6: Techniques of Antidifferentiation

The previous chapter introduced the antiderivative and connected it to signed areas under a curve through the Fundamental Theorem of Calculus. The next chapter explores more applications of definite integrals than just area. As evaluating definite integrals will become important, we will want to find antiderivatives of a variety of functions.

This chapter is devoted to exploring techniques of antidifferentiation. While not every function has an antiderivative in terms of elementary functions (a concept introduced in the section on Numerical Integration), we can still find antiderivatives of a wide variety of functions.

6.1 Substitution

We motivate this section with an example. Let $f(x) = (x^2+3x-5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2+3x-5)^9 \cdot (2x+3) = (20x+30)(x^2+3x-5)^9.$$

Now consider this: What is $\int (20x+30)(x^2+3x-5)^9\ dx$? We have the answer in front of us;

$$\int (20x+30)(x^2+3x-5)^9\ dx = (x^2+3x-5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution.* It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x)\ dx$ as a not–so–complicated integral $\int h(u)\ du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x+30)(x^2+3x-5)^9\ dx$. Arguably the most "complicated" part of the integrand is $(x^2+3x-5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2+3x-5$. Thus

$$(x^2+3x-5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x + 3)dx.$$

Keep in mind that $(2x+3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}
\int (20x + 30)(x^2 + 3x - 5)^9\, dx &= \int 10(2x + 3)(x^2 + 3x - 5)^9\, dx \\
&= \int 10(\underbrace{x^2 + 3x - 5}_{u})^9\, \underbrace{(2x + 3)\, dx}_{du} \\
&= \int 10u^9\, du \\
&= u^{10} + C \quad \text{(replace } u \text{ with } x^2 + 3x - 5) \\
&= (x^2 + 3x - 5)^{10} + C
\end{aligned}$$

One might well look at this and think "I (sort of) followed how that worked, but I could never come up with that on my own," but the process is learnable. This section contains numerous examples through which the reader will gain understanding and mathematical maturity enabling them to regard substitution as a natural tool when evaluating integrals.

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x)\, dx = F(g(x)) + C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u = g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F\big(u\big)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x)\, dx = \int F'(u)du = F(u) + C = F(g(x)) + C.$$

This concept is important so we restate it in the context of a theorem.

Notes:

Theorem 6.1.1 Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\,dx = \int F'(u)\,du = F(u) + C = F(g(x)) + C.$$

The point of substitution is to make the integration step easy. Indeed, the step $\int F'(u)\,du = F(u) + C$ looks easy, as the antiderivative of the derivative of F is just F, plus a constant. The "work" involved is making the proper substitution. There is not a step–by–step process that one can memorize; rather, experience will be one's guide. To gain experience, we now embark on many examples.

Example 6.1.1 Integrating by substitution

Evaluate $\displaystyle\int x\sin(x^2+5)\,dx$.

Solution Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of $\sin(x^2+5)$. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = x^2+5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int x\sin(x^2+5)\,dx &= \int \sin(\underbrace{x^2+5}_{u})\,\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}\sin u\,du\end{aligned}$$

Notes:

$$= -\frac{1}{2}\cos u + C \qquad \text{(now replace } u \text{ with } x^2+5\text{)}$$
$$= -\frac{1}{2}\cos(x^2+5) + C.$$

Thus $\int x\sin(x^2+5)\,dx = -\frac{1}{2}\cos(x^2+5) + C$. We can check our work by evaluating the derivative of the right hand side.

Example 6.1.2 Integrating by substitution

Evaluate $\displaystyle\int \cos(5x)\,dx$.

Solution Again let u replace the "inside" function. Letting $u = 5x$, we have $du = 5dx$. Since our integrand does not have a $5dx$ term, we can divide the previous equation by 5 to obtain $\frac{1}{5}du = dx$. We can now substitute.

$$\int \cos(5x)\,dx = \int \cos(\underbrace{5x}_{u})\,\underbrace{dx}_{\frac{1}{5}du}$$
$$= \int \frac{1}{5}\cos u\,du$$
$$= \frac{1}{5}\sin u + C$$
$$= \frac{1}{5}\sin(5x) + C.$$

We can again check our work through differentiation.

The previous example exhibited a common, and simple, type of substitution. The "inside" function was a linear function (in this case, $y = 5x$). When the inside function is linear, the resulting integration is very predictable, outlined here.

Key Idea 6.1.1 Substitution With A Linear Function

Consider $\int F'(ax+b)\,dx$, where $a \neq 0$ and b are constants. Letting $u = ax+b$ gives $du = a\cdot dx$, leading to the result

$$\int F'(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

Thus $\int \sin(7x-4)\,dx = -\frac{1}{7}\cos(7x-4) + C$. Our next example can use Key Idea 6.1.1, but we will only employ it after going through all of the steps.

Notes:

Example 6.1.3 Integrating by substituting a linear function

Evaluate $\displaystyle\int \frac{7}{-3x+1}\,dx$.

Solution View the integrand as the composition of functions $f(g(x))$, where $f(x) = 7/x$ and $g(x) = -3x+1$. Employing our understanding of substitution, we let $u = -3x+1$, the inside function. Thus $du = -3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3 = dx$. We can now evaluate the integral through substitution.

$$\begin{aligned}\int \frac{7}{-3x+1}\,dx &= \int \frac{7}{u}\frac{du}{-3}\\ &= \frac{-7}{3}\int\frac{du}{u}\\ &= \frac{-7}{3}\ln|u| + C\\ &= -\frac{7}{3}\ln|-3x+1| + C.\end{aligned}$$

Using Key Idea 6.1.1 is faster, recognizing that u is linear and $a = -3$. One may want to continue writing out all the steps until they are comfortable with this particular shortcut.

Not all integrals that benefit from substitution have a clear "inside" function. Several of the following examples will demonstrate ways in which this occurs.

Example 6.1.4 Integrating by substitution

Evaluate $\displaystyle\int \sin x\cos x\,dx$.

Solution There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* …" and see if this helps simplify the integral at all.

In this example, let's set $u = \sin x$. Then $du = \cos x\,dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x\cos x\,dx &= \int u\,du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

Notes:

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different.

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 6.1.5 Integrating by substitution

Evaluate $\displaystyle\int x\sqrt{x+3}\,dx$.

Solution Recognizing the composition of functions, set $u = x + 3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\,dx = \int x\sqrt{u}\,du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u-3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3}\,dx &= \int (u-3)u^{\frac{1}{2}}\,du\\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)du\\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C\\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem.

Example 6.1.6 Integrating by substitution

Evaluate $\displaystyle\int \frac{1}{x\ln x}\,dx$.

Solution This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 6.1.5 is useful here: choose something for u and consider what this implies du must

Notes:

be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\,dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\,dx$, which is part of the integrand. Thus:

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \underbrace{\frac{1}{\ln x}}_{u}\,\underbrace{\frac{1}{x}\,dx}_{du}\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln|\ln x| + C.\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct.

Integrals Involving Trigonometric Functions

Section 6.3 delves deeper into integrals of a variety of trigonometric functions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 6.1.7 Integration by substitution: antiderivatives of $\tan x$

Evaluate $\displaystyle\int \tan x\,dx$.

Solution The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have

Notes:

that $du = -\sin x\, dx$, hence $-du = \sin x\, dx$. We can integrate:

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x}\, dx \\ &= \int \underbrace{\frac{1}{\cos x}}_{u} \underbrace{\sin x\, dx}_{-du} \\ &= \int \frac{-1}{u}\, du \\ &= -\ln|u| + C \\ &= -\ln|\cos x| + C.\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C \\ &= \ln\left|\frac{1}{\cos x}\right| + C \\ &= \ln|\sec x| + C.\end{aligned}$$

Thus the result they give is $\int \tan x\, dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 6.1.8 Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\displaystyle\int \sec x\, dx$.

Solution This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}.$$

This may seem like it came out of left field, but it works beautifully. Consider:

$$\begin{aligned}\int \sec x\, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.\end{aligned}$$

Notes:

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x)\ dx$, which is our numerator. Thus:

$$\begin{aligned} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C. \end{aligned}$$

We can use similar techniques to those used in Examples 6.1.7 and 6.1.8 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 6.1.2 Antiderivatives of Trigonometric Functions

1. $\int \sin x\ dx = -\cos x + C$
2. $\int \cos x\ dx = \sin x + C$
3. $\int \tan x\ dx = -\ln|\cos x| + C$
4. $\int \csc x\ dx = -\ln|\csc x + \cot x| + C$
5. $\int \sec x\ dx = \ln|\sec x + \tan x| + C$
6. $\int \cot x\ dx = \ln|\sin x| + C$

We explore one more common trigonometric integral.

Example 6.1.9 Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\int \cos^2 x\ dx$.

Solution We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x\ dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\begin{aligned} \int \cos^2 x\ dx &= \int \frac{1 + \cos(2x)}{2}\ dx \\ &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\ dx. \end{aligned}$$

Notes:

Now use Key Idea 6.1.1:

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C$$
$$= \frac{1}{2}x + \frac{\sin(2x)}{4} + C.$$

We'll make significant use of this power–reducing technique in future sections.

Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 6.1.10 Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx$.

Solution One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

We skip the specifics of the steps, but note that when x^2+2x+1 is divided into x^3+4x^2+8x+5, it goes in $x+2$ times with a remainder of $3x+3$. Thus

$$\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2+\frac{3x+3}{x^2+2x+1}.$$

Integrating $x+2$ is simple. The fraction can be integrated by setting $u = x^2 + 2x+1$, giving $du = (2x+2)\,dx$. This is very similar to the numerator. Note that

Notes:

$du/2 = (x+1)\,dx$ and then consider the following:

$$\begin{aligned}\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx &= \int \left(x+2+\frac{3x+3}{x^2+2x+1}\right)\,dx\\ &= \int (x+2)\,dx + \int \frac{3(x+1)}{x^2+2x+1}\,dx\\ &= \frac{1}{2}x^2+2x+C_1+\int \frac{3}{u}\frac{du}{2}\\ &= \frac{1}{2}x^2+2x+C_1+\frac{3}{2}\ln|u|+C_2\\ &= \frac{1}{2}x^2+2x+\frac{3}{2}\ln|x^2+2x+1|+C.\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 6.1.11 Integration by alternate methods

Evaluate $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx$ with, and without, substitution.

Solution We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2+2x+3}{x^{1/2}} = x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}\int \frac{x^2+2x+3}{x^{1/2}}\,dx &= \int \left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}\right)\,dx\\ &= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C\end{aligned}$$

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.

Let $u=\sqrt{x}=x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\,dx \quad\Rightarrow\quad 2du = \frac{1}{\sqrt{x}}\,dx.$$

This gives us $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx = \int (x^2+2x+3)\cdot 2\,du$. What are we to do with the other x terms? Since $u=x^{\frac{1}{2}}$, $u^2=x$, etc. We can then replace x^2 and

Notes:

x with appropriate powers of u. We thus have

$$\begin{aligned}\int \frac{x^2+2x+3}{\sqrt{x}}\,dx &= \int (x^2+2x+3)\cdot 2\,du\\ &= \int 2(u^4+2u^2+3)\,du\\ &= \frac{2}{5}u^5+\frac{4}{3}u^3+6u+C\\ &= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C,\end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is.

Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1}x\big)=\frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1}5x\big)=\frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 6.1.12 Integrating by substitution: inverse trigonometric functions

Evaluate $\displaystyle\int \frac{1}{25+x^2}\,dx$.

Solution The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned}\frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})}\\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}\\ &= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}.\end{aligned}$$

Notes:

Thus

$$\int \frac{1}{25+x^2}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{5}\tan^{-1} u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Example 6.1.12 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 6.1.3 Integrals Involving Inverse Trigonometric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Let's practice using Theorem 6.1.3.

Example 6.1.13 Integrating by substitution: inverse trigonometric functions

Evaluate the given indefinite integrals.

$$1.\ \int \frac{1}{9+x^2}\,dx,\quad 2.\ \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx\quad 3.\ \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Notes:

Solution Each can be answered using a straightforward application of Theorem 6.1.3.

1. $\displaystyle\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$, as $a = 3$.

2. $\displaystyle\int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx = 10\sec^{-1}10x + C$, as $a = \frac{1}{10}$.

3. $\displaystyle\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C$, as $a = \sqrt{5}$.

Most applications of Theorem 6.1.3 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 6.1.14 Integrating by substitution: completing the square

Evaluate $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$.

Solution Initially, this integral seems to have nothing in common with the integrals in Theorem 6.1.3. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2+bx+c &= \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4} + c \\ &= \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned} \frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13} \\ &= \frac{1}{(x-2)^2+9} \end{aligned}$$

Notes:

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u = x - 2$, but we can employ Key Idea 6.1.1 instead. Thus we have

$$\int \frac{1}{x^2 - 4x + 13}\,dx = \int \frac{1}{(x-2)^2 + 9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

Example 6.1.15 Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 6.1.3; the second integral is handled by substitution, with $u = 16 - x^2$. We handle each separately.

$$\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16 - x^2$, so $du = -2xdx$ and $xdx = -du/2$. We have

$$\begin{aligned}
\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}} \\
&= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du \\
&= -\sqrt{u} + C \\
&= -\sqrt{16-x^2} + C.
\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

Substitution and Definite Integration

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

Notes:

1. Start with a definite integral $\displaystyle\int_a^b f(x)\,dx$ that requires substitution.

2. Ignore the bounds; use substitution to evaluate $\displaystyle\int f(x)\,dx$ and find an antiderivative $F(x)$.

3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b) - F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving).

At its heart, (using the notation of Theorem 6.1.1) substitution converts integrals of the form $\int F'(g(x))g'(x)\,dx$ into an integral of the form $\int F'(u)\,du$ with the substitution of $u = g(x)$. The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Theorem 6.1.4 Substitution with Definite Integrals

Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then

$$\int_a^b F'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} F'(u)\,du.$$

In effect, Theorem 6.1.4 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 6.1.16 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^2 \cos(3x-1)\,dx$ using Theorem 6.1.4.

Solution Observing the composition of functions, let $u = 3x - 1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x - 1$, we are implicitly stating that $g(x) = 3x - 1$. Theorem 6.1.4 states that the new lower bound is $g(0) = -1$; the new upper bound is

Notes:

$g(2) = 5$. We now evaluate the definite integral:

$$\begin{aligned}\int_0^2 \cos(3x-1)\,dx &= \int_{-1}^5 \cos u \frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^5\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \approx -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 6.1.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.

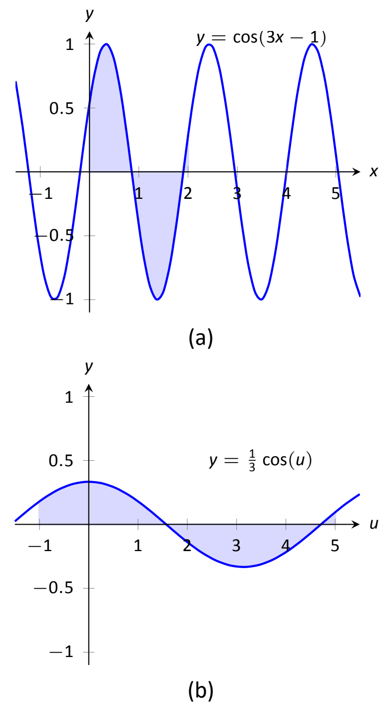


Figure 6.1.1: Graphing the areas defined by the definite integrals of Example 6.1.16.

Example 6.1.17 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x\cos x\,dx$ using Theorem 6.1.4.

Solution We saw the corresponding indefinite integral in Example 6.1.4. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\,dx$ and hence $\sin x\,dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x\cos x\,dx &= \int_1^0 -u\,du \qquad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u\,du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 6.1.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 6.1.4 guarantees that they have the same area.

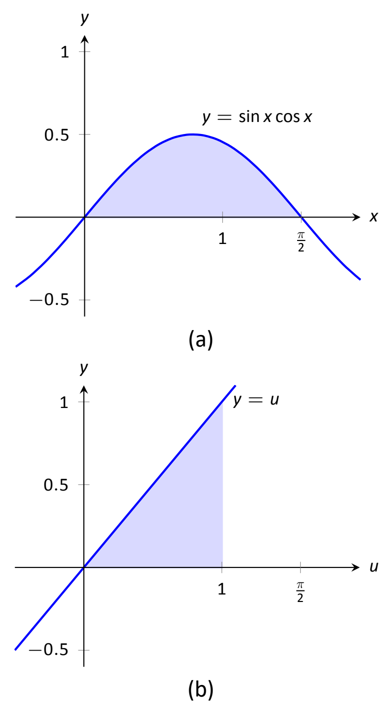


Figure 6.1.2: Graphing the areas defined by the definite integrals of Example 6.1.17.

Integration by substitution is a powerful and useful integration technique. The next section introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Notes:

Exercises 6.1

Terms and Concepts

06 01 ex 01 1. Substitution "undoes" what derivative rule?

06 01 ex 02 2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

06 01 exset 01 **In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.**

06 01 ex 03 3. $\displaystyle\int 3x^2\left(x^3-5\right)^7\,dx$

06 01 ex 04 4. $\displaystyle\int (2x-5)\left(x^2-5x+7\right)^3\,dx$

06 01 ex 05 5. $\displaystyle\int x\left(x^2+1\right)^8\,dx$

06 01 ex 06 6. $\displaystyle\int (12x+14)\left(3x^2+7x-1\right)^5\,dx$

06 01 ex 11 7. $\displaystyle\int \frac{1}{2x+7}dx$

06 01 ex 12 8. $\displaystyle\int \frac{1}{\sqrt{2x+3}}dx$

06 01 ex 13 9. $\displaystyle\int \frac{x}{\sqrt{x+3}}dx$

06 01 ex 17 10. $\displaystyle\int \frac{x^3-x}{\sqrt{x}}dx$

06 01 ex 18 11. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

06 01 ex 19 12. $\displaystyle\int \frac{x^4}{\sqrt{x^5+1}}dx$

06 01 ex 20 13. $\displaystyle\int \frac{\frac{1}{x}+1}{x^2}dx$

06 01 ex 21 14. $\displaystyle\int \frac{\ln(x)}{x}dx$

06 01 exset 02 **In Exercises 15 – 24, use Substitution to evaluate the indefinite integral involving trigonometric functions.**

06 01 ex 08 15. $\displaystyle\int \sin^2(x)\cos(x)dx$

06 01 ex 84 16. $\displaystyle\int \cos^3(x)\sin(x)dx$

06 01 ex 09 17. $\displaystyle\int \cos(3-6x)dx$

06 01 ex 10 18. $\displaystyle\int \sec^2(4-x)dx$

06 01 ex 16 19. $\displaystyle\int \sec(2x)dx$

06 01 ex 22 20. $\displaystyle\int \tan^2(x)\sec^2(x)dx$

06 01 ex 25 21. $\displaystyle\int x\cos\left(x^2\right)dx$

06 01 ex 27 22. $\displaystyle\int \tan^2(x)dx$

06 01 ex 82 23. $\displaystyle\int \cot x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

06 01 ex 83 24. $\displaystyle\int \csc x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

06 01 exset 03 **In Exercises 25 – 32, use Substitution to evaluate the indefinite integral involving exponential functions.**

06 01 ex 29 25. $\displaystyle\int e^{3x-1}dx$

06 01 ex 30 26. $\displaystyle\int e^{x^3}x^2dx$

06 01 ex 31 27. $\displaystyle\int e^{x^2-2x+1}(x-1)dx$

06 01 ex 32 28. $\displaystyle\int \frac{e^x+1}{e^x}dx$

06 01 ex 85 29. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

06 01 ex 33 30. $\displaystyle\int \frac{e^x-e^{-x}}{e^{2x}}dx$

06 01 ex 34 31. $\displaystyle\int 3^{3x}dx$

06 01 ex 35 32. $\displaystyle\int 4^{2x}dx$

06 01 exset 04 **In Exercises 33 – 36, use Substitution to evaluate the indefinite integral involving logarithmic functions.**

06 01 ex 36 33. $\displaystyle\int \frac{\ln x}{x}dx$

06 01 ex 37 34. $\displaystyle\int \frac{\left(\ln x\right)^2}{x}dx$

06 01 ex 38 35. $\displaystyle\int \frac{\ln\left(x^3\right)}{x}\,dx$

06 01 ex 39 36. $\displaystyle\int \frac{1}{x\ln\left(x^2\right)}\,dx$

06 01 exset 05 **In Exercises 37 – 42, use Substitution to evaluate the indefinite integral involving rational functions.**

06 01 ex 40 37. $\displaystyle\int \frac{x^2+3x+1}{x}\,dx$

06 01 ex 41 38. $\displaystyle\int \frac{x^3+x^2+x+1}{x}\,dx$

06 01 ex 42 39. $\displaystyle\int \frac{x^3-1}{x+1}\,dx$

06 01 ex 43 40. $\displaystyle\int \frac{x^2+2x-5}{x-3}\,dx$

06 01 ex 44 41. $\displaystyle\int \frac{3x^2-5x+7}{x+1}\,dx$

06 01 ex 45 42. $\displaystyle\int \frac{x^2+2x+1}{x^3+3x^2+3x}\,dx$

06 01 exset 06 **In Exercises 43 – 52, use Substitution to evaluate the indefinite integral involving inverse trigonometric functions.**

06 01 ex 50 43. $\displaystyle\int \frac{7}{x^2+7}\,dx$

06 01 ex 51 44. $\displaystyle\int \frac{3}{\sqrt{9-x^2}}\,dx$

06 01 ex 52 45. $\displaystyle\int \frac{14}{\sqrt{5-x^2}}\,dx$

06 01 ex 53 46. $\displaystyle\int \frac{2}{x\sqrt{x^2-9}}\,dx$

06 01 ex 54 47. $\displaystyle\int \frac{5}{\sqrt{x^4-16x^2}}\,dx$

06 01 ex 55 48. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

06 01 ex 60 49. $\displaystyle\int \frac{1}{x^2-2x+8}\,dx$

06 01 ex 61 50. $\displaystyle\int \frac{2}{\sqrt{-x^2+6x+7}}\,dx$

06 01 ex 62 51. $\displaystyle\int \frac{3}{\sqrt{-x^2+8x+9}}\,dx$

06 01 ex 63 52. $\displaystyle\int \frac{5}{x^2+6x+34}\,dx$

06 01 exset 07 **In Exercises 53 – 78, evaluate the indefinite integral.**

06 01 ex 14 53. $\displaystyle\int \frac{x^2}{\left(x^3+3\right)^2}\,dx$

06 01 ex 07 54. $\displaystyle\int \left(3x^2+2x\right)\left(5x^3+5x^2+2\right)^8\,dx$

06 01 ex 15 55. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

06 01 ex 26 56. $\displaystyle\int x^2\csc^2\left(x^3+1\right)\,dx$

06 01 ex 28 57. $\displaystyle\int \sin(x)\sqrt{\cos(x)}\,dx$

06 01 ex 86 58. $\displaystyle\int \sin\left(5x+1\right)\,dx$

06 01 ex 23 59. $\displaystyle\int \frac{1}{x-5}\,dx$

06 01 ex 24 60. $\displaystyle\int \frac{7}{3x+2}\,dx$

06 01 ex 46 61. $\displaystyle\int \frac{3x^3+4x^2+2x-22}{x^2+3x+5}\,dx$

06 01 ex 47 62. $\displaystyle\int \frac{2x+7}{x^2+7x+3}\,dx$

06 01 ex 48 63. $\displaystyle\int \frac{9(2x+3)}{3x^2+9x+7}\,dx$

06 01 ex 49 64. $\displaystyle\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}\,dx$

06 01 ex 56 65. $\displaystyle\int \frac{x}{x^4+81}\,dx$

06 01 ex 57 66. $\displaystyle\int \frac{2}{4x^2+1}\,dx$

06 01 ex 58 67. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

06 01 ex 59 68. $\displaystyle\int \frac{1}{\sqrt{16-9x^2}}\,dx$

06 01 ex 64 69. $\displaystyle\int \frac{3x-2}{x^2-2x+10}\,dx$

06 01 ex 65 70. $\displaystyle\int \frac{7-2x}{x^2+12x+61}\,dx$

06 01 ex 66 71. $\displaystyle\int \frac{x^2+5x-2}{x^2-10x+32}\,dx$

06 01 ex 67 72. $\displaystyle\int \frac{x^3}{x^2+9}\,dx$

06 01 ex 68 73. $\displaystyle\int \frac{x^3 - x}{x^2 + 4x + 9}\,dx$

06 01 ex 69 74. $\displaystyle\int \frac{\sin(x)}{\cos^2(x) + 1}\,dx$

06 01 ex 70 75. $\displaystyle\int \frac{\cos(x)}{\sin^2(x) + 1}\,dx$

06 01 ex 71 76. $\displaystyle\int \frac{\cos(x)}{1 - \sin^2(x)}\,dx$

06 01 ex 72 77. $\displaystyle\int \frac{3x - 3}{\sqrt{x^2 - 2x - 6}}\,dx$

06 01 ex 73 78. $\displaystyle\int \frac{x - 3}{\sqrt{x^2 - 6x + 8}}\,dx$

06 01 exset 08 **In Exercises 79 – 86, evaluate the definite integral.**

06 01 ex 74 79. $\displaystyle\int_1^3 \frac{1}{x - 5}\,dx$

06 01 ex 75 80. $\displaystyle\int_2^6 x\sqrt{x - 2}\,dx$

06 01 ex 76 81. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx$

06 01 ex 77 82. $\displaystyle\int_0^1 2x(1 - x^2)^4\,dx$

06 01 ex 78 83. $\displaystyle\int_{-2}^{-1} (x + 1)e^{x^2 + 2x + 1}\,dx$

06 01 ex 79 84. $\displaystyle\int_{-1}^1 \frac{1}{1 + x^2}\,dx$

06 01 ex 80 85. $\displaystyle\int_2^4 \frac{1}{x^2 - 6x + 10}\,dx$

06 01 ex 81 86. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}}\,dx$

6.2 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos x \, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)' \, dx = \int (u'v + uv') \, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

Solving for the second integral we have

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u \, dv = uv - \int v \, du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 6.2.1 Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u \, dv = uv - \int v \, du,$$

and

$$\int_{x=a}^{x=b} u \, dv = uv\Big|_a^b - \int_{x=a}^{x=b} v \, du.$$

Notes:

Let's try an example to understand our new technique.

Example 6.2.1 **Integrating using Integration by Parts**

Evaluate $\int x \cos x \, dx$.

Solution The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos x \, dx$.

It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 6.2.1; on the right we fill in the rest of what we need. If $u = x$, then $du = dx$. Since $dv = \cos x \, dx$, v is an antiderivative of $\cos x$. We choose $v = \sin x$.

$$\begin{array}{ll} u = x & v = \,? \\ du = \,? & dv = \cos x \, dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = x & v = \sin x \\ du = dx & dv = \cos x \, dx \end{array}$$

Figure 6.2.1: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x \cos x \, dx = x \sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x \sin x$. This product is what makes Integration by Parts necessary.

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v \, du$ will be simpler to integrate than the original integral $\int u \, dv$.

In the example above, we chose $u = x$ and $dv = \cos x \, dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x \cos x \, dx$, we could integrate $\sin x \, dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "LIATE," where

L = **L**ogarithmic, I = **I**nverse Trig., A = **A**lgebraic (polynomials),
T = **T**rigonometric, and E = **E**xponential.

Notes:

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by L coming before A in LIATE.

We now consider another example.

Example 6.2.2 Integrating using Integration by Parts

Evaluate $\displaystyle\int xe^x\,dx$.

SOLUTION The integrand contains an **A**lgebraic term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{cc} u = x & v = ? \\ du = ? & dv = e^x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x & v = e^x \\ du = dx & dv = e^x\,dx \end{array}$$

Figure 6.2.2: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term.

Example 6.2.3 Integrating using Integration by Parts

Evaluate $\displaystyle\int x^2\cos x\,dx$.

SOLUTION The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos x\,dx$. Then $du = 2x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = x^2 & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x^2 & v = \sin x \\ du = 2x\,dx & dv = \cos x\,dx \end{array}$$

Figure 6.2.3: Setting up Integration by Parts.

Notes:

The Integration by Parts formula gives

$$\int x^2 \cos x\, dx = x^2 \sin x - \int 2x \sin x\, dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $u = 2x$ and $dv = \sin x$ and fill in the rest below.

$$\begin{array}{cc} u = 2x & v = ? \\ du = ? & dv = \sin x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & v = -\cos x \\ du = 2\, dx & dv = \sin x\, dx \end{array}$$

Figure 6.2.4: Setting up Integration by Parts (again).

$$\int x^2 \cos x\, dx = x^2 \sin x - \left(-2x \cos x - \int -2 \cos x\, dx \right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2 \sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos x\, dx = x^2 \sin x + 2x \cos x - 2 \sin x + C.$$

Example 6.2.4 Integrating using Integration by Parts

Evaluate $\displaystyle\int e^x \cos x\, dx$.

Solution This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos x$ or e^x; to demonstrate that we do not have to follow LIATE, we choose $u = e^x$ and hence $dv = \cos x\, dx$. Then $du = e^x\, dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = e^x & v = ? \\ du = ? & dv = \cos x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & v = \sin x \\ du = e^x\, dx & dv = \cos x\, dx \end{array}$$

Figure 6.2.5: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x\, dx = e^x \sin x - \int e^x \sin x\, dx.$$

Notes:

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x\,dx$. This leads us to the following:

$$\begin{array}{ll} u = e^x & v = ? \\ du = ? & dv = \sin x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = e^x & v = -\cos x \\ du = e^x\,dx & dv = \sin x\,dx \end{array}$$

Figure 6.2.6: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\begin{aligned}\int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx.\end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\,dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\,dx$ to both sides. This gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

Now divide both sides by 2:

$$\int e^x \cos x\,dx = \frac{1}{2}\big(e^x \sin x + e^x \cos x\big).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x\,dx = \frac{1}{2}e^x\left(\sin x + \cos x\right) + C.$$

Example 6.2.5 Integrating using Integration by Parts: antiderivative of $\ln x$

Evaluate $\displaystyle\int \ln x\,dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a

Notes:

clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{array}{cc} u = \ln x & v = ? \\ du = ? & dv = dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = \ln x & v = x \\ du = 1/x\,dx & dv = dx \end{array}$$

Figure 6.2.7: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x\,dx = x\ln x - \int x\,\frac{1}{x}\,dx.$$

The new integral simplifies to $\int 1\,dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln x\,dx = x\ln x - x + C.$$

Example 6.2.6 Integrating using Int. by Parts: antiderivative of $\arctan x$

Evaluate $\displaystyle\int \arctan x\,dx$.

Solution The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1+x^2)\,dx$ and $v = x$. The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1 + x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\ln|u| + C$, which becomes $\ln(1+x^2) + C$ (we can drop the absolute values as $1 + x^2$ is always postive). Therefore, the answer is

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Notes:

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 6.2.7 Integration by Parts after substitution

Evaluate $\int \cos(\ln x)\, dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\, dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\, dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\, du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\, du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\, dx = \int e^u \cos u\, du.$$

We evaluated this integral in Example 6.2.4. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\, dx &= \int e^u \cos u\, du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem

Notes:

6.2.1 states. We do so in the next example.

Example 6.2.8 Definite integration using Integration by Parts

Evaluate $\displaystyle\int_1^2 x^2 \ln x\, dx$.

Solution Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{cc} u = \ln x & v = ? \\ du = ? & dv = x^2\, dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = \ln x & v = x^3/3 \\ du = 1/x\, dx & dv = x^2\, dx \end{array}$$

Figure 6.2.8: Setting up Integration by Parts.

The Integration by Parts formula then gives

$$\begin{aligned} \int_1^2 x^2 \ln x\, dx &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3}\,\frac{1}{x}\, dx \\ &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}\, dx \\ &= \frac{x^3}{3}\ln x\Big|_1^2 - \frac{x^3}{9}\Big|_1^2 \\ &= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\Big|_1^2 \\ &= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\ &= \frac{8}{3}\ln 2 - \frac{7}{9} \\ &\approx 1.07. \end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\, dx$ or $\int x^3 \sin x\, dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar–looking integrals

$$\int xe^x\, dx, \qquad \int xe^{x^2}\, dx \qquad \text{and} \qquad \int xe^{x^3}\, dx.$$

Notes:

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

Notes:

Exercises 6.2

Terms and Concepts

06 02 ex 01 1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.

06 02 ex 02 2. T/F: Integration by Parts can be thought of as the "opposite of the Chain Rule."

06 02 ex 03 3. For what is "LIATE" useful?

06 02 ex 50 4. T/F: If the integral that results from Integration by Parts appears to also need Integration by Parts, then a mistake was made in the orginal choice of "u".

Problems

06 02 exset 01 **In Exercises 5 – 34, evaluate the given indefinite integral.**

06 02 ex 04 5. $\int x \sin x \, dx$

06 02 ex 05 6. $\int xe^{-x} \, dx$

06 02 ex 06 7. $\int x^2 \sin x \, dx$

06 02 ex 07 8. $\int x^3 \sin x \, dx$

06 02 ex 08 9. $\int xe^{x^2} \, dx$

06 02 ex 09 10. $\int x^3 e^x \, dx$

06 02 ex 10 11. $\int xe^{-2x} \, dx$

06 02 ex 11 12. $\int e^x \sin x \, dx$

06 02 ex 12 13. $\int e^{2x} \cos x \, dx$

06 02 ex 13 14. $\int e^{2x} \sin(3x) \, dx$

06 02 ex 14 15. $\int e^{5x} \cos(5x) \, dx$

06 02 ex 15 16. $\int \sin x \cos x \, dx$

06 02 ex 16 17. $\int \sin^{-1} x \, dx$

06 02 ex 17 18. $\int \tan^{-1}(2x) \, dx$

06 02 ex 18 19. $\int x \tan^{-1} x \, dx$

06 02 ex 19 20. $\int \sin^{-1} x \, dx$

06 02 ex 22 21. $\int x \ln x \, dx$

06 02 ex 23 22. $\int (x-2) \ln x \, dx$

06 02 ex 24 23. $\int x \ln(x-1) \, dx$

06 02 ex 25 24. $\int x \ln(x^2) \, dx$

06 02 ex 26 25. $\int x^2 \ln x \, dx$

06 02 ex 27 26. $\int (\ln x)^2 \, dx$

06 02 ex 28 27. $\int (\ln(x+1))^2 \, dx$

06 02 ex 29 28. $\int x \sec^2 x \, dx$

06 02 ex 30 29. $\int x \csc^2 x \, dx$

06 02 ex 31 30. $\int x\sqrt{x-2} \, dx$

06 02 ex 32 31. $\int x\sqrt{x^2-2} \, dx$

06 02 ex 35 32. $\int \sec x \tan x \, dx$

06 02 ex 33 33. $\int x \sec x \tan x \, dx$

06 02 ex 34 34. $\int x \csc x \cot x \, dx$

06 02 exset 02 **In Exercises 35 – 40, evaluate the indefinite integral after first making a substitution.**

06 02 ex 36 35. $\int \sin(\ln x) \, dx$

06 02 ex 51 36. $\int e^{2x} \cos\left(e^x\right) \, dx$

06 02 ex 37 37. $\int \sin(\sqrt{x})\, dx$

06 02 ex 38 38. $\int \ln(\sqrt{x})\, dx$

06 02 ex 39 39. $\int e^{\sqrt{x}}\, dx$

06 02 ex 40 40. $\int e^{\ln x}\, dx$

06 02 exset 03 **In Exercises 41 – 49, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 5 – 13.**

06 02 ex 41 41. $\int_0^{\pi} x \sin x\, dx$

06 02 ex 42 42. $\int_{-1}^{1} xe^{-x}\, dx$

06 02 ex 43 43. $\int_{-\pi/4}^{\pi/4} x^2 \sin x\, dx$

06 02 ex 44 44. $\int_{-\pi/2}^{\pi/2} x^3 \sin x\, dx$

06 02 ex 45 45. $\int_0^{\sqrt{\ln 2}} xe^{x^2}\, dx$

06 02 ex 46 46. $\int_0^{1} x^3 e^x\, dx$

06 02 ex 47 47. $\int_1^{2} xe^{-2x}\, dx$

06 02 ex 48 48. $\int_0^{\pi} e^x \sin x\, dx$

06 02 ex 49 49. $\int_{-\pi/2}^{\pi/2} e^{2x} \cos x\, dx$

6.3 Trigonometric Integrals

Functions involving trigonometric functions are useful as they are good at describing periodic behavior. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\int \sin^m x \cos^n x \, dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x \, dx$ in Example 6.1.4. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x \, dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Key Idea 6.3.1 Integrals Involving Powers of Sine and Cosine

Consider $\displaystyle\int \sin^m x \cos^n x \, dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite

   $$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x.$$

   Then

   $$\int \sin^m x \cos^n x \, dx = \int (1 - \cos^2 x)^k \sin x \cos^n x \, dx = -\int (1 - u^2)^k u^n \, du,$$

   where $u = \cos x$ and $du = -\sin x \, dx$.

2. If n is odd, then using substitutions similar to that outlined above we have

   $$\int \sin^m x \cos^n x \, dx = \int u^m (1 - u^2)^k \, du,$$

   where $u = \sin x$ and $du = \cos x \, dx$.

3. If both m and n are even, use the power–reducing identities

   $$\cos^2 x = \frac{1 + \cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1 - \cos(2x)}{2}$$

   to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

Notes:

We practice applying Key Idea 6.3.1 in the next examples.

Example 6.3.1 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x \cos^8 x\, dx$.

Solution The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x.$$

Our integral is now $\displaystyle\int (1-\cos^2 x)^2 \cos^8 x \sin x\, dx$. Let $u = \cos x$, hence $du = -\sin x\, dx$. Making the substitution and expanding the integrand gives

$$\int (1-\cos^2)^2 \cos^8 x \sin x\, dx = -\int (1-u^2)^2 u^8\, du = -\int \big(1-2u^2+u^4\big) u^8\, du = -\int \big(u^8-2u^{10}+u^{12}\big)\, du.$$

This final integral is not difficult to evaluate, giving

$$\begin{aligned} -\int \big(u^8 - 2u^{10} + u^{12}\big)\, du &= -\frac{1}{9}u^9 + \frac{2}{11}u^{11} - \frac{1}{13}u^{13} + C \\ &= -\frac{1}{9}\cos^9 x + \frac{2}{11}\cos^{11} x - \frac{1}{13}\cos^{13} x + C. \end{aligned}$$

Example 6.3.2 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x \cos^9 x\, dx$.

Solution The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 6.3.1 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned} \cos^9 x &= \cos^8 x \cos x \\ &= (\cos^2 x)^4 \cos x \\ &= (1-\sin^2 x)^4 \cos x \\ &= (1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x) \cos x. \end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x \cos^9 x\, dx = \int \sin^5 x \big(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\big) \cos x\, dx.$$

Notes:

Now substitute and integrate, using $u = \sin x$ and $du = \cos x\,dx$.

$$\int \sin^5 x\left(1 - 4\sin^2 x + 6\sin^4 x - 4\sin^6 x + \sin^8 x\right)\cos x\,dx =$$

$$\begin{aligned}\int u^5(1 - 4u^2 + 6u^4 - 4u^6 + u^8)\,du &= \int (u^5 - 4u^7 + 6u^9 - 4u^{11} + u^{13})\,du\\ &= \frac{1}{6}u^6 - \frac{1}{2}u^8 + \frac{3}{5}u^{10} - \frac{1}{3}u^{12} + \frac{1}{14}u^{14} + C\\ &= \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x + \ldots\\ &\quad - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x + C.\end{aligned}$$

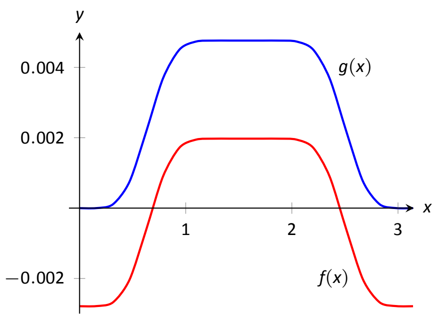


Figure 6.3.1: A plot of $f(x)$ and $g(x)$ from Example 6.3.2 and the Technology Note.

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x \cos^9 x\,dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384} - \frac{5\cos(4x)}{8192} + \frac{19\cos(6x)}{49152} + \frac{\cos(8x)}{4096} - \frac{\cos(10x)}{81920} - \frac{\cos(12x)}{24576} - \frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 6.3.2, which is

$$g(x) = \frac{1}{6}\sin^6 x - \frac{1}{2}\sin^8 x + \frac{3}{5}\sin^{10} x - \frac{1}{3}\sin^{12} x + \frac{1}{14}\sin^{14} x.$$

Figure 6.3.1 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

Example 6.3.3 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \cos^4 x \sin^2 x\,dx$.

Solution The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x\sin^2 x\,dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2\left(\frac{1-\cos(2x)}{2}\right)dx\\ &= \int \frac{1 + 2\cos(2x) + \cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\,dx\\ &= \int \frac{1}{8}\left(1 + \cos(2x) - \cos^2(2x) - \cos^3(2x)\right)dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea 6.1.1. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

Notes:

$$\int \cos(2x)\,dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\,dx = \int \frac{1+\cos(4x)}{2}\,dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\,dx$, hence

$$\begin{aligned}\int \cos^3(2x)\,dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\,dx\\ &= \int \frac{1}{2}(1-u^2)\,du\\ &= \frac{1}{2}\Big(u - \frac{1}{3}u^3\Big) + C\\ &= \frac{1}{2}\Big(\sin(2x) - \frac{1}{3}\sin^3(2x)\Big) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x\sin^2 x\,dx &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\,dx\\ &= \frac{1}{8}\Big[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big) - \frac{1}{2}\Big(\sin(2x)-\frac{1}{3}\sin^3(2x)\Big)\Big] + C\\ &= \frac{1}{8}\Big[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\Big] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, and $\int \sin(mx)\cos(nx)\,dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\,dx$$

Notes:

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\sin(mx)\sin(nx) = \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big]$$
$$\cos(mx)\cos(nx) = \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big]$$
$$\sin(mx)\cos(nx) = \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]$$

Example 6.3.4 **Integrating products of** $\sin(mx)$ **and** $\cos(nx)$

Evaluate $\displaystyle\int \sin(5x)\cos(2x)\,dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}\int \sin(5x)\cos(2x)\,dx &= \int \frac{1}{2}\Big[\sin(3x) + \sin(7x)\Big]\,dx\\ &= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C\end{aligned}$$

Integrals of the form $\displaystyle\int \tan^m x \sec^n x\,dx$.

When evaluating integrals of the form $\int \sin^m x \cos^n x\,dx$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vise–versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x \sec^n x\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x \tan x$, and
- $1 + \tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x \tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Notes:

Key Idea 6.3.2 Integrals Involving Powers of Tangent and Secant

Consider $\int \tan^m x \sec^n x\, dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as
$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1+\tan^2 x)^{k-1}\sec^2 x.$$
Then
$$\int \tan^m x \sec^n x\, dx = \int \tan^m x(1+\tan^2 x)^{k-1}\sec^2 x\, dx = \int u^m(1+u^2)^{k-1}\, du,$$
where $u = \tan x$ and $du = \sec^2 x\, dx$.

2. If m is odd, then $m = 2k+1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as
$$\tan^m x \sec^n x = \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x = (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.$$
Then
$$\int \tan^m x \sec^n x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx = \int (u^2-1)^k u^{n-1}\, du,$$
where $u = \sec x$ and $du = \sec x \tan x\, dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\, dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as
$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\sec^2 x - 1) = \tan^{m-2}\sec^2 x - \tan^{m-2} x.$$
So
$$\int \tan^m x\, dx = \underbrace{\int \tan^{m-2}\sec^2 x\, dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\, dx}_{\text{apply rule \#4 again}}.$$

The techniques described in items 1 and 2 of Key Idea 6.3.2 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Notes:

Example 6.3.5 Integrating powers of tangent and secant

Evaluate $\int \tan^2 x \sec^6 x\, dx$.

SOLUTION Since the power of secant is even, we use rule #1 from Key Idea 6.3.2 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x\, dx &= \int \tan^2 x \sec^4 x \sec^2 x\, dx \\ &= \int \tan^2 x \big(1+\tan^2 x\big)^2 \sec^2 x\, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x\, dx$.

$$= \int u^2 \big(1+u^2\big)^2\, du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

Example 6.3.6 Integrating powers of tangent and secant

Evaluate $\int \sec^3 x\, dx$.

SOLUTION We apply rule #3 from Key Idea 6.3.2 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x\, dx$, meaning that $u = \sec x$.

$$\begin{array}{ll} u = \sec x & v = ? \\ du = ? & dv = \sec^2 x\, dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \tan x\, dx & dv = \sec^2 x\, dx \end{array}$$

Figure 6.3.2: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned}\int \sec^3 x\, dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\, dx}_{dv} \\ &= \sec x \tan x - \int \sec x \tan^2 x\, dx.\end{aligned}$$

Notes:

This new integral also requires applying rule #3 of Key Idea 6.3.2:

$$\begin{aligned}
&= \sec x \tan x - \int \sec x\big(\sec^2 x - 1\big)\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\
&= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|
\end{aligned}$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x\,dx$ to both sides, giving:

$$\begin{aligned}
2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| \\
\int \sec^3 x\,dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C
\end{aligned}$$

We give one more example.

Example 6.3.7 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x\,dx$.

Solution We employ rule #4 of Key Idea 6.3.2.

$$\begin{aligned}
\int \tan^6 x\,dx &= \int \tan^4 x \tan^2 x\,dx \\
&= \int \tan^4 x\big(\sec^2 x - 1\big)\,dx \\
&= \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx
\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x\,dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x\big(\sec^2 x - 1\big)\,dx \\
&= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx
\end{aligned}$$

Notes:

Again, use substitution for the first integral and rule #4 for the second.

$$\begin{aligned} &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \left(\sec^2 x - 1\right)\,dx \\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C. \end{aligned}$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Notes:

Exercises 6.3

Terms and Concepts

06 03 ex 01 1. T/F: $\int \sin^2 x \cos^2 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.

06 03 ex 02 2. T/F: $\int \sin^3 x \cos^3 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.

06 03 ex 03 3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x \, dx$.

06 03 ex 34 4. T/F: Sometimes computer programs evaluate integrals involving trigonometric functions differently than one would using the techniques of this section. When this is the case, the techniques of this section have failed and one should only trust the answer given by the computer.

Problems

06 03 exset 01 **In Exercises 5 – 28, evaluate the indefinite integral.**

06 03 ex 04 5. $\int \sin x \cos^4 x \, dx$

06 03 ex 05 6. $\int \sin^3 x \cos x \, dx$

06 03 ex 06 7. $\int \sin^3 x \cos^2 x \, dx$

06 03 ex 07 8. $\int \sin^3 x \cos^3 x \, dx$

06 03 ex 08 9. $\int \sin^6 x \cos^5 x \, dx$

06 03 ex 09 10. $\int \sin^2 x \cos^7 x \, dx$

06 03 ex 10 11. $\int \sin^2 x \cos^2 x \, dx$

06 03 ex 35 12. $\int \sin x \cos x \, dx$

06 03 ex 11 13. $\int \sin(5x)\cos(3x) \, dx$

06 03 ex 12 14. $\int \sin(x)\cos(2x) \, dx$

06 03 ex 13 15. $\int \sin(3x)\sin(7x) \, dx$

06 03 ex 14 16. $\int \sin(\pi x)\sin(2\pi x) \, dx$

06 03 ex 15 17. $\int \cos(x)\cos(2x) \, dx$

06 03 ex 16 18. $\int \cos\left(\frac{\pi}{2}x\right)\cos(\pi x) \, dx$

06 03 ex 17 19. $\int \tan^4 x \sec^2 x \, dx$

06 03 ex 18 20. $\int \tan^2 x \sec^4 x \, dx$

06 03 ex 19 21. $\int \tan^3 x \sec^4 x \, dx$

06 03 ex 20 22. $\int \tan^3 x \sec^2 x \, dx$

06 03 ex 21 23. $\int \tan^3 x \sec^3 x \, dx$

06 03 ex 22 24. $\int \tan^5 x \sec^5 x \, dx$

06 03 ex 23 25. $\int \tan^4 x \, dx$

06 03 ex 24 26. $\int \sec^5 x \, dx$

06 03 ex 25 27. $\int \tan^2 x \sec x \, dx$

06 03 ex 26 28. $\int \tan^2 x \sec^3 x \, dx$

06 03 exset 02 **In Exercises 29 – 35, evaluate the definite integral. Note: the corresponding indefinite integrals appear in the previous set.**

06 03 ex 27 29. $\int_0^{\pi} \sin x \cos^4 x \, dx$

06 03 ex 28 30. $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$

06 03 ex 29 31. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x \, dx$

06 03 ex 30 32. $\int_0^{\pi/2} \sin(5x)\cos(3x) \, dx$

06 03 ex 31 33. $\int_{-\pi/2}^{\pi/2} \cos(x)\cos(2x) \, dx$

06 03 ex 32 34. $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

06 03 ex 33

35. $\displaystyle\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x \, dx$

6.4 Trigonometric Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2} \tag{6.1}$$

as we recognized that $f(x) = \sqrt{9-x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 6.1, though it can feel "backward." In Section 6.1, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the integral in (6.1). After the example, we will generalize the method and give more examples.

Example 6.4.1 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

Solution We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9 - 9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\,d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\,d\theta.
\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

Notes:

$$= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\, d\theta$$
$$= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big)\, d\theta$$
$$= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key Idea 6.4.1 Trigonometric Substitution

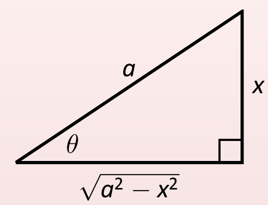
(a) For integrands containing $\sqrt{a^2-x^2}$:

Let $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.

On this interval, $\cos\theta \ge 0$, so

$\sqrt{a^2-x^2} = a\cos\theta$



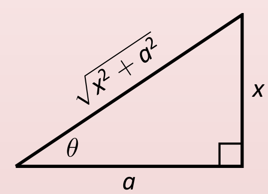
(b) For integrands containing $\sqrt{x^2+a^2}$:

Let $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2+a^2} = a\sec\theta$



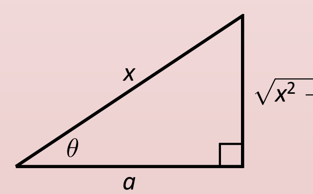
(c) For integrands containing $\sqrt{x^2-a^2}$:

Let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\, d\theta$

Thus $\theta = \sec^{-1}(x/a)$. If $x/a \ge 1$, then $0 \le \theta < \pi/2$; if $x/a \le -1$, then $\pi/2 < \theta \le \pi$.

We restrict our work to where $x \ge a$, so $x/a \ge 1$, and $0 \le \theta < \pi/2$. On this interval, $\tan\theta \ge 0$, so

$\sqrt{x^2-a^2} = a\tan\theta$



Notes:

Example 6.4.2 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx$.

Solution Using Key Idea 6.4.1(b), we recognize $a=\sqrt{5}$ and set $x=\sqrt{5}\tan\theta$. This makes $dx=\sqrt{5}\sec^2\theta\,d\theta$. We will use the fact that $\sqrt{5+x^2}=\sqrt{5+5\tan^2\theta}=\sqrt{5\sec^2\theta}=\sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\,d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln\big|\sec\theta+\tan\theta\big|+C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea 6.4.1(b) helps. With $x=\sqrt{5}\tan\theta$, we have

$$\tan\theta=\frac{x}{\sqrt{5}}\quad\text{and}\quad\sec\theta=\frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\big|\sec\theta+\tan\theta\big|+C\\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}}+\frac{x}{\sqrt{5}}\right|+C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right|+C\\ &= \ln\left|\frac{1}{\sqrt{5}}\right|+\ln\left|\sqrt{x^2+5}+x\right|+C\\ &= \ln\left|\sqrt{x^2+5}+x\right|+C,\end{aligned}$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. (In Section 6.6 we will learn another way of approaching this problem.)

Notes:

Example 6.4.3 Using Trigonometric Substitution

Evaluate $\int \sqrt{4x^2-1}\,dx$.

Solution We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)}\\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a=1/2$, and following Key Idea 6.4.1(c), we set $x=\frac{1}{2}\sec\theta$, and hence $dx=\frac{1}{2}\sec\theta\tan\theta\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx\\ &= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\,d\theta\\ &= \int\sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int\sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\,d\theta\\ &= \int\frac{1}{2}\tan^2\theta\sec\theta\,d\theta\\ &= \frac{1}{2}\int\Big(\sec^2\theta-1\Big)\sec\theta\,d\theta\\ &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta.\end{aligned}$$

We integrated $\sec^3\theta$ in Example 6.3.6, finding its antiderivatives to be

$$\int\sec^3\theta\,d\theta=\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Thus

$$\begin{aligned}\int\sqrt{4x^2-1}\,dx &= \frac{1}{2}\int\big(\sec^3\theta-\sec\theta\big)\,d\theta\\ &= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)-\ln|\sec\theta+\tan\theta|\right)+C\\ &= \frac{1}{4}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.\end{aligned}$$

Notes:

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a = 1/2$, and $x = \frac{1}{2}\sec\theta$, the reference triangle in Key Idea 6.4.1(c) shows that

$$\tan\theta = \sqrt{x^2 - 1/4}\Big/(1/2) = 2\sqrt{x^2 - 1/4} \quad \text{and} \quad \sec\theta = 2x.$$

Thus

$$\begin{aligned}\frac{1}{4}\Big(\sec\theta\tan\theta - \ln\left|\sec\theta + \tan\theta\right|\Big) + C &= \frac{1}{4}\Big(2x\cdot 2\sqrt{x^2-1/4} - \ln\left|2x + 2\sqrt{x^2-1/4}\right|\Big) + C \\ &= \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\left|2x + 2\sqrt{x^2-1/4}\right|\Big) + C.\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2-1}\,dx = \frac{1}{4}\Big(4x\sqrt{x^2-1/4} - \ln\left|2x + 2\sqrt{x^2-1/4}\right|\Big) + C.$$

Example 6.4.4 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Solution We use Key Idea 6.4.1(a) with $a = 2$, $x = 2\sin\theta$, $dx = 2\cos\theta$ and hence $\sqrt{4-x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta \\ &= \int \cot^2\theta\,d\theta \\ &= \int (\csc^2\theta - 1)\,d\theta \\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea 6.4.1(a), we have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Notes:

Example 6.4.5 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$.

Solution We know the answer already as $\tan^{-1} x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 6.4.1(b), let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and note that $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta\\ &= \int 1\,d\theta\\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1} x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$.

The next example is similar to the previous one in that it does not involve a square–root. It shows how several techniques and identities can be combined to obtain a solution.

Example 6.4.6 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\,d\theta$:

$$\begin{aligned}&= \int \frac{1}{(\tan^2\theta+1)^2}\sec^2\theta\,d\theta\\ &= \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\,d\theta\\ &= \int \cos^2\theta\,d\theta.\end{aligned}$$

Notes:

Applying a power reducing formula, we have

$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right)\,d\theta$$
$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \quad (6.2)$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea 6.4.1(b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}}\cdot\frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x+3$. We start with the expression in Equation (6.2):

$$\begin{aligned}\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C\\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.\end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 6.4.7 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx$.

Solution Using Key Idea 6.4.1(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\,d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The

Notes:

original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\begin{aligned}\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx &= \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta} 5\sec^2\theta\,d\theta \\ &= 25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta.\end{aligned}$$

We encountered this indefinite integral in Example 6.4.3 where we found

$$\int \tan^2\theta\sec\theta\,d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$\begin{aligned}25\int_0^{\pi/4} \tan^2\theta\sec\theta\,d\theta &= \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4} \\ &= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big) \\ &\approx 6.661.\end{aligned}$$

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Key Idea 6.4.2 Useful Equalities with Trigonometric Substitution

1. $\sin(2\theta) = 2\sin\theta\cos\theta$
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3. $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta + \ln\big|\sec\theta+\tan\theta\big|\Big) + C$
4. $\displaystyle\int \cos^2\theta\,d\theta = \int \frac{1}{2}\big(1+\cos(2\theta)\big)\,d\theta = \frac{1}{2}\big(\theta + \sin\theta\cos\theta\big) + C.$

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns “complicated” fractions into sums of “simpler” fractions, making integration easier.

Notes:

Exercises 6.4

Terms and Concepts

06 08 ex 01 1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".

06 08 ex 02 2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.

06 08 ex 03 3. Consider the Pythagorean Identity $\sin^2\theta + \cos^2\theta = 1$.

   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?

   (b) Use the new identity to simplify $9\tan^2\theta + 9$.

06 08 ex 04 4. Why does Key Idea 6.4.1(a) state that $\sqrt{a^2-x^2} = a\cos\theta$, and not $|a\cos\theta|$?

Problems

06 08 exset 01 **In Exercises 5 – 16, apply Trigonometric Substitution to evaluate the indefinite integrals.**

06 08 ex 05 5. $\displaystyle\int \sqrt{x^2+1}\, dx$

06 08 ex 06 6. $\displaystyle\int \sqrt{x^2+4}\, dx$

06 08 ex 07 7. $\displaystyle\int \sqrt{1-x^2}\, dx$

06 08 ex 08 8. $\displaystyle\int \sqrt{9-x^2}\, dx$

06 08 ex 09 9. $\displaystyle\int \sqrt{x^2-1}\, dx$

06 08 ex 10 10. $\displaystyle\int \sqrt{x^2-16}\, dx$

06 08 ex 11 11. $\displaystyle\int \sqrt{4x^2+1}\, dx$

06 08 ex 12 12. $\displaystyle\int \sqrt{1-9x^2}\, dx$

06 08 ex 13 13. $\displaystyle\int \sqrt{16x^2-1}\, dx$

06 08 ex 14 14. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\, dx$

06 08 ex 15 15. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\, dx$

06 08 ex 16 16. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\, dx$

06 08 exset 02 **In Exercises 17 – 26, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.**

06 08 ex 18 17. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\, dx$

06 08 ex 20 18. $\displaystyle\int \frac{1}{(x^2+1)^2}\, dx$

06 08 ex 17 19. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\, dx$

06 08 ex 24 20. $\displaystyle\int x^2\sqrt{1-x^2}\, dx$

06 08 ex 22 21. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\, dx$

06 08 ex 19 22. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\, dx$

06 08 ex 21 23. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\, dx$

06 08 ex 23 24. $\displaystyle\int x^2(1-x^2)^{-3/2}\, dx$

06 08 ex 25 25. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\, dx$

06 08 ex 26 26. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\, dx$

06 08 exset 03 **In Exercises 27 – 32, evaluate the definite integrals by making the proper trigonometric substitution *and* changing the bounds of integration. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)**

06 08 ex 27 27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\, dx$

06 08 ex 28 28. $\displaystyle\int_{4}^{8} \sqrt{x^2-16}\, dx$

06 08 ex 29 29. $\displaystyle\int_{0}^{2} \sqrt{x^2+4}\, dx$

06 08 ex 30 30. $\displaystyle\int_{-1}^{1} \frac{1}{(x^2+1)^2}\, dx$

06 08 ex 31 31. $\displaystyle\int_{-1}^{1} \sqrt{9-x^2}\, dx$

06 08 ex 32 32. $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\, dx$

6.5 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx \\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \frac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Notes:

Key Idea 6.5.1 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

The following examples will demonstrate how to put this Key Idea into practice. Example 6.5.1 stresses the decomposition aspect of the Key Idea.

Example 6.5.1 Decomposing into partial fractions

Decompose $f(x)=\dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

Notes:

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2}, \quad \frac{C}{(x-2)^2} \quad \text{and} \quad \frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7} \quad \text{and} \quad \frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\begin{aligned}\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} &= \frac{A}{x+5}+\frac{B}{x-2}+\frac{C}{(x-2)^2}+\frac{D}{(x-2)^3}+\\ &\quad \frac{Ex+F}{x^2+x+2}+\frac{Gx+H}{x^2+x+7}+\frac{Ix+J}{(x^2+x+7)^2}\end{aligned}$$

Solving for the coefficients A, $B \ldots J$ would be a bit tedious but not "hard."

Example 6.5.2 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator factors into two linear terms: $x^2-1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1}+\frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1=(x-1)(x+1)$:

$$\begin{aligned}1 &= \frac{A(x-1)(x+1)}{x-1}+\frac{B(x-1)(x+1)}{x+1}\\ &= A(x+1)+B(x-1)\\ &= Ax+A+Bx-B\end{aligned}$$

Now collect like terms.

$$= (A+B)x+(A-B).$$

The next step is key. Note the equality we have:

$$1 = (A+B)x+(A-B).$$

Notes:

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A + B)x + (A - B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A + B)$. Since both sides are equal, we must have that $0 = A + B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A - B)$. Therefore we have $1 = A - B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A + B = 0 \\ A - B = 1 \end{array} \Rightarrow \begin{array}{c} A = 1/2 \\ B = -1/2 \end{array}.$$

Thus

$$\frac{1}{x^2 - 1} = \frac{1/2}{x - 1} - \frac{1/2}{x + 1}.$$

Example 6.5.3 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx.$

Solution We decompose the integrand as follows, as described by Key Idea 6.5.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (6.3) \\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A + B = 0, \quad 4A + B + C = 0 \quad \text{and} \quad 4A - 2B - C = 1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Note: Equation 6.3 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x = 1$. However, when $x = 1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$. Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B.

Notes:

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$ (or by directly applying Key Idea 6.1.1 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 6.5.4 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx$.

Solution Key Idea 6.5.1 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x+2+\frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 6.5.1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x+30 &= A(x+3)+B(x-5) \\ &= (A+B)x + (3A-5B). \end{aligned}$$

This implies that:

$$\begin{aligned} 19 &= A+B \\ 30 &= 3A-5B. \end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned} 125/8 &= A \\ 27/8 &= B. \end{aligned}$$

Note: The values of A and B can be quickly found using the technique described in the margin of Example 6.5.3.

Notes:

We can now integrate.

$$\begin{aligned}\int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right)\,dx\\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C.\end{aligned}$$

Example 6.5.5 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx$.

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 6.5.1. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$\begin{aligned}7x^2+31x+54 &= A(x^2+6x+11)+(Bx+C)(x+1)\\ &= (A+B)x^2+(6A+B+C)x+(11A+C).\end{aligned}$$

This implies that:

$$\begin{aligned}7 &= A+B\\ 31 &= 6A+B+C\\ 54 &= 11A+C.\end{aligned}$$

Solving this system of linear equations gives the nice result of $A=5$, $B=2$ and $C=-1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int \left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\dfrac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u=x^2+6x+11$, so $du=(2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$

Notes:

term in the numerator by adding 0 in the form of "$7 - 7$."

$$\begin{aligned}\frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11}\\ &= \frac{2x+6}{x^2+6x+11} - \frac{7}{x^2+6x+11}.\end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 6.1.3 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx &= \int\left(\frac{5}{x+1} + \frac{2x-1}{x^2+6x+11}\right)\,dx\\ &= \int \frac{5}{x+1}\,dx + \int \frac{2x+6}{x^2+6x+11}\,dx - \int \frac{7}{x^2+6x+11}\,dx\\ &= 5\ln|x+1| + \ln|x^2+6x+11| - \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right) + C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

The next section introduces new functions, called the Hyperbolic Functions. They will allow us to make substitutions similar to those found when studying Trigonometric Substitution, allowing us to approach even more integration problems.

Notes:

Exercises 6.5

Terms and Concepts

06 04 ex 01 1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.

06 04 ex 02 2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

06 04 ex 03 3. Decompose $\dfrac{1}{x^2 - 3x}$ without solving for the coefficients, as done in Example 6.5.1.

06 04 ex 04 4. Decompose $\dfrac{7 - x}{x^2 - 9}$ without solving for the coefficients, as done in Example 6.5.1.

06 04 ex 05 5. Decompose $\dfrac{x - 3}{x^2 - 7}$ without solving for the coefficients, as done in Example 6.5.1.

06 04 ex 06 6. Decompose $\dfrac{2x + 5}{x^3 + 7x}$ without solving for the coefficients, as done in Example 6.5.1.

Problems

06 04 exset 01 **In Exercises 7 – 26, evaluate the indefinite integral.**

06 04 ex 07 7. $\displaystyle\int \frac{7x + 7}{x^2 + 3x - 10}\,dx$

06 04 ex 08 8. $\displaystyle\int \frac{7x - 2}{x^2 + x}\,dx$

06 04 ex 09 9. $\displaystyle\int \frac{-4}{3x^2 - 12}\,dx$

06 04 ex 30 10. $\displaystyle\int \frac{6x + 4}{3x^2 + 4x + 1}\,dx$

06 04 ex 10 11. $\displaystyle\int \frac{x + 7}{(x + 5)^2}\,dx$

06 04 ex 11 12. $\displaystyle\int \frac{-3x - 20}{(x + 8)^2}\,dx$

06 04 ex 12 13. $\displaystyle\int \frac{9x^2 + 11x + 7}{x(x + 1)^2}\,dx$

06 04 ex 13 14. $\displaystyle\int \frac{-12x^2 - x + 33}{(x - 1)(x + 3)(3 - 2x)}\,dx$

06 04 ex 14 15. $\displaystyle\int \frac{94x^2 - 10x}{(7x + 3)(5x - 1)(3x - 1)}\,dx$

06 04 ex 15 16. $\displaystyle\int \frac{x^2 + x + 1}{x^2 + x - 2}\,dx$

06 04 ex 16 17. $\displaystyle\int \frac{x^3}{x^2 - x - 20}\,dx$

06 04 ex 17 18. $\displaystyle\int \frac{2x^2 - 4x + 6}{x^2 - 2x + 3}\,dx$

06 04 ex 18 19. $\displaystyle\int \frac{1}{x^3 + 2x^2 + 3x}\,dx$

06 04 ex 19 20. $\displaystyle\int \frac{x^2 + x + 5}{x^2 + 4x + 10}\,dx$

06 04 ex 20 21. $\displaystyle\int \frac{12x^2 + 21x + 3}{(x + 1)(3x^2 + 5x - 1)}\,dx$

06 04 ex 21 22. $\displaystyle\int \frac{6x^2 + 8x - 4}{(x - 3)(x^2 + 6x + 10)}\,dx$

06 04 ex 22 23. $\displaystyle\int \frac{2x^2 + x + 1}{(x + 1)(x^2 + 9)}\,dx$

06 04 ex 23 24. $\displaystyle\int \frac{x^2 - 20x - 69}{(x - 7)(x^2 + 2x + 17)}\,dx$

06 04 ex 24 25. $\displaystyle\int \frac{9x^2 - 60x + 33}{(x - 9)(x^2 - 2x + 11)}\,dx$

06 04 ex 25 26. $\displaystyle\int \frac{6x^2 + 45x + 121}{(x + 2)(x^2 + 10x + 27)}\,dx$

06 04 exset 02 **In Exercises 27 – 30, evaluate the definite integral.**

06 04 ex 26 27. $\displaystyle\int_1^2 \frac{8x + 21}{(x + 2)(x + 3)}\,dx$

06 04 ex 27 28. $\displaystyle\int_0^5 \frac{14x + 6}{(3x + 2)(x + 4)}\,dx$

06 04 ex 28 29. $\displaystyle\int_{-1}^1 \frac{x^2 + 5x - 5}{(x - 10)(x^2 + 4x + 5)}\,dx$

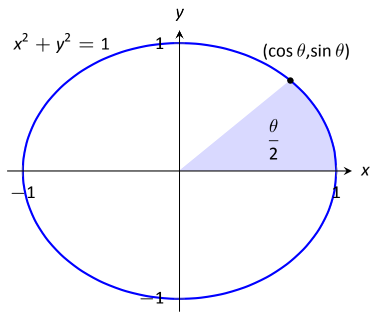
06 04 ex 29 30. $\displaystyle\int_0^1 \frac{x}{(x + 1)(x^2 + 2x + 1)}\,dx$

6.6 Hyperbolic Functions

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many, many connections between them and the standard trigonometric functions. Figure 6.6.1 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2+y^2=1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2-y^2=1$.

We begin with their definition.



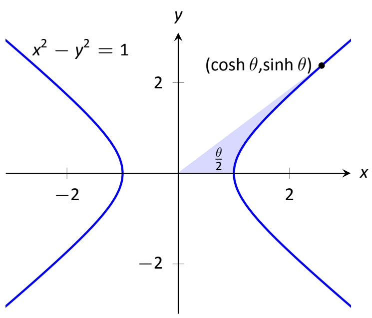


Figure 6.6.1: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

Definition 6.6.1 Hyperbolic Functions

1. $\cosh x = \dfrac{e^x+e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x-e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\operatorname{sech} x = \dfrac{1}{\cosh x}$
5. $\operatorname{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

These hyperbolic functions are graphed in Figure 6.6.2. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty,\infty)$, whereas both $\coth x$ and $\operatorname{csch} x$ have vertical asymptotes at $x=0$. Also note the ranges of these functions, especially $\tanh x$: as $x\to\infty$, both $\sinh x$ and $\cosh x$ approach $e^x/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Pronunciation Note:
"cosh" rhymes with "gosh,"
"sinh" rhymes with "pinch," and
"tanh" rhymes with "ranch."

Notes:

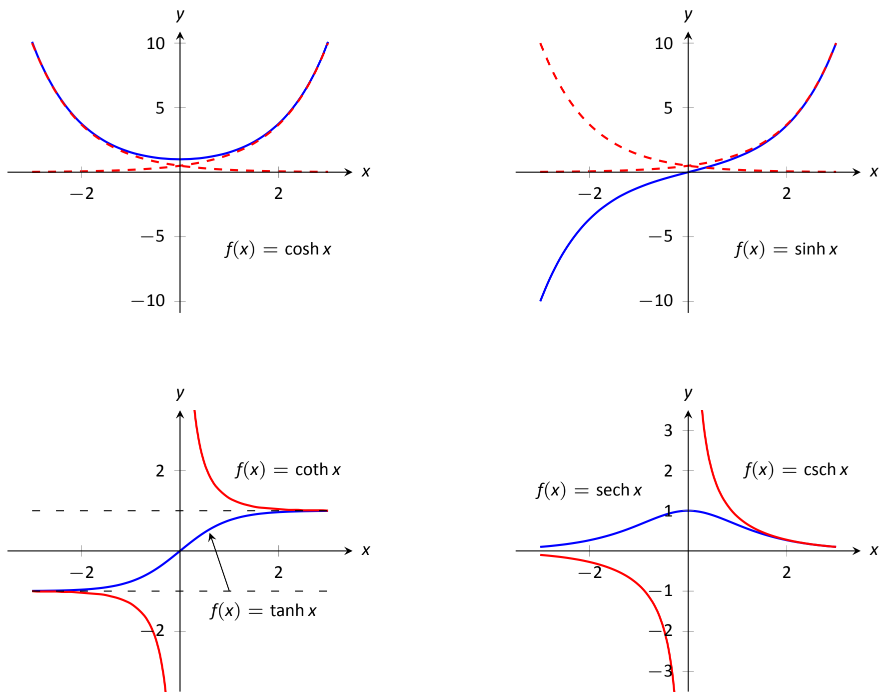


Figure 6.6.2: Graphs of the hyperbolic functions.

Example 6.6.1 Exploring properties of hyperbolic functions

Use Definition 6.6.1 to rewrite the following expressions.

1. $\cosh^2 x - \sinh^2 x$
2. $\tanh^2 x + \operatorname{sech}^2 x$
3. $2\cosh x \sinh x$
4. $\frac{d}{dx}\big(\cosh x\big)$
5. $\frac{d}{dx}\big(\sinh x\big)$
6. $\frac{d}{dx}\big(\tanh x\big)$

Solution

1.
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2e^xe^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^xe^{-x} + e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

Notes:

So $\cosh^2 x - \sinh^2 x = 1$.

2.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x}\\ &= \frac{\sinh^2 x + 1}{\cosh^2 x} \qquad \text{Now use identity from \#1.}\\ &= \frac{\cosh^2 x}{\cosh^2 x} = 1.\end{aligned}$$

So $\tanh^2 x + \operatorname{sech}^2 x = 1$.

3.
$$\begin{aligned}2\cosh x \sinh x &= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)\\ &= 2 \cdot \frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).\end{aligned}$$

Thus $2\cosh x \sinh x = \sinh(2x)$.

4.
$$\begin{aligned}\frac{d}{dx}\big(\cosh x\big) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^x - e^{-x}}{2}\\ &= \sinh x.\end{aligned}$$

So $\frac{d}{dx}\big(\cosh x\big) = \sinh x$.

5.
$$\begin{aligned}\frac{d}{dx}\big(\sinh x\big) &= \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right)\\ &= \frac{e^x + e^{-x}}{2}\\ &= \cosh x.\end{aligned}$$

So $\frac{d}{dx}\big(\sinh x\big) = \cosh x$.

6.
$$\begin{aligned}\frac{d}{dx}\big(\tanh x\big) &= \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right)\\ &= \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x}\\ &= \frac{1}{\cosh^2 x}\\ &= \operatorname{sech}^2 x.\end{aligned}$$

Notes:

So $\frac{d}{dx}\big(\tanh x\big) = \text{sech}^2 x$.

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 6.6.1.

Key Idea 6.6.1 Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \text{sech}^2 x = 1$
3. $\coth^2 x - \text{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2\sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
2. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
3. $\frac{d}{dx}\big(\tanh x\big) = \text{sech}^2 x$
4. $\frac{d}{dx}\big(\text{sech}\, x\big) = -\,\text{sech}\, x \tanh x$
5. $\frac{d}{dx}\big(\text{csch}\, x\big) = -\,\text{csch}\, x \coth x$
6. $\frac{d}{dx}\big(\coth x\big) = -\,\text{csch}^2 x$

Integrals

1. $\displaystyle\int \cosh x \, dx = \sinh x + C$
2. $\displaystyle\int \sinh x \, dx = \cosh x + C$
3. $\displaystyle\int \tanh x \, dx = \ln(\cosh x) + C$
4. $\displaystyle\int \coth x \, dx = \ln|\sinh x| + C$

We practice using Key Idea 6.6.1.

Example 6.6.2 Derivatives and integrals of hyperbolic functions
Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \text{sech}^2(7t-3)\, dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x \, dx$

Solution

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2\sinh 2x$.

   Just to demonstrate that it works, let's also use the Basic Identity found in

Notes:

Key Idea 6.6.1: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) = 2\cosh x \sinh x + 2\sinh x \cosh x$$
$$= 4\cosh x \sinh x.$$

Using another Basic Identity, we can see that $4\cosh x \sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t - 3$ and $du = 7dt$. Applying Key Ideas 6.1.1 and 6.6.1 we have:

$$\int \operatorname{sech}^2(7t-3)\,dt = \frac{1}{7}\tanh(7t-3) + C.$$

3. $$\int_0^{\ln 2} \cosh x\,dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

We can simplify this last expression as $\sinh x$ is based on exponentials:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}.$$

Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integrations, the inverse hyperbolic functions are useful with others. Figure 6.6.3 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 6.6.4.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 6.6.2. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln\big(x + \sqrt{x^2+1}\big)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

Notes:

omain	Range
$[0,\infty)$	$[1,\infty)$
$-\infty,\infty)$	$(-\infty,\infty)$
$-\infty,\infty)$	$(-1,1)$
$[0,\infty)$	$(0,1]$
$0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$0)\cup(0,\infty)$	$(-\infty,-1)\cup(1,\infty)$

Function	Domain	Range
$\cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\text{sech}^{-1} x$	$(0,1]$	$[0,\infty)$
$\text{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth^{-1} x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

e 6.6.3: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

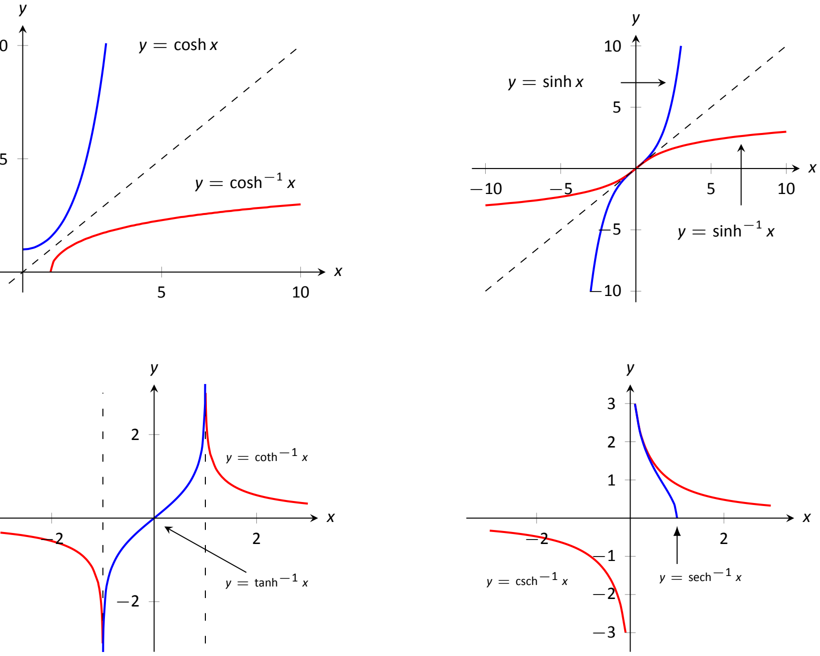


Figure 6.6.4: Graphs of the hyperbolic functions and their inverses.

a 6.6.2 Logarithmic definitions of Inverse Hyperbolic Functions

osh$^{-1} x = \ln\left(x+\sqrt{x^2-1}\right);\ x\ge 1$

4. $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$

nh$^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right);\ |x|<1$

5. $\coth^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{x+1}{x-1}\right);\ |x|>1$

ech$^{-1} x = \ln\left(\dfrac{1+\sqrt{1-x^2}}{x}\right);\ 0<x\le 1$

6. $\text{csch}^{-1} x = \ln\left(\dfrac{1}{x}+\dfrac{\sqrt{1+x^2}}{|x|}\right);\ x\neq 0$

Notes:

The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 6.6.4, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 6.6.2. Again, these latter functions are often more useful than the former. Note how inverse hyperbolic functions can be used to solve integrals we used Trigonometric Substitution to solve in Section 6.4.

Key Idea 6.6.3 Derivatives Involving Inverse Hyperbolic Functions

1. $\dfrac{d}{dx}\left(\cosh^{-1} x\right) = \dfrac{1}{\sqrt{x^2-1}};\ x > 1$
2. $\dfrac{d}{dx}\left(\sinh^{-1} x\right) = \dfrac{1}{\sqrt{x^2+1}}$
3. $\dfrac{d}{dx}\left(\tanh^{-1} x\right) = \dfrac{1}{1-x^2};\ |x| < 1$
4. $\dfrac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \dfrac{-1}{x\sqrt{1-x^2}};\ 0 < x < 1$
5. $\dfrac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \dfrac{-1}{|x|\sqrt{1+x^2}};\ x \neq 0$
6. $\dfrac{d}{dx}\left(\coth^{-1} x\right) = \dfrac{1}{1-x^2};\ |x| > 1$

Key Idea 6.6.4 Integrals Involving Inverse Hyperbolic Functions

1. $$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C;\ 0 < a < x \qquad = \ln\left|x + \sqrt{x^2-a^2}\right| +$$
2. $$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C;\ a > 0 \qquad = \ln\left|x + \sqrt{x^2+a^2}\right| +$$
3. $$\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C & x^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C & a^2 < x^2 \end{cases} \qquad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$
4. $$\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C;\ 0 < x < a \qquad = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right)$$
5. $$\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C;\ x \neq 0,\ a > 0 \qquad = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right|$$

We practice using the derivative and integral formulas in the following example.

Notes:

Example 6.6.3 Derivatives and integrals involving inverse hyperbolic functions

Evaluate the following.

1. $\dfrac{d}{dx}\left[\cosh^{-1}\left(\dfrac{3x-2}{5}\right)\right]$

2. $\displaystyle\int \frac{1}{x^2-1}\,dx$

3. $\displaystyle\int \frac{1}{\sqrt{9x^2+10}}\,dx$

Solution

1. Applying Key Idea 6.6.3 with the Chain Rule gives:
$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2-1}}\cdot\frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{-1}{1-x^2}\,dx$. The second integral can be solved with a direct application of item #3 from Key Idea 6.6.4, with $a=1$. Thus
$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx\\ &= \begin{cases} -\tanh^{-1}(x)+C & x^2<1\\ -\coth^{-1}(x)+C & 1<x^2\end{cases}\\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C\\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \end{aligned} \tag{6.4}$$
We should note that this exact problem was solved at the beginning of Section 6.5. In that example the answer was given as $\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$. Note that this is equivalent to the answer given in Equation 6.4, as $\ln(a/b)=\ln a-\ln b$.

3. This requires a substitution, then item #2 of Key Idea 6.6.4 can be applied. Let $u=3x$, hence $du=3dx$. We have
$$\int \frac{1}{\sqrt{9x^2+10}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+10}}\,du.$$

Notes:

Note $a^2 = 10$, hence $a = \sqrt{10}$. Now apply the integral rule.

$$= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{\sqrt{10}}\right) + C$$
$$= \frac{1}{3}\ln\left|3x + \sqrt{9x^2+10}\right| + C.$$

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses. Four Key Ideas were presented, each including quite a bit of information.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 6.6.4 contains perhaps the most useful information. Know the integration forms it helps evaluate and understand how to use the inverse hyperbolic answer and the logarithmic answer.

The next section takes a brief break from demonstrating new integration techniques. It instead demonstrates a technique of evaluating limits that return indeterminate forms. This technique will be useful in Section 6.8, where limits will arise in the evaluation of certain definite integrals.

Notes:

Exercises 6.6

Terms and Concepts

06 05 ex 01 1. In Key Idea 6.6.1, the equation $\int \tanh x\,dx = \ln(\cosh x) + C$ is given. Why is "$\ln|\cosh x|$" not used – i.e., why are absolute values not necessary?

06 05 ex 02 2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 6.6.1. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?

Problems

06 05 exset 01 **In Exercises 3 – 10, verify the given identity using Definition 6.6.1, as done in Example 6.6.1.**

06 05 ex 03 3. $\coth^2 x - \operatorname{csch}^2 x = 1$

06 05 ex 04 4. $\cosh 2x = \cosh^2 x + \sinh^2 x$

06 05 ex 05 5. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

06 05 ex 06 6. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

06 05 ex 07 7. $\dfrac{d}{dx}\left[\operatorname{sech} x\right] = -\operatorname{sech} x \tanh x$

06 05 ex 08 8. $\dfrac{d}{dx}\left[\coth x\right] = -\operatorname{csch}^2 x$

06 05 ex 09 9. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$

06 05 ex 10 10. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

06 05 exset 02 **In Exercises 11 – 22, find the derivative of the given function.**

06 05 ex 11 11. $f(x) = \sinh 2x$

06 05 ex 42 12. $f(x) = \cosh^2 x$

06 05 ex 12 13. $f(x) = \tanh(x^2)$

06 05 ex 13 14. $f(x) = \ln(\sinh x)$

06 05 ex 14 15. $f(x) = \sinh x \cosh x$

06 05 ex 15 16. $f(x) = x\sinh x - \cosh x$

06 05 ex 16 17. $f(x) = \operatorname{sech}^{-1}(x^2)$

06 05 ex 19 18. $f(x) = \sinh^{-1}(3x)$

06 05 ex 20 19. $f(x) = \cosh^{-1}(2x^2)$

06 05 ex 21 20. $f(x) = \tanh^{-1}(x + 5)$

06 05 ex 17 21. $f(x) = \tanh^{-1}(\cos x)$

06 05 ex 18 22. $f(x) = \cosh^{-1}(\sec x)$

06 05 exset 03 **In Exercises 23 – 28, find the equation of the line tangent to the function at the given x-value.**

06 05 ex 22 23. $f(x) = \sinh x$ at $x = 0$

06 05 ex 23 24. $f(x) = \cosh x$ at $x = \ln 2$

06 05 ex 43 25. $f(x) = \tanh x$ at $x = -\ln 3$

06 05 ex 24 26. $f(x) = \operatorname{sech}^2 x$ at $x = \ln 3$

06 05 ex 25 27. $f(x) = \sinh^{-1} x$ at $x = 0$

06 05 ex 26 28. $f(x) = \cosh^{-1} x$ at $x = \sqrt{2}$

06 05 exset 04 **In Exercises 29 – 44, evaluate the given indefinite integral.**

06 05 ex 27 29. $\displaystyle\int \tanh(2x)\,dx$

06 05 ex 28 30. $\displaystyle\int \cosh(3x - 7)\,dx$

06 05 ex 29 31. $\displaystyle\int \sinh x \cosh x\,dx$

06 02 ex 20 32. $\displaystyle\int x\cosh x\,dx$

06 02 ex 21 33. $\displaystyle\int x\sinh x\,dx$

06 05 ex 45 34. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,dx$

06 05 ex 46 35. $\displaystyle\int \frac{1}{\sqrt{x^2-9}}\,dx$

06 05 ex 30 36. $\displaystyle\int \frac{1}{9 - x^2}\,dx$

06 05 ex 31 37. $\displaystyle\int \frac{2x}{\sqrt{x^4-4}}\,dx$

06 05 ex 32 38. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1+x^3}}\,dx$

06 05 ex 33 39. $\displaystyle\int \frac{1}{x^4 - 16}\,dx$

06 05 ex 34 40. $\displaystyle\int \frac{1}{x^2 + x}\,dx$

06 05 ex 35 41. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

06 05 ex 36 42. $\displaystyle\int \sinh^{-1} x\,dx$

06 05 ex 37 43. $\displaystyle\int \tanh^{-1} x\,dx$

06 05 ex 38 44. $\displaystyle\int \operatorname{sech} x\,dx$ (Hint: multiply by $\frac{\cosh x}{\cosh x}$; set $u = \sinh x$.)

06 05 exset 05 **In Exercises 45 – 48, evaluate the given definite integral.**

06 05 ex 39 45. $\displaystyle\int_{-1}^{1} \sinh x\,dx$

06 05 ex 40 46. $\displaystyle\int_{-\ln 2}^{\ln 2} \cosh x\,dx$

06 05 ex 41 47. $\displaystyle\int_{0}^{1} \tanh^{-1} x\,dx$

06 05 ex 44 48. $\displaystyle\int_{0}^{2} \frac{1}{\sqrt{x^2+1}}\,dx$

6.7 L'Hôpital's Rule

While this chapter is devoted to learning techniques of integration, this section is not about integration. Rather, it is concerned with a technique of evaluating certain limits that will be useful in the following section, where integration is once more discussed.

Our treatment of limits exposed us to "0/0", an indeterminate form. If $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, we do not conclude that $\lim\limits_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression 0/0 has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces l'Hôpital's Rule, a method of resolving limits that produce the indeterminate forms 0/0 and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

Theorem 6.7.1 L'Hôpital's Rule, Part 1

Let $\lim\limits_{x\to c} f(x) = 0$ and $\lim\limits_{x\to c} g(x) = 0$, where f and g are differentiable functions on an open interval I containing c, and $g'(x) \neq 0$ on I except possibly at c. Then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}.$$

We demonstrate the use of l'Hôpital's Rule in the following examples; we will often use "LHR" as an abbreviation of "l'Hôpital's Rule."

Notes:

Example 6.7.1 Using l'Hôpital's Rule

Evaluate the following limits, using l'Hôpital's Rule as needed.

1. $\displaystyle \lim_{x\to 0} \frac{\sin x}{x}$
2. $\displaystyle \lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x}$
3. $\displaystyle \lim_{x\to 0} \frac{x^2}{1-\cos x}$
4. $\displaystyle \lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$

Solution

1. We proved this limit is 1 in Example 1.3.4 using the Squeeze Theorem. Here we use l'Hôpital's Rule to show its power.
$$\lim_{x\to 0} \frac{\sin x}{x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$
2. $$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x} \stackrel{\text{by LHR}}{=} \lim_{x\to 1} \frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$
3. $$\lim_{x\to 0} \frac{x^2}{1-\cos x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0} \frac{2x}{\sin x}.$$
This latter limit also evaluates to the 0/0 indeterminate form. To evaluate it, we apply l'Hôpital's Rule again.
$$\lim_{x\to 0} \frac{2x}{\sin x} \stackrel{\text{by LHR}}{=} \frac{2}{\cos x} = 2.$$
Thus $\displaystyle \lim_{x\to 0} \frac{x^2}{1-\cos x} = 2.$
4. We already know how to evaluate this limit; first factor the numerator and denominator. We then have:
$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2} \frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2} \frac{x+3}{x-1} = 5.$$
We now show how to solve this using l'Hôpital's Rule.
$$\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2} \stackrel{\text{by LHR}}{=} \lim_{x\to 2} \frac{2x+1}{2x-3} = 5.$$

Note that at each step where l'Hôpital's Rule was applied, it was *needed*: the initial limit returned the indeterminate form of "0/0." If the initial limit returns, for example, 1/2, then l'Hôpital's Rule does not apply.

Notes:

The following theorem extends our initial version of l'Hôpital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Theorem 6.7.2 L'Hôpital's Rule, Part 2

1. Let $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an open interval I containing a. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

2. Let f and g be differentiable functions on the open interval (a,∞) for some value a, where $g'(x) \neq 0$ on (a,∞) and $\lim_{x\to\infty} f(x)/g(x)$ returns either 0/0 or ∞/∞. Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

A similar statement can be made for limits where x approaches $-\infty$.

Example 6.7.2 Using l'Hôpital's Rule with limits involving ∞

Evaluate the following limits.

1. $\lim_{x\to\infty} \dfrac{3x^2 - 100x + 2}{4x^2 + 5x - 1000}$
2. $\lim_{x\to\infty} \dfrac{e^x}{x^3}$.

Solution

1. We can evaluate this limit already using Theorem 1.6.1; the answer is 3/4. We apply l'Hôpital's Rule to demonstrate its applicability.

$$\lim_{x\to\infty} \frac{3x^2 - 100x + 2}{4x^2 + 5x - 1000} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6x - 100}{8x + 5} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$$

2. $\lim_{x\to\infty} \dfrac{e^x}{x^3} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \dfrac{e^x}{3x^2} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \dfrac{e^x}{6x} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \dfrac{e^x}{6} = \infty.$

Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This

Notes:

has important implications in computing when considering efficiency of algorithms.)

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hôpital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0 \cdot \infty$ or $\infty - \infty$, we can sometimes apply algebra to rewrite the limit so that l'Hôpital's Rule can be applied. We demonstrate the general idea in the next example.

Example 6.7.3 Applying l'Hôpital's Rule to other indeterminate forms
Evaluate the following limits.

1. $\displaystyle\lim_{x\to 0^+} x \cdot e^{1/x}$
2. $\displaystyle\lim_{x\to 0^-} x \cdot e^{1/x}$
3. $\displaystyle\lim_{x\to \infty} \ln(x+1) - \ln x$
4. $\displaystyle\lim_{x\to \infty} x^2 - e^x$

Solution

1. As $x \to 0^+$, $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which l'Hôpital's Rule can be applied.
$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$
Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.

2. As $x \to 0^-$, $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that
$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$

3. This limit initially evaluates to the indeterminate form $\infty - \infty$. By applying a logarithmic rule, we can rewrite the limit as
$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$
As $x \to \infty$, the argument of the ln term approaches ∞/∞, to which we can apply l'Hôpital's Rule.
$$\lim_{x\to\infty} \frac{x+1}{x} \stackrel{\text{by LHR}}{=} \frac{1}{1} = 1.$$

Notes:

Since $x \to \infty$ implies $\dfrac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

4. The limit $\lim\limits_{x\to\infty} x^2 - e^x$ initially returns the indeterminate form $\infty - \infty$. We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \dfrac{e^x}{x^2}\right)$. We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\lim\limits_{x\to\infty} x^2 - e^x = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Key Idea 6.7.1 Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

If $\lim\limits_{x\to c} \ln\big(f(x)\big) = L$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} e^{\ln(f(x))} = e^L$.

Notes:

Example 6.7.4 Using l'Hôpital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

1. $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$ 2. $\displaystyle\lim_{x\to 0^+} x^x$.

Solution

1. This is equivalent to a special limit given in Theorem 1.3.5; these limits have important applications within mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate $\displaystyle\lim_{x\to\infty} f(x)$. Let's first evaluate $\displaystyle\lim_{x\to\infty} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty}\ln\big(f(x)\big) &= \lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right)^x\\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)\\ &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

This produces the indeterminate form 0/0, so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to\infty}\frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{(-1/x^2)}\\ &= \lim_{x\to\infty}\frac{1}{1+1/x}\\ &= 1.\end{aligned}$$

Thus $\displaystyle\lim_{x\to\infty}\ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 6.7.1.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider

Notes:

first $\lim_{x\to 0^+} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to 0^+} \ln\big(f(x)\big) &= \lim_{x\to 0^+} \ln\left(x^x\right)\\ &= \lim_{x\to 0^+} x\ln x\\ &= \lim_{x\to 0^+} \frac{\ln x}{1/x}.\end{aligned}$$

This produces the indeterminate form $-\infty/\infty$ so we apply l'Hôpital's Rule.

$$\begin{aligned}&= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}\\ &= \lim_{x\to 0^+} -x\\ &= 0.\end{aligned}$$

Thus $\lim_{x\to 0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 6.7.1.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 6.7.1.

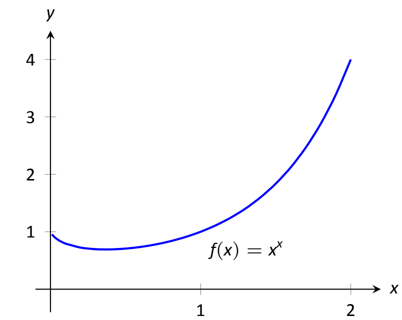


Figure 6.7.1: A graph of $f(x) = x^x$ supporting the fact that as $x \to 0^+$, $f(x) \to 1$.

Our brief revisit of limits will be rewarded in the next section where we consider *improper integration.* So far, we have only considered definite integrals where the bounds are finite numbers, such as $\int_0^1 f(x)\,dx$. Improper integration considers integrals where one, or both, of the bounds are "infinity." Such integrals have many uses and applications, in addition to generating ideas that are enlightening.

Notes:

Exercises 6.7

Terms and Concepts

06 06 ex 01 1. List the different indeterminate forms described in this section.

06 06 ex 02 2. T/F: l'Hôpital's Rule provides a faster method of computing derivatives.

06 06 ex 03 3. T/F: l'Hôpital's Rule states that $\dfrac{d}{dx}\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{f'(x)}{g'(x)}$.

06 06 ex 04 4. Explain what the indeterminate form "1^∞" means.

06 06 ex 05 5. Fill in the blanks:

The Quotient Rule is applied to $\dfrac{f(x)}{g(x)}$ when taking ________;

l'Hôpital's Rule is applied to $\dfrac{f(x)}{g(x)}$ when taking certain ________.

06 06 ex 06 6. Create (but do not evaluate!) a limit that returns "∞^0".

06 06 ex 07 7. Create a function $f(x)$ such that $\lim\limits_{x\to 1} f(x)$ returns "0^0".

06 06 ex 53 8. Create a function $f(x)$ such that $\lim\limits_{x\to\infty} f(x)$ returns "$0\cdot\infty$".

Problems

06 06 exset 01 **In Exercises 9 – 54, evaluate the given limit.**

06 06 ex 08 9. $\displaystyle\lim_{x\to 1} \frac{x^2+x-2}{x-1}$

06 06 ex 09 10. $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x^2-7x+10}$

06 06 ex 10 11. $\displaystyle\lim_{x\to \pi} \frac{\sin x}{x-\pi}$

06 06 ex 11 12. $\displaystyle\lim_{x\to \pi/4} \frac{\sin x-\cos x}{\cos(2x)}$

06 06 ex 12 13. $\displaystyle\lim_{x\to 0} \frac{\sin(5x)}{x}$

06 06 ex 13 14. $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{x+2}$

06 06 ex 14 15. $\displaystyle\lim_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

06 06 ex 15 16. $\displaystyle\lim_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$

06 06 ex 16 17. $\displaystyle\lim_{x\to 0^+} \frac{e^x-1}{x^2}$

06 06 ex 17 18. $\displaystyle\lim_{x\to 0^+} \frac{e^x-x-1}{x^2}$

06 06 ex 18 19. $\displaystyle\lim_{x\to 0^+} \frac{x-\sin x}{x^3-x^2}$

06 06 ex 19 20. $\displaystyle\lim_{x\to \infty} \frac{x^4}{e^x}$

06 06 ex 20 21. $\displaystyle\lim_{x\to \infty} \frac{\sqrt{x}}{e^x}$

06 06 ex 54 22. $\displaystyle\lim_{x\to \infty} \frac{1}{x^2}e^x$

06 06 ex 21 23. $\displaystyle\lim_{x\to \infty} \frac{e^x}{\sqrt{x}}$

06 06 ex 22 24. $\displaystyle\lim_{x\to \infty} \frac{e^x}{2^x}$

06 06 ex 23 25. $\displaystyle\lim_{x\to \infty} \frac{e^x}{3^x}$

06 06 ex 24 26. $\displaystyle\lim_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-7x^2+15x-9}$

06 06 ex 25 27. $\displaystyle\lim_{x\to -2} \frac{x^3+4x^2+4x}{x^3+7x^2+16x+12}$

06 06 ex 26 28. $\displaystyle\lim_{x\to \infty} \frac{\ln x}{x}$

06 06 ex 27 29. $\displaystyle\lim_{x\to \infty} \frac{\ln(x^2)}{x}$

06 06 ex 28 30. $\displaystyle\lim_{x\to \infty} \frac{\left(\ln x\right)^2}{x}$

06 06 ex 29 31. $\displaystyle\lim_{x\to 0^+} x\cdot\ln x$

06 06 ex 30 32. $\displaystyle\lim_{x\to 0^+} \sqrt{x}\cdot\ln x$

06 06 ex 52 33. $\displaystyle\lim_{x\to 0^+} xe^{1/x}$

06 06 ex 31 34. $\displaystyle\lim_{x\to \infty} x^3-x^2$

06 06 ex 32 35. $\displaystyle\lim_{x\to \infty} \sqrt{x}-\ln x$

06 06 ex 33 36. $\displaystyle\lim_{x\to -\infty} xe^x$

06 06 ex 34 37. $\displaystyle\lim_{x\to 0^+} \frac{1}{x^2}e^{-1/x}$

06 06 ex 35 38. $\displaystyle\lim_{x\to 0^+} (1+x)^{1/x}$

06 06 ex 36 39. $\lim_{x\to 0^+} (2x)^x$

06 06 ex 37 40. $\lim_{x\to 0^+} (2/x)^x$

06 06 ex 38 41. $\lim_{x\to 0^+} (\sin x)^x$ Hint: use the Squeeze Theorem.

06 06 ex 39 42. $\lim_{x\to 1^+} (1-x)^{1-x}$

06 06 ex 40 43. $\lim_{x\to \infty} (x)^{1/x}$

06 06 ex 41 44. $\lim_{x\to \infty} (1/x)^x$

06 06 ex 42 45. $\lim_{x\to 1^+} (\ln x)^{1-x}$

06 06 ex 43 46. $\lim_{x\to \infty} (1+x)^{1/x}$

06 06 ex 44 47. $\lim_{x\to \infty} (1+x^2)^{1/x}$

06 06 ex 45 48. $\lim_{x\to \pi/2} \tan x \cos x$

06 06 ex 46 49. $\lim_{x\to \pi/2} \tan x \sin(2x)$

06 06 ex 47 50. $\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$

06 06 ex 48 51. $\lim_{x\to 3^+} \frac{5}{x^2-9} - \frac{x}{x-3}$

06 06 ex 49 52. $\lim_{x\to \infty} x\tan(1/x)$

06 06 ex 50 53. $\lim_{x\to \infty} \frac{(\ln x)^3}{x}$

06 06 ex 51 54. $\lim_{x\to 1} \frac{x^2+x-2}{\ln x}$

6.8 Improper Integration

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608,$
- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698,$
- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \approx 1.5707.$

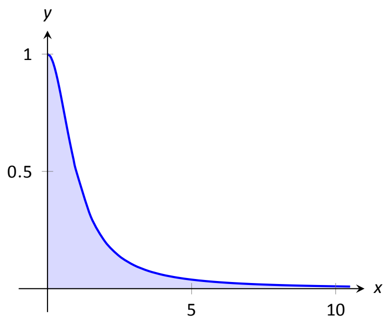


Figure 6.8.1: Graphing $f(x) = \dfrac{1}{1+x^2}$.

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 6.8.1). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a, b]$, was a finite interval, and
2. The function $f(x)$ was continuous on $[a, b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

Notes:

Improper Integrals with Infinite Bounds

Definition 6.8.1 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a, \infty)$. Define

$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty, b]$. Define

$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty} \int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty, \infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty} \int_a^c f(x)\,dx + \lim_{b\to\infty} \int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

Example 6.8.1 Evaluating improper integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution

1.
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty} \left.\frac{-1}{x}\right|_1^b$$
$$= \lim_{b\to\infty} \frac{-1}{b} + 1$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 6.8.2.

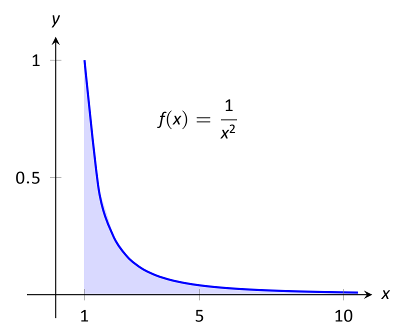


Figure 6.8.2: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.8.1.

Notes:

2. $$\begin{aligned}\int_1^\infty \frac{1}{x}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx\\ &= \lim_{b\to\infty} \ln|x|\Big|_1^b\\ &= \lim_{b\to\infty}\ln(b)\\ &= \infty.\end{aligned}$$

The limit does not exist, hence the improper integral $\int_1^\infty \frac{1}{x}\,dx$ diverges. Compare the graphs in Figures 6.8.2 and 6.8.3; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

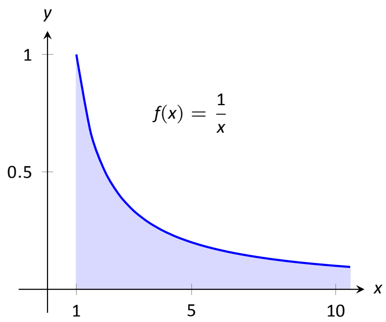


Figure 6.8.3: A graph of $f(x) = \frac{1}{x}$ in Example 6.8.1.

3. $$\begin{aligned}\int_{-\infty}^0 e^x\,dx &= \lim_{a\to-\infty}\int_a^0 e^x\,dx\\ &= \lim_{a\to-\infty} e^x\Big|_a^0\\ &= \lim_{a\to-\infty} e^0 - e^a\\ &= 1.\end{aligned}$$

A graph of the area defined by this integral is given in Figure 6.8.4.

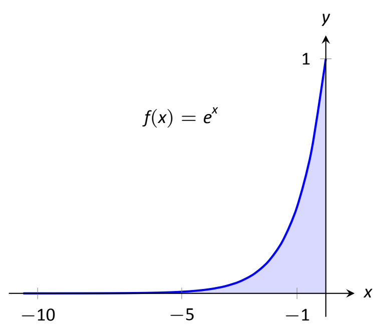


Figure 6.8.4: A graph of $f(x) = e^x$ in Example 6.8.1.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 6.8.1. Any value of c is fine; we choose $c = 0$.

$$\begin{aligned}\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx &= \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx\\ &= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty}\tan^{-1}x\Big|_0^b\\ &= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)\\ &= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).\end{aligned}$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 6.8.5.

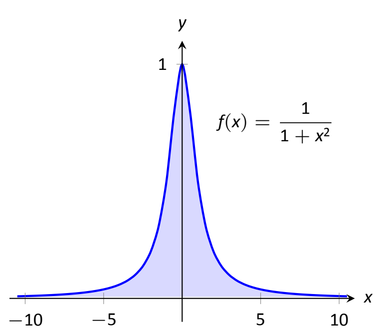


Figure 6.8.5: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 6.8.1.

Notes:

The previous section introduced l'Hôpital's Rule, a method of evaluating limits that return indeterminate forms. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

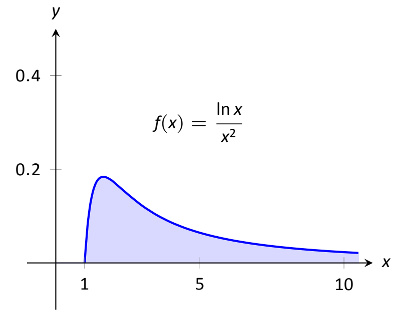


Figure 6.8.6: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 6.8.2.

Example 6.8.2 Improper integration and l'Hôpital's Rule

Evaluate the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\,dx$. Then

$$\begin{aligned}
\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx \\
&= \lim_{b\to\infty}\left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\right) \\
&= \lim_{b\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^b \\
&= \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1)\right).
\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\displaystyle\lim_{b\to\infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\displaystyle\lim_{b\to\infty}\frac{\ln b}{b}$ with l'Hôpital's Rule. We have:

$$\begin{aligned}
\lim_{b\to\infty}\frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty}\frac{1/b}{1} \\
&= 0.
\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Notes:

Note: In Definition 6.8.2, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Definition 6.8.2 Improper Integration with Infinite Range

Let $f(x)$ be a continuous function on $[a, b]$ except at c, $a \le c \le b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

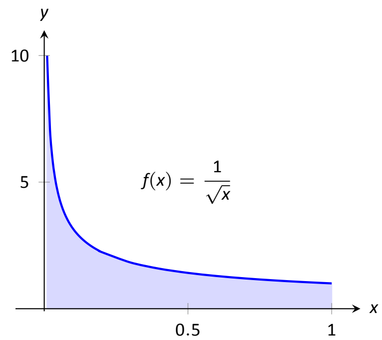


Figure 6.8.7: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 6.8.3.

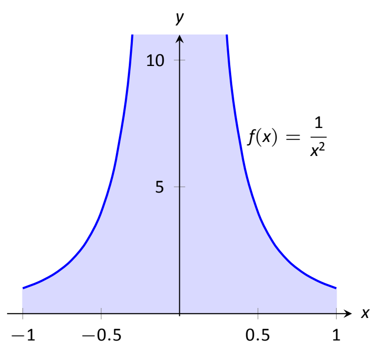


Figure 6.8.8: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.8.3.

Example 6.8.3 Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$
2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

SOLUTION

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 6.8.7. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 6.8.8, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= -\frac{1}{x}\Big|_{-1}^1\\ &= -1-(1)\\ &= -2!\end{aligned}$$

Notes:

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 6.8.2.

$$\begin{aligned}\int_{-1}^{1}\frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-1}^{t}\frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_{t}^{1}\frac{1}{x^2}\,dx\\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^{t} + \lim_{t\to 0^+} -\frac{1}{x}\Big|_{t}^{1}\\ &= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t}\\ &\Rightarrow \Big(\infty - 1\Big) + \Big(-1+\infty\Big).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behavior of functions of the form $\dfrac{1}{x^p}$.

Example 6.8.4 Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

Solution We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx\\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad (\text{assume } p\neq 1)\\ &= \lim_{b\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{1-p}\big(b^{1-p}-1^{1-p}\big).\end{aligned}$$

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1-p<0 \Rightarrow 1<p$.

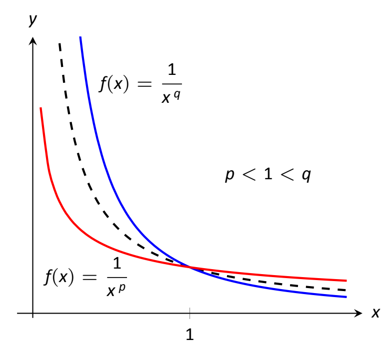


Figure 6.8.9: Plotting functions of the form $1/x^p$ in Example 6.8.4.

Notes:

Our analysis shows that if $p > 1$, then $\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p < 1$ the improper integral diverges; we showed in Example 6.8.1 that when $p = 1$ the integral also diverges.

Figure 6.8.9 graphs $y = 1/x$ with a dashed line, along with graphs of $y = 1/x^p$, $p < 1$, and $y = 1/x^q$, $q > 1$. Somehow the dashed line forms a dividing line between convergence and divergence.

The result of Example 6.8.4 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 6.8.1 Convergence of Improper Integrals $\int_1^\infty \frac{1}{x^p}\,dx$ and $\int_0^1 \frac{1}{x^p}\,dx$.

1. The improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p > 1$ and diverges when $p \leq 1$.
2. The improper integral $\int_0^1 \frac{1}{x^p}\,dx$ converges when $p < 1$ and diverges when $p \geq 1$.

Note: We used the upper and lower bound of "1" in Key Idea 6.8.1 for convenience. It can be replaced by any a where $a > 0$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare to as their convergence on certain intervals is known. This is described in the following theorem.

Theorem 6.8.1 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a,\infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a,\infty)$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Notes:

Example 6.8.5 Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\displaystyle\int_1^\infty e^{-x^2}\,dx$ 2. $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$

Solution

1. The function $f(x)=e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x)=1/x^2$, and as demonstrated in Figure 6.8.10, $e^{-x^2}<1/x^2$ on $[1,\infty)$. We know from Key Idea 6.8.1 that $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\displaystyle\int_1^\infty e^{-x^2}\,dx$ also converges.

2. Note that for large values of x, $\dfrac{1}{\sqrt{x^2-x}}\approx\dfrac{1}{\sqrt{x^2}}=\dfrac{1}{x}$. We know from Key Idea 6.8.1 and the subsequent note that $\displaystyle\int_3^\infty\frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x>0$, we have $x=\sqrt{x^2}>\sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving

   $$\frac{1}{x}<\frac{1}{\sqrt{x^2-x}}.$$

   Using Theorem 6.8.1, we conclude that since $\displaystyle\int_3^\infty\frac{1}{x}\,dx$ diverges, $\displaystyle\int_3^\infty\frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 6.8.11 illustrates this.

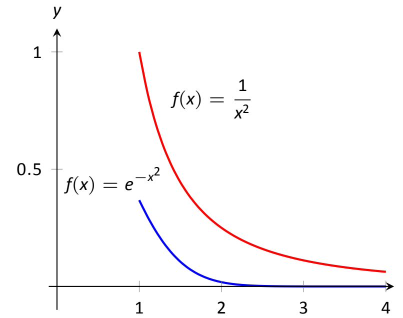


Figure 6.8.10: Graphs of $f(x)=e^{-x^2}$ and $f(x)=1/x^2$ in Example 6.8.5.

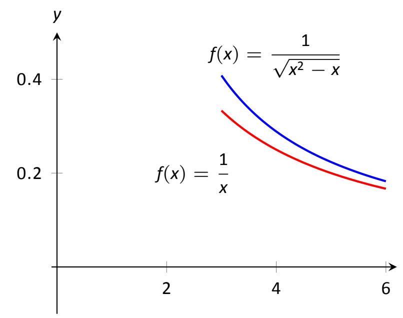


Figure 6.8.11: Graphs of $f(x)=1/\sqrt{x^2-x}$ and $f(x)=1/x$ in Example 6.8.5.

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\dfrac{1}{x}<\dfrac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\displaystyle\int_3^\infty\frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\dfrac{1}{x}>\dfrac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 6.8.1.

In cases like this (and many more) it is useful to employ the following theorem.

Notes:

Theorem 6.8.2 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a,\infty)$ where $f(x)>0$ and $g(x)>0$ for all x. If

$$\lim_{x\to\infty}\frac{f(x)}{g(x)}=L, \qquad 0<L<\infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 6.8.6 Determining convergence of improper integrals

Determine the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the quadratic inside the square root function will begin to behave much like $y=x$. So we compare $\dfrac{1}{\sqrt{x^2+2x+5}}$ to $\dfrac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty}\frac{1/\sqrt{x^2+2x+5}}{1/x}=\lim_{x\to\infty}\frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hôpital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hôpital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim_{x\to c} f(x)=L$, then $\lim_{x\to c} f(x)^2=L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty}\frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 6.8.12 graphs $f(x)=1/\sqrt{x^2+2x+5}$ and $f(x)=1/x$, illustrating that as x gets large, the functions become indistinguishable.

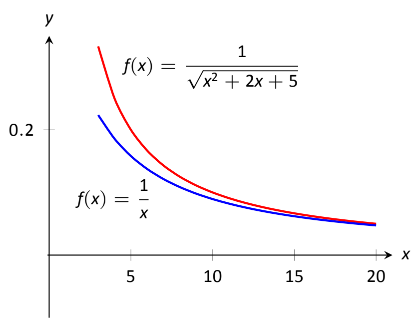


Figure 6.8.12: Graphing $f(x)=\frac{1}{\sqrt{x^2+2x+5}}$ and $f(x)=\frac{1}{x}$ in Example 6.8.6.

Notes:

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions and introduced the hyperbolic functions, which are closely related to the trigonometric functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Notes:

Exercises 6.8

Terms and Concepts

06 07 ex 03 1. The definite integral was defined with what two stipulations?

06 07 ex 01 2. If $\lim\limits_{b\to\infty} \int_0^b f(x)\,dx$ exists, then the integral $\int_0^\infty f(x)\,dx$ is said to ________________.

06 07 ex 02 3. If $\int_1^\infty f(x)\,dx = 10$, and $0 \le g(x) \le f(x)$ for all x, then we know that $\int_1^\infty g(x)\,dx$ ________________.

06 07 ex 04 4. For what values of p will $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

06 07 ex 05 5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\,dx$ converge?

06 07 ex 06 6. For what values of p will $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Problems

06 07 exset 01 **In Exercises 7 – 34, evaluate the given improper integral.**

06 07 ex 07 7. $\int_0^\infty e^{5-2x}\,dx$

06 07 ex 08 8. $\int_1^\infty \frac{1}{x^3}\,dx$

06 07 ex 09 9. $\int_1^\infty x^{-4}\,dx$

06 07 ex 10 10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

06 07 ex 11 11. $\int_{-\infty}^0 2^x\,dx$

06 07 ex 12 12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\,dx$

06 07 ex 13 13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

06 07 ex 14 14. $\int_3^\infty \frac{1}{x^2-4}\,dx$

06 07 ex 15 15. $\int_2^\infty \frac{1}{(x-1)^2}\,dx$

06 07 ex 16 16. $\int_1^2 \frac{1}{(x-1)^2}\,dx$

06 07 ex 17 17. $\int_2^\infty \frac{1}{x-1}\,dx$

06 07 ex 18 18. $\int_1^2 \frac{1}{x-1}\,dx$

06 07 ex 43 19. $\int_{-1}^1 \frac{1}{x}\,dx$

06 07 ex 19 20. $\int_1^3 \frac{1}{x-2}\,dx$

06 07 ex 20 21. $\int_0^\pi \sec^2 x\,dx$

06 07 ex 21 22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

06 07 ex 22 23. $\int_0^\infty xe^{-x}\,dx$

06 07 ex 23 24. $\int_0^\infty xe^{-x^2}\,dx$

06 07 ex 24 25. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

06 07 ex 25 26. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

06 07 ex 26 27. $\int_0^1 x\ln x\,dx$

06 07 ex 44 28. $\int_0^1 x^2\ln x\,dx$

06 07 ex 27 29. $\int_1^\infty \frac{\ln x}{x}\,dx$

06 07 ex 28 30. $\int_0^1 \ln x\,dx$

06 07 ex 29 31. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

06 07 ex 30 32. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$

06 07 ex 31 33. $\int_0^\infty e^{-x}\sin x\,dx$

06 07 ex 32 34. $\int_0^\infty e^{-x}\cos x\,dx$

06 07 exset 02

In Exercises 35 – 44, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

06 07 ex 33 35. $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

06 07 ex 34 36. $\displaystyle\int_{2}^{\infty} \frac{4}{\sqrt{7x^3-x}}\,dx$

06 07 ex 35 37. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

06 07 ex 36 38. $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\,dx$

06 07 ex 37 39. $\displaystyle\int_{5}^{\infty} e^{-x^2+3x+1}\,dx$

06 07 ex 38 40. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{e^x}\,dx$

06 07 ex 39 41. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+\sin x}\,dx$

06 07 ex 42 42. $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+\cos x}\,dx$

06 07 ex 40 43. $\displaystyle\int_{0}^{\infty} \frac{1}{x+e^x}\,dx$

06 07 ex 41 44. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x-x}\,dx$

A: Solutions To Selected Problems

Chapter 6

Section 6.1

06 01 ex 01 1. Chain Rule.

06 01 ex 02 2. T

06 01 ex 03 3. $\frac{1}{8}(x^3-5)^8+C$

06 01 ex 04 4. $\frac{1}{4}(x^2-5x+7)^4+C$

06 01 ex 05 5. $\frac{1}{18}\left(x^2+1\right)^9+C$

06 01 ex 06 6. $\frac{1}{3}(3x^2+7x-1)^6+C$

06 01 ex 11 7. $\frac{1}{2}\ln|2x+7|+C$

06 01 ex 12 8. $\sqrt{2x+3}+C$

06 01 ex 13 9. $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}+C=\frac{2}{3}(x-6)\sqrt{x+3}+C$

06 01 ex 17 10. $\frac{2}{21}x^{3/2}\left(3x^2-7\right)+C$

06 01 ex 18 11. $2e^{\sqrt{x}}+C$

06 01 ex 19 12. $\frac{2\sqrt{x^5+1}}{5}+C$

06 01 ex 20 13. $-\frac{1}{2x^2}-\frac{1}{x}+C$

06 01 ex 21 14. $\frac{\ln^2(x)}{2}+C$

06 01 ex 08 15. $\frac{\sin^3(x)}{3}+C$

06 01 ex 84 16. $-\frac{\cos^4(x)}{4}+C$

06 01 ex 09 17. $-\frac{1}{6}\sin(3-6x)+C$

06 01 ex 10 18. $-\tan(4-x)+C$

06 01 ex 16 19. $\frac{1}{2}\ln|\sec(2x)+\tan(2x)|+C$

06 01 ex 22 20. $\frac{\tan^3(x)}{3}+C$

06 01 ex 25 21. $\frac{\sin\left(x^2\right)}{2}+C$

06 01 ex 27 22. $\tan(x)-x+C$

06 01 ex 82 23. The key is to rewrite $\cot x$ as $\cos x/\sin x$, and let $u=\sin x$.

06 01 ex 83 24. The key is to multiply $\csc x$ by 1 in the form $(\csc x+\cot x)/(\csc x+\cot x)$.

06 01 ex 29 25. $\frac{1}{3}e^{3x-1}+C$

06 01 ex 30 26. $\frac{e^{x^3}}{3}+C$

06 01 ex 31 27. $\frac{1}{2}e^{(x-1)^2}+C$

06 01 ex 32 28. $x-e^{-x}+C$

06 01 ex 85 29. $\ln\left(e^x+1\right)+C$

06 01 ex 33 30. $\frac{e^{-3x}}{3}-e^{-x}+C$

06 01 ex 34 31. $\frac{27^x}{\ln 27}+C$

06 01 ex 35 32. $\frac{16^x}{\ln(16)}+C$

06 01 ex 36 33. $\frac{1}{2}\ln^2(x)+C$

06 01 ex 37 34. $\frac{\left(\ln x\right)^3}{3}+C$

06 01 ex 38 35. $\frac{3}{2}(\ln x)^2+C$

06 01 ex 39 36. $\frac{1}{2}\ln\left(\ln\left(x^2\right)\right)+C$

06 01 ex 40 37. $\frac{x^2}{2}+3x+\ln|x|+C$

06 01 ex 41 38. $\frac{x^3}{3}+\frac{x^2}{2}+x+\ln|x|+C$

06 01 ex 42 39. $\frac{x^3}{3}-\frac{x^2}{2}+x-2\ln|x+1|+C$

06 01 ex 43 40. $\frac{1}{2}\left(x^2+10x+20\ln|x-3|\right)+C$

06 01 ex 44 41. $\frac{3}{2}x^2-8x+15\ln|x+1|+C$

06 01 ex 45 42. $\frac{1}{3}\ln\left|x^2+3x+3\right|+\frac{\ln|x|}{3}+C$

06 01 ex 50 43. $\sqrt{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)+C$

06 01 ex 51 44. $3\sin^{-1}\left(\frac{x}{3}\right)+C$

06 01 ex 52 45. $14\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$

06 01 ex 53 46. $\frac{2}{3}\sec^{-1}(|x|/3)+C$

06 01 ex 54 47. $\frac{5}{4}\sec^{-1}(|x|/4)+C$

06 01 ex 55 48. $\frac{1}{2}\sin^{-1}\left(x^2\right)+C$

06 01 ex 60 49. $\frac{\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)}{\sqrt{7}}+C$

06 01 ex 61 50. $2\sin^{-1}\left(\frac{x-3}{4}\right)+C$

06 01 ex 62 51. $3\sin^{-1}\left(\frac{x-4}{5}\right)+C$

06 01 ex 63 52. $\tan^{-1}\left(\frac{x+3}{5}\right)+C$

06 01 ex 14 53. $-\frac{1}{3\left(x^3+3\right)}+C$

06 01 ex 07 54. $\frac{1}{45}(5x^3+5x^2+2)^9+C$

06 01 ex 15 55. $-\sqrt{1-x^2}+C$

06 01 ex 26 56. $-\frac{1}{3}\cot\left(x^3+1\right)+C$

06 01 ex 28 57. $-\frac{2}{3}\cos^{\frac{3}{2}}(x)+C$

06 01 ex 86 58. $-\cos\left(5x+1\right)/5+C$

06 01 ex 23 59. $\ln|x-5|+C$

06 01 ex 24 60. $\frac{7}{3}\ln|3x+2|+C$

06 01 ex 46 61. $\frac{3x^2}{2}+\ln\left|x^2+3x+5\right|-5x+C$

06 01 ex 47 62. $\ln\left|x^2+7x+3\right|+C$

06 01 ex 48 63. $3\ln\left|3x^2+9x+7\right|+C$

06 01 ex 49 64. $-\frac{x^2}{2}+2\ln\left|x^2-7x+1\right|+7x+C$

06 01 ex 56 65. $\frac{1}{18}\tan^{-1}\left(\frac{x^2}{9}\right)+C$

06 01 ex 57 66. $\tan^{-1}(2x)+C$

06 01 ex 58 67. $\sec^{-1}(|2x|)+C$

06 01 ex 59 68. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$

06 01 ex 64 69. $\frac{3}{2}\ln\left|x^2-2x+10\right|+\frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right)+C$

06 01 ex 65 70. $\frac{19}{5}\tan^{-1}\left(\frac{x+6}{5}\right)-\ln\left|x^2+12x+61\right|+C$

06 01 ex 66 71. $\frac{15}{2}\ln\left|x^2-10x+32\right|+x+\frac{41\tan^{-1}\left(\frac{x-5}{\sqrt{7}}\right)}{\sqrt{7}}+C$

06 01 ex 67 72. $\frac{x^2}{2} - \frac{9}{2}\ln\left|x^2+9\right| + C$

06 01 ex 68 73. $\frac{x^2}{2} + 3\ln\left|x^2+4x+9\right| - 4x + \frac{24\tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right)}{\sqrt{5}} + C$

06 01 ex 69 74. $-\tan^{-1}(\cos(x)) + C$

06 01 ex 70 75. $\tan^{-1}(\sin(x)) + C$

06 01 ex 71 76. $\ln|\sec x + \tan x| + C$ (integrand simplifies to $\sec x$)

06 01 ex 72 77. $3\sqrt{x^2-2x-6} + C$

06 01 ex 73 78. $\sqrt{x^2-6x+8} + C$

06 01 ex 74 79. $-\ln 2$

06 01 ex 75 80. $352/15$

06 01 ex 76 81. $2/3$

06 01 ex 77 82. $1/5$

06 01 ex 78 83. $(1-e)/2$

06 01 ex 79 84. $\pi/2$

06 01 ex 80 85. $\pi/2$

06 01 ex 81 86. $\pi/6$

Section 6.2

06 02 ex 01 1. T

06 02 ex 02 2. F

06 02 ex 03 3. Determining which functions in the integrand to set equal to "u" and which to set equal to "dv".

06 02 ex 50 4. F; it is not uncommon to need to use Integration by Parts several times to fully evaluate an integral.

06 02 ex 04 5. $\sin x - x\cos x + C$

06 02 ex 05 6. $-e^{-x} - xe^{-x} + C$

06 02 ex 06 7. $-x^2\cos x + 2x\sin x + 2\cos x + C$

06 02 ex 07 8. $-x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C$

06 02 ex 08 9. $1/2e^{x^2} + C$

06 02 ex 09 10. $x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$

06 02 ex 10 11. $-\frac{1}{2}xe^{-2x} - \frac{e^{-2x}}{4} + C$

06 02 ex 11 12. $1/2e^x(\sin x - \cos x) + C$

06 02 ex 12 13. $1/5e^{2x}(\sin x + 2\cos x) + C$

06 02 ex 13 14. $1/13e^{2x}(2\sin(3x) - 3\cos(3x)) + C$

06 02 ex 14 15. $1/10e^{5x}(\sin(5x) + \cos(5x)) + C$

06 02 ex 15 16. $-1/2\cos^2 x + C$

06 02 ex 16 17. $\sqrt{1-x^2} + x\sin^{-1}(x) + C$

06 02 ex 17 18. $x\tan^{-1}(2x) - \frac{1}{4}\ln\left|4x^2+1\right| + C$

06 02 ex 18 19. $\frac{1}{2}x^2\tan^{-1}(x) - \frac{x}{2} + \frac{1}{2}\tan^{-1}(x) + C$

06 02 ex 19 20. $\sqrt{1-x^2} + x\sin^{-1}x + C$

06 02 ex 22 21. $\frac{1}{2}x^2\ln|x| - \frac{x^2}{4} + C$

06 02 ex 23 22. $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x| + 2x - 2x\ln|x| + C$

06 02 ex 24 23. $-\frac{x^2}{4} + \frac{1}{2}x^2\ln|x-1| - \frac{x}{2} - \frac{1}{2}\ln|x-1| + C$

06 02 ex 25 24. $\frac{1}{2}x^2\ln\left(x^2\right) - \frac{x^2}{2} + C$

06 02 ex 26 25. $\frac{1}{3}x^3\ln|x| - \frac{x^3}{9} + C$

06 02 ex 27 26. $2x + x\left(\ln|x|\right)^2 - 2x\ln|x| + C$

06 02 ex 28 27. $2x + x\left(\ln|x+1|\right) + \left(\ln|x+1|\right)^2 - 2x\ln|x+1| - 2\ln|x+1| + 2 + C$

06 02 ex 29 28. $x\tan(x) + \ln|\cos(x)| + C$

06 02 ex 30 29. $\ln|\sin(x)| - x\cot(x) + C$

06 02 ex 31 30. $\frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$

06 02 ex 32 31. $\frac{1}{3}(x^2-2)^{3/2} + C$

06 02 ex 35 32. $\sec x + C$

06 02 ex 33 33. $x\sec x - \ln|\sec x + \tan x| + C$

06 02 ex 34 34. $-x\csc x - \ln|\csc x + \cot x| + C$

06 02 ex 36 35. $1/2x\big(\sin(\ln x) - \cos(\ln x)\big) + C$

06 02 ex 51 36. $\cos\left(e^x\right) + e^x\sin\left(e^x\right) + C$

06 02 ex 37 37. $2\sin\left(\sqrt{x}\right) - 2\sqrt{x}\cos\left(\sqrt{x}\right) + C$

06 02 ex 38 38. $\frac{1}{2}x\ln|x| - \frac{x}{2} + C$

06 02 ex 39 39. $2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + C$

06 02 ex 40 40. $1/2x^2 + C$

06 02 ex 41 41. π

06 02 ex 42 42. $-2/e$

06 02 ex 43 43. 0

06 02 ex 44 44. $\frac{3\pi^2}{2} - 12$

06 02 ex 45 45. $1/2$

06 02 ex 46 46. $6 - 2e$

06 02 ex 47 47. $\frac{3}{4e^2} - \frac{5}{4e^4}$

06 02 ex 48 48. $\frac{1}{2} + \frac{e^\pi}{2}$

06 02 ex 49 49. $1/5\left(e^\pi + e^{-\pi}\right)$

Section 6.3

06 03 ex 01 1. F

06 03 ex 02 2. F

06 03 ex 03 3. F

06 03 ex 34 4. F

06 03 ex 04 5. $-\frac{1}{5}\cos^5(x) + C$

06 03 ex 05 6. $\frac{1}{4}\sin^4(x) + C$

06 03 ex 06 7. $\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

06 03 ex 07 8. $\frac{1}{6}\cos^6 x - \frac{1}{4}\cos^4 x + C$

06 03 ex 08 9. $\frac{1}{11}\sin^{11}x - \frac{2}{9}\sin^9 x + \frac{1}{7}\sin^7 x + C$

06 03 ex 09 10. $-\frac{1}{9}\sin^9(x) + \frac{3\sin^7(x)}{7} - \frac{3\sin^5(x)}{5} + \frac{\sin^3(x)}{3} + C$

06 03 ex 10 11. $\frac{x}{8} - \frac{1}{32}\sin(4x) + C$

06 03 ex 35 12. $\frac{1}{2}\sin^2 x + C$ or $-\frac{1}{2}\cos^2 x + C$, depending on the choice of substitution

06 03 ex 11 13. $\frac{1}{2}\left(-\frac{1}{8}\cos(8x) - \frac{1}{2}\cos(2x)\right) + C$

06 03 ex 12 14. $\frac{1}{2}\left(-\frac{1}{3}\cos(3x) + \cos(-x)\right) + C$

06 03 ex 13 15. $\frac{1}{2}\left(\frac{1}{4}\sin(4x) - \frac{1}{10}\sin(10x)\right) + C$

06 03 ex 14 16. $\frac{1}{2}\left(\frac{1}{\pi}\sin(\pi x) - \frac{1}{3\pi}\sin(3\pi x)\right) + C$

06 03 ex 15 17. $\frac{1}{2}\left(\sin(x) + \frac{1}{3}\sin(3x)\right) + C$

06 03 ex 16 18. $\frac{1}{\pi}\sin(\frac{\pi}{2}x) + \frac{1}{3\pi}\sin(\pi x) + C$

06 03 ex 17 19. $\frac{\tan^5(x)}{5} + C$

06 03 ex 18 20. $\frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x + C$

06 03 ex 19 21. $\frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C$

06 03 ex 20 22. $\frac{\tan^4(x)}{4} + C$

06 03 ex 21 23. $\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

06 03 ex 22 24. $\frac{\sec^9(x)}{9} - \frac{2\sec^7(x)}{7} + \frac{\sec^5(x)}{5} + C$

06 03 ex 23 25. $\frac{1}{3}\tan^3 x - \tan x + x + C$

06 03 ex 24 26. $\frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$

06 03 ex 25 27. $\frac{1}{2}\left(\sec x\tan x - \ln|\sec x + \tan x|\right) + C$

06 03 ex 26 28. $\frac{1}{4}\tan x\sec^3 x - \frac{1}{8}\left(\sec x\tan x + \ln|\sec x + \tan x|\right) + C$

06 03 ex 27 29. $\frac{2}{5}$

06 03 ex 28 30. 0

06 03 ex 29 31. $32/315$

06 03 ex 30 32. $1/2$

06 03 ex 31 33. $2/3$

06 03 ex 32 34. $1/5$

06 03 ex 33 35. $16/15$

Section 6.4

06 08 ex 01 1. backwards

06 08 ex 02 2. $5\sin\theta$

06 08 ex 03 3.
   (a) $\tan^2\theta + 1 = \sec^2\theta$
   (b) $9\sec^2\theta$.

06 08 ex 04 4. Because we are considering $a > 0$ and $x = a\sin\theta$, which means $\theta = \sin^{-1}(x/a)$. The arcsine function has a domain of $-\pi/2 \le \theta \le \pi/2$; on this domain, $\cos\theta \ge 0$, so $a\cos\theta$ is always non-negative, allowing us to drop the absolute value signs.

06 08 ex 05 5. $\frac{1}{2}\left(x\sqrt{x^2+1} + \ln|\sqrt{x^2+1} + x|\right) + C$

06 08 ex 06 6. $2\left(\frac{x}{4}\sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right|\right) + C$

06 08 ex 07 7. $\frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + C$

06 08 ex 08 8. $\frac{1}{2}\left(9\sin^{-1}(x/3) + x\sqrt{9-x^2}\right) + C$

06 08 ex 09 9. $\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C$

06 08 ex 10 10. $\frac{1}{2}x\sqrt{x^2-16} - 8\ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + C$

06 08 ex 11 11. $x\sqrt{x^2+1/4} + \frac{1}{4}\ln|2\sqrt{x^2+1/4} + 2x| + C =$
$\frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln|\sqrt{4x^2+1} + 2x| + C$

06 08 ex 12 12. $\frac{1}{6}\sin^{-1}(3x) + \frac{3}{2}\sqrt{1/9 - x^2} + C =$
$\frac{1}{6}\sin^{-1}(3x) + \frac{1}{2}\sqrt{1-9x^2} + C$

06 08 ex 13 13. $4\left(\frac{1}{2}x\sqrt{x^2-1/16} - \frac{1}{32}\ln|4x + 4\sqrt{x^2-1/16}|\right) + C =$
$\frac{1}{2}x\sqrt{16x^2-1} - \frac{1}{8}\ln|4x + \sqrt{16x^2-1}| + C$

06 08 ex 14 14. $8\ln\left|\frac{\sqrt{x^2+2}}{\sqrt{2}} + \frac{x}{\sqrt{2}}\right| + C$; with Section 6.6, we can state the answer as $8\sinh^{-1}(x/\sqrt{2}) + C$.

06 08 ex 15 15. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$ (Trig. Subst. is not needed)

06 08 ex 16 16. $5\ln\left|\frac{x}{\sqrt{8}} + \frac{\sqrt{x^2-8}}{\sqrt{8}}\right| + C$

06 08 ex 18 17. $\sqrt{x^2-11} - \sqrt{11}\sec^{-1}(x/\sqrt{11}) + C$

06 08 ex 20 18. $\frac{1}{2}\left(\tan^{-1}x + \frac{x}{x^2+1}\right) + C$

06 08 ex 17 19. $\sqrt{x^2-3} + C$ (Trig. Subst. is not needed)

06 08 ex 24 20. $\frac{1}{8}\sin^{-1}x - \frac{1}{8}x\sqrt{1-x^2}(1-2x^2) + C$

06 08 ex 22 21. $-\frac{1}{\sqrt{x^2+9}} + C$ (Trig. Subst. is not needed)

06 08 ex 19 22. $\frac{5}{2}x\sqrt{x^2-10} + 25\ln\left|\frac{x}{\sqrt{10}} + \frac{\sqrt{x^2-10}}{\sqrt{10}}\right| + C$

06 08 ex 21 23. $\frac{1}{18}\frac{x+2}{x^2+4x+13} + \frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

06 08 ex 23 24. $\frac{x}{\sqrt{1-x^2}} - \sin^{-1}x + C$

06 08 ex 25 25. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x} - \sin^{-1}(x/\sqrt{5})\right) + C$

06 08 ex 26 26. $\frac{1}{2}x\sqrt{x^2+3} - \frac{3}{2}\ln\left|\frac{\sqrt{x^2+3}}{\sqrt{3}} + \frac{x}{\sqrt{3}}\right| + C$

06 08 ex 27 27. $\pi/2$

06 08 ex 28 28. $16\sqrt{3} - 8\ln(2+\sqrt{3})$

06 08 ex 29 29. $2\sqrt{2} + 2\ln(1+\sqrt{2})$

06 08 ex 30 30. $\pi/4 + 1/2)$

06 08 ex 31 31. $9\sin^{-1}(1/3) + \sqrt{8}$ Note: the new lower bound is $\theta = \sin^{-1}(-1/3)$ and the new upper bound is $\theta = \sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3) = -\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right) = \cos\left(\sin^{-1}(-1/3)\right) = \sqrt{8}/3$.

06 08 ex 32 32. $\pi/8$

Section 6.5

06 04 ex 01 1. rational

06 04 ex 02 2. T

06 04 ex 03 3. $\frac{A}{x} + \frac{B}{x-3}$

06 04 ex 04 4. $\frac{A}{x-3} + \frac{B}{x+3}$

06 04 ex 05 5. $\frac{A}{x-\sqrt{7}} + \frac{B}{x+\sqrt{7}}$

06 04 ex 06 6. $\frac{A}{x} + \frac{Bx+C}{x^2+7}$

06 04 ex 07 7. $3\ln|x-2| + 4\ln|x+5| + C$

06 04 ex 08 8. $9\ln|x+1| - 2\ln|x| + C$

06 04 ex 09 9. $\frac{1}{3}(\ln|x+2| - \ln|x-2|) + C$

06 04 ex 30 10. $\ln|x+1| + \ln|3x+1| + C$

06 04 ex 10 11. $\ln|x+5| - \frac{2}{x+5} + C$

06 04 ex 11 12. $-\frac{4}{x+8} - 3\ln|x+8| + C$

06 04 ex 12 13. $\frac{5}{x+1} + 7\ln|x| + 2\ln|x+1| + C$

06 04 ex 13 14. $-\ln|2x-3| + 5\ln|x-1| + 2\ln|x+3| + C$

06 04 ex 14 15. $-\frac{1}{5}\ln|5x-1| + \frac{2}{3}\ln|3x-1| + \frac{3}{7}\ln|7x+3| + C$

06 04 ex 15 16. $x + \ln|x-1| - \ln|x+2| + C$

06 04 ex 16 17. $\frac{x^2}{2} + x + \frac{125}{9}\ln|x-5| + \frac{64}{9}\ln|x+4| - \frac{35}{2} + C$

06 04 ex 17 18. $2x + C$

06 04 ex 18 19. $\frac{1}{6}\left(-\ln\left|x^2+2x+3\right| + 2\ln|x| - \sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right) + C$

06 04 ex 19 20. $-\frac{3}{2}\ln\left|x^2+4x+10\right| + x + \frac{\tan^{-1}\left(\frac{x+2}{\sqrt{6}}\right)}{\sqrt{6}} + C$

06 04 ex 20 21. $\ln\left|3x^2+5x-1\right| + 2\ln|x+1| + C$

06 04 ex 21 22. $2\ln|x-3| + 2\ln|x^2+6x+10| - 4\tan^{-1}(x+3) + C$

06 04 ex 22 23. $\frac{9}{10}\ln\left|x^2+9\right| + \frac{1}{5}\ln|x+1| - \frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right) + C$

06 04 ex 23 24. $\frac{1}{2}\left(3\ln\left|x^2+2x+17\right| - 4\ln|x-7| + \tan^{-1}\left(\frac{x+1}{4}\right)\right) + C$

06 04 ex 24 25. $3\left(\ln\left|x^2-2x+11\right| + \ln|x-9|\right) + 3\sqrt{\frac{2}{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right) + C$

06 04 ex 25 26. $\frac{1}{2}\ln\left|x^2+10x+27\right| + 5\ln|x+2| - 6\sqrt{2}\tan^{-1}\left(\frac{x+5}{\sqrt{2}}\right) + C$

06 04 ex 26 27. $\ln(2000/243) \approx 2.108$

06 04 ex 27 28. $5\ln(9/4) - \frac{1}{3}\ln(17/2) \approx 3.3413$

06 04 ex 28 29. $-\pi/4 + \tan^{-1}3 - \ln(11/9) \approx 0.263$

06 04 ex 29 30. $1/8$

Section 6.6

06 05 ex 01 1. Because $\cosh x$ is always positive.

06 05 ex 02 2. The points on the left hand side can be defined as $(-\cosh x, \sinh x)$.

06 05 ex 03 3.
$$\begin{aligned}\coth^2 x - \operatorname{csch}^2 x &= \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2\\ &= \frac{(e^{2x}+2+e^{-2x})-(4)}{e^{2x}-2+e^{-2x}}\\ &= \frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\ &= 1\end{aligned}$$

06 05 ex 04 4.
$$\begin{aligned}\cosh^2 x + \sinh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2 + \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4} + \frac{e^{2x}-2+e^{-2x}}{4}\\ &= \frac{2e^{2x}+2e^{-2x}}{4}\\ &= \frac{e^{2x}+e^{-2x}}{2}\\ &= \cosh 2x.\end{aligned}$$

06 05 ex 05 5.
$$\begin{aligned}\cosh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4}\\ &= \frac{1}{2}\frac{(e^{2x}+e^{-2x})+2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\ &= \frac{\cosh 2x+1}{2}.\end{aligned}$$

06 05 ex 06 6.
$$\begin{aligned}\sinh^2 x &= \left(\frac{e^x-e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}-2+e^{-2x}}{4}\\ &= \frac{1}{2}\frac{(e^{2x}+e^{-2x})-2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}-1\right)\\ &= \frac{\cosh 2x-1}{2}.\end{aligned}$$

06 05 ex 07 7.
$$\begin{aligned}\frac{d}{dx}[\operatorname{sech} x] &= \frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\ &= \frac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= -\frac{2(e^x-e^{-x})}{(e^x+e^{-x})(e^x+e^{-x})}\\ &= -\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\ &= -\operatorname{sech} x\tanh x\end{aligned}$$

06 05 ex 08 8.
$$\begin{aligned}\frac{d}{dx}[\coth x] &= \frac{d}{dx}\left[\frac{e^x+e^{-x}}{e^x-e^{-x}}\right]\\ &= \frac{(e^x-e^{-x})(e^x-e^{-x})-(e^x+e^{-x})(e^x+e^{-x})}{(e^x-e^{-x})^2}\\ &= \frac{e^{2x}+e^{-2x}-2-(e^{2x}+e^{-2x}+2)}{(e^x-e^{-x})^2}\\ &= -\frac{4}{(e^x-e^{-x})^2}\\ &= -\operatorname{csch}^2 x\end{aligned}$$

06 05 ex 09 9. $\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$

Let $u = \cosh x$; $du = (\sinh x)dx$
$$\begin{aligned}&= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln(\cosh x) + C.\end{aligned}$$

06 05 ex 10 10. $\int \coth x\,dx = \int \frac{\cosh x}{\sinh x}\,dx$

Let $u = \sinh x$; $du = (\cosh x)dx$
$$\begin{aligned}&= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln|\sinh x| + C.\end{aligned}$$

06 05 ex 11 11. $2\cosh 2x$

06 05 ex 42 12. Taking the derivative of $(\cosh x)^2$ directly, one gets $2\cosh x \sinh x$; using the identity $\cosh^2 x = \frac{1}{2}(\cosh 2x + 1)$ first, one gets $\sinh 2x$; by Key Idea 6.6.1, these are equal.

06 05 ex 12 13. $2x\sec^2(x^2)$

06 05 ex 13 14. $\coth x$

06 05 ex 14 15. $\sinh^2 x + \cosh^2 x$

06 05 ex 15 16. $x\cosh x$

06 05 ex 16 17. $\frac{-2x}{(x^2)\sqrt{1-x^4}}$

06 05 ex 19 18. $\frac{3}{\sqrt{9x^2+1}}$

06 05 ex 20 19. $\frac{4x}{\sqrt{4x^4-1}}$

06 05 ex 21 20. $\frac{1}{1-(x+5)^2}$

06 05 ex 17 21. $-\csc x$

06 05 ex 18 22. $\sec x$

06 05 ex 22 23. $y = x$

06 05 ex 23 24. $y = \frac{3}{4}(x - \ln 2) + \frac{5}{4}$

06 05 ex 43 25. $y = \frac{9}{25}(x + \ln 3) - \frac{4}{5}$

06 05 ex 24 26. $y = -\frac{72}{125}(x - \ln 3) + \frac{9}{25}$

06 05 ex 25 27. $y = x$

06 05 ex 26 28. $y = (x - \sqrt{2}) + \cosh^{-1}(\sqrt{2}) \approx (x - 1.414) + 0.881$

06 05 ex 27 29. $1/2\ln(\cosh(2x)) + C$

06 05 ex 28 30. $1/3\sinh(3x - 7) + C$

06 05 ex 29 31. $1/2\sinh^2 x + C$ or $1/2\cosh^2 x + C$

06 02 ex 20 32. $x\sinh(x) - \cosh(x) + C$

06 02 ex 21 33. $x\cosh(x) - \sinh(x) + C$

06 05 ex 45 34. $\sinh^{-1} x + C = \ln\left(x + \sqrt{x^2+1}\right) + C$

06 05 ex 46 35. $\cosh^{-1} x/3 + C = \ln\left(x + \sqrt{x^2-9}\right) + C$

06 05 ex 30 36. $\begin{cases} \frac{1}{3}\tanh^{-1}\left(\frac{x}{3}\right) + C & x^2 < 9 \\ \frac{1}{3}\coth^{-1}\left(\frac{x}{3}\right) + C & 9 < x^2 \end{cases} = \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x-1| + C$

06 05 ex 31 37. $\cosh^{-1}(x^2/2) + C = \ln(x^2 + \sqrt{x^4-4}) + C$

06 05 ex 32 38. $2/3\sinh^{-1} x^{3/2} + C = 2/3\ln(x^{3/2} + \sqrt{x^3+1}) + C$

06 05 ex 33 39. $\frac{1}{16}\tan^{-1}(x/2) + \frac{1}{32}\ln|x-2| + \frac{1}{32}\ln|x+2| + C$

06 05 ex 34 40. $\ln x - \ln|x+1| + C$

06 05 ex 35 41. $\tan^{-1}(e^x) + C$

06 05 ex 36 42. $x\sinh^{-1} x - \sqrt{x^2+1} + C$

06 05 ex 37 43. $x\tanh^{-1} x + 1/2\ln|x^2 - 1| + C$

06 05 ex 38 44. $\tan^{-1}(\sinh x) + C$

06 05 ex 39 45. 0

06 05 ex 40 46. $3/2$

06 05 ex 41 47. 2

06 05 ex 44 48. $\sinh^{-1} 2 = \ln\left(2 + \sqrt{5}\right) \approx 1.444$

Section 6.7

06 06 ex 01 1. $0/0, \infty/\infty, 0\cdot\infty, \infty - \infty, 0^0, 1^\infty, \infty^0$

06 06 ex 02 2. F

06 06 ex 03 3. F

06 06 ex 04 4. The base of an expression is approaching 1 while its power is growing without bound.

06 06 ex 05 5. derivatives; limits

06 06 ex 06 6. Answers will vary.

06 06 ex 07 7. Answers will vary.

06 06 ex 53 8. Answers will vary.

06 06 ex 08 9. 3

06 06 ex 09 10. $-5/3$

06 06 ex 10 11. -1

06 06 ex 11 12. $-\sqrt{2}/2$

06 06 ex 12 13. 5

06 06 ex 13 14. 0

06 06 ex 14 15. $2/3$

06 06 ex 15 16. a/b

06 06 ex 16 17. ∞

06 06 ex 17 18. $1/2$

06 06 ex 18 19. 0

06 06 ex 19 20. 0

06 06 ex 20 21. 0

06 06 ex 54 22. ∞

06 06 ex 21 23. ∞

06 06 ex 22 24. ∞

06 06 ex 23 25. 0

06 06 ex 24 26. 2

06 06 ex 25 27. -2

06 06 ex 26 28. 0

06 06 ex 27 29. 0

06 06 ex 28 30. 0

06 06 ex 29 31. 0

06 06 ex 30 32. 0

06 06 ex 52 33. ∞

06 06 ex 31 34. ∞

06 06 ex 32 35. ∞

06 06 ex 33 36. 0

06 06 ex 34 37. 0

06 06 ex 35 38. e

06 06 ex 36 39. 1

06 06 ex 37 40. 1

06 06 ex 38 41. 1

06 06 ex 39 42. 1

06 06 ex 40 43. 1

06 06 ex 41 44. 0

06 06 ex 42 45. 1

06 06 ex 43 46. 1

06 06 ex 44 47. 1

06 06 ex 45 48. 1

06 06 ex 46 49. 2

06 06 ex 47 50. $1/2$

06 06 ex 48 51. $-\infty$

06 06 ex 49 52. 1

06 06 ex 50 53. 0

06 06 ex 51 54. 3

Section 6.8

06 07 ex 03 1. The interval of integration is finite, and the integrand is continuous on that interval.

06 07 ex 01 2. converge

06 07 ex 02 3. converges; could also state < 10.

06 07 ex 04 4. $p > 1$

06 07 ex 05 5. $p > 1$

06 07 ex 06 6. $p < 1$

06 07 ex 07 7. $e^5/2$

06 07 ex 08 8. $1/2$

06 07 ex 09 9. $1/3$

06 07 ex 10 10. $\pi/3$

06 07 ex 11 11. $1/\ln 2$

06 07 ex 12 12. diverges

06 07 ex 13 13. diverges

06 07 ex 14 14. $-1/4\ln(1/5) = 1/2\tanh^{-1}(2/3) \approx 0.4024$

06 07 ex 15 15. 1

06 07 ex 16 16. diverges

06 07 ex 17 17. diverges

06 07 ex 18 18. diverges

06 07 ex 43 19. diverges

06 07 ex 19 20. diverges

06 07 ex 20 21. diverges

06 07 ex 21 22. $2 + 2\sqrt{2}$

06 07 ex 22 23. 1

06 07 ex 23 24. $1/2$

06 07 ex 24 25. 0

06 07 ex 25 26. $\pi/2$

06 07 ex 26 27. $-1/4$

06 07 ex 44 28. $-1/9$

06 07 ex 27 29. diverges

06 07 ex 28 30. -1

06 07 ex 29 31. 1

06 07 ex 30 32. diverges

06 07 ex 31 33. $1/2$

06 07 ex 32 34. $1/2$

06 07 ex 33 35. diverges; Limit Comparison Test with $1/x$.

06 07 ex 34 36. converges; Limit Comparison Test with $1/x^{3/2}$.

06 07 ex 35 37. diverges; Limit Comparison Test with $1/x$.

06 07 ex 36 38. converges; Direct Comparison Test with xe^{-x}.

06 07 ex 37 39. converges; Direct Comparison Test with e^{-x}.

06 07 ex 38 40. converges; Direct Comparison Test with xe^{-x}.

06 07 ex 39 41. converges; Direct Comparison Test with $1/(x^2 - 1)$.

06 07 ex 42 42. diverges; Direct Comparison Test with $x/(x^2 + 1)$.

06 07 ex 40 43. converges; Direct Comparison Test with $1/e^x$.

06 07 ex 41 44. converges; Limit Comparison Test with $1/e^x$.

Index