# 1: Vectors

This chapter introduces a new mathematical object, the **vector**. Defined in Section 10.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

## 1.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the *x*-*y* plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point $P$ in space can be represented with an ordered triple, $P = (a, b, c)$, where $a$, $b$ and $c$ represent the relative position of $P$ along the $x$-, $y$- and $z$-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive $x$-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive $y$-axis, then the extended thumb will point in the direction of the positive $z$-axis. (It may take some thought to verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 1.1.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the $x$-$y$ plane is drawn in its standard way, with the $z$-axis down to the left. The perspective is that the paper represents the $x$-$y$ plane and the positive $z$ axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

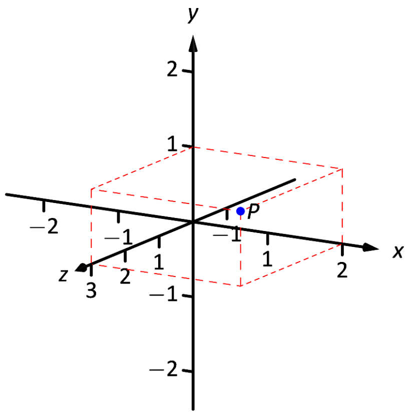


Figure 1.1.1: Plotting the point $P = (2, 1, 3)$ in space.

One can also consider the $x$-$y$ plane as being a horizontal plane in, say, a room, where the positive $z$-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 1.1.2. The same point $P$ is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

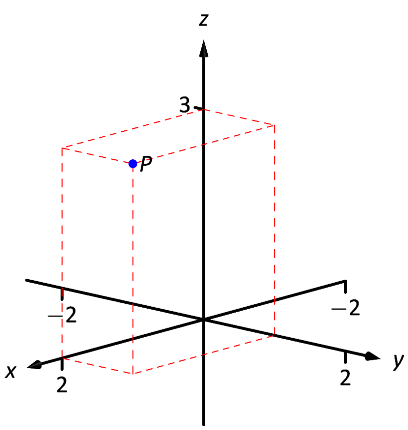


Figure 1.1.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

## Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

**Definition 1.1.1 Distance In Space**

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance $D$ between $P$ and $Q$ is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points $P$ and $Q$ in space as $\overline{PQ}$, and refer to the length of this segment as $||\overline{PQ}||$. The above distance formula allows us to compute the length of this segment.

**Example 1.1.1 Length of a line segment**

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

**Solution** The points $P$ and $Q$ are plotted in Figure 1.1.3; no special consideration need be made to draw the line segment connecting these two

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points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 1.1.1, we have

$$||\overline{PQ}|| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

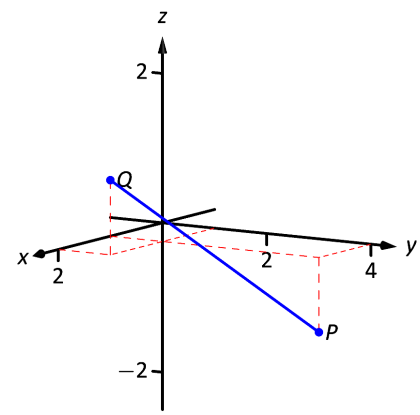


Figure 1.1.3: Plotting points $P$ and $Q$ in Example 1.1.1.

## Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 1.1.1 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius $r$. If a point $P = (x, y, z)$ lies on the sphere, then $P$ is $r$ units from $C$; that is,

$$||\overline{PC}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius $r$, as given in the following Key Idea.

**Key Idea 1.1.1 Standard Equation of a Sphere in Space**

The standard equation of the sphere with radius $r$, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

**Example 1.1.2 Equation of a sphere**

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

**SOLUTION** To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

The sphere is centered at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables $x$, $y$ and $z$ are all used. We now consider situations where surfaces are defined where one or two of these

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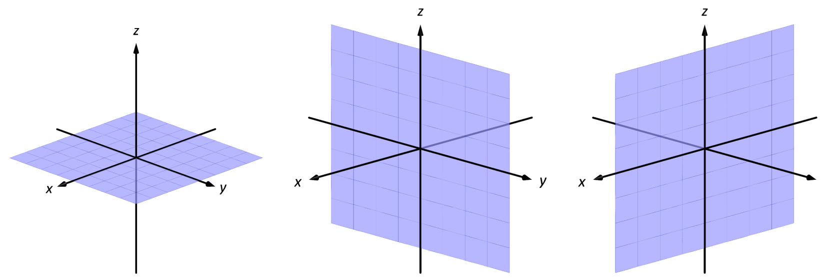
variables are absent.

## Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 1.1.4), the **coordinate planes**: the $x$-$y$ plane, the $y$-$z$ plane and the $x$-$z$ plane. The $x$-$y$ plane is characterized as the set of all points in space where the $z$-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the $x$-$z$ plane is all points where the $y$-value is 0, characterized by $y = 0$.




the $x$-$y$ plane the $y$-$z$ plane the $x$-$z$ plane

Figure 1.1.4: The coordinate planes.

The equation $x = 2$ describes all points in space where the $x$-value is 2. This is a plane, parallel to the $y$-$z$ coordinate plane, shown in Figure 1.1.5.

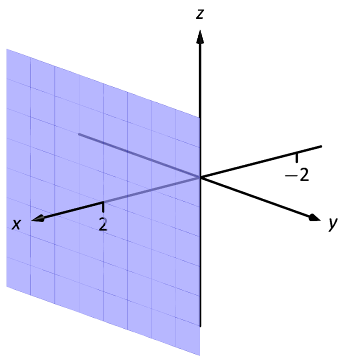



Figure 1.1.5: The plane $x = 2$.

**Example 1.1.3 Regions defined by planes**
Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

**SOLUTION** The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 1.1.6, which are parallel to the $x$-$z$ plane. Thus the region extends infinitely in the $x$ and $z$ directions, and is bounded by planes in the $y$ direction.

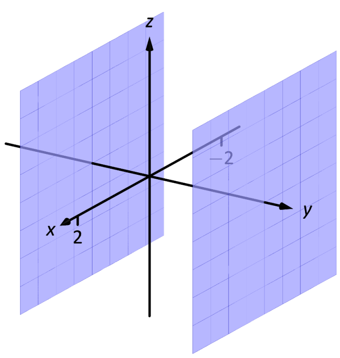



Figure 1.1.6: Sketching the boundaries of a region in Example 1.1.3.

## Cylinders

The equation $x = 1$ obviously lacks the $y$ and $z$ variables, meaning it defines points where the $y$ and $z$ coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space.* In *the plane*, this equation describes a circle of radius 1, centered at the origin. In space, the $z$ coordinate is not specified, meaning it can take on any value. In Figure 1.1.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant $z$-value of $-1.5$, the middle one has a $z$-value of 0 and the top circle has a

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$z$-value of 1. By plotting *all* possible $z$-values, we get the surface shown in Figure 1.1.8 (b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.

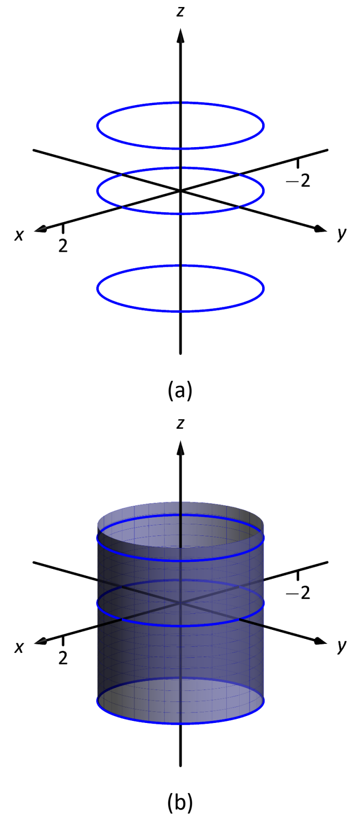


Figure 1.1.8: Sketching $x^2 + y^2 = 1$.

**Definition 1.1.2 Cylinder**

Let $C$ be a curve in a plane and let $L$ be a line not parallel to $C$. A **cylinder** is the set of all lines parallel to $L$ that pass through $C$. The curve $C$ is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves $C$ that lie in planes parallel to one of the coordinate planes, and lines $L$ that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3rd variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the $x$-$y$ plane is the directrix and the rulings are lines parallel to the $z$-axis. (Any circle shown in Figure 1.1.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

**Example 1.1.4 Graphing cylinders**

Graph the cylinder following cylinders.

1. $z = y^2$ 2. $x = \sin z$

**Solution**

1. We can view the equation $z = y^2$ as a parabola in the $y$-$z$ plane, as illustrated in Figure 1.1.7 (a). As $x$ does not appear in the equation, the rulings are lines through this parabola parallel to the $x$-axis, shown in (b). These rulings give a general idea as to what the surface looks like, drawn in (c).

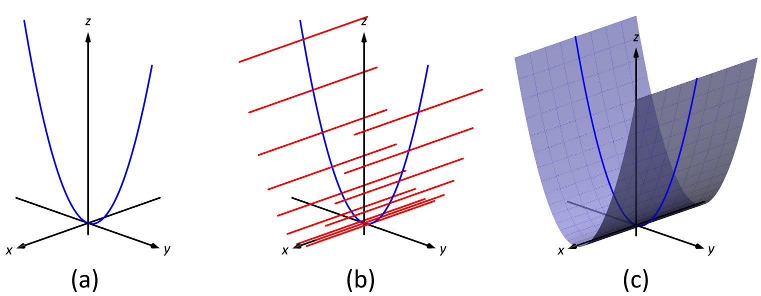


Figure 1.1.7: Sketching the cylinder defined by $z = y^2$.

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2. We can view the equation $x = \sin z$ as a sine curve that exists in the $x$-$z$ plane, as shown in Figure 1.1.9 (a). The rules are parallel to the $y$ axis as the variable $y$ does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

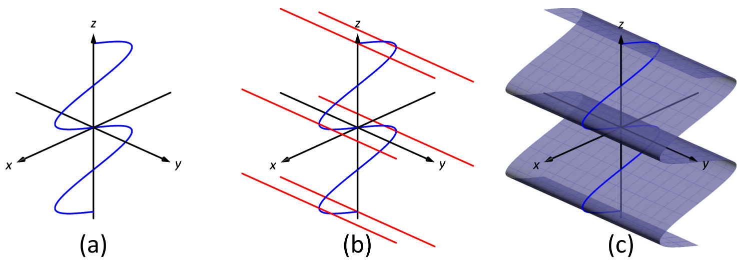


Figure 1.1.9: Sketching the cylinder defined by $x = \sin z$.

## Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the $x$-axis. Cross–sections of this surface parallel to the $y$-$z$ plane are circles, as shown in Figure 1.1.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius $r$. The radius is a function of $x$; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 1.1.10b is $y^2 + z^2 = (\sqrt{x})^2$.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

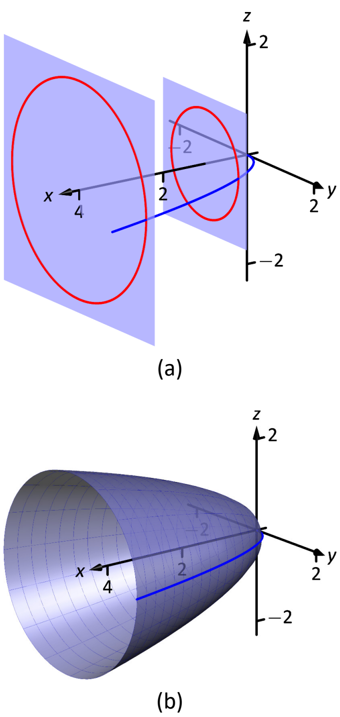


Figure 1.1.10: Introducing surfaces of revolution.

> **Key Idea 1.1.2 Surfaces of Revolution, Part 1**
>
> Let $r$ be a radius function.
>
> 1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the $x$-axis is $y^2 + z^2 = r(x)^2$.
> 2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the $y$-axis is $x^2 + z^2 = r(y)^2$.
> 3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the $z$-axis is $x^2 + y^2 = r(z)^2$.

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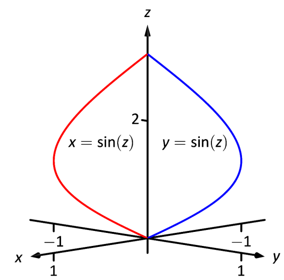
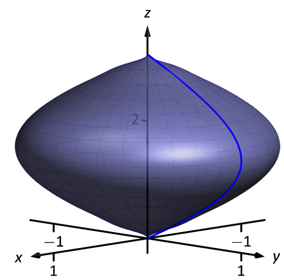
**Example 1.1.5 Finding equation of a surface of revolution**
Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the $z$-axis.

**Solution** Using Key Idea 1.1.2, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 1.1.11(a) and the surface is drawn in Figure 1.1.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 1.1.11(a).




(a)




(b)

Figure 1.1.11: Revolving $y = \sin z$ about the $z$-axis in Example 1.1.5.

This particular method of creating surfaces of revolution is limited. For instance, in Example 7.3.4 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the $y$-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the $x$-axis. Trying to rewrite $y = \sin x$ as a function of $y$ is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the $y$-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the $z$-axis. This will give us a more natural way of viewing the surface.

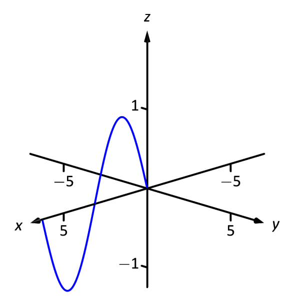
A value of $x$ is a measurement of distance from the $z$-axis. At the distance $r$, we plot a $z$-height of $f(r)$. When rotating $f(x)$ about the $z$-axis, we want all points a distance of $r$ from the $z$-axis in the $x$-$y$ plane to have a $z$-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing $r$ with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

**Key Idea 1.1.3 Surfaces of Revolution, Part 2**

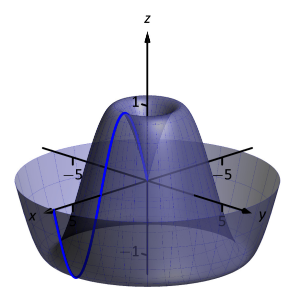
Let $z = f(x)$, $x \geq 0$, be a curve in the $x$-$z$ plane. The surface formed by revolving this curve about the $z$-axis has equation $z = f\big(\sqrt{x^2 + y^2}\big)$.

**Example 1.1.6 Finding equation of surface of revolution**
Find the equation of the surface found by revolving $z = \sin x$ about the $z$-axis.

**Solution** Using Key Idea 1.1.3, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 1.1.12.




(a)




(b)

Figure 1.1.12: Revolving $z = \sin x$ about the $z$-axis in Example 1.1.6.

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## Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

**Definition 1.1.3 Quadric Surface**

A **quadric surface** is the graph of the general second–degree equation in three variables:

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients $D$, $E$ or $F$ are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

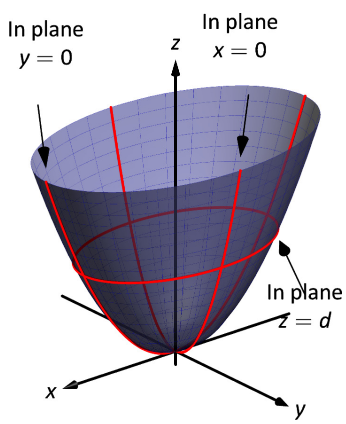


Figure 1.1.13: The elliptic paraboloid $z = x^2/4 + y^2$.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 1.1.13. If we intersect this shape with the plane $z = d$ (i.e., replace $z$ with $d$), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by $d$:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the $x$-$y$ coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the $x$-$z$ plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

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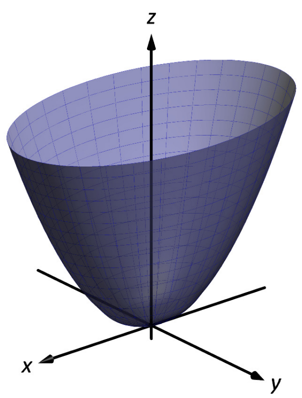
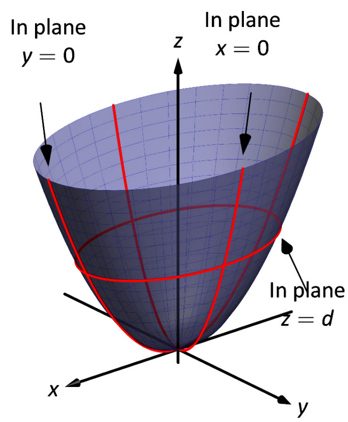
**Elliptic Paraboloid,** $z = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$




| Plane | Trace |
|---|---|
| $x = d$ | Parabola |
| $y = d$ | Parabola |
| $z = d$ | Ellipse |




One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the $z$ variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the $x$-axis.

Multiplying the right hand side by $(-1)$ defines an elliptic paraboloid that "opens" in the opposite direction.

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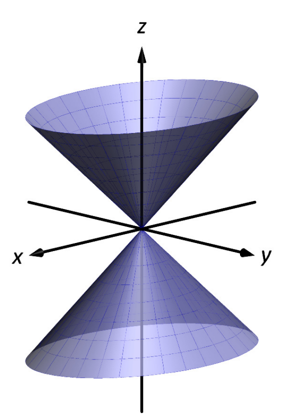
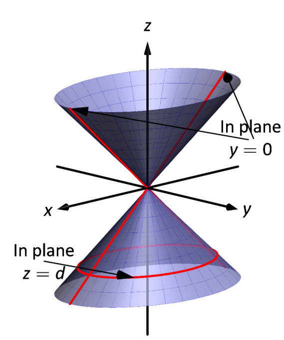
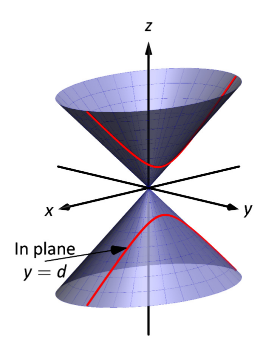
**Elliptic Cone,** $z^2 = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2}$



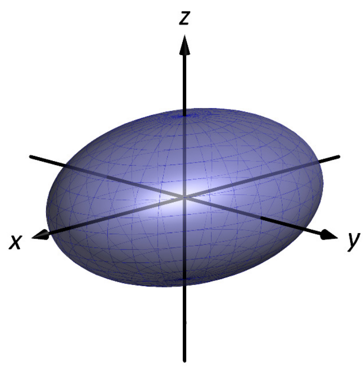

| Plane | Trace |
|---|---|
| $x = 0$ | Crossed Lines |
| $y = 0$ | Crossed Lines |
| $x = d$ | Hyperbola |
| $y = d$ | Hyperbola |
| $z = d$ | Ellipse |



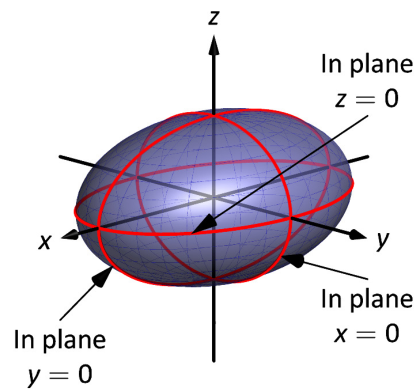




One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

**Ellipsoid,** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$




| Plane | Trace |
|---|---|
| $x=d$ | Ellipse |
| $y=d$ | Ellipse |
| $z=d$ | Ellipse |




If $a=b=c\neq 0$, the ellipsoid is a sphere with radius $a$; compare to Key Idea 1.1.1.

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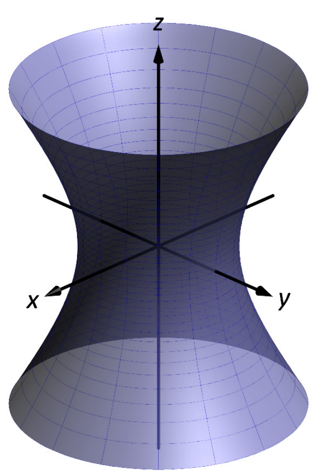
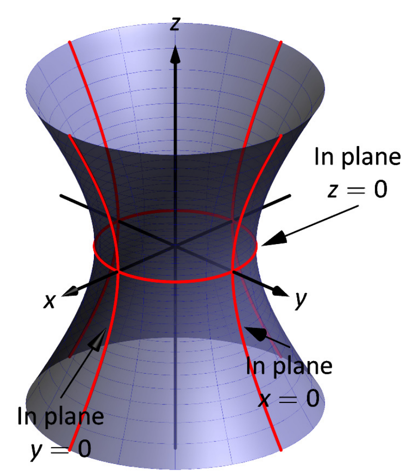
**Hyperboloid of One Sheet,** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

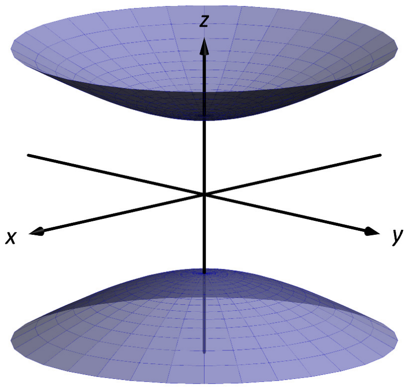



| Plane | Trace |
|---|---|
| $x=d$ | Hyperbola |
| $y=d$ | Hyperbola |
| $z=d$ | Ellipse |

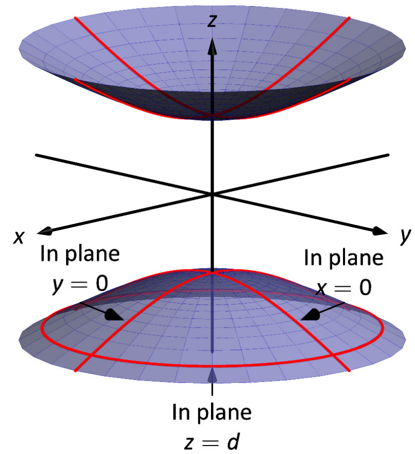



The one variable with a negative coefficient corresponds to the axis that the hyperboloid "opens" along.

**Hyperboloid of Two Sheets,** $\dfrac{z^2}{c^2} - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$



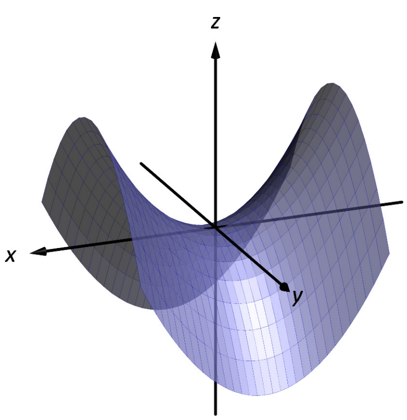

| Plane | Trace |
|---|---|
| $x = d$ | Hyperbola |
| $y = d$ | Hyperbola |
| $z = d$ | Ellipse |



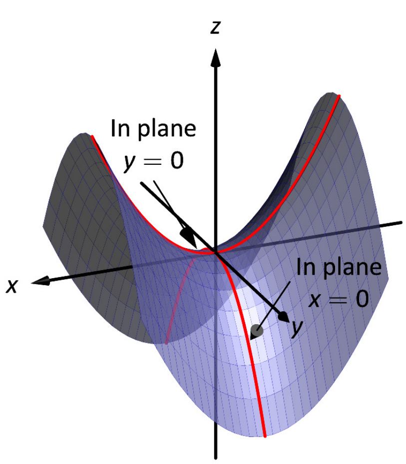

The one variable with a positive coefficient corresponds to the axis that the hyperboloid "opens" along. In the case illustrated, when $|d| < |c|$, there is no trace.

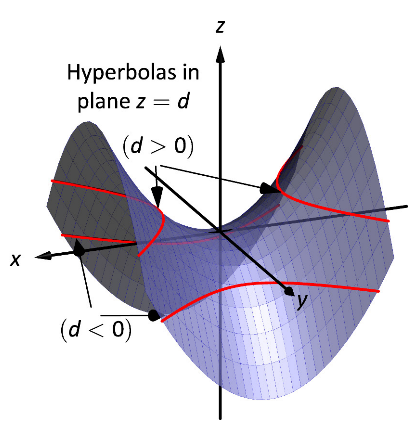
---

**Hyperbolic Paraboloid,** $z = \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2}$




| Plane | Trace |
|---|---|
| $x = d$ | Parabola |
| $y = d$ | Parabola |
| $z = d$ | Hyperbola |




The parabolic traces will open along the axis of the one variable that is raised to the first power.

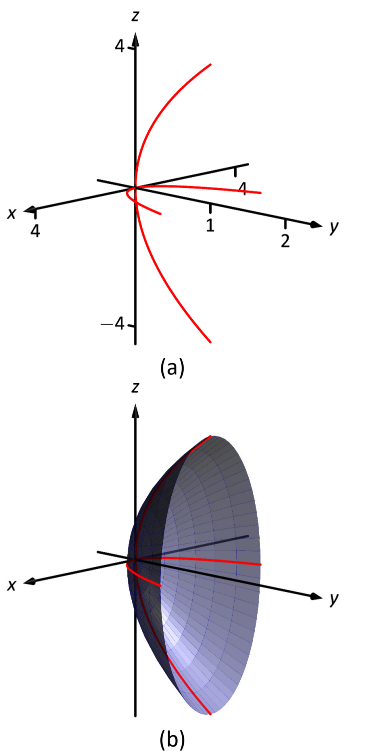



Figure 1.1.14: Sketching an elliptic paraboloid.

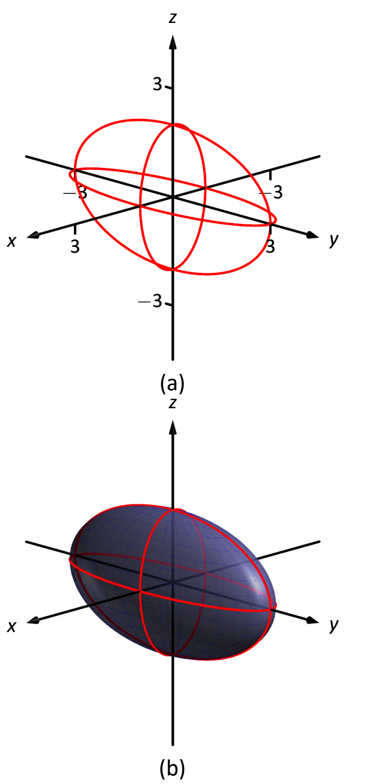



Figure 1.1.15: Sketching an ellipsoid.

**Example 1.1.7 Sketching quadric surfaces**

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$  2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$  3. $z = y^2 - x^2.$

**Solution**

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is $y$, we note the paraboloid opens along the $y$-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 1.1.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$ :

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the $y$–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the $z$-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the $z$-axis, and the minor axis has length 2 along the $x$-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the $y$-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 1.1.15(a). Filling in the surface gives Figure 1.1.15(b).

3. $z = y^2 - x^2$:

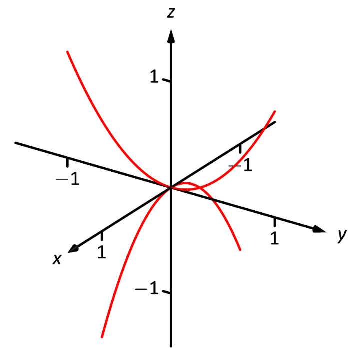
Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y-z$ and $x-z$ planes:

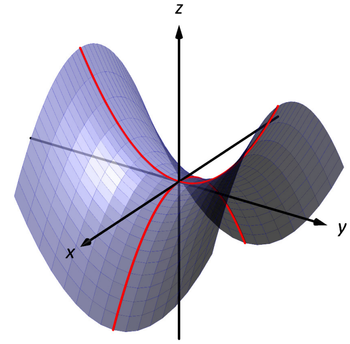
$x=0$: The trace is $z=y^2$, a parabola opening up in the $y-z$ plane.

$y=0$: The trace is $z=-x^2$, a parabola opening down in the $x-z$ plane.

Sketching these two parabolas gives a sketch like that in Figure 1.1.16 (a), and filling in the surface gives a sketch like (b).




(a)




(b)

Figure 1.1.16: Sketching a hyperbolic paraboloid.

**Example 1.1.8 Identifying quadric surfaces**

Consider the quadric surface shown in Figure 1.1.17. Which of the following equations best fits this surface?

(a) $x^2-y^2-\dfrac{z^2}{9}=0$ (c) $z^2-x^2-y^2=1$

(b) $x^2-y^2-z^2=1$ (d) $4x^2-y^2-\dfrac{z^2}{9}=1$

**Solution** The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2}-\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the $x$-axis, meaning $x$ must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the $z$-direction than in the $y$-direction, so we need an equation where $c>b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2-y^2-\frac{z^2}{9}=1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the $x$-direction and is wider in the $z$-direction than in the $y$. Now note the coefficient of the $x$-term. Rewriting $4x^2$ in standard form, we have: $4x^2=\dfrac{x^2}{(1/2)^2}$. Thus when $y=0$ and $z=0$, $x$ must be $1/2$; i.e., each hyperboloid "starts" at $x=1/2$. This matches our figure.

We conclude that $4x^2-y^2-\dfrac{z^2}{9}=1$ best fits the graph.

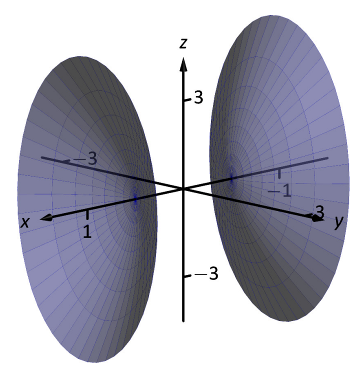



Figure 1.1.17: A possible equation of this quadric surface is found in Example 1.1.8.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Notes:

# Exercises 1.1

## *Terms and Concepts*

10 01 ex 08 1. Axes drawn in space must conform to the ________ ________ rule.

10 01 ex 01 2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.

10 01 ex 02 3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.

10 01 ex 03 4. Which quadric surface looks like a Pringles® chip?

10 01 ex 04 5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the $x$-axis, what quadric surface is formed?

10 01 ex 05 6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the $y$-axis, what quadric surface is formed?

## *Problems*

10 01 ex 06 7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

10 01 ex 07 8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

10 01 ex 09 9. Find the center and radius of the sphere defined by $x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10 01 ex 10 10. Find the center and radius of the sphere defined by $x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

10 01 exset 01 **In Exercises 11 – 14, describe the region in space defined by the inequalities.**

10 01 ex 11 11. $x^2 + y^2 + z^2 < 1$

10 01 ex 12 12. $0 \le x \le 3$

10 01 ex 13 13. $x \ge 0,\ y \ge 0,\ z \ge 0$

10 01 ex 14 14. $y \ge 3$

10 01 exset 02 **In Exercises 15 – 18, sketch the cylinder in space.**

10 01 ex 15 15. $z = x^3$

10 01 ex 16 16. $y = \cos z$

10 01 ex 17 17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

10 01 ex 18 18. $y = \dfrac{1}{x}$

10 01 exset 05 **In Exercises 19 – 22, give the equation of the surface of revolution described.**

10 01 ex 29 19. Revolve $z = \dfrac{1}{1 + y^2}$ about the $y$-axis.

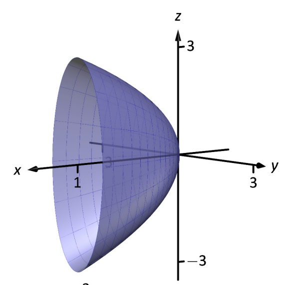
10 01 ex 30 20. Revolve $y = x^2$ about the $x$-axis.

10 01 ex 31 21. Revolve $z = x^2$ about the $z$-axis.

10 01 ex 32 22. Revolve $z = 1/x$ about the $z$-axis.

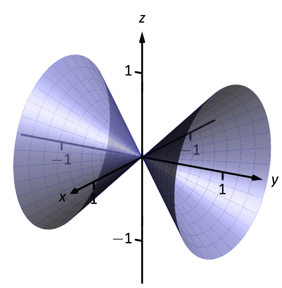
10 01 exset 03 **In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.**

10 01 ex 19 23.



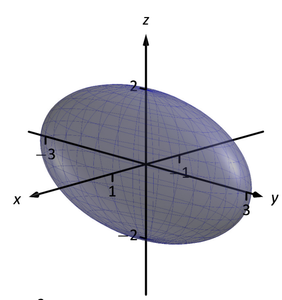

(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

10 01 ex 20 24.



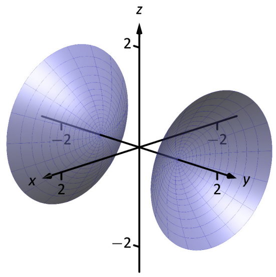

(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

10 01 ex 21 25.




(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

10 01 ex 22

26. 


(a) $y^2 - x^2 - z^2 = 1$

(b) $y^2 + x^2 - z^2 = 1$

10 01 exset 04

**In Exercises 27 – 32, sketch the quadric surface.**

10 01 ex 28

27. $z - y^2 + x^2 = 0$

10 01 ex 24

28. $z^2 = x^2 + \dfrac{y^2}{4}$

10 01 ex 23

29. $x = -y^2 - z^2$

10 01 ex 27

30. $16x^2 - 16y^2 - 16z^2 = 1$

10 01 ex 26

31. $\dfrac{x^2}{9} - y^2 + \dfrac{z^2}{25} = 1$

10 01 ex 25

32. $4x^2 + 2y^2 + z^2 = 4$

## 1.2 Lines

To find the equation of a line in the *x*-*y* plane, we need two pieces of information: a point and the slope. The slope conveys *direction* information. As vertical lines have an undefined slope, the following statement is more accurate:

> To define a line, one needs a point on the line and the direction of the line.

This holds true for lines in space.

Let $P$ be a point in space, let $\vec{p}$ be the vector with initial point at the origin and terminal point at $P$ (i.e., $\vec{p}$ "points" to $P$), and let $\vec{d}$ be a vector. Consider the points on the line through $P$ in the direction of $\vec{d}$.

Clearly one point on the line is $P$; we can say that the *vector* $\vec{p}$ lies at this point on the line. To find another point on the line, we can start at $\vec{p}$ and move in a direction parallel to $\vec{d}$. For instance, starting at $\vec{p}$ and traveling one length of $\vec{d}$ places one at another point on the line. Consider Figure 1.2.2 where certain points along the line are indicated.

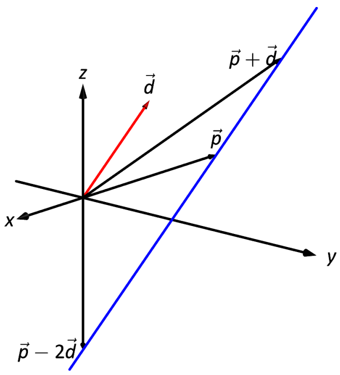


Figure 1.2.2: Defining a line in space.

The figure illustrates how every point on the line can be obtained by starting with $\vec{p}$ and moving a certain distance in the direction of $\vec{d}$. That is, we can define the line as a function of $t$:

$$\vec{\ell}(t) = \vec{p} + t\,\vec{d}. \tag{1.1}$$

In many ways, this is *not* a new concept. Compare Equation (1.1) to the familiar "$y = mx + b$" equation of a line:

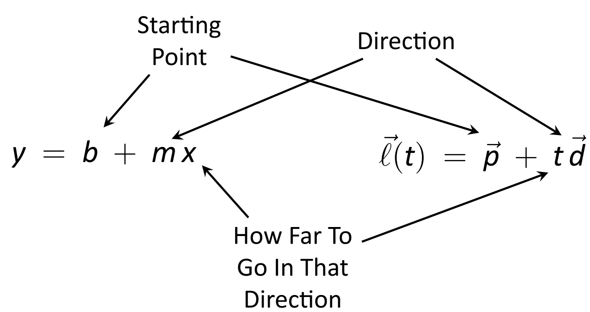


Figure 1.2.1: Understanding the vector equation of a line.

The equations exhibit the same structure: they give a starting point, define a direction, and state how far in that direction to travel.

Equation (1.1) is an example of a **vector–valued function**; the input of the function is a real number and the output is a vector. We will cover vector–valued functions extensively in the next chapter.

Notes:

There are other ways to represent a line. Let $\vec{p} = \langle x_0, y_0, z_0 \rangle$ and let $\vec{d} = \langle a, b, c \rangle$. Then the equation of the line through $\vec{p}$ in the direction of $\vec{d}$ is:

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t\vec{d} \\ &= \langle x_0, y_0, z_0 \rangle + t\,\langle a, b, c \rangle \\ &= \langle x_0 + at, y_0 + bt, z_0 + ct \rangle\,.\end{aligned}$$

The last line states that the $x$ values of the line are given by $x = x_0 + at$, the $y$ values are given by $y = y_0 + bt$, and the $z$ values are given by $z = z_0 + ct$. These three equations, taken together, are the **parametric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$.

Finally, each of the equations for $x$, $y$ and $z$ above contain the variable $t$. We can solve for $t$ in each equation:

$$\begin{aligned} x = x_0 + at \quad &\Rightarrow \quad t = \frac{x - x_0}{a}, \\ y = y_0 + bt \quad &\Rightarrow \quad t = \frac{y - y_0}{b}, \\ z = z_0 + ct \quad &\Rightarrow \quad t = \frac{z - z_0}{c}, \end{aligned}$$

assuming $a, b, c \neq 0$. Since $t$ is equal to each expression on the right, we can set these equal to each other, forming the **symmetric equations of the line** through $\vec{p}$ in the direction of $\vec{d}$:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

Each representation has its own advantages, depending on the context. We summarize these three forms in the following definition, then give examples of their use.

Notes:

**Definition 1.2.1 Equations of Lines in Space**

Consider the line in space that passes through $\vec{p} = \langle x_0, y_0, z_0 \rangle$ in the direction of $\vec{d} = \langle a, b, c \rangle$.

1. The **vector equation** of the line is
$$\vec{\ell}(t) = \vec{p} + t\vec{d}.$$

2. The **parametric equations** of the line are
$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct.$$

3. The **symmetric equations** of the line are
$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}.$$

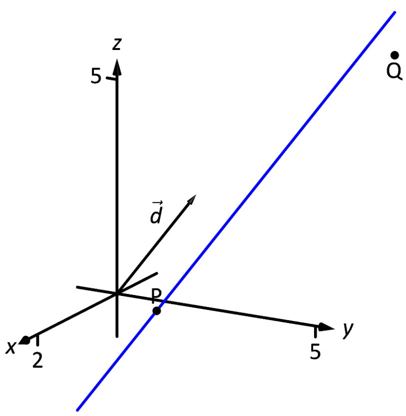



Figure 1.2.3: Graphing a line in Example 1.2.1.

**Example 1.2.1 Finding the equation of a line**

Give all three equations, as given in Definition 1.2.1, of the line through $P = (2, 3, 1)$ in the direction of $\vec{d} = \langle -1, 1, 2 \rangle$. Does the point $Q = (-1, 6, 6)$ lie on this line?

**Solution** We identify the point $P = (2, 3, 1)$ with the vector $\vec{p} = \langle 2, 3, 1 \rangle$. Following the definition, we have

- the vector equation of the line is $\vec{\ell}(t) = \langle 2, 3, 1 \rangle + t \langle -1, 1, 2 \rangle$;
- the parametric equations of the line are
$$x = 2 - t, \quad y = 3 + t, \quad z = 1 + 2t; \text{ and}$$
- the symmetric equations of the line are
$$\frac{x - 2}{-1} = \frac{y - 3}{1} = \frac{z - 1}{2}.$$

The first two equations of the line are useful when a $t$ value is given: one can immediately find the corresponding point on the line. These forms are good when calculating with a computer; most software programs easily handle equations in these formats. (For instance, the graphics program that made Figure 1.2.3 can be given the input "`(2-t,3+t,1+2*t)`" for $-1 \leq t \leq 3$.).

Does the point $Q = (-1, 6, 6)$ lie on the line? The graph in Figure 1.2.3 makes it clear that it does not. We can answer this question without the graph

Notes:

using any of the three equation forms. Of the three, the symmetric equations are probably best suited for this task. Simply plug in the values of $x$, $y$ and $z$ and see if equality is maintained:

$$\frac{-1-2}{-1} \stackrel{?}{=} \frac{6-3}{1} \stackrel{?}{=} \frac{6-1}{2} \quad \Rightarrow \quad 3 = 3 \neq 2.5.$$

We see that $Q$ does not lie on the line as it did not satisfy the symmetric equations.

**Example 1.2.2 Finding the equation of a line through two points**

Find the parametric equations of the line through the points $P = (2, -1, 2)$ and $Q = (1, 3, -1)$.

**Solution** Recall the statement made at the beginning of this section: to find the equation of a line, we need a point and a direction. We have *two* points; either one will suffice. The direction of the line can be found by the vector with initial point $P$ and terminal point $Q$: $\overrightarrow{PQ} = \langle -1, 4, -3\rangle$.

The parametric equations of the line $\ell$ through $P$ in the direction of $\overrightarrow{PQ}$ are:

$$\ell: \quad x = 2 - t \quad y = -1 + 4t \quad z = 2 - 3t.$$

A graph of the points and line are given in Figure 1.2.4. Note how in the given parametrization of the line, $t = 0$ corresponds to the point $P$, and $t = 1$ corresponds to the point $Q$. This relates to the understanding of the vector equation of a line described in Figure 1.2.1. The parametric equations "start" at the point $P$, and $t$ determines how far in the direction of $\overrightarrow{PQ}$ to travel. When $t = 0$, we travel 0 lengths of $\overrightarrow{PQ}$; when $t = 1$, we travel one length of $\overrightarrow{PQ}$, resulting in the point $Q$.

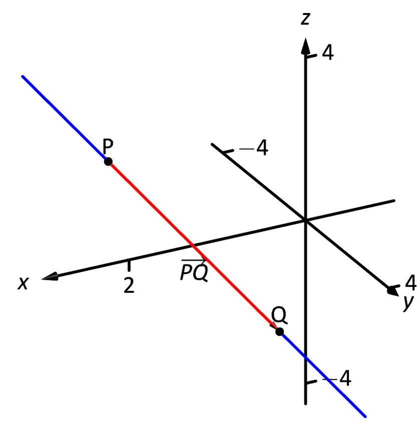



Figure 1.2.4: A graph of the line in Example 1.2.2.

## Parallel, Intersecting and Skew Lines

In the plane, two *distinct* lines can either be parallel or they will intersect at exactly one point. In space, given equations of two lines, it can sometimes be difficult to tell whether the lines are distinct or not (i.e., the same line can be represented in different ways). Given lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$, we have four possibilities: $\vec{\ell}_1$ and $\vec{\ell}_2$ are

| | |
|---|---|
| the same line | they share all points; |
| intersecting lines | share only 1 point; |
| parallel lines | $\vec{d}_1 \parallel \vec{d}_2$, no points in common; or |
| skew lines | $\vec{d}_1 \nparallel \vec{d}_2$, no points in common. |

Notes:

The next two examples investigate these possibilities.

**Example 1.2.3 Comparing lines**

Consider lines $\ell_1$ and $\ell_2$, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x & = & 1+3t \\ y & = & 2-t \\ z & = & t \end{array} \qquad \ell_2: \begin{array}{rcl} x & = & -2+4s \\ y & = & 3+s \\ z & = & 5+2s. \end{array}$$

Determine whether $\ell_1$ and $\ell_2$ are the same line, intersect, are parallel, or skew.

**Solution** We start by looking at the directions of each line. Line $\ell_1$ has the direction given by $\vec{d}_1 = \langle 3, -1, 1\rangle$ and line $\ell_2$ has the direction given by $\vec{d}_2 = \langle 4, 1, 2\rangle$. It should be clear that $\vec{d}_1$ and $\vec{d}_2$ are not parallel, hence $\ell_1$ and $\ell_2$ are not the same line, nor are they parallel. Figure 1.2.5 verifies this fact (where the points and directions indicated by the equations of each line are identified).

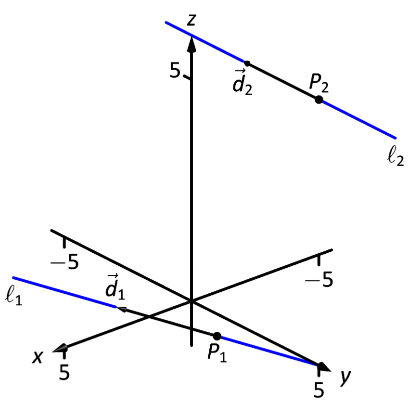



Figure 1.2.5: Sketching the lines from Example 1.2.3.

We next check to see if they intersect (if they do not, they are skew lines). To find if they intersect, we look for $t$ and $s$ values such that the respective $x$, $y$ and $z$ values are the same. That is, we want $s$ and $t$ such that:

$$\begin{array}{rcl} 1+3t & = & -2+4s \\ 2-t & = & 3+s \\ t & = & 5+2s. \end{array}$$

This is a relatively simple system of linear equations. Since the last equation is already solved for $t$, substitute that value of $t$ into the equation above it:

$$2-(5+2s) = 3+s \quad \Rightarrow \quad s=-2,\ t=1.$$

A key to remember is that we have *three* equations; we need to check if $s = -2,\ t = 1$ satisfies the first equation as well:

$$1+3(1) \neq -2+4(-2).$$

It does not. Therefore, we conclude that the lines $\ell_1$ and $\ell_2$ are skew.

**Example 1.2.4 Comparing lines**

Consider lines $\ell_1$ and $\ell_2$, given in parametric equation form:

$$\ell_1: \begin{array}{rcl} x & = & -0.7+1.6t \\ y & = & 4.2+2.72t \\ z & = & 2.3-3.36t \end{array} \qquad \ell_2: \begin{array}{rcl} x & = & 2.8-2.9s \\ y & = & 10.15-4.93s \\ z & = & -5.05+6.09s. \end{array}$$

Determine whether $\ell_1$ and $\ell_2$ are the same line, intersect, are parallel, or skew.

Notes:

**Solution** It is obviously very difficult to simply look at these equations and discern anything. This is done intentionally. In the "real world," most equations that are used do not have nice, integer coefficients. Rather, there are lots of digits after the decimal and the equations can look "messy."

We again start by deciding whether or not each line has the same direction. The direction of $\ell_1$ is given by $\vec{d}_1 = \langle 1.6, 2.72, -3.36\rangle$ and the direction of $\ell_2$ is given by $\vec{d}_2 = \langle -2.9, -4.93, 6.09\rangle$. When it is not clear through observation whether two vectors are parallel or not, the standard way of determining this is by comparing their respective unit vectors. Using a calculator, we find:

$$\vec{u}_1 = \frac{\vec{d}_1}{||\,\vec{d}_1\,||} = \langle 0.3471, 0.5901, -0.7289\rangle$$
$$\vec{u}_2 = \frac{\vec{d}_2}{||\,\vec{d}_2\,||} = \langle -0.3471, -0.5901, 0.7289\rangle\,.$$

The two vectors seem to be parallel (at least, their components are equal to 4 decimal places). In most situations, it would suffice to conclude that the lines are at least parallel, if not the same. One way to be sure is to rewrite $\vec{d}_1$ and $\vec{d}_2$ in terms of fractions, not decimals. We have

$$\vec{d}_1 = \left\langle \frac{16}{10}, \frac{272}{100}, -\frac{336}{100}\right\rangle \qquad \vec{d}_2 = \left\langle -\frac{29}{10}, -\frac{493}{100}, \frac{609}{100}\right\rangle.$$

One can then find the magnitudes of each vector in terms of fractions, then compute the unit vectors likewise. After a lot of manual arithmetic (or after briefly using a computer algebra system), one finds that

$$\vec{u}_1 = \left\langle \sqrt{\frac{10}{83}}, \frac{17}{\sqrt{830}}, -\frac{21}{\sqrt{830}}\right\rangle \qquad \vec{u}_2 = \left\langle -\sqrt{\frac{10}{83}}, -\frac{17}{\sqrt{830}}, \frac{21}{\sqrt{830}}\right\rangle.$$

We can now say without equivocation that these lines are parallel.

Are they the same line? The parametric equations for a line describe one point that lies on the line, so we know that the point $P_1 = (-0.7, 4.2, 2.3)$ lies on $\ell_1$. To determine if this point also lies on $\ell_2$, plug in the $x$, $y$ and $z$ values of $P_1$ into the symmetric equations for $\ell_2$:

$$\frac{(-0.7)-2.8}{-2.9} \stackrel{?}{=} \frac{(4.2)-10.15}{-4.93} \stackrel{?}{=} \frac{(2.3)-(-5.05)}{6.09} \quad\Rightarrow\quad 1.2069 = 1.2069 = 1.2069.$$

The point $P_1$ lies on both lines, so we conclude they are the same line, just parametrized differently. Figure 1.2.6 graphs this line along with the points and vectors described by the parametric equations. Note how $\vec{d}_1$ and $\vec{d}_2$ are parallel, though point in opposite directions (as indicated by their unit vectors above).

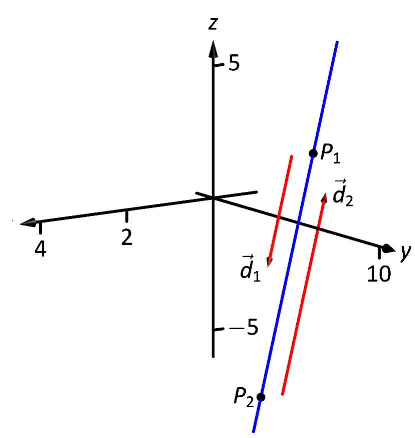


Figure 1.2.6: Graphing the lines in Example 1.2.4.

Notes:

## Distances

Given a point $Q$ and a line $\vec{\ell}(t) = \vec{p} + t\vec{d}$ in space, it is often useful to know the distance from the point to the line. (Here we use the standard definition of "distance," i.e., the length of the shortest line segment from the point to the line.) Identifying $\vec{p}$ with the point $P$, Figure 1.2.7 will help establish a general method of computing this distance $h$.

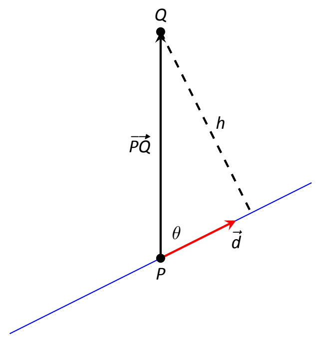


Figure 1.2.7: Establishing the distance from a point to a line.

From trigonometry, we know $h = \|\,\vec{PQ}\,\| \sin\theta$. We have a similar identity involving the cross product: $\|\,\vec{PQ} \times \vec{d}\,\| = \|\,\vec{PQ}\,\|\,\|\,\vec{d}\,\| \sin\theta$. Divide both sides of this latter equation by $\|\,\vec{d}\,\|$ to obtain $h$:

$$h = \frac{\|\,\vec{PQ} \times \vec{d}\,\|}{\|\,\vec{d}\,\|}. \tag{1.2}$$

It is also useful to determine the distance between lines, which we define as the length of the shortest line segment that connects the two lines (an argument from geometry shows that this line segments is perpendicular to both lines). Let lines $\vec{\ell}_1(t) = \vec{p}_1 + t\vec{d}_1$ and $\vec{\ell}_2(t) = \vec{p}_2 + t\vec{d}_2$ be given, as shown in Figure 1.2.8. To find the direction orthogonal to both $\vec{d}_1$ and $\vec{d}_2$, we take the cross product: $\vec{c} = \vec{d}_1 \times \vec{d}_2$. The magnitude of the orthogonal projection of $\vec{P_1P_2}$ onto $\vec{c}$ is the distance $h$ we seek:

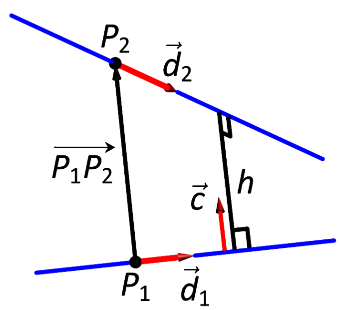


Figure 1.2.8: Establishing the distance between lines.

$$\begin{aligned} h &= \|\,\text{proj}_{\vec{c}}\,\vec{P_1P_2}\,\| \\ &= \left\|\,\frac{\vec{P_1P_2}\cdot\vec{c}}{\vec{c}\cdot\vec{c}}\vec{c}\,\right\| \\ &= \frac{|\vec{P_1P_2}\cdot\vec{c}|}{\|\,\vec{c}\,\|^2}\|\,\vec{c}\,\| \\ &= \frac{|\vec{P_1P_2}\cdot\vec{c}|}{\|\,\vec{c}\,\|}. \end{aligned}$$

A problem in the Exercise section is to show that this distance is 0 when the lines intersect. Note the use of the Triple Scalar Product: $\vec{P_1P_2}\cdot\vec{c} = \vec{P_1P_2}\cdot(\vec{d}_1 \times \vec{d}_2)$.

The following Key Idea restates these two distance formulas.

Notes:

**Key Idea 1.2.1 Distances to Lines**

1. Let $P$ be a point on a line $\ell$ that is parallel to $\vec{d}$. The distance $h$ from a point $Q$ to the line $\ell$ is:
$$h = \frac{\|\overrightarrow{PQ} \times \vec{d}\,\|}{\|\,\vec{d}\,\|}.$$

2. Let $P_1$ be a point on line $\ell_1$ that is parallel to $\vec{d_1}$, and let $P_2$ be a point on line $\ell_2$ parallel to $\vec{d_2}$, and let $\vec{c} = \vec{d_1} \times \vec{d_2}$, where lines $\ell_1$ and $\ell_2$ are not parallel. The distance $h$ between the two lines is:
$$h = \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{\|\,\vec{c}\,\|}.$$

**Example 1.2.5 Finding the distance from a point to a line**

Find the distance from the point $Q = (1, 1, 3)$ to the line $\vec{\ell}(t) = \langle 1, -1, 1\rangle + t\,\langle 2, 3, 1\rangle$ .

**Solution** The equation of the line gives us the point $P = (1, -1, 1)$ that lies on the line, hence $\overrightarrow{PQ} = \langle 0, 2, 2\rangle$. The equation also gives $\vec{d} = \langle 2, 3, 1\rangle$. Following Key Idea 1.2.1, we have the distance as

$$\begin{aligned} h &= \frac{\|\overrightarrow{PQ} \times \vec{d}\,\|}{\|\,\vec{d}\,\|} \\ &= \frac{\|\ \langle -4, 4, -4\rangle\ \|}{\sqrt{14}} \\ &= \frac{4\sqrt{3}}{\sqrt{14}} \approx 1.852. \end{aligned}$$

The point $Q$ is approximately 1.852 units from the line $\vec{\ell}(t)$.

**Example 1.2.6 Finding the distance between lines**

Find the distance between the lines

$$\ell_1: \begin{array}{rcl} x &=& 1+3t \\ y &=& 2-t \\ z &=& t \end{array} \qquad \ell_2: \begin{array}{rcl} x &=& -2+4s \\ y &=& 3+s \\ z &=& 5+2s. \end{array}$$

**Solution** These are the sames lines as given in Example 1.2.3, where

Notes:

we showed them to be skew. The equations allow us to identify the following points and vectors:

$$P_1 = (1, 2, 0) \quad P_2 = (-2, 3, 5) \quad \Rightarrow \quad \overrightarrow{P_1P_2} = \langle -3, 1, 5 \rangle .$$

$$\vec{d}_1 = \langle 3, -1, 1 \rangle \quad \vec{d}_2 = \langle 4, 1, 2 \rangle \quad \Rightarrow \quad \vec{c} = \vec{d}_1 \times \vec{d}_2 = \langle -3, -2, 7 \rangle .$$

From Key Idea 1.2.1 we have the distance $h$ between the two lines is

$$\begin{aligned} h &= \frac{|\overrightarrow{P_1P_2} \cdot \vec{c}|}{||\,\vec{c}\,||} \\ &= \frac{42}{\sqrt{62}} \approx 5.334. \end{aligned}$$

The lines are approximately 5.334 units apart.

One of the key points to understand from this section is this: to describe a line, we need a point and a direction. Whenever a problem is posed concerning a line, one needs to take whatever information is offered and glean point and direction information. Many questions can be asked (and *are* asked in the Exercise section) whose answer immediately follows from this understanding.

Lines are one of two fundamental objects of study in space. The other fundamental object is the *plane*, which we study in detail in the next section. Many complex three dimensional objects are studied by approximating their surfaces with lines and planes.

Notes:

# Exercises 1.2

## *Terms and Concepts*

10 05 ex 01 1. To find an equation of a line, what two pieces of information are needed?

10 05 ex 02 2. Two distinct lines in the plane can intersect or be ________.

10 05 ex 03 3. Two distinct lines in space can intersect, be ________ or be ________.

10 05 ex 04 4. Use your own words to describe what it means for two lines in space to be skew.

## *Problems*

10 05 exset 01 **In Exercises 5 – 14, write the vector, parametric and symmetric equations of the lines described.**

10 05 ex 05 5. Passes through $P = (2, -4, 1)$, parallel to $\vec{d} = \langle 9, 2, 5\rangle$.

10 05 ex 06 6. Passes through $P = (6, 1, 7)$, parallel to $\vec{d} = \langle -3, 2, 5\rangle$.

10 05 ex 07 7. Passes through $P = (2, 1, 5)$ and $Q = (7, -2, 4)$.

10 05 ex 08 8. Passes through $P = (1, -2, 3)$ and $Q = (5, 5, 5)$.

10 05 ex 09 9. Passes through $P = (0, 1, 2)$ and orthogonal to both $\vec{d}_1 = \langle 2, -1, 7\rangle$ and $\vec{d}_2 = \langle 7, 1, 3\rangle$.

10 05 ex 10 10. Passes through $P = (5, 1, 9)$ and orthogonal to both $\vec{d}_1 = \langle 1, 0, 1\rangle$ and $\vec{d}_2 = \langle 2, 0, 3\rangle$.

10 05 ex 11 11. Passes through the point of intersection of $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ and orthogonal to both lines, where $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\langle 5, 1, -2\rangle$ and $\vec{\ell}_2(t) = \langle -2, -1, 2\rangle + t\langle 3, 1, -1\rangle$.

10 05 ex 12 12. Passes through the point of intersection of $\ell_1(t)$ and $\ell_2(t)$ and orthogonal to both lines, where

$$\ell_1 = \begin{cases} x = t \\ y = -2 + 2t \\ z = 1 + t \end{cases} \quad \text{and} \quad \ell_2 = \begin{cases} x = 2 + t \\ y = 2 - t \\ z = 3 + 2t \end{cases}.$$

10 05 ex 13 13. Passes through $P = (1, 1)$, parallel to $\vec{d} = \langle 2, 3\rangle$.

10 05 ex 14 14. Passes through $P = (-2, 5)$, parallel to $\vec{d} = \langle 0, 1\rangle$.

10 05 exset 02 **In Exercises 15 – 22, determine if the described lines are the same line, parallel lines, intersecting or skew lines. If intersecting, give the point of intersection.**

10 05 ex 15 15. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\langle 2, -1, 1\rangle$, $\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\langle -4, 2, -2\rangle$.

10 05 ex 16 16. $\vec{\ell}_1(t) = \langle 2, 1, 1\rangle + t\langle 5, 1, 3\rangle$, $\vec{\ell}_2(t) = \langle 14, 5, 9\rangle + t\langle 1, 1, 1\rangle$.

10 05 ex 17 17. $\vec{\ell}_1(t) = \langle 3, 4, 1\rangle + t\langle 2, -3, 4\rangle$, $\vec{\ell}_2(t) = \langle -3, 3, -3\rangle + t\langle 3, -2, 4\rangle$.

10 05 ex 18 18. $\vec{\ell}_1(t) = \langle 1, 1, 1\rangle + t\langle 3, 1, 3\rangle$, $\vec{\ell}_2(t) = \langle 7, 3, 7\rangle + t\langle 6, 2, 6\rangle$.

10 05 ex 19 19. $\ell_1 = \begin{cases} x = 1 + 2t \\ y = 3 - 2t \\ z = t \end{cases}$ and $\ell_2 = \begin{cases} x = 3 - t \\ y = 3 + 5t \\ z = 2 + 7t \end{cases}$

10 05 ex 20 20. $\ell_1 = \begin{cases} x = 1.1 + 0.6t \\ y = 3.77 + 0.9t \\ z = -2.3 + 1.5t \end{cases}$ and $\ell_2 = \begin{cases} x = 3.11 + 3.4t \\ y = 2 + 5.1t \\ z = 2.5 + 8.5t \end{cases}$

10 05 ex 21 21. $\ell_1 = \begin{cases} x = 0.2 + 0.6t \\ y = 1.33 - 0.45t \\ z = -4.2 + 1.05t \end{cases}$ and $\ell_2 = \begin{cases} x = 0.86 + 9.2t \\ y = 0.835 - 6.9t \\ z = -3.045 + 16.1t \end{cases}$

10 05 ex 22 22. $\ell_1 = \begin{cases} x = 0.1 + 1.1t \\ y = 2.9 - 1.5t \\ z = 3.2 + 1.6t \end{cases}$ and $\ell_2 = \begin{cases} x = 4 - 2.1t \\ y = 1.8 + 7.2t \\ z = 3.1 + 1.1t \end{cases}$

10 05 exset 03 **In Exercises 23 – 26, find the distance from the point to the line.**

10 05 ex 23 23. $P = (1, 1, 1)$, $\quad \vec{\ell}(t) = \langle 2, 1, 3\rangle + t\langle 2, 1, -2\rangle$

10 05 ex 24 24. $P = (2, 5, 6)$, $\quad \vec{\ell}(t) = \langle -1, 1, 1\rangle + t\langle 1, 0, 1\rangle$

10 05 ex 25 25. $P = (0, 3)$, $\quad \vec{\ell}(t) = \langle 2, 0\rangle + t\langle 1, 1\rangle$

10 05 ex 26 26. $P = (1, 1)$, $\quad \vec{\ell}(t) = \langle 4, 5\rangle + t\langle -4, 3\rangle$

10 05 exset 04 **In Exercises 27 – 28, find the distance between the two lines.**

10 05 ex 27 27. $\vec{\ell}_1(t) = \langle 1, 2, 1\rangle + t\langle 2, -1, 1\rangle$, $\vec{\ell}_2(t) = \langle 3, 3, 3\rangle + t\langle 4, 2, -2\rangle$.

10 05 ex 28 28. $\vec{\ell}_1(t) = \langle 0, 0, 1\rangle + t\langle 1, 0, 0\rangle$, $\vec{\ell}_2(t) = \langle 0, 0, 3\rangle + t\langle 0, 1, 0\rangle$.

10 05 exset 05 **Exercises 29 – 31 explore special cases of the distance formulas found in Key Idea 1.2.1.**

10 05 ex 29 29. Let $Q$ be a point on the line $\vec{\ell}(t)$. Show why the distance formula correctly gives the distance from the point to the line as 0.

10 05 ex 30 30. Let lines $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ be intersecting lines. Show why the distance formula correctly gives the distance between these lines as 0.

10 05 ex 31

31. Let lines $\vec{\ell}_1(t)$ and $\vec{\ell}_2(t)$ be parallel.

   (a) Show why the distance formula for distance between lines cannot be used as stated to find the distance between the lines.

   (b) Show why letting $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ allows one to use the formula.

   (c) Show how one can use the formula for the distance between a point and a line to find the distance between parallel lines.

## 1.3 Extreme Values

Given a function $z = f(x, y)$, we are often interested in points where $z$ takes on the largest or smallest values. For instance, if $z$ represents a cost function, we would likely want to know what $(x, y)$ values minimize the cost. If $z$ represents the ratio of a volume to surface area, we would likely want to know where $z$ is greatest. This leads to the following definition.

**Definition 1.3.1 Relative and Absolute Extrema**

Let $z = f(x, y)$ be defined on a set $S$ containing the point $P = (x_0, y_0)$.

1. If there is an open disk $D$ containing $P$ such that $f(x_0, y_0) \geq f(x, y)$ for all $(x, y)$ in $D$, then $f$ has a **relative maximum** at $P$; if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y)$ in $D$, then $f$ has a **relative minimum** at $P$.

2. If $f(x_0, y_0) \geq f(x, y)$ for all $(x, y)$ in $S$, then $f$ has an **absolute maximum** at $P$; if $f(x_0, y_0) \leq f(x, y)$ for all $(x, y)$ in $S$, then $f$ has an **absolute minimum** at $P$.

3. If $f$ has a relative maximum or minimum at $P$, then $f$ has a **relative extrema** at $P$; if $f$ has an absolute maximum or minimum at $P$, then $f$ has an **absolute extrema** at $P$.

If $f$ has a relative or absolute maximum at $P = (x_0, y_0)$, it means every curve on the surface of $f$ through $P$ will also have a relative or absolute maximum at $P$. Recalling what we learned in Section 3.1, the slopes of the tangent lines to these curves at $P$ must be 0 or undefined. Since directional derivatives are computed using $f_x$ and $f_y$, we are led to the following definition and theorem.

**Definition 1.3.2 Critical Point**

Let $z = f(x, y)$ be continuous on an open set $S$. A **critical point** $P = (x_0, y_0)$ of $f$ is a point in $S$ such that

- $f_x(x_0, y_0) = 0$ and $f_y(x_0, y_0) = 0$, or
- $f_x(x_0, y_0)$ and/or $f_y(x_0, y_0)$ is undefined.

Notes:

**Theorem 1.3.1 Critical Points and Relative Extrema**

Let $z = f(x, y)$ be defined on an open set $S$ containing $P = (x_0, y_0)$. If $f$ has a relative extrema at $P$, then $P$ is a critical point of $f$.

Therefore, to find relative extrema, we find the critical points of $f$ and determine which correspond to relative maxima, relative minima, or neither. The following examples demonstrate this process.

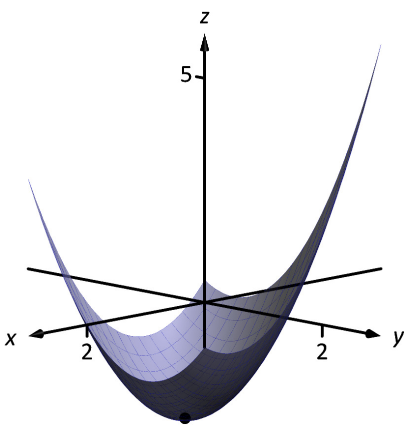



Figure 1.3.1: The surface in Example 1.3.1 with its absolute minimum indicated.

**Example 1.3.1 Finding critical points and relative extrema**

Let $f(x, y) = x^2 + y^2 - xy - x - 2$. Find the relative extrema of $f$.

**Solution** We start by computing the partial derivatives of $f$:

$$f_x(x, y) = 2x - y - 1 \qquad \text{and} \qquad f_y(x, y) = 2y - x.$$

Each is never undefined. A critical point occurs when $f_x$ and $f_y$ are simultaneously 0, leading us to solve the following system of linear equations:

$$2x - y - 1 = 0 \qquad \text{and} \qquad -x + 2y = 0.$$

This solution to this system is $x = 2/3$, $y = 1/3$. (Check that at $(2/3, 1/3)$, both $f_x$ and $f_y$ are 0.)

The graph in Figure 1.3.1 shows $f$ along with this critical point. It is clear from the graph that this is a relative minimum; further consideration of the function shows that this is actually the absolute minimum.

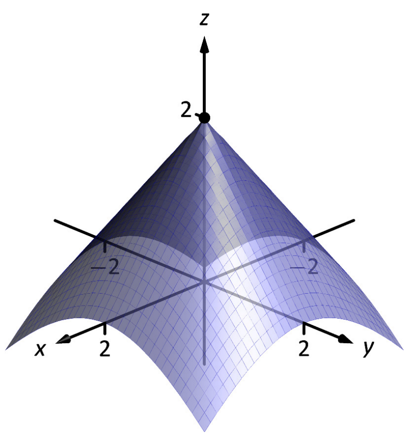



Figure 1.3.2: The surface in Example 1.3.2 with its absolute maximum indicated.

**Example 1.3.2 Finding critical points and relative extrema**

Let $f(x, y) = -\sqrt{x^2 + y^2} + 2$. Find the relative extrema of $f$.

**Solution** We start by computing the partial derivatives of $f$:

$$f_x(x, y) = \frac{-x}{\sqrt{x^2 + y^2}} \qquad \text{and} \qquad f_y(x, y) = \frac{-y}{\sqrt{x^2 + y^2}}.$$

It is clear that $f_x = 0$ when $x = 0$ & $y \neq 0$, and that $f_y = 0$ when $y = 0$ & $x \neq 0$. At $(0, 0)$, both $f_x$ and $f_y$ are *not* 0, but rather undefined. The point $(0, 0)$ is still a critical point, though, because the partial derivatives are undefined. This is the only critical point of $f$.

The surface of $f$ is graphed in Figure 1.3.2 along with the point $(0, 0, 2)$. The graph shows that this point is the absolute maximum of $f$.

Notes:

In each of the previous two examples, we found a critical point of $f$ and then determined whether or not it was a relative (or absolute) maximum or minimum by graphing. It would be nice to be able to determine whether a critical point corresponded to a max or a min without a graph. Before we develop such a test, we do one more example that sheds more light on the issues our test needs to consider.

**Example 1.3.3 Finding critical points and relative extrema**

Let $f(x,y) = x^3 - 3x - y^2 + 4y$. Find the relative extrema of $f$.

**Solution** Once again we start by finding the partial derivatives of $f$:

$$f_x(x,y) = 3x^2 - 3 \qquad \text{and} \qquad f_y(x,y) = -2y + 4.$$

Each is always defined. Setting each equal to 0 and solving for $x$ and $y$, we find

$$\begin{aligned} f_x(x,y) &= 0 \quad \Rightarrow x = \pm 1 \\ f_y(x,y) &= 0 \quad \Rightarrow y = 2. \end{aligned}$$

We have two critical points: $(-1,2)$ and $(1,2)$. To determine if they correspond to a relative maximum or minimum, we consider the graph of $f$ in Figure 1.3.3.

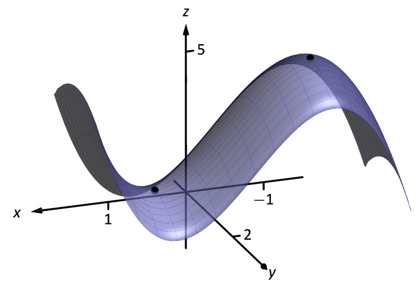



Figure 1.3.3: The surface in Example 1.3.3 with both critical points marked.

The critical point $(-1,2)$ clearly corresponds to a relative maximum. However, the critical point at $(1,2)$ is neither a maximum nor a minimum, displaying a different, interesting characteristic.

If one walks parallel to the $y$-axis towards this critical point, then this point becomes a relative maximum along this path. But if one walks towards this point parallel to the $x$-axis, this point becomes a relative minimum along this path. A point that seems to act as both a max and a min is a **saddle point**. A formal definition follows.

**Definition 1.3.3 Saddle Point**

Let $P = (x_0, y_0)$ be in the domain of $f$ where $f_x = 0$ and $f_y = 0$ at $P$. $P$ is a **saddle point** of $f$ if, for every open disk $D$ containing $P$, there are points $(x_1, y_1)$ and $(x_2, y_2)$ in $D$ such that $f(x_0, y_0) > f(x_1, y_1)$ and $f(x_0, y_0) < f(x_2, y_2)$.

At a saddle point, the instantaneous rate of change in all directions is 0 and there are points nearby with $z$-values both less than and greater than the $z$-value of the saddle point.

Notes:

Before Example 1.3.3 we mentioned the need for a test to differentiate between relative maxima and minima. We now recognize that our test also needs to account for saddle points. To do so, we consider the second partial derivatives of $f$.

Recall that with single variable functions, such as $y = f(x)$, if $f''(c) > 0$, then if $f$ is concave up at $c$, and if $f'(c) = 0$, then $f$ has a relative minimum at $x = c$. (We called this the Second Derivative Test.) Note that at a saddle point, it seems the graph is "both" concave up and concave down, depending on which direction you are considering.

It would be nice if the following were true:

$$\begin{array}{rcc} f_{xx} \text{ and } f_{yy} > 0 & \Rightarrow & \text{relative minimum} \\ f_{xx} \text{ and } f_{yy} < 0 & \Rightarrow & \text{relative maximum} \\ f_{xx} \text{ and } f_{yy} \text{ have opposite signs} & \Rightarrow & \text{saddle point.} \end{array}$$

However, this is not the case. Functions $f$ exist where $f_{xx}$ and $f_{yy}$ are both positive but a saddle point still exists. In such a case, while the concavity in the $x$-direction is up (i.e., $f_{xx} > 0$) and the concavity in the $y$-direction is also up (i.e., $f_{yy} > 0$), the concavity switches somewhere in between the $x$- and $y$-directions.

To account for this, consider $D = f_{xx}f_{yy} - f_{xy}f_{yx}$. Since $f_{xy}$ and $f_{yx}$ are equal when continuous (refer back to Theorem 12.3.1), we can rewrite this as $D = f_{xx}f_{yy} - f_{xy}^2$. $D$ can be used to test whether the concavity at a point changes depending on direction. If $D > 0$, the concavity does not switch (i.e., at that point, the graph is concave up or down in all directions). If $D < 0$, the concavity does switch. If $D = 0$, our test fails to determine whether concavity switches or not. We state the use of $D$ in the following theorem.

**Theorem 1.3.2 Second Derivative Test**

Let $R$ be an open set on which a function $z = f(x, y)$ and all its first and second partial derivatives are defined, let $P = (x_0, y_0)$ be a critical point of $f$ in $R$, and let

$$D = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0).$$

1. If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f$ has a relative minimum at $P$.
2. If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f$ has a relative maximum at $P$.
3. If $D < 0$, then $f$ has a saddle point at $P$.
4. If $D = 0$, the test is inconclusive.

Notes:

We first practice using this test with the function in the previous example, where we visually determined we had a relative maximum and a saddle point.

**Example 1.3.4 Using the Second Derivative Test**

Let $f(x,y) = x^3 - 3x - y^2 + 4y$ as in Example 1.3.3. Determine whether the function has a relative minimum, maximum, or saddle point at each critical point.

**SOLUTION** We determined previously that the critical points of $f$ are $(-1,2)$ and $(1,2)$. To use the Second Derivative Test, we must find the second partial derivatives of $f$:

$$f_{xx} = 6x; \qquad f_{yy} = -2; \qquad f_{xy} = 0.$$

Thus $D(x,y) = -12x$.

At $(-1,2)$: $D(-1,2) = 12 > 0$, and $f_{xx}(-1,2) = -6$. By the Second Derivative Test, $f$ has a relative maximum at $(-1,2)$.

At $(1,2)$: $D(1,2) = -12 < 0$. The Second Derivative Test states that $f$ has a saddle point at $(1,2)$.

The Second Derivative Test confirmed what we determined visually.

**Example 1.3.5 Using the Second Derivative Test**

Find the relative extrema of $f(x,y) = x^2y + y^2 + xy$.

**SOLUTION** We start by finding the first and second partial derivatives of $f$:

$$\begin{array}{cc} f_x = 2xy + y & f_y = x^2 + 2y + x \\ f_{xx} = 2y & f_{yy} = 2 \\ f_{xy} = 2x + 1 & f_{yx} = 2x + 1. \end{array}$$

We find the critical points by finding where $f_x$ and $f_y$ are simultaneously 0 (they are both never undefined). Setting $f_x = 0$, we have:

$$f_x = 0 \quad \Rightarrow \quad 2xy + y = 0 \quad \Rightarrow \quad y(2x+1) = 0.$$

This implies that for $f_x = 0$, either $y = 0$ or $2x + 1 = 0$.

Assume $y = 0$ then consider $f_y = 0$:

$$\begin{aligned} f_y &= 0 \\ x^2 + 2y + x &= 0, \qquad \text{and since } y = 0, \text{ we have} \\ x^2 + x &= 0 \\ x(x+1) &= 0. \end{aligned}$$

Thus if $y = 0$, we have either $x = 0$ or $x = -1$, giving two critical points: $(-1,0)$ and $(0,0)$.

Notes:

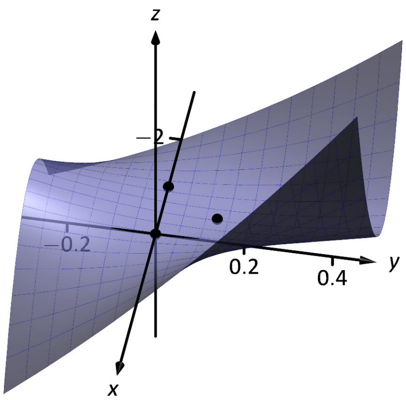



Figure 1.3.4: Graphing $f$ from Example 1.3.5 and its relative extrema.

Going back to $f_x$, now assume $2x+1=0$, i.e., that $x=-1/2$, then consider $f_y=0$:

$$\begin{aligned} f_y &= 0 \\ x^2+2y+x &= 0, \qquad \text{and since } x=-1/2, \text{ we have} \\ 1/4+2y-1/2 &= 0 \\ y &= 1/8. \end{aligned}$$

Thus if $x=-1/2$, $y=1/8$ giving the critical point $(-1/2,1/8)$.

With $D=4y-(2x+1)^2$, we apply the Second Derivative Test to each critical point.

At $(-1,0)$, $D<0$, so $(-1,0)$ is a saddle point.

At $(0,0)$, $D<0$, so $(0,0)$ is also a saddle point.

At $(-1/2,1/8)$, $D>0$ and $f_{xx}>0$, so $(-1/2,1/8)$ is a relative minimum.

Figure 1.3.4 shows a graph of $f$ and the three critical points. Note how this function does not vary much near the critical points – that is, visually it is difficult to determine whether a point is a saddle point or relative minimum (or even a critical point at all!). This is one reason why the Second Derivative Test is so important to have.

## Constrained Optimization

When optimizing functions of one variable such as $y=f(x)$, we made use of Theorem 3.1.1, the Extreme Value Theorem, that said that over a closed interval $I$, a continuous function has both a maximum and minimum value. To find these maximum and minimum values, we evaluated $f$ at all critical points in the interval, as well as at the endpoints (the "boundary") of the interval.

A similar theorem and procedure applies to functions of two variables. A continuous function over a closed set also attains a maximum and minimum value (see the following theorem). We can find these values by evaluating the function at the critical values in the set and over the boundary of the set. After formally stating this extreme value theorem, we give examples.

**Theorem 1.3.3 Extreme Value Theorem**

Let $z=f(x,y)$ be a continuous function on a closed, bounded set $S$. Then $f$ has a maximum and minimum value on $S$.

**Example 1.3.6 Finding extrema on a closed set**

Let $f(x,y)=x^2-y^2+5$ and let $S$ be the triangle with vertices $(-1,-2)$, $(0,1)$ and $(2,-2)$. Find the maximum and minimum values of $f$ on $S$.

Notes:

**Solution** It can help to see a graph of $f$ along with the set $S$. In Figure 1.3.5(a) the triangle defining $S$ is shown in the $x$-$y$ plane in a dashed line. Above it is the surface of $f$; we are only concerned with the portion of $f$ enclosed by the "triangle" on its surface.

We begin by finding the critical points of $f$. With $f_x = 2x$ and $f_y = -2y$, we find only one critical point, at $(0, 0)$.

We now find the maximum and minimum values that $f$ attains along the boundary of $S$, that is, along the edges of the triangle. In Figure 1.3.5(b) we see the triangle sketched in the plane with the equations of the lines forming its edges labeled.

Start with the bottom edge, along the line $y = -2$. If $y$ is $-2$, then on the surface, we are considering points $f(x, -2)$; that is, our function reduces to $f(x, -2) = x^2 - (-2)^2 + 5 = x^2 + 1 = f_1(x)$. We want to maximize/minimize $f_1(x) = x^2 + 1$ on the interval $[-1, 2]$. To do so, we evaluate $f_1(x)$ at its critical points and at the endpoints.

The critical points of $f_1$ are found by setting its derivative equal to 0:

$$f_1'(x) = 0 \qquad \Rightarrow x = 0.$$

Evaluating $f_1$ at this critical point, and at the endpoints of $[-1, 2]$ gives:

$$\begin{aligned} f_1(-1) = 2 &\qquad \Rightarrow \qquad f(-1, -2) = 2 \\ f_1(0) = 1 &\qquad \Rightarrow \qquad f(0, -2) = 1 \\ f_1(2) = 5 &\qquad \Rightarrow \qquad f(2, -2) = 5. \end{aligned}$$

Notice how evaluating $f_1$ at a point is the same as evaluating $f$ at its corresponding point.

We need to do this process twice more, for the other two edges of the triangle.

Along the left edge, along the line $y = 3x + 1$, we substitute $3x + 1$ in for $y$ in $f(x, y)$:

$$f(x, y) = f(x, 3x + 1) = x^2 - (3x + 1)^2 + 5 = -8x^2 - 6x + 4 = f_2(x).$$

We want the maximum and minimum values of $f_2$ on the interval $[-1, 0]$, so we evaluate $f_2$ at its critical points and the endpoints of the interval. We find the critical points:

$$f_2'(x) = -16x - 6 = 0 \qquad \Rightarrow \qquad x = -3/8.$$

Evaluate $f_2$ at its critical point and the endpoints of $[-1, 0]$:

$$\begin{aligned} f_2(-1) = 2 &\qquad \Rightarrow \qquad f(-1, -2) = 2 \\ f_2(-3/8) = 41/8 = 5.125 &\qquad \Rightarrow \qquad f(-3/8, -0.125) = 5.125 \\ f_2(0) = 4 &\qquad \Rightarrow \qquad f(0, 1) = 4. \end{aligned}$$

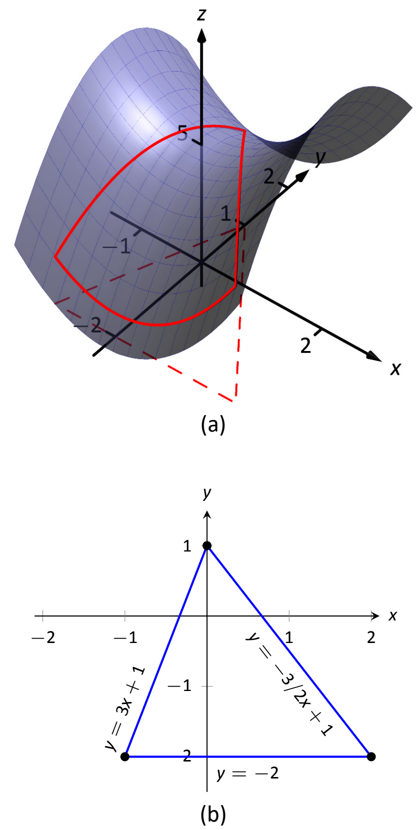


Figure 1.3.5: Plotting the surface of $f$ along with the restricted domain $S$ in Example 1.3.6.

Notes:

Finally, we evaluate $f$ along the right edge of the triangle, where $y = -3/2x + 1$.

$$f(x, y) = f(x, -3/2x + 1) = x^2 - (-3/2x + 1)^2 + 5 = -\frac{5}{4}x^2 + 3x + 4 = f_3(x).$$

The critical points of $f_3(x)$ are:

$$f_3'(x) = 0 \qquad \Rightarrow \qquad x = 6/5 = 1.2.$$

We evaluate $f_3$ at this critical point and at the endpoints of the interval $[0, 2]$:

$$\begin{aligned} f_3(0) &= 4 & \Rightarrow \quad & f(0,1) = 4 \\ f_3(1.2) &= 5.8 & \Rightarrow \quad & f(1.2, -0.8) = 5.8 \\ f_3(2) &= 5 & \Rightarrow \quad & f(2, -2) = 5. \end{aligned}$$

One last point to test: the critical point of $f$, $(0, 0)$. We find $f(0, 0) = 5$.

We have evaluated $f$ at a total of 7 different places, all shown in Figure 1.3.6. We checked each vertex of the triangle twice, as each showed up as the end-point of an interval twice. Of all the $z$-values found, the maximum is 5.8, found at $(1.2, -0.8)$; the minimum is 1, found at $(0, -2)$.

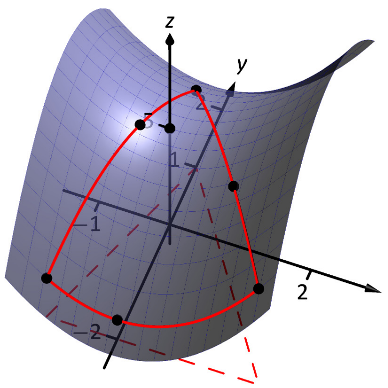



Figure 1.3.6: The surface of $f$ along with important points along the boundary of $S$ and the interior in Example 1.3.6.

This portion of the text is entitled "Constrained Optimization" because we want to optimize a function (i.e., find its maximum and/or minimum values) subject to a *constraint* – some limit to what values the function can attain. In the previous example, we constrained ourselves by considering a function only within the boundary of a triangle. This was largely arbitrary; the function and the boundary were chosen just as an example, with no real "meaning" behind the function or the chosen constraint.

However, solving constrained optimization problems is a very important topic in applied mathematics. The techniques developed here are the basis for solving larger problems, where more than two variables are involved.

We illustrate the technique once more with a classic problem.

**Example 1.3.7 Constrained Optimization**

The U.S. Postal Service states that the girth+length of Standard Post Package must not exceed 130". Given a rectangular box, the "length" is the longest side, and the "girth" is twice the width+height.

Given a rectangular box where the width and height are equal, what are the dimensions of the box that give the maximum volume subject to the constraint of the size of a Standard Post Package?

**Solution** Let $w$, $h$ and $\ell$ denote the width, height and length of a rectangular box; we assume here that $w = h$. The girth is then $2(w + h) = 4w$. The

Notes:

volume of the box is $V(w,\ell) = wh\ell = w^2\ell$. We wish to maximize this volume subject to the constraint $4w + \ell \leq 130$, or $\ell \leq 130 - 4w$. (Common sense also indicates that $\ell > 0, w > 0$.)

We begin by finding the critical values of $V$. We find that $V_w = 2w\ell$ and $V_\ell = w^2$; these are simultaneously 0 only at $(0,0)$. This gives a volume of 0, so we can ignore this critical point.

We now consider the volume along the constraint $\ell = 130 - 4w$. Along this line, we have:

$$V(w,\ell) = V(w, 130-4w) = w^2(130-4w) = 130w^2 - 4w^3 = V_1(w).$$

The constraint is applicable on the $w$-interval $[0, 32.5]$ as indicated in the figure. Thus we want to maximize $V_1$ on $[0, 32.5]$.

Finding the critical values of $V_1$, we take the derivative and set it equal to 0:

$$V_1'(w) = 260w - 12w^2 = 0 \quad\Rightarrow\quad w(260-12w) = 0 \quad\Rightarrow\quad w = 0, \frac{260}{12} \approx 21.67.$$

We found two critical values: when $w = 0$ and when $w = 21.67$. We again ignore the $w = 0$ solution; the maximum volume, subject to the constraint, comes at $w = h = 21.67$, $\ell = 130 - 4(21.6) = 43.33$. This gives a volume of $V(21.67, 43.33) \approx 19{,}408\text{in}^3$.

The volume function $V(w,\ell)$ is shown in Figure 1.3.7 along with the constraint $\ell = 130 - 4w$. As done previously, the constraint is drawn dashed in the $x$-$y$ plane and also along the surface of the function. The point where the volume is maximized is indicated.

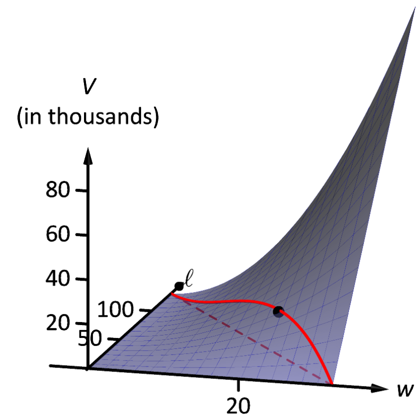



Figure 1.3.7: Graphing the volume of a box with girth $4w$ and length $\ell$, subject to a size constraint.

It is hard to overemphasize the importance of optimization. In "the real world," we routinely seek to make *something* better. By expressing the *something* as a mathematical function, "making *something* better" means "optimize *some function.*"

The techniques shown here are only the beginning of an incredibly important field. Many functions that we seek to optimize are incredibly complex, making the step of "find the gradient and set it equal to $\vec{0}$" highly nontrivial. Mastery of the principles here are key to being able to tackle these more complicated problems.

Notes:

# Exercises 1.3

## *Terms and Concepts*

12 07 ex 01 1. T/F: Theorem 1.3.1 states that if $f$ has a critical point at $P$, then $f$ has a relative extrema at $P$.

12 07 ex 02 2. T/F: A point $P$ is a critical point of $f$ if $f_x$ and $f_y$ are both 0 at $P$.

12 07 ex 03 3. T/F: A point $P$ is a critical point of $f$ if $f_x$ or $f_y$ are undefined at $P$.

12 07 ex 04 4. Explain what it means to "solve a constrained optimization" problem.

## *Problems*

12 07 exset 01 **In Exercises 5 – 14, find the critical points of the given function. Use the Second Derivative Test to determine if each critical point corresponds to a relative maximum, minimum, or saddle point.**

12 07 ex 07 5. $f(x, y) = \frac{1}{2}x^2 + 2y^2 - 8y + 4x$

12 07 ex 05 6. $f(x, y) = x^2 + 4x + y^2 - 9y + 3xy$

12 07 ex 06 7. $f(x, y) = x^2 + 3y^2 - 6y + 4xy$

12 07 ex 08 8. $f(x, y) = \dfrac{1}{x^2 + y^2 + 1}$

12 07 ex 09 9. $f(x, y) = x^2 + y^3 - 3y + 1$

12 07 ex 12 10. $f(x, y) = \dfrac{1}{3}x^3 - x + \dfrac{1}{3}y^3 - 4y$

12 07 ex 11 11. $f(x, y) = x^2y^2$

12 07 ex 10 12. $f(x, y) = x^4 - 2x^2 + y^3 - 27y - 15$

12 07 ex 17 13. $f(x, y) = \sqrt{16 - (x - 3)^2 - y^2}$

12 07 ex 18 14. $f(x, y) = \sqrt{x^2 + y^2}$

12 07 exset 02 **In Exercises 15 – 18, find the absolute maximum and minimum of the function subject to the given constraint.**

12 07 ex 13 15. $f(x, y) = x^2 + y^2 + y + 1$, constrained to the triangle with vertices $(0, 1)$, $(-1, -1)$ and $(1, -1)$.

12 07 ex 14 16. $f(x, y) = 5x - 7y$, constrained to the region bounded by $y = x^2$ and $y = 1$.

12 07 ex 15 17. $f(x, y) = x^2 + 2x + y^2 + 2y$, constrained to the region bounded by the circle $x^2 + y^2 = 4$.

12 07 ex 16 18. $f(x, y) = 3y - 2x^2$, constrained to the region bounded by the parabola $y = x^2 + x - 1$ and the line $y = x$.

# A: Solutions To Selected Problems

## Chapter 1

### Section 1.1

10 01 ex 08 1. right hand

10 01 ex 01 2. line; plane

10 01 ex 02 3. curve (a parabola); surface (a cylinder)

10 01 ex 03 4. a hyperbolic paraboloid

10 01 ex 04 5. a hyperboloid of two sheets

10 01 ex 05 6. a hyperboloid of one sheet

10 01 ex 06 7. $\| \overline{AB} \| = \sqrt{6}$; $\| \overline{BC} \| = \sqrt{17}$; $\| \overline{AC} \| = \sqrt{11}$. Yes, it is a right triangle as $\| \overline{AB} \|^2 + \| \overline{AC} \|^2 = \| \overline{BC} \|^2$.

10 01 ex 07 8. Yes, as opposite sides have equal length. $\| \overline{AB} \| = \sqrt{21} = \| \overline{CD} \|$; $\| \overline{BC} \| = \sqrt{6} = \| \overline{AD} \|$.

10 01 ex 09 9. Center at $(4, -1, 0)$; radius = 3

10 01 ex 10 10. Center at $(-2, 1, 2)$; radius = $\sqrt{5}$

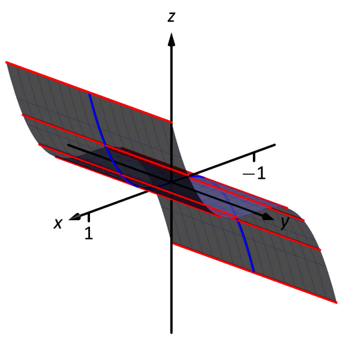
10 01 ex 11 11. Interior of a sphere with radius 1 centered at the origin.

10 01 ex 12 12. Region bounded between the planes $x = 0$ (the $y - z$ coordinate plane) and $x = 3$.

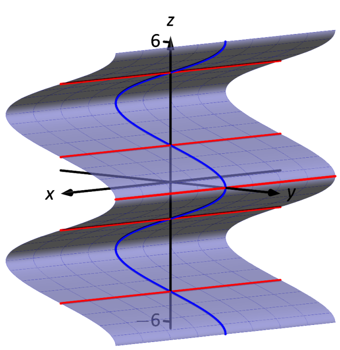
10 01 ex 13 13. The first octant of space; all points $(x, y, z)$ where each of $x$, $y$ and $z$ are non-negative. (Analogous to the first quadrant in the plane.)

10 01 ex 14 14. All points in space where the $y$ value is greater than 3; viewing space as often depicted in this text, this is the region "to the right" of the plane $y = 3$ (which is parallel to the $x - z$ coordinate plane.)

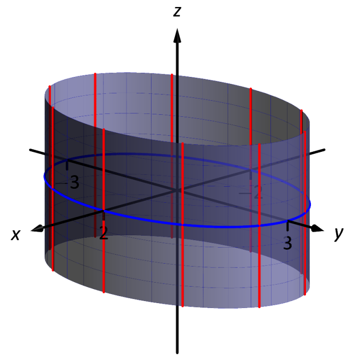
10 01 ex 15 15.



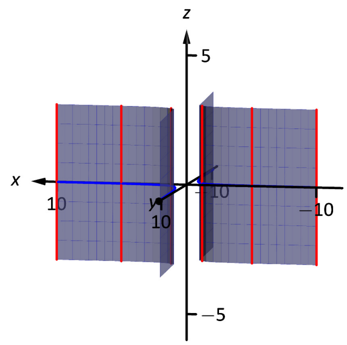

10 01 ex 16 16.




10 01 ex 17 17.




10 01 ex 18 18.




10 01 ex 29 19. $x^2 + z^2 = \frac{1}{(1+y^2)^2}$

10 01 ex 30 20. $y^2 + z^2 = x^4$

10 01 ex 31 21. $z = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$

10 01 ex 32 22. $z = \frac{1}{\sqrt{x^2+y^2}}$

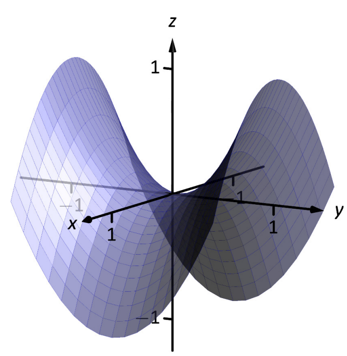
10 01 ex 19 23. (a) $x = y^2 + \dfrac{z^2}{9}$

10 01 ex 20 24. (b) $x^2 - y^2 + z^2 = 0$

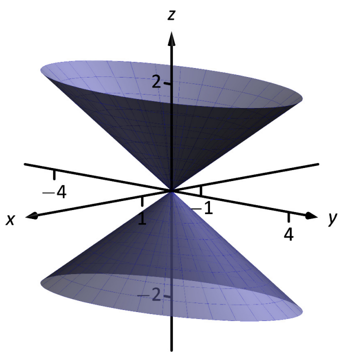
10 01 ex 21 25. (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

10 01 ex 22 26. (a) $y^2 - x^2 - z^2 = 1$

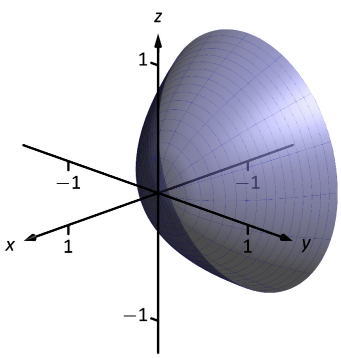
10 01 ex 28 27.



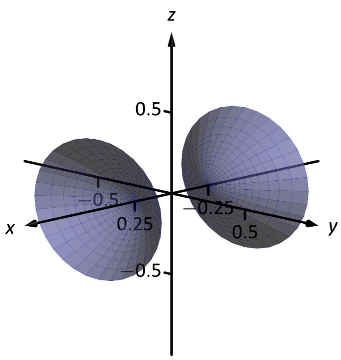
10 01 ex 24 28.



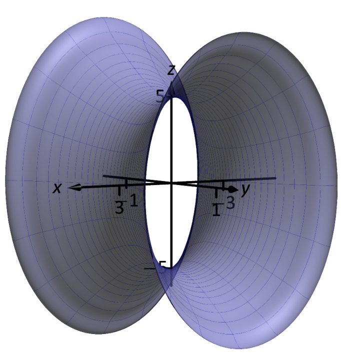

10 01 ex 23 29.



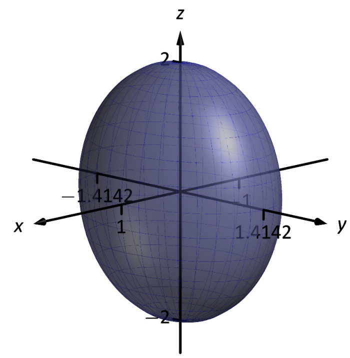

10 01 ex 27 30.




10 01 ex 26 31.




10 01 ex 25 32.




**Section 1.2**

10 05 ex 01 1. A point on the line and the direction of the line.

10 05 ex 02 2. parallel

10 05 ex 03 3. parallel, skew

10 05 ex 04 4. Answers will vary

10 05 ex 05 5. vector: $\ell(t) = \langle 2, -4, 1\rangle + t\,\langle 9, 2, 5\rangle$
parametric: $x = 2 + 9t$, $y = -4 + 2t$, $z = 1 + 5t$
symmetric: $(x-2)/9 = (y+4)/2 = (z-1)/5$

10 05 ex 06 6. vector: $\ell(t) = \langle 6, 1, 7\rangle + t\,\langle -3, 2, 5\rangle$
parametric: $x = 6 - 3t$, $y = 1 + 2t$, $z = 7 + 5t$
symmetric: $-(x-6)/3 = (y-1)/2 = (z-7)/5$

10 05 ex 07 7. Answers can vary: vector: $\ell(t) = \langle 2, 1, 5\rangle + t\,\langle 5, -3, -1\rangle$
parametric: $x = 2 + 5t$, $y = 1 - 3t$, $z = 5 - t$
symmetric: $(x-2)/5 = -(y-1)/3 = -(z-5)$

10 05 ex 08 8. Answers can vary: vector: $\ell(t) = \langle 1, -2, 3\rangle + t\,\langle 4, 7, 2\rangle$
parametric: $x = 1 + 4t$, $y = -2 + 7t$, $z = 3 + 2t$
symmetric: $(x-1)/4 = (y+2)/7 = (z-3)/2$

10 05 ex 09 9. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 0, 1, 2\rangle + t\,\langle -10, 43, 9\rangle$
parametric: $x = -10t$, $y = 1 + 43t$, $z = 2 + 9t$
symmetric: $-x/10 = (y-1)/43 = (z-2)/9$

10 05 ex 10 10. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 5, 1, 9\rangle + t\,\langle 0, -1, 0\rangle$
parametric: $x = 5$, $y = 1 - t$, $z = 9$
symmetric: not defined, as some components of the direction are 0.

10 05 ex 11 11. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 7, 2, -1\rangle + t\,\langle 1, -1, 2\rangle$
parametric: $x = 7 + t$, $y = 2 - t$, $z = -1 + 2t$
symmetric: $x - 7 = 2 - y = (z+1)/2$

10 05 ex 12 12. Answers can vary; here the direction is given by $\vec{d}_1 \times \vec{d}_2$:
vector: $\ell(t) = \langle 2, 2, 3\rangle + t\,\langle 5, -1, -3\rangle$
parametric: $x = 2 + 5t$, $y = 2 - t$, $z = 3 - 3t$
symmetric: $(x-2)/5 = -(y-2) = -(z-3)/3$

10 05 ex 13 13. vector: $\ell(t) = \langle 1, 1\rangle + t\,\langle 2, 3\rangle$
parametric: $x = 1 + 2t$, $y = 1 + 3t$
symmetric: $(x-1)/2 = (y-1)/3$

10 05 ex 14 14. vector: $\ell(t) = \langle -2, 5\rangle + t\,\langle 0, 1\rangle$
parametric: $x = -2$, $y = 5 + t$
symmetric: not defined

10 05 ex 15 15. parallel

10 05 ex 16 16. intersecting; $\ell_1(2) = \ell_2(-2) = \langle 12, 3, 7\rangle$

10 05 ex 17 17. intersecting; $\vec{\ell}_1(3) = \vec{\ell}_2(4) = \langle 9, -5, 13\rangle$

10 05 ex 18 18. same

10 05 ex 19 19. skew

10 05 ex 20 20. parallel

10 05 ex 21 21. same

10 05 ex 22 22. skew

10 05 ex 23 23. $\sqrt{41}/3$

10 05 ex 24 24. $3\sqrt{2}$

10 05 ex 25 25. $5\sqrt{2}/2$

10 05 ex 26 26. 5

10 05 ex 27 27. $3/\sqrt{2}$

10 05 ex 28 28. 2

10 05 ex 29 29. Since both $P$ and $Q$ are on the line, $\overrightarrow{PQ}$ is parallel to $\vec{d}$. Thus $\overrightarrow{PQ} \times \vec{d} = \vec{0}$, giving a distance of 0.

10 05 ex 30 30. (Note: this solution is easier once one has studied Section 10.6.) Since the two lines intersect, we can state $P_2 = P_1 + a\vec{d}_1 + b\vec{d}_2$ for some scalars $a$ and $b$. (Here we abuse notation slightly and add points to vectors.) Thus $\overrightarrow{P_1P_2} = a\vec{d}_1 + b\vec{d}_2$. Vector $\vec{c}$ is the cross product of $\vec{d}_1$ and $\vec{d}_2$, hence is orthogonal to both, and hence is orthogonal to $\overrightarrow{P_1P_2}$. Thus $\overrightarrow{P_1P_2} \cdot \vec{c} = 0$, and the distance between lines is 0.

10 05 ex 31 31.

(a) The distance formula cannot be used because since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\vec{c}$ is $\vec{0}$ and we cannot divide by $||\,\vec{0}\,||$.

(b) Since $\vec{d}_1$ and $\vec{d}_2$ are parallel, $\overrightarrow{P_1P_2}$ lies in the plane formed by the two lines. Thus $\overrightarrow{P_1P_2} \times \vec{d}_2$ is orthogonal to this plane, and $\vec{c} = (\overrightarrow{P_1P_2} \times \vec{d}_2) \times \vec{d}_2$ is parallel to the plane, but still orthogonal to both $\vec{d}_1$ and $\vec{d}_2$. We desire the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{c}$, which is what the formula provides.

(c) Since the lines are parallel, one can measure the distance between the lines at any location on either line (just as to find the distance between straight railroad tracks, one can use a measuring tape anywhere along the track, not just at one specific place.) Let $P = P_1$ and $Q = P_2$ as given by the equations of the lines, and apply the formula for distance between a point and a line.

**Section 1.3**

12 07 ex 01 1. F; it is the "other way around."

12 07 ex 02 2. T

12 07 ex 03 3. T

12 07 ex 04 4. Answers will vary. A good answer will state that we are optimizing a function subject to a constraint, or limit, on the domain of the function. We are looking to maximize/minimize the function while "looking" at only a certain part of the domain.

12 07 ex 07 5. One critical point at $(-4, 2)$; $f_{xx} = 1$ and $D = 4$, so this point corresponds to a relative minimum.

12 07 ex 05 6. One critical point at $(7, -6)$; $D = -5$, so this point corresponds to a saddle point.

12 07 ex 06 7. One critical point at $(6, -3)$; $D = -4$, so this point corresponds to a saddle point.

12 07 ex 08 8. One critical point at $(0, 0)$; $f_{xx} = -2$ and $D = 4$, so this point corresponds to a relative maximum.

12 07 ex 09 9. Two critical points: at $(0, -1)$; $f_{xx} = 2$ and $D = -12$, so this point corresponds to a saddle point;
at $(0, 1)$, $f_{xx} = 2$ and $D = 12$, so this corresponds to a relative minimum.

12 07 ex 12 10. There are 4 critical points:
$(-1, -2)$, rel. max; $(1, -2)$, saddle point;
$(-1, 2)$, saddle point; $(1, 2)$, rel. min.,
where $f_{xx} = 2x$ and $D = 4xy$.

12 07 ex 11 11. There are infinite critical points, whenever $x = 0$ or $y = 0$. With $D = -12x^2y^2$, at each critical point $D = 0$ and the test is inconclusive. (Some elementary thought shows that each is an absolute minimum.)

12 07 ex 10 12. Six critical points: $f_x = 0$ when $x = -1, 0$ and $1$; $f_y = 0$ when $y = -3, 3$. Together, we get the points:
$(-1, -3)$ saddle point; $(-1, 3)$ rel. min
$(0, -3)$ rel. max; $(0, 3)$ saddle point
$(1, -3)$ saddle point; $(1, 3)$ relative min
where $f_{xx} = 12x^2 - 4$ and $D = 24y(3x^2 - 1)$.

12 07 ex 17 13. One critical point: $f_x = 0$ when $x = 3$; $f_y = 0$ when $y = 0$, so one critical point at $(3, 0)$, which is a relative maximum, where $f_{xx} = \frac{y^2-16}{(16-(x-3)^2-y^2)^{3/2}}$ and $D = \frac{16}{(16-(x-3)^2-y^2)^2}$.
Both $f_x$ and $f_y$ are undefined along the circle $(x-3)^2 + y^2 = 16$; at any point along this curve, $f(x, y) = 0$, the absolute minimum of the function.

12 07 ex 18 14. One critical point: $f_x = 0$ when $x = 0$; $f_y = 0$ when $y = 0$, so one critical point at $(0, 0)$ (although it should be noted that at $(0, 0)$, both $f_x$ and $f_y$ are undefined.) The Second Derivative Test fails at $(0, 0)$, with $D = 0$. A graph, or simple calculation, shows that $(0, 0)$ is the absolute minimum of $f$.

12 07 ex 13 15. The triangle is bound by the lines $y = -1$, $y = 2x + 1$ and $y = -2x + 1$.
Along $y = -1$, there is a critical point at $(0, -1)$.
Along $y = 2x + 1$, there is a critical point at $(-3/5, -1/5)$.
Along $y = -2x + 1$, there is a critical point at $(3/5, -1/5)$.
The function $f$ has one critical point, irrespective of the constraint, at $(0, -1/2)$.
Checking the value of $f$ at these four points, along with the three vertices of the triangle, we find the absolute maximum is at $(0, 1, 3)$ and the absolute minimum is at $(0, -1/2, 3/4)$.

12 07 ex 14 16. The region has two "corners" at $(1, 1)$ and $(-1, 1)$.
Along $y = 1$, there is no critical point.
Along $y = x^2$, there is a critical point at $(5/14, 25/196) \approx (0.357, 0.128)$.
The function $f$ itself has no critical points. Checking the value of $f$ at the corners $(1, 1)$, $(-1, 1)$ and the critical point $(5/14, 25/196)$, we find the absolute maximum is at

$(5/14, 25/196, 25/28) \approx (0.357, 0.128, 0.893)$ and the absolute minimum is at $(-1, 1, -12)$.

12 07 ex 15

17. The region has no "corners" or "vertices," just a smooth edge.
To find critical points along the circle $x^2 + y^2 = 4$, we solve for $y^2$: $y^2 = 4 - x^2$. We can go further and state $y = \pm\sqrt{4 - x^2}$.
We can rewrite $f$ as
$f(x) = x^2 + 2x + (4 - x^2) + 2\sqrt{4 - x^2} = 2x + 4 + 2\sqrt{4 - x^2}$.
(We will return and use $-\sqrt{4 - x^2}$ later.) Solving $f'(x) = 0$, we get $x = \sqrt{2} \Rightarrow y = \sqrt{2}$. $f'(x)$ is also undefined at $x = \pm 2$, where $y = 0$.
Using $y = -\sqrt{4 - x^2}$, we rewrite $f(x, y)$ as
$f(x) = 2x + 4 - 2\sqrt{4 - x^2}$. Solving $f'(x) = 0$, we get $x = -\sqrt{2},\ y = -\sqrt{2}$. Again, $f'(x)$ is undefined at $x = \pm 2$.
The function $z = f(x, y)$ itself has a critical point at $(-1, -1)$.
Checking the value of $f$ at $(-1, -1)$, $(\sqrt{2}, \sqrt{2})$, $(-\sqrt{2}, -\sqrt{2})$, $(2, 0)$ and $(-2, 0)$, we find the absolute maximum is at $(\sqrt{2}, \sqrt{2}, 4 + 4\sqrt{2})$ and the absolute minimum is at $(-1, -1, -2)$.

12 07 ex 16

18. The region has two "corners" at $(-1, -1)$ and $(1, 1)$.
Along the line $y = x$, $f(x, y)$ becomes $f(x) = 3x - 2x^2$.
Along this line, we have a critical point at $(3/4, 3/4)$.
Along the curve $y = x^2 + x - 1$, $f(x, y)$ becomes $f(x) = x^2 + 3x - 3$. There is a critical point along this curve at $(-3/2, -1/4)$. Since $x = -3/2$ lies outside our bounded region, we ignore this critical point.
The function $f$ itself has no critical points.
Checking the value of $f$ at $(-1, -1)$, $(1, 1)$, $(3/4, 3/4)$, we find the absolute maximum is at $(3/4, 3/4, 9/8)$ and the absolute minimum is at $(-1, -1, -5)$.