1: Functions of Several Variables

1.1 Differentiability and the Total Differential

We studied **differentials** in Section 4.4, where Definition 18 states that if $y = f(x)$ and f is differentiable, then $dy = f'(x)dx$. One important use of this differential is in Integration by Substitution. Another important application is approximation. Let $\Delta x = dx$ represent a change in x. When dx is small, $dy \approx \Delta y$, the change in y resulting from the change in x. Fundamental in this understanding is this: as dx gets small, the difference between Δy and dy goes to 0. Another way of stating this: as dx goes to 0, the *error* in approximating Δy with dy goes to 0.

We extend this idea to functions of two variables. Let $z = f(x, y)$, and let $\Delta x = dx$ and $\Delta y = dy$ represent changes in x and y, respectively. Let $\Delta z = f(x + dx, y + dy) - f(x, y)$ be the change in z over the change in x and y. Recalling that f_x and f_y give the instantaneous rates of z-change in the x- and y-directions, respectively, we can approximate Δz with $dz = f_x dx + f_y dy$; in words, the total change in z is approximately the change caused by changing x plus the change caused by changing y. In a moment we give an indication of whether or not this approximation is any good. First we give a name to dz.

Definition 1 Total Differential

Let $z = f(x, y)$ be continuous on an open set S. Let dx and dy represent changes in x and y, respectively. Where the partial derivatives f_x and f_y exist, the **total differential of** z is

$$dz = f_x(x, y)dx + f_y(x, y)dy.$$

Example 1 Finding the total differential

Let $z = x^4 e^{3y}$. Find dz.

Solution We compute the partial derivatives: $f_x = 4x^3 e^{3y}$ and $f_y = 3x^4 e^{3y}$. Following Definition 1, we have

$$dz = 4x^3 e^{3y} dx + 3x^4 e^{3y} dy.$$

We *can* approximate Δz with dz, but as with all approximations, there is error involved. A good approximation is one in which the error is small. At a given point (x_0, y_0), let E_x and E_y be functions of dx and dy such that $E_x dx + E_y dy$ describes this error. Then

$$\begin{aligned}\Delta z &= dz + E_x dx + E_y dy\\ &= f_x(x_0, y_0)dx + f_y(x_0, y_0)dy + E_x dx + E_y dy.\end{aligned}$$

If the approximation of Δz by dz is good, then as dx and dy get small, so does $E_x dx + E_y dy$. The approximation of Δz by dz is even better if, as dx and dy go to 0, so do E_x and E_y. This leads us to our definition of differentiability.

Definition 2 Multivariable Differentiability

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0) where $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist. Let dz be the total differential of z at (x_0, y_0), let $\Delta z = f(x_0 + dx, y_0 + dy) - f(x_0, y_0)$, and let E_x and E_y be functions of dx and dy such that

$$\Delta z = dz + E_x dx + E_y dy.$$

1. f is **differentiable at** (x_0, y_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $||\langle dx, dy\rangle|| < \delta$, then $||\langle E_x, E_y\rangle|| < \varepsilon$. That is, as dx and dy go to 0, so do E_x and E_y.
2. f is **differentiable on** S if f is differentiable at every point in S. If f is differentiable on $\mathbb{R}^2$, we say that f is **differentiable everywhere**.

Example 2 Showing a function is differentiable

Show $f(x, y) = xy + 3y^2$ is differentiable using Definition 2.

Solution We begin by finding $f(x + dx, y + dy)$, Δz, f_x and f_y.

$$\begin{aligned}f(x + dx, y + dy) &= (x + dx)(y + dy) + 3(y + dy)^2\\ &= xy + xdy + ydx + dxdy + 3y^2 + 6ydy + 3dy^2.\end{aligned}$$

$\Delta z = f(x + dx, y + dy) - f(x, y)$, so

$$\Delta z = xdy + ydx + dxdy + 6ydy + 3dy^2.$$

Notes:

It is straightforward to compute $f_x = y$ and $f_y = x + 6y$. Consider once more Δz:

$$\begin{aligned}\Delta z &= xdy + ydx + dxdy + 6ydy + 3dy^2 \qquad \text{(now reorder)}\\ &= ydx + xdy + 6ydy + dxdy + 3dy^2\\ &= \underbrace{(y)}_{f_x} dx + \underbrace{(x+6y)}_{f_y} dy + \underbrace{(dy)}_{E_x} dx + \underbrace{(3dy)}_{E_y} dy\\ &= f_x dx + f_y dy + E_x dx + E_y dy.\end{aligned}$$

With $E_x = dy$ and $E_y = 3dy$, it is clear that as dx and dy go to 0, E_x and E_y also go to 0. Since this did not depend on a specific point (x_0, y_0), we can say that $f(x, y)$ is differentiable for all pairs (x, y) in $\mathbb{R}^2$, or, equivalently, that f is differentiable everywhere.

Our intuitive understanding of differentiability of functions $y = f(x)$ of one variable was that the graph of f was "smooth." A similar intuitive understanding of functions $z = f(x, y)$ of two variables is that the surface defined by f is also "smooth," not containing cusps, edges, breaks, etc. The following theorem states that differentiable functions are continuous, followed by another theorem that provides a more tangible way of determining whether a great number of functions are differentiable or not.

Theorem 1 Continuity and Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Theorem 2 Differentiability of Multivariable Functions

Let $z = f(x, y)$ be defined on an open set S containing (x_0, y_0). If f_x and f_y are both continuous on S, then f is differentiable on S.

The theorems assure us that essentially all functions that we see in the course of our studies here are differentiable (and hence continuous) on their natural domains. There is a difference between Definition 2 and Theorem 2, though: it is possible for a function f to be differentiable yet f_x and/or f_y is *not* continuous. Such strange behavior of functions is a source of delight for many mathematicians.

Notes:

When f_x and f_y exist at a point but are not continuous at that point, we need to use other methods to determine whether or not f is differentiable at that point.

For instance, consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We can find $f_x(0,0)$ and $f_y(0,0)$ using Definition 83:

$$\begin{aligned} f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0; \\ f_y(0,0) &= \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{0}{h^2} = 0. \end{aligned}$$

Both f_x and f_y *exist* at $(0,0)$, but they are not continuous at $(0,0)$, as

$$f_x(x,y) = \frac{y(y^2-x^2)}{(x^2+y^2)^2} \qquad \text{and} \qquad f_y(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

are not continuous at $(0,0)$. (Take the limit of f_x as $(x,y) \to (0,0)$ along the x- and y-axes; they give different results.) So even though f_x and f_y *exist* at every point in the x-y plane, they are not continuous. Therefore it is possible, by Theorem 2, for f to not be differentiable.

Indeed, it is not. One can show that f is not continuous at $(0,0)$ (see Example 400), and by Theorem 1, this means f is not differentiable at $(0,0)$.

Approximating with the Total Differential

By the definition, when f is differentiable dz is a good approximation for Δz when dx and dy are small. We give some simple examples of how this is used here.

Example 3 **Approximating with the total differential**

Let $z = \sqrt{x}\sin y$. Approximate $f(4.1, 0.8)$.

Solution Recognizing that $\pi/4 \approx 0.785 \approx 0.8$, we can approximate $f(4.1, 0.8)$ using $f(4, \pi/4)$. We can easily compute $f(4,\pi/4) = \sqrt{4}\sin(\pi/4) = 2\left(\frac{\sqrt{2}}{2}\right) = \sqrt{2} \approx 1.414$. Without calculus, this is the best approximation we

Notes:

could reasonably come up with. The total differential gives us a way of adjusting this initial approximation to hopefully get a more accurate answer.

We let $\Delta z = f(4.1, 0.8) - f(4, \pi/4)$. The total differential dz is approximately equal to Δz, so

$$f(4.1, 0.8) - f(4, \pi/4) \approx dz \quad \Rightarrow \quad f(4.1, 0.8) \approx dz + f(4, \pi/4). \tag{1.1}$$

To find dz, we need f_x and f_y.

$$\begin{aligned} f_x(x, y) = \frac{\sin y}{2\sqrt{x}} \quad \Rightarrow \qquad f_x(4, \pi/4) &= \frac{\sin \pi/4}{2\sqrt{4}} \\ &= \frac{\sqrt{2}/2}{4} = \sqrt{2}/8. \\ f_y(x, y) = \sqrt{x}\cos y \quad \Rightarrow \qquad f_y(4, \pi/4) &= \sqrt{4}\frac{\sqrt{2}}{2} \\ &= \sqrt{2}. \end{aligned}$$

Approximating 4.1 with 4 gives $dx = 0.1$; approximating 0.8 with $\pi/4$ gives $dy \approx 0.015$. Thus

$$\begin{aligned} dz(4, \pi/4) &= f_x(4, \pi/4)(0.1) + f_y(4, \pi/4)(0.015) \\ &= \frac{\sqrt{2}}{8}(0.1) + \sqrt{2}(0.015) \\ &\approx 0.039. \end{aligned}$$

Returning to Equation (1.1), we have

$$f(4.1, 0.8) \approx 0.039 + 1.414 = 1.4531.$$

We, of course, can compute the actual value of $f(4.1, 0.8)$ with a calculator; the actual value, accurate to 5 places after the decimal, is 1.45254. Obviously our approximation is quite good.

The point of the previous example was *not* to develop an approximation method for known functions. After all, we can very easily compute $f(4.1, 0.8)$ using readily available technology. Rather, it serves to illustrate how well this method of approximation works, and to reinforce the following concept:

"New position = old position $+$ amount of change," so
"New position $\approx$ old position + approximate amount of change."

In the previous example, we could easily compute $f(4, \pi/4)$ and could approximate the amount of z-change when computing $f(4.1, 0.8)$, letting us approximate the new z-value.

Notes:

It may be surprising to learn that it is not uncommon to know the values of f, f_x and f_y at a particular point without actually knowing the function f. The total differential gives a good method of approximating f at nearby points.

Example 4 Approximating an unknown function
Given that $f(2,-3) = 6$, $f_x(2,-3) = 1.3$ and $f_y(2,-3) = -0.6$, approximate $f(2.1,-3.03)$.

Solution The total differential approximates how much f changes from the point $(2,-3)$ to the point $(2.1,-3.03)$. With $dx = 0.1$ and $dy = -0.03$, we have

$$\begin{aligned} dz &= f_x(2,-3)dx + f_y(2,-3)dy \\ &= 1.3(0.1) + (-0.6)(-0.03) \\ &= 0.148. \end{aligned}$$

The change in z is approximately 0.148, so we approximate $f(2.1,-3.03) \approx 6.148$.

Error/Sensitivity Analysis

The total differential gives an approximation of the change in z given small changes in x and y. We can use this to approximate error propagation; that is, if the input is a little off from what it should be, how far from correct will the output be? We demonstrate this in an example.

Example 5 Sensitivity analysis
A cylindrical steel storage tank is to be built that is 10ft tall and 4ft across in diameter. It is known that the steel will expand/contract with temperature changes; is the overall volume of the tank more sensitive to changes in the diameter or in the height of the tank?

Solution A cylindrical solid with height h and radius r has volume $V = \pi r^2 h$. We can view V as a function of two variables, r and h. We can compute partial derivatives of V:

$$\frac{\partial V}{\partial r} = V_r(r,h) = 2\pi rh \qquad \text{and} \qquad \frac{\partial V}{\partial h} = V_h(r,h) = \pi r^2.$$

The total differential is $dV = (2\pi rh)dr + (\pi r^2)dh$. When $h = 10$ and $r = 2$, we have $dV = 40\pi dr + 4\pi dh$. Note that the coefficient of dr is $40\pi \approx 125.7$; the coefficient of dh is a tenth of that, approximately 12.57. A small change in radius will be multiplied by 125.7, whereas a small change in height will be multiplied

Notes:

by 12.57. Thus the volume of the tank is more sensitive to changes in radius than in height.

The previous example showed that the volume of a particular tank was more sensitive to changes in radius than in height. Keep in mind that this analysis only applies to a tank of those dimensions. A tank with a height of 1ft and radius of 5ft would be more sensitive to changes in height than in radius.

One could make a chart of small changes in radius and height and find exact changes in volume given specific changes. While this provides exact numbers, it does not give as much insight as the error analysis using the total differential.

Differentiability of Functions of Three Variables

The definition of differentiability for functions of three variables is very similar to that of functions of two variables. We again start with the total differential.

Definition 3 Total Differential

Let $w = f(x, y, z)$ be continuous on an open set S. Let dx, dy and dz represent changes in x, y and z, respectively. Where the partial derivatives f_x, f_y and f_z exist, the **total differential of** w is

$$dz = f_x(x, y, z)dx + f_y(x, y, z)dy + f_z(x, y, z)dz.$$

This differential can be a good approximation of the change in w when $w = f(x, y, z)$ is **differentiable**.

Definition 4 Multivariable Differentiability

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0) where $f_x(x_0, y_0, z_0)$, $f_y(x_0, y_0, z_0)$ and $f_z(x_0, y_0, z_0)$ exist. Let dw be the total differential of w at (x_0, y_0, z_0), let $\Delta w = f(x_0 + dx, y_0 + dy, z_0 + dz) - f(x_0, y_0, z_0)$, and let E_x, E_y and E_z be functions of dx, dy and dz such that

$$\Delta w = dw + E_x dx + E_y dy + E_z dz.$$

1. f is **differentiable at** (x_0, y_0, z_0) if, given $\varepsilon > 0$, there is a $\delta > 0$ such that if $|| \langle dx, dy, dz \rangle || < \delta$, then $|| \langle E_x, E_y, E_z \rangle || < \varepsilon$.
2. f is **differentiable on** B if f is differentiable at every point in B. If f is differentiable on $\mathbb{R}^3$, we say that f is **differentiable everywhere**.

Notes:

Just as before, this definition gives a rigorous statement about what it means to be differentiable that is not very intuitive. We follow it with a theorem similar to Theorem 2.

Theorem 3 Continuity and Differentiability of Functions of Three Variables

Let $w = f(x, y, z)$ be defined on an open ball B containing (x_0, y_0, z_0).

1. If f is differentiable at (x_0, y_0, z_0), then f is continuous at (x_0, y_0, z_0).
2. If f_x, f_y and f_z are continuous on B, then f is differentiable on B.

This set of definition and theorem extends to functions of any number of variables. The theorem again gives us a simple way of verifying that most functions that we enounter are differentiable on their natural domains.

This section has given us a formal definition of what it means for a functions to be "differentiable," along with a theorem that gives a more accessible understanding. The following sections return to notions prompted by our study of partial derivatives that make use of the fact that most functions we encounter are differentiable.

Notes:

Exercises 1.1

Terms and Concepts

12 04 ex 01 1. T/F: If $f(x,y)$ is differentiable on S, the f is continuous on S.

12 04 ex 02 2. T/F: If f_x and f_y are continuous on S, then f is differentiable on S.

12 04 ex 03 3. T/F: If $z = f(x,y)$ is differentiable, then the change in z over small changes dx and dy in x and y is approximately dz.

12 04 ex 04 4. Finish the sentence: "The new z-value is approximately the old z-value plus the approximate ________."

Problems

12 04 exset 01 **In Exercises 5 – 8, find the total differential dz.**

12 04 ex 05 5. $z = x\sin y + x^2$

12 04 ex 06 6. $z = (2x^2 + 3y)^2$

12 04 ex 07 7. $z = 5x - 7y$

12 04 ex 08 8. $z = xe^{x+y}$

12 04 exset 02 **In Exercises 9 – 12, a function $z = f(x,y)$ is given. Give the indicated approximation using the total differential.**

12 04 ex 09 9. $f(x,y) = \sqrt{x^2 + y}$. Approximate $f(2.95, 7.1)$ knowing $f(3,7) = 4$.

12 04 ex 10 10. $f(x,y) = \sin x \cos y$. Approximate $f(0.1, -0.1)$ knowing $f(0,0) = 0$.

12 04 ex 11 11. $f(x,y) = x^2y - xy^2$. Approximate $f(2.04, 3.06)$ knowing $f(2,3) = -6$.

12 04 ex 12 12. $f(x,y) = \ln(x - y)$. Approximate $f(5.1, 3.98)$ knowing $f(5,4) = 0$.

12 04 exset 04 **Exercises 13 – 16 ask a variety of questions dealing with approximating error and sensitivity analysis.**

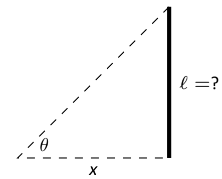
12 04 ex 15 13. A cylindrical storage tank is to be 2ft tall with a radius of 1ft. Is the volume of the tank more sensitive to changes in the radius or the height?

12 04 ex 16 14. **Projectile Motion:** The x-value of an object moving under the principles of projectile motion is $x(\theta, v_0, t) = (v_0\cos\theta)t$. A particular projectile is fired with an initial velocity of $v_0 = 250$ft/s and an angle of elevation of $\theta = 60^\circ$. It travels a distance of 375ft in 3 seconds.

Is the projectile more sensitive to errors in initial speed or angle of elevation?

12 04 ex 17 15. The length ℓ of a long wall is to be approximated. The angle θ, as shown in the diagram (not to scale), is measured to be 85°, and the distance x is measured to be 30'. Assume that the triangle formed is a right triangle.

Is the measurement of the length of ℓ more sensitive to errors in the measurement of x or in θ?



12 04 ex 18 16. It is "common sense" that it is far better to measure a long distance with a long measuring tape rather than a short one. A measured distance D can be viewed as the product of the length ℓ of a measuring tape times the number n of times it was used. For instance, using a 3' tape 10 times gives a length of 30'. To measure the same distance with a 12' tape, we would use the tape 2.5 times. (I.e., $30 = 12 \times 2.5$.) Thus $D = n\ell$.

Suppose each time a measurement is taken with the tape, the recorded distance is within 1/16" of the actual distance. (I.e., $d\ell = 1/16'' \approx 0.005$ft). Using differentials, show why common sense proves correct in that it is better to use a long tape to measure long distances.

12 04 exset 05 **In Exercises 17 – 18, find the total differential dw.**

12 04 ex 19 17. $w = x^2yz^3$

12 04 ex 20 18. $w = e^x \sin y \ln z$

12 04 exset 03 **In Exercises 19 – 22, use the information provided and the total differential to make the given approximation.**

12 04 ex 13 19. $f(3,1) = 7$, $f_x(3,1) = 9$, $f_y(3,1) = -2$. Approximate $f(3.05, 0.9)$.

12 04 ex 14 20. $f(-4,2) = 13$, $f_x(-4,2) = 2.6$, $f_y(-4,2) = 5.1$. Approximate $f(-4.12, 2.07)$.

12 04 ex 21 21. $f(2,4,5) = -1$, $f_x(2,4,5) = 2$, $f_y(2,4,5) = -3$, $f_z(2,4,5) = 3.7$. Approximate $f(2.5, 4.1, 4.8)$.

12 04 ex 22 22. $f(3,3,3) = 5$, $f_x(3,3,3) = 2$, $f_y(3,3,3) = 0$, $f_z(3,3,3) = -2$. Approximate $f(3.1, 3.1, 3.1)$.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

12 04 ex 01 1. T

12 04 ex 02 2. T

12 04 ex 03 3. T

12 04 ex 04 4. amount of change

12 04 ex 05 5. $dz = (\sin y + 2x)dx + (x \cos y)dy$

12 04 ex 06 6. $dz = 8x(2x^2 + 3y)dx + 6(2x^2 + 3y)dy$

12 04 ex 07 7. $dz = 5dx - 7dy$

12 04 ex 08 8. $dz = (e^{x+y} + xe^{x+y})dx + xe^{x+y}dy$

12 04 ex 09 9. $dz = \frac{x}{\sqrt{x^2+y}}dx + \frac{1}{2\sqrt{x^2+y}}dy$, with $dx = -0.05$ and $dy = .1$. At $(3, 7)$, $dz = 3/4(-0.05) + 1/8(.1) = -0.025$, so $f(2.95, 7.1) \approx -0.025 + 4 = 3.975$.

12 04 ex 10 10. $dz = (\cos x \cos y)dx - (\sin x \sin y)dy$, with $dx = 0.1$ and $dy = -0.1$. At $(0, 0)$, $dz = 1(.1) - (0)(-0.1) = 0.1$, so $f(0.1, -0.1) \approx 0.1 + 0 = 0.1$.

12 04 ex 11 11. $dz = (2xy - y^2)dx + (x^2 - 2xy)dy$, with $dx = 0.04$ and $dy = 0.06$. At $(2, 3)$, $dz = 3(0.04) + (-8)(0.06) = -0.36$, so $f(2.04, 3.06) \approx -0.36 - 6 = -6.36$.

12 04 ex 12 12. $dz = \frac{1}{x-y}dx - \frac{1}{x-y}dy$, with $dx = 0.1$ and $dy = -0.02$. At $(5, 4)$, $dz = 1(0.1) + (-1)(-0.02) = 0.12$, so $f(5.1, 3.98) \approx 0.12 + 0 = 0.12$.

12 04 ex 15 13. The total differential of volume is $dV = 4\pi dr + \pi dh$. The coefficient of dr is greater than the coefficient of dh, so the volume is more sensitive to changes in the radius.

12 04 ex 16 14. Distance of the projectile is a function of two variables (leaving $t = 3$): $D(v_0, \theta) = 3v_0 \cos\theta$. The total differential of D is $dD = 3\cos\theta dv_0 - 3v_0 \sin\theta d\theta$. The coefficient of $d\theta$ has a much greater magnitude than the coefficient of dv_0, so a small change in the angle of elevation has a much greater effect on distance traveled than a small change in initial velocity.

12 04 ex 17 15. Using trigonometry, $\ell = x\tan\theta$, so $d\ell = \tan\theta dx + x\sec^2\theta d\theta$. With $\theta = 85^\circ$ and $x = 30$, we have $d\ell = 11.43dx + 3949.38d\theta$. The measured length of the wall is much more sensitive to errors in θ than in x. While it can be difficult to compare sensitivities between measuring feet and measuring degrees (it is somewhat like "comparing apples to oranges"), here the coefficients are so different that the result is clear: a small error in degree has a much greater impact than a small error in distance.

12 04 ex 18 16. With $D = n\ell$, the total differential is $dD = \ell\, dn + n\, d\ell$. If one measures with a short tape, n must be large and hence $n\, d\ell$ is going to be greater than when a large tape is used (wherein n will be small).

12 04 ex 19 17. $dw = 2xyz^3\, dx + x^2z^3\, dy + 3x^2yz^2\, dz$

12 04 ex 20 18. $dw = e^x \sin y \ln z\, dx + e^x \cos y \ln z\, dy + e^x \sin y \frac{1}{z}\, dz$

12 04 ex 13 19. $dx = 0.05$, $dy = -0.1$. $dz = 9(.05) + (-2)(-0.1) = 0.65$. So $f(3.05, 0.9) \approx 7 + 0.65 = 7.65$.

12 04 ex 14 20. $dx = -0.12$, $dy = 0.07$. $dz = 2.6(-.12) + (5.1)(0.07) = 0.045$. So $f(-4.12, 2.07) \approx 13 + 0.045 = 13.045$.

12 04 ex 21 21. $dx = 0.5$, $dy = 0.1$, $dz = -0.2$. $dw = 2(0.5) + (-3)(0.1) + 3.7(-0.2) = -0.04$, so $f(2.5, 4.1, 4.8) \approx -1 - 0.04 = -1.04$.

12 04 ex 22 22. $dx = 0.1$, $dy = 0.1$, $dz = 0.1$. $dw = 2(0.1) + (0)(0.1) + (-2)(.1) = 0$, so $f(3.1, 3.1, 3.1) \approx 5 + 0 = 5$.