1: INTEGRATION

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute areas, volumes, mass, force, pressure, work, and much more.

1.1 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 1 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of f**, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them.

We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 1 Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$. Then there exists a constant C such that

$$G(x) = F(x) + C.$$

Given a function f and one of its antiderivatives F, we know *all* antiderivatives of f have the form $F(x) + C$ for some constant C. Using Definition 1, we can say that

$$\int f(x)\,dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

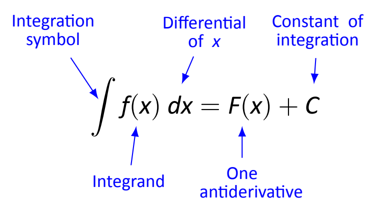


Figure 1.1: Understanding the indefinite integral notation.

Figure 1.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Example 1 Evaluating indefinite integrals

Evaluate $\displaystyle\int \sin x\,dx$.

Notes:

Solution We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x \, dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x \, dx$$

presents us with a differential, $dy = \sin x \, dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x \, dx$, rewrite

$$\int \sin x \, dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 2 **Evaluating indefinite integrals**

Evaluate $\displaystyle\int (3x^2 + 4x + 5) \, dx$.

Solution We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Notes:

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\ dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\ dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

Theorem 24 gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Notes:

Theorem 2 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) =$ $f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\,dx =$ $\int f(x)\,dx \pm \int g(x)\,dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\,dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\,dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\,dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\,dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\,dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\,dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\,dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\,dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

We highlight a few important points from Theorem 2:

- Rule #1 states $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

$$\int 5\cos x\,dx = 5 \cdot \int \cos x\,dx = 5 \cdot (\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by

Notes:

5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 2. So:

$$\begin{aligned}\int (3x^2+4x+5)\,dx &= \int 3x^2\,dx + \int 4x\,dx + \int 5\,dx \\ &= 3\int x^2\,dx + 4\int x\,dx + \int 5\,dx \\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C \\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section 2.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinitely many antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

Notes:

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 3 **Solving initial value problems**

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\, dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 4 **Solving initial value problems**

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\, dt = \int \cos t\, dt = \sin t + C = f'(t).$$

Notes:

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\,dt = \int (\sin t + 3)\,dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 5.4, we will see how areas and antiderivatives are closely tied together.

Notes:

Exercises 1.1

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a ________ function.

7. The antiderivative of an acceleration function is a ________ function.

Problems

In Exercises 8 – 26, evaluate the given indefinite integral.

8. $\int 3x^3\, dx$

9. $\int x^8\, dx$

10. $\int (10x^2 - 2)\, dx$

11. $\int dt$

12. $\int 1\, ds$

13. $\int \frac{1}{3t^2}\, dt$

14. $\int \frac{3}{t^2}\, dt$

15. $\int \frac{1}{\sqrt{x}}\, dx$

16. $\int \sec^2\theta\, d\theta$

17. $\int \sin\theta\, d\theta$

18. $\int (\sec x \tan x + \csc x \cot x)\, dx$

19. $\int 5e^{\theta}\, d\theta$

20. $\int 3^t\, dt$

21. $\int \frac{5^t}{2}\, dt$

22. $\int (2t+3)^2\, dt$

23. $\int (t^2+3)(t^3-2t)\, dt$

24. $\int x^2x^3\, dx$

25. $\int e^{\pi}\, dx$

26. $\int a\, dx$

27. This problem investigates why Theorem 2 states that $\int \frac{1}{x}\, dx = \ln|x| + C$.

   (a) What is the domain of $y = \ln x$?

   (b) Find $\frac{d}{dx}\big(\ln x\big)$.

   (c) What is the domain of $y = \ln(-x)$?

   (d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.

   (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\int \frac{1}{x}\, dx$ that takes these different domains into account, and explain your answer.

In Exercises 28 – 38, find $f(x)$ described by the given initial value problem.

28. $f'(x) = \sin x$ and $f(0) = 2$

29. $f'(x) = 5e^x$ and $f(0) = 10$

30. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

31. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

32. $f'(x) = 7^x$ and $f(2) = 1$

33. $f''(x) = 5$ and $f'(0) = 7$, $f(0) = 3$

34. $f''(x) = 7x$ and $f'(1) = -1$, $f(1) = 10$

35. $f''(x) = 5e^x$ and $f'(0) = 3$, $f(0) = 5$

36. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2$, $f(\pi) = 4$

37. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5, f(0) = 0$

38. $f''(x) = 0$ and $f'(1) = 3, f(1) = 1$

Review

39. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

40. Given $y = x^2 e^x \cos x$, find dy.

1.2 The Definite Integral

We start with an easy problem. An object travels in a straight line at a constant velocity of 5 ft/s for 10 seconds. How far away from its starting point is the object?

We approach this problem with the familiar "Distance $=$ Rate $\times$ Time" equation. In this case, Distance = 5ft/s $\times$ 10s $=$ 50 feet.

It is interesting to note that this solution of 50 feet can be represented graphically. Consider Figure 1.2, where the constant velocity of 5ft/s is graphed on the axes. Shading the area under the line from $t = 0$ to $t = 10$ gives a rectangle with an area of 50 square units; when one considers the units of the axes, we can say this area represents 50 ft.

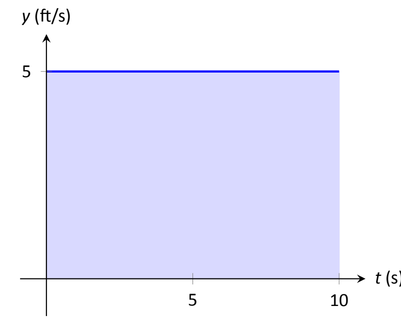


Figure 1.2: The area under a constant velocity function corresponds to distance traveled.

Now consider a slightly harder situation (and not particularly realistic): an object travels in a straight line with a constant velocity of 5ft/s for 10 seconds, then instantly reverses course at a rate of 2ft/s for 4 seconds. (Since the object is traveling in the opposite direction when reversing course, we say the velocity is a constant -2ft/s.) How far away from the starting point is the object – what is its *displacement*?

Here we use "Distance $=$ Rate$_1$ $\times$ Time$_1$ + Rate$_2$ $\times$ Time$_2$," which is

$$\text{Distance} \quad = 5 \cdot 10 + (-2) \cdot 4 = 42 \text{ ft.}$$

Hence the object is 42 feet from its starting location.

We can again depict this situation graphically. In Figure 1.3 we have the velocities graphed as straight lines on $[0, 10]$ and $[10, 14]$, respectively. The displacement of the object is

$$\text{"Area above the } t\text{–axis} \quad - \quad \text{Area below the } t\text{–axis,"}$$

which is easy to calculate as $50 - 8 = 42$ feet.

Now consider a more difficult problem.

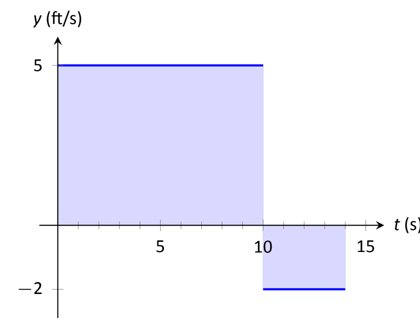


Figure 1.3: The total displacement is the area above the t–axis minus the area below the t–axis.

Example 5 Finding position using velocity

The velocity of an object moving straight up/down under the acceleration of gravity is given as $v(t) = -32t + 48$, where time t is given in seconds and velocity is in ft/s. When $t = 0$, the object had a height of 0 ft.

1. What was the initial velocity of the object?
2. What was the maximum height of the object?
3. What was the height of the object at time $t = 2$?

SOLUTION It is straightforward to find the initial velocity; at time $t = 0$, $v(0) = -32 \cdot 0 + 48 = 48$ ft/s.

Notes:

To answer questions about the height of the object, we need to find the object's position function $s(t)$. This is an initial value problem, which we studied in the previous section. We are told the initial height is 0, i.e., $s(0) = 0$. We know $s'(t) = v(t) = -32t + 48$. To find s, we find the indefinite integral of $v(t)$:

$$\int v(t)\,dt = \int (-32t + 48)\,dt = -16t^2 + 48t + C = s(t).$$

Since $s(0) = 0$, we conclude that $C = 0$ and $s(t) = -16t^2 + 48t$.

To find the maximum height of the object, we need to find the maximum of s. Recalling our work finding extreme values, we find the critical points of s by setting its derivative equal to 0 and solving for t:

$$s'(t) = -32t + 48 = 0 \quad \Rightarrow \quad t = 48/32 = 1.5\text{s}.$$

(Notice how we ended up just finding when the velocity was 0ft/s!) The first derivative test shows this is a maximum, so the maximum height of the object is found at

$$s(1.5) = -16(1.5)^2 + 48(1.5) = 36\text{ft}.$$

The height at time $t = 2$ is now straightforward to compute: it is $s(2) = 32$ft.

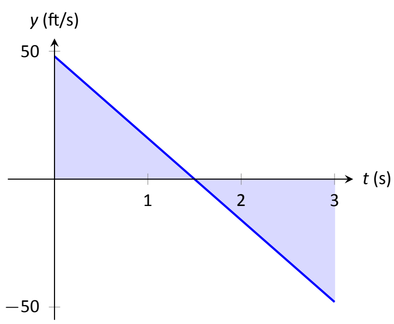


Figure 1.4: A graph of $v(t) = -32t + 48$; the shaded areas help determine displacement.

While we have answered all three questions, let's look at them again graphically, using the concepts of area that we explored earlier.

Figure 1.4 shows a graph of $v(t)$ on axes from $t = 0$ to $t = 3$. It is again straightforward to find $v(0)$. How can we use the graph to find the maximum height of the object?

Recall how in our previous work that the displacement of the object (in this case, its height) was found as the area under the velocity curve, as shaded in the figure. Moreover, the area between the curve and the t–axis that is below the t–axis counted as "negative" area. That is, it represents the object coming back toward its starting position. So to find the maximum distance from the starting point – the maximum height – we find the area under the velocity line that is above the t–axis, i.e., from $t = 0$ to $t = 1.5$. This region is a triangle; its area is

$$\text{Area } = \frac{1}{2}\text{Base} \times \text{Height} = \frac{1}{2} \times 1.5\text{s} \times 48\text{ft/s} = 36\text{ft},$$

which matches our previous calculation of the maximum height.

Finally, to find the height of the object at time $t = 2$ we calculate the total signed area under the velocity function from $t = 0$ to $t = 2$. This signed area is equal to $s(2)$, the displacement (i.e., signed distance) from the starting position at $t = 0$ to the position at time $t = 2$. That is,

$$\text{Displacement = Area above the } t\text{–axis} - \text{Area below } t\text{–axis}.$$

Notes:

The regions are triangles, and we find

$$\text{Displacement} = \frac{1}{2}(1.5\text{s})(48\text{ft/s}) - \frac{1}{2}(.5\text{s})(16\text{ft/s}) = 32\text{ft}.$$

This also matches our previous calculation of the height at $t = 2$.

Notice how we answered each question in this example in two ways. Our first method was to manipulate equations using our understanding of antiderivatives and derivatives. Our second method was geometric: we answered questions looking at a graph and finding the areas of certain regions of this graph.

The above example does not *prove* a relationship between area under a velocity function and displacement, but it does imply a relationship exists. Section 5.4 will fully establish fact that the area under a velocity function is displacement.

Given a graph of a function $y = f(x)$, we will find that there is great use in computing the area between the curve $y = f(x)$ and the x-axis. Because of this, we need to define some terms.

Definition 2 The Definite Integral, Total Signed Area

Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from $x = a$ to $x = b$ under f** is:

(area under f and above the x–axis on $[a, b]$) $-$ (area above f and under the x–axis on $[a, b]$).

The **definite integral of f on** $[a, b]$ is the total signed area of f on $[a, b]$, denoted

$$\int_a^b f(x)\, dx,$$

where a and b are the **bounds of integration.**

By our definition, the definite integral gives the "signed area under f." We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve."

The previous section introduced the indefinite integral, which related to antiderivatives. We have now defined the definite integral, which relates to areas under a function. The two are very much related, as we'll see when we learn the Fundamental Theorem of Calculus in Section 5.4. Recall that earlier we said that the "$\int$" symbol was an "elongated S" that represented finding a "sum." In the context of the definite integral, this notation makes a bit more sense, as we are adding up areas under the function f.

Notes:

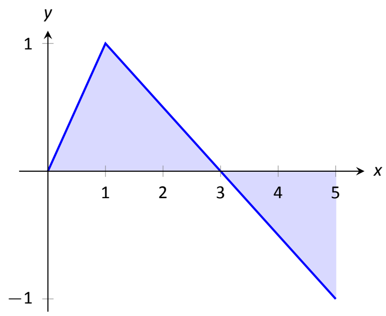


Figure 1.5: A graph of $f(x)$ in Example 6.

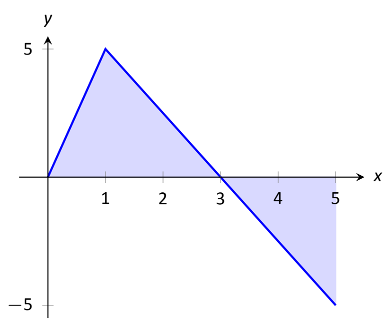


Figure 1.6: A graph of $5f$ in Example 6. (Yes, it looks just like the graph of f in Figure 1.5, just with a different y-scale.)

We practice using this notation.

Example 6 **Evaluating definite integrals**

Consider the function f given in Figure 1.5.

Find:

1. $\displaystyle\int_0^3 f(x)\,dx$
2. $\displaystyle\int_3^5 f(x)\,dx$
3. $\displaystyle\int_0^5 f(x)\,dx$
4. $\displaystyle\int_0^3 5f(x)\,dx$
5. $\displaystyle\int_1^1 f(x)\,dx$

Solution

1. $\int_0^3 f(x)\,dx$ is the area under f on the interval $[0,3]$. This region is a triangle, so the area is $\int_0^3 f(x)\,dx = \frac{1}{2}(3)(1) = 1.5$.

2. $\int_3^5 f(x)\,dx$ represents the area of the triangle found under the x–axis on $[3,5]$. The area is $\frac{1}{2}(2)(1) = 1$; since it is found *under* the x–axis, this is "negative area." Therefore $\int_3^5 f(x)\,dx = -1$.

3. $\int_0^5 f(x)\,dx$ is the total signed area under f on $[0,5]$. This is $1.5+(-1) = 0.5$.

4. $\int_0^3 5f(x)\,dx$ is the area under $5f$ on $[0,3]$. This is sketched in Figure 1.6. Again, the region is a triangle, with height 5 times that of the height of the original triangle. Thus the area is $\int_0^3 5f(x)\,dx = 15/2 = 7.5$.

5. $\int_1^1 f(x)\,dx$ is the area under f on the "interval" $[1,1]$. This describes a line segment, not a region; it has no width. Therefore the area is 0.

This example illustrates some of the properties of the definite integral, given here.

Notes:

Theorem 3 Properties of the Definite Integral

Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant. The following hold:

1. $\displaystyle\int_a^a f(x)\, dx = 0$
2. $\displaystyle\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$
3. $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$
4. $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$
5. $\displaystyle\int_a^b k\cdot f(x)\, dx = k\cdot\int_a^b f(x)\, dx$

We give a brief justification of Theorem 3 here.

1. As demonstrated in Example 6, there is no "area under the curve" when the region has no width; hence this definite integral is 0.
2. This states that total area is the sum of the areas of subregions. It is easily considered when we let $a < b < c$. We can break the interval $[a, c]$ into two subintervals, $[a, b]$ and $[b, c]$. The total area over $[a, c]$ is the area over $[a, b]$ plus the area over $[b, c]$.

   It is important to note that this still holds true even if $a < b < c$ is not true. We discuss this in the next point.
3. This property can be viewed a merely a convention to make other properties work well. (Later we will see how this property has a justification all its own, not necessarily in support of other properties.) Suppose $b < a < c$. The discussion from the previous point clearly justifies

   $$\int_b^a f(x)\, dx + \int_a^c f(x)\, dx = \int_b^c f(x)\, dx. \tag{1.1}$$

   However, we still claim that, as originally stated,

   $$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx. \tag{1.2}$$

Notes:

How do Equations (1.1) and (1.2) relate? Start with Equation (1.1):

$$\int_b^a f(x)\,dx + \int_a^c f(x)\,dx = \int_b^c f(x)\,dx$$

$$\int_a^c f(x)\,dx = -\int_b^a f(x)\,dx + \int_b^c f(x)\,dx$$

Property (3) justifies changing the sign and switching the bounds of integration on the $-\displaystyle\int_b^a f(x)\,dx$ term; when this is done, Equations (1.1) and (1.2) are equivalent.

The conclusion is this: by adopting the convention of Property (3), Property (2) holds no matter the order of a, b and c. Again, in the next section we will see another justification for this property.

4,5. Each of these may be non–intuitive. Property (5) states that when one scales a function by, for instance, 7, the area of the enclosed region also is scaled by a factor of 7. Both Properties (4) and (5) can be proved using geometry. The details are not complicated but are not discussed here.

Example 7 **Evaluating definite integrals using Theorem 3.**

Consider the graph of a function $f(x)$ shown in Figure 1.7. Answer the following:

1. Which value is greater: $\displaystyle\int_a^b f(x)\,dx$ or $\displaystyle\int_b^c f(x)\,dx$?
2. Is $\displaystyle\int_a^c f(x)\,dx$ greater or less than 0?
3. Which value is greater: $\displaystyle\int_a^b f(x)\,dx$ or $\displaystyle\int_c^b f(x)\,dx$?

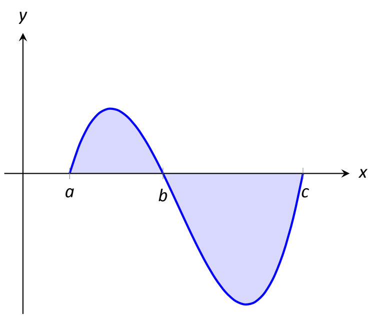


Figure 1.7: A graph of a function in Example 7.

Solution

1. $\int_a^b f(x)\,dx$ has a positive value (since the area is above the x–axis) whereas $\int_b^c f(x)\,dx$ has a negative value. Hence $\int_a^b f(x)\,dx$ is bigger.
2. $\int_a^c f(x)\,dx$ is the total signed area under f between $x = a$ and $x = c$. Since the region below the x–axis looks to be larger than the region above, we conclude that the definite integral has a value less than 0.
3. Note how the second integral has the bounds "reversed." Therefore $\int_c^b f(x)\,dx$ represents a positive number, greater than the area described by the first definite integral. Hence $\int_c^b f(x)\,dx$ is greater.

Notes:

The area definition of the definite integral allows us to use geometry to compute the definite integral of some simple functions.

Example 8 **Evaluating definite integrals using geometry**

Evaluate the following definite integrals:

$$1.\ \int_{-2}^{5} (2x-4)\, dx \qquad 2.\ \int_{-3}^{3} \sqrt{9-x^2}\, dx.$$

Solution

1. It is useful to sketch the function in the integrand, as shown in Figure 1.8(a). We see we need to compute the areas of two regions, which we have labeled R_1 and R_2. Both are triangles, so the area computation is straightforward:

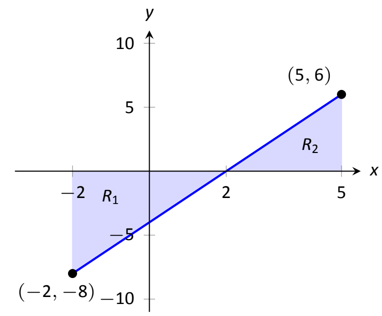
$$R_1: \frac{1}{2}(4)(8) = 16 \qquad R_2: \frac{1}{2}(3)6 = 9.$$

Region R_1 lies under the x–axis, hence it is counted as negative area (we can think of the triangle's height as being "-8"), so

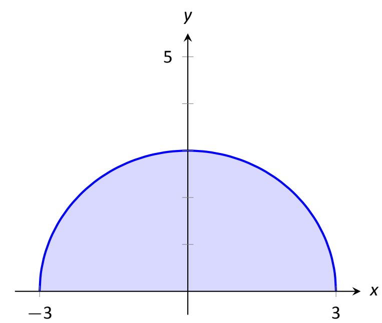
$$\int_{-2}^{5} (2x-4)\, dx = -16 + 9 = -7.$$

2. Recognize that the integrand of this definite integral describes a half circle, as sketched in Figure 1.8(b), with radius 3. Thus the area is:

$$\int_{-3}^{3} \sqrt{9-x^2}\, dx = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi.$$



(a)



(b)

Figure 1.8: A graph of $f(x) = 2x - 4$ in (a) and $f(x) = \sqrt{9-x^2}$ in (b), from Example 8.

Example 9 **Understanding motion given velocity**

Consider the graph of a velocity function of an object moving in a straight line, given in Figure 1.9, where the numbers in the given regions gives the area of that region. Assume that the definite integral of a velocity function gives displacement. Find the maximum speed of the object and its maximum displacement from its starting position.

Solution Since the graph gives velocity, finding the maximum speed is simple: it looks to be 15ft/s.

At time $t = 0$, the displacement is 0; the object is at its starting position. At time $t = a$, the object has moved backward 11 feet. Between times $t = a$ and

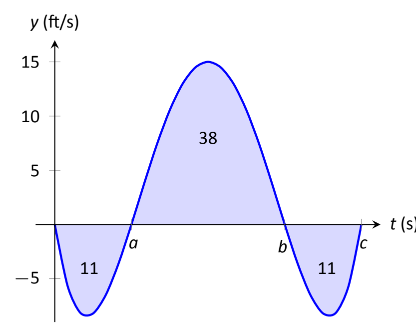


Figure 1.9: A graph of a velocity in Example 9.

Notes:

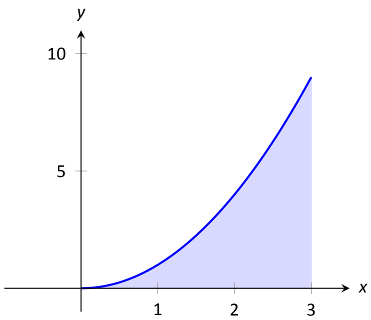


Figure 1.10: What is the area below $y = x^2$ on $[0, 3]$? The region is not a usual geometric shape.

$t = b$, the object moves forward 38 feet, bringing it into a position 27 feet forward of its starting position. From $t = b$ to $t = c$ the object is moving backwards again, hence its maximum displacement is 27 feet from its starting position.

In our examples, we have either found the areas of regions that have nice geometric shapes (such as rectangles, triangles and circles) or the areas were given to us. Consider Figure 1.10, where a region below $y = x^2$ is shaded. What is its area? The function $y = x^2$ is relatively simple, yet the shape it defines has an area that is not simple to find geometrically.

In the next section we will explore how to find the areas of such regions.

Notes:

Exercises 1.2

Terms and Concepts

1. What is "total signed area"?

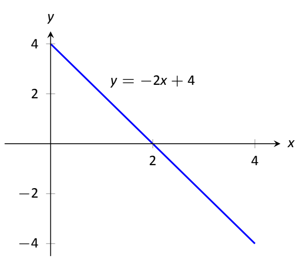
2. What is "displacement"?

3. What is $\int_3^3 \sin x \, dx$?

4. Give a single definite integral that has the same value as $\int_0^1 (2x+3)\,dx + \int_1^2 (2x+3)\,dx$.

Problems

In Exercises 5 – 9, a graph of a function $f(x)$ is given. Using the geometry of the graph, evaluate the definite integrals.

5. 

(a) $\int_0^1 (-2x+4)\,dx$

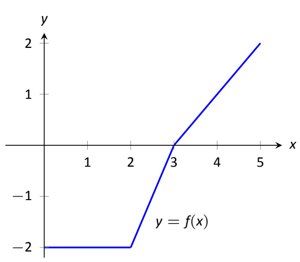
(b) $\int_0^2 (-2x+4)\,dx$

(c) $\int_0^3 (-2x+4)\,dx$

(d) $\int_1^3 (-2x+4)\,dx$

(e) $\int_2^4 (-2x+4)\,dx$

(f) $\int_0^1 (-6x+12)\,dx$

6. 

(a) $\int_0^2 f(x)\,dx$

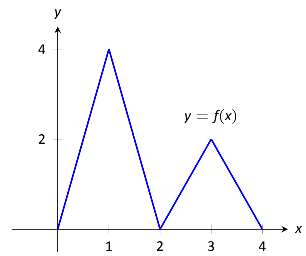
(b) $\int_0^3 f(x)\,dx$

(c) $\int_0^5 f(x)\,dx$

(d) $\int_2^5 f(x)\,dx$

(e) $\int_5^3 f(x)\,dx$

(f) $\int_0^3 -2f(x)\,dx$

7. 

(a) $\int_0^2 f(x)\,dx$

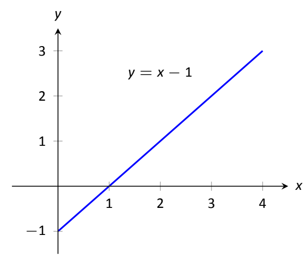
(b) $\int_2^4 f(x)\,dx$

(c) $\int_2^4 2f(x)\,dx$

(d) $\int_0^1 4x\,dx$

(e) $\int_2^3 (2x-4)\,dx$

(f) $\int_2^3 (4x-8)\,dx$

8. 

(a) $\int_0^1 (x-1)\,dx$

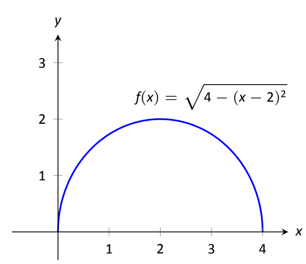
(b) $\int_0^2 (x-1)\,dx$

(c) $\int_0^3 (x-1)\,dx$

(d) $\int_2^3 (x-1)\,dx$

(e) $\int_1^4 (x-1)\,dx$

(f) $\int_1^4 \big((x-1)+1\big)\,dx$

9. 

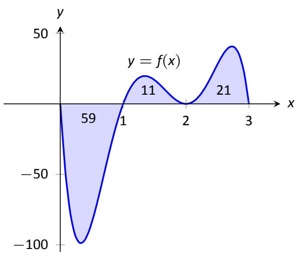
(a) $\int_0^2 f(x)\,dx$

(b) $\int_2^4 f(x)\,dx$

(c) $\int_0^4 f(x)\,dx$

(d) $\int_0^4 5f(x)\,dx$

In Exercises 10 – 13, a graph of a function $f(x)$ is given; the numbers inside the shaded regions give the area of that region. Evaluate the definite integrals using this area information.

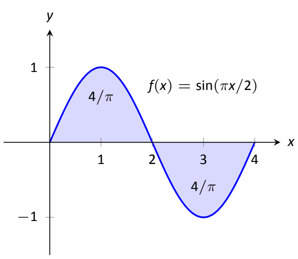
10. 

(a) $\displaystyle\int_0^1 f(x)\,dx$

(b) $\displaystyle\int_0^2 f(x)\,dx$

(c) $\displaystyle\int_0^3 f(x)\,dx$

(d) $\displaystyle\int_1^2 -3f(x)\,dx$

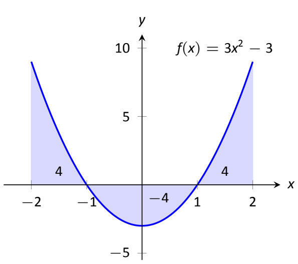
11. 

(a) $\displaystyle\int_0^2 f(x)\,dx$

(b) $\displaystyle\int_2^4 f(x)\,dx$

(c) $\displaystyle\int_0^4 f(x)\,dx$

(d) $\displaystyle\int_0^1 f(x)\,dx$

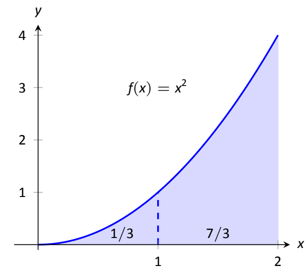
12. 

(a) $\displaystyle\int_{-2}^{-1} f(x)\,dx$

(b) $\displaystyle\int_1^2 f(x)\,dx$

(c) $\displaystyle\int_{-1}^1 f(x)\,dx$

(d) $\displaystyle\int_0^1 f(x)\,dx$

13. 

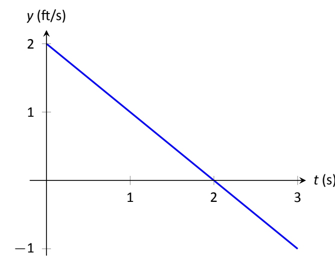
(a) $\displaystyle\int_0^2 5x^2\,dx$

(b) $\displaystyle\int_0^2 (x^2 + 3)\,dx$

(c) $\displaystyle\int_1^3 (x-1)^2\,dx$

(d) $\displaystyle\int_2^4 \left((x-2)^2 + 5\right)\,dx$

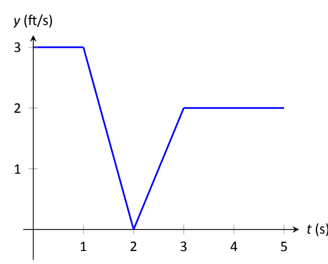
In Exercises 14 – 15, a graph of the velocity function of an object moving in a straight line is given. Answer the questions based on that graph.

14. 

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0, 3]$?

15. 

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0, 5]$?

16. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 64$, where t is in seconds, from a height of 48 feet.

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) When does the maximum displacement occur?

(d) When will the object reach a height of 0? (Hint: find when the displacement is -48ft.)

17. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 96$, where t is in seconds, from a height of 64 feet.

    (a) What is the object's initial velocity?

    (b) When is the object's displacement 0?

    (c) How long does it take for the object to return to its initial height?

    (d) When will the object reach a height of 210 feet?

In Exercises 18 – 21, let

- $\displaystyle\int_0^2 f(x)\,dx = 5,$
- $\displaystyle\int_0^3 f(x)\,dx = 7,$
- $\displaystyle\int_0^2 g(x)\,dx = -3,$ **and**
- $\displaystyle\int_2^3 g(x)\,dx = 5.$

Use these values to evaluate the given definite integrals.

18. $\displaystyle\int_0^2 \big(f(x) + g(x)\big)\,dx$

19. $\displaystyle\int_0^3 \big(f(x) - g(x)\big)\,dx$

20. $\displaystyle\int_2^3 \big(3f(x) + 2g(x)\big)\,dx$

21. Find values for a and b such that
$\displaystyle\int_0^3 \big(af(x) + bg(x)\big)\,dx = 0$

In Exercises 22 – 25, let

- $\displaystyle\int_0^3 s(t)\,dt = 10,$
- $\displaystyle\int_3^5 s(t)\,dt = 8,$
- $\displaystyle\int_3^5 r(t)\,dt = -1,$ **and**
- $\displaystyle\int_0^5 r(t)\,dt = 11.$

Use these values to evaluate the given definite integrals.

22. $\displaystyle\int_0^3 \big(s(t) + r(t)\big)\,dt$

23. $\displaystyle\int_5^0 \big(s(t) - r(t)\big)\,dt$

24. $\displaystyle\int_3^3 \big(\pi s(t) - 7r(t)\big)\,dt$

25. Find values for a and b such that
$\displaystyle\int_0^5 \big(ar(t) + bs(t)\big)\,dt = 0$

Review

In Exercises 26 – 29, evaluate the given indefinite integral.

26. $\displaystyle\int \big(x^3 - 2x^2 + 7x - 9\big)\,dx$

27. $\displaystyle\int \big(\sin x - \cos x + \sec^2 x\big)\,dx$

28. $\displaystyle\int \big(\sqrt[3]{t} + \frac{1}{t^2} + 2^t\big)\,dt$

29. $\displaystyle\int \left(\frac{1}{x} - \csc x \cot x\right)\,dx$

2: Techniques of Antidifferentiation

The previous chapter introduced the antiderivative and connected it to signed areas under a curve through the Fundamental Theorem of Calculus. The next chapter explores more applications of definite integrals than just area. As evaluating definite integrals will become important, we will want to find antiderivatives of a variety of functions.

This chapter is devoted to exploring techniques of antidifferentiation. While not every function has an antiderivative in terms of elementary functions (a concept introduced in the section on Numerical Integration), we can still find antiderivatives of a wide variety of functions.

2.1 Substitution

We motivate this section with an example. Let $f(x) = (x^2+3x-5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2+3x-5)^9 \cdot (2x+3) = (20x+30)(x^2+3x-5)^9.$$

Now consider this: What is $\int (20x+30)(x^2+3x-5)^9\ dx$? We have the answer in front of us;

$$\int (20x+30)(x^2+3x-5)^9\ dx = (x^2+3x-5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution.* It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x)\ dx$ as a not–so–complicated integral $\int h(u)\ du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x+30)(x^2+3x-5)^9\ dx$. Arguably the most "complicated" part of the integrand is $(x^2+3x-5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2+3x-5$. Thus

$$(x^2+3x-5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x+3)dx.$$

Keep in mind that $(2x+3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}
\int (20x+30)(x^2+3x-5)^9\,dx &= \int 10(2x+3)(x^2+3x-5)^9\,dx \\
&= \int 10(\underbrace{x^2+3x-5}_{u})^9\,\underbrace{(2x+3)\,dx}_{du} \\
&= \int 10u^9\,du \\
&= u^{10}+C \quad \text{(replace } u \text{ with } x^2+3x-5) \\
&= (x^2+3x-5)^{10}+C
\end{aligned}$$

One might well look at this and think "I (sort of) followed how that worked, but I could never come up with that on my own," but the process is learnable. This section contains numerous examples through which the reader will gain understanding and mathematical maturity enabling them to regard substitution as a natural tool when evaluating integrals.

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x)\,dx = F(g(x))+C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u=g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F\big(u\big)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x)\,dx = \int F'(u)du = F(u)+C = F(g(x))+C.$$

This concept is important so we restate it in the context of a theorem.

Notes:

Theorem 4 Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\,dx = \int F'(u)\,du = F(u) + C = F(g(x)) + C.$$

The point of substitution is to make the integration step easy. Indeed, the step $\int F'(u)\,du = F(u) + C$ looks easy, as the antiderivative of the derivative of F is just F, plus a constant. The "work" involved is making the proper substitution. There is not a step–by–step process that one can memorize; rather, experience will be one's guide. To gain experience, we now embark on many examples.

Example 10 Integrating by substitution

Evaluate $\displaystyle\int x\sin(x^2+5)\,dx$.

Solution Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of $\sin(x^2+5)$. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = x^2 + 5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int x\sin(x^2+5)\,dx &= \int \sin(\underbrace{x^2+5}_{u})\,\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}\sin u\,du\end{aligned}$$

Notes:

$$= -\frac{1}{2}\cos u + C \qquad \text{(now replace } u \text{ with } x^2+5\text{)}$$
$$= -\frac{1}{2}\cos(x^2+5) + C.$$

Thus $\int x\sin(x^2+5)\,dx = -\frac{1}{2}\cos(x^2+5) + C$. We can check our work by evaluating the derivative of the right hand side.

Example 11 **Integrating by substitution**

Evaluate $\displaystyle\int \cos(5x)\,dx$.

Solution Again let u replace the "inside" function. Letting $u = 5x$, we have $du = 5dx$. Since our integrand does not have a $5dx$ term, we can divide the previous equation by 5 to obtain $\frac{1}{5}du = dx$. We can now substitute.

$$\int \cos(5x)\,dx = \int \cos(\underbrace{5x}_{u})\,\underbrace{dx}_{\frac{1}{5}du}$$
$$= \int \frac{1}{5}\cos u\,du$$
$$= \frac{1}{5}\sin u + C$$
$$= \frac{1}{5}\sin(5x) + C.$$

We can again check our work through differentiation.

The previous example exhibited a common, and simple, type of substitution. The "inside" function was a linear function (in this case, $y = 5x$). When the inside function is linear, the resulting integration is very predictable, outlined here.

Key Idea 1 **Substitution With A Linear Function**

Consider $\int F'(ax+b)\,dx$, where $a \neq 0$ and b are constants. Letting $u = ax+b$ gives $du = a\cdot dx$, leading to the result

$$\int F'(ax+b)\,dx = \frac{1}{a}F(ax+b) + C.$$

Thus $\int \sin(7x-4)\,dx = -\frac{1}{7}\cos(7x-4) + C$. Our next example can use Key Idea 1, but we will only employ it after going through all of the steps.

Notes:

Example 12 **Integrating by substituting a linear function**

Evaluate $\displaystyle\int \frac{7}{-3x+1}\,dx$.

Solution View the integrand as the composition of functions $f(g(x))$, where $f(x) = 7/x$ and $g(x) = -3x + 1$. Employing our understanding of substitution, we let $u = -3x+1$, the inside function. Thus $du = -3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3 = dx$. We can now evaluate the integral through substitution.

$$\begin{aligned}
\int \frac{7}{-3x+1}\,dx &= \int \frac{7}{u}\frac{du}{-3} \\
&= \frac{-7}{3}\int \frac{du}{u} \\
&= \frac{-7}{3}\ln|u| + C \\
&= -\frac{7}{3}\ln|-3x+1| + C.
\end{aligned}$$

Using Key Idea 1 is faster, recognizing that u is linear and $a = -3$. One may want to continue writing out all the steps until they are comfortable with this particular shortcut.

Not all integrals that benefit from substitution have a clear "inside" function. Several of the following examples will demonstrate ways in which this occurs.

Example 13 **Integrating by substitution**

Evaluate $\displaystyle\int \sin x\cos x\,dx$.

Solution There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* …" and see if this helps simplify the integral at all.

In this example, let's set $u = \sin x$. Then $du = \cos x\,dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}
\int \sin x\cos x\,dx &= \int u\,du \\
&= \frac{1}{2}u^2 + C \\
&= \frac{1}{2}\sin^2 x + C.
\end{aligned}$$

Notes:

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different.

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 14 **Integrating by substitution**

Evaluate $\displaystyle\int x\sqrt{x+3}\,dx$.

Solution Recognizing the composition of functions, set $u = x + 3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\,dx = \int x\sqrt{u}\,du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u-3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3}\,dx &= \int (u-3)u^{\frac{1}{2}}\,du\\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right)du\\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C\\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem.

Example 15 **Integrating by substitution**

Evaluate $\displaystyle\int \frac{1}{x\ln x}\,dx$.

Solution This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 14 is useful here: choose something for u and consider what this implies du must

Notes:

be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\,dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\,dx$, which is part of the integrand. Thus:

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \underbrace{\frac{1}{\ln x}}_{1/u}\underbrace{\frac{1}{x}\,dx}_{du}\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln|\ln x| + C.\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct.

Integrals Involving Trigonometric Functions

Section 6.3 delves deeper into integrals of a variety of trigonometric functions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 16 **Integration by substitution: antiderivatives of** $\tan x$

Evaluate $\displaystyle\int \tan x\,dx$.

Solution The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have

Notes:

that $du = -\sin x\,dx$, hence $-du = \sin x\,dx$. We can integrate:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= \int \underbrace{\frac{1}{\cos x}}_{u}\underbrace{\sin x\,dx}_{-du}\\ &= \int \frac{-1}{u}\,du\\ &= -\ln|u| + C\\ &= -\ln|\cos x| + C.\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C\\ &= \ln\left|\frac{1}{\cos x}\right| + C\\ &= \ln|\sec x| + C.\end{aligned}$$

Thus the result they give is $\int \tan x\,dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 17 **Integrating by substitution: antiderivatives of** $\sec x$

Evaluate $\displaystyle\int \sec x\,dx$.

Solution This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}.$$

This may seem like it came out of left field, but it works beautifully. Consider:

$$\begin{aligned}\int \sec x\,dx &= \int \sec x\cdot\frac{\sec x + \tan x}{\sec x + \tan x}\,dx\\ &= \int \frac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\,dx.\end{aligned}$$

Notes:

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x)\,dx$, which is our numerator. Thus:

$$\begin{aligned} &= \int \frac{du}{u} \\ &= \ln|u| + C \\ &= \ln|\sec x + \tan x| + C. \end{aligned}$$

We can use similar techniques to those used in Examples 16 and 17 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 5 Antiderivatives of Trigonometric Functions

1. $\displaystyle\int \sin x\,dx = -\cos x + C$
2. $\displaystyle\int \cos x\,dx = \sin x + C$
3. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C$
4. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
5. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
6. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

We explore one more common trigonometric integral.

Example 18 Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\displaystyle\int \cos^2 x\,dx$.

Solution We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x\,dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\begin{aligned} \int \cos^2 x\,dx &= \int \frac{1+\cos(2x)}{2}\,dx \\ &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\,dx. \end{aligned}$$

Notes:

Now use Key Idea 1:

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C$$
$$= \frac{1}{2}x + \frac{\sin(2x)}{4} + C.$$

We'll make significant use of this power–reducing technique in future sections.

Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 19 Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx$.

Solution One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

We skip the specifics of the steps, but note that when x^2+2x+1 is divided into x^3+4x^2+8x+5, it goes in $x+2$ times with a remainder of $3x+3$. Thus

$$\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2+\frac{3x+3}{x^2+2x+1}.$$

Integrating $x+2$ is simple. The fraction can be integrated by setting $u = x^2 + 2x+1$, giving $du = (2x+2)\,dx$. This is very similar to the numerator. Note that

Notes:

$du/2 = (x+1)\,dx$ and then consider the following:

$$\begin{aligned}\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx &= \int \left(x+2+\frac{3x+3}{x^2+2x+1}\right)\,dx \\ &= \int (x+2)\,dx + \int \frac{3(x+1)}{x^2+2x+1}\,dx \\ &= \frac{1}{2}x^2+2x+C_1+\int \frac{3}{u}\frac{du}{2} \\ &= \frac{1}{2}x^2+2x+C_1+\frac{3}{2}\ln|u|+C_2 \\ &= \frac{1}{2}x^2+2x+\frac{3}{2}\ln|x^2+2x+1|+C.\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 20 **Integration by alternate methods**

Evaluate $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx$ with, and without, substitution.

Solution We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2+2x+3}{x^{1/2}} = x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}\int \frac{x^2+2x+3}{x^{1/2}}\,dx &= \int \left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}\right)\,dx \\ &= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C\end{aligned}$$

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.

Let $u=\sqrt{x}=x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\,dx \quad \Rightarrow \quad 2du = \frac{1}{\sqrt{x}}\,dx.$$

This gives us $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx = \int (x^2+2x+3)\cdot 2\,du$. What are we to do with the other x terms? Since $u=x^{\frac{1}{2}}$, $u^2=x$, etc. We can then replace x^2 and

Notes:

x with appropriate powers of u. We thus have

$$\begin{aligned}\int \frac{x^2+2x+3}{\sqrt{x}}\,dx &= \int (x^2+2x+3)\cdot 2\,du\\ &= \int 2(u^4+2u^2+3)\,du\\ &= \frac{2}{5}u^5+\frac{4}{3}u^3+6u+C\\ &= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C,\end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is.

Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1}x\big) = \frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1}5x\big) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 21 **Integrating by substitution: inverse trigonometric functions**

Evaluate $\displaystyle\int \frac{1}{25+x^2}\,dx$.

Solution The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned}\frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})}\\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}\\ &= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}.\end{aligned}$$

Notes:

Thus

$$\int \frac{1}{25+x^2}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{5}\tan^{-1} u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Example 21 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 6 Integrals Involving Inverse Trigonomentric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Let's practice using Theorem 6.

Example 22 Integrating by substitution: inverse trigonometric functions

Evaluate the given indefinite integrals.

$$\int \frac{1}{9+x^2}\,dx, \quad \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx \quad \text{and} \quad \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Notes:

Solution Each can be answered using a straightforward application of Theorem 6.

$$\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C, \text{ as } a = 3.$$

$$\int \frac{1}{x\sqrt{x^2 - \frac{1}{100}}}\,dx = 10\sec^{-1}10x + C, \text{ as } a = \tfrac{1}{10}.$$

$$\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C, \text{ as } a = \sqrt{5}.$$

Most applications of Theorem 6 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 23 **Integrating by substitution: completing the square**

Evaluate $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$.

Solution Initially, this integral seems to have nothing in common with the integrals in Theorem 6. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2+bx+c &= \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4} + c \\ &= \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned} \frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13} \\ &= \frac{1}{(x-2)^2+9} \end{aligned}$$

Notes:

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u = x - 2$, but we can employ Key Idea 1 instead. Thus we have

$$\int \frac{1}{x^2 - 4x + 13}\,dx = \int \frac{1}{(x-2)^2 + 9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

Example 24 Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 6; the second integral is handled by substitution, with $u = 16 - x^2$. We handle each separately.

$$\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16 - x^2$, so $du = -2xdx$ and $xdx = -du/2$. We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}}\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\sqrt{u} + C\\ &= -\sqrt{16-x^2} + C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

Substitution and Definite Integration

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

Notes:

1. Start with a definite integral $\int_a^b f(x)\,dx$ that requires substitution.

2. Ignore the bounds; use substitution to evaluate $\int f(x)\,dx$ and find an antiderivative $F(x)$.

3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b) - F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving).

At its heart, (using the notation of Theorem 4) substitution converts integrals of the form $\int F'(g(x))g'(x)\,dx$ into an integral of the form $\int F'(u)\,du$ with the substitution of $u = g(x)$. The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Theorem 7 Substitution with Definite Integrals

Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then

$$\int_a^b F'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} F'(u)\,du.$$

In effect, Theorem 7 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 25 Definite integrals and substitution: changing the bounds

Evaluate $\int_0^2 \cos(3x-1)\,dx$ using Theorem 7.

Solution Observing the composition of functions, let $u = 3x - 1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x - 1$, we are implicitly stating that $g(x) = 3x - 1$. Theorem 7 states that the new lower bound is $g(0) = -1$; the new upper bound is $g(2) =$

Notes:

5. We now evaluate the definite integral:

$$\begin{aligned}\int_0^2 \cos(3x-1)\,dx &= \int_{-1}^{5} \cos u \frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^{5}\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \approx -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 2.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.

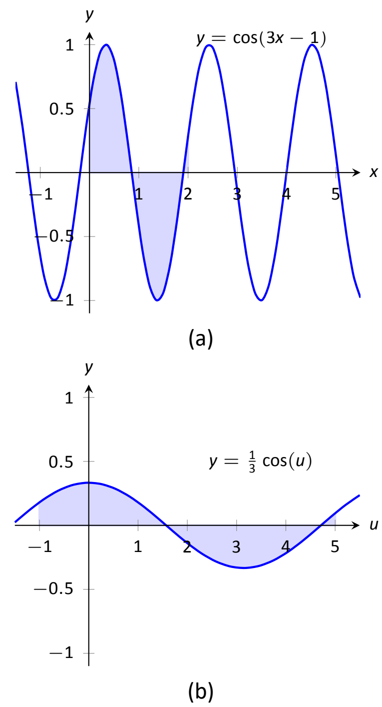


Figure 2.1: Graphing the areas defined by the definite integrals of Example 25.

Example 26 **Definite integrals and substitution: changing the bounds**

Evaluate $\displaystyle\int_0^{\pi/2} \sin x \cos x\,dx$ using Theorem 7.

Solution We saw the corresponding indefinite integral in Example 13. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x\,dx$ and hence $\sin x\,dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x \cos x\,dx &= \int_1^{0} -u\,du \qquad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u\,du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 2.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 7 guarantees that they have the same area.

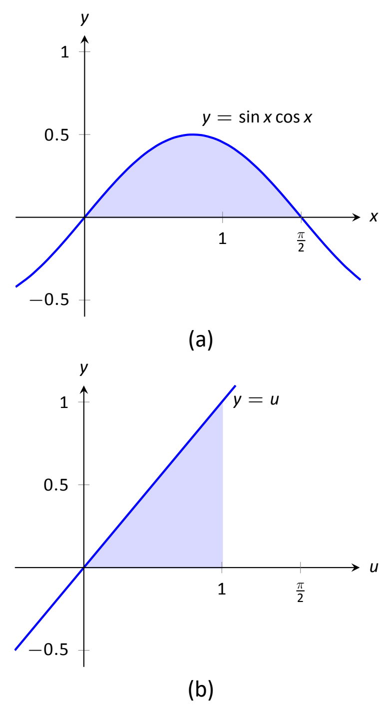


Figure 2.2: Graphing the areas defined by the definite integrals of Example 26.

Integration by substitution is a powerful and useful integration technique. The next section introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Notes:

Exercises 2.1

Terms and Concepts

1. Substitution "undoes" what derivative rule?

2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.

3. $\displaystyle\int 3x^2\left(x^3-5\right)^7\,dx$

4. $\displaystyle\int (2x-5)\left(x^2-5x+7\right)^3\,dx$

5. $\displaystyle\int x\left(x^2+1\right)^8\,dx$

6. $\displaystyle\int (12x+14)\left(3x^2+7x-1\right)^5\,dx$

7. $\displaystyle\int \frac{1}{2x+7}dx$

8. $\displaystyle\int \frac{1}{\sqrt{2x+3}}dx$

9. $\displaystyle\int \frac{x}{\sqrt{x+3}}dx$

10. $\displaystyle\int \frac{x^3-x}{\sqrt{x}}dx$

11. $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

12. $\displaystyle\int \frac{x^4}{\sqrt{x^5+1}}dx$

13. $\displaystyle\int \frac{\frac{1}{x}+1}{x^2}dx$

14. $\displaystyle\int \frac{\ln(x)}{x}dx$

In Exercises 15 – 23, use Substitution to evaluate the indefinite integral involving trigonometric functions.

15. $\displaystyle\int \sin^2(x)\cos(x)dx$

16. $\displaystyle\int \cos(3-6x)dx$

17. $\displaystyle\int \sec^2(4-x)dx$

18. $\displaystyle\int \sec(2x)dx$

19. $\displaystyle\int \tan^2(x)\sec^2(x)dx$

20. $\displaystyle\int x\cos\left(x^2\right)\,dx$

21. $\displaystyle\int \tan^2(x)dx$

22. $\displaystyle\int \cot x\,dx$. Do not just refer to Theorem 5 for the answer; justify it through Substitution.

23. $\displaystyle\int \csc x\,dx$. Do not just refer to Theorem 5 for the answer; justify it through Substitution.

In Exercises 24 – 30, use Substitution to evaluate the indefinite integral involving exponential functions.

24. $\displaystyle\int e^{3x-1}dx$

25. $\displaystyle\int e^{x^3}x^2dx$

26. $\displaystyle\int e^{x^2-2x+1}(x-1)dx$

27. $\displaystyle\int \frac{e^x+1}{e^x}dx$

28. $\displaystyle\int \frac{e^x-e^{-x}}{e^{2x}}dx$

29. $\displaystyle\int 3^{3x}dx$

30. $\displaystyle\int 4^{2x}dx$

In Exercises 31 – 34, use Substitution to evaluate the indefinite integral involving logarithmic functions.

31. $\displaystyle\int \frac{\ln x}{x}dx$

32. $\displaystyle\int \frac{\left(\ln x\right)^2}{x}dx$

33. $\displaystyle\int \frac{\ln\left(x^3\right)}{x}dx$

34. $\displaystyle\int \frac{1}{x\ln\left(x^2\right)}\,dx$

In Exercises 35 – 40, use Substitution to evaluate the indefinite integral involving rational functions.

35. $\displaystyle\int \frac{x^2+3x+1}{x}\,dx$

36. $\displaystyle\int \frac{x^3+x^2+x+1}{x}\,dx$

37. $\displaystyle\int \frac{x^3-1}{x+1}\,dx$

38. $\displaystyle\int \frac{x^2+2x-5}{x-3}\,dx$

39. $\displaystyle\int \frac{3x^2-5x+7}{x+1}\,dx$

40. $\displaystyle\int \frac{x^2+2x+1}{x^3+3x^2+3x}\,dx$

In Exercises 41 – 50, use Substitution to evaluate the indefinite integral involving inverse trigonometric functions.

41. $\displaystyle\int \frac{7}{x^2+7}\,dx$

42. $\displaystyle\int \frac{3}{\sqrt{9-x^2}}\,dx$

43. $\displaystyle\int \frac{14}{\sqrt{5-x^2}}\,dx$

44. $\displaystyle\int \frac{2}{x\sqrt{x^2-9}}\,dx$

45. $\displaystyle\int \frac{5}{\sqrt{x^4-16x^2}}\,dx$

46. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

47. $\displaystyle\int \frac{1}{x^2-2x+8}\,dx$

48. $\displaystyle\int \frac{2}{\sqrt{-x^2+6x+7}}\,dx$

49. $\displaystyle\int \frac{3}{\sqrt{-x^2+8x+9}}\,dx$

50. $\displaystyle\int \frac{5}{x^2+6x+34}\,dx$

In Exercises 51 – 75, evaluate the indefinite integral.

51. $\displaystyle\int \frac{x^2}{\left(x^3+3\right)^2}\,dx$

52. $\displaystyle\int \left(3x^2+2x\right)\left(5x^3+5x^2+2\right)^8\,dx$

53. $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$

54. $\displaystyle\int x^2\csc^2\left(x^3+1\right)\,dx$

55. $\displaystyle\int \sin(x)\sqrt{\cos(x)}\,dx$

56. $\displaystyle\int \frac{1}{x-5}\,dx$

57. $\displaystyle\int \frac{7}{3x+2}\,dx$

58. $\displaystyle\int \frac{3x^3+4x^2+2x-22}{x^2+3x+5}\,dx$

59. $\displaystyle\int \frac{2x+7}{x^2+7x+3}\,dx$

60. $\displaystyle\int \frac{9(2x+3)}{3x^2+9x+7}\,dx$

61. $\displaystyle\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}\,dx$

62. $\displaystyle\int \frac{x}{x^4+81}\,dx$

63. $\displaystyle\int \frac{2}{4x^2+1}\,dx$

64. $\displaystyle\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

65. $\displaystyle\int \frac{1}{\sqrt{16-9x^2}}\,dx$

66. $\displaystyle\int \frac{3x-2}{x^2-2x+10}\,dx$

67. $\displaystyle\int \frac{7-2x}{x^2+12x+61}\,dx$

68. $\displaystyle\int \frac{x^2+5x-2}{x^2-10x+32}\,dx$

69. $\displaystyle\int \frac{x^3}{x^2+9}\,dx$

70. $\displaystyle\int \frac{x^3-x}{x^2+4x+9}\,dx$

71. $\displaystyle\int \frac{\sin(x)}{\cos^2(x)+1}\,dx$

72. $\displaystyle\int \frac{\cos(x)}{\sin^2(x)+1}\,dx$

73. $\displaystyle\int \frac{\cos(x)}{1-\sin^2(x)}\,dx$

74. $\displaystyle\int \frac{3x-3}{\sqrt{x^2-2x-6}}\,dx$

75. $\displaystyle\int \frac{x-3}{\sqrt{x^2-6x+8}}\,dx$

In Exercises 76 – 83, evaluate the definite integral.

76. $\displaystyle\int_1^3 \frac{1}{x-5}\,dx$

77. $\displaystyle\int_2^6 x\sqrt{x-2}\,dx$

78. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx$

79. $\displaystyle\int_0^1 2x(1-x^2)^4\,dx$

80. $\displaystyle\int_{-2}^{-1} (x+1)e^{x^2+2x+1}\,dx$

81. $\displaystyle\int_{-1}^{1} \frac{1}{1+x^2}\,dx$

82. $\displaystyle\int_2^4 \frac{1}{x^2-6x+10}\,dx$

83. $\displaystyle\int_1^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$

A: Solutions To Selected Problems

Chapter 1

Section 1.1

1. Answers will vary.
2. "an"
3. Answers will vary.
4. opposite; opposite
5. Answers will vary.
6. velocity
7. velocity
8. $3/4x^4 + C$
9. $1/9x^9 + C$
10. $10/3x^3 - 2x + C$
11. $t + C$
12. $s + C$
13. $-1/(3t) + C$
14. $-3/(t) + C$
15. $2\sqrt{x} + C$
16. $\tan\theta + C$
17. $-\cos\theta + C$
18. $\sec x - \csc x + C$
19. $5e^{\theta} + C$
20. $3^t/\ln 3 + C$
21. $\frac{5^t}{2\ln 5} + C$
22. $4/3t^3 + 6t^2 + 9t + C$
23. $t^6/6 + t^4/4 - 3t^2 + C$
24. $x^6/6 + C$
25. $e^{\pi}x + C$
26. $ax + C$
27. (a) $x > 0$
    (b) $1/x$
    (c) $x < 0$
    (d) $1/x$
    (e) $\ln|x| + C$. Explanations will vary.
28. $-\cos x + 3$
29. $5e^x + 5$
30. $x^4 - x^3 + 7$
31. $\tan x + 4$
32. $7^x/\ln 7 + 1 - 49/\ln 7$
33. $5/2x^2 + 7x + 3$
34. $\frac{7x^3}{6} - \frac{9x}{2} + \frac{40}{3}$
35. $5e^x - 2x$
36. $\theta - \sin(\theta) - \pi + 4$
37. $2x^4 + \cos x + \frac{2^x}{(\ln 2)^2} + (5 - \frac{1}{\ln 2})x - 1 - \frac{1}{(\ln 2)^2}$
38. $3x - 2$
39. No answer provided.
40. $dy = (2xe^x\cos x + x^2e^x\cos x - x^2e^x\sin x)dx$

Section 1.2

1. Answers will vary.
2. Answers will vary.
3. 0
4. $\int 0^2(2x+3)\,dx$
5. (a) 3
   (b) 4
   (c) 3
   (d) 0
   (e) -4
   (f) 9
6. (a) -4
   (b) -5
   (c) -3
   (d) 1
   (e) -2
   (f) 10
7. (a) 4
   (b) 2
   (c) 4
   (d) 2
   (e) 1
   (f) 2
8. (a) $-1/2$
   (b) 0
   (c) $3/2$
   (d) $3/2$
   (e) $9/2$
   (f) $15/2$
9. (a) π
   (b) π
   (c) 2π
   (d) 10π
10. (a) -59
    (b) -48
    (c) -27
    (d) -33
11. (a) $4/\pi$
    (b) $-4/\pi$
    (c) 0
    (d) $2/\pi$
12. (a) 4
    (b) 4
    (c) -4
    (d) -2

13. (a) $40/3$
    (b) $26/3$
    (c) $8/3$
    (d) $38/3$
14. (a) 2ft/s
    (b) 2ft
    (c) 1.5ft
15. (a) 3ft/s
    (b) 9.5ft
    (c) 9.5ft
16. (a) 64ft/s
    (b) 64ft
    (c) $t = 2$
    (d) $t = 2 + \sqrt{7} \approx 4.65$ seconds
17. (a) 96ft/s
    (b) 6 seconds
    (c) 6 seconds
    (d) Never; the maximum height is 208ft.
18. 2
19. 5
20. 16
21. Answers can vary; one solution is $a = -2$, $b = 7$
22. 24
23. -7
24. -7
25. Answers can vary; one solution is $a = -11$, $b = 18$
26. $1/4x^4 - 2/3x^3 + 7/2x^2 - 9x + C$
27. $-\cos x - \sin x + \tan x + C$
28. $3/4t^{4/3} - 1/t + 2^t/\ln 2 + C$
29. $\ln|x| + \csc x + C$

Chapter 2

Section 2.1

1. Chain Rule.
2. T
3. $\frac{1}{8}(x^3 - 5)^8 + C$
4. $\frac{1}{4}(x^2 - 5x + 7)^4 + C$
5. $\frac{1}{18}\left(x^2 + 1\right)^9 + C$
6. $\frac{1}{3}(3x^2 + 7x - 1)^6 + C$
7. $\frac{1}{2}\ln|2x + 7| + C$
8. $\sqrt{2x + 3} + C$
9. $\frac{2}{3}(x + 3)^{3/2} - 6(x + 3)^{1/2} + C = \frac{2}{3}(x - 6)\sqrt{x + 3} + C$
10. $\frac{2}{21}x^{3/2}\left(3x^2 - 7\right) + C$
11. $2e^{\sqrt{x}} + C$
12. $\frac{2\sqrt{x^5+1}}{5} + C$
13. $-\frac{1}{2x^2} - \frac{1}{x} + C$
14. $\frac{\ln^2(x)}{2} + C$
15. $\frac{\sin^3(x)}{3} + C$
16. $-\frac{1}{6}\sin(3 - 6x) + C$
17. $-\tan(4 - x) + C$
18. $\frac{1}{2}\ln|\sec(2x) + \tan(2x)| + C$
19. $\frac{\tan^3(x)}{3} + C$
20. $\frac{\sin(x^2)}{2} + C$
21. $\tan(x) - x + C$
22. The key is to rewrite $\cot x$ as $\cos x/\sin x$, and let $u = \sin x$.
23. The key is to multiply $\csc x$ by 1 in the form $(\csc x + \cot x)/(\csc x + \cot x)$.
24. $\frac{1}{3}e^{3x-1} + C$
25. $\frac{e^{x^3}}{3} + C$
26. $\frac{1}{2}e^{(x-1)^2} + C$
27. $x - e^{-x} + C$
28. $\frac{e^{-3x}}{3} - e^{-x} + C$
29. $\frac{27^x}{\ln 27} + C$
30. $\frac{16^x}{\ln(16)} + C$
31. $\frac{1}{2}\ln^2(x) + C$
32. $\frac{\left(\ln x\right)^3}{3} + C$
33. $\frac{1}{6}\ln^2\left(x^3\right) + C$
34. $\frac{1}{2}\ln\left(\ln\left(x^2\right)\right) + C$
35. $\frac{x^2}{2} + 3x + \ln|x| + C$
36. $\frac{x^3}{3} + \frac{x^2}{2} + x + \ln|x| + C$
37. $\frac{x^3}{3} - \frac{x^2}{2} + x - 2\ln|x + 1| + C$
38. $\frac{1}{2}\left(x^2 + 10x + 20\ln|x - 3|\right) + C$
39. $\frac{3}{2}x^2 - 8x + 15\ln|x + 1| + C$
40. $\frac{1}{3}\ln\left|x^2 + 3x + 3\right| + \frac{\ln|x|}{3} + C$
41. $\sqrt{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$
42. $3\sin^{-1}\left(\frac{x}{3}\right) + C$
43. $14\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + C$
44. $\frac{2}{3}\sec^{-1}(|x|/3) + C$
45. $\frac{5}{4}\sec^{-1}(|x|/4) + C$
46. $\frac{1}{2}\sin^{-1}\left(x^2\right) + C$
47. $\frac{\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)}{\sqrt{7}} + C$
48. $2\sin^{-1}\left(\frac{x-3}{4}\right) + C$
49. $3\sin^{-1}\left(\frac{x-4}{5}\right) + C$
50. $\tan^{-1}\left(\frac{x+3}{5}\right) + C$
51. $-\frac{1}{3\left(x^3+3\right)} + C$
52. $\frac{1}{45}(5x^3 + 5x^2 + 2)^9 + C$
53. $-\sqrt{1 - x^2} + C$

54. $-\frac{1}{3}\cot\left(x^3+1\right)+C$

55. $-\frac{2}{3}\cos^{\frac{3}{2}}(x)+C$

56. $\ln|x-5|+C$

57. $\frac{7}{3}\ln|3x+2|+C$

58. $\frac{3x^2}{2}+\ln\left|x^2+3x+5\right|-5x+C$

59. $\ln\left|x^2+7x+3\right|+C$

60. $3\ln\left|3x^2+9x+7\right|+C$

61. $-\frac{x^2}{2}+2\ln\left|x^2-7x+1\right|+7x+C$

62. $\frac{1}{18}\tan^{-1}\left(\frac{x^2}{9}\right)+C$

63. $\tan^{-1}(2x)+C$

64. $\sec^{-1}(|2x|)+C$

65. $\frac{1}{3}\sin^{-1}\left(\frac{3x}{4}\right)+C$

66. $\frac{3}{2}\ln\left|x^2-2x+10\right|+\frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right)+C$

67. $\frac{19}{5}\tan^{-1}\left(\frac{x+6}{5}\right)-\ln\left|x^2+12x+61\right|+C$

68. $\frac{15}{2}\ln\left|x^2-10x+32\right|+x+\frac{41\tan^{-1}\left(\frac{x-5}{\sqrt{7}}\right)}{\sqrt{7}}+C$

69. $\frac{x^2}{2}-\frac{9}{2}\ln\left|x^2+9\right|+C$

70. $\frac{x^2}{2}+3\ln\left|x^2+4x+9\right|-4x+\frac{24\tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right)}{\sqrt{5}}+C$

71. $-\tan^{-1}(\cos(x))+C$

72. $\tan^{-1}(\sin(x))+C$

73. $\ln|\sec x+\tan x|+C$ (integrand simplifies to $\sec x$)

74. $3\sqrt{x^2-2x-6}+C$

75. $\sqrt{x^2-6x+8}+C$

76. $-\ln 2$

77. $352/15$

78. $2/3$

79. $1/5$

80. $(1-e)/2$

81. $\pi/2$

82. $\pi/2$

83. $\pi/6$

Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$
   $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C,\ n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

22. $\displaystyle\int \frac{1}{\sqrt{a^2 - x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$

23. $\displaystyle\int \frac{1}{x\sqrt{x^2 - a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

24. $\displaystyle\int \cosh x\, dx = \sinh x + C$

25. $\displaystyle\int \sinh x\, dx = \cosh x + C$

26. $\displaystyle\int \tanh x\, dx = \ln(\cosh x) + C$

27. $\displaystyle\int \coth x\, dx = \ln|\sinh x| + C$

28. $\displaystyle\int \frac{1}{\sqrt{x^2 - a^2}}\, dx = \ln\left|x + \sqrt{x^2 - a^2}\right| + C$

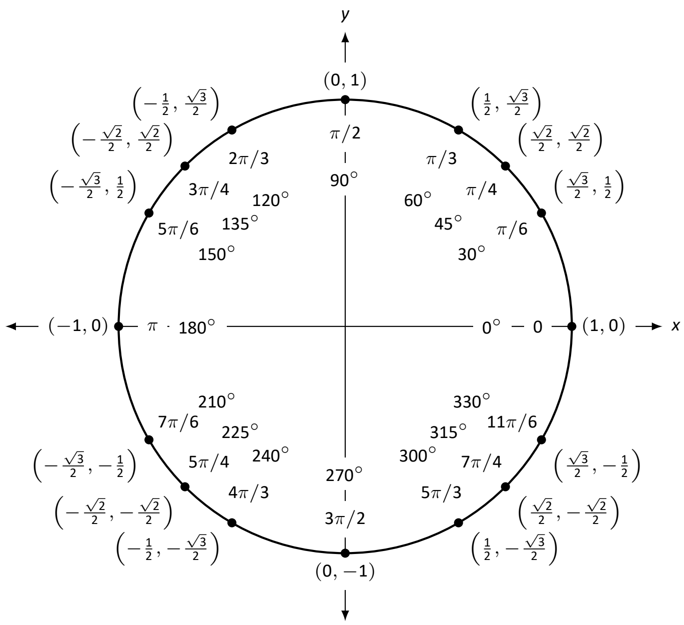
29. $\displaystyle\int \frac{1}{\sqrt{x^2 + a^2}}\, dx = \ln\left|x + \sqrt{x^2 + a^2}\right| + C$

30. $\displaystyle\int \frac{1}{a^2 - x^2}\, dx = \frac{1}{2a}\ln\left|\frac{a + x}{a - x}\right| + C$

31. $\displaystyle\int \frac{1}{x\sqrt{a^2 - x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a + \sqrt{a^2 - x^2}}\right) + C$

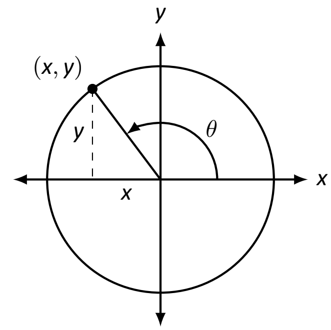
32. $\displaystyle\int \frac{1}{x\sqrt{x^2 + a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a + \sqrt{x^2 + a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

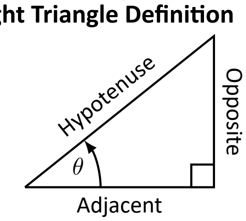


$\sin\theta = y \qquad \cos\theta = x$

$\csc\theta = \dfrac{1}{y} \qquad \sec\theta = \dfrac{1}{x}$

$\tan\theta = \dfrac{y}{x} \qquad \cot\theta = \dfrac{x}{y}$

Right Triangle Definition



$\sin\theta = \dfrac{\text{O}}{\text{H}} \qquad \csc\theta = \dfrac{\text{H}}{\text{O}}$

$\cos\theta = \dfrac{\text{A}}{\text{H}} \qquad \sec\theta = \dfrac{\text{H}}{\text{A}}$

$\tan\theta = \dfrac{\text{O}}{\text{A}} \qquad \cot\theta = \dfrac{\text{A}}{\text{O}}$

Common Trigonometric Identities

Pythagorean Identities

$\sin^2 x + \cos^2 x = 1$

$\tan^2 x + 1 = \sec^2 x$

$1 + \cot^2 x = \csc^2 x$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= 2\cos^2 x - 1\\ &= 1 - 2\sin^2 x\end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\sec(-x) = \sec x$

$\cot(-x) = -\cot x$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

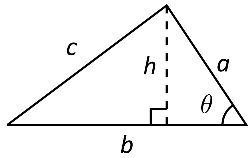
Areas and Volumes

Triangles

$h = a \sin\theta$

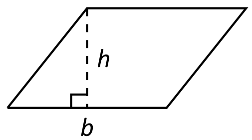
Area = $\frac{1}{2}bh$

Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



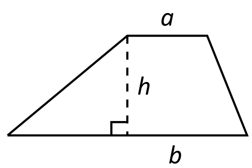
Parallelograms

Area = bh



Trapezoids

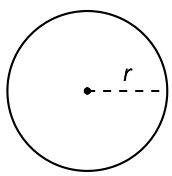
Area = $\frac{1}{2}(a+b)h$



Circles

Area = πr^2

Circumference = $2\pi r$

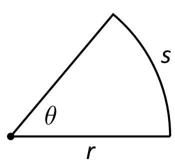


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

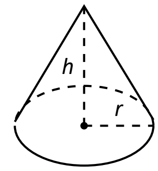
$s = r\theta$



Right Circular Cone

Volume = $\frac{1}{3}\pi r^2 h$

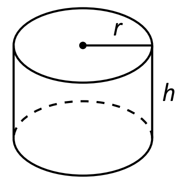
Surface Area =
$\pi r\sqrt{r^2+h^2} + \pi r^2$



Right Circular Cylinder

Volume = $\pi r^2 h$

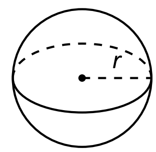
Surface Area =
$2\pi rh + 2\pi r^2$



Sphere

Volume = $\frac{4}{3}\pi r^3$

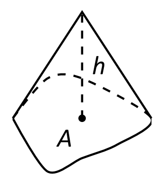
Surface Area =$4\pi r^2$



General Cone

Area of Base = A

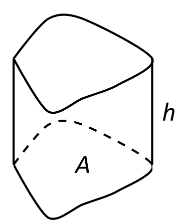
Volume = $\frac{1}{3}Ah$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \leq b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cs + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$ab + ac = a(b + c)$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

$$\frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$a^0 = 1, \ \ a \neq 0$

$(ab)^x = a^x b^x$

$a^x a^y = a^{x+y}$

$\sqrt{a} = a^{1/2}$

$$\frac{a^x}{a^y} = a^{x-y}$$

$\sqrt[n]{a} = a^{1/n}$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

$\sqrt[n]{a^m} = a^{m/n}$

$$a^{-x} = \frac{1}{a^x}$$

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

$(a^x)^y = a^{xy}$

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\,\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\,\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\ \ dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\ \ dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\ \ dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\ \ dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\, d(y)\, \ell(y)\ \ dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\|r\| < 1$	$\|r\| \geq 1$	$\text{Sum} = \dfrac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		$\text{Sum} = \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \dfrac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive. Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive. Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$