

## 12.1 Triple Integration with Cylindrical and Spherical Coordinates

Just as polar coordinates gave us a new way of describing curves in the plane, in this section we will see how *cylindrical* and *spherical* coordinates give us new ways of describing surfaces and regions in space.

### Cylindrical Coordinates

In short, cylindrical coordinates can be thought of as a combination of the polar and rectangular coordinate systems. One can identify a point  $(x_0, y_0, z_0)$ , given in rectangular coordinates, with the point  $(r_0, \theta_0, z_0)$ , given in cylindrical coordinates, where the  $z$ -value in both systems is the same, and the point  $(x_0, y_0)$  in the  $x$ - $y$  plane is identified with the polar point  $P(r_0, \theta_0)$ ; see Figure 12.1.1. So that each point in space that does not lie on the  $z$ -axis is defined uniquely, we will restrict  $r \geq 0$  and  $0 \leq \theta \leq 2\pi$ .

We use the identity  $z = z$  along with the identities found in Key Idea ?? to convert between the rectangular coordinate  $(x, y, z)$  and the cylindrical coordinate  $(r, \theta, z)$ , namely

$$\begin{aligned} \text{From rectangular to cylindrical: } & r = \sqrt{x^2 + y^2}, \quad \tan \theta = y/x \quad \text{and} \quad z = z; \\ \text{From cylindrical to rectangular: } & x = r \cos \theta \quad y = r \sin \theta \quad \text{and} \quad z = z. \end{aligned}$$

These identities, along with conversions related to spherical coordinates, are given later in Key Idea 12.1.1.

**Example 12.1.1 Converting between rectangular and cylindrical coordinates**  
Convert the rectangular point  $(2, -2, 1)$  to cylindrical coordinates, and convert the cylindrical point  $(4, 3\pi/4, 5)$  to rectangular.

**SOLUTION** Following the identities given above (and, later in Key Idea 12.1.1), we have  $r = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$ . Using  $\tan \theta = y/x$ , we find  $\theta = \tan^{-1}(-2/2) = -\pi/4$ . As we restrict  $\theta$  to being between 0 and  $2\pi$ , we set  $\theta = 7\pi/4$ . Finally,  $z = 1$ , giving the cylindrical point  $(2\sqrt{2}, 7\pi/4, 1)$ .

In converting the cylindrical point  $(4, 3\pi/4, 5)$  to rectangular, we have  $x = 4 \cos(3\pi/4) = -2\sqrt{2}$ ,  $y = 4 \sin(3\pi/4) = 2\sqrt{2}$  and  $z = 5$ , giving the rectangular point  $(-2\sqrt{2}, 2\sqrt{2}, 5)$ .

Setting each of  $r$ ,  $\theta$  and  $z$  equal to a constant defines a surface in space, as illustrated in the following example.

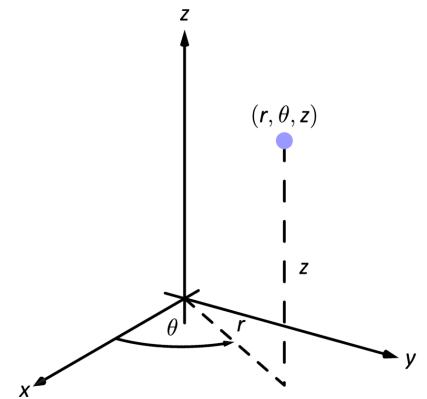


Figure 12.1.1: Illustrating the principles behind cylindrical coordinates.

**Note:** Our rectangular to polar conversion formulas used  $r^2 = x^2 + y^2$ , allowing for negative  $r$  values. Since we now restrict  $r \geq 0$ , we can use  $r = \sqrt{x^2 + y^2}$ .

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Notes:

### Example 12.1.2 Canonical surfaces in cylindrical coordinates

Describe the surfaces  $r = 1$ ,  $\theta = \pi/3$  and  $z = 2$ , given in cylindrical coordinates.

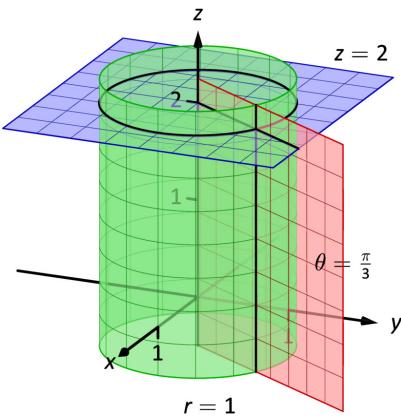


Figure 12.1.2: Graphing the canonical surfaces in cylindrical coordinates from Example 12.1.2.

**SOLUTION** The equation  $r = 1$  describes all points in space that are 1 unit away from the  $z$ -axis. This surface is a “tube” or “cylinder” of radius 1, centered on the  $z$ -axis, as graphed in Figure ?? (which describes the cylinder  $x^2 + y^2 = 1$  in space).

The equation  $\theta = \pi/3$  describes the plane formed by extending the line  $\theta = \pi/3$ , as given by polar coordinates in the  $x$ - $y$  plane, parallel to the  $z$ -axis.

The equation  $z = 2$  describes the plane of all points in space that are 2 units above the  $x$ - $y$  plane. This plane is the same as the plane described by  $z = 2$  in rectangular coordinates.

All three surfaces are graphed in Figure 12.1.2. Note how their intersection uniquely defines the point  $P = (1, \pi/3, 2)$ .

Cylindrical coordinates are useful when describing certain domains in space, allowing us to evaluate triple integrals over these domains more easily than if we used rectangular coordinates.

Theorem ?? shows how to evaluate  $\iiint_D h(x, y, z) dV$  using rectangular coordinates. In that evaluation, we use  $dV = dz dy dx$  (or one of the other five orders of integration). Recall how, in this order of integration, the bounds on  $y$  are “curve to curve” and the bounds on  $x$  are “point to point”: these bounds describe a region  $R$  in the  $x$ - $y$  plane. We could describe  $R$  using polar coordinates as done in Section ???. In that section, we saw how we used  $dA = r dr d\theta$  instead of  $dA = dy dx$ .

Considering the above thoughts, we have  $dV = dz(r dr d\theta) = r dz dr d\theta$ . We set bounds on  $z$  as “surface to surface” as done in the previous section, and then use “curve to curve” and “point to point” bounds on  $r$  and  $\theta$ , respectively. Finally, using the identities given above, we replace the integrand  $h(x, y, z)$  to  $h(r, \theta, z)$ .

This process should sound plausible; the following theorem states it is truly a way of evaluating a triple integral.

#### Theorem 12.1.1 Triple Integration in Cylindrical Coordinates

Let  $w = h(r, \theta, z)$  be a continuous function on a closed, bounded region  $D$  in space, bounded in cylindrical coordinates by  $\alpha \leq \theta \leq \beta$ ,  $g_1(\theta) \leq r \leq g_2(\theta)$  and  $f_1(r, \theta) \leq z \leq f_2(r, \theta)$ . Then

$$\iiint_D h(r, \theta, z) dV = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} \int_{f_1(r, \theta)}^{f_2(r, \theta)} h(r, \theta, z) r dz dr d\theta.$$

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Notes:

**Example 12.1.3 Evaluating a triple integral with cylindrical coordinates**

Find the mass of the solid represented by the region in space bounded by  $z = 0$ ,  $z = \sqrt{4 - x^2 - y^2} + 3$  and the cylinder  $x^2 + y^2 = 4$  (as shown in Figure 12.1.3), with density function  $\delta(x, y, z) = x^2 + y^2 + z + 1$ , using a triple integral in cylindrical coordinates. Distances are measured in centimeters and density is measured in grams/cm<sup>3</sup>.

**SOLUTION** We begin by describing this region of space with cylindrical coordinates. The plane  $z = 0$  is left unchanged; with the identity  $r = \sqrt{x^2 + y^2}$ , we convert the hemisphere of radius 2 to the equation  $z = \sqrt{4 - r^2}$ ; the cylinder  $x^2 + y^2 = 4$  is converted to  $r^2 = 4$ , or, more simply,  $r = 2$ . We also convert the density function:  $\delta(r, \theta, z) = r^2 + z + 1$ .

To describe this solid with the bounds of a triple integral, we bound  $z$  with  $0 \leq z \leq \sqrt{4 - r^2} + 3$ ; we bound  $r$  with  $0 \leq r \leq 2$ ; we bound  $\theta$  with  $0 \leq \theta \leq 2\pi$ .

Using Definition ?? and Theorem 12.1.1, we have the mass of the solid is

$$\begin{aligned} M &= \iiint_D \delta(x, y, z) dV = \int_0^{2\pi} \int_0^2 \int_0^{\sqrt{4-r^2}+3} (r^2 + z + 1) r dz dr d\theta \\ &= \int_0^{2\pi} \int_0^2 ((r^3 + 4r)\sqrt{4 - r^2} + \frac{5}{2}r^3 + \frac{19}{2}r) dr d\theta \\ &= \frac{1318\pi}{15} \approx 276.04 \text{ gm}, \end{aligned}$$

where we leave the details of the remaining double integral to the reader.

**Example 12.1.4 Finding the center of mass using cylindrical coordinates**

Find the center of mass of the solid with constant density whose base can be described by the polar curve  $r = \cos(3\theta)$  and whose top is defined by the plane  $z = 1 - x + 0.1y$ , where distances are measured in feet, as seen in Figure 12.1.4. (The volume of this solid was found in Example ??.)

**SOLUTION** We convert the equation of the plane to use cylindrical coordinates:  $z = 1 - r \cos \theta + 0.1r \sin \theta$ . Thus the region in space is bounded by  $0 \leq z \leq 1 - r \cos \theta + 0.1r \sin \theta$ ,  $0 \leq r \leq \cos(3\theta)$ ,  $0 \leq \theta \leq \pi$  (recall that the rose curve  $r = \cos(3\theta)$  is traced out once on  $[0, \pi]$ ).

Since density is constant, we set  $\delta = 1$  and finding the mass is equivalent to finding the volume of the solid. We set up the triple integral to compute this but do not evaluate it; we leave it to the reader to confirm it evaluates to the same result found in Example ??.

$$M = \iiint_D \delta dV = \int_0^\pi \int_0^{\cos(3\theta)} \int_0^{1-r\cos\theta+0.1r\sin\theta} r dz dr d\theta \approx 0.785.$$

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Notes:

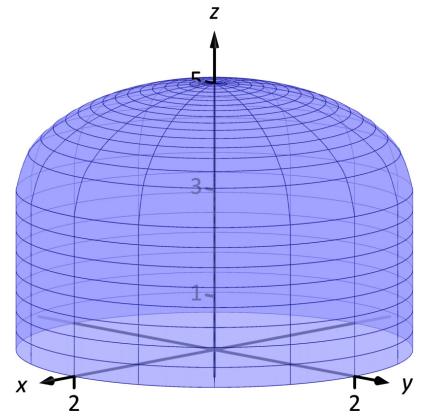


Figure 12.1.3: Visualizing the solid used in Example 12.1.3.

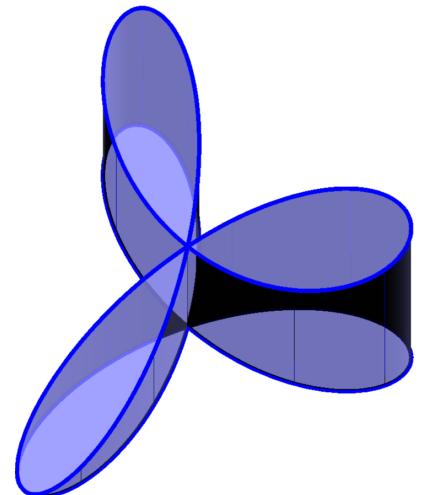


Figure 12.1.4: Visualizing the solid used in Example 12.1.4.

From Definition ?? we set up the triple integrals to compute the moments about the three coordinate planes. The computation of each is left to the reader (using technology is recommended):

$$M_{yz} = \iiint_D x \, dV = \int_0^\pi \int_0^{\cos(3\theta)} \int_0^{1-r\cos\theta+0.1r\sin\theta} (r\cos\theta)r \, dz \, dr \, d\theta \\ = -0.147.$$

$$M_{xz} = \iiint_D y \, dV = \int_0^\pi \int_0^{\cos(3\theta)} \int_0^{1-r\cos\theta+0.1r\sin\theta} (r\sin\theta)r \, dz \, dr \, d\theta \\ = 0.015.$$

$$M_{xy} = \iiint_D z \, dV = \int_0^\pi \int_0^{\cos(3\theta)} \int_0^{1-r\cos\theta+0.1r\sin\theta} (z)r \, dz \, dr \, d\theta \\ = 0.467.$$

The center of mass, in rectangular coordinates, is located at  $(-0.147, 0.015, 0.467)$ , which lies outside the bounds of the solid.

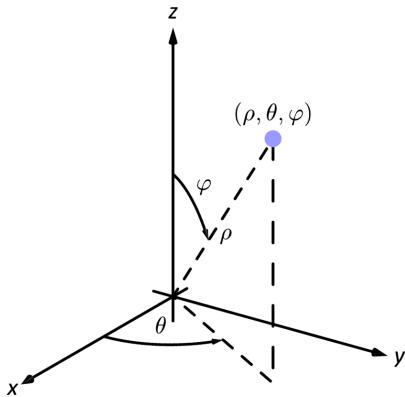


Figure 12.1.5: Illustrating the principles behind spherical coordinates.

**Note:** The symbol  $\rho$  is the Greek letter “rho.” Traditionally it is used in the spherical coordinate system, while  $r$  is used in the polar and cylindrical coordinate systems.

## Spherical Coordinates

In short, spherical coordinates can be thought of as a “double application” of the polar coordinate system. In spherical coordinates, a point  $P$  is identified with  $(\rho, \theta, \varphi)$ , where  $\rho$  is the distance from the origin to  $P$ ,  $\theta$  is the same angle as would be used to describe  $P$  in the cylindrical coordinate system, and  $\varphi$  is the angle between the positive  $z$ -axis and the ray from the origin to  $P$ ; see Figure 12.1.5. So that each point in space that does not lie on the  $z$ -axis is defined uniquely, we will restrict  $\rho \geq 0$ ,  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi$ .

The following Key Idea gives conversions to/from our three spatial coordinate systems.

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Notes:

**Key Idea 12.1.1 Converting Between Rectangular, Cylindrical and Spherical Coordinates****Rectangular and Cylindrical**

$$\begin{aligned} r^2 &= x^2 + y^2, & \tan \theta &= y/x, & z &= z \\ x &= r \cos \theta, & y &= r \sin \theta, & z &= z \end{aligned}$$

**Rectangular and Spherical**

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2 + z^2}, & \tan \theta &= y/x, & \cos \varphi &= z/\sqrt{x^2 + y^2 + z^2} \\ x &= \rho \sin \varphi \cos \theta, & y &= \rho \sin \varphi \sin \theta, & z &= \rho \cos \varphi \end{aligned}$$

**Cylindrical and Spherical**

$$\begin{aligned} \rho &= \sqrt{r^2 + z^2}, & \theta &= \theta, & \tan \varphi &= r/z \\ r &= \rho \sin \varphi, & \theta &= \theta, & z &= \rho \cos \varphi \end{aligned}$$

**Note:** The role of  $\theta$  and  $\varphi$  in spherical coordinates differs between mathematicians and physicists. When reading about physics in spherical coordinates, be careful to note how that particular author uses these variables and recognize that these identities will no longer be valid.

**Example 12.1.5 Converting between rectangular and spherical coordinates**

Convert the rectangular point  $(2, -2, 1)$  to spherical coordinates, and convert the spherical point  $(6, \pi/3, \pi/2)$  to rectangular and cylindrical coordinates.

**SOLUTION** This rectangular point is the same as used in Example 12.1.1. Using Key Idea 12.1.1, we find  $\rho = \sqrt{2^2 + (-1)^2 + 1^2} = 3$ . Using the same logic as in Example 12.1.1, we find  $\theta = 7\pi/4$ . Finally,  $\cos \varphi = 1/3$ , giving  $\varphi = \cos^{-1}(1/3) \approx 1.23$ , or about  $70.53^\circ$ . Thus the spherical coordinates are approximately  $(3, 7\pi/4, 1.23)$ .

Converting the spherical point  $(6, \pi/3, \pi/2)$  to rectangular, we have  $x = 6 \sin(\pi/2) \cos(\pi/3) = 3$ ,  $y = 6 \sin(\pi/2) \sin(\pi/3) = 3\sqrt{3}$  and  $z = 6 \cos(\pi/2) = 0$ . Thus the rectangular coordinates are  $(3, 3\sqrt{3}, 0)$ .

To convert this spherical point to cylindrical, we have  $r = 6 \sin(\pi/2) = 6$ ,  $\theta = \pi/3$  and  $z = 6 \cos(\pi/2) = 0$ , giving the cylindrical point  $(6, \pi/3, 0)$ .

**Example 12.1.6 Canonical surfaces in spherical coordinates**

Describe the surfaces  $\rho = 1$ ,  $\theta = \pi/3$  and  $\varphi = \pi/6$ , given in spherical coordinates.

**SOLUTION** The equation  $\rho = 1$  describes all points in space that are 1 unit away from the origin: this is the sphere of radius 1, centered at the origin.

The equation  $\theta = \pi/3$  describes the same surface in spherical coordinates as it does in cylindrical coordinates: beginning with the line  $\theta = \pi/3$  in the  $x$ - $y$  plane as given by polar coordinates, extend the line parallel to the  $z$ -axis, forming

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Notes:

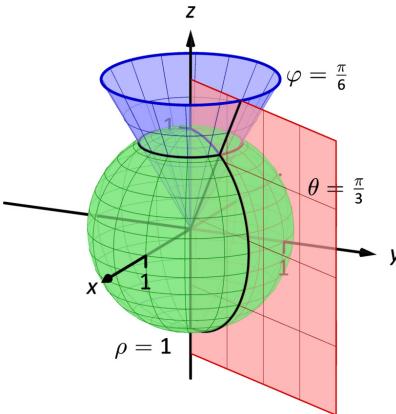


Figure 12.1.6: Graphing the canonical surfaces in spherical coordinates from Example 12.1.6.

a plane.

The equation  $\varphi = \pi/6$  describes all points  $P$  in space where the ray from the origin to  $P$  makes an angle of  $\pi/6$  with the positive  $z$ -axis. This describes a cone, with the positive  $z$ -axis its axis of symmetry, with point at the origin.

All three surfaces are graphed in Figure 12.1.6. Note how their intersection uniquely defines the point  $P = (1, \pi/3, \pi/6)$ .

Spherical coordinates are useful when describing certain domains in space, allowing us to evaluate triple integrals over these domains more easily than if we used rectangular coordinates or cylindrical coordinates. The crux of setting up a triple integral in spherical coordinates is appropriately describing the “small amount of volume,”  $dV$ , used in the integral.

Considering Figure 12.1.7, we can make a small “spherical wedge” by varying  $\rho$ ,  $\theta$  and  $\varphi$  each a small amount,  $\Delta\rho$ ,  $\Delta\theta$  and  $\Delta\varphi$ , respectively. This wedge is approximately a rectangular solid when the change in each coordinate is small, giving a volume of about

$$\Delta V \approx \Delta\rho \times \rho\Delta\varphi \times \rho\sin(\varphi)\Delta\theta.$$

Given a region  $D$  in space, we can approximate the volume of  $D$  with many such wedges. As the size of each of  $\Delta\rho$ ,  $\Delta\theta$  and  $\Delta\varphi$  goes to zero, the number of wedges increases to infinity and the volume of  $D$  is more accurately approximated, giving

$$dV = d\rho \times \rho d\varphi \times \rho\sin(\varphi)d\theta = \rho^2\sin(\varphi)d\rho d\theta d\varphi.$$

Again, this development of  $dV$  should sound reasonable, and the following theorem states it is the appropriate manner by which triple integrals are to be evaluated in spherical coordinates.

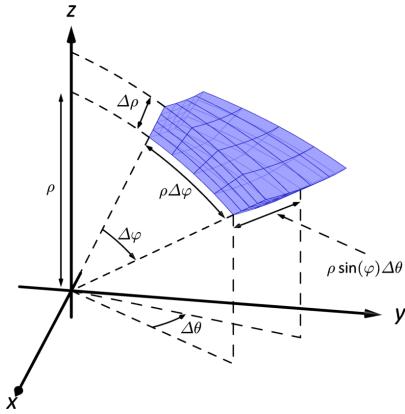


Figure 12.1.7: Approximating the volume of a standard region in space using spherical coordinates.

### Theorem 12.1.2 Triple Integration in Spherical Coordinates

Let  $w = h(\rho, \theta, \varphi)$  be a continuous function on a closed, bounded region  $D$  in space, bounded in spherical coordinates by  $\alpha_1 \leq \varphi \leq \alpha_2$ ,  $\beta_1 \leq \theta \leq \beta_2$  and  $f_1(\theta, \varphi) \leq \rho \leq f_2(\theta, \varphi)$ . Then

$$\iiint_D h(\rho, \theta, \varphi) dV = \int_{\alpha_1}^{\alpha_2} \int_{\beta_1}^{\beta_2} \int_{f_1(\theta, \varphi)}^{f_2(\theta, \varphi)} h(\rho, \theta, \varphi) \rho^2 \sin(\varphi) d\rho d\theta d\varphi.$$

### Example 12.1.7 Establishing the volume of a sphere

Let  $D$  be the region in space bounded by the sphere, centered at the origin, of radius  $r$ . Use a triple integral in spherical coordinates to find the volume  $V$  of  $D$ .

Notes:

**SOLUTION** The sphere of radius  $r$ , centered at the origin, has equation  $\rho = r$ . To obtain the full sphere, the bounds on  $\theta$  and  $\varphi$  are  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi$ . This leads us to:

$$\begin{aligned} V &= \iiint_D dV \\ &= \int_0^\pi \int_0^{2\pi} \int_0^r (\rho^2 \sin(\varphi)) d\rho d\theta d\varphi \\ &= \int_0^\pi \int_0^{2\pi} \left( \frac{1}{3} \rho^3 \sin(\varphi) \Big|_0^r \right) d\theta d\varphi \\ &= \int_0^\pi \int_0^{2\pi} \left( \frac{1}{3} r^3 \sin(\varphi) \right) d\theta d\varphi \\ &= \int_0^\pi \left( \frac{2\pi}{3} r^3 \sin(\varphi) \right) d\varphi \\ &= \left( -\frac{2\pi}{3} r^3 \cos(\varphi) \right) \Big|_0^\pi \\ &= \frac{4\pi}{3} r^3, \end{aligned}$$

the familiar formula for the volume of a sphere. Note how the integration steps were easy, not using square-roots nor integration steps such as Substitution.

#### Example 12.1.8 Finding the center of mass using spherical coordinates

Find the center of mass of the solid with constant density whose surface can be represented by  $\rho = 4$ ,  $\varphi = \pi/6$ , where  $\theta$  ranges from 0 to  $2\pi$ , as illustrated in Figure 12.1.8.

**SOLUTION** We will set up the four triple integrals needed to find the center of mass (i.e., to compute  $M$ ,  $M_{yz}$ ,  $M_{xz}$  and  $M_{xy}$ ) and leave it to the reader to evaluate each integral. Because of symmetry, we expect the  $x$  and  $y$  coordinates of the center of mass to be 0.

While the surfaces describing the solid are given in the statement of the problem, to describe the full solid  $D$ , we use the following bounds:  $0 \leq \rho \leq 4$ ,  $0 \leq \theta \leq 2\pi$  and  $0 \leq \varphi \leq \pi/6$ . Since density  $\delta$  is constant, we assume  $\delta = 1$ .

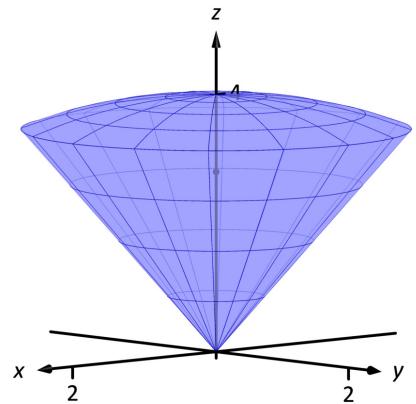


Figure 12.1.8: Graphing the solid, and its center of mass, from Example 12.1.8.

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Notes:

The mass of the solid:

$$\begin{aligned} M &= \iiint_D dm = \iiint_D dV \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 (\rho^2 \sin(\varphi)) d\rho d\theta d\varphi \\ &= \frac{64}{3} (2 - \sqrt{3}) \pi \approx 17.958. \end{aligned}$$

To compute  $M_{yz}$ , the integrand is  $x$ ; using Key Idea 12.1.1, we have  $x = \rho \sin \varphi \cos \theta$ . This gives:

$$\begin{aligned} M_{yz} &= \iiint_D x dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 ((\rho \sin(\varphi) \cos(\theta)) \rho^2 \sin(\varphi)) d\rho d\theta d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 (\rho^3 \sin^2(\varphi) \cos(\theta)) d\rho d\theta d\varphi \\ &= 0, \end{aligned}$$

which we expected as we expect  $\bar{x} = 0$ .

To compute  $M_{xz}$ , the integrand is  $y$ ; using Key Idea 12.1.1, we have  $y = \rho \sin \varphi \sin \theta$ . This gives:

$$\begin{aligned} M_{xz} &= \iiint_D y dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 ((\rho \sin(\varphi) \sin(\theta)) \rho^2 \sin(\varphi)) d\rho d\theta d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 (\rho^3 \sin^2(\varphi) \sin(\theta)) d\rho d\theta d\varphi \\ &= 0, \end{aligned}$$

which we also expected as we expect  $\bar{y} = 0$ .

To compute  $M_{xy}$ , the integrand is  $z$ ; using Key Idea 12.1.1, we have  $z =$

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Notes:

$\rho \cos \varphi$ . This gives:

$$\begin{aligned} M_{xy} &= \iiint_D z \, dm \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 ((\rho \cos(\varphi)) \rho^2 \sin(\varphi)) \, d\rho \, d\theta \, d\varphi \\ &= \int_0^{\pi/6} \int_0^{2\pi} \int_0^4 (\rho^3 \cos(\varphi) \sin(\varphi)) \, d\rho \, d\theta \, d\varphi \\ &= 16\pi \approx 50.266. \end{aligned}$$

Thus the center of mass is  $(0, 0, M_{xy}/M) \approx (0, 0, 2.799)$ , as indicated in Figure 12.1.8.

This section has provided a brief introduction into two new coordinate systems useful for identifying points in space. Each can be used to define a variety of surfaces in space beyond the canonical surfaces graphed as each system was introduced.

The usefulness of these coordinate systems does not lie in the variety of surfaces that they can describe nor the regions in space these surfaces may enclose. Rather, cylindrical coordinates are mostly used to describe cylinders and spherical coordinates are mostly used to describe spheres. These shapes are of special interest in the sciences, especially in physics, and computations on/inside these shapes is difficult using rectangular coordinates.

For instance, in the study of electricity and magnetism, one often studies the effects of an electrical current passing through a wire; that wire is essentially a cylinder, described well by cylindrical coordinates.

This chapter investigated the natural follow-on to partial derivatives: iterated integration. We learned how to use the bounds of a double integral to describe a region in the plane using both rectangular and polar coordinates, then later expanded to use the bounds of a triple integral to describe a region in space. We used double integrals to find volumes under surfaces, surface area, and the center of mass of lamina; we used triple integrals as an alternate method of finding volumes of space regions and also to find the center of mass of a region in space.

Integration does not stop here. We could continue to iterate our integrals, next investigating “quadruple integrals” whose bounds describe a region in 4-dimensional space (which are very hard to visualize). We can also look back to “regular” integration where we found the area under a curve in the plane. A natural analogue to this is finding the “area under a curve,” where the curve is in space, not in a plane. These are just two of many avenues to explore under the heading of “integration.”

Notes:

# Exercises 12.1

## Terms and Concepts

- Explain the difference between the roles  $r$ , in cylindrical coordinates, and  $\rho$ , in spherical coordinates, play in determining the location of a point.
- Why are points on the  $z$ -axis not determined uniquely when using cylindrical and spherical coordinates?
- What surfaces are naturally defined using cylindrical coordinates?
- What surfaces are naturally defined using spherical coordinates?

## Problems

**In Exercises 5 – 6, a point is given in either the rectangular, cylindrical or spherical coordinate systems. Find the coordinates of the point in the other systems.**

- (a) Points in rectangular coordinates:  
 $(2, 2, 1)$  and  $(-\sqrt{3}, 1, 0)$   
(b) Points in cylindrical coordinates:  
 $(2, \pi/4, 2)$  and  $(3, 3\pi/2, -4)$   
(c) Points in spherical coordinates:  
 $(2, \pi/4, \pi/4)$  and  $(1, 0, 0)$
- (a) Points in rectangular coordinates:  
 $(0, 1, 1)$  and  $(-1, 0, 1)$   
(b) Points in cylindrical coordinates:  
 $(0, \pi, 1)$  and  $(2, 4\pi/3, 0)$   
(c) Points in spherical coordinates:  
 $(2, \pi/6, \pi/2)$  and  $(3, \pi, \pi)$

**In Exercises 7 – 8, describe the curve, surface or region in space determined by the given bounds.**

- Bounds in cylindrical coordinates:  
(a)  $r = 1$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq z \leq 1$   
(b)  $1 \leq r \leq 2$ ,  $0 \leq \theta \leq \pi$ ,  $0 \leq z \leq 1$   
Bounds in spherical coordinates:  
(c)  $\rho = 3$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \varphi \leq \pi/2$

(d)  $2 \leq \rho \leq 3$ ,  $0 \leq \theta \leq 2\pi$ ,  $0 \leq \varphi \leq \pi$

- Bounds in cylindrical coordinates:

(a)  $1 \leq r \leq 2$ ,  $\theta = \pi/2$ ,  $0 \leq z \leq 1$

(b)  $r = 2$ ,  $0 \leq \theta \leq 2\pi$ ,  $z = 5$

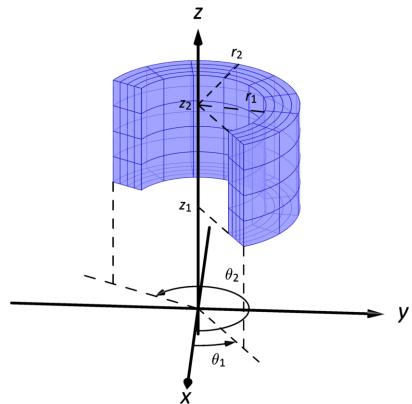
Bounds in spherical coordinates:

(c)  $0 \leq \rho \leq 2$ ,  $0 \leq \theta \leq \pi$ ,  $\varphi = \pi/4$

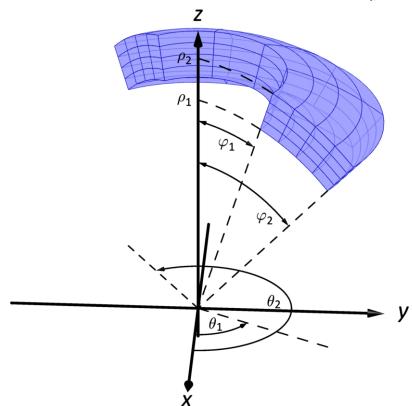
(d)  $\rho = 2$ ,  $0 \leq \theta \leq 2\pi$ ,  $\varphi = \pi/6$

**In Exercises 9 – 10, standard regions in space, as defined by cylindrical and spherical coordinates, are shown. Set up the triple integral that integrates the given function over the graphed region.**

- Cylindrical coordinates, integrating  $h(r, \theta, z)$ :



- Cylindrical coordinates, integrating  $h(\rho, \theta, \varphi)$ :



# A: SOLUTIONS TO SELECTED PROBLEMS

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## Chapter 12

### Section 12.1

1. In cylindrical,  $r$  determines how far from the origin one goes in the  $x$ - $y$  plane before considering the  $z$ -component. Equivalently, if on projects a point in cylindrical coordinates onto the  $x$ - $y$  plane,  $r$  will be the distance of this projection from the origin.  
In spherical,  $\rho$  is the distance from the origin to the point.
2. If  $r = 0$  or  $\rho = 0$ , then the point in each coordinate system lies on the  $z$ -axis regardless of the value of  $\theta$ .
3. Cylinders (tubes) centered at the origin, parallel to the  $z$ -axis; planes parallel to the  $z$ -axis that intersect the  $z$ -axis; planes parallel to the  $x$ - $y$  plane.
4. Spheres centered at the origin; planes parallel to the  $z$ -axis that intersect the  $z$ -axis; cones centered on the  $z$ -axis with point at the origin.
5. (a) Cylindrical:  $(2\sqrt{2}, \pi/4, 1)$  and  $(2, 5\pi/6, 0)$   
Spherical:  $(3, \pi/4, \cos^{-1}(1/3))$  and  $(2, 5\pi/6, \pi/2)$   
(b) Rectangular:  $(\sqrt{2}, \sqrt{2}, 2)$  and  $(0, -3, -4)$   
Spherical:  $(2\sqrt{2}, \pi/4, \pi/4)$  and  $(5, 3\pi/2, \pi - \tan^{-1}(3/4))$   
(c) Rectangular:  $(1, 1, \sqrt{2})$  and  $(0, 0, 1)$   
Cylindrical:  $(\sqrt{2}, \pi/4, \sqrt{2})$  and  $(0, 0, 1)$
6. (a) Cylindrical:  $(1, \pi/2, 1)$  and  $(1, \pi, 1)$   
Spherical:  $(\sqrt{2}, \pi/2, \pi/4)$  and  $(\sqrt{2}, \pi, \pi/4)$   
(b) Rectangular:  $(0, 0, 1)$  and  $(-1, -\sqrt{3}, 0)$   
Spherical:  $(1, \pi, 0)$  and  $(2, 4\pi/3, \pi/2)$

- (c) Rectangular:  $(\sqrt{3}, 1, 0)$  and  $(0, 0, -3)$   
Cylindrical:  $(2, \pi/6, 0)$  and  $(0, \pi, -3)$
7. (a) A cylindrical surface or tube, centered along the  $z$ -axis of radius 1, extending from the  $x$ - $y$  plane up to the plane  $z = 1$  (i.e., the tube has a length of 1).  
(b) This is a region of space, being half of a tube with “thick” walls of inner radius 1 and outer radius 2, centered along the  $z$ -axis with a length of 1, where the half “below” the  $x$ - $z$  plane is removed.  
(c) This is upper half of the sphere of radius 3 centered at the origin (i.e., the upper hemisphere).  
(d) This is a region of space, where the ball of radius 2, centered at the origin, is removed from the ball of radius 3, centered at the origin.
8. (a) A square portion of the  $y$ - $z$  plane with corners at  $(0, 1, 0)$ ,  $(0, 1, 1)$ ,  $(0, 2, 1)$  and  $(0, 2, 0)$ .  
(b) This is a curve, a circle of radius 2, centered at  $(0, 0, 5)$ , lying parallel to the  $x$ - $y$  plane (i.e., in the plane  $z = 5$ ).  
(c) This is a region of space, a half of a solid cone with rounded top, where the rounded top is a portion of the ball of radius 2 centered at the origin and the sides of the cone make an angle of  $\pi/4$  with the positive  $z$ -axis. The bounds on  $\theta$  mean only the portion “above” the  $x$ - $z$  plane are retained.  
(d) This is a curve, a circle of radius 1 centered at  $(0, 0, \sqrt{3})$ , lying parallel to the  $x$ - $y$  plane.
9. 
$$\int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} \int_{z_1}^{z_2} h(r, \theta, z) r dz dr d\theta$$
10. 
$$\int_{\varphi_1}^{\varphi_2} \int_{\theta_1}^{\theta_2} \int_{\rho_1}^{\rho_2} h(\rho, \theta, \varphi) \rho^2 \sin(\varphi) d\rho d\theta d\varphi$$

