1: The Graphical Behavior of Functions

Our study of limits led to continuous functions, which is a certain class of functions that behave in a particularly nice way. Limits then gave us an even nicer class of functions, functions that are differentiable.

This chapter explores many of the ways we can take advantage of the information that continuous and differentiable functions provide.

1.1 Extreme Values

Given any quantity described by a function, we are often interested in the largest and/or smallest values that quantity attains. For instance, if a function describes the speed of an object, it seems reasonable to want to know the fastest/slowest the object traveled. If a function describes the value of a stock, we might want to know the highest/lowest values the stock attained over the past year. We call such values *extreme values.*

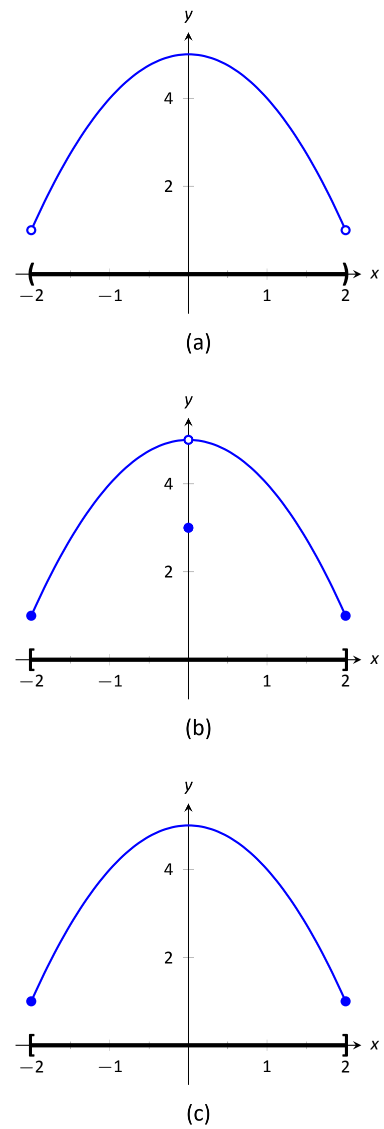

Figure 1.1: Graphs of functions with and without extreme values.

Definition 1 Extreme Values

Let f be defined on an interval I containing c.

1. $f(c)$ is the **minimum** (also, **absolute minimum**) of f on I if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the **maximum** (also, **absolute maximum**) of f on I if $f(c) \geq f(x)$ for all x in I.

The maximum and minimum values are the **extreme values**, or **extrema**, of f on I.

Consider Figure 1.1. The function displayed in (a) has a maximum, but no minimum, as the interval over which the function is defined is open. In (b), the function has a minimum, but no maximum; there is a discontinuity in the "natural" place for the maximum to occur. Finally, the function shown in (c) has both a maximum and a minimum; note that the function is continuous and the interval on which it is defined is closed.

It is possible for discontinuous functions defined on an open interval to have both a maximum and minimum value, but we have just seen examples where they did not. On the other hand, continuous functions on a closed interval *always* have a maximum and minimum value.

Note: The extreme values of a function are "y" values, values the function attains, not the input values.

Theorem 1 The Extreme Value Theorem

Let f be a continuous function defined on a closed interval I. Then f has both a maximum and minimum value on I.

This theorem states that f has extreme values, but it does not offer any advice about how/where to find these values. The process can seem to be fairly easy, as the next example illustrates. After the example, we will draw on lessons learned to form a more general and powerful method for finding extreme values.

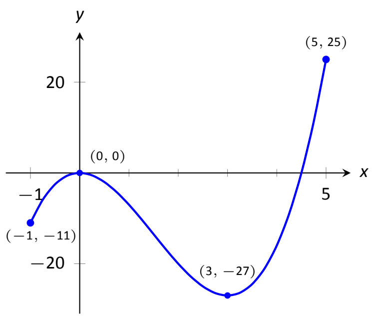

Figure 1.2: A graph of $f(x) = 2x^3 - 9x^2$ as in Example 1.

Example 1 Approximating extreme values

Consider $f(x) = 2x^3 - 9x^2$ on $I = [-1, 5]$, as graphed in Figure 1.2. Approximate the extreme values of f.

SOLUTION The graph is drawn in such a way to draw attention to certain points. It certainly seems that the smallest y value is -27, found when $x = 3$. It also seems that the largest y value is 25, found at the endpoint of I, $x = 5$. We use the word *seems*, for by the graph alone we cannot be sure the smallest value is not less than -27. Since the problem asks for an approximation, we approximate the extreme values to be 25 and -27.

Notice how the minimum value came at "the bottom of a hill," and the maximum value came at an endpoint. Also note that while 0 is not an extreme value, it would be if we narrowed our interval to $[-1, 4]$. The idea that the point $(0, 0)$ is the location of an extreme value for some interval is important, leading us to a definition.

Definition 2 Relative Minimum and Relative Maximum

Let f be defined on an interval I containing c.

1. If there is an open interval containing c such that $f(c)$ is the minimum value, then $f(c)$ is a **relative minimum** of f. We also say that f has a relative minimum at $(c, f(c))$.
2. If there is an open interval containing c such that $f(c)$ is the maximum value, then $f(c)$ is a **relative maximum** of f. We also say that f has a relative maximum at $(c, f(c))$.

The relative maximum and minimum values comprise the **relative extrema** of f.

Note: The terms *local minimum* and *local maximum* are often used as synonyms for *relative minimum* and *relative maximum*.

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We briefly practice using these definitions.

Example 2 **Approximating relative extrema**

Consider $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$, as shown in Figure 1.3. Approximate the relative extrema of f. At each of these points, evaluate f'.

SOLUTION We still do not have the tools to exactly find the relative extrema, but the graph does allow us to make reasonable approximations. It seems f has relative minima at $x = -1$ and $x = 2$, with values of $f(-1) = 0$ and $f(2) = -5.4$. It also seems that f has a relative maximum at the point $(0, 1)$.

We approximate the relative minima to be 0 and -5.4; we approximate the relative maximum to be 1.

It is straightforward to evaluate $f'(x) = \frac{1}{5}(12x^3 - 12x^2 - 24x)$ at $x = 0, 1$ and 2. In each case, $f'(x) = 0$.

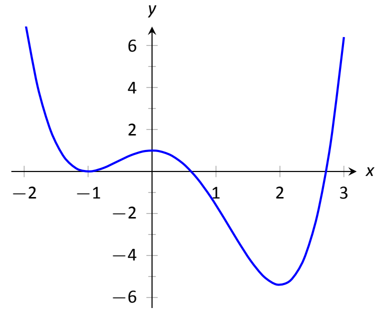

Figure 1.3: A graph of $f(x) = (3x^4 - 4x^3 - 12x^2 + 5)/5$ as in Example 2.

Example 3 **Approximating relative extrema**

Approximate the relative extrema of $f(x) = (x - 1)^{2/3} + 2$, shown in Figure 1.4. At each of these points, evaluate f'.

SOLUTION The figure implies that f does not have any relative maxima, but has a relative minimum at $(1, 2)$. In fact, the graph suggests that not only is this point a relative minimum, $y = f(1) = 2$ is *the* minimum value of the function.

We compute $f'(x) = \frac{2}{3}(x - 1)^{-1/3}$. When $x = 1$, f' is undefined.

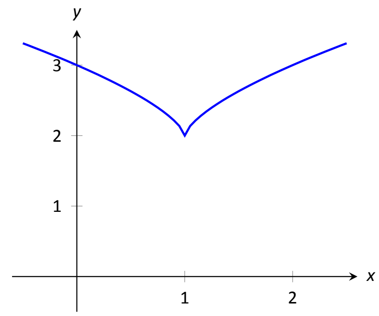

Figure 1.4: A graph of $f(x) = (x-1)^{2/3}+2$ as in Example 3.

What can we learn from the previous two examples? We were able to visually approximate relative extrema, and at each such point, the derivative was either 0 or it was not defined. This observation holds for all functions, leading to a definition and a theorem.

Definition 3 **Critical Numbers and Critical Points**

Let f be defined at c. The value c is a **critical number** (or **critical value**) of f if $f'(c) = 0$ or $f'(c)$ is not defined.

If c is a critical number of f, then the point $(c, f(c))$ is a **critical point** of f.

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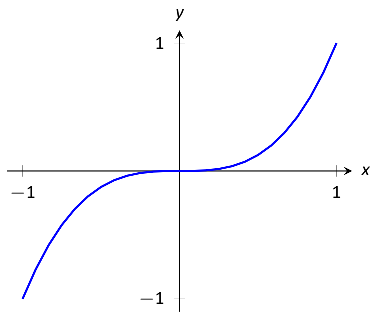

Figure 1.5: A graph of $f(x) = x^3$ which has a critical value of $x = 0$, but no relative extrema.

Theorem 2 Relative Extrema and Critical Points

Let a function f have a relative extrema at the point $(c, f(c))$. Then c is a critical number of f.

Be careful to understand that this theorem states "All relative extrema occur at critical points." It does not say "All critical numbers produce relative extrema." For instance, consider $f(x) = x^3$. Since $f'(x) = 3x^2$, it is straightforward to determine that $x = 0$ is a critical number of f. However, f has no relative extrema, as illustrated in Figure 1.5.

Theorem 1 states that a continuous function on a closed interval will have absolute extrema, that is, both an absolute maximum and an absolute minimum. These extrema occur either at the endpoints or at critical values in the interval. We combine these concepts to offer a strategy for finding extrema.

Key Idea 1 Finding Extrema on a Closed Interval

Let f be a continuous function defined on a closed interval $[a, b]$. To find the maximum and minimum values of f on $[a, b]$:

1. Evaluate f at the endpoints a and b of the interval.
2. Find the critical numbers of f in $[a, b]$.
3. Evaluate f at each critical number.
4. The absolute maximum of f is the largest of these values, and the absolute minimum of f is the least of these values.

We practice these ideas in the next examples.

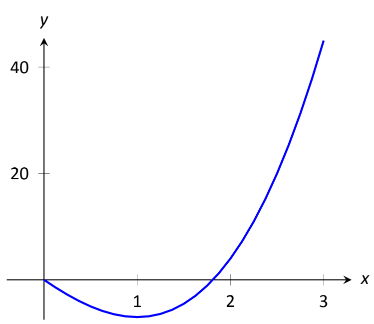

Figure 1.6: A graph of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$ as in Example 4.

Example 4 Finding extreme values

Find the extreme values of $f(x) = 2x^3 + 3x^2 - 12x$ on $[0, 3]$, graphed in Figure 1.6.

Solution We follow the steps outlined in Key Idea 1. We first evaluate f at the endpoints:

$$f(0) = 0 \quad \text{and} \quad f(3) = 45.$$

Next, we find the critical values of f on $[0, 3]$. $f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1)$; therefore the critical values of f are $x = -2$ and $x = 1$. Since $x = -2$

Notes:

does not lie in the interval $[0, 3]$, we ignore it. Evaluating f at the only critical number in our interval gives: $f(1) = -7$.

The table in Figure 1.7 gives f evaluated at the "important" x values in $[0, 3]$. We can easily see the maximum and minimum values of f: the maximum value is 45 and the minimum value is -7.

x	$f(x)$
0	0
1	-7
3	45

Figure 1.7: Finding the extreme values of f in Example 4.

Note that all this was done without the aid of a graph; this work followed an analytic algorithm and did not depend on any visualization. Figure 1.6 shows f and we can confirm our answer, but it is important to understand that these answers can be found without graphical assistance.

We practice again.

Example 5 Finding extreme values

Find the maximum and minimum values of f on $[-4, 2]$, where

$$f(x) = \begin{cases} (x-1)^2 & x \le 0 \\ x+1 & x > 0 \end{cases}.$$

Solution Here f is piecewise–defined, but we can still apply Key Idea 1 as it is continuous on $[-4, 2]$ (one should check to verify that $\lim\limits_{x\to 0} f(x) = f(0)$). Evaluating f at the endpoints gives:

$$f(-4) = 25 \quad \text{and} \quad f(2) = 3.$$

We now find the critical numbers of f. We have to define f' in a piecewise manner; it is

$$f'(x) = \begin{cases} 2(x-1) & x < 0 \\ 1 & x > 0 \end{cases}.$$

Note that while f is defined for all of $[-4, 2]$, f' is not, as the derivative of f does not exist when $x = 0$. (From the left, the derivative approaches -2; from the right the derivative is 1.) Thus one critical number of f is $x = 0$.

We now set $f'(x) = 0$. When $x > 0$, $f'(x)$ is never 0. When $x < 0$, $f'(x)$ is also never 0. (We may be tempted to say that $f'(x) = 0$ when $x = 1$. However, this is nonsensical, for we only consider $f'(x) = 2(x-1)$ when $x < 0$, so we will ignore a solution that says $x = 1$.)

So we have three important x values to consider: $x = -4, 2$ and 0. Evaluating f at each gives, respectively, 25, 3 and 1, shown in Figure 1.8. Thus the absolute minimum of f is 1; the absolute maximum of f is 25. Our answer is confirmed by the graph of f in Figure 1.9.

x	$f(x)$
-4	25
0	1
2	3

Figure 1.8: Finding the extreme values of f in Example 5.

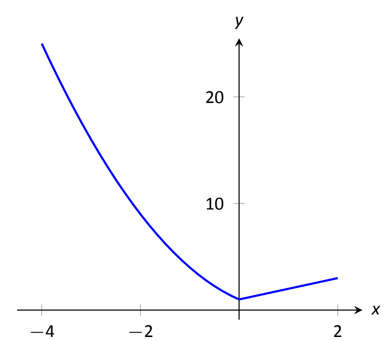

Figure 1.9: A graph of $f(x)$ on $[-4, 2]$ as in Example 5.

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Example 6 Finding extreme values

Find the extrema of $f(x) = \cos(x^2)$ on $[-2, 2]$.

Solution We again use Key Idea 1. Evaluating f at the endpoints of the interval gives: $f(-2) = f(2) = \cos(4) \approx -0.6536$. We now find the critical values of f.

Applying the Chain Rule, we find $f'(x) = -2x\sin(x^2)$. Set $f'(x) = 0$ and solve for x to find the critical values of f.

We have $f'(x) = 0$ when $x = 0$ and when $\sin(x^2) = 0$. In general, $\sin t = 0$ when $t = \ldots -2\pi, -\pi, 0, \pi, \ldots$ Thus $\sin(x^2) = 0$ when $x^2 = 0, \pi, 2\pi, \ldots$ (x^2 is always positive so we ignore $-\pi$, etc.) So $\sin(x^2) = 0$ when $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi}$, etc. The only values to fall in the given interval of $[-2, 2]$ are 0 and $\pm\sqrt{\pi}$, where $\sqrt{\pi} \approx 1.77$.

We again construct a table of important values in Figure 1.10. In this example we have 5 values to consider: $x = 0, \pm 2, \pm\sqrt{\pi}$.

From the table it is clear that the maximum value of f on $[-2, 2]$ is 1; the minimum value is -1. The graph in Figure 1.11 confirms our results.

x	$f(x)$
-2	-0.65
$-\sqrt{\pi}$	-1
0	1
$\sqrt{\pi}$	-1
2	-0.65

Figure 1.10: Finding the extrema of $f(x) = \cos(x^2)$ in Example 6.

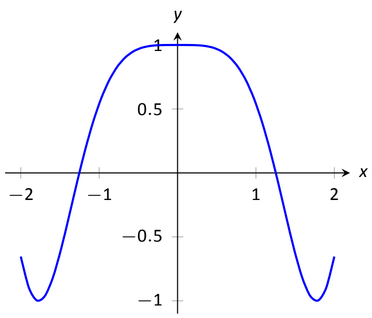


Figure 1.11: A graph of $f(x) = \cos(x^2)$ on $[-2, 2]$ as in Example 6.

We consider one more example.

Example 7 Finding extreme values

Find the extreme values of $f(x) = \sqrt{1-x^2}$.

Solution A closed interval is not given, so we find the extreme values of f on its domain. f is defined whenever $1 - x^2 \geq 0$; thus the domain of f is $[-1, 1]$. Evaluating f at either endpoint returns 0.

Using the Chain Rule, we find $f'(x) = \dfrac{-x}{\sqrt{1-x^2}}$. The critical points of f are found when $f'(x) = 0$ or when f' is undefined. It is straightforward to find that $f'(x) = 0$ when $x = 0$, and f' is undefined when $x = \pm 1$, the endpoints of the interval. The table of important values is given in Figure 1.13. The maximum value is 1, and the minimum value is 0.

x	$f(x)$
-1	0
0	1
1	0

Figure 1.13: Finding the extrema of the half–circle in Example 7.

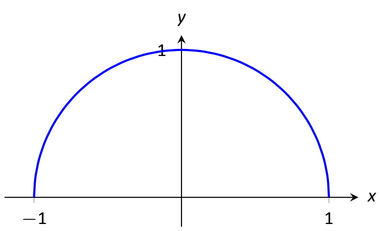


Figure 1.12: A graph of $f(x) = \sqrt{1-x^2}$ on $[-1, 1]$ as in Example 7.

We have seen that continuous functions on closed intervals always have a maximum and minimum value, and we have also developed a technique to find these values. In the next section, we further our study of the information we can glean from "nice" functions with the Mean Value Theorem. On a closed interval, we can find the *average rate of change* of a function (as we did at the beginning of Chapter 2). We will see that differentiable functions always have a point at which their *instantaneous* rate of change is same as the *average* rate of change. This is surprisingly useful, as we'll see.

Note: We implicitly found the derivative of $x^2 + y^2 = 1$, the unit circle, in Example 71 as $\frac{dy}{dx} = -x/y$. In Example 7, half of the unit circle is given as $y = f(x) = \sqrt{1-x^2}$. We found $f'(x) = \frac{-x}{\sqrt{1-x^2}}$. Recognize that the denominator of this fraction is y; that is, we again found $f'(x) = \frac{dy}{dx} = -x/y$.

Notes:

Exercises 1.1

Terms and Concepts

1. Describe what an "extreme value" of a function is in your own words.

2. Sketch the graph of a function f on $(-1, 1)$ that has both a maximum and minimum value.

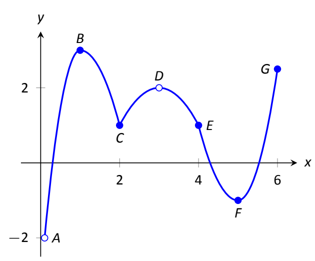
3. Describe the difference between absolute and relative maxima in your own words.

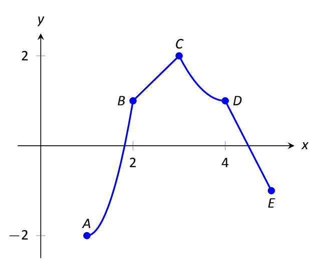
4. Sketch the graph of a function f where f has a relative maximum at $x = 1$ and $f'(1)$ is undefined.

5. T/F: If c is a critical value of a function f, then f has either a relative maximum or relative minimum at $x = c$.

Problems

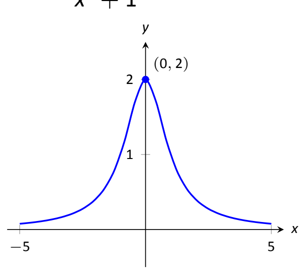
In Exercises 6 – 7, identify each of the marked points as being an absolute maximum or minimum, a relative maximum or minimum, or none of the above.

6. 

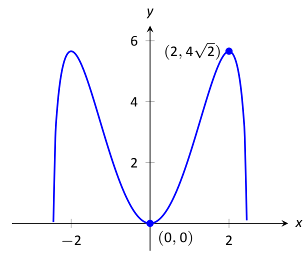
7. 

In Exercises 8 – 14, evaluate $f'(x)$ at the points indicated in the graph.

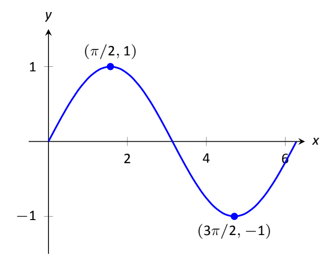
8. $f(x) = \dfrac{2}{x^2+1}$



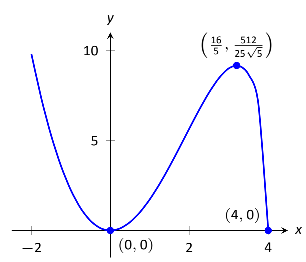
9. $f(x) = x^2\sqrt{6-x^2}$



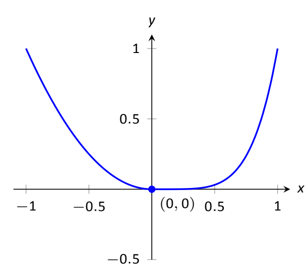
10. $f(x) = \sin x$



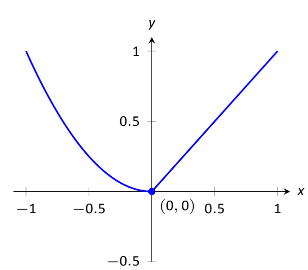
11. $f(x) = x^2\sqrt{4-x}$



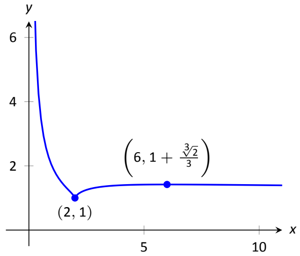
12. $f(x) = \begin{cases} x^2 & x \le 0 \\ x^5 & x > 0 \end{cases}$



13. $f(x) = \begin{cases} x^2 & x \le 0 \\ x & x > 0 \end{cases}$



14. $f(x) = \dfrac{(x-2)^{2/3}}{x}$



In Exercises 15 – 24, find the extreme values of the function on the given interval.

15. $f(x) = x^2 + x + 4$ on $[-1, 2]$.

16. $f(x) = x^3 - \dfrac{9}{2}x^2 - 30x + 3$ on $[0, 6]$.

17. $f(x) = 3\sin x$ on $[\pi/4, 2\pi/3]$.

18. $f(x) = x^2\sqrt{4-x^2}$ on $[-2, 2]$.

19. $f(x) = x + \dfrac{3}{x}$ on $[1, 5]$.

20. $f(x) = \dfrac{x^2}{x^2+5}$ on $[-3, 5]$.

21. $f(x) = e^x \cos x$ on $[0, \pi]$.

22. $f(x) = e^x \sin x$ on $[0, \pi]$.

23. $f(x) = \dfrac{\ln x}{x}$ on $[1, 4]$.

24. $f(x) = x^{2/3} - x$ on $[0, 2]$.

Review

25. Find $\frac{dy}{dx}$, where $x^2y - y^2x = 1$.

26. Find the equation of the line tangent to the graph of $x^2 + y^2 + xy = 7$ at the point $(1, 2)$.

27. Let $f(x) = x^3 + x$.

Evaluate $\displaystyle\lim_{s\to 0} \frac{f(x+s) - f(x)}{s}$.

1.2 The Mean Value Theorem

We motivate this section with the following question: Suppose you leave your house and drive to your friend's house in a city 100 miles away, completing the trip in two hours. At any point during the trip do you necessarily have to be going 50 miles per hour?

In answering this question, it is clear that the *average* speed for the entire trip is 50 mph (i.e. 100 miles in 2 hours), but the question is whether or not your *instantaneous* speed is ever exactly 50 mph. More simply, does your speedometer ever read exactly 50 mph?. The answer, under some very reasonable assumptions, is "yes."

Let's now see why this situation is in a calculus text by translating it into mathematical symbols.

First assume that the function $y = f(t)$ gives the distance (in miles) traveled from your home at time t (in hours) where $0 \leq t \leq 2$. In particular, this gives $f(0) = 0$ and $f(2) = 100$. The slope of the secant line connecting the starting and ending points $(0, f(0))$ and $(2, f(2))$ is therefore

$$\frac{\Delta f}{\Delta t} = \frac{f(2) - f(0)}{2 - 0} = \frac{100 - 0}{2} = 50 \text{ mph}.$$

The slope at any point on the graph itself is given by the derivative $f'(t)$. So, since the answer to the question above is "yes," this means that at some time during the trip, the derivative takes on the value of 50 mph. Symbolically,

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = 50$$

for some time $0 \leq c \leq 2$.

How about more generally? Given any function $y = f(x)$ and a range $a \leq x \leq b$ does the value of the derivative at some point between a and b have to match the slope of the secant line connecting the points $(a, f(a))$ and $(b, f(b))$? Or equivalently, does the equation $f'(c) = \frac{f(b)-f(a)}{b-a}$ have to hold for some $a < c < b$?

Let's look at two functions in an example.

Example 8 Comparing average and instantaneous rates of change

Consider functions

$$f_1(x) = \frac{1}{x^2} \quad \text{and} \quad f_2(x) = |x|$$

with $a = -1$ and $b = 1$ as shown in Figure 1.14(a) and (b), respectively. Both functions have a value of 1 at a and b. Therefore the slope of the secant line

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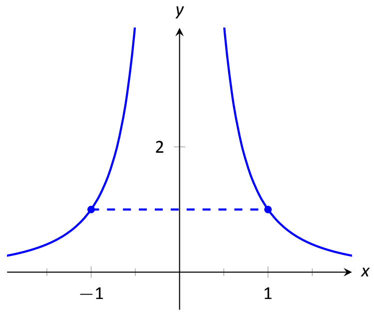
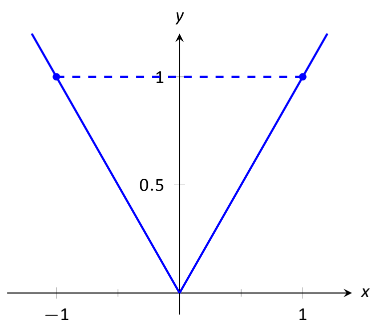
connecting the end points is 0 in each case. But if you look at the plots of each, you can see that there are no points on either graph where the tangent lines have slope zero. Therefore we have found that there is no c in $[-1,1]$ such that

$$f'(c) = \frac{f(1)-f(-1)}{1-(-1)} = 0.$$

So what went "wrong"? It may not be surprising to find that the discontinuity of f_1 and the corner of f_2 play a role. If our functions had been continuous and differentiable, would we have been able to find that special value c? This is our motivation for the following theorem.



(a)



(b)

Figure 1.14: A graph of $f_1(x) = 1/x^2$ and $f_2(x) = |x|$ in Example 8.

Theorem 3 The Mean Value Theorem of Differentiation

Let $y = f(x)$ be continuous function on the closed interval $[a,b]$ and differentiable on the open interval (a,b). There exists a value c, $a < c < b$, such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

That is, there is a value c in (a,b) where the instantaneous rate of change of f at c is equal to the average rate of change of f on $[a,b]$.

Note that the reasons that the functions in Example 8 fail are indeed that f_1 has a discontinuity on the interval $[-1,1]$ and f_2 is not differentiable at the origin.

We will give a proof of the Mean Value Theorem below. To do so, we use a fact, called Rolle's Theorem, stated here.

Theorem 4 Rolle's Theorem

Let f be continuous on $[a,b]$ and differentiable on (a,b), where $f(a) = f(b)$. There is some c in (a,b) such that $f'(c) = 0$.

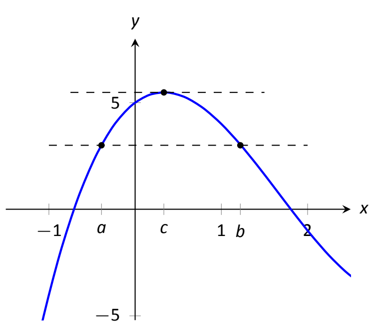


Figure 1.15: A graph of $f(x) = x^3 - 5x^2 + 3x + 5$, where $f(a) = f(b)$. Note the existence of c, where $a < c < b$, where $f'(c) = 0$.

Consider Figure 1.15 where the graph of a function f is given, where $f(a) = f(b)$. It should make intuitive sense that if f is differentiable (and hence, continuous) that there would be a value c in (a,b) where $f'(c) = 0$; that is, there would be a relative maximum or minimum of f in (a,b). Rolle's Theorem guarantees at least one; there may be more.

Rolle's Theorem is really just a special case of the Mean Value Theorem. If $f(a) = f(b)$, then the *average* rate of change on (a,b) is 0, and the theorem

Notes:

guarantees some c where $f'(c) = 0$. We will prove Rolle's Theorem, then use it to prove the Mean Value Theorem.

Proof of Rolle's Theorem

Let f be differentiable on (a, b) where $f(a) = f(b)$. We consider two cases.
Case 1: Consider the case when f is constant on $[a, b]$; that is, $f(x) = f(a) = f(b)$ for all x in $[a, b]$. Then $f'(x) = 0$ for all x in $[a, b]$, showing there is at least one value c in (a, b) where $f'(c) = 0$.
Case 2: Now assume that f is not constant on $[a, b]$. The Extreme Value Theorem guarantees that f has a maximal and minimal value on $[a, b]$, found either at the endpoints or at a critical value in (a, b). Since $f(a) = f(b)$ and f is not constant, it is clear that the maximum and minimum cannot *both* be found at the endpoints. Assume, without loss of generality, that the maximum of f is not found at the endpoints. Therefore there is a c in (a, b) such that $f(c)$ is the maximum value of f. By Theorem 2, c must be a critical number of f; since f is differentiable, we have that $f'(c) = 0$, completing the proof of the theorem. □

We can now prove the Mean Value Theorem.

Proof of the Mean Value Theorem

Define the function

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}x.$$

We know g is differentiable on (a, b) and continuous on $[a, b]$ since f is. We can show $g(a) = g(b)$ (it is actually easier to show $g(b) - g(a) = 0$, which suffices). We can then apply Rolle's theorem to guarantee the existence of $c \in (a, b)$ such that $g'(c) = 0$. But note that

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \; ;$$

hence

$$f'(c) = \frac{f(b) - f(a)}{b - a},$$

which is what we sought to prove. □

Going back to the very beginning of the section, we see that the only assumption we would need about our distance function $f(t)$ is that it be continuous and differentiable for t from 0 to 2 hours (both reasonable assumptions). By the Mean Value Theorem, we are guaranteed a time during the trip where our instantaneous speed is 50 mph. This fact is used in practice. Some law enforcement agencies monitor traffic speeds while in aircraft. They do not measure

Notes:

speed with radar, but rather by timing individual cars as they pass over lines painted on the highway whose distances apart are known. The officer is able to measure the *average* speed of a car between the painted lines; if that average speed is greater than the posted speed limit, the officer is assured that the driver exceeded the speed limit at some time.

Note that the Mean Value Theorem is an *existence* theorem. It states that a special value c *exists*, but it does not give any indication about how to find it. It turns out that when we need the Mean Value Theorem, existence is all we need.

Example 9 Using the Mean Value Theorem

Consider $f(x) = x^3 + 5x + 5$ on $[-3, 3]$. Find c in $[-3, 3]$ that satisfies the Mean Value Theorem.

SOLUTION The average rate of change of f on $[-3, 3]$ is:

$$\frac{f(3) - f(-3)}{3 - (-3)} = \frac{84}{6} = 14.$$

We want to find c such that $f'(c) = 14$. We find $f'(x) = 3x^2 + 5$. We set this equal to 14 and solve for x.

$$\begin{aligned} f'(x) &= 14 \\ 3x^2 + 5 &= 14 \\ x^2 &= 3 \\ x &= \pm\sqrt{3} \approx \pm 1.732 \end{aligned}$$

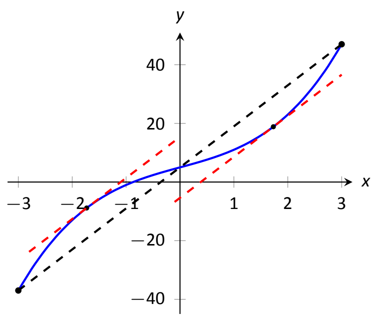


Figure 1.16: Demonstrating the Mean Value Theorem in Example 9.

We have found 2 values c in $[-3, 3]$ where the instantaneous rate of change is equal to the average rate of change; the Mean Value Theorem guaranteed at least one. In Figure 1.16 f is graphed with a dashed line representing the average rate of change; the lines tangent to f at $x = \pm\sqrt{3}$ are also given. Note how these lines are parallel (i.e., have the same slope) as the dashed line.

While the Mean Value Theorem has practical use (for instance, the speed monitoring application mentioned before), it is mostly used to advance other theory. We will use it in the next section to relate the shape of a graph to its derivative.

Notes:

Exercises 1.2

Terms and Concepts

1. Explain in your own words what the Mean Value Theorem states.

2. Explain in your own words what Rolle's Theorem states.

Problems

In Exercises 3 – 10, a function $f(x)$ and interval $[a, b]$ are given. Check if Rolle's Theorem can be applied to f on $[a, b]$; if so, find c in $[a, b]$ such that $f'(c) = 0$.

3. $f(x) = 6$ on $[-1, 1]$.

4. $f(x) = 6x$ on $[-1, 1]$.

5. $f(x) = x^2 + x - 6$ on $[-3, 2]$.

6. $f(x) = x^2 + x - 2$ on $[-3, 2]$.

7. $f(x) = x^2 + x$ on $[-2, 2]$.

8. $f(x) = \sin x$ on $[\pi/6, 5\pi/6]$.

9. $f(x) = \cos x$ on $[0, \pi]$.

10. $f(x) = \dfrac{1}{x^2 - 2x + 1}$ on $[0, 2]$.

In Exercises 11 – 20, a function $f(x)$ and interval $[a, b]$ are given. Check if the Mean Value Theorem can be applied to f on $[a, b]$; if so, find a value c in $[a, b]$ guaranteed by the Mean Value Theorem.

11. $f(x) = x^2 + 3x - 1$ on $[-2, 2]$.

12. $f(x) = 5x^2 - 6x + 8$ on $[0, 5]$.

13. $f(x) = \sqrt{9 - x^2}$ on $[0, 3]$.

14. $f(x) = \sqrt{25 - x}$ on $[0, 9]$.

15. $f(x) = \dfrac{x^2 - 9}{x^2 - 1}$ on $[0, 2]$.

16. $f(x) = \ln x$ on $[1, 5]$.

17. $f(x) = \tan x$ on $[-\pi/4, \pi/4]$.

18. $f(x) = x^3 - 2x^2 + x + 1$ on $[-2, 2]$.

19. $f(x) = 2x^3 - 5x^2 + 6x + 1$ on $[-5, 2]$.

20. $f(x) = \sin^{-1} x$ on $[-1, 1]$.

Review

21. Find the extreme values of $f(x) = x^2 - 3x + 9$ on $[-2, 5]$.

22. Describe the critical points of $f(x) = \cos x$.

23. Describe the critical points of $f(x) = \tan x$.

1.3 Increasing and Decreasing Functions

Our study of "nice" functions f in this chapter has so far focused on individual points: points where f is maximal/minimal, points where $f'(x) = 0$ or f' does not exist, and points c where $f'(c)$ is the average rate of change of f on some interval.

In this section we begin to study how functions behave *between* special points; we begin studying in more detail the shape of their graphs.

We start with an intuitive concept. Given the graph in Figure 1.17, where would you say the function is *increasing*? *Decreasing*? Even though we have not defined these terms mathematically, one likely answered that f is increasing when $x > 1$ and decreasing when $x < 1$. We formally define these terms here.

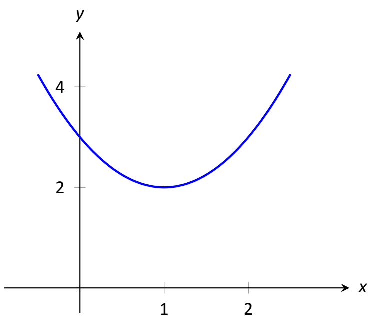


Figure 1.17: A graph of a function f used to illustrate the concepts of *increasing* and *decreasing*.

Definition 4 Increasing and Decreasing Functions

Let f be a function defined on an interval I.

1. f is **increasing** on I if for every $a < b$ in I, $f(a) \leq f(b)$.
2. f is **decreasing** on I if for every $a < b$ in I, $f(a) \geq f(b)$.

A function is **strictly increasing** when $a < b$ in I implies $f(a) < f(b)$, with a similar definition holding for **strictly decreasing**.

Informally, a function is increasing if as x gets larger (i.e., looking left to right) $f(x)$ gets larger.

Our interest lies in finding intervals in the domain of f on which f is either increasing or decreasing. Such information should seem useful. For instance, if f describes the speed of an object, we might want to know when the speed was increasing or decreasing (i.e., when the object was accelerating vs. decelerating). If f describes the population of a city, we should be interested in when the population is growing or declining.

To find such intervals, we again consider secant lines. Let f be an increasing, differentiable function on an open interval I, such as the one shown in Figure 1.18, and let $a < b$ be given in I. The secant line on the graph of f from $x = a$ to $x = b$ is drawn; it has a slope of $(f(b) - f(a))/(b - a)$. But note:

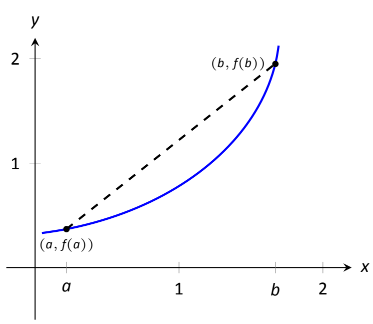


Figure 1.18: Examining the secant line of an increasing function.

$$\frac{f(b)-f(a)}{b-a} \Rightarrow \frac{\text{numerator } > 0}{\text{denominator } > 0} \Rightarrow \begin{array}{c}\text{slope of the}\\ \text{secant line} > 0\end{array} \Rightarrow \begin{array}{c}\text{Average rate of}\\ \text{change of } f \text{ on}\\ [a,b] \text{ is} > 0.\end{array}$$

We have shown mathematically what may have already been obvious: when f is increasing, its secant lines will have a positive slope. Now recall the Mean

Notes:

Value Theorem guarantees that there is a number c, where $a < c < b$, such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} > 0.$$

By considering all such secant lines in I, we strongly imply that $f'(x) \geq 0$ on I. A similar statement can be made for decreasing functions.

Our above logic can be summarized as "If f is increasing, then f' is probably positive." Theorem 5 below turns this around by stating "If f' is postive, then f is increasing." This leads us to a method for finding when functions are increasing and decreasing.

Theorem 5 Test For Increasing/Decreasing Functions

Let f be a continuous function on $[a, b]$ and differentiable on (a, b).

1. If $f'(c) > 0$ for all c in (a, b), then f is increasing on $[a, b]$.
2. If $f'(c) < 0$ for all c in (a, b), then f is decreasing on $[a, b]$.
3. If $f'(c) = 0$ for all c in (a, b), then f is constant on $[a, b]$.

Note: Parts 1 & 2 of Theorem 5 also hold if $f'(c) = 0$ for a finite number of values of c in I.

Let a and b be in I where $f'(a) > 0$ and $f'(b) < 0$. It follows from the Intermediate Value Theorem that there must be some value c between a and b where $f'(c) = 0$. This leads us to the following method for finding intervals on which a function is increasing or decreasing.

Key Idea 2 Finding Intervals on Which f is Increasing or Decreasing

Let f be a differentiable function on an interval I. To find intervals on which f is increasing and decreasing:

1. Find the critical values of f. That is, find all c in I where $f'(c) = 0$ or f' is not defined.
2. Use the critical values to divide I into subintervals.
3. Pick any point p in each subinterval, and find the sign of $f'(p)$.
   (a) If $f'(p) > 0$, then f is increasing on that subinterval.
   (b) If $f'(p) < 0$, then f is decreasing on that subinterval.

Notes:

We demonstrate using this process in the following example.

Example 10 Finding intervals of increasing/decreasing

Let $f(x) = x^3 + x^2 - x + 1$. Find intervals on which f is increasing or decreasing.

Solution Using Key Idea 2, we first find the critical values of f. We have $f'(x) = 3x^2 + 2x - 1 = (3x-1)(x+1)$, so $f'(x) = 0$ when $x = -1$ and when $x = 1/3$. f' is never undefined.

Since an interval was not specified for us to consider, we consider the entire domain of f which is $(-\infty, \infty)$. We thus break the whole real line into three subintervals based on the two critical values we just found: $(-\infty, -1)$, $(-1, 1/3)$ and $(1/3, \infty)$. This is shown in Figure 1.19.

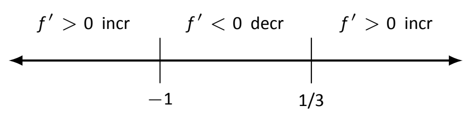


Figure 1.19: Number line for f in Example 10.

We now pick a value p in each subinterval and find the sign of $f'(p)$. All we care about is the sign, so we do not actually have to fully compute $f'(p)$; pick "nice" values that make this simple.

Subinterval 1, $(-\infty, -1)$: We (arbitrarily) pick $p = -2$. We can compute $f'(-2)$ directly: $f'(-2) = 3(-2)^2 + 2(-2) - 1 = 7 > 0$. We conclude that f is increasing on $(-\infty, -1)$.

Note we can arrive at the same conclusion without computation. For instance, we could choose $p = -100$. The first term in $f'(-100)$, i.e., $3(-100)^2$ is clearly positive and very large. The other terms are small in comparison, so we know $f'(-100) > 0$. All we need is the sign.

Subinterval 2, $(-1, 1/3)$: We pick $p = 0$ since that value seems easy to deal with. $f'(0) = -1 < 0$. We conclude f is decreasing on $(-1, 1/3)$.

Subinterval 3, $(1/3, \infty)$: Pick an arbitrarily large value for $p > 1/3$ and note that $f'(p) = 3p^2 + 2p - 1 > 0$. We conclude that f is increasing on $(1/3, \infty)$.

We can verify our calculations by considering Figure 1.20, where f is graphed. The graph also presents f'; note how $f' > 0$ when f is increasing and $f' < 0$ when f is decreasing.

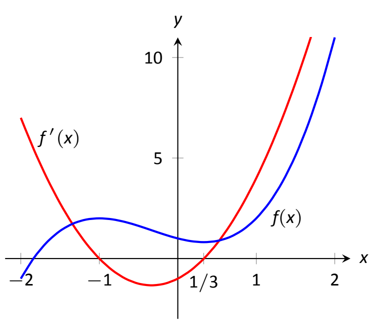


Figure 1.20: A graph of $f(x)$ in Example 10, showing where f is increasing and decreasing.

One is justified in wondering why so much work is done when the graph seems to make the intervals very clear. We give three reasons why the above work is worthwhile.

First, the points at which f switches from increasing to decreasing are not precisely known given a graph. The graph shows us something significant hap-

Notes:

pens near $x = -1$ and $x = 0.3$, but we cannot determine exactly where from the graph.

One could argue that just finding critical values is important; once we know the significant points are $x = -1$ and $x = 1/3$, the graph shows the increasing/decreasing traits just fine. That is true. However, the technique prescribed here helps reinforce the relationship between increasing/decreasing and the sign of f'. Once mastery of this concept (and several others) is obtained, one finds that either (a) just the critical points are computed and the graph shows all else that is desired, or (b) a graph is never produced, because determining increasing/decreasing using f' is straightforward and the graph is unnecessary. So our second reason why the above work is worthwhile is this: once mastery of a subject is gained, one has *options* for finding needed information. We are working to develop mastery.

Finally, our third reason: many problems we face "in the real world" are very complex. Solutions are tractable only through the use of computers to do many calculations for us. Computers do not solve problems "on their own," however; they need to be taught (i.e., *programmed*) to do the right things. It would be beneficial to give a function to a computer and have it return maximum and minimum values, intervals on which the function is increasing and decreasing, the locations of relative maxima, etc. The work that we are doing here is easily programmable. It is hard to teach a computer to "look at the graph and see if it is going up or down." It is easy to teach a computer to "determine if a number is greater than or less than 0."

In Section 1.1 we learned the definition of relative maxima and minima and found that they occur at critical points. We are now learning that functions can switch from increasing to decreasing (and vice–versa) at critical points. This new understanding of increasing and decreasing creates a great method of determining whether a critical point corresponds to a maximum, minimum, or neither. Imagine a function increasing until a critical point at $x = c$, after which it decreases. A quick sketch helps confirm that $f(c)$ must be a relative maximum. A similar statement can be made for relative minimums. We formalize this concept in a theorem.

Theorem 6 First Derivative Test

Let f be differentiable on an interval I and let c be a critical number in I.

1. If the sign of f' switches from positive to negative at c, then $f(c)$ is a relative maximum of f.
2. If the sign of f' switches from negative to positive at c, then $f(c)$ is a relative minimum of f.
3. If the sign of f' does not change at c, then $f(c)$ is not a relative extrema of f.

Notes:

Example 11 **Using the First Derivative Test**

Find the intervals on which f is increasing and decreasing, and use the First Derivative Test to determine the relative extrema of f, where

$$f(x) = \frac{x^2+3}{x-1}.$$

Solution We start by noting the domain of f: $(-\infty, 1) \cup (1, \infty)$. Key Idea 2 describes how to find intervals where f is increasing and decreasing *when the domain of f is an interval.* Since the domain of f in this example is the union of two intervals, we apply the techniques of Key Idea 2 to both intervals of the domain of f.

Since f is not defined at $x = 1$, the increasing/decreasing nature of f could switch at this value. We do not formally consider $x = 1$ to be a critical value of f, but we will include it in our list of critical values that we find next.

Using the Quotient Rule, we find

$$f'(x) = \frac{x^2-2x-3}{(x-1)^2}.$$

We need to find the critical values of f; we want to know when $f'(x) = 0$ and when f' is not defined. That latter is straightforward: when the denominator of $f'(x)$ is 0, f' is undefined. That occurs when $x = 1$, which we've already recognized as an important value.

$f'(x) = 0$ when the numerator of $f'(x)$ is 0. That occurs when $x^2 - 2x - 3 = (x-3)(x+1) = 0$; i.e., when $x = -1, 3$.

We have found that f has two critical numbers, $x = -1, 3$, and at $x = 1$ something important might also happen. These three numbers divide the real number line into 4 subintervals:

$$(-\infty, -1), \quad (-1, 1), \quad (1, 3) \quad \text{and} \quad (3, \infty).$$

Pick a number p from each subinterval and test the sign of f' at p to determine whether f is increasing or decreasing on that interval. Again, we do well to avoid complicated computations; notice that the denominator of f' is *always* positive so we can ignore it during our work.

Interval 1, $(-\infty, -1)$: Choosing a very small number (i.e., a negative number with a large magnitude) p returns $p^2 - 2p - 3$ in the numerator of f'; that will be positive. Hence f is increasing on $(-\infty, -1)$.

Interval 2, $(-1, 1)$: Choosing 0 seems simple: $f'(0) = -3 < 0$. We conclude f is decreasing on $(-1, 1)$.

Notes:

Interval 3, $(1, 3)$: Choosing 2 seems simple: $f'(2) = -3 < 0$. Again, f is decreasing.

Interval 4, $(3, \infty)$: Choosing an very large number p from this subinterval will give a positive numerator and (of course) a positive denominator. So f is increasing on $(3, \infty)$.

In summary, f is increasing on the set $(-\infty, -1) \cup (3, \infty)$ and is decreasing on the set $(-1, 1) \cup (1, 3)$. Since at $x = -1$, the sign of f' switched from positive to negative, Theorem 6 states that $f(-1)$ is a relative maximum of f. At $x = 3$, the sign of f' switched from negative to positive, meaning $f(3)$ is a relative minimum. At $x = 1$, f is not defined, so there is no relative extrema at $x = 1$.

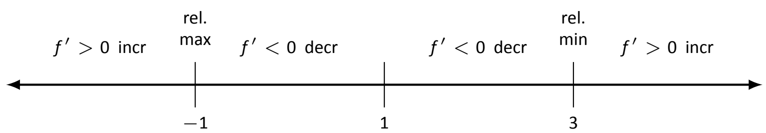


Figure 1.21: Number line for f in Example 11.

This is summarized in the number line shown in Figure 1.21. Also, Figure 1.22 shows a graph of f, confirming our calculations. This figure also shows f', again demonstrating that f is increasing when $f' > 0$ and decreasing when $f' < 0$.

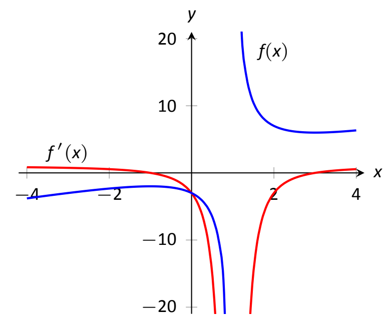


Figure 1.22: A graph of $f(x)$ in Example 11, showing where f is increasing and decreasing.

One is often tempted to think that functions always alternate "increasing, decreasing, increasing, decreasing,. . ." around critical values. Our previous example demonstrated that this is not always the case. While $x = 1$ was not technically a critical value, it was an important value we needed to consider. We found that f was decreasing on "both sides of $x = 1$."

We examine one more example.

Example 12 Using the First Derivative Test

Find the intervals on which $f(x) = x^{8/3} - 4x^{2/3}$ is increasing and decreasing and identify the relative extrema.

Solution We again start with taking derivatives. Since we know we want to solve $f'(x) = 0$, we will do some algebra after taking derivatives.

$$\begin{aligned} f(x) &= x^{\frac{8}{3}} - 4x^{\frac{2}{3}} \\ f'(x) &= \frac{8}{3}x^{\frac{5}{3}} - \frac{8}{3}x^{-\frac{1}{3}} \\ &= \frac{8}{3}x^{-\frac{1}{3}}\left(x^{\frac{6}{3}} - 1\right) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x^2 - 1) \\ &= \frac{8}{3}x^{-\frac{1}{3}}(x-1)(x+1). \end{aligned}$$

Notes:

This derivation of f' shows that $f'(x) = 0$ when $x = \pm 1$ and f' is not defined when $x = 0$. Thus we have 3 critical values, breaking the number line into 4 subintervals as shown in Figure 1.23.

Interval 1, $(\infty, -1)$: We choose $p = -2$; we can easily verify that $f'(-2) < 0$. So f is decreasing on $(-\infty, -1)$.
Interval 2, $(-1, 0)$: Choose $p = -1/2$. Once more we practice finding the sign of $f'(p)$ without computing an actual value. We have $f'(p) = (8/3)p^{-1/3}(p-1)(p+1)$; find the sign of each of the three terms.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{<0} \cdot \underbrace{(p-1)}_{<0} \underbrace{(p+1)}_{>0}.$$

We have a "negative $\times$ negative $\times$ positive" giving a positive number; f is increasing on $(-1, 0)$.
Interval 3, $(0, 1)$: We do a similar sign analysis as before, using p in $(0, 1)$.

$$f'(p) = \frac{8}{3} \cdot \underbrace{p^{-\frac{1}{3}}}_{>0} \cdot \underbrace{(p-1)}_{<0} \underbrace{(p+1)}_{>0}.$$

We have 2 positive factors and one negative factor; $f'(p) < 0$ and so f is decreasing on $(0, 1)$.
Interval 4, $(1, \infty)$: Similar work to that done for the other three intervals shows that $f'(x) > 0$ on $(1, \infty)$, so f is increasing on this interval.

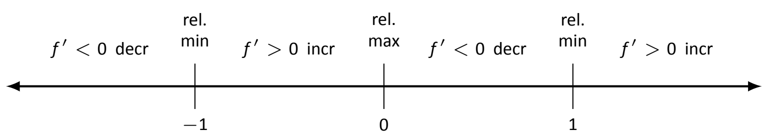


Figure 1.23: Number line for f in Example 12.

We conclude by stating that f is increasing on $(-1, 0) \cup (1, \infty)$ and decreasing on $(-\infty, -1) \cup (0, 1)$. The sign of f' changes from negative to positive around $x = -1$ and $x = 1$, meaning by Theorem 6 that $f(-1)$ and $f(1)$ are relative minima of f. As the sign of f' changes from positive to negative at $x = 0$, we have a relative maximum at $f(0)$. Figure 1.24 shows a graph of f, confirming our result. We also graph f', highlighting once more that f is increasing when $f' > 0$ and is decreasing when $f' < 0$.

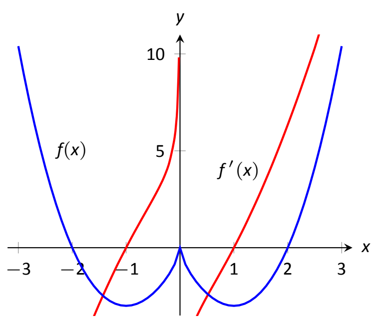


Figure 1.24: A graph of $f(x)$ in Example 12, showing where f is increasing and decreasing.

We have seen how the first derivative of a function helps determine when the function is going "up" or "down." In the next section, we will see how the second derivative helps determine how the graph of a function curves.

Notes:

Exercises 1.3

Terms and Concepts

1. In your own words describe what it means for a function to be increasing.

2. What does a decreasing function "look like"?

3. Sketch a graph of a function on $[0,2]$ that is increasing but not strictly increasing.

4. Give an example of a function describing a situation where it is "bad" to be increasing and "good" to be decreasing.

5. A function f has derivative $f'(x)=(\sin x+2)e^{x^2+1}$, where $f'(x)>1$ for all x. Is f increasing, decreasing, or can we not tell from the given information?

Problems

In Exercises 6 – 13, a function $f(x)$ is given.

(a) Compute $f'(x)$.

(b) Graph f and f' on the same axes (using technology is permitted) and verify Theorem 5.

6. $f(x)=2x+3$

7. $f(x)=x^2-3x+5$

8. $f(x)=\cos x$

9. $f(x)=\tan x$

10. $f(x)=x^3-5x^2+7x-1$

11. $f(x)=2x^3-x^2+x-1$

12. $f(x)=x^4-5x^2+4$

13. $f(x)=\dfrac{1}{x^2+1}$

In Exercises 14 – 23, a function $f(x)$ is given.

(a) Give the domain of f.

(b) Find the critical numbers of f.

(c) Create a number line to determine the intervals on which f is increasing and decreasing.

(d) Use the First Derivative Test to determine whether each critical point is a relative maximum, minimum, or neither.

14. $f(x)=x^2+2x-3$

15. $f(x)=x^3+3x^2+3$

16. $f(x)=2x^3+x^2-x+3$

17. $f(x)=x^3-3x^2+3x-1$

18. $f(x)=\dfrac{1}{x^2-2x+2}$

19. $f(x)=\dfrac{x^2-4}{x^2-1}$

20. $f(x)=\dfrac{x}{x^2-2x-8}$

21. $f(x)=\dfrac{(x-2)^{2/3}}{x}$

22. $f(x)=\sin x\cos x$ on $(-\pi,\pi)$.

23. $f(x)=x^5-5x$

Review

24. Consider $f(x)=x^2-3x+5$ on $[-1,2]$; find c guaranteed by the Mean Value Theorem.

25. Consider $f(x)=\sin x$ on $[-\pi/2,\pi/2]$; find c guaranteed by the Mean Value Theorem.

1.4 Concavity and the Second Derivative

Our study of “nice” functions continues. The previous section showed how the first derivative of a function, f', can relay important information about f. We now apply the same technique to f' itself, and learn what this tells us about f.

The key to studying f' is to consider its derivative, namely f'', which is the second derivative of f. When $f'' > 0$, f' is increasing. When $f'' < 0$, f' is decreasing. f' has relative maxima and minima where $f'' = 0$ or is undefined.

This section explores how knowing information about f'' gives information about f.

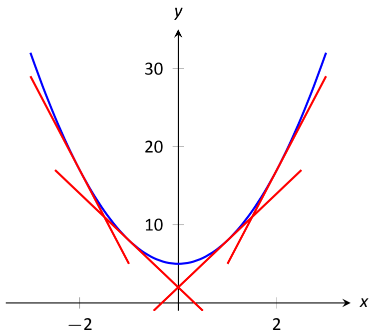


Figure 1.25: A function f with a concave up graph. Notice how the slopes of the tangent lines, when looking from left to right, are increasing.

Concavity

We begin with a definition, then explore its meaning.

Note: We often state that “f is concave up” instead of “the graph of f is concave up” for simplicity.

> **Definition 5 Concave Up and Concave Down**
>
> Let f be differentiable on an interval I. The graph of f is **concave up** on I if f' is increasing. The graph of f is **concave down** on I if f' is decreasing. If f' is constant then the graph of f is said to have **no concavity**.

Note: A mnemonic for remembering what concave up/down means is: “Concave up is like a cup; concave down is like a frown.” It is admittedly terrible, but it works.

The graph of a function f is *concave up* when f' is *increasing.* That means as one looks at a concave up graph from left to right, the slopes of the tangent lines will be increasing. Consider Figure 1.25, where a concave up graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, downward, corresponding to a small value of f'. On the right, the tangent line is steep, upward, corresponding to a large value of f'.

If a function is decreasing and concave up, then its rate of decrease is slowing; it is “leveling off.” If the function is increasing and concave up, then the *rate* of increase is increasing. The function is increasing at a faster and faster rate.

Now consider a function which is concave down. We essentially repeat the above paragraphs with slight variation.

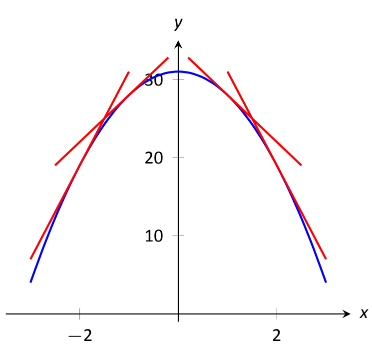


Figure 1.26: A function f with a concave down graph. Notice how the slopes of the tangent lines, when looking from left to right, are decreasing.

The graph of a function f is *concave down* when f' is *decreasing.* That means as one looks at a concave down graph from left to right, the slopes of the tangent lines will be decreasing. Consider Figure 1.26, where a concave down graph is shown along with some tangent lines. Notice how the tangent line on the left is steep, upward, corresponding to a large value of f'. On the right, the tangent line is steep, downward, corresponding to a small value of f'.

If a function is increasing and concave down, then its rate of increase is slowing; it is “leveling off.” If the function is decreasing and concave down, then the *rate* of decrease is decreasing. The function is decreasing at a faster and faster rate.

Notes:

Our definition of concave up and concave down is given in terms of when the first derivative is increasing or decreasing. We can apply the results of the previous section and to find intervals on which a graph is concave up or down. That is, we recognize that f' is increasing when $f'' > 0$, etc.

Theorem 7 Test for Concavity

Let f be twice differentiable on an interval I. The graph of f is concave up if $f'' > 0$ on I, and is concave down if $f'' < 0$ on I.

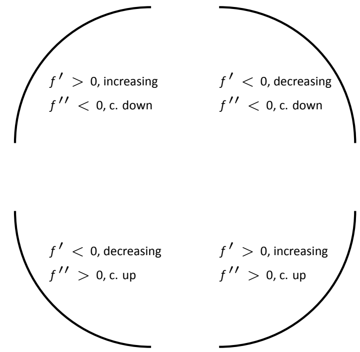


Figure 1.27: Demonstrating the 4 ways that concavity interacts with increasing/decreasing, along with the relationships with the first and second derivatives.

If knowing where a graph is concave up/down is important, it makes sense that the places where the graph changes from one to the other is also important. This leads us to a definition.

Definition 6 Point of Inflection

A **point of inflection** is a point on the graph of f at which the concavity of f changes.

Note: Geometrically speaking, a function is concave up if its graph lies above its tangent lines. A function is concave down if its graph lies below its tangent lines.

Figure 1.28 shows a graph of a function with inflection points labeled.

If the concavity of f changes at a point $(c, f(c))$, then f' is changing from increasing to decreasing (or, decreasing to increasing) at $x = c$. That means that the sign of f'' is changing from positive to negative (or, negative to positive) at $x = c$. This leads to the following theorem.

Theorem 8 Points of Inflection

If $(c, f(c))$ is a point of inflection on the graph of f, then either $f''(c) = 0$ or f'' is not defined at c.

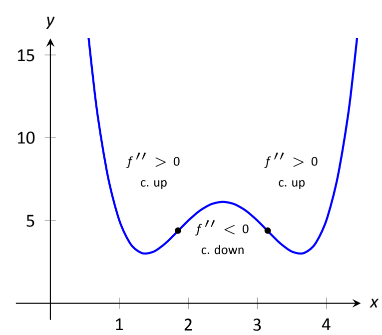


Figure 1.28: A graph of a function with its inflection points marked. The intervals where concave up/down are also indicated.

We have identified the concepts of concavity and points of inflection. It is now time to practice using these concepts; given a function, we should be able to find its points of inflection and identify intervals on which it is concave up or down. We do so in the following examples.

Example 13 Finding intervals of concave up/down, inflection points

Let $f(x) = x^3 - 3x + 1$. Find the inflection points of f and the intervals on which it is concave up/down.

Notes:

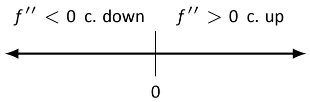


Figure 1.29: A number line determining the concavity of f in Example 13.

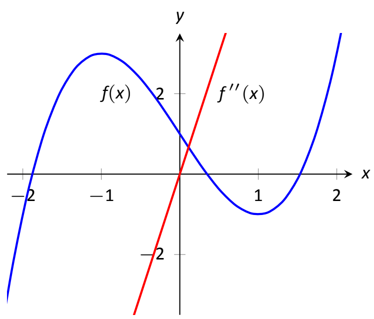


Figure 1.30: A graph of $f(x)$ used in Example 13.

Solution We start by finding $f'(x) = 3x^2 - 3$ and $f''(x) = 6x$. To find the inflection points, we use Theorem 8 and find where $f''(x) = 0$ or where f'' is undefined. We find f'' is always defined, and is 0 only when $x = 0$. So the point $(0, 1)$ is the only possible point of inflection.

This possible inflection point divides the real line into two intervals, $(-\infty, 0)$ and $(0, \infty)$. We use a process similar to the one used in the previous section to determine increasing/decreasing. Pick any $c < 0$; $f''(c) < 0$ so f is concave down on $(-\infty, 0)$. Pick any $c > 0$; $f''(c) > 0$ so f is concave up on $(0, \infty)$. Since the concavity changes at $x = 0$, the point $(0, 1)$ is an inflection point.

The number line in Figure 1.29 illustrates the process of determining concavity; Figure 1.30 shows a graph of f and f'', confirming our results. Notice how f is concave down precisely when $f''(x) < 0$ and concave up when $f''(x) > 0$.

Example 14 **Finding intervals of concave up/down, inflection points**

Let $f(x) = x/(x^2 - 1)$. Find the inflection points of f and the intervals on which it is concave up/down.

Solution We need to find f' and f''. Using the Quotient Rule and simplifying, we find

$$f'(x) = \frac{-(1 + x^2)}{(x^2 - 1)^2} \quad \text{and} \quad f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3}.$$

To find the possible points of inflection, we seek to find where $f''(x) = 0$ and where f'' is not defined. Solving $f''(x) = 0$ reduces to solving $2x(x^2 + 3) = 0$; we find $x = 0$. We find that f'' is not defined when $x = \pm 1$, for then the denominator of f'' is 0. We also note that f itself is not defined at $x = \pm 1$, having a domain of $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$. Since the domain of f is the union of three intervals, it makes sense that the concavity of f could switch across intervals. We technically cannot say that f has a point of inflection at $x = \pm 1$ as they are not part of the domain, but we must still consider these x-values to be important and will include them in our number line.

The important x-values at which concavity might switch are $x = -1$, $x = 0$ and $x = 1$, which split the number line into four intervals as shown in Figure 1.31. We determine the concavity on each. Keep in mind that all we are concerned with is the *sign* of f'' on the interval.

Interval 1, $(-\infty, -1)$: Select a number c in this interval with a large magnitude (for instance, $c = -100$). The denominator of $f''(x)$ will be positive. In the numerator, the $(c^2 + 3)$ will be positive and the $2c$ term will be negative. Thus the numerator is negative and $f''(c)$ is negative. We conclude f is concave down on $(-\infty, -1)$.

Notes:

Interval 2, $(-1, 0)$: For any number c in this interval, the term $2c$ in the numerator will be negative, the term $(c^2 + 3)$ in the numerator will be positive, and the term $(c^2 - 1)^3$ in the denominator will be negative. Thus $f''(c) > 0$ and f is concave up on this interval.
Interval 3, $(0, 1)$: Any number c in this interval will be positive and "small." Thus the numerator is positive while the denominator is negative. Thus $f''(c) < 0$ and f is concave down on this interval.
Interval 4, $(1, \infty)$: Choose a large value for c. It is evident that $f''(c) > 0$, so we conclude that f is concave up on $(1, \infty)$.

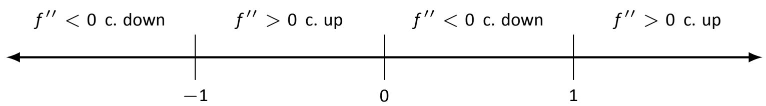


Figure 1.31: Number line for f in Example 14.

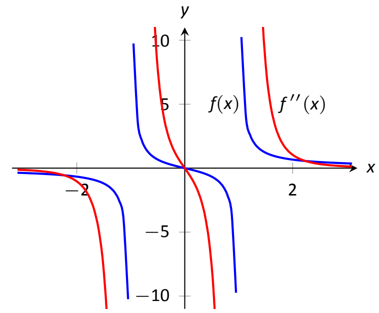


Figure 1.32: A graph of $f(x)$ and $f''(x)$ in Example 14.

We conclude that f is concave up on $(-1, 0) \cup (1, \infty)$ and concave down on $(-\infty, -1) \cup (0, 1)$. There is only one point of inflection, $(0, 0)$, as f is not defined at $x = \pm 1$. Our work is confirmed by the graph of f in Figure 1.32. Notice how f is concave up whenever f'' is positive, and concave down when f'' is negative.

Recall that relative maxima and minima of f are found at critical points of f; that is, they are found when $f'(x) = 0$ or when f' is undefined. Likewise, the relative maxima and minima of f' are found when $f''(x) = 0$ or when f'' is undefined; note that these are the inflection points of f.

What does a "relative maximum of f'" *mean*? The derivative measures the rate of change of f; maximizing f' means finding the where f is increasing the most – where f has the steepest tangent line. A similar statement can be made for minimizing f'; it corresponds to where f has the steepest negatively–sloped tangent line.

We utilize this concept in the next example.

Example 15 Understanding inflection points
The sales of a certain product over a three-year span are modeled by $S(t) = t^4 - 8t^2 + 20$, where t is the time in years, shown in Figure 1.33. Over the first two years, sales are decreasing. Find the point at which sales are decreasing at their greatest rate.

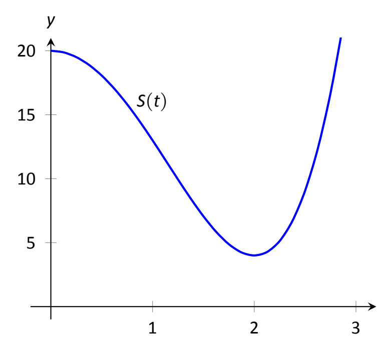


Figure 1.33: A graph of $S(t)$ in Example 15, modeling the sale of a product over time.

Solution We want to maximize the rate of decrease, which is to say, we want to find where S' has a minimum. To do this, we find where S'' is 0. We find $S'(t) = 4t^3 - 16t$ and $S''(t) = 12t^2 - 16$. Setting $S''(t) = 0$ and solving, we get $t = \sqrt{4/3} \approx 1.16$ (we ignore the negative value of t since it does not lie in the domain of our function S).

This is both the inflection point and the point of maximum decrease. This

Notes:

is the point at which things first start looking up for the company. After the inflection point, it will still take some time before sales start to increase, but at least sales are not decreasing quite as quickly as they had been.

A graph of $S(t)$ and $S'(t)$ is given in Figure 1.34. When $S'(t) < 0$, sales are decreasing; note how at $t \approx 1.16$, $S'(t)$ is minimized. That is, sales are decreasing at the fastest rate at $t \approx 1.16$. On the interval of $(1.16, 2)$, S is decreasing but concave up, so the decline in sales is "leveling off."

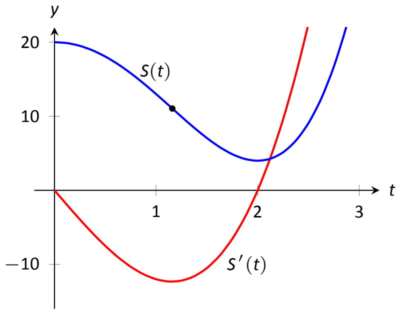


Figure 1.34: A graph of $S(t)$ in Example 15 along with $S'(t)$.

Not every critical point corresponds to a relative extrema; $f(x) = x^3$ has a critical point at $(0, 0)$ but no relative maximum or minimum. Likewise, just because $f''(x) = 0$ we cannot conclude concavity changes at that point. We were careful before to use terminology "*possible* point of inflection" since we needed to check to see if the concavity changed. The canonical example of $f''(x) = 0$ *without* concavity changing is $f(x) = x^4$. At $x = 0$, $f''(x) = 0$ but f is always concave up, as shown in Figure 1.35.

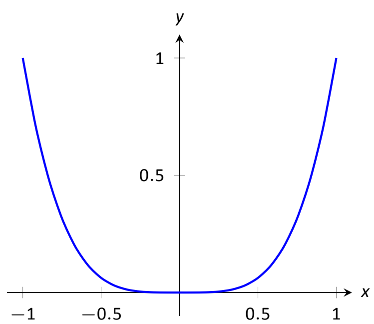


Figure 1.35: A graph of $f(x) = x^4$. Clearly f is always concave up, despite the fact that $f''(x) = 0$ when $x = 0$. It this example, the *possible* point of inflection $(0, 0)$ is not a point of inflection.

The Second Derivative Test

The first derivative of a function gave us a test to find if a critical value corresponded to a relative maximum, minimum, or neither. The second derivative gives us another way to test if a critical point is a local maximum or minimum. The following theorem officially states something that is intuitive: if a critical value occurs in a region where a function f is concave up, then that critical value must correspond to a relative minimum of f, etc. See Figure 1.36 for a visualization of this.

Theorem 9 The Second Derivative Test

Let c be a critical value of f where $f''(c)$ is defined.

1. If $f''(c) > 0$, then f has a local minimum at $(c, f(c))$.
2. If $f''(c) < 0$, then f has a local maximum at $(c, f(c))$.

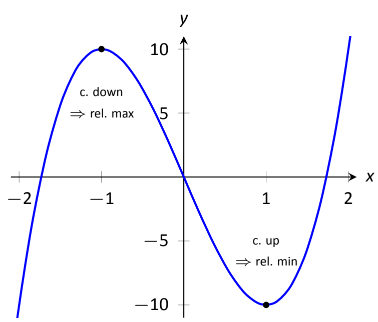


Figure 1.36: Demonstrating the fact that relative maxima occur when the graph is concave down and relative minima occur when the graph is concave up.

The Second Derivative Test relates to the First Derivative Test in the following way. If $f''(c) > 0$, then the graph is concave up at a critical point c and f' itself is growing. Since $f'(c) = 0$ and f' is growing at c, then it must go from negative to positive at c. This means the function goes from decreasing to increasing, indicating a local minimum at c.

Notes:

Example 16 Using the Second Derivative Test
Let $f(x) = 100/x + x$. Find the critical points of f and use the Second Derivative Test to label them as relative maxima or minima.

Solution We find $f'(x) = -100/x^2 + 1$ and $f''(x) = 200/x^3$. We set $f'(x) = 0$ and solve for x to find the critical values (note that f' is not defined at $x = 0$, but neither is f so this is not a critical value.) We find the critical values are $x = \pm 10$. Evaluating f'' at $x = 10$ gives $0.1 > 0$, so there is a local minimum at $x = 10$. Evaluating $f''(-10) = -0.1 < 0$, determining a relative maximum at $x = -10$. These results are confirmed in Figure 1.37.

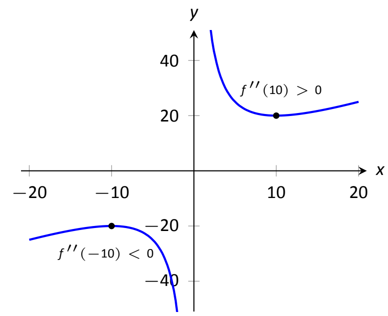


Figure 1.37: A graph of $f(x)$ in Example 16. The second derivative is evaluated at each critical point. When the graph is concave up, the critical point represents a local minimum; when the graph is concave down, the critical point represents a local maximum.

We have been learning how the first and second derivatives of a function relate information about the graph of that function. We have found intervals of increasing and decreasing, intervals where the graph is concave up and down, along with the locations of relative extrema and inflection points. In Chapter 1 we saw how limits explained asymptotic behavior. In the next section we combine all of this information to produce accurate sketches of functions.

Notes:

Exercises 1.4

Terms and Concepts

1. Sketch a graph of a function $f(x)$ that is concave up on $(0, 1)$ and is concave down on $(1, 2)$.

2. Sketch a graph of a function $f(x)$ that is:

   (a) Increasing, concave up on $(0, 1)$,

   (b) increasing, concave down on $(1, 2)$,

   (c) decreasing, concave down on $(2, 3)$ and

   (d) increasing, concave down on $(3, 4)$.

3. Is is possible for a function to be increasing and concave down on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

4. Is is possible for a function to be increasing and concave up on $(0, \infty)$ with a horizontal asymptote of $y = 1$? If so, give a sketch of such a function.

Problems

In Exercises 5 – 15, a function $f(x)$ is given.

(a) Compute $f''(x)$.

(b) Graph f and f'' on the same axes (using technology is permitted) and verify Theorem 7.

5. $f(x) = -7x + 3$

6. $f(x) = -4x^2 + 3x - 8$

7. $f(x) = 4x^2 + 3x - 8$

8. $f(x) = x^3 - 3x^2 + x - 1$

9. $f(x) = -x^3 + x^2 - 2x + 5$

10. $f(x) = \cos x$

11. $f(x) = \sin x$

12. $f(x) = \tan x$

13. $f(x) = \dfrac{1}{x^2 + 1}$

14. $f(x) = \dfrac{1}{x}$

15. $f(x) = \dfrac{1}{x^2}$

In Exercises 16 – 28, a function $f(x)$ is given.

(a) Find the possible points of inflection of f.

(b) Create a number line to determine the intervals on which f is concave up or concave down.

16. $f(x) = x^2 - 2x + 1$

17. $f(x) = -x^2 - 5x + 7$

18. $f(x) = x^3 - x + 1$

19. $f(x) = 2x^3 - 3x^2 + 9x + 5$

20. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

21. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

22. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

23. $f(x) = \dfrac{1}{x^2 + 1}$

24. $f(x) = \dfrac{x}{x^2 - 1}$

25. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

26. $f(x) = x^2 e^x$

27. $f(x) = x^2 \ln x$

28. $f(x) = e^{-x^2}$

In Exercises 29 – 41, a function $f(x)$ is given. Find the critical points of f and use the Second Derivative Test, when possible, to determine the relative extrema. (Note: these are the same functions as in Exercises 16 – 28.)

29. $f(x) = x^2 - 2x + 1$

30. $f(x) = -x^2 - 5x + 7$

31. $f(x) = x^3 - x + 1$

32. $f(x) = 2x^3 - 3x^2 + 9x + 5$

33. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

34. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

35. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

36. $f(x) = \dfrac{1}{x^2 + 1}$

37. $f(x) = \dfrac{x}{x^2 - 1}$

38. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

39. $f(x) = x^2 e^x$

40. $f(x) = x^2 \ln x$

41. $f(x) = e^{-x^2}$

In Exercises 42 – 54, a function $f(x)$ is given. Find the x values where $f'(x)$ has a relative maximum or minimum. (Note: these are the same functions as in Exercises 16 – 28.)

42. $f(x) = x^2 - 2x + 1$

43. $f(x) = -x^2 - 5x + 7$

44. $f(x) = x^3 - x + 1$

45. $f(x) = 2x^3 - 3x^2 + 9x + 5$

46. $f(x) = \dfrac{x^4}{4} + \dfrac{x^3}{3} - 2x + 3$

47. $f(x) = -3x^4 + 8x^3 + 6x^2 - 24x + 2$

48. $f(x) = x^4 - 4x^3 + 6x^2 - 4x + 1$

49. $f(x) = \dfrac{1}{x^2 + 1}$

50. $f(x) = \dfrac{x}{x^2 - 1}$

51. $f(x) = \sin x + \cos x$ on $(-\pi, \pi)$

52. $f(x) = x^2 e^x$

53. $f(x) = x^2 \ln x$

54. $f(x) = e^{-x^2}$

1.5 Curve Sketching

We have been learning how we can understand the behavior of a function based on its first and second derivatives. While we have been treating the properties of a function separately (increasing and decreasing, concave up and concave down, etc.), we combine them here to produce an accurate graph of the function without plotting lots of extraneous points.

Why bother? Graphing utilities are very accessible, whether on a computer, a hand–held calculator, or a smartphone. These resources are usually very fast and accurate. We will see that our method is not particularly fast – it will require time (but it is not *hard*). So again: why bother?

We are attempting to understand the behavior of a function f based on the information given by its derivatives. While all of a function's derivatives relay information about it, it turns out that "most" of the behavior we care about is explained by f' and f''. Understanding the interactions between the graph of f and f' and f'' is important. To gain this understanding, one might argue that all that is needed is to look at lots of graphs. This is true to a point, but is somewhat similar to stating that one understands how an engine works after looking only at pictures. It is true that the basic ideas will be conveyed, but "hands–on" access increases understanding.

The following Key Idea summarizes what we have learned so far that is applicable to sketching graphs of functions and gives a framework for putting that information together. It is followed by several examples.

Key Idea 3 Curve Sketching

To produce an accurate sketch a given function f, consider the following steps.

1. Find the domain of f. Generally, we assume that the domain is the entire real line then find restrictions, such as where a denominator is 0 or where negatives appear under the radical.
2. Find the critical values of f.
3. Find the possible points of inflection of f.
4. Find the location of any vertical asymptotes of f (usually done in conjunction with item 1 above).
5. Consider the limits $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$ to determine the end behavior of the function.

(continued)

Notes:

Key Idea 3 Curve Sketching – Continued

6. Create a number line that includes all critical points, possible points of inflection, and locations of vertical asymptotes. For each interval created, determine whether f is increasing or decreasing, concave up or down.

7. Evaluate f at each critical point and possible point of inflection. Plot these points on a set of axes. Connect these points with curves exhibiting the proper concavity. Sketch asymptotes and x and y intercepts where applicable.

Example 17 Curve sketching

Use Key Idea 3 to sketch $f(x) = 3x^3 - 10x^2 + 7x + 5$.

Solution We follow the steps outlined in the Key Idea.

1. The domain of f is the entire real line; there are no values x for which $f(x)$ is not defined.

2. Find the critical values of f. We compute $f'(x) = 9x^2 - 20x + 7$. Use the Quadratic Formula to find the roots of f':

$$x = \frac{20 \pm \sqrt{(-20)^2 - 4(9)(7)}}{2(9)} = \frac{1}{9}\left(10 \pm \sqrt{37}\right) \Rightarrow x \approx 0.435, 1.787.$$

3. Find the possible points of inflection of f. Compute $f''(x) = 18x - 20$. We have

$$f''(x) = 0 \Rightarrow x = 10/9 \approx 1.111.$$

4. There are no vertical asymptotes.

5. We determine the end behavior using limits as x approaches $\pm$infinity.

$$\lim_{x\to-\infty} f(x) = -\infty \qquad \lim_{x\to\infty} f(x) = \infty.$$

   We do not have any horizontal asymptotes.

6. We place the values $x = (10 \pm \sqrt{37})/9$ and $x = 10/9$ on a number line, as shown in Figure 1.38. We mark each subinterval as increasing or

Notes:

decreasing, concave up or down, using the techniques used in Sections 1.3 and 1.4.

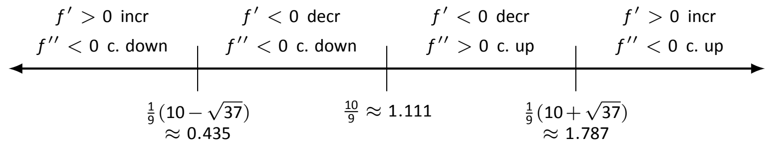
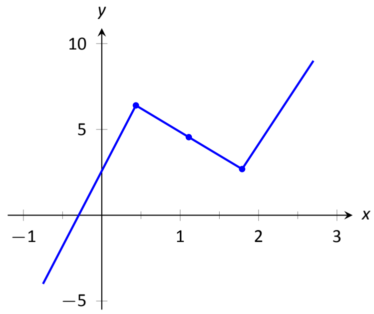
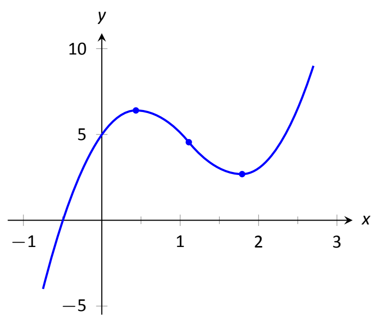


Figure 1.38: Number line for f in Example 17.

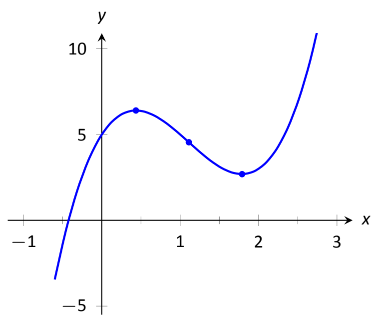
7. We plot the appropriate points on axes as shown in Figure 1.39(a) and connect the points with straight lines. In Figure 1.39(b) we adjust these lines to demonstrate the proper concavity. Our curve crosses the y axis at $y = 5$ and crosses the x axis near $x = -0.424$. In Figure 1.39(c) we show a graph of f drawn with a computer program, verifying the accuracy of our sketch.



(a)



(b)



(c)

Figure 1.39: Sketching f in Example 17.

Example 18 **Curve sketching**

Sketch $f(x) = \dfrac{x^2 - x - 2}{x^2 - x - 6}$.

Solution We again follow the steps outlined in Key Idea 3.

1. In determining the domain, we assume it is all real numbers and looks for restrictions. We find that at $x = -2$ and $x = 3$, $f(x)$ is not defined. So the domain of f is $D = \{\text{real numbers } x \mid x \neq -2, 3\}$.

2. To find the critical values of f, we first find $f'(x)$. Using the Quotient Rule, we find
$$f'(x) = \frac{-8x + 4}{(x^2 + x - 6)^2} = \frac{-8x + 4}{(x-3)^2(x+2)^2}.$$
$f'(x) = 0$ when $x = 1/2$, and f' is undefined when $x = -2, 3$. Since f' is undefined only when f is, these are not critical values. The only critical value is $x = 1/2$.

3. To find the possible points of inflection, we find $f''(x)$, again employing the Quotient Rule:
$$f''(x) = \frac{24x^2 - 24x + 56}{(x-3)^3(x+2)^3}.$$
We find that $f''(x)$ is never 0 (setting the numerator equal to 0 and solving for x, we find the only roots to this quadratic are imaginary) and f'' is

Notes:

undefined when $x = -2, 3$. Thus concavity will possibly only change at $x = -2$ and $x = 3$.

4. The vertical asymptotes of f are at $x = -2$ and $x = 3$, the places where f is undefined.

5. There is a horizontal asymptote of $y = 1$, as $\lim\limits_{x\to-\infty} f(x) = 1$ and $\lim\limits_{x\to\infty} f(x) = 1$.

6. We place the values $x = 1/2$, $x = -2$ and $x = 3$ on a number line as shown in Figure 1.40. We mark in each interval whether f is increasing or decreasing, concave up or down. We see that f has a relative maximum at $x = 1/2$; concavity changes only at the vertical asymptotes.

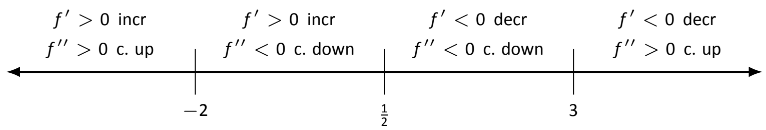


Figure 1.40: Number line for f in Example 18.

7. In Figure 1.41(a), we plot the points from the number line on a set of axes and connect the points with straight lines to get a general idea of what the function looks like (these lines effectively only convey increasing/decreasing information). In Figure 1.41(b), we adjust the graph with the appropriate concavity. We also show f crossing the x axis at $x = -1$ and $x = 2$.

Figure 1.41(c) shows a computer generated graph of f, which verifies the accuracy of our sketch.

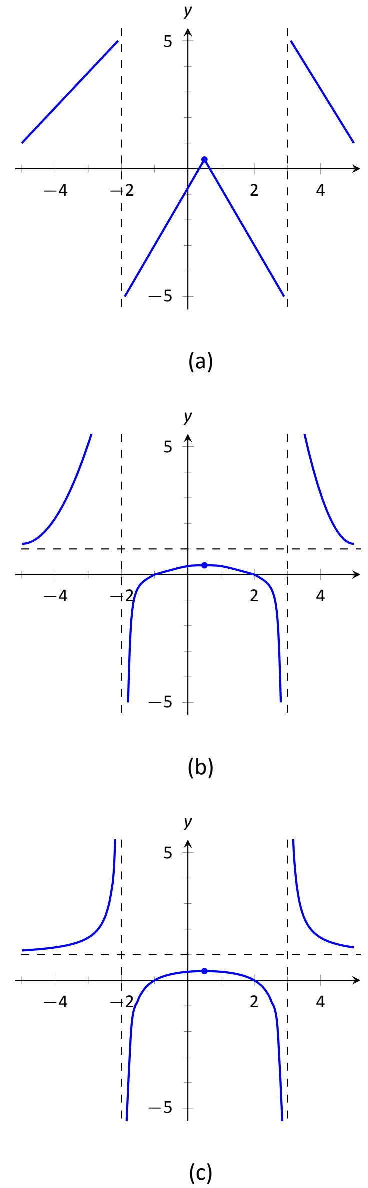


Figure 1.41: Sketching f in Example 18.

Example 19 Curve sketching

Sketch $f(x) = \dfrac{5(x-2)(x+1)}{x^2+2x+4}$.

Solution We again follow Key Idea 3.

1. We assume that the domain of f is all real numbers and consider restrictions. The only restrictions come when the denominator is 0, but this never occurs. Therefore the domain of f is all real numbers, $\mathbb{R}$.

2. We find the critical values of f by setting $f'(x) = 0$ and solving for x. We find

$$f'(x) = \frac{15x(x+4)}{(x^2+2x+4)^2} \quad \Rightarrow \quad f'(x) = 0 \text{ when } x = -4, 0.$$

Notes:

3. We find the possible points of inflection by solving $f''(x) = 0$ for x. We find
$$f''(x) = -\frac{30x^3 + 180x^2 - 240}{(x^2 + 2x + 4)^3}.$$
The cubic in the numerator does not factor very "nicely." We instead approximate the roots at $x = -5.759$, $x = -1.305$ and $x = 1.064$.

4. There are no vertical asymptotes.

5. We have a horizontal asymptote of $y = 5$, as $\displaystyle\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x) = 5$.

6. We place the critical points and possible points on a number line as shown in Figure 1.42 and mark each interval as increasing/decreasing, concave up/down appropriately.

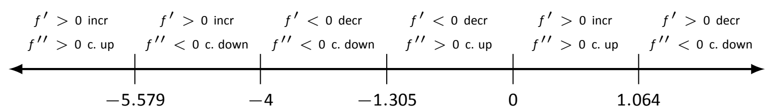


Figure 1.42: Number line for f in Example 19.

7. In Figure 1.43(a) we plot the significant points from the number line as well as the two roots of f, $x = -1$ and $x = 2$, and connect the points with straight lines to get a general impression about the graph. In Figure 1.43(b), we add concavity. Figure 1.43(c) shows a computer generated graph of f, affirming our results.

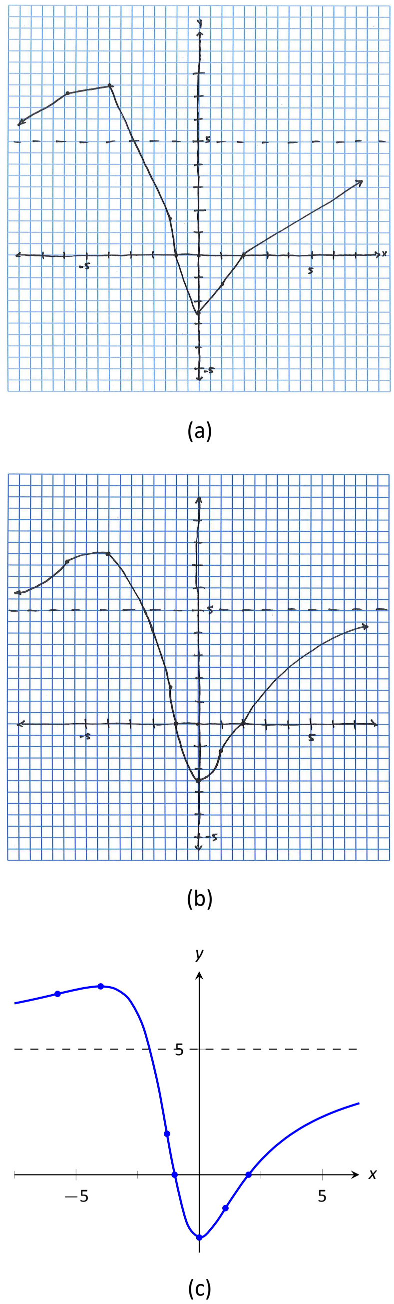


Figure 1.43: Sketching f in Example 19.

In each of our examples, we found a few, significant points on the graph of f that corresponded to changes in increasing/decreasing or concavity. We connected these points with straight lines, then adjusted for concavity, and finished by showing a very accurate, computer generated graph.

Why are computer graphics so good? It is not because computers are "smarter" than we are. Rather, it is largely because computers are much faster at computing than we are. In general, computers graph functions much like most students do when first learning to draw graphs: they plot equally spaced points, then connect the dots using lines. By using lots of points, the connecting lines are short and the graph looks smooth.

This does a fine job of graphing in most cases (in fact, this is the method used for many graphs in this text). However, in regions where the graph is very "curvy," this can generate noticeable sharp edges on the graph unless a large number of points are used. High quality computer algebra systems, such as

Notes:

Mathematica, use special algorithms to plot lots of points only where the graph is "curvy."

In Figure 1.44, a graph of $y = \sin x$ is given, generated by *Mathematica*. The small points represent each of the places *Mathematica* sampled the function. Notice how at the "bends" of $\sin x$, lots of points are used; where $\sin x$ is relatively straight, fewer points are used. (Many points are also used at the endpoints to ensure the "end behavior" is accurate.)

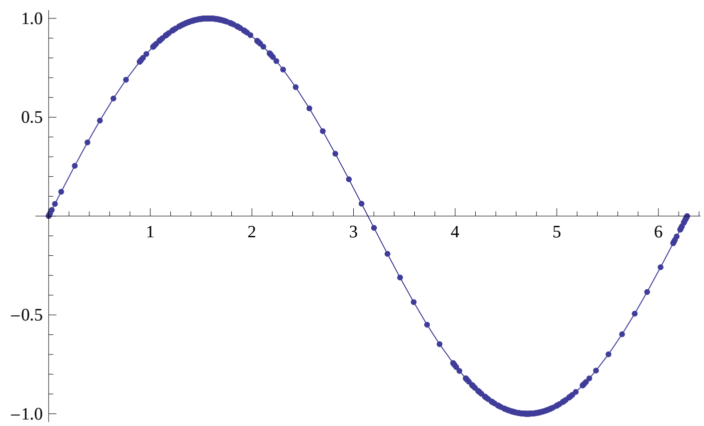


Figure 1.44: A graph of $y = \sin x$ generated by *Mathematica*.

How does *Mathematica* know where the graph is "curvy"? Calculus. When we study *curvature* in a later chapter, we will see how the first and second derivatives of a function work together to provide a measurement of "curviness." *Mathematica* employs algorithms to determine regions of "high curvature" and plots extra points there.

Again, the goal of this section is not "How to graph a function when there is no computer to help." Rather, the goal is "Understand that the shape of the graph of a function is largely determined by understanding the behavior of the function at a few key places." In Example 19, we were able to accurately sketch a complicated graph using only 5 points and knowledge of asymptotes!

There are many applications of our understanding of derivatives beyond curve sketching. The next chapter explores some of these applications, demonstrating just a few kinds of problems that can be solved with a basic knowledge of differentiation.

Notes:

Exercises 1.5

Terms and Concepts

1. Why is sketching curves by hand beneficial even though technology is ubiquitous?

2. What does "ubiquitous" mean?

3. T/F: When sketching graphs of functions, it is useful to find the critical points.

4. T/F: When sketching graphs of functions, it is useful to find the possible points of inflection.

5. T/F: When sketching graphs of functions, it is useful to find the horizontal and vertical asymptotes.

Problems

In Exercises 6 – 11, practice using Key Idea 3 by applying the principles to the given functions with familiar graphs.

6. $f(x) = 2x + 4$

7. $f(x) = -x^2 + 1$

8. $f(x) = \sin x$

9. $f(x) = e^x$

10. $f(x) = \dfrac{1}{x}$

11. $f(x) = \dfrac{1}{x^2}$

In Exercises 12 – 25, sketch a graph of the given function using Key Idea 3. Show all work; check your answer with technology.

12. $f(x) = x^3 - 2x^2 + 4x + 1$

13. $f(x) = -x^3 + 5x^2 - 3x + 2$

14. $f(x) = x^3 + 3x^2 + 3x + 1$

15. $f(x) = x^3 - x^2 - x + 1$

16. $f(x) = (x-2)\ln(x-2)$

17. $f(x) = (x-2)^2\ln(x-2)$

18. $f(x) = \dfrac{x^2-4}{x^2}$

19. $f(x) = \dfrac{x^2-4x+3}{x^2-6x+8}$

20. $f(x) = \dfrac{x^2-2x+1}{x^2-6x+8}$

21. $f(x) = x\sqrt{x+1}$

22. $f(x) = x^2e^x$

23. $f(x) = \sin x\cos x$ on $[-\pi, \pi]$

24. $f(x) = (x-3)^{2/3} + 2$

25. $f(x) = \dfrac{(x-1)^{2/3}}{x}$

In Exercises 26 – 28, a function with the parameters *a* and *b* are given. Describe the critical points and possible points of inflection of *f* in terms of *a* and *b*.

26. $f(x) = \dfrac{a}{x^2+b^2}$

27. $f(x) = \sin(ax + b)$

28. $f(x) = (x-a)(x-b)$

29. Given $x^2 + y^2 = 1$, use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Use this information to justify the sketch of the unit circle.

A: Solutions To Selected Problems

Chapter 1

Section 1.1

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. Answers will vary.
5. F
6. A: none B: abs. max & rel. max C: rel. min D: none E: none F: rel. min G: none
7. A: abs. min B: none C: abs. max D: none E: none
8. $f'(0) = 0$
9. $f'(0) = 0\, f'(2) = 0$
10. $f'(\pi/2) = 0\, f'(3\pi/2) = 0$
11. $f'(0) = 0\, f'(3.2) = 0\, f'(4)$ is undefined
12. $f'(0) = 0$
13. $f'(0)$ is not defined
14. $f'(2)$ is not defined $f'(6) = 0$
15. min: $(-0.5, 3.75)$
max: $(2, 10)$
16. min: $(5, -134.5)$
max: $(0, 3)$
17. min: $(\pi/4, 3\sqrt{2}/2)$
max: $(\pi/2, 3)$
18. min: $(0, 0)$ and $(\pm 2, 0)$
max: $(\pm 2\sqrt{2/3}, 16\sqrt{3}/9)$
19. min: $(\sqrt{3}, 2\sqrt{3})$
max: $(5, 28/5)$
20. min: $(0, 0)$
max: $(5, 5/6)$
21. min: $(\pi, -e^{\pi})$
max: $(\pi/4, \frac{\sqrt{2}e^{\pi/4}}{2})$
22. min: $(0, 0)$ and $(\pi, 0)$
max: $(3\pi/4, \frac{\sqrt{2}e^{3\pi/4}}{2})$
23. min: $(1, 0)$
max: $(e, 1/e)$
24. min: $(2, 2^{2/3} - 2)$
max: $(8/27, 4/27)$
25. $\frac{dy}{dx} = \frac{y(y-2x)}{x(x-2y)}$
26. $y = -4/5(x - 1) + 2$
27. $3x^2 + 1$

Section 1.2

1. Answers will vary.
2. Answers will vary.
3. Any c in $[-1, 1]$ is valid.
4. Rolle's Thm. does not apply.
5. $c = -1/2$
6. $c = -1/2$
7. Rolle's Thm. does not apply.
8. $c = \pi/2$
9. Rolle's Thm. does not apply.
10. Rolle's Theorem does not apply.
11. $c = 0$
12. $c = 5/2$
13. $c = 3/\sqrt{2}$
14. $c = 19/4$
15. The Mean Value Theorem does not apply.
16. $c = 4/\ln 5$
17. $c = \pm\sec^{-1}(2/\sqrt{\pi})$
18. $c = -2/3$
19. $c = \frac{5\pm 7\sqrt{7}}{6}$
20. $c = \frac{\pm\sqrt{\pi^2-4}}{\pi}$
21. Max value of 19 at $x = -2$ and $x = 5$; min value of 6.75 at $x = 1.5$.
22. They are the odd, integer valued multiples of $\pi/2$ (such as $0, \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, etc.)
23. They are the odd, integer valued multiples of $\pi/2$ (such as $0, \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2$, etc.)

Section 1.3

1. Answers will vary.
2. Answers will vary.
3. Answers will vary.
4. Answers will vary.
5. Increasing
6. Graph and verify.
7. Graph and verify.
8. Graph and verify.
9. Graph and verify.
10. Graph and verify.
11. Graph and verify.
12. Graph and verify.
13. Graph and verify.
14. domain: $(-\infty, \infty)$
c.p. at $c = -1$;
decreasing on $(-\infty, -1)$;
increasing on $(-1, \infty)$;
rel. min at $x = -1$.
15. domain=$(-\infty, \infty)$
c.p. at $c = -2, 0$;
increasing on $(-\infty, -2) \cup (0, \infty)$;
decreasing on $(-2, 0)$;
rel. min at $x = 0$;
rel. max at $x = -2$.

16. domain=$(-\infty,\infty)$
c.p. at $c=\frac{1}{6}(-1\pm\sqrt{7})$;
decreasing on $(\frac{1}{6}(-1-\sqrt{7}),\frac{1}{6}(-1+\sqrt{7})))$;
increasing on $(-\infty,\frac{1}{6}(-1-\sqrt{7}))\cup(\frac{1}{6}(-1+\sqrt{7}),\infty)$;
rel. min at $x=\frac{1}{6}(-1+\sqrt{7})$;
rel. max at $x=\frac{1}{6}(-1-\sqrt{7})$.
17. domain=$(-\infty,\infty)$
c.p. at $c=1$;
increasing on $(-\infty,\infty)$;
18. domain=$(-\infty,\infty)$
c.p. at $c=1$;
decreasing on $(1,\infty)$
increasing on $(-\infty,1)$;
rel. max at $x=1$.
19. domain=$(-\infty,-1)\cup(-1,1)\cup(1,\infty)$
c.p. at $c=0$;
decreasing on $(-\infty,-1)\cup(-1,0)$;
increasing on $(0,1)\cup(1,\infty)$;
rel. min at $x=0$;
20. domain=$(-\infty,-2)\cup(-2,4)\cup(4,\infty)$
no c.p.;
decreasing on entire domain, $(-\infty,-2)\cup(-2,4)\cup(4,\infty)$
21. domain=$(-\infty,0)\cup(0,\infty)$;
c.p. at $c=2,6$;
decreasing on $(-\infty,0)\cup(0,2)\cup(6,\infty)$;
increasing on $(2,6)$;
rel. min at $x=2$; rel. max at $x=6$.
22. domain=$(-\infty,\infty)$
c.p. at $c=-3\pi/4,-\pi/4,\pi/4,3\pi/4$;
decreasing on $(-3\pi/4,-\pi/4)\cup(\pi/4,3\pi/4)$;
increasing on $(-\pi,-3\pi/4)\cup(-\pi/4,\pi/4)\cup(3\pi/4,\pi)$;
rel. min at $x=-\pi/4,3\pi/4$;
rel. max at $x=-3\pi/4,\pi/4$.
23. domain = $(-\infty,\infty)$;
c.p. at $c=-1,1$;
decreasing on $(-1,1)$;
increasing on $(-\infty,-1)\cup(1,\infty)$;
rel. min at $x=1$;
rel. max at $x=-1$
24. $c=1/2$
25. $c=\pm\cos^{-1}(2/\pi)$

Section 1.4

1. Answers will vary.
2. Answers will vary.
3. Yes; Answers will vary.
4. No.
5. Graph and verify.
6. Graph and verify.
7. Graph and verify.
8. Graph and verify.
9. Graph and verify.
10. Graph and verify.
11. Graph and verify.
12. Graph and verify.
13. Graph and verify.
14. Graph and verify.
15. Graph and verify.
16. Possible points of inflection: none; concave up on $(-\infty,\infty)$
17. Possible points of inflection: none; concave down on $(-\infty,\infty)$
18. Possible points of inflection: $x=0$; concave down on $(-\infty,0)$; concave up on $(0,\infty)$
19. Possible points of inflection: $x=1/2$; concave down on $(-\infty,1/2)$; concave up on $(1/2,\infty)$
20. Possible points of inflection: $x=-2/3,0$; concave down on $(-2/3,0)$; concave up on $(-\infty,-2/3)\cup(0,\infty)$
21. Possible points of inflection: $x=(1/3)(2\pm\sqrt{7})$; concave up on $((1/3)(2-\sqrt{7}),(1/3)(2+\sqrt{7}))$; concave down on $(-\infty,(1/3)(2-\sqrt{7}))\cup((1/3)(2+\sqrt{7}),\infty)$
22. Possible points of inflection: $x=1$; concave up on $(-\infty,\infty)$
23. Possible points of inflection: $x=\pm1/\sqrt{3}$; concave down on $(-1/\sqrt{3},1/\sqrt{3})$; concave up on $(-\infty,-1/\sqrt{3})\cup(1/\sqrt{3},\infty)$
24. Possible points of inflection: $x=0,\pm1$; concave down on $(-\infty,-1)\cup(0,1)$ concave up on $(-1,0)\cup(1,\infty)$
25. Possible points of inflection: $x=-\pi/4,3\pi/4$; concave down on $(-\pi/4,3\pi/4)$ concave up on $(-\pi,-\pi/4)\cup(3\pi/4,\pi)$
26. Possible points of inflection: $x=-2\pm\sqrt{2}$; concave down on $(-2-\sqrt{2},-2+\sqrt{2})$ concave up on $(-\infty,-2-\sqrt{2})\cup(-2+\sqrt{2},\infty)$
27. Possible points of inflection: $x=1/e^{3/2}$; concave down on $(0,1/e^{3/2})$ concave up on $(1/e^{3/2},\infty)$
28. Possible points of inflection: $x=\pm1/\sqrt{2}$; concave down on $(-1/\sqrt{2},1/\sqrt{2})$ concave up on $(-\infty,-1/\sqrt{2})\cup(1/\sqrt{2},\infty)$
29. min: $x=1$
30. max: $x=-5/2$
31. max: $x=-1/\sqrt{3}$ min: $x=1/\sqrt{3}$
32.
33. min: $x=1$
34. max: $x=-1,2$; min: $x=1$
35. min: $x=1$
36. max: $x=0$
37. critical values: $x=-1,1$; no max/min
38. max: $x=\pi/4$; min: $x=-3\pi/4$
39. max: $x=-2$; min: $x=0$
40. min: $x=1/\sqrt{e}$
41. max: $x=0$
42. f' has no maximal or minimal value.
43. f' has no maximal or minimal value
44. f' has a minimal value at $x=0$
45. f' has a minimal value at $x=1/2$
46. Possible points of inflection: $x=-2/3,0$; f' has a relative min at: $x=0$; relative max at: $x=-2/3$
47. f' has a relative max at: $x=(1/3)(2+\sqrt{7})$ relative min at: $x=(1/3)(2-\sqrt{7})$
48. f' has no relative extrema
49. f' has a relative max at $x=-1/\sqrt{3}$; relative min at $x=1/\sqrt{3}$

50. f' has a relative max at $x = 0$
51. f' has a relative min at $x = 3\pi/4$; relative max at $x = -\pi/4$
52. f' has a relative max at $x = -2 - \sqrt{2}$; relative min at $x = -2 + \sqrt{2}$
53. f' has a relative min at $x = 1/\sqrt{e^3} = e^{-3/2}$
54. f' has a relative max at $x = -1/\sqrt{2}$; a relative min at $x = 1/\sqrt{2}$

Section 1.5

1. Answers will vary.
2. Found everywhere.
3. T
4. T
5. T
6. A good sketch will include the *x* and *y* intercepts and draw the appropriate line.
7. A good sketch will include the *x* and *y* intercepts..
8. Use technology to verify sketch.
9. Use technology to verify sketch.
10. Use technology to verify sketch.
11. Use technology to verify sketch.
12. Use technology to verify sketch.
13. Use technology to verify sketch.
14. Use technology to verify sketch.
15. Use technology to verify sketch.
16. Use technology to verify sketch.
17. Use technology to verify sketch.
18. Use technology to verify sketch.
19. Use technology to verify sketch.
20. Use technology to verify sketch.
21. Use technology to verify sketch.
22. Use technology to verify sketch.
23. Use technology to verify sketch.
24. Use technology to verify sketch.
25. Use technology to verify sketch.
26. Critical point: $x = 0$ Points of inflection: $\pm b/\sqrt{3}$
27. Critical points: $x = \frac{n\pi/2-b}{a}$, where *n* is an odd integer Points of inflection: $(n\pi - b)/a$, where *n* is an integer.
28. Critical point: $x = (a+b)/2$ Points of inflection: none
29. $\frac{dy}{dx} = -x/y$, so the function is increasing in second and fourth quadrants, decreasing in the first and third quadrants.
    $\frac{d^2y}{dx^2} = -1/y - x^2/y^3$, which is positive when $y < 0$ and is negative when $y > 0$. Hence the function is concave down in the first and second quadrants and concave up in the third and fourth quadrants.

Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$ $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

22. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

23. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

24. $\displaystyle\int \cosh x\,dx = \sinh x + C$

25. $\displaystyle\int \sinh x\,dx = \cosh x + C$

26. $\displaystyle\int \tanh x\,dx = \ln(\cosh x) + C$

27. $\displaystyle\int \coth x\,dx = \ln|\sinh x| + C$

28. $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$

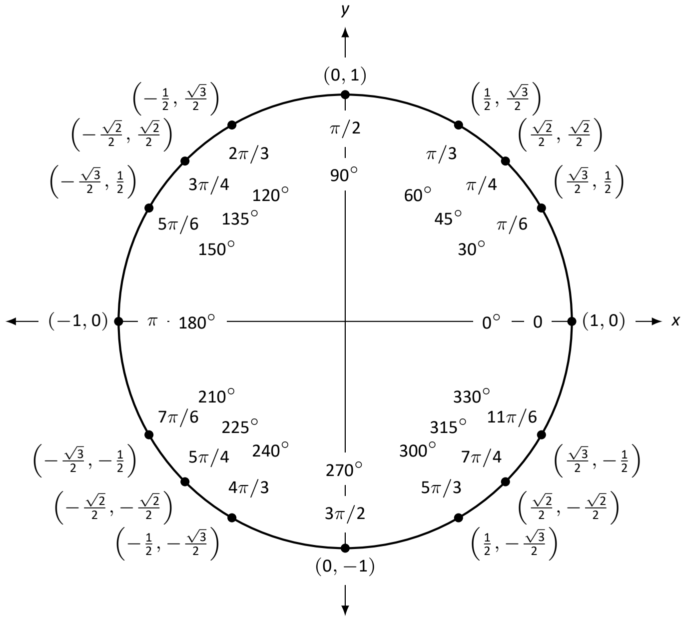
29. $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$

30. $\displaystyle\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$

31. $\displaystyle\int \frac{1}{x\sqrt{a^2 - x^2}}\,dx = \frac{1}{a}\ln\left(\frac{x}{a + \sqrt{a^2 - x^2}}\right) + C$

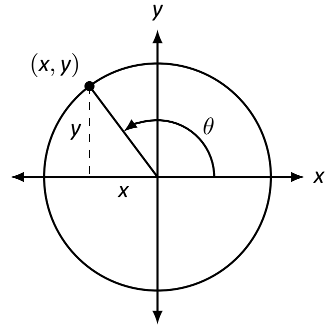
32. $\displaystyle\int \frac{1}{x\sqrt{x^2 + a^2}}\,dx = \frac{1}{a}\ln\left|\frac{x}{a + \sqrt{x^2 + a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

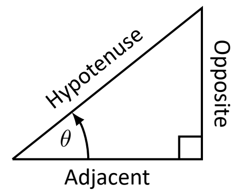


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$

$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$

Right Triangle Definition



$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$

$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$

$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$

$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$

$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$\sin 2x = 2\sin x\cos x$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$

$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

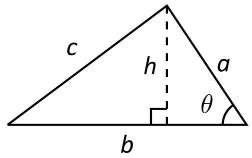
Areas and Volumes

Triangles

$h = a \sin\theta$

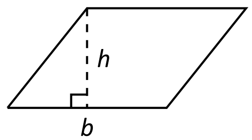
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



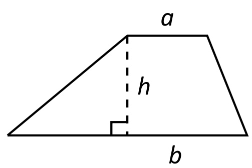
Parallelograms

Area = bh



Trapezoids

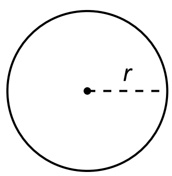
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

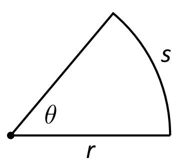


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

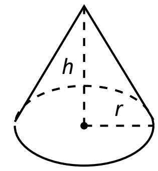
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

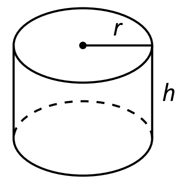
Surface Area =
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

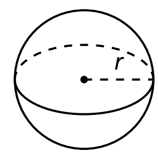
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

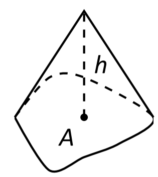
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

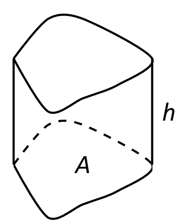
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$ $\qquad$ $x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$ $\qquad$ $x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$ $\qquad$ $(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ $\qquad$ $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$ $\qquad$ $(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$$a^0 = 1,\ \ a \neq 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;\,dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;\,dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;\,dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;\,dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;\,dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	$\text{Sum} = \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		$\text{Sum} = \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$