. $\dfrac{\sin(x)+y}{\cos(y)+x} = 1$

02 06 ex 21 24. $\ln(x^2 + y^2) = e$

02 06 ex 22 25. $\ln(x^2 + xy + y^2) = 1$

02 06 ex 20 26. Show that $\dfrac{dy}{dx}$ is the same for each of the following implicitly defined functions.

(a) $xy = 1$

(b) $x^2y^2 = 1$

(c) $\sin(xy) = 1$

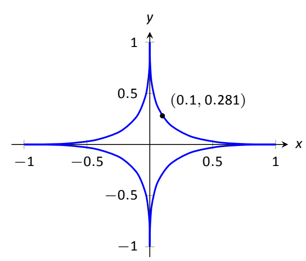
(d) $\ln(xy) = 1$

02 06 exset 03 **In Exercises 27 – 32, find the equation of the tangent line to the graph of the implicitly defined function at the indicated points. As a visual aid, each function is graphed.**

02 06 ex 23 27. $x^{2/5} + y^{2/5} = 1$

(a) At $(1, 0)$.

(b) At $(0.1, 0.281)$ (which does not *exactly* lie on the curve, but is very close).

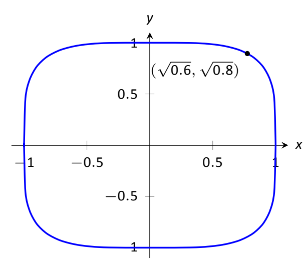


02 06 ex 24 28. $x^4 + y^4 = 1$

(a) At $(1, 0)$.

(b) At $(\sqrt{0.6}, \sqrt{0.8})$.

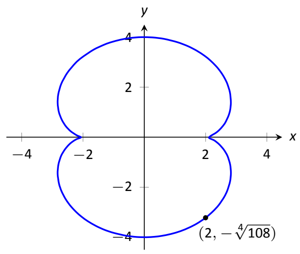
(c) At $(0, 1)$.



02 06 ex 25 29. $(x^2+y^2-4)^3=108y^2$

(a) At $(0,4)$.

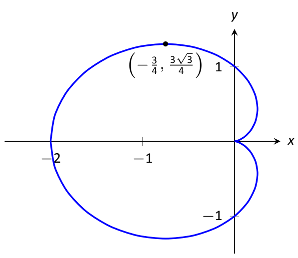
(b) At $(2,-\sqrt[4]{108})$.



02 06 ex 26 30. $(x^2+y^2+x)^2=x^2+y^2$

(a) At $(0,1)$.

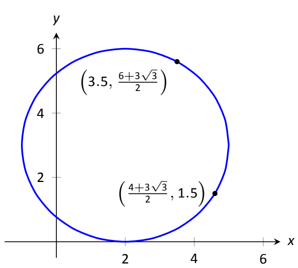
(b) At $\left(-\dfrac{3}{4},\dfrac{3\sqrt{3}}{4}\right)$.



02 06 ex 27 31. $(x-2)^2+(y-3)^2=9$

(a) At $\left(\dfrac{7}{2},\dfrac{6+3\sqrt{3}}{2}\right)$.

(b) At $\left(\dfrac{4+3\sqrt{3}}{2},\dfrac{3}{2}\right)$.

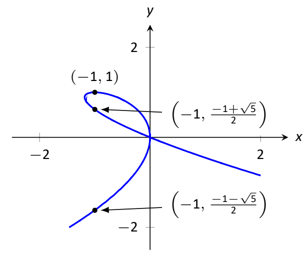


02 06 ex 38 32. $x^2+y^3+2xy=0$

(a) At $(-1,1)$.

(b) At $\left(-1,\dfrac{1}{2}(-1+\sqrt{5})\right)$.

(c) At $\left(-1,\dfrac{1}{2}(-1-\sqrt{5})\right)$.



02 06 exset 04 **In Exercises 33 – 36, an implicitly defined function is given. Find $\dfrac{d^2y}{dx^2}$. Note: these are the same problems used in Exercises 13 through 16.**

02 06 ex 28 33. $x^4+y^2+y=7$

02 06 ex 29 34. $x^{2/5}+y^{2/5}=1$

02 06 ex 30 35. $\cos x+\sin y=1$

02 06 ex 31 36. $\dfrac{x}{y}=10$

02 06 exset 05 **In Exercises 37 – 42, use logarithmic differentiation to find $\dfrac{dy}{dx}$, then find the equation of the tangent line at the indicated x–value.**

02 06 ex 32 37. $y=(1+x)^{1/x},\quad x=1$

02 06 ex 33 38. $y=(2x)^{x^2},\quad x=1$

02 06 ex 34 39. $y=\dfrac{x^x}{x+1},\quad x=1$

02 06 ex 35 40. $y=x^{\sin(x)+2},\quad x=\pi/2$

02 06 ex 36 41. $y=\dfrac{x+1}{x+2},\quad x=1$

02 06 ex 37 42. $y=\dfrac{(x+1)(x+2)}{(x+3)(x+4)},\quad x=0$

1.7 Derivatives of Inverse Functions

Recall that a function $y = f(x)$ is said to be *one to one* if it passes the horizontal line test; that is, for two different x values x_1 and x_2, we do *not* have $f(x_1) = f(x_2)$. In some cases the domain of f must be restricted so that it is one to one. For instance, consider $f(x) = x^2$. Clearly, $f(-1) = f(1)$, so f is not one to one on its regular domain, but by restricting f to $(0, \infty)$, f is one to one.

Now recall that one to one functions have *inverses*. That is, if f is one to one, it has an inverse function, denoted by f^{-1}, such that if $f(a) = b$, then $f^{-1}(b) = a$. The domain of f^{-1} is the range of f, and vice-versa. For ease of notation, we set $g = f^{-1}$ and treat g as a function of x.

Since $f(a) = b$ implies $g(b) = a$, when we compose f and g we get a nice result:

$$f\big(g(b)\big) = f(a) = b.$$

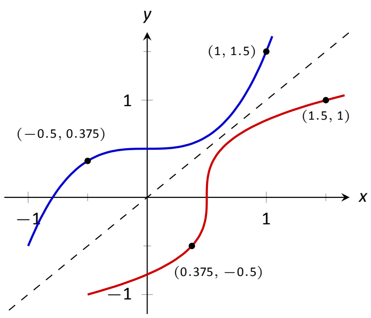


Figure 1.7.1: A function f along with its inverse f^{-1}. (Note how it does not matter which function we refer to as f; the other is f^{-1}.)

In general, $f\big(g(x)\big) = x$ and $g\big(f(x)\big) = x$. This gives us a convenient way to check if two functions are inverses of each other: compose them and if the result is x, then they are inverses (on the appropriate domains.)

When the point (a, b) lies on the graph of f, the point (b, a) lies on the graph of g. This leads us to discover that the graph of g is the reflection of f across the line $y = x$. In Figure 1.7.1 we see a function graphed along with its inverse. See how the point $(1, 1.5)$ lies on one graph, whereas $(1.5, 1)$ lies on the other. Because of this relationship, whatever we know about f can quickly be transferred into knowledge about g.

For example, consider Figure 1.7.2 where the tangent line to f at the point (a, b) is drawn. That line has slope $f'(a)$. Through reflection across $y = x$, we can see that the tangent line to g at the point (b, a) should have slope $\dfrac{1}{f'(a)}$. This then tells us that $g'(b) = \dfrac{1}{f'(a)}$.

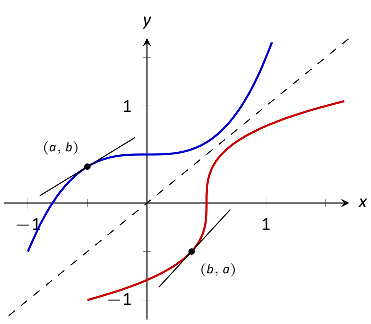


Figure 1.7.2: Corresponding tangent lines drawn to f and f^{-1}.

Consider:

Information about f	Information about $g = f^{-1}$
$(-0.5, 0.375)$ lies on f	$(0.375, -0.5)$ lies on g
Slope of tangent line to f at $x = -0.5$ is $3/4$	Slope of tangent line to g at $x = 0.375$ is $4/3$
$f'(-0.5) = 3/4$	$g'(0.375) = 4/3$

We have discovered a relationship between f' and g' in a mostly graphical way. We can realize this relationship analytically as well. Let $y = g(x)$, where again $g = f^{-1}$. We want to find y'. Since $y = g(x)$, we know that $f(y) = x$. Using the Chain Rule and Implicit Differentiation, take the derivative of both sides of

Notes:

this last equality.

$$\begin{aligned}
\frac{d}{dx}\Big(f(y)\Big) &= \frac{d}{dx}\Big(x\Big)\\
f'(y)\cdot y' &= 1\\
y' &= \frac{1}{f'(y)}\\
y' &= \frac{1}{f'(g(x))}.
\end{aligned}$$

This leads us to the following theorem.

Theorem 1.7.1 Derivatives of Inverse Functions

Let f be differentiable and one to one on an open interval I, where $f'(x) \neq 0$ for all x in I, let J be the range of f on I, let g be the inverse function of f, and let $f(a) = b$ for some a in I. Then g is a differentiable function on J, and in particular,

1. $\left(f^{-1}\right)'(b) = g'(b) = \dfrac{1}{f'(a)}$ and 2. $\left(f^{-1}\right)'(x) = g'(x) = \dfrac{1}{f'(g(x))}$

The results of Theorem 1.7.1 are not trivial; the notation may seem confusing at first. Careful consideration, along with examples, should earn understanding.

In the next example we apply Theorem 1.7.1 to the arcsine function.

Example 1.7.1 Finding the derivative of an inverse trigonometric function

Let $y = \arcsin x = \sin^{-1} x$. Find y' using Theorem 1.7.1.

Solution Adopting our previously defined notation, let $g(x) = \arcsin x$ and $f(x) = \sin x$. Thus $f'(x) = \cos x$. Applying the theorem, we have

$$\begin{aligned}
g'(x) &= \frac{1}{f'(g(x))}\\
&= \frac{1}{\cos(\arcsin x)}.
\end{aligned}$$

This last expression is not immediately illuminating. Drawing a figure will help, as shown in Figure 1.7.4. Recall that the sine function can be viewed as taking in an angle and returning a ratio of sides of a right triangle, specifically, the ratio "opposite over hypotenuse." This means that the arcsine function takes as input a ratio of sides and returns an angle. The equation $y = \arcsin x$ can be rewritten as $y = \arcsin(x/1)$; that is, consider a right triangle where the

Notes:

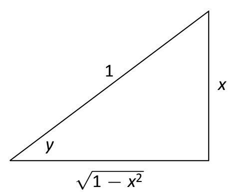


Figure 1.7.4: A right triangle defined by $y = \sin^{-1}(x/1)$ with the length of the third leg found using the Pythagorean Theorem.

hypotenuse has length 1 and the side opposite of the angle with measure y has length x. This means the final side has length $\sqrt{1-x^2}$, using the Pythagorean Theorem.

Therefore $\cos(\sin^{-1} x) = \cos y = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$, resulting in

$$\frac{d}{dx}\big(\arcsin x\big) = g'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Remember that the input x of the arcsine function is a ratio of a side of a right triangle to its hypotenuse; the absolute value of this ratio will never be greater than 1. Therefore the inside of the square root will never be negative.

In order to make $y = \sin x$ one to one, we restrict its domain to $[-\pi/2, \pi/2]$; on this domain, the range is $[-1, 1]$. Therefore the domain of $y = \arcsin x$ is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$. When $x = \pm 1$, note how the derivative of the arcsine function is undefined; this corresponds to the fact that as $x \to \pm 1$, the tangent lines to arcsine approach vertical lines with undefined slopes.

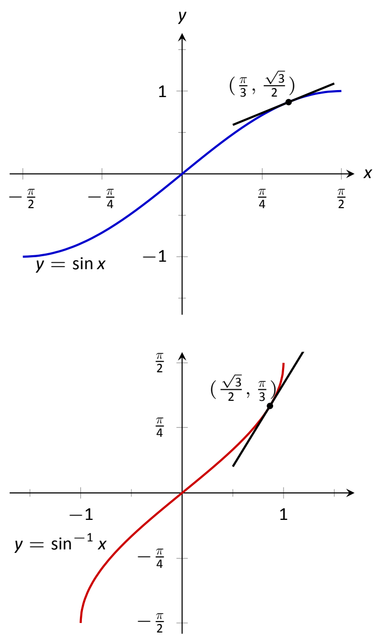


Figure 1.7.5: Graphs of $\sin x$ and $\sin^{-1} x$ along with corresponding tangent lines.

In Figure 1.7.5 we see $f(x) = \sin x$ and $f^{-1}(x) = \sin^{-1} x$ graphed on their respective domains. The line tangent to $\sin x$ at the point $(\pi/3, \sqrt{3}/2)$ has slope $\cos \pi/3 = 1/2$. The slope of the corresponding point on $\sin^{-1} x$, the point $(\sqrt{3}/2, \pi/3)$, is

$$\frac{1}{\sqrt{1-(\sqrt{3}/2)^2}} = \frac{1}{\sqrt{1-3/4}} = \frac{1}{\sqrt{1/4}} = \frac{1}{1/2} = 2,$$

verifying yet again that at corresponding points, a function and its inverse have reciprocal slopes.

Using similar techniques, we can find the derivatives of all the inverse trigonometric functions. In Figure 1.7.3 we show the restrictions of the domains of the standard trigonometric functions that allow them to be invertible.

Notes:

Function	Domain	Range	Inverse Function	Domain	Range
$\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$	$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos x$	$[0, \pi]$	$[-1, 1]$	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan x$	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$	$\tan^{-1} x$	$(-\infty, \infty)$	$(-\pi/2, \pi/2)$
$\csc x$	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$	$\csc^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[-\pi/2, 0) \cup (0, \pi/2]$
$\sec x$	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$
$\cot x$	$(0, \pi)$	$(-\infty, \infty)$	$\cot^{-1} x$	$(-\infty, \infty)$	$(0, \pi)$

Figure 1.7.3: Domains and ranges of the trigonometric and inverse trigonometric functions.

Theorem 1.7.2 Derivatives of Inverse Trigonometric Functions

The inverse trigonometric functions are differentiable on all open sets contained in their domains (as listed in Figure 1.7.3) and their derivatives are as follows:

1. $\dfrac{d}{dx}\big(\sin^{-1} x\big) = \dfrac{1}{\sqrt{1-x^2}}$
2. $\dfrac{d}{dx}\big(\sec^{-1} x\big) = \dfrac{1}{|x|\sqrt{x^2-1}}$
3. $\dfrac{d}{dx}\big(\tan^{-1} x\big) = \dfrac{1}{1+x^2}$
4. $\dfrac{d}{dx}\big(\cos^{-1} x\big) = -\dfrac{1}{\sqrt{1-x^2}}$
5. $\dfrac{d}{dx}\big(\csc^{-1} x\big) = -\dfrac{1}{|x|\sqrt{x^2-1}}$
6. $\dfrac{d}{dx}\big(\cot^{-1} x\big) = -\dfrac{1}{1+x^2}$

Note how the last three derivatives are merely the opposites of the first three, respectively. Because of this, the first three are used almost exclusively throughout this text.

In Section 1.3, we stated without proof or explanation that $\dfrac{d}{dx}\big(\ln x\big) = \dfrac{1}{x}$. We can justify that now using Theorem 1.7.1, as shown in the example.

Example 1.7.2 Finding the derivative of $y = \ln x$

Use Theorem 1.7.1 to compute $\dfrac{d}{dx}\big(\ln x\big)$.

Solution View $y = \ln x$ as the inverse of $y = e^x$. Therefore, using our standard notation, let $f(x) = e^x$ and $g(x) = \ln x$. We wish to find $g'(x)$. Theorem

---

Notes:

1.7.1 gives:

$$\begin{aligned} g'(x) &= \frac{1}{f'(g(x))} \\ &= \frac{1}{e^{\ln x}} \\ &= \frac{1}{x}. \end{aligned}$$

In this chapter we have defined the derivative, given rules to facilitate its computation, and given the derivatives of a number of standard functions. We restate the most important of these in the following theorem, intended to be a reference for further work.

Theorem 1.7.3 Glossary of Derivatives of Elementary Functions

Let u and v be differentiable functions, and let a, c and n be real numbers, $a > 0$, $n \neq 0$.

1. $\frac{d}{dx}\big(cu\big) = cu'$
2. $\frac{d}{dx}\big(u \pm v\big) = u' \pm v'$
3. $\frac{d}{dx}\big(u \cdot v\big) = uv' + u'v$
4. $\frac{d}{dx}\big(\frac{u}{v}\big) = \frac{u'v - uv'}{v^2}$
5. $\frac{d}{dx}\big(u(v)\big) = u'(v)v'$
6. $\frac{d}{dx}\big(c\big) = 0$
7. $\frac{d}{dx}\big(x\big) = 1$
8. $\frac{d}{dx}\big(x^n\big) = nx^{n-1}$
9. $\frac{d}{dx}\big(e^x\big) = e^x$
10. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$
11. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$
12. $\frac{d}{dx}\big(\log_a x\big) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}\big(\sin x\big) = \cos x$
14. $\frac{d}{dx}\big(\cos x\big) = -\sin x$
15. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$
16. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$
17. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$
18. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$
19. $\frac{d}{dx}\big(\sin^{-1} x\big) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\big(\cos^{-1} x\big) = -\frac{1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\big(\sec^{-1} x\big) = \frac{1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\big(\csc^{-1} x\big) = -\frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\big(\cot^{-1} x\big) = -\frac{1}{1+x^2}$

Notes:

Exercises 1.7

Terms and Concepts

02 07 ex 01 1. T/F: Every function has an inverse.

02 07 ex 02 2. In your own words explain what it means for a function to be "one to one."

02 07 ex 03 3. If $(1, 10)$ lies on the graph of $y = f(x)$, what can be said about the graph of $y = f^{-1}(x)$?

02 07 ex 04 4. If $(1, 10)$ lies on the graph of $y = f(x)$ and $f'(1) = 5$, what can be said about $y = f^{-1}(x)$?

Problems

02 07 exset 01 **In Exercises 5 – 8, verify that the given functions are inverses.**

02 07 ex 05 5. $f(x) = 2x + 6$ and $g(x) = \frac{1}{2}x - 3$

02 07 ex 06 6. $f(x) = x^2 + 6x + 11$, $x \geq 3$ and
$g(x) = \sqrt{x - 2} - 3$, $x \geq 2$

02 07 ex 07 7. $f(x) = \dfrac{3}{x - 5}$, $x \neq 5$ and
$g(x) = \dfrac{3 + 5x}{x}$, $x \neq 0$

02 07 ex 08 8. $f(x) = \dfrac{x + 1}{x - 1}$, $x \neq 1$ and $g(x) = f(x)$

02 07 exset 02 **In Exercises 9 – 14, an invertible function $f(x)$ is given along with a point that lies on its graph. Using Theorem 1.7.1, evaluate $\left(f^{-1}\right)'(x)$ at the indicated value.**

02 07 ex 09 9. $f(x) = 5x + 10$
Point$= (2, 20)$
Evaluate $\left(f^{-1}\right)'(20)$

02 07 ex 10 10. $f(x) = x^2 - 2x + 4$, $x \geq 1$
Point$= (3, 7)$
Evaluate $\left(f^{-1}\right)'(7)$

02 07 ex 11 11. $f(x) = \sin 2x$, $-\pi/4 \leq x \leq \pi/4$
Point$= (\pi/6, \sqrt{3}/2)$
Evaluate $\left(f^{-1}\right)'(\sqrt{3}/2)$

02 07 ex 12 12. $f(x) = x^3 - 6x^2 + 15x - 2$
Point$= (1, 8)$
Evaluate $\left(f^{-1}\right)'(8)$

02 07 ex 13 13. $f(x) = \dfrac{1}{1 + x^2}$, $x \geq 0$
Point$= (1, 1/2)$
Evaluate $\left(f^{-1}\right)'(1/2)$

02 07 ex 14 14. $f(x) = 6e^{3x}$
Point$= (0, 6)$
Evaluate $\left(f^{-1}\right)'(6)$

02 07 exset 03 **In Exercises 15 – 24, compute the derivative of the given function.**

02 07 ex 15 15. $h(t) = \sin^{-1}(2t)$

02 07 ex 16 16. $f(t) = \sec^{-1}(2t)$

02 07 ex 17 17. $g(x) = \tan^{-1}(2x)$

02 07 ex 18 18. $f(x) = x \sin^{-1} x$

02 07 ex 19 19. $g(t) = \sin t \cos^{-1} t$

02 07 ex 20 20. $f(t) = \ln te^t$

02 07 ex 21 21. $h(x) = \dfrac{\sin^{-1} x}{\cos^{-1} x}$

02 07 ex 22 22. $g(x) = \tan^{-1}(\sqrt{x})$

02 07 ex 23 23. $f(x) = \sec^{-1}(1/x)$

02 07 ex 24 24. $f(x) = \sin(\sin^{-1} x)$

02 07 exset 04 **In Exercises 25 – 26, compute the derivative of the given function in two ways:**

(a) By simplifying first, then taking the derivative, and

(b) by using the Chain Rule first then simplifying.

Verify that the two answers are the same.

02 07 ex 24 25. $f(x) = \sin(\sin^{-1} x)$

02 07 ex 25 26. $f(x) = \tan^{-1}(\tan x)$

02 07 exset 05 **In Exercises 27 – 28, find the equation of the line tangent to the graph of f at the indicated x value.**

02 07 ex 27 27. $f(x) = \sin^{-1} x \quad \text{at} \quad x = \frac{\sqrt{2}}{2}$

02 07 ex 28 28. $f(x) = \cos^{-1}(2x) \quad \text{at} \quad x = \frac{\sqrt{3}}{4}$

Review

02 07 ex 29 29. Find $\frac{dy}{dx}$, where $x^2y - y^2x = 1$.

02 07 ex 30 30. Find the equation of the line tangent to the graph of $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.

02 07 ex 31 31. Let $f(x) = x^3 + x$.
Evaluate $\displaystyle\lim_{s \to 0} \frac{f(x + s) - f(x)}{s}$.

A: SOLUTIONS TO SELECTED PROBLEMS

Chapter 1

Section 1.1

02 01 ex 01 1. T

02 01 ex 02 2. T

02 01 ex 03 3. Answers will vary.

02 01 ex 04 4. Answers will vary.

02 01 ex 05 5. Answers will vary.

02 01 ex 35 6. The two lines have opposite–reciprocal slopes.

02 01 ex 10 7. $f'(x) = 0$

02 01 ex 11 8. $f'(x) = 2$

02 01 ex 12 9. $f'(t) = -3$

02 01 ex 13 10. $g'(x) = 2x$

02 01 ex 36 11. $h'(x) = 3x^2$

02 01 ex 14 12. $f''(x) = 6x - 1$

02 01 ex 15 13. $r'(x) = \frac{-1}{x^2}$

02 01 ex 16 14. $r'(s) = \frac{-1}{(s-2)^2}$

02 01 ex 17 15.

(a) $y = 6$

(b) $x = -2$

02 01 ex 18 16.

(a) $y = 2x$

(b) $y = -1/2x$

02 01 ex 19 17.

(a) $y = -3x + 4$

(b) $y = 1/3(x - 7) - 17$

02 01 ex 20 18.

(a) $y = 4(x - 2) + 4$

(b) $y = -1/4(x - 2) + 4$

02 01 ex 37 19.

(a) $y = 48(x - 4) + 64$

(b) $y = -\frac{1}{48}(x - 4) + 64$

02 01 ex 21 20.

(a) $y = -7(x + 1) + 8$

(b) $y = 1/7(x + 1) + 8$

02 01 ex 22 21.

(a) $y = -1/4(x + 2) - 1/2$

(b) $y = 4(x + 2) - 1/2$

02 01 ex 23 22.

(a) $y = -1(x - 3) + 1$

(b) $y = 1(x - 3) + 1$

02 01 ex 06 23. $y = 8.1(x - 3) + 16$

02 01 ex 07 24. $y = -0.099(x - 9) + 1$

02 01 ex 08 25. $y = 7.77(x - 2) + e^2$, or $y = 7.77(x - 2) + 7.39$.

02 01 ex 09 26. $y = -0.05x + 1$

02 01 ex 24 27.

(a) Approximations will vary; they should match (c) closely.

(b) $f'(x) = 2x$

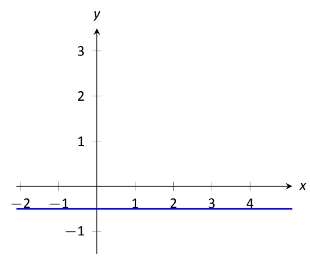
(c) At $(-1, 0)$, slope is -2. At $(0, -1)$, slope is 0. At $(2, 3)$, slope is 4.

02 01 ex 25 28.

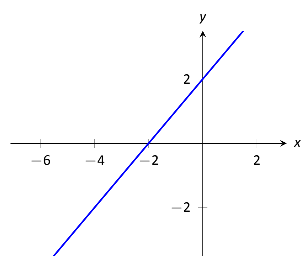
(a) Approximations will vary; they should match (c) closely.

(b) $f'(x) = -1/(x + 1)^2$

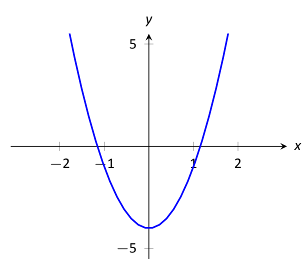
(c) At $(0, 1)$, slope is -1. At $(1, 0.5)$, slope is $-1/4$.



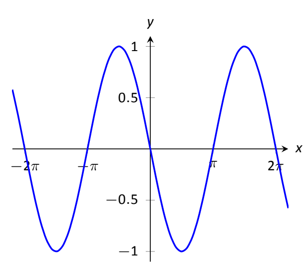
02 01 ex 26 29.



02 01 ex 27 30.



02 01 ex 28 31.



02 01 ex 29 32.

02 01 ex 30 33.

(a) Approximately on $(-1.5, 1.5)$.

(b) Approximately on $(-\infty, -1.5) \cup (1.5, \infty)$.

(c) Approximately at $x = \pm 1.5$.

(d) On $(-\infty, -1) \cup (0, 1)$.

(e) On $(-1, 0) \cup (1, \infty)$.

(f) At $x = \pm 1$.

02 01 ex 31 34. Approximately 24.

02 01 ex 32 35. Approximately 0.54.

02 01 ex 33 36.

(a) $(-\infty, \infty)$

(b) $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$

(c) $(-\infty, 5]$

(d) $[-\sqrt{5}, \sqrt{5}]$

02 01 ex 34 37.

(a) 1

(b) 3

(c) Does not exist

(d) $(-\infty, -3) \cup (3, \infty)$

Section 1.2

02 02 ex 01 1. Velocity

02 02 ex 02 2. Answers will vary.

02 02 ex 03 3. Linear functions.

02 02 ex 04 4. 12

02 02 ex 05 5. -17

02 02 ex 06 6. 102

02 02 ex 07 7. $f(10.1)$ is likely most accurate, as accuracy is lost the farther from $x = 10$ we go.

02 02 ex 08 8. -4

02 02 ex 09 9. 6

02 02 ex 10 10. decibels per person

02 02 ex 11 11. ft/s^2

02 02 ex 12 12. ft/h

02 02 ex 13 13.

(a) thousands of dollars per car

(b) It is likely that $P(0) < 0$. That is, negative profit for not producing any cars.

02 02 ex 14 14.

(a) degrees Fahrenheit per hour

(b) It is likely that $T'(8) > 0$ since at 8 in the morning, the temperature is likely rising.

(c) It is very likely that $T(8) > 0$, as at 8 in the morning on July 4, we would expect the temperature to be well above 0.

02 02 ex 15 15. $f(x) = g'(x)$

02 02 ex 16 16. $g(x) = f'(x)$

02 02 ex 17 17. Either $g(x) = f'(x)$ or $f(x) = g'(x)$ is acceptable. The actual answer is $g(x) = f'(x)$, but is very hard to show that $f(x) \neq g'(x)$ given the level of detail given in the graph.

02 02 ex 18 18. $g(x) = f'(x)$

02 02 ex 19 19. $f'(x) = 10x$

02 02 ex 20 20. $f'(x) = 3x^2 - 12x + 12$

02 02 ex 21 21. $f'(\pi) \approx 0$.

02 02 ex 22 22. $f'(9) \approx 0.1667$.

Section 1.3

02 03 ex 01 1. Power Rule.

02 03 ex 02 2. $1/x$

02 03 ex 03 3. One answer is $f(x) = 10e^x$.

02 03 ex 04 4. One answer is $f(x) = 10$.

02 03 ex 05 5. $g(x)$ and $h(x)$

02 03 ex 06 6. Answers will vary.

02 03 ex 07 7. One possible answer is $f(x) = 17x - 205$.

02 03 ex 08 8. Answers will vary.

02 03 ex 09 9. $f'(x)$ is a velocity function, and $f''(x)$ is acceleration.

02 03 ex 10 10. lbs/ft^2.

02 03 ex 11 11. $f'(x) = 14x - 5$

02 03 ex 12 12. $g'(x) = 42x^2 + 14x + 11$

02 03 ex 13 13. $m'(t) = 45t^4 - \frac{3}{8}t^2 + 3$

02 03 ex 14 14. $f'(\theta) = 9\cos\theta - 10\sin\theta$

02 03 ex 15 15. $f'(r) = 6e^r$

02 03 ex 16 16. $g'(t) = 40t^3 + \sin t + 7\cos t$

02 03 ex 17 17. $f'(x) = \frac{2}{x} - 1$

02 03 ex 18 18. $p'(s) = s^3 + s^2 + s + 1$

02 03 ex 19 19. $h'(t) = e^t - \cos t + \sin t$

02 03 ex 20 20. $f'(x) = \frac{2}{x}$

02 03 ex 21 21. $f'(t) = 0$

02 03 ex 22 22. $g'(t) = 18t + 6$

02 03 ex 23 23. $g'(x) = 24x^2 - 120x + 150$

02 03 ex 24 24. $f'(x) = -3x^2 + 6x - 3$

02 03 ex 25 25. $f'(x) = 18x - 12$

02 03 ex 50 26.

(a) $\log_{10} x = \frac{\ln x}{\ln 10}$.

(b) $\frac{d}{dx}\left(\log_{10} x\right) = \frac{d}{dx}\left(\frac{1}{\ln 10}\ln x\right) = \frac{1}{\ln 10}\frac{1}{x}$.

(c) $\frac{d}{dx}\left(\log_a x\right) = \frac{1}{\ln a}\frac{1}{x}$

02 03 ex 26 27. $f'(x) = 6x^5$ $f''(x) = 30x^4$ $f'''(x) = 120x^3$ $f^{(4)}(x) = 360x^2$

02 03 ex 26 28. $g'(x) = -2\sin x$ $g''(x) = -2\cos x$ $g'''(x) = 2\sin x$ $g^{(4)}(x) = 2\cos x$

02 03 ex 28 29. $h'(t) = 2t - e^t$ $h''(t) = 2 - e^t$ $h'''(t) = -e^t$ $h^{(4)}(t) = -e^t$

02 03 ex 29 30. $p'(\theta) = 4\theta^3 - 3\theta^2$ $p''(\theta) = 12\theta^2 - 6\theta$ $p'''(\theta) = 24\theta - 6$ $p^{(4)}(\theta) = 24$

02 03 ex 30 31. $f'(\theta) = \cos\theta + \sin\theta$ $f''(\theta) = -\sin\theta + \cos\theta$ $f'''(\theta) = -\cos\theta - \sin\theta$ $f^{(4)}(\theta) = \sin\theta - \cos\theta$

02 03 ex 31 32. $f'(x) = f''(x) = f'''(x) = f^{(4)}(x) = 0$

02 03 ex 32 33. Tangent line: $y = 2(x-1)$
Normal line: $y = -1/2(x-1)$

02 03 ex 33 34. Tangent line: $y = t + 4$
Normal line: $y = -t + 4$

02 03 ex 34 35. Tangent line: $y = x - 1$
Normal line: $y = -x + 1$

02 03 ex 35 36. Tangent line: $y = 4$
Normal line: $x = \pi/2$

02 03 ex 36 37. Tangent line: $y = \sqrt{2}(x - \frac{\pi}{4}) - \sqrt{2}$
Normal line: $y = \frac{-1}{\sqrt{2}}(x - \frac{\pi}{4}) - \sqrt{2}$

02 03 ex 37 38. Tangent line: $y = 2x + 3$
Normal line: $y = -1/2(x-5) + 13$

02 03 ex 39 39. The tangent line to $f(x) = e^x$ at $x = 0$ is $y = x + 1$; thus $e^{0.1} \approx y(0.1) = 1.1$.

Section 1.4

02 04 ex 01 1. F

02 04 ex 02 2. F

02 04 ex 03 3. T

02 04 ex 04 4. Quotient Rule

02 04 ex 05 5. F

02 04 ex 06 6. Answers will vary.

02 04 ex 07 7.
(a) $f'(x) = (x^2 + 3x) + x(2x + 3)$
(b) $f'(x) = 3x^2 + 6x$
(c) They are equal.

02 04 ex 08 8.
(a) $g'(x) = 4x(5x^3) + 2x^2(15x^2)$
(b) $g'(x) = 50x^4$
(c) They are equal.

02 04 ex 09 9.
(a) $h'(s) = 2(s + 4) + (2s - 1)(1)$
(b) $h'(s) = 4s + 7$
(c) They are equal.

02 04 ex 10 10.
(a) $f'(x) = 2x(3 - x^3) + (x^2 + 5)(-3x^2)$
(b) $f'(x) = -5x^4 - 15x^2 + 6x$
(c) They are equal.

02 04 ex 11 11.
(a) $f'(x) = \frac{x(2x) - (x^2+3)1}{x^2}$
(b) $f'(x) = 1 - \frac{3}{x^2}$
(c) They are equal.

02 04 ex 12 12.
(a) $g'(x) = \frac{2x^2(3x^2-4x) - (x^3-2x^2)(4x)}{4x^4}$
(b) $g'(x) = 1/2$
(c) They are equal.

02 04 ex 13 13.
(a) $h'(s) = \frac{4s^3(0) - 3(12s^2)}{16s^6}$
(b) $h'(s) = -9/4s^{-4}$
(c) They are equal.

02 04 ex 14 14.
(a) $f'(t) = \frac{(t+1)(2t) - (t^2-1)(1)}{(t+1)^2}$
(b) $f(t) = t - 1$ when $t \neq -1$, so $f'(t) = 1$.
(c) They are equal.

02 04 ex 16 15. $f'(x) = \sin x + x\cos x$

02 04 ex 48 16. $f'(x) = 2x\cos x - x^2 \sin x$

02 04 ex 49 17. $f'(x) = e^x \ln x + e^x \frac{1}{x}$

02 04 ex 17 18. $f'(t) = \frac{-2}{t^3}(\csc t - 4) + \frac{1}{t^2}(-\csc t \cot t)$

02 04 ex 18 19. $g'(x) = \frac{-12}{(x-5)^2}$

02 04 ex 19 20. $g'(t) = \frac{(\cos t - 2t^2)(5t^4) - (t^5)(-\sin t - 4t)}{(\cos t - 2t^2)^2}$

02 04 ex 20 21. $h'(x) = -\csc^2 x - e^x$

02 04 ex 50 22. $f'(x) = \big(\sec^2 x\big)\ln x + \big(\tan x\big)\frac{1}{x}$

02 04 ex 21 23. $h'(t) = 14t + 6$

02 04 ex 15 24.
(a) $f'(x) = \frac{(x+2)(4x^3+6x^2) - (x^4+2x^3)(1)}{(x+2)^2}$
(b) $f(x) = x^3$ when $x \neq -2$, so $f'(x) = 3x^2$.
(c) They are equal.

02 04 ex 51 25. $f'(x) = \big(6x + 8\big)e^x + \big(3x^2 + 8x + 7\big)e^x$

02 04 ex 52 26. $g'(t) = \frac{e^t(5t^4 - 3t^2) - (t^5 - t^3)e^t}{(e^t)^2}$

02 04 ex 22 27. $f'(x) = 7$

02 04 ex 23 28. $f'(t) = 5t^4(\sec t + e^t) + t^5(\sec t \tan t + e^t)$

02 04 ex 46 29. $f'(x) = \frac{\sin^2(x) + \cos^2(x) + 3\cos(x)}{(\cos(x)+3)^2}$

02 04 ex 53 30. $f'(\theta) = 3\theta^2 \sin\theta + \theta^3 \cos\theta + \frac{\theta^3\cos\theta - (\sin\theta)(3\theta^2)}{\theta^6}$

02 04 ex 54 31. $f'(x) = \frac{-x\sin x - \cos x}{x^2} + \frac{\tan x - x\sec^2 x}{\tan^2 x}$

02 04 ex 24 32. $g'(x) = 0$

02 04 ex 25 33. $g'(t) = 12t^2 e^t + 4t^3 e^t - \cos^2 t + \sin^2 t$

02 04 ex 47 34. $f'(x) = \frac{(t^2\cos t + 2)(2t\sin t + t^2\cos t) - (t^2\sin t + 3)(2t\cos t - t^2\sin t)}{(t^2\cos t + 2)^2}$

02 04 ex 26 35. $f'(x) = 2xe^x \tan x = x^2 e^x \tan x + x^2 e^x \sec^2 x$

02 04 ex 27 36. $g'(x) = 2\sin x \sec x + 2x\cos x \sec x + 2x\sin x \sec x \tan x = 2\tan x + 2x + 2x\tan^2 x = 2\tan x + 2x\sec^2 x$

02 04 ex 28 37. Tangent line: $y = 2x + 2$
Normal line: $y = -1/2x + 2$

02 04 ex 29 38. Tangent line: $y = -(x - \frac{3\pi}{2}) - \frac{3\pi}{2} = -x$
Normal line: $y = (x - \frac{3\pi}{2}) - \frac{3\pi}{2} = x - 3\pi$

02 04 ex 30 39. Tangent line: $y = 4$
Normal line: $x = 2$

02 04 ex 31 40. Tangent line: $y = -9x + 1$
Normal line: $y = 1/9x + 1$

02 04 ex 32 41. $x = 3/2$

02 04 ex 33 42. $x = 0$

02 04 ex 34 43. $f'(x)$ is never 0.

02 04 ex 35 44. $x = -2, 0$

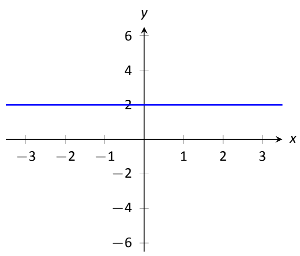
02 04 ex 36 45. $f''(x) = 2\cos x - x\sin x$

02 04 ex 37 46. $f^{(4)}(x) = -4\cos x + x\sin x$

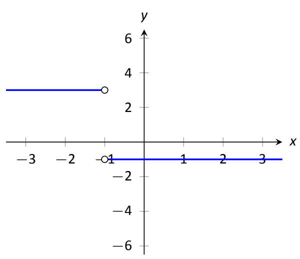
02 04 ex 38 47. $f''(x) = \cot^2 x\csc x + \csc^3 x$

02 04 ex 39 48. $f^{(8)} = 0$

02 04 ex 42 49.



02 04 ex 43 50.



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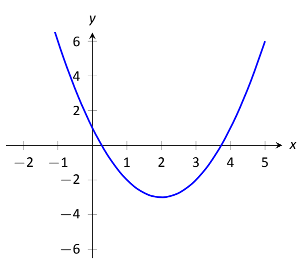
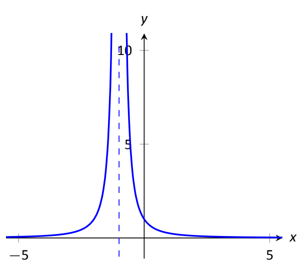


02 04 ex 45 52.



Section 1.5

02 05 ex 05 1. T

02 05 ex 01 2. F

02 05 ex 02 3. F

02 05 ex 03 4. T

02 05 ex 04 5. T

02 05 ex 53 6. $f'(x) = 3\big(\ln x + x^2\big)2(\frac{1}{x} + 2x)$

02 05 ex 06 7.
$f'(x) = 10(4x^3 - x)^9 \cdot (12x^2 - 1) = (120x^2 - 10)(4x^3 - x)^9$

02 05 ex 07 8. $f'(t) = 15(3t - 2)^4$

02 05 ex 08 9. $g'(\theta) = 3(\sin\theta + \cos\theta)^2(\cos\theta - \sin\theta)$

02 05 ex 09 10. $h'(t) = (6t + 1)e^{3t^2+t-1}$

02 05 ex 53 11. $f'(x) = 3\big(\ln x + x^2\big)2(\frac{1}{x} + 2x)$

02 05 ex 54 12. $f'(x) = (\ln 2)(2^{x^3+3x})(3x^2 + 3)$

02 05 ex 10 13. $f'(x) = 4\big(x + \frac{1}{x}\big)^3\big(1 - \frac{1}{x^2}\big)$

02 05 ex 11 14. $f'(x) = -3\sin(3x)$

02 05 ex 12 15. $g'(x) = 5\sec^2(5x)$

02 05 ex 55 16. $h'(\theta) = \sec^2\big(\theta^2 + 4\theta\big)(2\theta + 4)$

02 05 ex 56 17. $g'(t) = \cos\big(t^5 + \frac{1}{t}\big)\big(5t^4 - \frac{1}{t^2}\big)$

02 05 ex 13 18. $h'(t) = 8\sin^3(2t)\cos(2t)$

02 05 ex 14 19. $p'(t) = -3\cos^2(t^2 + 3t + 1)\sin(t^2 + 3t + 1)(2t + 3)$

02 05 ex 15 20. $f'(x) = -\tan x$

02 05 ex 16 21. $f'(x) = 2/x$

02 05 ex 17 22. $f'(x) = 2/x$

02 05 ex 18 23. $g'(r) = \ln 4 \cdot 4^r$

02 05 ex 19 24. $g'(t) = -\ln 5 \cdot 5^{\cos t}\sin t$

02 05 ex 20 25. $g'(t) = 0$

02 05 ex 21 26. $m'(w) = \ln(3/2)(3/2)^w$

02 05 ex 42 27. $f'(x) = \frac{(3^t+2)\left((\ln 2)2^t\right)-(2^t+3)\left((\ln 3)3^t\right)}{(3^t+2)^2}$

02 05 ex 22 28. $m'(w) = \frac{2^w\left(\ln 3\cdot 3^w - \ln 2\cdot(3^w+1)\right)}{2^{2w}}$

02 05 ex 23 29. $f'(x) = \frac{2^{x^2}(\ln 3\cdot 3^{x^2}2x+1)-(3^{x^2}+x)(\ln 2\cdot 2^{x^2}2x)}{2^{2x^2}}$

02 05 ex 24 30. $f'(x) = 5x^2\cos(5x) + 2x\sin(5x)$

02 05 ex 58 31. $f'(x) = 5(x^2 + x)^4(2x + 1)(3x^4 + 2x)^3 + 3(x^2 + x)^5(3x^4 + 2x)^2(12x^3 + 2)$

02 05 ex 25 32. $g'(t) =$
$5\cos(t^2 + 3t)\cos(5t - 7) - (2t + 3)\sin(t^2 + 3t)\sin(5t - 7)$

02 05 ex 57 33.
$f'(x) = 3\cos(3x + 4)\cos(5 - 2x) + 2\sin(3x + 4)\sin(5 - 2x)$

02 05 ex 26 34. $g'(t) = 10t\cos(\frac{1}{t})e^{5t^2} + \frac{1}{t^2}\sin(\frac{1}{t})e^{5t^2}$

02 05 ex 59 35. $f'(x) = \frac{4(5x-9)^3\cos(4x+1) - 15\sin(4x+1)(5x-9)^2}{(5x-9)^6}$

02 05 ex 60 36. $f'(x) = \frac{8\tan(5x)(4x+1) - 5(4x+1)^2\sec^2(5x)}{\tan^2(5x)}$

02 05 ex 27 37. Tangent line: $y = 0$
Normal line: $x = 0$

02 05 ex 28 38. Tangent line: $y = 15(t-1)+1$
Normal line: $y = -1/15(t-1)+1$

02 05 ex 29 39. Tangent line: $y = -3(\theta - \pi/2)+1$
Normal line: $y = 1/3(\theta - \pi/2)+1$

02 05 ex 30 40. Tangent line: $y = -5e(t+1)+e$
Normal line: $y = 1/(5e)(t+1)+e$

02 05 ex 31 41. In both cases the derivative is the same: $1/x$.

02 05 ex 32 42. In both cases the derivative is the same: k/x.

02 05 ex 33 43.

(a) $^\circ$ F/mph

(b) The sign would be negative; when the wind is blowing at 10 mph, any increase in wind speed will make it feel colder, i.e., a lower number on the Fahrenheit scale.

02 05 ex 34 44.

(a) $2xe^x \cot x + x^2e^x \cot x - x^2e^x \csc^2 x$

(b) $\ln(48)48^x$

Section 1.6

02 06 ex 01 1. Answers will vary.

02 06 ex 02 2. The Chain Rule.

02 06 ex 03 3. T

02 06 ex 04 4. T

02 05 ex 50 5. $f'(x) = \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{x^3}}$

02 05 ex 51 6. $f'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-1/3} = \frac{1}{3\sqrt[3]{x^2}} + \frac{2}{3\sqrt[3]{x}}$

02 06 ex 06 7. $f'(t) = \frac{-t}{\sqrt{1-t^2}}$

02 06 ex 07 8. $g'(t) = \sqrt{t}\cos t + \frac{\sin t}{2\sqrt{t}}$

02 06 ex 08 9. $h'(x) = 1.5x^{0.5} = 1.5\sqrt{x}$

02 05 ex 52 10. $f'(x) = \pi x^{\pi-1} + 1.9x^{0.9}$

02 05 ex 40 11. $g'(x) = \frac{\sqrt{x}(1)-(x+7)(1/2x^{-1/2})}{x} = \frac{1}{2\sqrt{x}} - \frac{7}{2\sqrt{x^3}}$

02 05 ex 41 12. $f'(t) = \frac{1}{5}x^{-4/5}(\sec t + e^t) + \sqrt[5]{t}(\sec t \tan t + e^t)$

02 06 ex 09 13. $\frac{dy}{dx} = \frac{-4x^3}{2y+1}$

02 06 ex 10 14. $\frac{dy}{dx} = -\frac{y^{3/5}}{x^{3/5}}$

02 06 ex 11 15. $\frac{dy}{dx} = \sin(x)\sec(y)$

02 06 ex 12 16. $\frac{dy}{dx} = \frac{y}{x}$

02 06 ex 13 17. $\frac{dy}{dx} = \frac{y}{x}$

02 06 ex 14 18. $\frac{dy}{dx} = -\frac{e^x x(x+2)2^{-y}}{\ln|2|}$

02 06 ex 15 19. $\frac{dy}{dx} = -\frac{2\sin(y)\cos(y)}{x}$

02 06 ex 16 20. $\frac{dy}{dx} = -\frac{x}{y^2}$

02 06 ex 17 21. $\frac{dy}{dx} = \frac{1}{2y+2}$

02 06 ex 18 22. If one takes the derivative of the equation, as shown, using the Quotient Rule, one finds $\frac{dy}{dx} = \frac{x^2+2xy^2-y}{2x^2y-x+y^2}$.
If one first clears the denominator and writes $x^2+y = 17(x+y^2)$ then takes the derivative of both sides, one finds $\frac{dy}{dx} = \frac{2x-17}{34y-1}$.
These expressions, by themselves, are not equal.
However, for values of x and y that satisfy the original equation (i.e, for x and y such that $\frac{x^2+y}{x+y^2} = 17$), these expressions are equal.

02 06 ex 19 23. If one takes the derivative of the equation, as shown, using the Quotient Rule, one finds
$\frac{dy}{dx} = \frac{-\cos(x)(x+\cos(y))+\sin(x)+y}{\sin(y)(\sin(x)+y)+x+\cos(y)}$.
If one first clears the denominator and writes $\sin(x)+y = \cos(y)+x$ then takes the derivative of both sides, one finds $\frac{dy}{dx} = \frac{1-\cos(x)}{1+\sin(y)}$.
These expressions, by themselves, are not equal.
However, for values of x and y that satisfy the original equation (i.e, for x and y such that $\frac{\sin(x)+y)}{\cos(y)+x)} = 1$), these expressions are equal.

02 06 ex 21 24. $\frac{dy}{dx} = -\frac{x}{y}$

02 06 ex 22 25. $\frac{dy}{dx} = -\frac{2x+y}{2y+x}$

02 06 ex 20 26. In each, $\frac{dy}{dx} = -\frac{y}{x}$.

02 06 ex 23 27.

(a) $y = 0$

(b) $y = -1.859(x-0.1)+0.281$

02 06 ex 24 28.

(a) $x = 1$

(b) $y = -\frac{3\sqrt{3}}{8}(x-\sqrt{.6})+\sqrt{.8} \approx -0.65(x-0.775)+0.894$

(c) $y = 1$

02 06 ex 25 29.

(a) $y = 4$

(b) $y = 0.93(x-2)+\sqrt[4]{108}$

02 06 ex 26 30.

(a) $y = -1/3x+1$

(b) $y = 3\sqrt{3}/4$

02 06 ex 27 31.

(a) $y = -\frac{1}{\sqrt{3}}(x-\frac{7}{2})+\frac{6+3\sqrt{3}}{2}$

(b) $y = \sqrt{3}(x-\frac{4+3\sqrt{3}}{2})+\frac{3}{2}$

02 06 ex 38 32.

(a) $y = 1$

(b) $y = -\frac{2}{\sqrt{5}}(x+1)+\frac{1}{2}(-1+\sqrt{5})$

(c) $y = \frac{2}{\sqrt{5}}(x+1)+\frac{1}{2}(-1-\sqrt{5})$

02 06 ex 28 33. $\frac{d^2y}{dx^2} = \frac{(2y+1)(-12x^2)+4x^3\left(2\frac{-4x^3}{2y+1}\right)}{(2y+1)^2}$

02 06 ex 29 34. $\frac{d^2y}{dx^2} = \frac{3}{5}\frac{y^{3/5}}{x^{8/5}} + \frac{3}{5}\frac{1}{yx^{6/5}}$

02 06 ex 30 35. $\frac{d^2y}{dx^2} = \frac{\cos x\cos y+\sin^2 x\tan y}{\cos^2 y}$

02 06 ex 31 36. $\frac{d^2y}{dx^2} = 0$

02 06 ex 32 37. $y' = (1+x)^{1/x}\left(\frac{1}{x(x+1)} - \frac{\ln(1+x)}{x^2}\right)$
Tangent line: $y = (1-2\ln 2)(x-1)+2$

02 06 ex 33 38. $y' = (2x)^{x^2}\left(2x\ln(2x)+x\right)$
Tangent line: $y = (2+4\ln 2)(x-1)+2$

02 06 ex 34 39. $y' = \frac{x^x}{x+1}\left(\ln x + 1 - \frac{1}{x+1}\right)$
Tangent line: $y = (1/4)(x-1)+1/2$

02 06 ex 35 40. $y' = x^{\sin(x)+2}\left(\cos x \ln x + \frac{\sin x + 2}{x}\right)$
Tangent line: $y = (3\pi^2/4)(x-\pi/2)+(\pi/2)^3$

02 06 ex 36 41. $y' = \frac{x+1}{x+2}\left(\frac{1}{x+1} - \frac{1}{x+2}\right)$
Tangent line: $y = 1/9(x-1)+2/3$

02 06 ex 37 42. $y' = \frac{(x+1)(x+2)}{(x+3)(x+4)}\left(\frac{1}{x+1}+\frac{1}{x+2}-\frac{1}{x+3}-\frac{1}{x+4}\right)$
Tangent line: $y = 11/72x+1/6$

Section 1.7

02 07 ex 01 1. F

02 07 ex 02 2. Answers will vary.

02 07 ex 03 3. The point $(10,1)$ lies on the graph of $y = f^{-1}(x)$ (assuming f is invertible).

02 07 ex 04 4. The point $(10,1)$ lies on the graph of $y = f^{-1}(x)$ (assuming f is invertible) and $(f^{-1})'(10) = 1/5$.

02 07 ex 05 5. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

02 07 ex 06 6. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

02 07 ex 07 7. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

02 07 ex 08 8. Compose $f(g(x))$ and $g(f(x))$ to confirm that each equals x.

02 07 ex 09 9. $\left(f^{-1}\right)'(20) = \frac{1}{f'(2)} = 1/5$

02 07 ex 10 10. $\left(f^{-1}\right)'(7) = \frac{1}{f'(3)} = 1/4$

02 07 ex 11 11. $\left(f^{-1}\right)'(\sqrt{3}/2) = \frac{1}{f'(\pi/6)} = 1$

02 07 ex 12 12. $\left(f^{-1}\right)'(8) = \frac{1}{f'(1)} = 1/6$

02 07 ex 13 13. $\left(f^{-1}\right)'(1/2) = \frac{1}{f'(1)} = -2$

02 07 ex 14 14. $\left(f^{-1}\right)'(6) = \frac{1}{f'(0)} = 1/6$

02 07 ex 15 15. $h'(t) = \frac{2}{\sqrt{1-4t^2}}$

02 07 ex 16 16. $f'(t) = \frac{1}{|t|\sqrt{4t^2+1}}$

02 07 ex 17 17. $g'(x) = \frac{2}{1+4x^2}$

02 07 ex 18 18. $f'(x) = \frac{x}{\sqrt{1-x^2}} + \sin^{-1}(x)$

02 07 ex 19 19. $g'(t) = \cos^{-1}(t)\cos(t) - \frac{\sin(t)}{\sqrt{1-t^2}}$

02 07 ex 20 20. $f'(t) = \frac{e^t}{t} + \ln t e^t$

02 07 ex 21 21. $h'(x) = \frac{\sin^{-1}(x)+\cos^{-1}(x)}{\sqrt{1-x^2}\cos^{-1}(x)^2}$

02 07 ex 22 22. $g'(x) = \frac{1}{\sqrt{x}(2x+2)}$

02 07 ex 23 23. $f'(x) = -\frac{1}{\sqrt{1-x^2}}$

02 07 ex 24 24.

(a) $f(x) = x$, so $f'(x) = 1$

(b) $f'(x) = \cos(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} = 1.$

02 07 ex 24 25.

(a) $f(x) = x$, so $f'(x) = 1$

(b) $f'(x) = \cos(\sin^{-1} x)\frac{1}{\sqrt{1-x^2}} = 1.$

02 07 ex 25 26.

(a) $f(x) = x$, so $f'(x) = 1$

(b) $f'(x) = \frac{1}{1+\tan^2 x}\sec^2 x = 1$

02 07 ex 27 27. $y = \sqrt{2}(x-\sqrt{2}/2)+\pi/4$

02 07 ex 28 28. $y = -4(x-\sqrt{3}/4)+\pi/6$

02 07 ex 29 29. $\frac{dy}{dx} = \frac{y(y-2x)}{x(x-2y)}$

02 07 ex 30 30. $y = -4/5(x-1)+2$

02 07 ex 31 31. $3x^2+1$

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