1: Limits

Calculus means “a method of calculation or reasoning.” When one computes the sales tax on a purchase, one employs a simple calculus. When one finds the area of a polygonal shape by breaking it up into a set of triangles, one is using another calculus. Proving a theorem in geometry employs yet another calculus.

Despite the wonderful advances in mathematics that had taken place into the first half of the 17$^{\text{th}}$ century, mathematicians and scientists were keenly aware of what they *could not do.* (This is true even today.) In particular, two important concepts eluded mastery by the great thinkers of that time: area and rates of change.

Area seems innocuous enough; areas of circles, rectangles, parallelograms, etc., are standard topics of study for students today just as they were then. However, the areas of *arbitrary* shapes could not be computed, even if the boundary of the shape could be described exactly.

Rates of change were also important. When an object moves at a constant rate of change, then “distance = rate $\times$ time.” But what if the rate is not constant – can distance still be computed? Or, if distance is known, can we discover the rate of change?

It turns out that these two concepts were related. Two mathematicians, Sir Isaac Newton and Gottfried Leibniz, are credited with independently formulating a system of computing that solved the above problems and showed how they were connected. Their system of reasoning was “a” calculus. However, as the power and importance of their discovery took hold, it became known to many as “the” calculus. Today, we generally shorten this to discuss “calculus.”

The foundation of “the calculus” is the *limit.* It is a tool to describe a particular behavior of a function. This chapter begins our study of the limit by approximating its value graphically and numerically. After a formal definition of the limit, properties are established that make “finding limits” tractable. Once the limit is understood, then the problems of area and rates of change can be approached.

1.1 An Introduction To Limits

We begin our study of *limits* by considering examples that demonstrate key concepts that will be explained as we progress.

Consider the function $y = \frac{\sin x}{x}$. When x is near the value 1, what value (if any) is y near?

While our question is not precisely formed (what constitutes “near the value

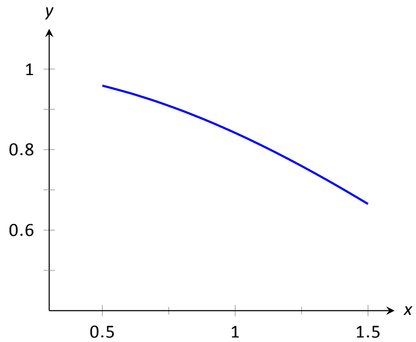


Figure 1.1.1: $\sin(x)/x$ near $x = 1$.

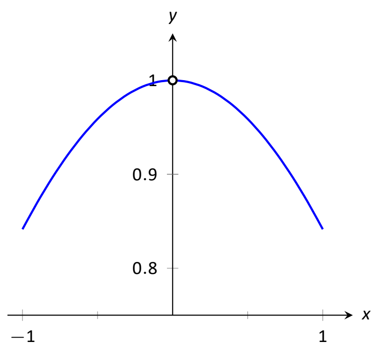


Figure 1.1.2: $\sin(x)/x$ near $x = 0$.

x	$\sin(x)/x$
0.9	0.870363
0.99	0.844471
0.999	0.841772
1	**0.841471**
1.001	0.84117
1.01	0.838447
1.1	0.810189

Figure 1.1.3: Values of $\sin(x)/x$ with x near 1.

1"?), the answer does not seem difficult to find. One might think first to look at a graph of this function to approximate the appropriate y values. Consider Figure 1.1.1, where $y = \frac{\sin x}{x}$ is graphed. For values of x near 1, it seems that y takes on values near 0.85. In fact, when $x = 1$, then $y = \frac{\sin 1}{1} \approx 0.84$, so it makes sense that when x is "near" 1, y will be "near" 0.84.

Consider this again at a different value for x. When x is near 0, what value (if any) is y near? By considering Figure 1.1.2, one can see that it seems that y takes on values near 1. But what happens when $x = 0$? We have

$$y \rightarrow \frac{\sin 0}{0} \rightarrow \text{“}\frac{0}{0}\text{”}.$$

The expression "0/0" has no value; it is *indeterminate.* Such an expression gives no information about what is going on with the function nearby. We cannot find out how y behaves near $x = 0$ for this function simply by letting $x = 0$.

Finding a limit entails understanding how a function behaves near a particular value of x. Before continuing, it will be useful to establish some notation. Let $y = f(x)$; that is, let y be a function of x for some function f. The expression "the limit of y as x approaches 1" describes a number, often referred to as L, that y nears as x nears 1. We write all this as

$$\lim_{x\to 1} y = \lim_{x\to 1} f(x) = L.$$

This is not a complete definition (that will come in the next section); this is a pseudo-definition that will allow us to explore the idea of a limit.

Above, where $f(x) = \sin(x)/x$, we approximated

$$\lim_{x\to 1} \frac{\sin x}{x} \approx 0.84 \quad \text{and} \quad \lim_{x\to 0} \frac{\sin x}{x} \approx 1.$$

(We *approximated* these limits, hence used the "$\approx$" symbol, since we are working with the pseudo-definition of a limit, not the actual definition.)

Once we have the true definition of a limit, we will find limits *analytically*; that is, exactly using a variety of mathematical tools. For now, we will *approximate* limits both graphically and numerically. Graphing a function can provide a good approximation, though often not very precise. Numerical methods can provide a more accurate approximation. We have already approximated limits graphically, so we now turn our attention to numerical approximations.

Consider again $\lim_{x\to 1} \sin(x)/x$. To approximate this limit numerically, we can create a table of x and $f(x)$ values where x is "near" 1. This is done in Figure 1.1.3.

Notice that for values of x near 1, we have $\sin(x)/x$ near 0.841. The $x = 1$ row is in bold to highlight the fact that when considering limits, we are *not* concerned

Notes:

with the value of the function at that particular x value; we are only concerned with the values of the function when x is *near* 1.

Now approximate $\lim_{x\to 0} \sin(x)/x$ numerically. We already approximated the value of this limit as 1 graphically in Figure 1.1.2. The table in Figure 1.1.4 shows the value of $\sin(x)/x$ for values of x near 0. Ten places after the decimal point are shown to highlight how close to 1 the value of $\sin(x)/x$ gets as x takes on values very near 0. We include the $x = 0$ row in bold again to stress that we are not concerned with the value of our function at $x = 0$, only on the behavior of the function *near* 0.

x	$\sin(x)/x$
-0.1	0.9983341665
-0.01	0.9999833334
-0.001	0.9999998333
0	**not defined**
0.001	0.9999998333
0.01	0.9999833334
0.1	0.9983341665

Figure 1.1.4: Values of $\sin(x)/x$ with x near 0.

This numerical method gives confidence to say that 1 is a good approximation of $\lim_{x\to 0} \sin(x)/x$; that is,

$$\lim_{x\to 0} \sin(x)/x \approx 1.$$

Later we will be able to prove that the limit is *exactly* 1.

We now consider several examples that allow us explore different aspects of the limit concept.

Example 1.1.1 Approximating the value of a limit

Use graphical and numerical methods to approximate

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3}.$$

SOLUTION To graphically approximate the limit, graph

$$y = (x^2 - x - 6)/(6x^2 - 19x + 3)$$

on a small interval that contains 3. To numerically approximate the limit, create a table of values where the x values are near 3. This is done in Figures 1.1.5 and 1.1.6, respectively.

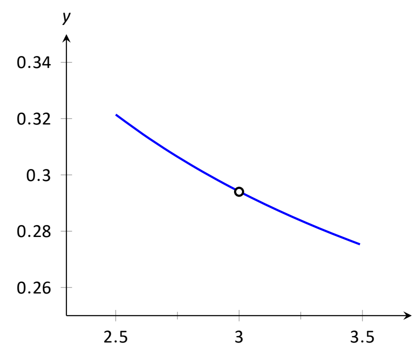


Figure 1.1.5: Graphically approximating a limit in Example 1.1.1.

x	$\frac{x^2-x-6}{6x^2-19x+3}$
2.9	0.29878
2.99	0.294569
2.999	0.294163
3	**not defined**
3.001	0.294073
3.01	0.293669
3.1	0.289773

Figure 1.1.6: Numerically approximating a limit in Example 1.1.1.

The graph shows that when x is near 3, the value of y is very near 0.3. By considering values of x near 3, we see that $y = 0.294$ is a better approximation. The graph and the table imply that

$$\lim_{x\to 3} \frac{x^2 - x - 6}{6x^2 - 19x + 3} \approx 0.294.$$

This example may bring up a few questions about approximating limits (and the nature of limits themselves).

1. If a graph does not produce as good an approximation as a table, why bother with it?

2. How many values of x in a table are "enough?" In the previous example, could we have just used $x = 3.001$ and found a fine approximation?

Notes:

Graphs are useful since they give a visual understanding concerning the behavior of a function. Sometimes a function may act "erratically" near certain x values which is hard to discern numerically but very plain graphically. Since graphing utilities are very accessible, it makes sense to make proper use of them.

Since tables and graphs are used only to *approximate* the value of a limit, there is not a firm answer to how many data points are "enough." Include enough so that a trend is clear, and use values (when possible) both less than and greater than the value in question. In Example 1.1.1, we used both values less than and greater than 3. Had we used just $x = 3.001$, we might have been tempted to conclude that the limit had a value of 0.3. While this is not far off, we could do better. Using values "on both sides of 3" helps us identify trends.

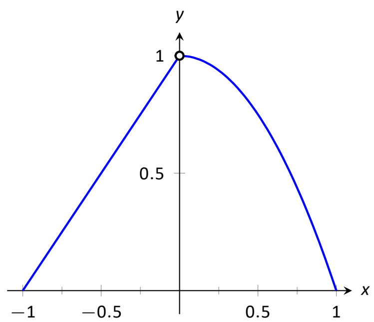


Figure 1.1.7: Graphically approximating a limit in Example 1.1.2.

Example 1.1.2 Approximating the value of a limit

Graphically and numerically approximate the limit of $f(x)$ as x approaches 0, where

$$f(x) = \begin{cases} x+1 & x < 0 \\ -x^2+1 & x > 0 \end{cases}.$$

SOLUTION Again we graph $f(x)$ and create a table of its values near $x = 0$ to approximate the limit. Note that this is a piecewise defined function, so it behaves differently on either side of 0. Figure 1.1.7 shows a graph of $f(x)$, and on either side of 0 it seems the y values approach 1. Note that $f(0)$ is not actually defined, as indicated in the graph with the open circle.

The table shown in Figure 1.1.8 shows values of $f(x)$ for values of x near 0. It is clear that as x takes on values very near 0, $f(x)$ takes on values very near 1. It turns out that if we let $x = 0$ for either "piece" of $f(x)$, 1 is returned; this is significant and we'll return to this idea later.

The graph and table allow us to say that $\lim_{x\to 0} f(x) \approx 1$; in fact, we are probably very sure it *equals* 1.

x	$f(x)$
-0.1	0.9
-0.01	0.99
-0.001	0.999
0.001	0.999999
0.01	0.9999
0.1	0.99

Figure 1.1.8: Numerically approximating a limit in Example 1.1.2.

Identifying When Limits Do Not Exist

A function may not have a limit for all values of x. That is, we cannot say $\lim_{x\to c} f(x) = L$ for some numbers L for all values of c, for there may not be a number that $f(x)$ is approaching. There are three common ways in which a limit may fail to exist.

1. The function $f(x)$ may approach different values on either side of c.
2. The function may grow without upper or lower bound as x approaches c.
3. The function may oscillate as x approaches c without approaching a specific value.

Notes:

We'll explore each of these in turn.

Example 1.1.3 Different Values Approached From Left and Right

Explore why $\lim_{x\to 1} f(x)$ does not exist, where

$$f(x) = \begin{cases} x^2 - 2x + 3 & x \leq 1 \\ x & x > 1 \end{cases}.$$

Solution A graph of $f(x)$ around $x = 1$ and a table are given in Figures 1.1.9 and 1.1.10, respectively. It is clear that as x approaches 1, $f(x)$ does not seem to approach a single number. Instead, it seems as though $f(x)$ approaches two different numbers. When considering values of x less than 1 (approaching 1 from the left), it seems that $f(x)$ is approaching 2; when considering values of x greater than 1 (approaching 1 from the right), it seems that $f(x)$ is approaching 1. Recognizing this behavior is important; we'll study this in greater depth later. Right now, it suffices to say that the limit does not exist since $f(x)$ is not approaching one value as x approaches 1.

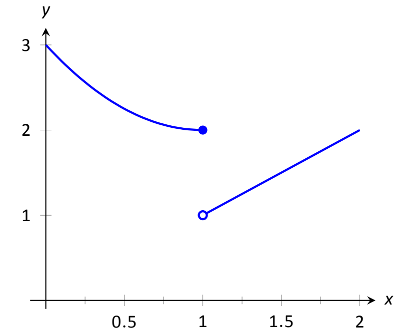


Figure 1.1.9: Observing no limit as $x \to 1$ in Example 1.1.3.

x	$f(x)$
0.9	2.01
0.99	2.0001
0.999	2.000001
1.001	1.001
1.01	1.01
1.1	1.1

Figure 1.1.10: Values of $f(x)$ near $x = 1$ in Example 1.1.3.

Example 1.1.4 The Function Grows Without Bound

Explore why $\lim_{x\to 1} 1/(x-1)^2$ does not exist.

Solution A graph and table of $f(x) = 1/(x-1)^2$ are given in Figures 1.1.11 and 1.1.12, respectively. Both show that as x approaches 1, $f(x)$ grows larger and larger.

We can deduce this on our own, without the aid of the graph and table. If x is near 1, then $(x-1)^2$ is very small, and:

$$\frac{1}{\text{very small number}} = \text{very large number}.$$

Since $f(x)$ is not approaching a single number, we conclude that

$$\lim_{x\to 1} \frac{1}{(x-1)^2}$$

does not exist.

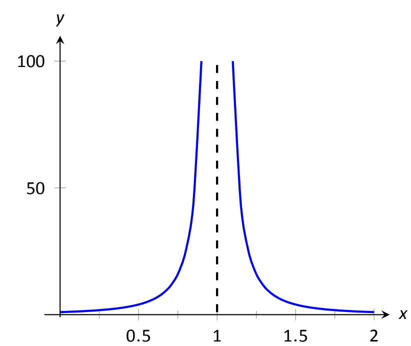


Figure 1.1.11: Observing no limit as $x \to 1$ in Example 1.1.4.

x	$f(x)$
0.9	100.
0.99	10000.
0.999	$1. \times 10^6$
1.001	$1. \times 10^6$
1.01	10000.
1.1	100.

Figure 1.1.12: Values of $f(x)$ near $x = 1$ in Example 1.1.4.

Example 1.1.5 The Function Oscillates

Explore why $\lim_{x\to 0} \sin(1/x)$ does not exist.

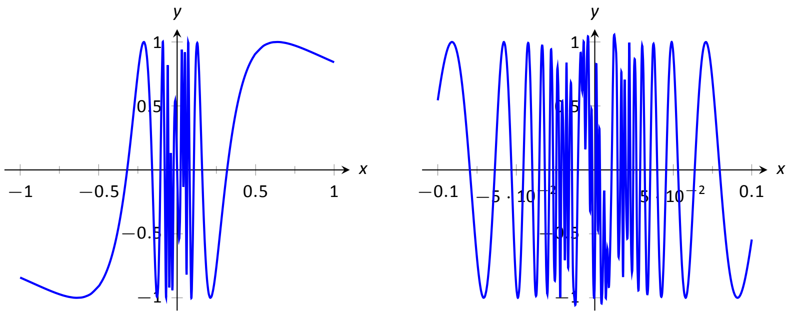
Solution Two graphs of $f(x) = \sin(1/x)$ are given in Figures 1.1.13. Figure 1.1.13(a) shows $f(x)$ on the interval $[-1, 1]$; notice how $f(x)$ seems to oscillate near $x = 0$. One might think that despite the oscillation, as x approaches

Notes:

0, $f(x)$ approaches 0. However, Figure 1.1.13(b) zooms in on $\sin(1/x)$, on the interval $[-0.1, 0.1]$. Here the oscillation is even more pronounced. Finally, in the table in Figure 1.1.13(c), we see $\sin(x)/x$ evaluated for values of x near 0. As x approaches 0, $f(x)$ does not appear to approach any value.

It can be shown that in reality, as x approaches 0, $\sin(1/x)$ takes on all values between -1 and 1 infinite times! Because of this oscillation,

$\lim_{x\to 0} \sin(1/x)$ does not exist.



x	$\sin(1/x)$
0.1	-0.544021
0.01	-0.506366
0.001	0.82688
0.0001	-0.305614
$1. \times 10^{-5}$	0.0357488
$1. \times 10^{-6}$	-0.349994
$1. \times 10^{-7}$	0.420548

(a) (b) (c)

Figure 1.1.13: Observing that $f(x) = \sin(1/x)$ has no limit as $x \to 0$ in Example 1.1.5.

Limits of Difference Quotients

We have approximated limits of functions as x approached a particular number. We will consider another important kind of limit after explaining a few key ideas.

Let $f(x)$ represent the position function, in feet, of some particle that is moving in a straight line, where x is measured in seconds. Let's say that when $x = 1$, the particle is at position 10 ft., and when $x = 5$, the particle is at 20 ft. Another way of expressing this is to say

$$f(1) = 10 \quad \text{and} \quad f(5) = 20.$$

Since the particle traveled 10 feet in 4 seconds, we can say the particle's *average velocity* was 2.5 ft/s. We write this calculation using a "quotient of differences," or, a *difference quotient*:

$$\frac{f(5) - f(1)}{5 - 1} = \frac{10}{4} = 2.5\text{ft/s}.$$

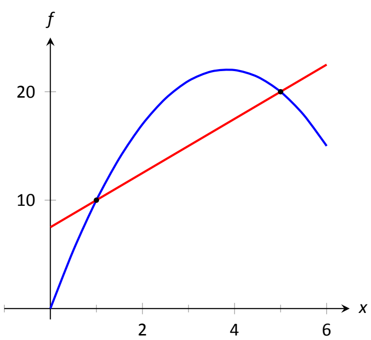


Figure 1.1.14: Interpreting a difference quotient as the slope of a secant line.

This difference quotient can be thought of as the familiar "rise over run" used to compute the slopes of lines. In fact, that is essentially what we are doing:

Notes:

given two points on the graph of f, we are finding the slope of the *secant line* through those two points. See Figure 1.1.14.

Now consider finding the average speed on another time interval. We again start at $x = 1$, but consider the position of the particle h seconds later. That is, consider the positions of the particle when $x = 1$ and when $x = 1 + h$. The difference quotient is now

$$\frac{f(1+h) - f(1)}{(1+h) - 1} = \frac{f(1+h) - f(1)}{h}.$$

Let $f(x) = -1.5x^2 + 11.5x$; note that $f(1) = 10$ and $f(5) = 20$, as in our discussion. We can compute this difference quotient for all values of h (even negative values!) except $h = 0$, for then we get "0/0," the indeterminate form introduced earlier. For all values $h \neq 0$, the difference quotient computes the average velocity of the particle over an interval of time of length h starting at $x = 1$.

For small values of h, i.e., values of h close to 0, we get average velocities over very short time periods and compute secant lines over small intervals. See Figure 1.1.15. This leads us to wonder what the limit of the difference quotient is as h approaches 0. That is,

$$\lim_{h \to 0} \frac{f(1+h) - f(1)}{h} = \; ?$$

As we do not yet have a true definition of a limit nor an exact method for computing it, we settle for approximating the value. While we could graph the difference quotient (where the x-axis would represent h values and the y-axis would represent values of the difference quotient) we settle for making a table. See Figure 1.1.16. The table gives us reason to assume the value of the limit is about 8.5.

Proper understanding of limits is key to understanding calculus. With limits, we can accomplish seemingly impossible mathematical things, like adding up an infinite number of numbers (and not get infinity) and finding the slope of a line between two points, where the "two points" are actually the same point. These are not just mathematical curiosities; they allow us to link position, velocity and acceleration together, connect cross-sectional areas to volume, find the work done by a variable force, and much more.

In the next section we give the formal definition of the limit and begin our study of finding limits analytically. In the following exercises, we continue our introduction and approximate the value of limits.

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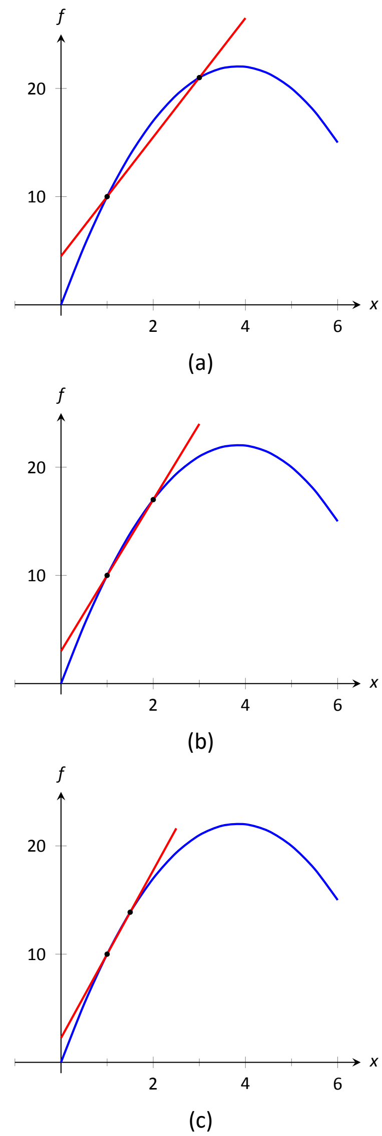


Figure 1.1.15: Secant lines of $f(x)$ at $x = 1$ and $x = 1 + h$, for shrinking values of h (i.e., $h \to 0$).

h	$\frac{f(1+h)-f(1)}{h}$
-0.5	9.25
-0.1	8.65
-0.01	8.515
0.01	8.485
0.1	8.35
0.5	7.75

Figure 1.1.16: The difference quotient evaluated at values of h near 0.

Exercises 1.1

Terms and Concepts

01 01 ex 19 1. In your own words, what does it mean to "find the limit of $f(x)$ as x approaches 3"?

01 01 ex 20 2. An expression of the form $\frac{0}{0}$ is called ____.

01 01 ex 21 3. T/F: The limit of $f(x)$ as x approaches 5 is $f(5)$.

01 01 ex 22 4. Describe three situations where $\lim\limits_{x\to c} f(x)$ does not exist.

01 01 ex 23 5. In your own words, what is a difference quotient?

01 01 ex 24 6. When x is near 0, $\dfrac{\sin x}{x}$ is near what value?

Problems

01 01 exset 01 **In Exercises 7 – 16, approximate the given limits both numerically and graphically.**

01 01 ex 01 7. $\lim\limits_{x\to 1} x^2 + 3x - 5$

01 01 ex 02 8. $\lim\limits_{x\to 0} x^3 - 3x^2 + x - 5$

01 01 ex 03 9. $\lim\limits_{x\to 0} \dfrac{x+1}{x^2+3x}$

01 01 ex 04 10. $\lim\limits_{x\to 3} \dfrac{x^2-2x-3}{x^2-4x+3}$

01 01 ex 05 11. $\lim\limits_{x\to -1} \dfrac{x^2+8x+7}{x^2+6x+5}$

01 01 ex 06 12. $\lim\limits_{x\to 2} \dfrac{x^2+7x+10}{x^2-4x+4}$

01 01 ex 07 13. $\lim\limits_{x\to 2} f(x)$, where

$$f(x) = \begin{cases} x+2 & x \leq 2 \\ 3x-5 & x > 2 \end{cases}.$$

01 01 ex 08 14. $\lim\limits_{x\to 3} f(x)$, where

$$f(x) = \begin{cases} x^2-x+1 & x \leq 3 \\ 2x+1 & x > 3 \end{cases}.$$

01 01 ex 09 15. $\lim\limits_{x\to 0} f(x)$, where

$$f(x) = \begin{cases} \cos x & x \leq 0 \\ x^2+3x+1 & x > 0 \end{cases}.$$

01 01 ex 10 16. $\lim\limits_{x\to \pi/2} f(x)$, where

$$f(x) = \begin{cases} \sin x & x \leq \pi/2 \\ \cos x & x > \pi/2 \end{cases}.$$

01 01 exset 02 **In Exercises 17 – 24, a function f and a value a are given. Approximate the limit of the difference quotient, $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$, using $h = \pm 0.1, \pm 0.01$.**

01 01 ex 11 17. $f(x) = -7x + 2, \quad a = 3$

01 01 ex 12 18. $f(x) = 9x + 0.06, \quad a = -1$

01 01 ex 13 19. $f(x) = x^2 + 3x - 7, \quad a = 1$

01 01 ex 14 20. $f(x) = \dfrac{1}{x+1}, \quad a = 2$

01 01 ex 15 21. $f(x) = -4x^2 + 5x - 1, \quad a = -3$

01 01 ex 16 22. $f(x) = \ln x, \quad a = 5$

01 01 ex 17 23. $f(x) = \sin x, \quad a = \pi$

01 01 ex 18 24. $f(x) = \cos x, \quad a = \pi$

1.2 Epsilon-Delta Definition of a Limit

This section introduces the formal definition of a limit. Many refer to this as "the epsilon-delta," definition, referring to the letters ε and δ of the Greek alphabet.

Note: the common phrase "the ε-δ definition" is read aloud as "the epsilon delta definition." The hyphen between ε and δ is not a minus sign.

Before we give the actual definition, let's consider a few informal ways of describing a limit. Given a function $y = f(x)$ and an x-value, c, we say that "the limit of the function f, as x approaches c, is a value L":

1. if "y tends to L" as "x tends to c."

2. if "y approaches L" as "x approaches c."

3. if "y is near L" whenever "x is near c."

The problem with these definitions is that the words "tends," "approach," and especially "near" are not exact. In what way does the variable x tend to, or approach, c? How near do x and y have to be to c and L, respectively?

The definition we describe in this section comes from formalizing **3**. A quick restatement gets us closer to what we want:

3′. If x is within a certain *tolerance level* of c, then the corresponding value $y = f(x)$ is within a certain *tolerance level* of L.

The traditional notation for the x-tolerance is the lowercase Greek letter delta, or δ, and the y-tolerance is denoted by lowercase epsilon, or ε. One more rephrasing of **3′** nearly gets us to the actual definition:

3″. If x is within δ units of c, then the corresponding value of y is within ε units of L.

We can write "x is within δ units of c" mathematically as

$$|x - c| < \delta, \qquad \text{which is equivalent to} \qquad c - \delta < x < c + \delta.$$

Letting the symbol "$\longrightarrow$" represent the word "implies," we can rewrite **3″** as

$$|x-c| < \delta \longrightarrow |y-L| < \varepsilon \qquad \text{or} \qquad c-\delta < x < c+\delta \longrightarrow L-\varepsilon < y < L+\varepsilon.$$

The point is that δ and ε, being tolerances, can be any positive (but typically small) values. Finally, we have the formal definition of the limit with the notation seen in the previous section.

Notes:

Definition 1.2.1 The Limit of a Function f

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c. The **limit of** $f(x)$**, as** x **approaches** c**, is** L, denoted by

$$\lim_{x\to c} f(x) = L,$$

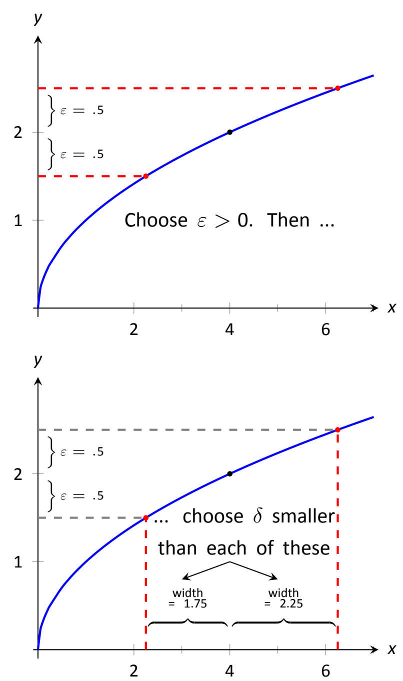
means that given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

(Mathematicians often enjoy writing ideas without using any words. Here is the wordless definition of the limit:

$$\lim_{x\to c} f(x) = L \iff \forall\, \varepsilon > 0, \exists\, \delta > 0 \text{ s.t. } 0 < |x - c| < \delta \longrightarrow |f(x) - L| < \varepsilon.)$$

Note the order in which ε and δ are given. In the definition, the y-tolerance ε is given *first* and then the limit will exist **if** we can find an x-tolerance δ that works.

An example will help us understand this definition. Note that the explanation is long, but it will take one through all steps necessary to understand the ideas.



With $\varepsilon = 0.5$, we pick any $\delta < 1.75$.

Figure 1.2.1: Illustrating the $\varepsilon - \delta$ process.

Example 1.2.1 Evaluating a limit using the definition

Show that $\lim\limits_{x\to 4} \sqrt{x} = 2.$

Solution Before we use the formal definition, let's try some numerical tolerances. What if the y tolerance is 0.5, or $\varepsilon = 0.5$? How close to 4 does x have to be so that y is within 0.5 units of 2, i.e., $1.5 < y < 2.5$? In this case, we can proceed as follows:

$$\begin{aligned} 1.5 <& \;\; y &< 2.5 \\ 1.5 <& \sqrt{x} &< 2.5 \\ 1.5^2 <& \;\; x &< 2.5^2 \\ 2.25 <& \;\; x &< 6.25. \end{aligned}$$

So, what is the desired x tolerance? Remember, we want to find a symmetric interval of x values, namely $4 - \delta < x < 4 + \delta$. The lower bound of 2.25 is 1.75 units from 4; the upper bound of 6.25 is 2.25 units from 4. We need the smaller of these two distances; we must have $\delta < 1.75$. See Figure 1.2.1.

Notes:

Given the y tolerance $\varepsilon = 0.5$, we have found an x tolerance, $\delta < 1.75$, such that whenever x is within δ units of 4, then y is within ε units of 2. That's what we were trying to find.

Let's try another value of ε.

What if the y tolerance is 0.01, i.e., $\varepsilon = 0.01$? How close to 4 does x have to be in order for y to be within 0.01 units of 2 (or $1.99 < y < 2.01$)? Again, we just square these values to get $1.99^2 < x < 2.01^2$, or

$$3.9601 < x < 4.0401.$$

What is the desired x tolerance? In this case we must have $\delta < 0.0399$, which is the minimum distance from 4 of the two bounds given above.

What we have so far: if $\varepsilon = 0.5$, then $\delta < 1.75$ and if $\varepsilon = 0.01$, then $\delta < 0.0399$. A pattern is not easy to see, so we switch to general ε try to determine δ symbolically. We start by assuming $y = \sqrt{x}$ is within ε units of 2:

$$\begin{aligned}
|y-2| &< \varepsilon \\
-\varepsilon < y-2 &< \varepsilon && \text{(Definition of absolute value)} \\
-\varepsilon < \sqrt{x}-2 &< \varepsilon && (y=\sqrt{x}) \\
2-\varepsilon < \sqrt{x} &< 2+\varepsilon && \text{(Add 2)} \\
(2-\varepsilon)^2 < x &< (2+\varepsilon)^2 && \text{(Square all)} \\
4-4\varepsilon+\varepsilon^2 < x &< 4+4\varepsilon+\varepsilon^2 && \text{(Expand)} \\
4-(4\varepsilon-\varepsilon^2) < x &< 4+(4\varepsilon+\varepsilon^2). && \text{(Rewrite in the desired form)}
\end{aligned}$$

The "desired form" in the last step is "$4 - \textit{something} < x < 4 + \textit{something}$." Since we want this last interval to describe an x tolerance around 4, we have that either $\delta < 4\varepsilon - \varepsilon^2$ or $\delta < 4\varepsilon + \varepsilon^2$, whichever is smaller:

$$\delta < \min\{4\varepsilon - \varepsilon^2, 4\varepsilon + \varepsilon^2\}.$$

Since $\varepsilon > 0$, the minimum is $\delta < 4\varepsilon - \varepsilon^2$. That's the formula: given an ε, set $\delta < 4\varepsilon - \varepsilon^2$.

We can check this for our previous values. If $\varepsilon = 0.5$, the formula gives $\delta < 4(0.5) - (0.5)^2 = 1.75$ and when $\varepsilon = 0.01$, the formula gives $\delta < 4(0.01) - (0.01)^2 = 0.399$.

So given any $\varepsilon > 0$, set $\delta < 4\varepsilon - \varepsilon^2$. Then if $|x-4| < \delta$ (and $x \neq 4$), then $|f(x) - 2| < \varepsilon$, satisfying the definition of the limit. We have shown formally (and finally!) that $\lim\limits_{x\to 4} \sqrt{x} = 2$.

Notes:

The previous example was a little long in that we sampled a few specific cases of ε before handling the general case. Normally this is not done. The previous example is also a bit unsatisfying in that $\sqrt{4} = 2$; why work so hard to prove something so obvious? Many ε-δ proofs are long and difficult to do. In this section, we will focus on examples where the answer is, frankly, obvious, because the non–obvious examples are even harder. In the next section we will learn some theorems that allow us to evaluate limits *analytically*, that is, without using the ε-δ definition.

Example 1.2.2 Evaluating a limit using the definition
Show that $\lim\limits_{x\to 2} x^2 = 4$.

Solution Let's do this example symbolically from the start. Let $\varepsilon > 0$ be given; we want $|y-4| < \varepsilon$, i.e., $|x^2-4| < \varepsilon$. How do we find δ such that when $|x-2| < \delta$, we are guaranteed that $|x^2-4| < \varepsilon$?

This is a bit trickier than the previous example, but let's start by noticing that $|x^2-4| = |x-2| \cdot |x+2|$. Consider:

$$|x^2-4| < \varepsilon \longrightarrow |x-2| \cdot |x+2| < \varepsilon \longrightarrow |x-2| < \frac{\varepsilon}{|x+2|}. \tag{1.1}$$

Could we not set $\delta = \dfrac{\varepsilon}{|x+2|}$?

We are close to an answer, but the catch is that δ must be a *constant* value (so it can't contain x). There is a way to work around this, but we do have to make an assumption. Remember that ε is supposed to be a small number, which implies that δ will also be a small value. In particular, we can (probably) assume that $\delta < 1$. If this is true, then $|x-2| < \delta$ would imply that $|x-2| < 1$, giving $1 < x < 3$.

Now, back to the fraction $\dfrac{\varepsilon}{|x+2|}$. If $1 < x < 3$, then $3 < x+2 < 5$ (add 2 to all terms in the inequality). Taking reciprocals, we have

$$\begin{aligned}
\frac{1}{5} &< \frac{1}{|x+2|} < \frac{1}{3} && \text{which implies} \\
\frac{1}{5} &< \frac{1}{|x+2|} && \text{which implies} \\
\frac{\varepsilon}{5} &< \frac{\varepsilon}{|x+2|}.
\end{aligned} \tag{1.2}$$

This suggests that we set $\delta < \dfrac{\varepsilon}{5}$. To see why, let consider what follows when we assume $|x-2| < \delta$:

Notes:

$$
\begin{aligned}
|x-2| &< \delta \\
|x-2| &< \frac{\varepsilon}{5} && \text{(Our choice of } \delta\text{)} \\
|x-2|\cdot|x+2| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Multiply by } |x+2|\text{)} \\
|x^2-4| &< |x+2|\cdot\frac{\varepsilon}{5} && \text{(Combine left side)} \\
|x^2-4| &< |x+2|\cdot\frac{\varepsilon}{5} < |x+2|\cdot\frac{\varepsilon}{|x+2|} = \varepsilon && \text{(Using (1.2) as long as } \delta < 1\text{)}
\end{aligned}
$$

We have arrived at $|x^2-4| < \varepsilon$ as desired. Note again, in order to make this happen we needed δ to first be less than 1. That is a safe assumption; we want ε to be arbitrarily small, forcing δ to also be small.

We have also picked δ to be smaller than "necessary." We could get by with a slightly larger δ, as shown in Figure 1.2.2. The dashed outer lines show the boundaries defined by our choice of ε. The dotted inner lines show the boundaries defined by setting $\delta = \varepsilon/5$. Note how these dotted lines are within the dashed lines. That is perfectly fine; by choosing x within the dotted lines we are guaranteed that $f(x)$ will be within ε of 4.

In summary, given $\varepsilon > 0$, set $\delta = \varepsilon/5$. Then $|x-2| < \delta$ implies $|x^2-4| < \varepsilon$ (i.e. $|y-4| < \varepsilon$) as desired. This shows that $\lim\limits_{x\to 2} x^2 = 4$. Figure 1.2.2 gives a visualization of this; by restricting x to values within $\delta = \varepsilon/5$ of 2, we see that $f(x)$ is within ε of 4.

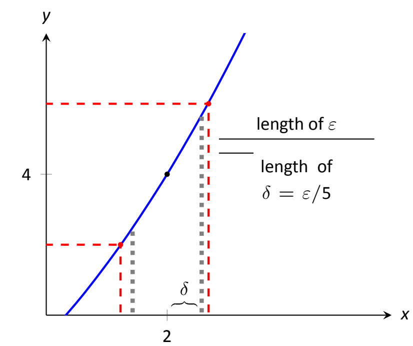


Figure 1.2.2: Choosing $\delta = \varepsilon/5$ in Example 1.2.2.

Make note of the general pattern exhibited in these last two examples. In some sense, each starts out "backwards." That is, while we want to

1. start with $|x-c| < \delta$ and conclude that
2. $|f(x)-L| < \varepsilon$,

we actually start by assuming

1. $|f(x)-L| < \varepsilon$, then perform some algebraic manipulations to give an inequality of the form
2. $|x-c| <$ *something*.

When we have properly done this, the *something* on the "greater than" side of the inequality becomes our δ. We can refer to this as the "scratch–work" phase of our proof. Once we have δ, we can formally start with $|x-c| < \delta$ and use algebraic manipulations to conclude that $|f(x)-L| < \varepsilon$, usually by using the same steps of our "scratch–work" in reverse order.

Notes:

We highlight this process in the following example.

Example 1.2.3 Evaluating a limit using the definition
Prove that $\lim_{x\to 1} x^3 - 2x = -1$.

Solution We start our scratch–work by considering $|f(x) - (-1)| < \varepsilon$:

$$\begin{aligned} |f(x) - (-1)| &< \varepsilon \\ |x^3 - 2x + 1| &< \varepsilon && \text{(Now factor)} \\ |(x-1)(x^2 + x - 1)| &< \varepsilon \\ |x - 1| &< \frac{\varepsilon}{|x^2 + x - 1|}. && (1.3) \end{aligned}$$

We are at the phase of saying that $|x - 1| < $ *something*, where *something*$=$ $\varepsilon/|x^2 + x - 1|$. We want to turn that *something* into δ.

Since x is approaching 1, we are safe to assume that x is between 0 and 2. So

$$\begin{aligned} &0 < x < 2 \\ &0 < x^2 < 4. && \text{(squared each term)} \end{aligned}$$

Since $0 < x < 2$, we can add 0, x and 2, respectively, to each part of the inequality and maintain the inequality.

$$\begin{aligned} 0 &< x^2 + x < 6 \\ -1 &< x^2 + x - 1 < 5. && \text{(subtracted 1 from each part)} \end{aligned}$$

In Equation (1.3), we wanted $|x - 1| < \varepsilon/|x^2 + x - 1|$. The above shows that given any x in $[0, 2]$, we know that

$$\begin{aligned} x^2 + x - 1 &< 5 && \text{which implies that} \\ \frac{1}{5} &< \frac{1}{x^2 + x - 1} && \text{which implies that} \\ \frac{\varepsilon}{5} &< \frac{\varepsilon}{x^2 + x - 1}. && (1.4) \end{aligned}$$

So we set $\delta < \varepsilon/5$. This ends our scratch–work, and we begin the formal proof (which also helps us understand why this was a good choice of δ).

Given ε, let $\delta < \varepsilon/5$. We want to show that when $|x - 1| < \delta$, then $|(x^3 -$

Notes:

$2x) - (-1)| < \varepsilon$. We start with $|x-1| < \delta$:

$$\begin{aligned}
|x-1| &< \delta \\
|x-1| &< \frac{\varepsilon}{5} \\
|x-1| &< \frac{\varepsilon}{5} < \frac{\varepsilon}{|x^2+x-1|} \quad \text{(for } x \text{ near 1, from Equation (1.4))} \\
|x-1| \cdot |x^2+x-1| &< \varepsilon \\
|x^3-2x+1| &< \varepsilon \\
|(x^3-2x)-(-1)| &< \varepsilon,
\end{aligned}$$

which is what we wanted to show. Thus $\lim_{x\to 1} x^3 - 2x = -1$.

We illustrate evaluating limits once more.

Example 1.2.4 Evaluating a limit using the definition

Prove that $\lim_{x\to 0} e^x = 1$.

Solution Symbolically, we want to take the equation $|e^x - 1| < \varepsilon$ and unravel it to the form $|x-0| < \delta$. Here is our scratch–work:

$$\begin{aligned}
|e^x - 1| &< \varepsilon \\
-\varepsilon < e^x - 1 &< \varepsilon & \text{(Definition of absolute value)} \\
1-\varepsilon < e^x &< 1+\varepsilon & \text{(Add 1)} \\
\ln(1-\varepsilon) < x &< \ln(1+\varepsilon) & \text{(Take natural logs)}
\end{aligned}$$

Note: Recall $\ln 1 = 0$ and $\ln x < 0$ when $0 < x < 1$. So $\ln(1-\varepsilon) < 0$, hence we consider its absolute value.

Making the safe assumption that $\varepsilon < 1$ ensures the last inequality is valid (i.e., so that $\ln(1-\varepsilon)$ is defined). We can then set δ to be the minimum of $|\ln(1-\varepsilon)|$ and $\ln(1+\varepsilon)$; i.e.,

$$\delta = \min\{|\ln(1-\varepsilon)|, \ln(1+\varepsilon)\} = \ln(1+\varepsilon).$$

Now, we work through the actual the proof:

$$\begin{aligned}
|x-0| &< \delta \\
-\delta < x &< \delta & \text{(Definition of absolute value)} \\
-\ln(1+\varepsilon) < x &< \ln(1+\varepsilon). \\
\ln(1-\varepsilon) < x &< \ln(1+\varepsilon). & \text{(since } \ln(1-\varepsilon) < -\ln(1+\varepsilon))
\end{aligned}$$

Notes:

The above line is true by our choice of δ and by the fact that since $|\ln(1-\varepsilon)| > \ln(1+\varepsilon)$ and $\ln(1-\varepsilon) < 0$, we know $\ln(1-\varepsilon) < -\ln(1+\varepsilon)$.

$$1-\varepsilon < e^x < 1+\varepsilon \qquad \text{(Exponentiate)}$$
$$-\varepsilon < e^x - 1 < \varepsilon \qquad \text{(Subtract 1)}$$

In summary, given $\varepsilon > 0$, let $\delta = \ln(1+\varepsilon)$. Then $|x-0| < \delta$ implies $|e^x - 1| < \varepsilon$ as desired. We have shown that $\lim\limits_{x\to 0} e^x = 1$.

We note that we could actually show that $\lim_{x\to c} e^x = e^c$ for any constant c. We do this by factoring out e^c from both sides, leaving us to show $\lim_{x\to c} e^{x-c} = 1$ instead. By using the substitution $u = x-c$, this reduces to showing $\lim_{u\to 0} e^u = 1$ which we just did in the last example. As an added benefit, this shows that in fact the function $f(x) = e^x$ is *continuous* at all values of x, an important concept we will define in Section 1.5.

This formal definition of the limit is not an easy concept grasp. Our examples are actually "easy" examples, using "simple" functions like polynomials, square–roots and exponentials. It is very difficult to prove, using the techniques given above, that $\lim\limits_{x\to 0}(\sin x)/x = 1$, as we approximated in the previous section.

There is hope. The next section shows how one can evaluate complicated limits using certain basic limits as building blocks. While limits are an incredibly important part of calculus (and hence much of higher mathematics), rarely are limits evaluated using the definition. Rather, the techniques of the following section are employed.

Notes:

Exercises 1.2

Terms and Concepts

01 02 ex 01 1. What is wrong with the following "definition" of a limit?

   "The limit of $f(x)$, as x approaches a, is K" means that given any $\delta > 0$ there exists $\varepsilon > 0$ such that whenever $|f(x) - K| < \varepsilon$, we have $|x - a| < \delta$.

01 02 ex 02 2. Which is given first in establishing a limit, the x–tolerance or the y–tolerance?

01 02 ex 09 3. T/F: ε must always be positive.

01 02 ex 10 4. T/F: δ must always be positive.

Problems

01 02 exset 02 **In Exercises 5 – 14, prove the given limit using an $\varepsilon - \delta$ proof.**

01 02 ex 12 5. $\lim\limits_{x\to 4} 2x + 5 = 13$

01 02 ex 04 6. $\lim\limits_{x\to 5} 3 - x = -2$

01 02 ex 05 7. $\lim\limits_{x\to 3} x^2 - 3 = 6$

01 02 ex 11 8. $\lim\limits_{x\to 4} x^2 + x - 5 = 15$

01 02 ex 13 9. $\lim\limits_{x\to 1} 2x^2 + 3x + 1 = 6$

01 02 ex 06 10. $\lim\limits_{x\to 2} x^3 - 1 = 7$

01 02 ex 03 11. $\lim\limits_{x\to 2} 5 = 5$

01 02 ex 07 12. $\lim\limits_{x\to 0} e^{2x} - 1 = 0$

01 02 ex 14 13. $\lim\limits_{x\to 1} \dfrac{1}{x} = 1$

01 02 ex 08 14. $\lim\limits_{x\to 0} \sin x = 0$ (Hint: use the fact that $|\sin x| \leq |x|$, with equality only when $x = 0$.)

1.3 Finding Limits Analytically

In Section 1.1 we explored the concept of the limit without a strict definition, meaning we could only make approximations. In the previous section we gave the definition of the limit and demonstrated how to use it to verify our approximations were correct. Thus far, our method of finding a limit is 1) make a really good approximation either graphically or numerically, and 2) verify our approximation is correct using a ε-δ proof.

Recognizing that ε-δ proofs are cumbersome, this section gives a series of theorems which allow us to find limits much more quickly and intuitively.

Suppose that $\lim_{x\to 2} f(x) = 2$ and $\lim_{x\to 2} g(x) = 3$. What is $\lim_{x\to 2}(f(x) + g(x))$? Intuition tells us that the limit should be 5, as we expect limits to behave in a nice way. The following theorem states that already established limits do behave nicely.

Theorem 1.3.1 Basic Limit Properties

Let b, c, L and K be real numbers, let n be a positive integer, and let f and g be functions defined on an open interval I containing c with the following limits:

$$\lim_{x\to c} f(x) = L \text{ and } \lim_{x\to c} g(x) = K.$$

The following limits hold.

1. Constants: $\lim_{x\to c} b = b$
2. Identity $\lim_{x\to c} x = c$
3. Sums/Differences: $\lim_{x\to c}(f(x) \pm g(x)) = L \pm K$
4. Scalar Multiples: $\lim_{x\to c} b \cdot f(x) = bL$
5. Products: $\lim_{x\to c} f(x) \cdot g(x) = LK$
6. Quotients: $\lim_{x\to c} f(x)/g(x) = L/K$, $(K \neq 0)$
7. Powers: $\lim_{x\to c} f(x)^n = L^n$
8. Roots: $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L}$
   (If n is even then require $f(x) \geq 0$ on I.)
9. Compositions: Adjust our previously given limit situation to:
   $$\lim_{x\to c} f(x) = L,\ \lim_{x\to L} g(x) = K \text{ and } g(L) = K.$$
   Then $\lim_{x\to c} g(f(x)) = K$.

Notes:

We make a note about Property #8: when n is even, L must be greater than 0. If n is odd, then the statement is true for all L.

We apply the theorem to an example.

Example 1.3.1 Using basic limit properties

Let

$$\lim_{x\to 2} f(x) = 2, \quad \lim_{x\to 2} g(x) = 3 \quad \text{and} \quad p(x) = 3x^2 - 5x + 7.$$

Find the following limits:

1. $\lim\limits_{x\to 2} \big(f(x) + g(x)\big)$
2. $\lim\limits_{x\to 2} \big(5f(x) + g(x)^2\big)$
3. $\lim\limits_{x\to 2} p(x)$

Solution

1. Using the Sum/Difference rule, we know that $\lim\limits_{x\to 2} \big(f(x)+g(x)\big) = 2+3 = 5$.
2. Using the Scalar Multiple and Sum/Difference rules, we find that $\lim\limits_{x\to 2} \big(5f(x)+g(x)^2\big) = 5\cdot 2 + 3^2 = 19$.
3. Here we combine the Power, Scalar Multiple, Sum/Difference and Constant Rules. We show quite a few steps, but in general these can be omitted:

$$\begin{aligned}
\lim_{x\to 2} p(x) &= \lim_{x\to 2}(3x^2 - 5x + 7)\\
&= \lim_{x\to 2} 3x^2 - \lim_{x\to 2} 5x + \lim_{x\to 2} 7\\
&= 3\cdot 2^2 - 5\cdot 2 + 7\\
&= 9
\end{aligned}$$

Part 3 of the previous example demonstrates how the limit of a quadratic polynomial can be determined using the properties of Theorem 1.3.1. Not only that, recognize that

$$\lim_{x\to 2} p(x) = 9 = p(2);$$

i.e., the limit at 2 was found just by plugging 2 into the function. This holds true for all polynomials, and also for rational functions (which are quotients of polynomials), as stated in the following theorem.

Notes:

Theorem 1.3.2 Limits of Polynomial and Rational Functions

Let $p(x)$ and $q(x)$ be polynomials and c a real number. Then:

1. $\lim\limits_{x\to c} p(x) = p(c)$

2. $\lim\limits_{x\to c} \dfrac{p(x)}{q(x)} = \dfrac{p(c)}{q(c)}$, where $q(c) \neq 0$.

Example 1.3.2 Finding a limit of a rational function

Using Theorem 1.3.2, find

$$\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3}.$$

Solution Using Theorem 1.3.2, we can quickly state that

$$\begin{aligned}\lim_{x\to -1} \frac{3x^2-5x+1}{x^4-x^2+3} &= \frac{3(-1)^2-5(-1)+1}{(-1)^4-(-1)^2+3}\\ &= \frac{9}{3} = 3.\end{aligned}$$

It was likely frustrating in Section 1.2 to do a lot of work to prove that

$$\lim_{x\to 2} x^2 = 4$$

as it seemed fairly obvious. The previous theorems state that many functions behave in such an "obvious" fashion, as demonstrated by the rational function in Example 1.3.2.

Polynomial and rational functions are not the only functions to behave in such a predictable way. The following theorem gives a list of functions whose behavior is particularly "nice" in terms of limits. In the next section, we will give a formal name to these functions that behave "nicely."

Theorem 1.3.3 Special Limits

Let c be a real number in the domain of the given function and let n be a positive integer. The following limits hold:

1. $\lim\limits_{x\to c} \sin x = \sin c$
2. $\lim\limits_{x\to c} \cos x = \cos c$
3. $\lim\limits_{x\to c} \tan x = \tan c$
4. $\lim\limits_{x\to c} \csc x = \csc c$
5. $\lim\limits_{x\to c} \sec x = \sec c$
6. $\lim\limits_{x\to c} \cot x = \cot c$
7. $\lim\limits_{x\to c} a^x = a^c$ $(a > 0)$
8. $\lim\limits_{x\to c} \ln x = \ln c$
9. $\lim\limits_{x\to c} \sqrt[n]{x} = \sqrt[n]{c}$

Notes:

Example 1.3.3 Evaluating limits analytically

Evaluate the following limits.

1. $\lim_{x\to\pi} \cos x$
2. $\lim_{x\to 3} (\sec^2 x - \tan^2 x)$
3. $\lim_{x\to\pi/2} \cos x \sin x$
4. $\lim_{x\to 1} e^{\ln x}$
5. $\lim_{x\to 0} \dfrac{\sin x}{x}$

Solution

1. This is a straightforward application of Theorem 1.3.3. $\lim_{x\to\pi} \cos x = \cos\pi = -1$.
2. We can approach this in at least two ways. First, by directly applying Theorem 1.3.3, we have:
$$\lim_{x\to 3}(\sec^2 x - \tan^2 x) = \sec^2 3 - \tan^2 3.$$
Using the Pythagorean Theorem, this last expression is 1; therefore
$$\lim_{x\to 3}(\sec^2 x - \tan^2 x) = 1.$$
We can also use the Pythagorean Theorem from the start.
$$\lim_{x\to 3}(\sec^2 x - \tan^2 x) = \lim_{x\to 3} 1 = 1,$$
using the Constant limit rule. Either way, we find the limit is 1.
3. Applying the Product limit rule of Theorem 1.3.1 and Theorem 1.3.3 gives
$$\lim_{x\to\pi/2} \cos x \sin x = \cos(\pi/2)\sin(\pi/2) = 0 \cdot 1 = 0.$$
4. Again, we can approach this in two ways. First, we can use the exponential/logarithmic identity that $e^{\ln x} = x$ and evaluate $\lim_{x\to 1} e^{\ln x} = \lim_{x\to 1} x = 1$.
We can also use the limit Composition Rule of Theorem 1.3.1. Using Theorem 1.3.3, we have $\lim_{x\to 1} \ln x = \ln 1 = 0$ and $\lim_{x\to 0} e^x = e^0 = 1$, satisfying the conditions of the Composition Rule. Applying this rule,
$$\lim_{x\to 1} e^{\ln x} = \lim_{x\to 0} e^x = e^0 = 1.$$
Both approaches are valid, giving the same result.

Notes:

5. We encountered this limit in Section 1.1. Applying our theorems, we attempt to find the limit as

$$\lim_{x\to 0}\frac{\sin x}{x} \to \frac{\sin 0}{0} \to \text{“}\frac{0}{0}\text{”}.$$

This, of course, violates a condition of Theorem 1.3.1, as the limit of the denominator is not allowed to be 0. Therefore, we are still unable to evaluate this limit with tools we currently have at hand.

The section could have been titled “Using Known Limits to Find Unknown Limits.” By knowing certain limits of functions, we can find limits involving sums, products, powers, etc., of these functions. We further the development of such comparative tools with the Squeeze Theorem, a clever and intuitive way to find the value of some limits.

Before stating this theorem formally, suppose we have functions f, g and h where g always takes on values between f and h; that is, for all x in an interval,

$$f(x) \leq g(x) \leq h(x).$$

If f and h have the same limit at c, and g is always “squeezed” between them, then g must have the same limit as well. That is what the Squeeze Theorem states.

Theorem 1.3.4 Squeeze Theorem

Let f, g and h be functions on an open interval I containing c such that for all x in I,

$$f(x) \leq g(x) \leq h(x).$$

If

$$\lim_{x\to c} f(x) = L = \lim_{x\to c} h(x),$$

then

$$\lim_{x\to c} g(x) = L.$$

It can take some work to figure out appropriate functions by which to “squeeze” a given function. However, that is generally the only place where work is necessary; the theorem makes the “evaluating the limit part” very simple.

We use the Squeeze Theorem in the following example to finally prove that $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$.

Notes:

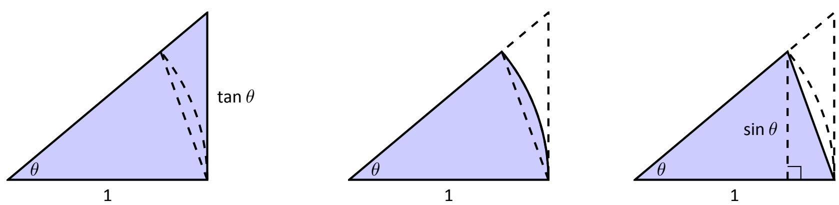
Example 1.3.4 Using the Squeeze Theorem

Use the Squeeze Theorem to show that

$$\lim_{x\to 0} \frac{\sin x}{x} = 1.$$

Solution We begin by considering the unit circle. Each point on the unit circle has coordinates $(\cos\theta, \sin\theta)$ for some angle θ as shown in Figure 1.3.1. Using similar triangles, we can extend the line from the origin through the point to the point $(1, \tan\theta)$, as shown. (Here we are assuming that $0 \leq \theta \leq \pi/2$. Later we will show that we can also consider $\theta \leq 0$.)

Figure 1.3.1 shows three regions have been constructed in the first quadrant, two triangles and a sector of a circle, which are also drawn below. The area of the large triangle is $\frac{1}{2}\tan\theta$; the area of the sector is $\theta/2$; the area of the triangle contained inside the sector is $\frac{1}{2}\sin\theta$. It is then clear from the diagram that



$$\frac{\tan\theta}{2} \geq \frac{\theta}{2} \geq \frac{\sin\theta}{2}$$

Multiply all terms by $\dfrac{2}{\sin\theta}$, giving

$$\frac{1}{\cos\theta} \geq \frac{\theta}{\sin\theta} \geq 1.$$

Taking reciprocals reverses the inequalities, giving

$$\cos\theta \leq \frac{\sin\theta}{\theta} \leq 1.$$

(These inequalities hold for all values of θ near 0, even negative values, since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$.)

Now take limits.

$$\lim_{\theta\to 0} \cos\theta \leq \lim_{\theta\to 0} \frac{\sin\theta}{\theta} \leq \lim_{\theta\to 0} 1$$

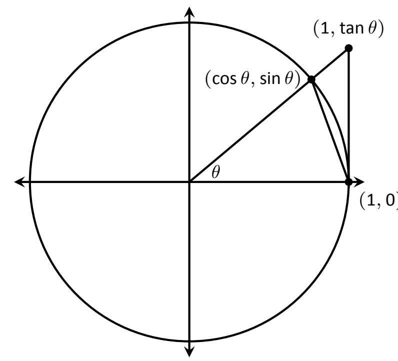


Figure 1.3.1: The unit circle and related triangles.

Notes:

$$\cos 0 \leq \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \leq 1$$

$$1 \leq \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \leq 1$$

Clearly this means that $\lim_{\theta \to 0} \dfrac{\sin \theta}{\theta} = 1$.

Two notes about the previous example are worth mentioning. First, one might be discouraged by this application, thinking "I would *never* have come up with that on my own. This is too hard!" Don't be discouraged; within this text we will guide you in your use of the Squeeze Theorem. As one gains mathematical maturity, clever proofs like this are easier and easier to create.

Second, this limit tells us more than just that as x approaches 0, $\sin(x)/x$ approaches 1. Both x and $\sin x$ are approaching 0, but the *ratio* of x and $\sin x$ approaches 1, meaning that they are approaching 0 in essentially the same way. Another way of viewing this is: for small x, the functions $y = x$ and $y = \sin x$ are essentially indistinguishable.

We include this special limit, along with three others, in the following theorem.

Theorem 1.3.5 Special Limits

1. $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$
2. $\lim_{x \to 0} \dfrac{\cos x - 1}{x} = 0$
3. $\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
4. $\lim_{x \to 0} \dfrac{e^x - 1}{x} = 1$

A short word on how to interpret the latter three limits. We know that as x goes to 0, $\cos x$ goes to 1. So, in the second limit, both the numerator and denominator are approaching 0. However, since the limit is 0, we can interpret this as saying that "$\cos x$ is approaching 1 faster than x is approaching 0."

In the third limit, inside the parentheses we have an expression that is approaching 1 (though never equaling 1), and we know that 1 raised to any power is still 1. At the same time, the power is growing toward infinity. What happens to a number near 1 raised to a very large power? In this particular case, the result approaches Euler's number, e, approximately 2.718.

In the fourth limit, we see that as $x \to 0$, e^x approaches 1 "just as fast" as $x \to 0$, resulting in a limit of 1.

Notes:

Our final theorem for this section will be motivated by the following example.

Example 1.3.5 Using algebra to evaluate a limit
Evaluate the following limit:

$$\lim_{x\to 1} \frac{x^2-1}{x-1}.$$

Solution We begin by attempting to apply Theorem 1.3.2 and substituting 1 for x in the quotient. This gives:

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \text{“}\frac{0}{0}\text{”},$$

an indeterminate form. We cannot apply the theorem.

By graphing the function, as in Figure 1.3.2, we see that the function seems to be linear, implying that the limit should be easy to evaluate. Recognize that the numerator of our quotient can be factored:

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1}.$$

The function is not defined when $x = 1$, but for all other x,

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}} = x+1.$$

Clearly $\lim\limits_{x\to 1} x+1 = 2$. Recall that when considering limits, we are not concerned with the value of the function at 1, only the value the function approaches as x approaches 1. Since $(x^2-1)/(x-1)$ and $x+1$ are the same at all points except $x = 1$, they both approach the same value as x approaches 1. Therefore we can conclude that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2.$$

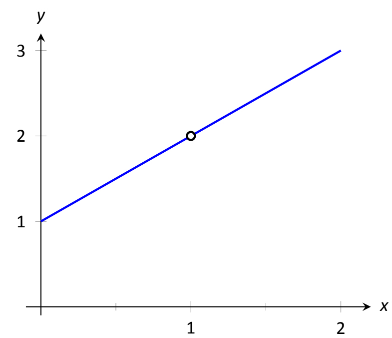


Figure 1.3.2: Graphing f in Example 1.3.5 to understand a limit.

The key to the above example is that the functions $y = (x^2-1)/(x-1)$ and $y = x+1$ are identical except at $x = 1$. Since limits describe a value the function is approaching, not the value the function actually attains, the limits of the two functions are always equal.

Notes:

Theorem 1.3.6 Limits of Functions Equal At All But One Point

Let $g(x) = f(x)$ for all x in an open interval, except possibly at c, and let $\lim\limits_{x\to c} g(x) = L$ for some real number L. Then

$$\lim_{x\to c} f(x) = L.$$

The Fundamental Theorem of Algebra tells us that when dealing with a rational function of the form $g(x)/f(x)$ and directly evaluating the limit $\lim\limits_{x\to c} \dfrac{g(x)}{f(x)}$ returns "0/0", then $(x-c)$ is a factor of both $g(x)$ and $f(x)$. One can then use algebra to factor this term out, cancel, then apply Theorem 1.3.6. We demonstrate this once more.

Example 1.3.6 Evaluating a limit using Theorem 1.3.6

Evaluate $\lim\limits_{x\to 3} \dfrac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15}$.

Solution We attempt to apply Theorem 1.3.2 by substituting 3 for x. This returns the familiar indeterminate form of "0/0". Since the numerator and denominator are each polynomials, we know that $(x-3)$ is factor of each. Using whatever method is most comfortable to you, factor out $(x-3)$ from each (using polynomial division, synthetic division, a computer algebra system, etc.). We find that

$$\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} = \frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}.$$

We can cancel the $(x-3)$ terms as long as $x \neq 3$. Using Theorem 1.3.6 we conclude:

$$\begin{aligned}\lim_{x\to 3}\frac{x^3-2x^2-5x+6}{2x^3+3x^2-32x+15} &= \lim_{x\to 3}\frac{(x-3)(x^2+x-2)}{(x-3)(2x^2+9x-5)}\\ &= \lim_{x\to 3}\frac{(x^2+x-2)}{(2x^2+9x-5)}\\ &= \frac{10}{40} = \frac{1}{4}.\end{aligned}$$

We end this section by revisiting a limit first seen in Section 1.1, a limit of a difference quotient. Let $f(x) = -1.5x^2 + 11.5x$; we approximated the limit $\lim\limits_{h\to 0}\dfrac{f(1+h)-f(1)}{h} \approx 8.5$. We formally evaluate this limit in the following example.

Notes:

Example 1.3.7 Evaluating the limit of a difference quotient

Let $f(x) = -1.5x^2 + 11.5x$; find $\lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$.

Solution Since f is a polynomial, our first attempt should be to employ Theorem 1.3.2 and substitute 0 for h. However, we see that this gives us "$0/0$." Knowing that we have a rational function hints that some algebra will help. Consider the following steps:

$$\begin{aligned}
\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} &= \lim_{h\to 0}\frac{-1.5(1+h)^2+11.5(1+h)-\big(-1.5(1)^2+11.5(1)\big)}{h}\\
&= \lim_{h\to 0}\frac{-1.5(1+2h+h^2)+11.5+11.5h-10}{h}\\
&= \lim_{h\to 0}\frac{-1.5h^2+8.5h}{h}\\
&= \lim_{h\to 0}\frac{h(-1.5h+8.5)}{h}\\
&= \lim_{h\to 0}(-1.5h+8.5) \quad \big(\text{using Theorem 1.3.6, as } h\neq 0\big)\\
&= 8.5 \quad \big(\text{using Theorem 1.3.3}\big)
\end{aligned}$$

This matches our previous approximation.

This section contains several valuable tools for evaluating limits. One of the main results of this section is Theorem 1.3.3; it states that many functions that we use regularly behave in a very nice, predictable way. In Section 1.5 we give a name to this nice behavior; we label such functions as *continuous.* Defining that term will require us to look again at what a limit is and what causes limits to not exist.

Notes:

Exercises 1.3

Terms and Concepts

01 03 ex 01

1. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} b = b$.

01 03 ex 02

2. Explain in your own words, without using ε-δ formality, why $\lim\limits_{x\to c} x = c$.

01 03 ex 03

3. What does the text mean when it says that certain functions' "behavior is 'nice' in terms of limits"? What, in particular, is "nice"?

01 03 ex 04

4. Sketch a graph that visually demonstrates the Squeeze Theorem.

01 03 ex 05

5. You are given the following information:

   (a) $\lim\limits_{x\to 1} f(x) = 0$

   (b) $\lim\limits_{x\to 1} g(x) = 0$

   (c) $\lim\limits_{x\to 1} f(x)/g(x) = 2$

   What can be said about the relative sizes of $f(x)$ and $g(x)$ as x approaches 1?

01 03 ex 44

6. T/F: $\lim\limits_{x\to 1} \ln x = 0$. Use a theorem to defend your answer.

Problems

01 03 exset 02

In Exercises 7 – 14, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim\limits_{x\to 9} f(x) = 6$, $\lim\limits_{x\to 6} f(x) = 9$, $f(9) = 6$
- $\lim\limits_{x\to 9} g(x) = 3$, $\lim\limits_{x\to 6} g(x) = 3$, $g(6) = 9$

01 03 ex 06

7. $\lim\limits_{x\to 9}\big(f(x) + g(x)\big)$

01 03 ex 07

8. $\lim\limits_{x\to 9}\big(3f(x)/g(x)\big)$

01 03 ex 08

9. $\lim\limits_{x\to 9}\left(\dfrac{f(x) - 2g(x)}{g(x)}\right)$

01 03 ex 09

10. $\lim\limits_{x\to 6}\left(\dfrac{f(x)}{3 - g(x)}\right)$

01 03 ex 10

11. $\lim\limits_{x\to 9} g\big(f(x)\big)$

01 03 ex 11

12. $\lim\limits_{x\to 6} f\big(g(x)\big)$

01 03 ex 12

13. $\lim\limits_{x\to 6} g\big(f(f(x))\big)$

01 03 ex 13

14. $\lim\limits_{x\to 6} f(x)g(x) - f^2(x) + g^2(x)$

01 03 exset 03

In Exercises 15 – 18, use the following information to evaluate the given limit, when possible. If it is not possible to determine the limit, state why not.

- $\lim\limits_{x\to 1} f(x) = 2$, $\lim\limits_{x\to 10} f(x) = 1$, $f(1) = 1/5$
- $\lim\limits_{x\to 1} g(x) = 0$, $\lim\limits_{x\to 10} g(x) = \pi$, $g(10) = \pi$

01 03 ex 14

15. $\lim\limits_{x\to 1} f(x)^{g(x)}$

01 03 ex 15

16. $\lim\limits_{x\to 10} \cos\big(g(x)\big)$

01 03 ex 16

17. $\lim\limits_{x\to 1} f(x)g(x)$

01 03 ex 17

18. $\lim\limits_{x\to 1} g\big(5f(x)\big)$

01 03 exset 04

In Exercises 19 – 34, evaluate the given limit.

01 03 ex 18

19. $\lim\limits_{x\to 3} x^2 - 3x + 7$

01 03 ex 21

20. $\lim\limits_{x\to \pi}\left(\dfrac{x-3}{x-5}\right)^7$

01 03 ex 22

21. $\lim\limits_{x\to \pi/4} \cos x \sin x$

01 03 ex 45

22. $\lim\limits_{x\to 1} \dfrac{2x-2}{x+4}$

01 03 ex 23

23. $\lim\limits_{x\to 0} \ln x$

01 03 ex 24

24. $\lim\limits_{x\to 3} 4^{x^3-8x}$

01 03 ex 25

25. $\lim\limits_{x\to \pi/6} \csc x$

01 03 ex 26

26. $\lim\limits_{x\to 0} \ln(1+x)$

01 03 ex 20

27. $\lim\limits_{x\to \pi} \dfrac{x^2+3x+5}{5x^2-2x-3}$

01 03 ex 19

28. $\lim\limits_{x\to \pi} \dfrac{3x+1}{1-x}$

01 03 ex 27

29. $\lim\limits_{x\to 6} \dfrac{x^2-4x-12}{x^2-13x+42}$

01 03 ex 28

30. $\lim\limits_{x\to 0} \dfrac{x^2+2x}{x^2-2x}$

01 03 ex 29

31. $\lim\limits_{x\to 2} \dfrac{x^2+6x-16}{x^2-3x+2}$

01 03 ex 30

32. $\lim\limits_{x\to 2} \dfrac{x^2-10x+16}{x^2-x-2}$

01 03 ex 31

33. $\lim\limits_{x\to -2} \dfrac{x^2-5x-14}{x^2+10x+16}$

01 03 ex 32 34. $\lim_{x\to -1} \dfrac{x^2+9x+8}{x^2-6x-7}$

01 03 exset 06 **Use the Squeeze Theorem in Exercises 35 – 38, where appropriate, to evaluate the given limit.**

01 03 ex 38 35. $\lim_{x\to 0} x\sin\left(\dfrac{1}{x}\right)$

01 03 ex 40 36. $\lim_{x\to 0} \sin x\cos\left(\dfrac{1}{x^2}\right)$

01 03 ex 42 37. $\lim_{x\to 1} f(x)$, where $3x-2 \leq f(x) \leq x^3$.

01 03 ex 41 38. $\lim_{x\to 3} f(x)$, where $6x-9 \leq f(x) \leq x^2$.

01 03 exset 05 **Exercises 39 – 43 challenge your understanding of limits but can be evaluated using the knowledge gained in this section.**

01 03 ex 33 39. $\lim_{x\to 0} \dfrac{\sin 3x}{x}$

01 03 ex 34 40. $\lim_{x\to 0} \dfrac{\sin 5x}{8x}$

01 03 ex 35 41. $\lim_{x\to 0} \dfrac{\ln(1+x)}{x}$

01 03 ex 36 42. $\lim_{x\to 0} \dfrac{\sin x}{x}$, where x is measured in degrees, not radians.

01 03 ex 43 43. Let $f(x) = 0$ and $g(x) = \dfrac{x}{x}$.

(a) Show why $\lim_{x\to 2} f(x) = 0$.

(b) Show why $\lim_{x\to 0} g(x) = 1$.

(c) Show why $\lim_{x\to 2} g\big(f(x)\big)$ does not exist.

(d) Show why the answer to part (c) does not violate the Composition Rule of Theorem 1.3.1.

1.4 One Sided Limits

We introduced the concept of a limit gently, approximating their values graphically and numerically. Next came the rigorous definition of the limit, along with an admittedly tedious method for evaluating them. The previous section gave us tools (which we call theorems) that allow us to compute limits with greater ease. Chief among the results were the facts that polynomials and rational, trigonometric, exponential and logarithmic functions (and their sums, products, etc.) all behave "nicely." In this section we rigorously define what we mean by "nicely."

In Section 1.1 we explored the three ways in which limits of functions failed to exist:

1. The function approached different values from the left and right,
2. The function grows without bound, and
3. The function oscillates.

In this section we explore in depth the concepts behind #1 by introducing the *one-sided limit*. We begin with formal definitions that are very similar to the definition of the limit given in Section 1.2, but the notation is slightly different and "$x \neq c$" is replaced with either "$x < c$" or "$x > c$."

Definition 1.4.1 One Sided Limits: Left- and Right-Hand Limits

Left-Hand Limit
Let f be a function defined on (a, c) for some $a < c$ and let L be a real number.
The **limit of $f(x)$, as x approaches c from the left, is L**, or, **the left-hand limit of f at c is L**, denoted by

$$\lim_{x \to c^-} f(x) = L,$$

means given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $a < x < c$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Right-Hand Limit
Let f be a function defined on (c, b) for some $b > c$ and let L be a real number.
The **limit of $f(x)$, as x approaches c from the right, is L**, or, **the right-hand limit of f at c is L**, denoted by

$$\lim_{x \to c^+} f(x) = L,$$

means given any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $c < x < b$, if $|x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

Notes:

Practically speaking, when evaluating a left-hand limit, we consider only values of x "to the left of c," i.e., where $x < c$. The admittedly imperfect notation $x \to c^-$ is used to imply that we look at values of x to the left of c. The notation has nothing to do with positive or negative values of either x or c. A similar statement holds for evaluating right-hand limits; there we consider only values of x to the right of c, i.e., $x > c$. We can use the theorems from previous sections to help us evaluate these limits; we just restrict our view to one side of c.

We practice evaluating left- and right-hand limits through a series of examples.

Example 1.4.1 Evaluating one sided limits

Let $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 3-x & 1 < x < 2 \end{cases}$, as shown in Figure 1.4.1. Find each of the following:

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

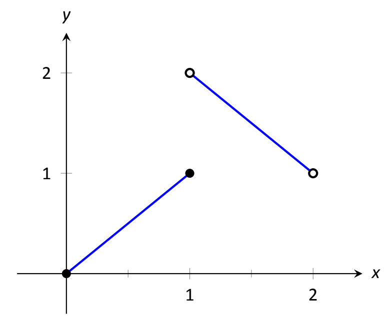


Figure 1.4.1: A graph of f in Example 1.4.1.

SOLUTION For these problems, the visual aid of the graph is likely more effective in evaluating the limits than using f itself. Therefore we will refer often to the graph.

1. As x goes to 1 *from the left*, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x\to 1^-} f(x) = 1$.
2. As x goes to 1 *from the right*, we see that $f(x)$ is approaching the value of 2. Recall that it does not matter that there is an "open circle" there; we are evaluating a limit, not the value of the function. Therefore $\lim\limits_{x\to 1^+} f(x) = 2$.
3. *The* limit of f as x approaches 1 does not exist, as discussed in the first section. The function does not approach one particular value, but two different values from the left and the right.
4. Using the definition and by looking at the graph we see that $f(1) = 1$.
5. As x goes to 0 from the right, we see that $f(x)$ is also approaching 0. Therefore $\lim\limits_{x\to 0^+} f(x) = 0$. Note we cannot consider a left-hand limit at 0 as f is not defined for values of $x < 0$.

Notes:

6. Using the definition and the graph, $f(0) = 0$.
7. As x goes to 2 from the left, we see that $f(x)$ is approaching the value of 1. Therefore $\lim\limits_{x\to 2^-} f(x) = 1$.
8. The graph and the definition of the function show that $f(2)$ is not defined.

Note how the left and right-hand limits were different at $x = 1$. This, of course, causes *the* limit to not exist. The following theorem states what is fairly intuitive: *the* limit exists precisely when the left and right-hand limits are equal.

Theorem 1.4.1 Limits and One Sided Limits

Let f be a function defined on an open interval I containing c. Then

$$\lim_{x\to c} f(x) = L$$

if, and only if,

$$\lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

The phrase "if, and only if" means the two statements are *equivalent*: they are either both true or both false. If the limit equals L, then the left and right hand limits both equal L. If the limit is not equal to L, then at least one of the left and right-hand limits is not equal to L (it may not even exist).

One thing to consider in Examples 1.4.1 – 1.4.4 is that the value of the function may/may not be equal to the value(s) of its left/right-hand limits, even when these limits agree.

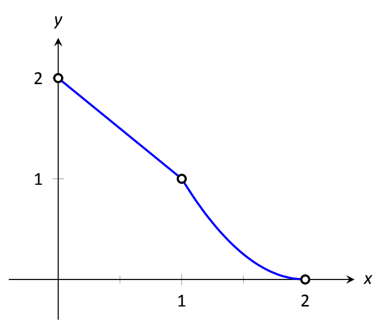


Figure 1.4.2: A graph of f from Example 1.4.2

Example 1.4.2 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} 2-x & 0 < x < 1 \\ (x-2)^2 & 1 < x < 2 \end{cases}$, as shown in Figure 1.4.2. Evaluate the following.

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$
5. $\lim\limits_{x\to 0^+} f(x)$
6. $f(0)$
7. $\lim\limits_{x\to 2^-} f(x)$
8. $f(2)$

Notes:

Solution Again we will evaluate each using both the definition of f and its graph.

1. As x approaches 1 from the left, we see that $f(x)$ approaches 1. Therefore $\lim_{x\to 1^-} f(x) = 1$.
2. As x approaches 1 from the right, we see that again $f(x)$ approaches 1. Therefore $\lim_{x\to 1+} f(x) = 1$.
3. *The* limit of f as x approaches 1 exists and is 1, as f approaches 1 from both the right and left. Therefore $\lim_{x\to 1} f(x) = 1$.
4. $f(1)$ is not defined. Note that 1 is not in the domain of f as defined by the problem, which is indicated on the graph by an open circle when $x = 1$.
5. As x goes to 0 from the right, $f(x)$ approaches 2. So $\lim_{x\to 0^+} f(x) = 2$.
6. $f(0)$ is not defined as 0 is not in the domain of f.
7. As x goes to 2 from the left, $f(x)$ approaches 0. So $\lim_{x\to 2^-} f(x) = 0$.
8. $f(2)$ is not defined as 2 is not in the domain of f.

Example 1.4.3 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} (x-1)^2 & 0 \leq x \leq 2, x \neq 1 \\ 1 & x = 1 \end{cases}$, as shown in Figure 1.4.3. Evaluate the following.

1. $\lim_{x\to 1^-} f(x)$
2. $\lim_{x\to 1^+} f(x)$
3. $\lim_{x\to 1} f(x)$
4. $f(1)$

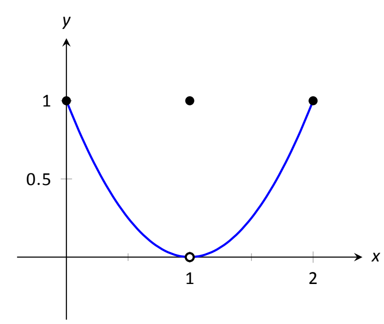


Figure 1.4.3: Graphing f in Example 1.4.3

Solution It is clear by looking at the graph that both the left and right-hand limits of f, as x approaches 1, are 0. Thus it is also clear that *the* limit is 0; i.e., $\lim_{x\to 1} f(x) = 0$. It is also clearly stated that $f(1) = 1$.

Example 1.4.4 Evaluating limits of a piecewise–defined function

Let $f(x) = \begin{cases} x^2 & 0 \leq x \leq 1 \\ 2-x & 1 < x \leq 2 \end{cases}$, as shown in Figure 1.4.4. Evaluate the following.

Notes:

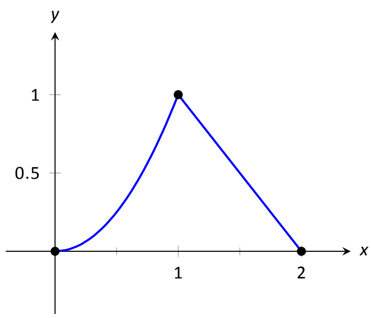


Figure 1.4.4: Graphing f in Example 1.4.4

1. $\lim\limits_{x\to 1^-} f(x)$
2. $\lim\limits_{x\to 1^+} f(x)$
3. $\lim\limits_{x\to 1} f(x)$
4. $f(1)$

Solution It is clear from the definition of the function and its graph that all of the following are equal:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x) = f(1) = 1.$$

In Examples 1.4.1 – 1.4.4 we were asked to find both $\lim\limits_{x\to 1} f(x)$ and $f(1)$. Consider the following table:

	$\lim\limits_{x\to 1} f(x)$	$f(1)$
Example 1.4.1	does not exist	1
Example 1.4.2	1	not defined
Example 1.4.3	0	1
Example 1.4.4	1	1

Only in Example 1.4.4 do both the function and the limit exist and agree. This seems "nice;" in fact, it seems "normal." This is in fact an important situation which we explore in the next section, entitled "Continuity." In short, a *continuous function* is one in which when a function approaches a value as $x \to c$ (i.e., when $\lim\limits_{x\to c} f(x) = L$), it actually *attains* that value at c. Such functions behave nicely as they are very predictable.

Notes:

Exercises 1.4

Terms and Concepts

01 04 ex 01 1. What are the three ways in which a limit may fail to exist?

01 04 ex 02 2. T/F: If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1} f(x) = 5$

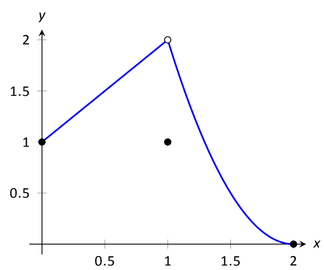
01 04 ex 03 3. T/F: If $\lim\limits_{x\to 1^-} f(x) = 5$, then $\lim\limits_{x\to 1^+} f(x) = 5$

01 04 ex 04 4. T/F: If $\lim\limits_{x\to 1} f(x) = 5$, then $\lim\limits_{x\to 1^-} f(x) = 5$

Problems

01 04 exset 02 **In Exercises 5 – 12, evaluate each expression using the given graph of $f(x)$.**

01 04 ex 05 5.



(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

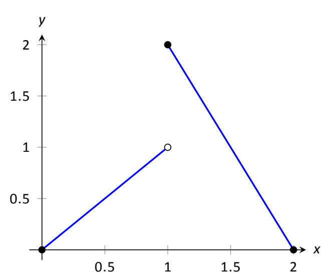
(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 0^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

01 04 ex 06 6.



(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

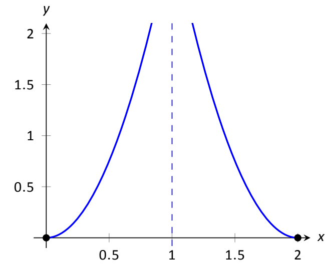
(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 2^+} f(x)$

01 04 ex 07 7.



(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

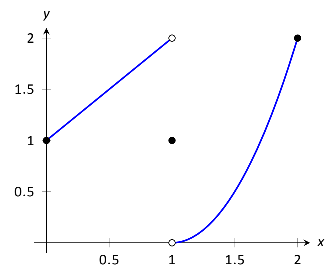
(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

(e) $\lim\limits_{x\to 2^-} f(x)$

(f) $\lim\limits_{x\to 0^+} f(x)$

01 04 ex 08 8.



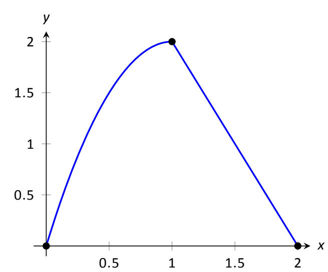
(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

01 04 ex 09 9.



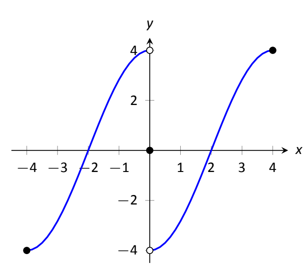
(a) $\lim\limits_{x\to 1^-} f(x)$

(b) $\lim\limits_{x\to 1^+} f(x)$

(c) $\lim\limits_{x\to 1} f(x)$

(d) $f(1)$

01 04 ex 10 10.



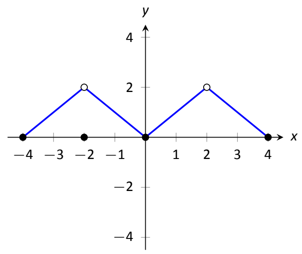
(a) $\lim\limits_{x\to 0^-} f(x)$

(b) $\lim\limits_{x\to 0^+} f(x)$

(c) $\lim\limits_{x\to 0} f(x)$

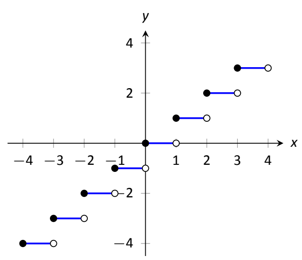
(d) $f(0)$

01 04 ex 11 11.



(a) $\lim\limits_{x\to -2^-} f(x)$
(b) $\lim\limits_{x\to -2^+} f(x)$
(c) $\lim\limits_{x\to -2} f(x)$
(d) $f(-2)$
(e) $\lim\limits_{x\to 2^-} f(x)$
(f) $\lim\limits_{x\to 2^+} f(x)$
(g) $\lim\limits_{x\to 2} f(x)$
(h) $f(2)$

01 04 ex 12 12.



Let $-3 \leq a \leq 3$ be an integer.

(a) $\lim\limits_{x\to a^-} f(x)$
(b) $\lim\limits_{x\to a^+} f(x)$
(c) $\lim\limits_{x\to a} f(x)$
(d) $f(a)$

01 04 exset 03 **In Exercises 13 – 21, evaluate the given limits of the piecewise defined functions f.**

01 04 ex 13 13. $f(x) = \begin{cases} x+1 & x \leq 1 \\ x^2 - 5 & x > 1 \end{cases}$

(a) $\lim\limits_{x\to 1^-} f(x)$
(b) $\lim\limits_{x\to 1^+} f(x)$
(c) $\lim\limits_{x\to 1} f(x)$
(d) $f(1)$

01 04 ex 14 14. $f(x) = \begin{cases} 2x^2 + 5x - 1 & x < 0 \\ \sin x & x \geq 0 \end{cases}$

(a) $\lim\limits_{x\to 0^-} f(x)$
(b) $\lim\limits_{x\to 0^+} f(x)$
(c) $\lim\limits_{x\to 0} f(x)$
(d) $f(0)$

01 04 ex 15 15. $f(x) = \begin{cases} x^2 - 1 & x < -1 \\ x^3 + 1 & -1 \leq x \leq 1 \\ x^2 + 1 & x > 1 \end{cases}$

(a) $\lim\limits_{x\to -1^-} f(x)$
(b) $\lim\limits_{x\to -1^+} f(x)$
(c) $\lim\limits_{x\to -1} f(x)$
(d) $f(-1)$
(e) $\lim\limits_{x\to 1^-} f(x)$
(f) $\lim\limits_{x\to 1^+} f(x)$
(g) $\lim\limits_{x\to 1} f(x)$
(h) $f(1)$

01 04 ex 16 16. $f(x) = \begin{cases} \cos x & x < \pi \\ \sin x & x \geq \pi \end{cases}$

(a) $\lim\limits_{x\to \pi^-} f(x)$
(b) $\lim\limits_{x\to \pi^+} f(x)$
(c) $\lim\limits_{x\to \pi} f(x)$
(d) $f(\pi)$

01 04 ex 17 17. $f(x) = \begin{cases} 1 - \cos^2 x & x < a \\ \sin^2 x & x \geq a \end{cases}$,
where a is a real number.

(a) $\lim\limits_{x\to a^-} f(x)$
(b) $\lim\limits_{x\to a^+} f(x)$
(c) $\lim\limits_{x\to a} f(x)$
(d) $f(a)$

01 04 ex 18 18. $f(x) = \begin{cases} x+1 & x < 1 \\ 1 & x = 1 \\ x-1 & x > 1 \end{cases}$

(a) $\lim\limits_{x\to 1^-} f(x)$
(b) $\lim\limits_{x\to 1^+} f(x)$
(c) $\lim\limits_{x\to 1} f(x)$
(d) $f(1)$

01 04 ex 19 19. $f(x) = \begin{cases} x^2 & x < 2 \\ x+1 & x = 2 \\ -x^2 + 2x + 4 & x > 2 \end{cases}$

(a) $\lim\limits_{x\to 2^-} f(x)$
(b) $\lim\limits_{x\to 2^+} f(x)$
(c) $\lim\limits_{x\to 2} f(x)$
(d) $f(2)$

01 04 ex 20 20. $f(x) = \begin{cases} a(x-b)^2 + c & x < b \\ a(x-b) + c & x \geq b \end{cases}$,
where a, b and c are real numbers.

(a) $\lim\limits_{x\to b^-} f(x)$
(b) $\lim\limits_{x\to b^+} f(x)$
(c) $\lim\limits_{x\to b} f(x)$
(d) $f(b)$

01 04 ex 21 21. $f(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$

(a) $\lim\limits_{x\to 0^-} f(x)$
(b) $\lim\limits_{x\to 0^+} f(x)$
(c) $\lim\limits_{x\to 0} f(x)$
(d) $f(0)$

Review

01 04 ex 22 22. Evaluate the limit: $\lim\limits_{x\to -1} \dfrac{x^2 + 5x + 4}{x^2 - 3x - 4}$.

01 04 ex 23 23. Evaluate the limit: $\lim\limits_{x\to -4} \dfrac{x^2 - 16}{x^2 - 4x - 32}$.

01 04 ex 24 24. Evaluate the limit: $\lim\limits_{x\to -6} \dfrac{x^2 - 15x + 54}{x^2 - 6x}$.

01 04 ex 26 25. Approximate the limit numerically: $\lim\limits_{x\to 0.4} \dfrac{x^2 - 4.4x + 1.6}{x^2 - 0.4x}$.

01 04 ex 27 26. Approximate the limit numerically: $\lim\limits_{x\to 0.2} \dfrac{x^2 + 5.8x - 1.2}{x^2 - 4.2x + 0.8}$.

1.5 Continuity

As we have studied limits, we have gained the intuition that limits measure "where a function is heading." That is, if $\lim_{x \to 1} f(x) = 3$, then as x is close to 1, $f(x)$ is close to 3. We have seen, though, that this is not necessarily a good indicator of what $f(1)$ actually is. This can be problematic; functions can tend to one value but attain another. This section focuses on functions that *do not* exhibit such behavior.

Definition 1.5.1 Continuous Function

Let f be a function defined on an open interval I containing c.

1. f is **continuous at** c if $\lim_{x \to c} f(x) = f(c)$.
2. f is **continuous on** I if f is continuous at c for all values of c in I. If f is continuous on $(-\infty, \infty)$, we say f is **continuous everywhere**.

A useful way to establish whether or not a function f is continuous at c is to verify the following three things:

1. $\lim_{x \to c} f(x)$ exists,
2. $f(c)$ is defined, and
3. $\lim_{x \to c} f(x) = f(c)$.

Example 1.5.1 Finding intervals of continuity

Let f be defined as shown in Figure 1.5.1. Give the interval(s) on which f is continuous.

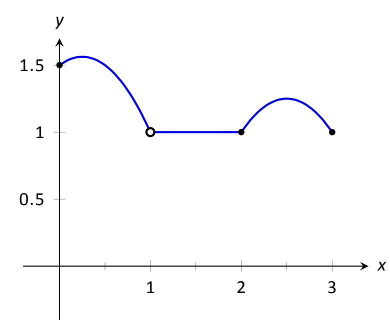


Figure 1.5.1: A graph of f in Example 1.5.1.

Solution We proceed by examining the three criteria for continuity.

1. The limits $\lim_{x \to c} f(x)$ exists for all c between 0 and 3.
2. $f(c)$ is defined for all c between 0 and 3, *except for* $c = 1$. We know immediately that f cannot be continuous at $x = 1$.
3. The limit $\lim_{x \to c} f(x) = f(c)$ for all c between 0 and 3, except, of course, for $c = 1$.

We conclude that f is continuous at every point of $(0, 3)$ except at $x = 1$. Therefore f is continuous on $(0, 1)$ and $(1, 3)$.

Our definition of continuity (currently) only applies to open intervals. After Definition 1.5.2, we'll be able to say that f is continuous on $[0, 1)$ and $(1, 3]$.

Notes:

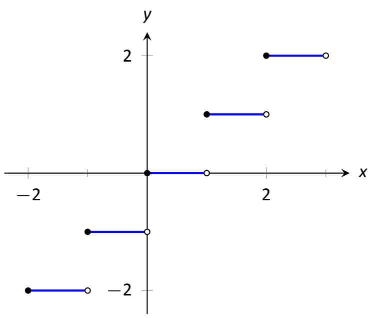


Figure 1.5.2: A graph of the step function in Example 1.5.2.

Example 1.5.2 Finding intervals of continuity

The *floor function*, $f(x) = \lfloor x \rfloor$, returns the largest integer smaller than, or equal to, the input x. (For example, $f(\pi) = \lfloor \pi \rfloor = 3$.) The graph of f in Figure 1.5.2 demonstrates why this is often called a "step function."

Give the intervals on which f is continuous.

SOLUTION We examine the three criteria for continuity.

1. The limits $\lim_{x\to c} f(x)$ do not exist at the jumps from one "step" to the next, which occur at all integer values of c. Therefore the limits exist for all c except when c is an integer.
2. The function is defined for all values of c.
3. The limit $\lim\limits_{x\to c} f(x) = f(c)$ for all values of c where the limit exist, since each step consists of just a line.

We conclude that f is continuous everywhere except at integer values of c. So the intervals on which f is continuous are

$$\ldots, (-2,-1), (-1,0), (0,1), (1,2), \ldots.$$

Our definition of continuity on an interval specifies the interval is an open interval. We can extend the definition of continuity to closed intervals by considering the appropriate one-sided limits at the endpoints.

Definition 1.5.2 Continuity on Closed Intervals

Let f be defined on the closed interval $[a,b]$ for some real numbers $a < b$. f is **continuous on** $[a,b]$ if:

1. f is continuous on (a,b),
2. $\lim\limits_{x\to a^+} f(x) = f(a)$ and
3. $\lim\limits_{x\to b^-} f(x) = f(b)$.

We can make the appropriate adjustments to talk about continuity on half–open intervals such as $[a,b)$ or $(a,b]$ if necessary.

Using this new definition, we can adjust our answer in Example 1.5.1 by stating that f is continuous on $[0,1)$ and $(1,3]$, as mentioned in that example. We

Notes:

can also revisit Example 1.5.2 and state that the floor function is continuous on the following half–open intervals

$$\ldots, [-2,-1), [-1,0), [0,1), [1,2), \ldots.$$

This can tempt us to conclude that f is continuous everywhere; after all, if f is continuous on $[0,1)$ and $[1,2)$, isn't f also continuous on $[0,2)$? Of course, the answer is *no*, and the graph of the floor function immediately confirms this.

Continuous functions are important as they behave in a predictable fashion: functions attain the value they approach. Because continuity is so important, most of the functions you have likely seen in the past are continuous on their domains. This is demonstrated in the following example where we examine the intervals of continuity of a variety of common functions.

Example 1.5.3 Determining intervals on which a function is continuous

For each of the following functions, give the domain of the function and the interval(s) on which it is continuous.

1. $f(x) = 1/x$
2. $f(x) = \sin x$
3. $f(x) = \sqrt{x}$
4. $f(x) = \sqrt{1-x^2}$
5. $f(x) = |x|$

Solution We examine each in turn.

1. The domain of $f(x) = 1/x$ is $(-\infty,0)\cup(0,\infty)$. As it is a rational function, we apply Theorem 1.3.2 to recognize that f is continuous on all of its domain.
2. The domain of $f(x) = \sin x$ is all real numbers, or $(-\infty,\infty)$. Applying Theorem 1.3.3 shows that $\sin x$ is continuous everywhere.
3. The domain of $f(x) = \sqrt{x}$ is $[0,\infty)$. Applying Theorem 1.3.3 shows that $f(x) = \sqrt{x}$ is continuous on its domain of $[0,\infty)$.
4. The domain of $f(x) = \sqrt{1-x^2}$ is $[-1,1]$. Applying Theorems 1.3.1 and 1.3.3 shows that f is continuous on all of its domain, $[-1,1]$.
5. The domain of $f(x) = |x|$ is $(-\infty,\infty)$. We can define the absolute value function as $f(x) = \begin{cases} -x & x<0 \\ x & x\geq 0 \end{cases}$. Each "piece" of this piecewise defined function is continuous on all of its domain, giving that f is continuous on $(-\infty,0)$ and $[0,\infty)$. We cannot assume this implies that f is continuous on $(-\infty,\infty)$; we need to check that $\lim\limits_{x\to 0} f(x) = f(0)$, as $x=0$ is the point where f transitions from one "piece" of its definition to the other. It is easy to verify that this is indeed true, hence we conclude that $f(x) = |x|$ is continuous everywhere.

Notes:

Continuity is inherently tied to the properties of limits. Because of this, the properties of limits found in Theorems 1.3.1 and 1.3.2 apply to continuity as well. Further, now knowing the definition of continuity we can re–read Theorem 1.3.3 as giving a list of functions that are continuous on their domains. The following theorem states how continuous functions can be combined to form other continuous functions, followed by a theorem which formally lists functions that we know are continuous on their domains.

Theorem 1.5.1 Properties of Continuous Functions

Let f and g be continuous functions on an interval I, let c be a real number and let n be a positive integer. The following functions are continuous on I.

1. Sums/Differences: $f \pm g$
2. Constant Multiples: $c \cdot f$
3. Products: $f \cdot g$
4. Quotients: f/g (as long as $g \neq 0$ on I)
5. Powers: f^n
6. Roots: $\sqrt[n]{f}$ (If n is even then require $f(x) \geq 0$ on I.)
7. Compositions: Adjust the definitions of f and g to: Let f be continuous on I, where the range of f on I is J, and let g be continuous on J. Then $g \circ f$, i.e., $g(f(x))$, is continuous on I.

Theorem 1.5.2 Continuous Functions

Let n be a positive integer. The following functions are continuous on their domains.

1. $f(x) = \sin x$
2. $f(x) = \cos x$
3. $f(x) = \tan x$
4. $f(x) = \csc x$
5. $f(x) = \sec x$
6. $f(x) = \cot x$
7. $f(x) = a^x$ ($a > 0$)
8. $f(x) = \ln x$
9. $f(x) = \sqrt[n]{x}$

We apply these theorems in the following Example.

Notes:

Example 1.5.4 Determining intervals on which a function is continuous

State the interval(s) on which each of the following functions is continuous.

1. $f(x) = \sqrt{x-1} + \sqrt{5-x}$
2. $f(x) = x\sin x$
3. $f(x) = \tan x$
4. $f(x) = \sqrt{\ln x}$

Solution We examine each in turn, applying Theorems 1.5.1 and 1.5.2 as appropriate.

1. The square–root terms are continuous on the intervals $[1, \infty)$ and $(-\infty, 5]$, respectively. As f is continuous only where each term is continuous, f is continuous on $[1, 5]$, the intersection of these two intervals. A graph of f is given in Figure 1.5.3.

2. The functions $y = x$ and $y = \sin x$ are each continuous everywhere, hence their product is, too.

3. Theorem 1.5.2 states that $f(x) = \tan x$ is continuous "on its domain." Its domain includes all real numbers except odd multiples of $\pi/2$. Thus the intervals on which $f(x) = \tan x$ is continuous are
$$\ldots\left(-\frac{3\pi}{2}, -\frac{\pi}{2}\right), \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \left(\frac{\pi}{2}, \frac{3\pi}{2}\right), \ldots,.$$

4. The domain of $y = \sqrt{x}$ is $[0, \infty)$. The range of $y = \ln x$ is $(-\infty, \infty)$, but if we restrict its domain to $[1, \infty)$ its range is $[0, \infty)$. So restricting $y = \ln x$ to the domain of $[1, \infty)$ restricts its output is $[0, \infty)$, on which $y = \sqrt{x}$ is defined. Thus the domain of $f(x) = \sqrt{\ln x}$ is $[1, \infty)$.

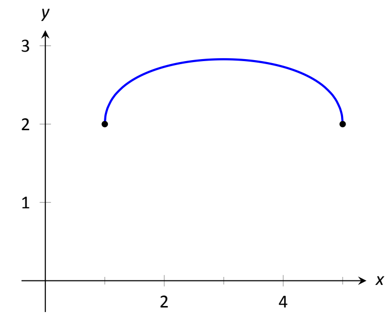


Figure 1.5.3: A graph of f in Example 1.5.4(1).

A common way of thinking of a continuous function is that "its graph can be sketched without lifting your pencil." That is, its graph forms a "continuous" curve, without holes, breaks or jumps. While beyond the scope of this text, this pseudo–definition glosses over some of the finer points of continuity. Very strange functions are continuous that one would be hard pressed to actually sketch by hand.

This intuitive notion of continuity does help us understand another important concept as follows. Suppose f is defined on $[1, 2]$ and $f(1) = -10$ and $f(2) = 5$. If f is continuous on $[1, 2]$ (i.e., its graph can be sketched as a continuous curve from $(1, -10)$ to $(2, 5)$) then we know intuitively that somewhere on $[1, 2]$ f must be equal to -9, and -8, and -7, -6, $\ldots$, 0, $1/2$, etc. In short, f takes on all *intermediate* values between -10 and 5. It may take on more values; f may actually equal 6 at some time, for instance, but we are guaranteed all values between -10 and 5.

Notes:

While this notion seems intuitive, it is not trivial to prove and its importance is profound. Therefore the concept is stated in the form of a theorem.

Theorem 1.5.3 Intermediate Value Theorem

Let f be a continuous function on $[a, b]$ and, without loss of generality, let $f(a) < f(b)$. Then for every value y, where $f(a) < y < f(b)$, there is at least one value c in (a, b) such that $f(c) = y$.

One important application of the Intermediate Value Theorem is root finding. Given a function f, we are often interested in finding values of x where $f(x) = 0$. These roots may be very difficult to find exactly. Good approximations can be found through successive applications of this theorem. Suppose through direct computation we find that $f(a) < 0$ and $f(b) > 0$, where $a < b$. The Intermediate Value Theorem states that there is at least one c in (a, b) such that $f(c) = 0$. The theorem does not give us any clue as to where to find such a value in the interval (a, b), just that at least one such value exists.

There is a technique that produces a good approximation of c. Let d be the midpoint of the interval $[a, b]$ and consider $f(d)$. There are three possibilities:

1. $f(d) = 0$: We got lucky and stumbled on the actual value. We stop as we found a root.

2. $f(d) < 0$: Then we know there is a root of f on the interval $[d, b]$ – we have halved the size of our interval, hence are closer to a good approximation of the root.

3. $f(d) > 0$: Then we know there is a root of f on the interval $[a, d]$ – again,we have halved the size of our interval, hence are closer to a good approximation of the root.

Successively applying this technique is called the **Bisection Method** of root finding. We continue until the interval is sufficiently small. We demonstrate this in the following example.

Example 1.5.5 Using the Bisection Method

Approximate the root of $f(x) = x - \cos x$, accurate to three places after the decimal.

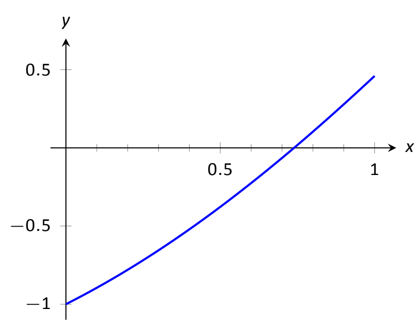


Figure 1.5.4: Graphing a root of $f(x) = x - \cos x$.

Solution Consider the graph of $f(x) = x - \cos x$, shown in Figure 1.5.4. It is clear that the graph crosses the x-axis somewhere near $x = 0.8$. To start the Bisection Method, pick an interval that contains 0.8. We choose $[0.7, 0.9]$. Note that all we care about are signs of $f(x)$, not their actual value, so this is all we display.

Notes:

Iteration 1: $f(0.7) < 0, f(0.9) > 0$, and $f(0.8) > 0$. So replace 0.9 with 0.8 and repeat.

Iteration 2: $f(0.7) < 0, f(0.8) > 0$, and at the midpoint, 0.75, we have $f(0.75) > 0$. So replace 0.8 with 0.75 and repeat. Note that we don't need to continue to check the endpoints, just the midpoint. Thus we put the rest of the iterations in Figure 1.5.5.

Notice that in the 12^{th} iteration we have the endpoints of the interval each starting with 0.739. Thus we have narrowed the zero down to an accuracy of the first three places after the decimal. Using a computer, we have

$$f(0.7390) = -0.00014, \quad f(0.7391) = 0.000024.$$

Either endpoint of the interval gives a good approximation of where f is 0. The Intermediate Value Theorem states that the actual zero is still within this interval. While we do not know its exact value, we know it starts with 0.739.

This type of exercise is rarely done by hand. Rather, it is simple to program a computer to run such an algorithm and stop when the endpoints differ by a preset small amount. One of the authors did write such a program and found the zero of f, accurate to 10 places after the decimal, to be 0.7390851332. While it took a few minutes to write the program, it took less than a thousandth of a second for the program to run the necessary 35 iterations. In less than 8 hundredths of a second, the zero was calculated to 100 decimal places (with less than 200 iterations).

Iteration #	Interval	Midpoint Sign
1	$[0.7, 0.9]$	$f(0.8) > 0$
2	$[0.7, 0.8]$	$f(0.75) > 0$
3	$[0.7, 0.75]$	$f(0.725) < 0$
4	$[0.725, 0.75]$	$f(0.7375) < 0$
5	$[0.7375, 0.75]$	$f(0.7438) > 0$
6	$[0.7375, 0.7438]$	$f(0.7407) > 0$
7	$[0.7375, 0.7407]$	$f(0.7391) > 0$
8	$[0.7375, 0.7391]$	$f(0.7383) < 0$
9	$[0.7383, 0.7391]$	$f(0.7387) < 0$
10	$[0.7387, 0.7391]$	$f(0.7389) < 0$
11	$[0.7389, 0.7391]$	$f(0.7390) < 0$
12	$[0.7390, 0.7391]$	

Figure 1.5.5: Iterations of the Bisection Method of Root Finding

It is a simple matter to extend the Bisection Method to solve problems similar to "Find x, where $f(x) = 0$." For instance, we can find x, where $f(x) = 1$. It actually works very well to define a new function g where $g(x) = f(x) - 1$. Then use the Bisection Method to solve $g(x) = 0$.

Similarly, given two functions f and g, we can use the Bisection Method to solve $f(x) = g(x)$. Once again, create a new function h where $h(x) = f(x) - g(x)$ and solve $h(x) = 0$.

In Section 4.1 another equation solving method will be introduced, called Newton's Method. In many cases, Newton's Method is much faster. It relies on more advanced mathematics, though, so we will wait before introducing it.

This section formally defined what it means to be a continuous function. "Most" functions that we deal with are continuous, so often it feels odd to have to formally define this concept. Regardless, it is important, and forms the basis of the next chapter.

In the next section we examine one more aspect of limits: limits that involve infinity.

Notes:

Exercises 1.5

Terms and Concepts

01 05 ex 06 1. In your own words, describe what it means for a function to be continuous.

01 05 ex 08 2. In your own words, describe what the Intermediate Value Theorem states.

01 05 ex 09 3. What is a "root" of a function?

01 05 ex 10 4. Given functions f and g on an interval I, how can the Bisection Method be used to find a value c where $f(c) = g(c)$?

01 05 ex 01 5. T/F: If f is defined on an open interval containing c, and $\lim\limits_{x\to c} f(x)$ exists, then f is continuous at c.

01 05 ex 02 6. T/F: If f is continuous at c, then $\lim\limits_{x\to c} f(x)$ exists.

01 05 ex 03 7. T/F: If f is continuous at c, then $\lim\limits_{x\to c^+} f(x) = f(c)$.

01 05 ex 04 8. T/F: If f is continuous on $[a, b]$, then $\lim\limits_{x\to a^-} f(x) = f(a)$.

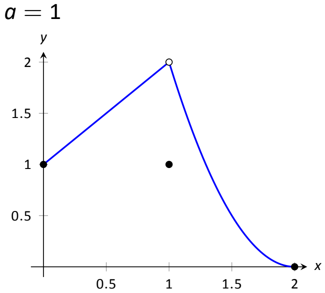
01 05 ex 05 9. T/F: If f is continuous on $[0, 1)$ and $[1, 2)$, then f is continuous on $[0, 2)$.

01 05 ex 07 10. T/F: The sum of continuous functions is also continuous.

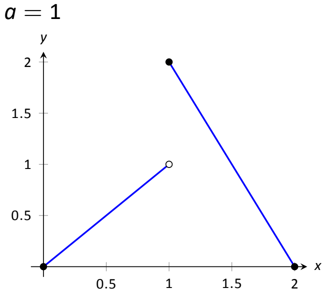
Problems

01 05 exset 01 **In Exercises 11 – 18, a graph of a function f is given along with a value a. Determine if f is continuous at a; if it is not, state why it is not.**

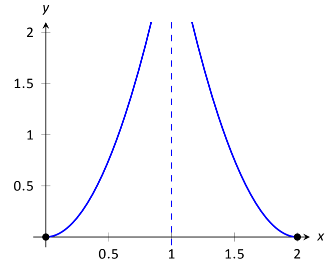
01 05 ex 11 11. $a = 1$



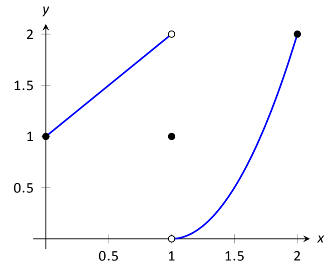
01 05 ex 12 12. $a = 1$



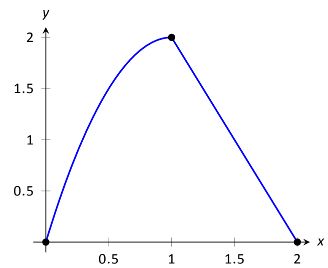
01 05 ex 13 13. $a = 1$



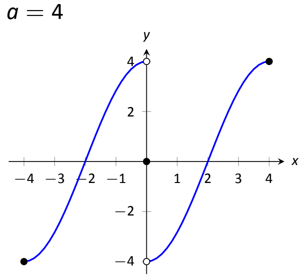
01 05 ex 14 14. $a = 0$



01 05 ex 15 15. $a = 1$



01 05 ex 16 16. $a = 4$

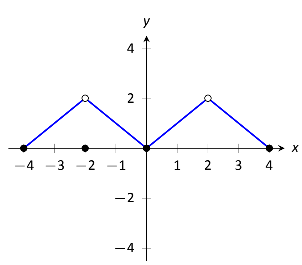


01 05 ex 17 17.

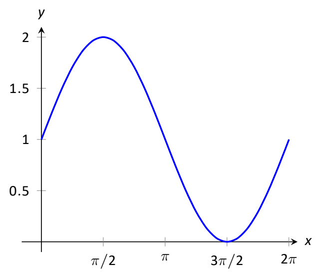
(a) $a = -2$

(b) $a = 0$

(c) $a = 2$



01 05 ex 44 18. $a = 3\pi/2$



01 05 exset 02 **In Exercises 19 – 22, determine if f is continuous at the indicated values. If not, explain why.**

01 05 ex 18 19. $f(x) = \begin{cases} 1 & x = 0 \\ \frac{\sin x}{x} & x > 0 \end{cases}$

(a) $x = 0$

(b) $x = \pi$

01 05 ex 19 20. $f(x) = \begin{cases} x^3 - x & x < 1 \\ x - 2 & x \geq 1 \end{cases}$

(a) $x = 0$

(b) $x = 1$

01 05 ex 20 21. $f(x) = \begin{cases} \frac{x^2+5x+4}{x^2+3x+2} & x \neq -1 \\ 3 & x = -1 \end{cases}$

(a) $x = -1$

(b) $x = 10$

01 05 ex 21 22. $f(x) = \begin{cases} \frac{x^2-64}{x^2-11x+24} & x \neq 8 \\ 5 & x = 8 \end{cases}$

(a) $x = 0$

(b) $x = 8$

01 05 exset 03 **In Exercises 23 – 34, give the intervals on which the given function is continuous.**

01 05 ex 22 23. $f(x) = x^2 - 3x + 9$

01 05 ex 23 24. $g(x) = \sqrt{x^2 - 4}$

01 05 ex 45 25. $g(x) = \sqrt{4 - x^2}$

01 05 ex 24 26. $h(k) = \sqrt{1-k} + \sqrt{k+1}$

01 05 ex 25 27. $f(t) = \sqrt{5t^2 - 30}$

01 05 ex 26 28. $g(t) = \dfrac{1}{\sqrt{1-t^2}}$

01 05 ex 27 29. $g(x) = \dfrac{1}{1+x^2}$

01 05 ex 28 30. $f(x) = e^x$

01 05 ex 29 31. $g(s) = \ln s$

01 05 ex 30 32. $h(t) = \cos t$

01 05 ex 31 33. $f(k) = \sqrt{1 - e^k}$

01 05 ex 32 34. $f(x) = \sin(e^x + x^2)$

01 05 exset 05 **Exercises 35 – 38 test your understanding of the Intermediate Value Theorem.**

01 05 ex 33 35. Let f be continuous on $[1, 5]$ where $f(1) = -2$ and $f(5) = -10$. Does a value $1 < c < 5$ exist such that $f(c) = -9$? Why/why not?

01 05 ex 34 36. Let g be continuous on $[-3, 7]$ where $g(0) = 0$ and $g(2) = 25$. Does a value $-3 < c < 7$ exist such that $g(c) = 15$? Why/why not?

01 05 ex 35 37. Let f be continuous on $[-1, 1]$ where $f(-1) = -10$ and $f(1) = 10$. Does a value $-1 < c < 1$ exist such that $f(c) = 11$? Why/why not?

01 05 ex 36 38. Let h be a function on $[-1, 1]$ where $h(-1) = -10$ and $h(1) = 10$. Does a value $-1 < c < 1$ exist such that $h(c) = 0$? Why/why not?

01 05 exset 04 **In Exercises 39 – 42, use the Bisection Method to approximate, accurate to two decimal places, the value of the root of the given function in the given interval.**

01 05 ex 37 39. $f(x) = x^2 + 2x - 4$ on $[1, 1.5]$.

01 05 ex 38 40. $f(x) = \sin x - 1/2$ on $[0.5, 0.55]$

01 05 ex 39 41. $f(x) = e^x - 2$ on $[0.65, 0.7]$.

01 05 ex 40 42. $f(x) = \cos x - \sin x$ on $[0.7, 0.8]$.

Review

01 05 ex 41 43. Let $f(x) = \begin{cases} x^2 - 5 & x < 5 \\ 5x & x \geq 5 \end{cases}$.

(a) $\lim\limits_{x\to 5^-} f(x)$

(b) $\lim\limits_{x\to 5^+} f(x)$

(c) $\lim\limits_{x\to 5} f(x)$

(d) $f(5)$

01 05 ex 42 44. Numerically approximate the following limits:

(a) $\lim\limits_{x\to -4/5^+} \dfrac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$

(b) $\lim\limits_{x\to -4/5^-} \dfrac{x^2 - 8.2x - 7.2}{x^2 + 5.8x + 4}$

01 05 ex 43 45. Give an example of function $f(x)$ for which $\lim\limits_{x\to 0} f(x)$ does not exist.

1.6 Limits Involving Infinity

In Definition 1.2.1 we stated that in the equation $\lim_{x\to c} f(x) = L$, both c and L were numbers. In this section we relax that definition a bit by considering situations when it makes sense to let c and/or L be "infinity."

As a motivating example, consider $f(x) = 1/x^2$, as shown in Figure 1.6.1. Note how, as x approaches 0, $f(x)$ grows very, very large – in fact, it grows without bound. It seems appropriate, and descriptive, to state that

$$\lim_{x\to 0} \frac{1}{x^2} = \infty.$$

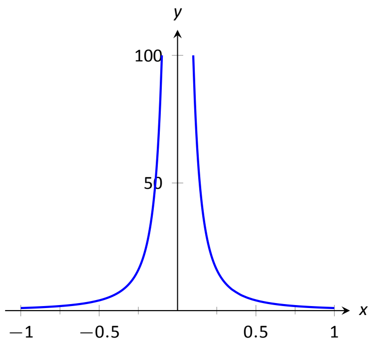


Figure 1.6.1: Graphing $f(x) = 1/x^2$ for values of x near 0.

Also note that as x gets very large, $f(x)$ gets very, very small. We could represent this concept with notation such as

$$\lim_{x\to \infty} \frac{1}{x^2} = 0.$$

We explore both types of use of ∞ in turn.

Definition 1.6.1 Limit of Infinity, ∞

Let I be an open interval containing c, and let f be a function defined on I, except possibly at c.

- The **limit of $f(x)$, as x approaches c, is infinity**, denoted by

$$\lim_{x\to c} f(x) = \infty,$$

  means that given any $M > 0$, there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $f(x) > M$.

- The **limit of $f(x)$, as x approaches c, is negative infinity**, denoted by

$$\lim_{x\to c} f(x) = -\infty,$$

  means that given any $M < 0$, there exists $\delta > 0$ such that for all x in I, where $x \neq c$, if $|x - c| < \delta$, then $f(x) < M$.

The first definition is similar to the ε–δ definition from Section 1.2. In that definition, given any (small) value ε, if we let x get close enough to c (within δ units of c) then $f(x)$ is guaranteed to be within ε of L. Here, given any (large) value M, if we let x get close enough to c (within δ units of c), then $f(x)$ will be

Notes:

at least as large as M. In other words, if we get close enough to c, then we can make $f(x)$ as large as we want. We define limits equal to $-\infty$ in a similar way.

It is important to note that by saying $\lim_{x\to c} f(x) = \infty$ we are implicitly stating that *the* limit of $f(x)$, as x approaches c, *does not exist.* A limit only exists when $f(x)$ approaches an actual numeric value. We use the concept of limits that approach infinity because it is helpful and descriptive.

We define one-sided limits that approach infinity in a similar way.

Definition 1.6.2 One-Sided Limit of Infinity

- Let f be a function defined on (a, c) for some $a < c$.

  The **limit of $f(x)$, as x approaches c from the left, is infinity**, or, **the left-hand limit of f at c is infinity**, denoted by

  $$\lim_{x\to c^-} f(x) = \infty,$$

  means given any $M > 0$, there exists $\delta > 0$ such that for all $a < x < c$, if $|x - c| < \delta$, then $f(x) > M$.

- Let f be a function defined on (c, b) for some $b > c$.

  The **limit of $f(x)$, as x approaches c from the right, is infinity**, or, **the right-hand limit of f at c is infinity**, denoted by

  $$\lim_{x\to c^+} f(x) = \infty,$$

  means given any $M > 0$, there exists $\delta > 0$ such that for all $c < x < b$, if $|x - c| < \delta$, then $f(x) > M$.

- The term **left- (or, right-) hand limit of f at c is negative infinity** is defined in a manner similar to Definition 1.6.1.

Example 1.6.1 Evaluating limits involving infinity

Find $\displaystyle\lim_{x\to 1} \frac{1}{(x-1)^2}$ as shown in Figure 1.6.2.

Solution In Example 1.1.4 of Section 1.1, by inspecting values of x close to 1 we concluded that this limit does not exist. That is, it cannot equal any real number. But the limit could be infinite. And in fact, we see that the function does appear to be growing larger and larger, as $f(.99) = 10^4$, $f(.999) = 10^6$, $f(.9999) = 10^8$. A similar thing happens on the other side of 1. In general, let a "large" value M be given. Let $\delta = 1/\sqrt{M}$. If x is within δ of 1, i.e., if

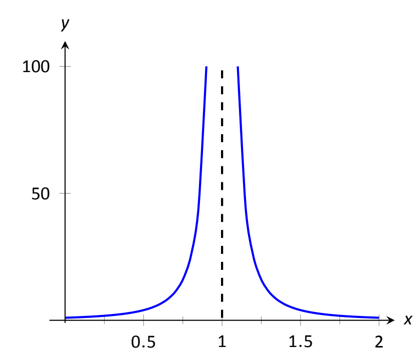


Figure 1.6.2: Observing infinite limit as $x \to 1$ in Example 1.6.1.

Notes:

$|x-1| < 1/\sqrt{M}$, then:

$$|x-1| < \frac{1}{\sqrt{M}}$$
$$(x-1)^2 < \frac{1}{M}$$
$$\frac{1}{(x-1)^2} > M,$$

which is what we wanted to show. So we may say $\lim\limits_{x\to 1} 1/(x-1)^2 = \infty$.

Example 1.6.2 Evaluating limits involving infinity

Find $\lim\limits_{x\to 0} \frac{1}{x}$, as shown in Figure 1.6.3.

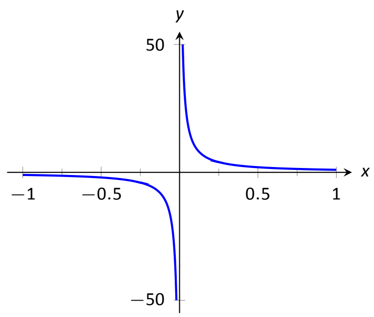


Figure 1.6.3: Evaluating $\lim\limits_{x\to 0} \frac{1}{x}$.

Solution It is easy to see that the function grows without bound near 0, but it does so in different ways on different sides of 0. Since its behavior is not consistent, we cannot say that $\lim\limits_{x\to 0} \frac{1}{x} = \infty$. However, we can make a statement about one–sided limits. We can state that $\lim\limits_{x\to 0^+} \frac{1}{x} = \infty$ and $\lim\limits_{x\to 0^-} \frac{1}{x} = -\infty$.

Vertical asymptotes

If the limit of $f(x)$ as x approaches c from either the left or right (or both) is ∞ or $-\infty$, we say the function has a **vertical asymptote** at c.

Example 1.6.3 Finding vertical asymptotes

Find the vertical asymptotes of $f(x) = \frac{3x}{x^2-4}$.

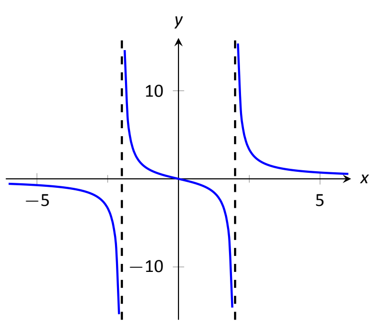


Figure 1.6.4: Graphing $f(x) = \frac{3x}{x^2-4}$.

Solution Vertical asymptotes occur where the function grows without bound; this can occur at values of c where the denominator is 0. When x is near c, the denominator is small, which in turn can make the function take on large values. In the case of the given function, the denominator is 0 at $x = \pm 2$. Substituting in values of x close to 2 and -2 seems to indicate that the function tends toward ∞ or $-\infty$ at those points. We can graphically confirm this by looking at Figure 1.6.4. Thus the vertical asymptotes are at $x = \pm 2$.

When a rational function has a vertical asymptote at $x = c$, we can conclude that the denominator is 0 at $x = c$. However, just because the denominator is 0 at a certain point does not mean there is a vertical asymptote there. For

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instance, $f(x) = (x^2 - 1)/(x - 1)$ does not have a vertical asymptote at $x = 1$, as shown in Figure 1.6.5. While the denominator does get small near $x = 1$, the numerator gets small too, matching the denominator step for step. In fact, factoring the numerator, we get

$$f(x) = \frac{(x-1)(x+1)}{x-1}.$$

Canceling the common term, we get that $f(x) = x + 1$ for $x \neq 1$. So there is clearly no asymptote, rather a hole exists in the graph at $x = 1$.

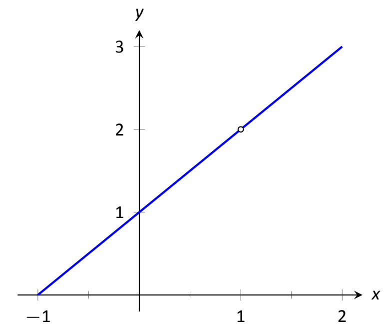


Figure 1.6.5: Graphically showing that $f(x) = \dfrac{x^2 - 1}{x - 1}$ does not have an asymptote at $x = 1$.

The above example may seem a little contrived. Another example demonstrating this important concept is $f(x) = (\sin x)/x$. We have considered this function several times in the previous sections. We found that $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$; i.e., there is no vertical asymptote. No simple algebraic cancellation makes this fact obvious; we used the Squeeze Theorem in Section 1.3 to prove this.

If the denominator is 0 at a certain point but the numerator is not, then there will usually be a vertical asymptote at that point. On the other hand, if the numerator and denominator are both zero at that point, then there may or may not be a vertical asymptote at that point. This case where the numerator and denominator are both zero returns us to an important topic.

Indeterminate Forms

We have seen how the limits

$$\lim_{x\to 0} \frac{\sin x}{x} \quad \text{and} \quad \lim_{x\to 1} \frac{x^2 - 1}{x - 1}$$

each return the indeterminate form "0/0" when we blindly plug in $x = 0$ and $x = 1$, respectively. However, $0/0$ is not a valid arithmetical expression. It gives no indication that the respective limits are 1 and 2.

With a little cleverness, one can come up with $0/0$ expressions which have a limit of ∞, 0, or any other real number. That is why this expression is called *indeterminate*.

A key concept to understand is that such limits do not really return $0/0$. Rather, keep in mind that we are taking *limits*. What is really happening is that the numerator is shrinking to 0 while the denominator is also shrinking to 0. The respective rates at which they do this are very important and determine the actual value of the limit.

An indeterminate form indicates that one needs to do more work in order to compute the limit. That work may be algebraic (such as factoring and canceling)

Notes:

or it may require a tool such as the Squeeze Theorem. In a later section we will learn a technique called l'Hôspital's Rule that provides another way to handle indeterminate forms.

Some other common indeterminate forms are $\infty - \infty$, $\infty \cdot 0$, ∞/∞, 0^0, ∞^0 and 1^∞. Again, keep in mind that these are the "blind" results of evaluating a limit, and each, in and of itself, has no meaning. The expression $\infty - \infty$ does not really mean "subtract infinity from infinity." Rather, it means "One quantity is subtracted from the other, but both are growing without bound." What is the result? It is possible to get every value between $-\infty$ and ∞

Note that $1/0$ and $\infty/0$ are not indeterminate forms, though they are not exactly valid mathematical expressions, either. In each, the function is growing without bound, indicating that the limit will be ∞, $-\infty$, or simply not exist if the left- and right-hand limits do not match.

Limits at Infinity and Horizontal Asymptotes

At the beginning of this section we briefly considered what happens to $f(x) = 1/x^2$ as x grew very large. Graphically, it concerns the behavior of the function to the "far right" of the graph. We make this notion more explicit in the following definition.

Definition 1.6.3 Limits at Infinity and Horizontal Asymptote

1. We say $\lim\limits_{x\to\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M > 0$ such that if $x \geq M$, then $|f(x) - L| < \varepsilon$.

2. We say $\lim\limits_{x\to-\infty} f(x) = L$ if for every $\varepsilon > 0$ there exists $M < 0$ such that if $x \leq M$, then $|f(x) - L| < \varepsilon$.

3. If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, we say that $y = L$ is a **horizontal asymptote** of f.

We can also define limits such as $\lim\limits_{x\to\infty} f(x) = \infty$ by combining this definition with Definition 1.6.1.

Notes:

Example 1.6.4 Approximating horizontal asymptotes

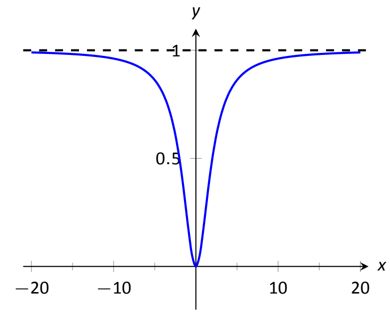
Approximate the horizontal asymptote(s) of $f(x) = \dfrac{x^2}{x^2+4}$.

Solution We will approximate the horizontal asymptotes by approximating the limits

$$\lim_{x\to-\infty}\frac{x^2}{x^2+4} \quad\text{and}\quad \lim_{x\to\infty}\frac{x^2}{x^2+4}.$$

Figure 1.6.6(a) shows a sketch of f, and part (b) gives values of $f(x)$ for large magnitude values of x. It seems reasonable to conclude from both of these sources that f has a horizontal asymptote at $y = 1$.

Later, we will show how to determine this analytically.



(a)

x	$f(x)$
10	0.9615
100	0.9996
10000	0.999996
-10	0.9615
-100	0.9996
-10000	0.999996

(b)

Figure 1.6.6: Using a graph and a table to approximate a horizontal asymptote in Example 1.6.4.

Horizontal asymptotes can take on a variety of forms. Figure 1.6.7(a) shows that $f(x) = x/(x^2+1)$ has a horizontal asymptote of $y = 0$, where 0 is approached from both above and below.

Figure 1.6.7(b) shows that $f(x) = x/\sqrt{x^2+1}$ has two horizontal asymptotes; one at $y = 1$ and the other at $y = -1$.

Figure 1.6.7(c) shows that $f(x) = (\sin x)/x$ has even more interesting behavior than at just $x = 0$; as x approaches $\pm\infty$, $f(x)$ approaches 0, but oscillates as it does this.

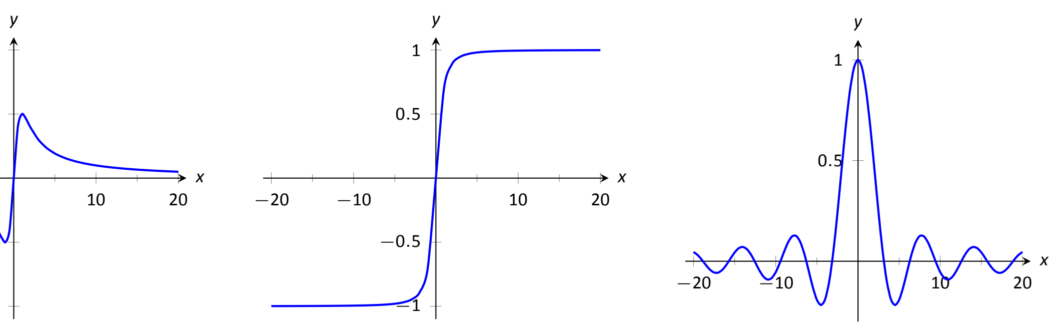


Figure 1.6.7: Considering different types of horizontal asymptotes.

We can analytically evaluate limits at infinity for rational functions once we understand $\lim\limits_{x\to\infty} 1/x$. As x gets larger and larger, the $1/x$ gets smaller and smaller, approaching 0. We can, in fact, make $1/x$ as small as we want by choosing a large

Notes:

enough value of x. Given ε, we can make $1/x < \varepsilon$ by choosing $x > 1/\varepsilon$. Thus we have $\lim_{x\to\infty} 1/x = 0$.

It is now not much of a jump to conclude the following:

$$\lim_{x\to\infty} \frac{1}{x^n} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Now suppose we need to compute the following limit:

$$\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9}.$$

A good way of approaching this is to divide through the numerator and denominator by x^3 (hence dividing by 1), which is the largest power of x to appear in the function. Doing this, we get

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} &= \lim_{x\to\infty} \frac{1/x^3}{1/x^3} \cdot \frac{x^3 + 2x + 1}{4x^3 - 2x^2 + 9} \\
&= \lim_{x\to\infty} \frac{x^3/x^3 + 2x/x^3 + 1/x^3}{4x^3/x^3 - 2x^2/x^3 + 9/x^3} \\
&= \lim_{x\to\infty} \frac{1 + 2/x^2 + 1/x^3}{4 - 2/x + 9/x^3}.
\end{aligned}$$

Then using the rules for limits (which also hold for limits at infinity), as well as the fact about limits of $1/x^n$, we see that the limit becomes

$$\frac{1+0+0}{4-0+0} = \frac{1}{4}.$$

This procedure works for any rational function. In fact, it gives us the following theorem.

Notes:

Theorem 1.6.1 Limits of Rational Functions at Infinity

Let $f(x)$ be a rational function of the following form:

$$f(x) = \frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0},$$

where any of the coefficients may be 0 except for a_n and b_m.

1. If $n = m$, then $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \frac{a_n}{b_m}$.
2. If $n < m$, then $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$.
3. If $n > m$, then $\displaystyle\lim_{x\to\infty} f(x)$ and $\displaystyle\lim_{x\to-\infty} f(x)$ are both infinite.

We can see why this is true. If the highest power of x is the same in both the numerator and denominator (i.e. $n = m$), we will be in a situation like the example above, where we will divide by x^n and in the limit all the terms will approach 0 except for $a_n x^n/x^n$ and $b_m x^m/x^n$. Since $n = m$, this will leave us with the limit a_n/b_m. If $n < m$, then after dividing through by x^m, all the terms in the numerator will approach 0 in the limit, leaving us with $0/b_m$ or 0. If $n > m$, and we try dividing through by x^n, we end up with all the terms in the denominator tending toward 0, while the x^n term in the numerator does not approach 0. This is indicative of some sort of infinite limit.

Intuitively, as x gets very large, all the terms in the numerator are small in comparison to $a_n x^n$, and likewise all the terms in the denominator are small compared to $b_n x^m$. If $n = m$, looking only at these two important terms, we have $(a_n x^n)/(b_n x^m)$. This reduces to a_n/b_m. If $n < m$, the function behaves like $a_n/(b_m x^{m-n})$, which tends toward 0. If $n > m$, the function behaves like $a_n x^{n-m}/b_m$, which will tend to either ∞ or $-\infty$ depending on the values of n, m, a_n, b_m and whether you are looking for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$.

With care, we can quickly evaluate limits at infinity for a large number of functions by considering the largest powers of x. For instance, consider again $\displaystyle\lim_{x\to\pm\infty} \frac{x}{\sqrt{x^2+1}}$, graphed in Figure 1.6.7(b). When x is very large, $x^2 + 1 \approx x^2$. Thus

$$\sqrt{x^2+1} \approx \sqrt{x^2} = |x|, \quad \text{and} \quad \frac{x}{\sqrt{x^2+1}} \approx \frac{x}{|x|}.$$

This expression is 1 when x is positive and -1 when x is negative. Hence we get asymptotes of $y = 1$ and $y = -1$, respectively.

Notes:

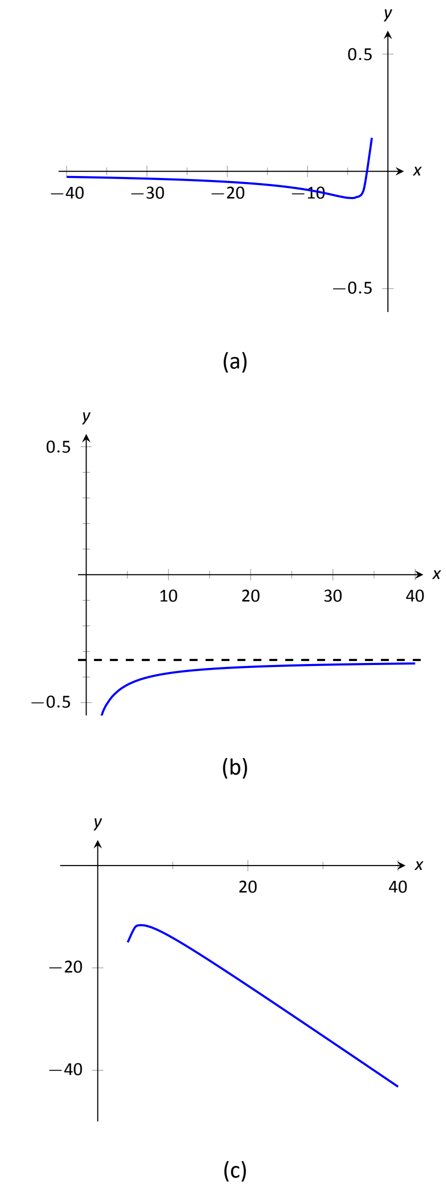


Figure 1.6.8: Visualizing the functions in Example 1.6.6.

Example 1.6.5 Finding a limit of a rational function

Confirm analytically that $y = 1$ is the horizontal asymptote of $f(x) = \dfrac{x^2}{x^2+4}$, as approximated in Example 1.6.4.

SOLUTION Before using Theorem 1.6.1, let's use the technique of evaluating limits at infinity of rational functions that led to that theorem. The largest power of x in f is 2, so divide the numerator and denominator of f by x^2, then take limits.

$$\begin{aligned}\lim_{x\to\infty}\frac{x^2}{x^2+4} &= \lim_{x\to\infty}\frac{x^2/x^2}{x^2/x^2+4/x^2}\\ &= \lim_{x\to\infty}\frac{1}{1+4/x^2}\\ &= \frac{1}{1+0}\\ &= 1.\end{aligned}$$

We can also use Theorem 1.6.1 directly; in this case $n = m$ so the limit is the ratio of the leading coefficients of the numerator and denominator, i.e., 1/1 = 1.

Example 1.6.6 Finding limits of rational functions

Use Theorem 1.6.1 to evaluate each of the following limits.

1. $\displaystyle\lim_{x\to-\infty}\frac{x^2+2x-1}{x^3+1}$
2. $\displaystyle\lim_{x\to\infty}\frac{x^2+2x-1}{1-x-3x^2}$
3. $\displaystyle\lim_{x\to\infty}\frac{x^2-1}{3-x}$

SOLUTION

1. The highest power of x is in the denominator. Therefore, the limit is 0; see Figure 1.6.8(a).
2. The highest power of x is x^2, which occurs in both the numerator and denominator. The limit is therefore the ratio of the coefficients of x^2, which is $-1/3$. See Figure 1.6.8(b).
3. The highest power of x is in the numerator so the limit will be ∞ or $-\infty$. To see which, consider only the dominant terms from the numerator and denominator, which are x^2 and $-x$. The expression in the limit will behave like $x^2/(-x) = -x$ for large values of x. Therefore, the limit is $-\infty$. See Figure 1.6.8(c).

Notes:

Notes:

Chapter Summary

In this chapter we:

- defined the limit,
- found accessible ways to approximate their values numerically and graphically,
- developed a not–so–easy method of proving the value of a limit (ε-δ proofs),
- explored when limits do not exist,
- defined continuity and explored properties of continuous functions, and
- considered limits that involved infinity.

Why? Mathematics is famous for building on itself and calculus proves to be no exception. In the next chapter we will be interested in "dividing by 0." That is, we will want to divide a quantity by a smaller and smaller number and see what value the quotient approaches. In other words, we will want to find a limit. These limits will enable us to, among other things, determine *exactly* how fast something is moving when we are only given position information.

Later, we will want to add up an infinite list of numbers. We will do so by first adding up a finite list of numbers, then take a limit as the number of things we are adding approaches infinity. Surprisingly, this sum often is finite; that is, we can add up an infinite list of numbers and get, for instance, 42.

These are just two quick examples of why we are interested in limits. Many students dislike this topic when they are first introduced to it, but over time an appreciation is often formed based on the scope of its applicability.

Notes:

Exercises 1.6

Terms and Concepts

01 06 ex 01 1. T/F: If $\lim_{x\to 5} f(x) = \infty$, then we are implicitly stating that the limit exists.

01 06 ex 02 2. T/F: If $\lim_{x\to \infty} f(x) = 5$, then we are implicitly stating that the limit exists.

01 06 ex 03 3. T/F: If $\lim_{x\to 1^-} f(x) = -\infty$, then $\lim_{x\to 1^+} f(x) = \infty$

01 06 ex 04 4. T/F: If $\lim_{x\to 5} f(x) = \infty$, then f has a vertical asymptote at $x = 5$.

01 06 ex 05 5. T/F: $\infty/0$ is not an indeterminate form.

01 06 ex 06 6. List 5 indeterminate forms.

01 06 ex 07 7. Construct a function with a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

01 06 ex 08 8. Let $\lim_{x\to 7} f(x) = \infty$. Explain how we know that f is/is not continuous at $x = 7$.

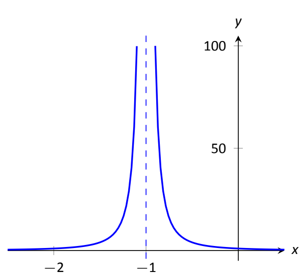
Problems

01 06 exset 01 **In Exercises 9 – 14, evaluate the given limits using the graph of the function.**

01 06 ex 09 9. $f(x) = \dfrac{1}{(x+1)^2}$

(a) $\lim_{x\to -1^-} f(x)$

(b) $\lim_{x\to -1^+} f(x)$



01 06 ex 10 10. $f(x) = \dfrac{1}{(x-3)(x-5)^2}$.

(a) $\lim_{x\to 3^-} f(x)$

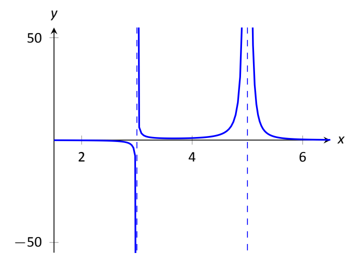
(b) $\lim_{x\to 3^+} f(x)$

(c) $\lim_{x\to 3} f(x)$

(d) $\lim_{x\to 5^-} f(x)$

(e) $\lim_{x\to 5^+} f(x)$

(f) $\lim_{x\to 5} f(x)$



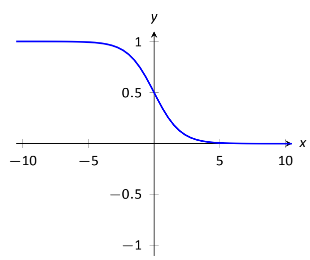
01 06 ex 11 11. $f(x) = \dfrac{1}{e^x + 1}$

(a) $\lim_{x\to -\infty} f(x)$

(b) $\lim_{x\to \infty} f(x)$

(c) $\lim_{x\to 0^-} f(x)$

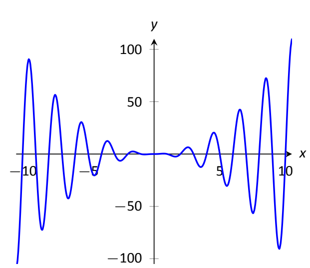
(d) $\lim_{x\to 0^+} f(x)$



01 06 ex 12 12. $f(x) = x^2 \sin(\pi x)$

(a) $\lim_{x\to -\infty} f(x)$

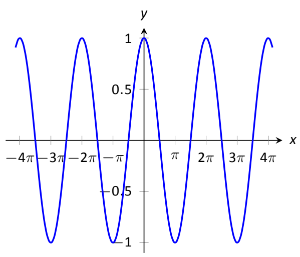
(b) $\lim_{x\to \infty} f(x)$



01 06 ex 13 13. $f(x) = \cos(x)$

(a) $\lim\limits_{x\to-\infty} f(x)$

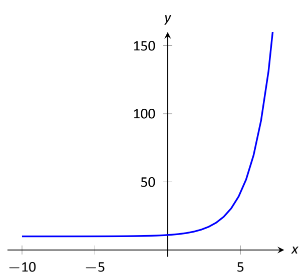
(b) $\lim\limits_{x\to\infty} f(x)$



01 06 ex 40 14. $f(x) = 2^x + 10$

(a) $\lim\limits_{x\to-\infty} f(x)$

(b) $\lim\limits_{x\to\infty} f(x)$



01 06 exset 02 **In Exercises 15 – 18, numerically approximate the following limits:**

(a) $\lim\limits_{x\to3^-} f(x)$

(b) $\lim\limits_{x\to3^+} f(x)$

(c) $\lim\limits_{x\to3} f(x)$

01 06 ex 14 15. $f(x) = \dfrac{x^2-1}{x^2-x-6}$

01 06 ex 15 16. $f(x) = \dfrac{x^2+5x-36}{x^3-5x^2+3x+9}$

01 06 ex 16 17. $f(x) = \dfrac{x^2-11x+30}{x^3-4x^2-3x+18}$

01 06 ex 17 18. $f(x) = \dfrac{x^2-9x+18}{x^2-x-6}$

01 06 exset 03 **In Exercises 19 – 24, identify the horizontal and vertical asymptotes, if any, of the given function.**

01 06 ex 18 19. $f(x) = \dfrac{2x^2-2x-4}{x^2+x-20}$

01 06 ex 19 20. $f(x) = \dfrac{-3x^2-9x-6}{5x^2-10x-15}$

01 06 ex 20 21. $f(x) = \dfrac{x^2+x-12}{7x^3-14x^2-21x}$

01 06 ex 21 22. $f(x) = \dfrac{x^2-9}{9x-9}$

01 06 ex 22 23. $f(x) = \dfrac{x^2-9}{9x+27}$

01 06 ex 41 24. $f(x) = \dfrac{x^2-1}{-x^2-1}$

01 06 exset 04 **In Exercises 25 – 28, evaluate the given limit.**

01 06 ex 23 25. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{x-5}$

01 06 ex 24 26. $\lim\limits_{x\to\infty} \dfrac{x^3+2x^2+1}{5-x}$

01 06 ex 25 27. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{x^2-5}$

01 06 ex 26 28. $\lim\limits_{x\to-\infty} \dfrac{x^3+2x^2+1}{5-x^2}$

Review

01 06 ex 27 29. Use an $\varepsilon-\delta$ proof to show that $\lim\limits_{x\to1} 5x-2 = 3$.

01 06 ex 28 30. Let $\lim\limits_{x\to2} f(x) = 3$ and $\lim\limits_{x\to2} g(x) = -1$. Evaluate the following limits.

(a) $\lim\limits_{x\to2} (f+g)(x)$

(b) $\lim\limits_{x\to2} (fg)(x)$

(c) $\lim\limits_{x\to2} (f/g)(x)$

(d) $\lim\limits_{x\to2} f(x)^{g(x)}$

01 06 ex 29 31. Let $f(x) = \begin{cases} x^2-1 & x<3 \\ x+5 & x\geq 3 \end{cases}$.
Is f continuous everywhere?

01 06 ex 30 32. Evaluate the limit: $\lim\limits_{x\to e} \ln x$.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

01 01 ex 19 1. Answers will vary.

01 01 ex 20 2. An indeterminate form.

01 01 ex 21 3. F

01 01 ex 22 4. The function may approach different values from the left and right, the function may grow without bound, or the function might oscillate.

01 01 ex 23 5. Answers will vary.

01 01 ex 24 6. 1

01 01 ex 01 7. -1

01 01 ex 02 8. -5

01 01 ex 03 9. Limit does not exist

01 01 ex 04 10. 2

01 01 ex 05 11. 1.5

01 01 ex 06 12. Limit does not exist.

01 01 ex 07 13. Limit does not exist.

01 01 ex 08 14. 7

01 01 ex 09 15. 1

01 01 ex 10 16. Limit does not exist.

01 01 ex 11 17.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-7
-0.01	-7
0.01	-7
0.1	-7

The limit seems to be exactly 7.

01 01 ex 12 18.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	9
-0.01	9
0.01	9
0.1	9

The limit seems to be exactly 9.

01 01 ex 13 19.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	4.9
-0.01	4.99
0.01	5.01
0.1	5.1

The limit is approx. 5.

01 01 ex 14 20.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.114943
-0.01	-0.111483
0.01	-0.110742
0.1	-0.107527

The limit is approx. -0.11.

01 01 ex 15 21.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	29.4
-0.01	29.04
0.01	28.96
0.1	28.6

The limit is approx. 29.

01 01 ex 16 22.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	0.202027
-0.01	0.2002
0.01	0.1998
0.1	0.198026

The limit is approx. 0.2.

01 01 ex 17 23.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.998334
-0.01	-0.999983
0.01	-0.999983
0.1	-0.998334

The limit is approx. -1.

01 01 ex 18 24.

h	$\frac{f(a+h)-f(a)}{h}$
-0.1	-0.0499583
-0.01	-0.00499996
0.01	0.00499996
0.1	0.0499583

The limit is approx. 0.005.

Section 1.2

01 02 ex 01 1. ε should be given first, and the restriction $|x-a|<\delta$ implies $|f(x)-K|<\varepsilon$, not the other way around.

01 02 ex 02 2. The y–tolerance.

01 02 ex 09 3. T

01 02 ex 10 4. T

01 02 ex 12 5. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-4|<\delta$, $|f(x)-13|<\varepsilon$.
Consider $|f(x)-13|<\varepsilon$:

$$\begin{gathered}|f(x)-13|<\varepsilon\\ |(2x+5)-13|<\varepsilon\\ |2x-8|<\varepsilon\\ 2|x-4|<\varepsilon\\ -\varepsilon/2<x-4<\varepsilon/2.\end{gathered}$$

This implies we can let $\delta=\varepsilon/2$. Then:

$$\begin{gathered}|x-4|<\delta\\ -\delta<x-4<\delta\\ -\varepsilon/2<x-4<\varepsilon/2\\ -\varepsilon<2x-8<\varepsilon\\ -\varepsilon<(2x+5)-13<\varepsilon\\ |(2x+5)-13|<\varepsilon,\end{gathered}$$

which is what we wanted to prove.

01 02 ex 04 6. Let $\varepsilon>0$ be given. We wish to find $\delta>0$ such that when $|x-5|<\delta$, $|f(x)-(-2)|<\varepsilon$.
Consider $|f(x)-(-2)|<\varepsilon$:

$$\begin{gathered}|f(x)+2|<\varepsilon\\ |(3-x)+2|<\varepsilon\\ |5-x|<\varepsilon\\ -\varepsilon<5-x<\varepsilon\\ -\varepsilon<x-5<\varepsilon.\end{gathered}$$

This implies we can let $\delta = \varepsilon$. Then:

$$\begin{aligned} |x-5| &< \delta \\ -\delta < x-5 &< \delta \\ -\varepsilon < x-5 &< \varepsilon \\ -\varepsilon < (x-3)-2 &< \varepsilon \\ -\varepsilon < (-x+3)-(-2) &< \varepsilon \\ |3-x-(-2)| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

01 02 ex 05 7. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-3| < \delta$, $|f(x)-6| < \varepsilon$.

Consider $|f(x)-6| < \varepsilon$, keeping in mind we want to make a statement about $|x-3|$:

$$\begin{aligned} |f(x)-6| &< \varepsilon \\ |x^2-3-6| &< \varepsilon \\ |x^2-9| &< \varepsilon \\ |x-3|\cdot|x+3| &< \varepsilon \\ |x-3| &< \varepsilon/|x+3| \end{aligned}$$

Since x is near 3, we can safely assume that, for instance, $2 < x < 4$. Thus

$$\begin{aligned} 2+3 < x+3 &< 4+3 \\ 5 < x+3 &< 7 \\ \frac{1}{7} < \frac{1}{x+3} &< \frac{1}{5} \\ \frac{\varepsilon}{7} < \frac{\varepsilon}{x+3} &< \frac{\varepsilon}{5} \end{aligned}$$

Let $\delta = \frac{\varepsilon}{7}$. Then:

$$\begin{aligned} |x-3| &< \delta \\ |x-3| &< \frac{\varepsilon}{7} \\ |x-3| &< \frac{\varepsilon}{x+3} \\ |x-3|\cdot|x+3| &< \frac{\varepsilon}{x+3}\cdot|x+3| \end{aligned}$$

Assuming x is near 3, $x+3$ is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-3|\cdot|x+3| &< \frac{\varepsilon}{x+3}\cdot(x+3) \\ |x^2-9| &< \varepsilon \\ |(x^2-3)-6| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

01 02 ex 11 8. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-4| < \delta$, $|f(x)-15| < \varepsilon$.

Consider $|f(x)-15| < \varepsilon$, keeping in mind we want to make a statement about $|x-4|$:

$$\begin{aligned} |f(x)-15| &< \varepsilon \\ |x^2+x-5-15| &< \varepsilon \\ |x^2+x-20| &< \varepsilon \\ |x-4|\cdot|x+5| &< \varepsilon \\ |x-4| &< \varepsilon/|x+5| \end{aligned}$$

Since x is near 4, we can safely assume that, for instance, $3 < x < 5$. Thus

$$\begin{aligned} 3+5 < x+5 &< 5+5 \\ 8 < x+5 &< 10 \\ \frac{1}{10} < \frac{1}{x+5} &< \frac{1}{8} \\ \frac{\varepsilon}{10} < \frac{\varepsilon}{x+5} &< \frac{\varepsilon}{8} \end{aligned}$$

Let $\delta = \frac{\varepsilon}{10}$. Then:

$$\begin{aligned} |x-4| &< \delta \\ |x-4| &< \frac{\varepsilon}{10} \\ |x-4| &< \frac{\varepsilon}{x+5} \\ |x-4|\cdot|x+5| &< \frac{\varepsilon}{x+5}\cdot|x+5| \end{aligned}$$

Assuming x is near 4, $x+5$ is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-4|\cdot|x+5| &< \frac{\varepsilon}{x+5}\cdot(x+5) \\ |x^2+x-20| &< \varepsilon \\ |(x^2+x-5)-15| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

01 02 ex 13 9. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-1| < \delta$, $|f(x)-6| < \varepsilon$.

Consider $|f(x)-6| < \varepsilon$, keeping in mind we want to make a statement about $|x-1|$:

$$\begin{aligned} |f(x)-6| &< \varepsilon \\ |(2x^2+3x+1)-6| &< \varepsilon \\ |2x^2+3x-5| &< \varepsilon \\ |2x+5|\cdot|x-1| &< \varepsilon \\ |x-1| &< \varepsilon/|2x+5| \end{aligned}$$

Since x is near 1, we can safely assume that, for instance,

$0 < x < 2$. Thus

$$\begin{aligned} 0+5 < 2x+5 &< 4+5 \\ 5 < 2x+5 &< 9 \\ \frac{1}{9} < \frac{1}{2x+5} &< \frac{1}{5} \\ \frac{\varepsilon}{9} < \frac{\varepsilon}{2x+5} &< \frac{\varepsilon}{5} \end{aligned}$$

Let $\delta = \frac{\varepsilon}{9}$. Then:

$$\begin{aligned} |x-1| &< \delta \\ |x-1| &< \frac{\varepsilon}{9} \\ |x-1| &< \frac{\varepsilon}{2x+5} \\ |x-1| \cdot |2x+5| &< \frac{\varepsilon}{2x+5} \cdot |2x+5| \end{aligned}$$

Assuming x is near 1, $2x+5$ is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-1| \cdot |2x+5| &< \frac{\varepsilon}{2x+5} \cdot (2x+5) \\ |2x^2+3x-5| &< \varepsilon \\ |(2x^2+3x+1)-6| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

01 02 ex 06 10. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-7| < \varepsilon$.
Consider $|f(x)-7| < \varepsilon$, keeping in mind we want to make a statement about $|x-2|$:

$$\begin{aligned} |f(x)-7| &< \varepsilon \\ |x^3-1-7| &< \varepsilon \\ |x^3-8| &< \varepsilon \\ |x-2| \cdot |x^2+2x+4| &< \varepsilon \\ |x-3| &< \varepsilon/|x^2+2x+4| \end{aligned}$$

Since x is near 2, we can safely assume that, for instance, $1 < x < 3$. Thus

$$\begin{aligned} 1^2+2\cdot 1+4 < x^2+2x+4 &< 3^2+2\cdot 3+4 \\ 7 < x^2+2x+4 &< 19 \\ \frac{1}{19} < \frac{1}{x^2+2x+4} &< \frac{1}{7} \\ \frac{\varepsilon}{19} < \frac{\varepsilon}{x^2+2x+4} &< \frac{\varepsilon}{7} \end{aligned}$$

Let $\delta = \frac{\varepsilon}{19}$. Then:

$$\begin{aligned} |x-2| &< \delta \\ |x-2| &< \frac{\varepsilon}{19} \\ |x-2| &< \frac{\varepsilon}{x^2+2x+4} \\ |x-2| \cdot |x^2+2x+4| &< \frac{\varepsilon}{x^2+2x+4} \cdot |x^2+2x+4| \end{aligned}$$

Assuming x is near 2, x^2+2x+4 is positive and we can drop the absolute value signs on the right.

$$\begin{aligned} |x-2| \cdot |x^2+2x+4| &< \frac{\varepsilon}{x^2+2x+4} \cdot (x^2+2x+4) \\ |x^3-8| &< \varepsilon \\ |(x^3-1)-7| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

01 02 ex 03 11. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-2| < \delta$, $|f(x)-5| < \varepsilon$. However, since $f(x) = 5$, a constant function, the latter inequality is simply $|5-5| < \varepsilon$, which is always true. Thus we can choose any δ we like; we arbitrarily choose $\delta = \varepsilon$.

01 02 ex 07 12. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0| < \delta$, $|f(x)-0| < \varepsilon$.
Consider $|f(x)-0| < \varepsilon$, keeping in mind we want to make a statement about $|x-0|$ (i.e., $|x|$):

$$\begin{aligned} |f(x)-0| &< \varepsilon \\ |e^{2x}-1| &< \varepsilon \\ -\varepsilon < e^{2x}-1 &< \varepsilon \\ 1-\varepsilon < e^{2x} &< 1+\varepsilon \\ \ln(1-\varepsilon) < 2x &< \ln(1+\varepsilon) \\ \frac{\ln(1-\varepsilon)}{2} < x &< \frac{\ln(1+\varepsilon)}{2} \end{aligned}$$

Let $\delta = \min\left\{\left|\frac{\ln(1-\varepsilon)}{2}\right|, \frac{\ln(1+\varepsilon)}{2}\right\} = \frac{\ln(1+\varepsilon)}{2}$.
Thus:

$$\begin{aligned} |x| &< \delta \\ |x| < \frac{\ln(1+\varepsilon)}{2} &< \left|\frac{\ln(1-\varepsilon)}{2}\right| \\ \frac{\ln(1-\varepsilon)}{2} < x &< \frac{\ln(1+\varepsilon)}{2} \\ \ln(1-\varepsilon) < 2x &< \ln(1+\varepsilon) \\ 1-\varepsilon < e^{2x} &< 1+\varepsilon \\ -\varepsilon < e^{2x}-1 &< \varepsilon \\ |e^{2x}-1-(0)| &< \varepsilon, \end{aligned}$$

which is what we wanted to prove.

01 02 ex 14 13. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-1| < \delta$, $|f(x)-1| < \varepsilon$.
Consider $|f(x)-1| < \varepsilon$, keeping in mind we want to make a statement about $|x-1|$:

$$\begin{aligned} |f(x)-1| &< \varepsilon \\ |1/x-1| &< \varepsilon \\ |(1-x)/x| &< \varepsilon \\ |x-1|/|x| &< \varepsilon \\ |x-1| &< \varepsilon \cdot |x| \end{aligned}$$

Since x is near 1, we can safely assume that, for instance, $1/2 < x < 3/2$. Thus $\varepsilon/2 < \varepsilon \cdot x$.
Let $\delta = \frac{\varepsilon}{2}$. Then:

$$|x-1| < \delta$$
$$|x-1| < \frac{\varepsilon}{2}$$
$$|x-1| < \varepsilon \cdot x$$
$$|x-1|/x < \varepsilon$$

Assuming x is near 1, x is positive and we can bring it into the absolute value signs on the left.

$$|(x-1)/x| < \varepsilon$$
$$|1-1/x| < \varepsilon$$
$$|(1/x)-1| < \varepsilon,$$

which is what we wanted to prove.

01 02 ex 08 14. Let $\varepsilon > 0$ be given. We wish to find $\delta > 0$ such that when $|x-0| < \delta$, $|f(x)-0| < \varepsilon$. In simpler terms, we want to show that when $|x| < \delta$, $|\sin x| < \varepsilon$.
Set $\delta = \varepsilon$. We start with assuming that $|x| < \delta$. Using the hint, we have that $|\sin x| < |x| < \delta = \varepsilon$. Hence if $|x| < \delta$, we know immediately that $|\sin x| < \varepsilon$.

Section 1.3

01 03 ex 01 1. Answers will vary.

01 03 ex 02 2. Answers will vary.

01 03 ex 03 3. Answers will vary.

01 03 ex 04 4. Answers will vary.

01 03 ex 05 5. As x is near 1, both f and g are near 0, but f is approximately twice the size of g. (I.e., $f(x) \approx 2g(x)$.)

01 03 ex 44 6. T; by Theorem 1.3.3, $\lim_{x\to 1} \ln x = \ln 1 = 0$.

01 03 ex 06 7. 9

01 03 ex 07 8. 6

01 03 ex 08 9. 0

01 03 ex 09 10. Limit does not exist.

01 03 ex 10 11. 3

01 03 ex 11 12. Not possible to know; as x approaches 6, $g(x)$ approaches 3, but we know nothing of the behavior of $f(x)$ when x is near 3.

01 03 ex 12 13. 3

01 03 ex 13 14. -45

01 03 ex 14 15. 1

01 03 ex 15 16. -1

01 03 ex 16 17. 0

01 03 ex 17 18. π

01 03 ex 18 19. 7

01 03 ex 21 20. $-0.000000015 \approx 0$

01 03 ex 22 21. $1/2$

01 03 ex 45 22. 0

01 03 ex 23 23. Limit does not exist

01 03 ex 24 24. 64

01 03 ex 25 25. 2

01 03 ex 26 26. 0

01 03 ex 20 27. $\frac{\pi^2+3\pi+5}{5\pi^2-2\pi-3} \approx 0.6064$

01 03 ex 19 28. $\frac{3\pi+1}{1-\pi}$

01 03 ex 27 29. -8

01 03 ex 28 30. -1

01 03 ex 29 31. 10

01 03 ex 30 32. -2

01 03 ex 31 33. $-3/2$

01 03 ex 32 34. $-7/8$

01 03 ex 38 35. 0

01 03 ex 40 36. 0

01 03 ex 42 37. 1

01 03 ex 41 38. 9

01 03 ex 33 39. 3

01 03 ex 34 40. $5/8$

01 03 ex 35 41. 1

01 03 ex 36 42. $\pi/180$

01 03 ex 43 43.

(a) Apply Part 1 of Theorem 1.3.1.

(b) Apply Theorem 1.3.6; $g(x) = \frac{x}{x}$ is the same as $g(x) = 1$ everywhere except at $x = 0$. Thus $\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} 1 = 1$.

(c) The function $f(x)$ is always 0, so $g\big(f(x)\big)$ is never defined as $g(x)$ is not defined at $x = 0$. Therefore the limit does not exist.

(d) The Composition Rule requires that $\lim\limits_{x\to 0} g(x)$ be equal to $g(0)$. They are not equal, so the conditions of the Composition Rule are not satisfied, and hence the rule is not violated.

Section 1.4

01 04 ex 01 1. The function approaches different values from the left and right; the function grows without bound; the function oscillates.

01 04 ex 02 2. F

01 04 ex 03 3. F

01 04 ex 04 4. T

01 04 ex 05 5.

(a) 2

(b) 2

(c) 2

(d) 1

(e) As f is not defined for $x < 0$, this limit is not defined.

(f) 1

01 04 ex 06 6.

(a) 1
(b) 2
(c) Does not exist.
(d) 2
(e) 0
(f) As f is not defined for $x > 2$, this limit is not defined.

01 04 ex 07 7.

(a) Does not exist.
(b) Does not exist.
(c) Does not exist.
(d) Not defined.
(e) 0
(f) 0

01 04 ex 08 8.

(a) 2
(b) 0
(c) Does not exist.
(d) 1

01 04 ex 09 9.

(a) 2
(b) 2
(c) 2
(d) 2

01 04 ex 10 10.

(a) 4
(b) -4
(c) Does not exist.
(d) 0

01 04 ex 11 11.

(a) 2
(b) 2
(c) 2
(d) 0
(e) 2
(f) 2
(g) 2
(h) Not defined

01 04 ex 12 12.

(a) $a - 1$
(b) a
(c) Does not exist.
(d) a

01 04 ex 13 13.

(a) 2
(b) -4
(c) Does not exist.
(d) 2

01 04 ex 14 14.

(a) -1
(b) 0
(c) Does not exist.
(d) 0

01 04 ex 15 15.

(a) 0
(b) 0
(c) 0
(d) 0
(e) 2
(f) 2
(g) 2
(h) 2

01 04 ex 16 16.

(a) -1
(b) 0
(c) Does not exist.
(d) 0

01 04 ex 17 17.

(a) $1 - \cos^2 a = \sin^2 a$
(b) $\sin^2 a$
(c) $\sin^2 a$
(d) $\sin^2 a$

01 04 ex 18 18.

(a) 2
(b) 0
(c) Does not exist
(d) 1

01 04 ex 19 19.

(a) 4
(b) 4
(c) 4
(d) 3

01 04 ex 20 20.

(a) c
(b) c
(c) c
(d) c

01 04 ex 21 21.

   (a) -1
   (b) 1
   (c) Does not exist
   (d) 0

01 04 ex 22 22. $-3/5$

01 04 ex 23 23. $2/3$

01 04 ex 24 24. 2.5

01 04 ex 26 25. -9

01 04 ex 27 26. -1.63

Section 1.5

01 05 ex 06 1. Answers will vary.

01 05 ex 08 2. Answers will vary.

01 05 ex 09 3. A root of a function f is a value c such that $f(c) = 0$.

01 05 ex 10 4. Consider the function $h(x) = g(x) - f(x)$, and use the Bisection Method to find a root of h.

01 05 ex 01 5. F

01 05 ex 02 6. T

01 05 ex 03 7. T

01 05 ex 04 8. F

01 05 ex 05 9. F

01 05 ex 07 10. T

01 05 ex 11 11. No; $\lim\limits_{x\to 1} f(x) = 2$, while $f(1) = 1$.

01 05 ex 12 12. No; $\lim\limits_{x\to 1} f(x)$ does not exist.

01 05 ex 13 13. No; $f(1)$ does not exist.

01 05 ex 14 14. Yes

01 05 ex 15 15. Yes

01 05 ex 16 16. Yes

01 05 ex 17 17.

   (a) No; $\lim\limits_{x\to -2} f(x) \neq f(-2)$
   (b) Yes
   (c) No; $f(2)$ is not defined.

01 05 ex 44 18. Yes; $\lim\limits_{x\to 3\pi/2} \sin x + 1 = 0$, and $\sin(3\pi/2) + 1 = 0$.

01 05 ex 18 19.

   (a) Yes
   (b) Yes

01 05 ex 19 20.

   (a) Yes
   (b) No; the left and right hand limits at 1 are not equal.

01 05 ex 20 21.

   (a) Yes
   (b) Yes

01 05 ex 21 22.

   (a) Yes
   (b) No. $\lim_{x\to 8} f(x) = 16/5 \neq f(8) = 5$.

01 05 ex 22 23. $(-\infty, \infty)$

01 05 ex 23 24. $(-\infty, -2]$ and $[2, \infty)$

01 05 ex 45 25. $[-2, 2]$

01 05 ex 24 26. $[-1, 1]$

01 05 ex 25 27. $(-\infty, -\sqrt{6}]$ and $[\sqrt{6}, \infty)$

01 05 ex 26 28. $(-1, 1)$

01 05 ex 27 29. $(-\infty, \infty)$

01 05 ex 28 30. $(-\infty, \infty)$

01 05 ex 29 31. $(0, \infty)$

01 05 ex 30 32. $(-\infty, \infty)$

01 05 ex 31 33. $(-\infty, 0]$

01 05 ex 32 34. $(-\infty, \infty)$

01 05 ex 33 35. Yes, by the Intermediate Value Theorem.

01 05 ex 34 36. Yes, by the Intermediate Value Theorem. In fact, we can be more specific and state such a value c exists in $(0, 2)$, not just in $(-3, 7)$.

01 05 ex 35 37. We cannot say; the Intermediate Value Theorem only applies to function values between -10 and 10; as 11 is outside this range, we do not know.

01 05 ex 36 38. We cannot say; the Intermediate Value Theorem only applies to continuous functions. As we do know know if h is continuous, we cannot say.

01 05 ex 37 39. Approximate root is $x = 1.23$. The intervals used are:
$[1, 1.5]$ $[1, 1.25]$ $[1.125, 1.25]$
$[1.1875, 1.25]$ $[1.21875, 1.25]$ $[1.234375, 1.25]$
$[1.234375, 1.2421875]$ $[1.234375, 1.2382813]$

01 05 ex 38 40. Approximate root is $x = 0.52$. The intervals used are:
$[0.5, 0.55]$ $[0.5, 0.525]$ $[0.5125, 0.525]$
$[0.51875, 0.525]$ $[0.521875, 0.525]$

01 05 ex 39 41. Approximate root is $x = 0.69$. The intervals used are:
$[0.65, 0.7]$ $[0.675, 0.7]$ $[0.6875, 0.7]$
$[0.6875, 0.69375]$ $[0.690625, 0.69375]$

01 05 ex 40 42. Approximate root is $x = 0.78$. The intervals used are:
$[0.7, 0.8]$ $[0.75, 0.8]$ $[0.775, 0.8]$
$[0.775, 0.7875]$ $[0.78125, 0.7875]$
(A few more steps show that 0.79 is better as the root is $\pi/4 \approx 0.78539$.)

01 05 ex 41 43.

   (a) 20
   (b) 25
   (c) Limit does not exist
   (d) 25

01 05 ex 42 44.

x	$f(x)$
-0.81	-2.34129
-0.801	-2.33413
-0.79	-2.32542
-0.799	-2.33254

The top two lines give an approximation of the limit from the left: -2.33. The bottom two lines give an approximation from the right: -2.33 as well.

01 05 ex 43 45. Answers will vary.

Section 1.6

01 06 ex 01 1. F

01 06 ex 02 2. T

01 06 ex 03 3. F

01 06 ex 04 4. T

01 06 ex 05 5. T

01 06 ex 06 6. Answers will vary.

01 06 ex 07 7. Answers will vary.

01 06 ex 08 8. The limit of f as x approaches 7 does not exist, hence f is not continuous. (Note: f could be defined at 7!)

01 06 ex 09 9.

(a) ∞

(b) ∞

01 06 ex 10 10.

(a) $-\infty$

(b) ∞

(c) Limit does not exist

(d) ∞

(e) ∞

(f) ∞

01 06 ex 11 11.

(a) 1

(b) 0

(c) $1/2$

(d) $1/2$

01 06 ex 12 12.

(a) Limit does not exist

(b) Limit does not exist

01 06 ex 13 13.

(a) Limit does not exist

(b) Limit does not exist

01 06 ex 40 14.

(a) 10

(b) ∞

01 06 ex 14 15. Tables will vary.

(a)

x	$f(x)$
2.9	-15.1224
2.99	-159.12
2.999	-1599.12

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	16.8824
3.01	160.88
3.001	1600.88

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x)$ does not exist.

01 06 ex 15 16. Tables will vary.

(a)

x	$f(x)$
2.9	-335.64
2.99	-30350.6

It seems $\lim_{x\to 3^-} f(x) = -\infty$.

(b)

x	$f(x)$
3.1	-265.61
3.01	-29650.6

It seems $\lim_{x\to 3^+} f(x) = -\infty$.

(c) It seems $\lim_{x\to 3} f(x) = -\infty$.

01 06 ex 16 17. Tables will vary.

(a)

x	$f(x)$
2.9	132.857
2.99	12124.4

It seems $\lim_{x\to 3^-} f(x) = \infty$.

(b)

x	$f(x)$
3.1	108.039
3.01	11876.4

It seems $\lim_{x\to 3^+} f(x) = \infty$.

(c) It seems $\lim_{x\to 3} f(x) = \infty$.

01 06 ex 17 18. Tables will vary.

(a)

x	$f(x)$
2.9	-0.632
2.99	-0.6032
2.999	-0.60032

It seems $\lim_{x\to 3^-} f(x) = -0.6$.

(b)

x	$f(x)$
3.1	-0.5686
3.01	-0.5968
3.001	-0.59968

It seems $\lim_{x\to 3^+} f(x) = -0.6$.

(c) It seems $\lim_{x\to 3} f(x) = -0.6$.

01 06 ex 18 19. Horizontal asymptote at $y = 2$; vertical asymptotes at $x = -5, 4$.

01 06 ex 19 20. Horizontal asymptote at $y = -3/5$; vertical asymptote at $x = 3$.

01 06 ex 20 21. Horizontal asymptote at $y = 0$; vertical asymptotes at $x = -1, 0$.

01 06 ex 21 22. No horizontal asymptote; vertical asymptote at $x = 1$.

01 06 ex 22 23. No horizontal or vertical asymptotes.

01 06 ex 41 24. Horizontal asymptote at $y = -1$; no vertical asymptotes

01 06 ex 23 25. ∞

01 06 ex 24 26. $-\infty$

01 06 ex 25 27. $-\infty$

01 06 ex 26 28. ∞

01 06 ex 27 29. Solution omitted.

01 06 ex 28 30.

(a) 2

(b) -3

(c) -3

(d) $1/3$

01 06 ex 29 31. Yes. The only “questionable” place is at $x = 3$, but the left and right limits agree.

01 06 ex 30 32. 1

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