1: INTEGRATION

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute areas, volumes, mass, force, pressure, work, and much more.

1.1 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 1.1.1 Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of** f, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them.

We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 1.1.1 Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$ on an interval I. Then there exists a constant C such that, on I,

$$G(x) = F(x) + C.$$

Given a function f defined on an interval I and one of its antiderivatives F, we know *all* antiderivatives of f on I have the form $F(x) + C$ for some constant C. Using Definition 1.1.1, we can say that

$$\int f(x)\,dx = F(x) + C.$$

Let's analyze this indefinite integral notation.

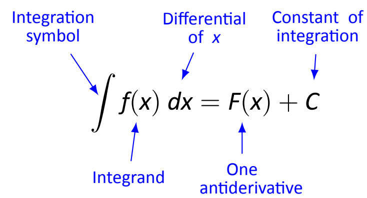


Figure 1.1.1: Understanding the indefinite integral notation.

Figure 1.1.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Notes:

Example 1.1.1 Evaluating indefinite integrals

Evaluate $\int \sin x \, dx$.

SOLUTION We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x \, dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x \, dx$$

presents us with a differential, $dy = \sin x \, dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x \, dx$, rewrite

$$\int \sin x \, dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 1.1.2 Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5) \, dx$.

SOLUTION We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

Notes:

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\ dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\ dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\ dx\right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

Theorem 2.7.3 gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Notes:

Theorem 1.1.2 Derivatives and Antiderivatives

Common Differentiation Rules	Common Indefinite Integral Rules
1. $\frac{d}{dx}\big(cf(x)\big) = c \cdot f'(x)$	1. $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$
2. $\frac{d}{dx}\big(f(x) \pm g(x)\big) = f'(x) \pm g'(x)$	2. $\int \big(f(x) \pm g(x)\big)\,dx = \int f(x)\,dx \pm \int g(x)\,dx$
3. $\frac{d}{dx}\big(C\big) = 0$	3. $\int 0\,dx = C$
4. $\frac{d}{dx}\big(x\big) = 1$	4. $\int 1\,dx = \int dx = x + C$
5. $\frac{d}{dx}\big(x^n\big) = n \cdot x^{n-1}$	5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
6. $\frac{d}{dx}\big(\sin x\big) = \cos x$	6. $\int \cos x\,dx = \sin x + C$
7. $\frac{d}{dx}\big(\cos x\big) = -\sin x$	7. $\int \sin x\,dx = -\cos x + C$
8. $\frac{d}{dx}\big(\tan x\big) = \sec^2 x$	8. $\int \sec^2 x\,dx = \tan x + C$
9. $\frac{d}{dx}\big(\csc x\big) = -\csc x \cot x$	9. $\int \csc x \cot x\,dx = -\csc x + C$
10. $\frac{d}{dx}\big(\sec x\big) = \sec x \tan x$	10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\frac{d}{dx}\big(\cot x\big) = -\csc^2 x$	11. $\int \csc^2 x\,dx = -\cot x + C$
12. $\frac{d}{dx}\big(e^x\big) = e^x$	12. $\int e^x\,dx = e^x + C$
13. $\frac{d}{dx}\big(a^x\big) = \ln a \cdot a^x$	13. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$
14. $\frac{d}{dx}\big(\ln x\big) = \frac{1}{x}$	14. $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$

We highlight a few important points from Theorem 1.1.2:

- Rule #1 states $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\big(3x^2\big)$ is just as easy to compute as $\frac{d}{dx}\big(x^2\big)$). An example:

  $$\int 5\cos x\,dx = 5 \cdot \int \cos x\,dx = 5 \cdot (\sin x + C) = 5\sin x + C.$$

  In the last step we can consider the constant as also being multiplied by

Notes:

5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 1.1.2. So:

$$\begin{aligned}\int (3x^2+4x+5)\,dx &= \int 3x^2\,dx + \int 4x\,dx + \int 5\,dx\\ &= 3\int x^2\,dx + 4\int x\,dx + \int 5\,dx\\ &= 3\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + 5x + C\\ &= x^3 + 2x^2 + 5x + C\end{aligned}$$

  In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\,dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.
  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     > "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section 2.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinitely many antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

Notes:

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 1.1.3 Solving initial value problems

The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

Solution We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and
- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\,dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$\begin{aligned} v(t) &= -32t + C \\ v(3) &= -10 \\ -32(3) + C &= -10 \\ C &= 86 \end{aligned}$$

Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.

When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 1.1.4 Solving initial value problems

Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

Solution We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\,dt = \int \cos t\,dt = \sin t + C = f'(t).$$

Notes:

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\, dt = \int (\sin t + 3)\, dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$\begin{aligned} -\cos 0 + 3(0) + C &= 5 \\ -1 + C &= 5 \\ C &= 6 \end{aligned}$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a position function given a velocity function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 5.4, we will see how areas and antiderivatives are closely tied together.

Notes:

Exercises 1.1

Terms and Concepts

05 01 ex 01 1. Define the term "antiderivative" in your own words.

05 01 ex 02 2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

05 01 ex 03 3. Use your own words to define the indefinite integral of $f(x)$.

05 01 ex 04 4. Fill in the blanks: "Inverse operations do the ________ things in the ________ order."

05 01 ex 05 5. What is an "initial value problem"?

05 01 ex 06 6. The derivative of a position function is a ________ function.

05 01 ex 07 7. The antiderivative of an acceleration function is a ________ function.

05 01 ex 42 8. If $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$, give an antiderivative of $f(x) + g(x)$.

Problems

05 01 exset 01 **In Exercises 9 – 27, evaluate the given indefinite integral.**

05 01 ex 08 9. $\displaystyle\int 3x^3\,dx$

05 01 ex 09 10. $\displaystyle\int x^8\,dx$

05 01 ex 10 11. $\displaystyle\int (10x^2 - 2)\,dx$

05 01 ex 11 12. $\displaystyle\int dt$

05 01 ex 12 13. $\displaystyle\int 1\,ds$

05 01 ex 13 14. $\displaystyle\int \frac{1}{3t^2}\,dt$

05 01 ex 14 15. $\displaystyle\int \frac{3}{t^2}\,dt$

05 01 ex 15 16. $\displaystyle\int \frac{1}{\sqrt{x}}\,dx$

05 01 ex 16 17. $\displaystyle\int \sec^2\theta\,d\theta$

05 01 ex 17 18. $\displaystyle\int \sin\theta\,d\theta$

05 01 ex 18 19. $\displaystyle\int (\sec x\tan x + \csc x\cot x)\,dx$

05 01 ex 19 20. $\displaystyle\int 5e^\theta\,d\theta$

05 01 ex 20 21. $\displaystyle\int 3^t\,dt$

05 01 ex 21 22. $\displaystyle\int \frac{5^t}{2}\,dt$

05 01 ex 22 23. $\displaystyle\int (2t+3)^2\,dt$

05 01 ex 23 24. $\displaystyle\int (t^2+3)(t^3-2t)\,dt$

05 01 ex 24 25. $\displaystyle\int x^2x^3\,dx$

05 01 ex 25 26. $\displaystyle\int e^\pi\,dx$

05 01 ex 26 27. $\displaystyle\int a\,dx$

05 01 ex 39 28. This problem investigates why Theorem 1.1.2 states that $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$.

(a) What is the domain of $y = \ln x$?

(b) Find $\frac{d}{dx}\big(\ln x\big)$.

(c) What is the domain of $y = \ln(-x)$?

(d) Find $\frac{d}{dx}\big(\ln(-x)\big)$.

(e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\displaystyle\int \frac{1}{x}\,dx$ that takes these different domains into account, and explain your answer.

05 01 exset 02 **In Exercises 29 – 39, find $f(x)$ described by the given initial value problem.**

05 01 ex 28 29. $f'(x) = \sin x$ and $f(0) = 2$

05 01 ex 29 30. $f'(x) = 5e^x$ and $f(0) = 10$

05 01 ex 30 31. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

05 01 ex 31 32. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

05 01 ex 32 33. $f'(x) = 7^x$ and $f(2) = 1$

05 01 ex 33 34. $f''(x) = 5$ and $f'(0) = 7$, $f(0) = 3$

05 01 ex 34 35. $f''(x) = 7x$ and $f'(1) = -1$, $f(1) = 10$

05 01 ex 35 36. $f''(x) = 5e^x$ and $f'(0) = 3$, $f(0) = 5$

05 01 ex 36 37. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2$, $f(\pi) = 4$

05 01 ex 37 38. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5$, $f(0) = 0$

05 01 ex 38 39. $f''(x) = 0$ and $f'(1) = 3$, $f(1) = 1$

Review

05 01 ex 40 40. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

05 01 ex 41 41. Given $y = x^2 e^x \cos x$, find dy.

1.2 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the function in the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomials, n^{th} roots, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 1.2.1.

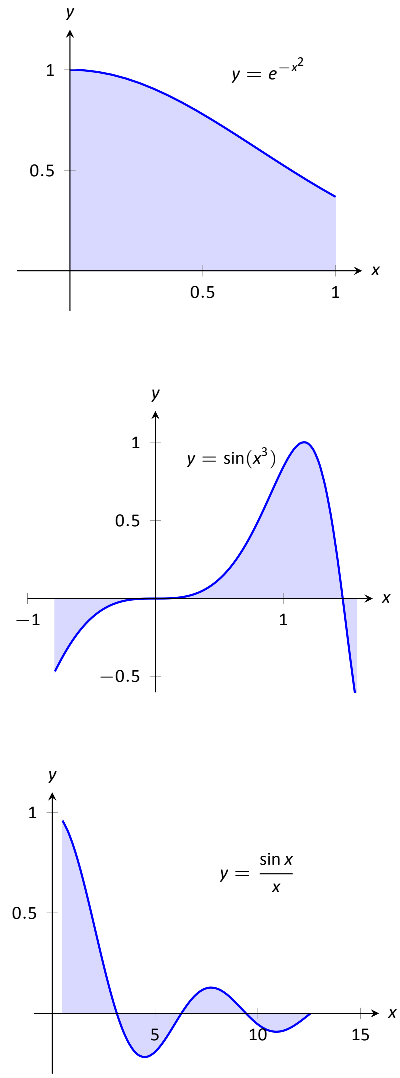


Figure 1.2.1: Graphically representing three definite integrals that cannot be evaluated using antiderivatives.

The Left and Right Hand Rule Methods

In Section 5.3 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a, b]$. We wish to approximate $\int_a^b f(x)\,dx$. We partition $[a, b]$ into n equally spaced subintervals, each of length $\Delta x = \frac{b-a}{n}$. The endpoints of these

Notes:

subintervals are labeled as

$$x_1 = a,\ x_2 = a + \Delta x,\ x_3 = a + 2\Delta x,\ \ldots,\ x_i = a + (i-1)\Delta x,\ \ldots,\ x_{n+1} = b.$$

Key Idea 5.3.1 states that to use the Left Hand Rule we use the summation $\sum_{i=1}^{n} f(x_i)\Delta x$ and to use the Right Hand Rule we use $\sum_{i=1}^{n} f(x_{i+1})\Delta x$. We review the use of these rules in the context of examples.

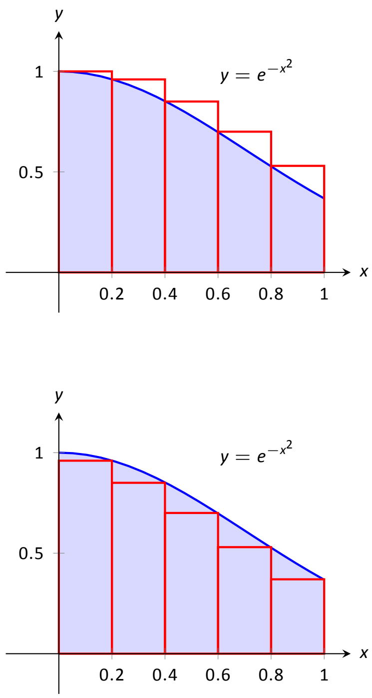


Figure 1.2.2: Approximating $\int_0^1 e^{-x^2}\,dx$ in Example 1.2.1.

Example 1.2.1 Approximating definite integrals with rectangles

Approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

Solution We begin by partitioning the interval $[0,1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

$$x_1 = 0,\ x_2 = 0.2,\ x_3 = 0.4,\ x_4 = 0.6,\ x_5 = 0.8,\ \text{and } x_6 = 1.$$

Using the Left Hand Rule, we have:

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\big)\Delta x \\
&= \big(f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)\big)\Delta x \\
&\approx \big(1 + 0.961 + 0.852 + 0.698 + 0.527\big)(0.2) \\
&\approx 0.808.
\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}
\sum_{i=1}^{n} f(x_{i+1})\Delta x &= \big(f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\big)\Delta x \\
&= \big(f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)\big)\Delta x \\
&\approx \big(0.961 + 0.852 + 0.698 + 0.527 + 0.368\big)(0.2) \\
&\approx 0.681.
\end{aligned}$$

Figure 1.2.2 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles, as we did in

Notes:

Section 5.3. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.808 + 0.681}{2} = 0.7445.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

Example 1.2.2 Approximating definite integrals with rectangles

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Left and Right Hand Rules with 10 equally spaced subintervals.

SOLUTION We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2 - (-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 1.2.3, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

x_i	Exact	Approx.	$\sin(x_i^3)$
x_1	$-\pi/4$	-0.785	-0.466
x_2	$-7\pi/40$	-0.550	-0.165
x_3	$-\pi/10$	-0.314	-0.031
x_4	$-\pi/40$	-0.0785	0
x_5	$\pi/20$	0.157	0.004
x_6	$\pi/8$	0.393	0.061
x_7	$\pi/5$	0.628	0.246
x_8	$11\pi/40$	0.864	0.601
x_9	$7\pi/20$	1.10	0.971
x_{10}	$17\pi/40$	1.34	0.690
x_{11}	$\pi/2$	1.57	-0.670

Figure 1.2.3: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 1.2.2.

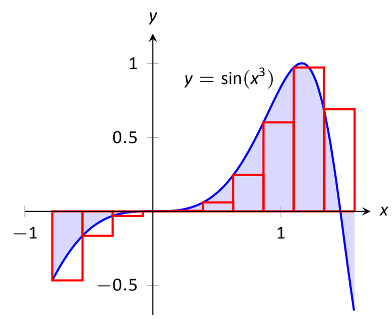
Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

Left Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.91)(0.236) = 0.451.$

Right Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.71)(0.236) = 0.404.$

Average of the Left and Right Hand Rules: 0.4275.

The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 1.2.4. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.



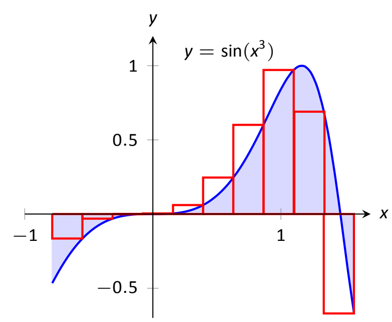


Figure 1.2.4: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 1.2.2.

The Trapezoidal Rule

In Example 1.2.1 we approximated the value of $\int_0^1 e^{-x^2}\,dx$ with 5 rectangles of equal width. Figure 1.2.2 shows the rectangles used in the Left and Right

Notes:

Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 1.2.5, we show the region under $f(x) = e^{-x^2}$ on $[0,1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\,dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

The formula for the area of a trapezoid is given in Figure 1.2.6. We approximate $\int_0^1 e^{-x^2}\,dx$ with these trapezoids in the following example.

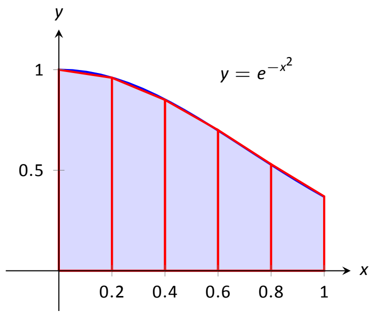


Figure 1.2.5: Approximating $\int_0^1 e^{-x^2}\,dx$ using 5 trapezoids of equal widths.

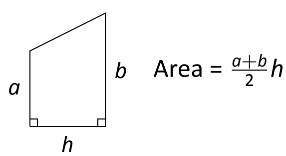


Figure 1.2.6: The area of a trapezoid.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 1.2.7: A table of values of e^{-x^2}.

Example 1.2.3 Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\,dx$.

Solution To compute the areas of the 5 trapezoids in Figure 1.2.5, it will again be useful to create a table of values as shown in Figure 1.2.7.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961+0.852}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\frac{1+0.961}{2}(0.2)+\frac{0.961+0.852}{2}(0.2)+\frac{0.852+0.698}{2}(0.2)+$$
$$\frac{0.698+0.527}{2}(0.2)+\frac{0.527+0.368}{2}(0.2) = 0.7445.$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\,dx \approx 0.7445$.

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+(0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$

Notes:

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1 + 2(0.961 + 0.852 + 0.698 + 0.527) + 0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, wherein a definite integral $\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints x_1, x_2, ..., x_{n+1}, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \sum_{i=1}^{n} \frac{f(x_i)+f(x_{i+1})}{2}\Delta x\\ &= \frac{\Delta x}{2}\sum_{i=1}^{n}\big(f(x_i)+f(x_{i+1})\big)\\ &= \frac{\Delta x}{2}\Big[f(x_1) + 2\sum_{i=2}^{n} f(x_i) + f(x_{n+1})\Big].\end{aligned}$$

Example 1.2.4 Using the Trapezoidal Rule

Revisit Example 1.2.2 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

Solution We refer back to Figure 1.2.3 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\begin{aligned}\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx &\approx \frac{0.236}{2}\Big[-0.466 + 2\Big(-0.165 + (-0.031) + \ldots + 0.69\Big) + (-0.67)\Big]\\ &= 0.4275.\end{aligned}$$

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively

Notes:

renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function f with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x–value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose x–values are equally spaced and $x_1 < x_2 < x_3$. Let f be the quadratic function that goes through these three points. It is not hard to show that

$$\int_{x_1}^{x_3} f(x)\,dx = \frac{x_3 - x_1}{6}\big(y_1 + 4y_2 + y_3\big). \tag{1.1}$$

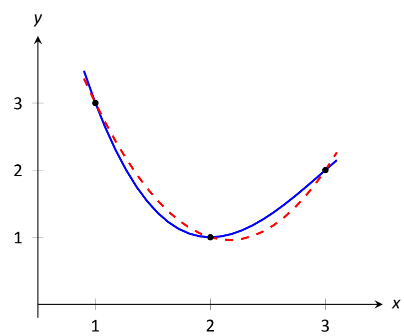


Figure 1.2.8: A graph of a function f and a parabola that approximates it well on $[1, 3]$.

Consider Figure 1.2.8. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_1^3 g(x)\,dx = \frac{3-1}{6}\big(3 + 4(1) + 2\big) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that

$$\int_1^3 f(x)\,dx \approx 3.$$

Notes:

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\int_a^b f(x)\,dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b-a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (1.1) to compute the area under these parabolas. Adding up these areas gives the formula:

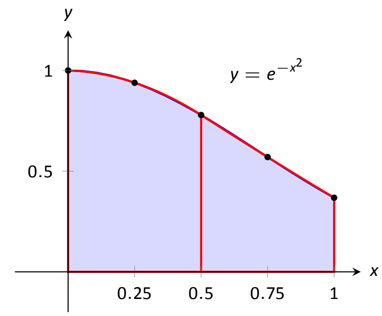
$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+\ldots+2f(x_{n-1})+4f(x_n)+f(x_{n+1})\Big].$$

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, . . ., 2, 4, 1.

Let's demonstrate Simpson's Rule with a concrete example.

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)



(b)

Figure 1.2.9: A table of values to approximate $\int_0^1 e^{-x^2}\,dx$, along with a graph of the function.

Example 1.2.5 Using Simpson's Rule

Approximate $\int_0^1 e^{-x^2}\,dx$ using Simpson's Rule and 4 equally spaced subintervals.

Solution We begin by making a table of values as we have in the past, as shown in Figure 1.2.9(a). Simpson's Rule states that

$$\int_0^1 e^{-x^2}\,dx \approx \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.7468\overline{3}.$$

Recall in Example 1.2.1 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 1.2.9(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function.

Example 1.2.6 Using Simpson's Rule

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using Simpson's Rule and 10 equally spaced intervals.

Solution Figure 1.2.10 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.

x_i	$\sin(x_i^3)$
-0.785	-0.466
-0.550	-0.165
-0.314	-0.031
-0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	-0.670

Figure 1.2.10: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 1.2.6.

Notes:

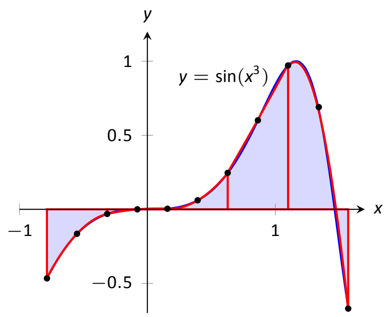


Figure 1.2.11: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx$ in Example 1.2.6 with Simpson's Rule and 10 equally spaced intervals.

Simpson's Rule states that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\ dx \approx \frac{0.236}{3}\Big[(-0.466) + 4(-0.165) + 2(-0.031) + \ldots$$
$$\ldots + 2(0.971) + 4(0.69) + (-0.67)\Big]$$
$$= 0.4701$$

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within one $1/100^{\text{th}}$ of the correct value. The graph in Figure 1.2.11 shows how closely the parabolas match the shape of the graph.

Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Key Idea 1.2.1 Numerical Integration

Let f be a continuous function on $[a, b]$, let n be a positive integer, and let $\Delta x = \dfrac{b-a}{n}$.
Set $x_1 = a$, $x_2 = a + \Delta x, \ldots, x_i = a + (i-1)\Delta x, x_{n+1} = b$.
Consider $\displaystyle\int_a^b f(x)\ dx$.

Left Hand Rule: $\displaystyle\int_a^b f(x)\ dx \approx \Delta x\Big[f(x_1) + f(x_2) + \ldots + f(x_n)\Big].$

Right Hand Rule: $\displaystyle\int_a^b f(x)\ dx \approx \Delta x\Big[f(x_2) + f(x_3) + \ldots + f(x_{n+1})\Big].$

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\ dx \approx \frac{\Delta x}{2}\Big[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\Big].$

Simpson's Rule: $\displaystyle\int_a^b f(x)\ dx \approx \frac{\Delta x}{3}\Big[f(x_1) + 4f(x_2) + 2f(x_3) + \ldots + 4f(x_n) + f(x_{n+1})\Big]$ (n even).

In our examples, we approximated the value of a definite integral using a given method then compared it to the "right" answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?
2. If the right answer can be found, what is the point of approximating?
3. If there is value to approximating, how are we supposed to know if the approximation is any good?

Notes:

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example 1.2.1, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more subintervals. Keep in mind that trial and error is never foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem 1.2.1 states what these bounds are.

Theorem 1.2.1 Error Bounds in the Trapezoidal and Simpson's Rules

1. Let E_T be the error in approximating $\displaystyle\int_a^b f(x)\,dx$ using the Trapezoidal Rule.

   If f has a continuous 2[nd] derivative on $[a,b]$ and M is any upper bound of $\left|f''(x)\right|$ on $[a,b]$, then

   $$E_T \leq \frac{(b-a)^3}{12n^2}M.$$

2. Let E_S be the error in approximating $\displaystyle\int_a^b f(x)\,dx$ using Simpson's Rule.

   If f has a continuous 4[th] derivative on $[a,b]$ and M is any upper bound of $\left|f^{(4)}\right|$ on $[a,b]$, then

   $$E_S \leq \frac{(b-a)^5}{180n^4}M.$$

There are some key things to note about this theorem.

Notes:

1. The larger the interval, the larger the error. This should make sense intuitively.
2. The error shrinks as more subintervals are used (i.e., as n gets larger).
3. The error in Simpson's Rule has a term relating to the 4th derivative of f. Consider a cubic polynomial: it's 4th derivative is 0. Therefore, the error in approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 – Simpson's Rule computes the exact answer!

We revisit Examples 1.2.3 and 1.2.5 and compute the error bounds using Theorem 1.2.1 in the following example.

Example 1.2.7 Computing error bounds

Find the error bounds when approximating $\int_0^1 e^{-x^2}\,dx$ using the Trapezoidal Rule and 5 subintervals, and using Simpson's Rule with 4 subintervals.

Solution

Trapezoidal Rule with $n = 5$:

We start by computing the 2nd derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

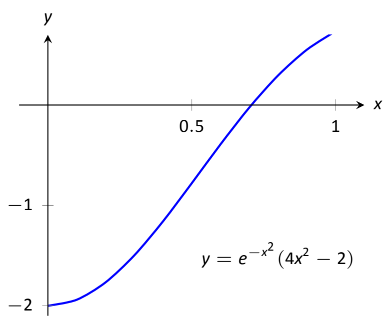


Figure 1.2.12: Graphing $f''(x)$ in Example 1.2.7 to help establish error bounds.

Figure 1.2.12 shows a graph of $f''(x)$ on $[0, 1]$. It is clear that the largest value of f'', in absolute value, is 2. Thus we let $M = 2$ and apply the error formula from Theorem 1.2.1.

$$E_T = \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Our error estimation formula states that our approximation of 0.7445 found in Example 1.2.3 is within 0.0067 of the correct answer, hence we know that

$$0.7445 - 0.0067 = .7378 \leq \int_0^1 e^{-x^2}\,dx \leq 0.7512 = 0.7445 + 0.0067.$$

We had earlier computed the exact answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem 1.2.1.

Simpson's Rule with $n = 4$:

We start by computing the 4th derivative of $f(x) = e^{-x^2}$:

$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

Notes:

Figure 1.2.13 shows a graph of $f^{(4)}(x)$ on $[0, 1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $M = 12$ and apply the error formula from Theorem 1.2.1.

$$E_s = \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 1.2.5 is within 0.00026 of the correct answer, hence we know that

$$0.74683 - 0.00026 = .74657 \le \int_0^1 e^{-x^2}\,dx \le 0.74709 = 0.74683 + 0.00026.$$

Once again we affirm the validity of Theorem 1.2.1.

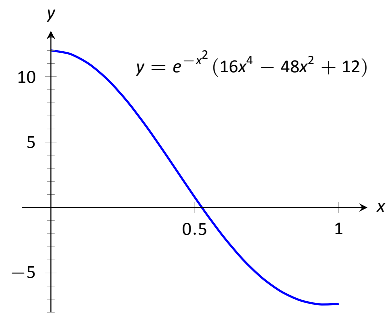


Figure 1.2.13: Graphing $f^{(4)}(x)$ in Example 1.2.7 to help establish error bounds.

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. "Most of the time" we observe behavior but do not know "the" function that describes it. We instead collect data about the behavior and make approximations based off of this data. We demonstrate this in an example.

Example 1.2.8 Approximating distance traveled

One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 1.2.14. Approximate the distance they traveled.

Solution Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\int_a^b v(t)\,dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in 30 second increments.

We need to compute $\Delta x = (b-a)/n$. Clearly, $n = 24$. What are a and b? Since we start at time $t = 0$, we have that $a = 0$. The final recorded time came after 24 periods of 30 seconds, which is 12 minutes or 1/5 of an hour. Thus we have

$$\Delta x = \frac{b-a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \qquad \frac{\Delta x}{3} = \frac{1}{360}.$$

Time	Speed (mph)
0	0
1	25
2	22
3	19
4	39
5	0
6	43
7	59
8	54
9	51
10	43
11	35
12	40
13	43
14	30
15	0
16	0
17	28
18	40
19	42
20	40
21	39
22	40
23	23
24	0

Figure 1.2.14: Speed data collected at 30 second intervals for Example 1.2.8.

Notes:

Thus the distance traveled is approximately:

$$\begin{aligned}
\int_0^{0.2} v(t)\,dt &\approx \frac{1}{360}\Big[f(x_1)+4f(x_2)+2f(x_3)+\cdots+4f(x_n)+f(x_{n+1})\Big]\\
&= \frac{1}{360}\Big[0+4\cdot 25+2\cdot 22+\cdots+2\cdot 40+4\cdot 23+0\Big]\\
&\approx 6.2167 \text{ miles.}
\end{aligned}$$

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

We started this chapter learning about antiderivatives and indefinite integrals. We then seemed to change focus by looking at areas between the graph of a function and the x-axis. We defined these areas as the definite integral of the function, using a notation very similar to the notation of the indefinite integral. The Fundamental Theorem of Calculus tied these two seemingly separate concepts together: we can find areas under a curve, i.e., we can evaluate a definite integral, using antiderivatives.

We ended the chapter by noting that antiderivatives are sometimes more than difficult to find: they are impossible. Therefore we developed numerical techniques that gave us good approximations of definite integrals.

We used the definite integral to compute areas, and also to compute displacements and distances traveled. There is far more we can do than that. In Chapter 7 we'll see more applications of the definite integral. Before that, in Chapter 6 we'll learn advanced techniques of integration, analogous to learning rules like the Product, Quotient and Chain Rules of differentiation.

Notes:

Exercises 1.2

Terms and Concepts

05 05 ex 23 1. T/F: Simpson's Rule is a method of approximating antiderivatives.

05 05 ex 24 2. What are the two basic situations where approximating the value of a definite integral is necessary?

05 05 ex 25 3. Why are the Left and Right Hand Rules rarely used?

05 05 ex 26 4. Simpson's Rule is based on approximating portions of a function with what type of function?

Problems

05 05 exset 02 **In Exercises 5 – 12, a definite integral is given.**

(a) Approximate the definite integral with the Trapezoidal Rule and $n = 4$.

(b) Approximate the definite integral with Simpson's Rule and $n = 4$.

(c) Find the exact value of the integral.

05 05 ex 03 5. $\displaystyle\int_{-1}^{1} x^2\,dx$

05 05 ex 04 6. $\displaystyle\int_{0}^{10} 5x\,dx$

05 05 ex 05 7. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

05 05 ex 06 8. $\displaystyle\int_{0}^{4} \sqrt{x}\,dx$

05 05 ex 07 9. $\displaystyle\int_{0}^{3} (x^3 + 2x^2 - 5x + 7)\,dx$

05 05 ex 08 10. $\displaystyle\int_{0}^{1} x^4\,dx$

05 05 ex 09 11. $\displaystyle\int_{0}^{2\pi} \cos x\,dx$

05 05 ex 10 12. $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\,dx$

05 05 exset 03 **In Exercises 13 – 20, approximate the definite integral with the Trapezoidal Rule and Simpson's Rule, with $n = 6$.**

05 05 ex 11 13. $\displaystyle\int_{0}^{1} \cos\left(x^2\right)\,dx$

05 05 ex 12 14. $\displaystyle\int_{-1}^{1} e^{x^2}\,dx$

05 05 ex 13 15. $\displaystyle\int_{0}^{5} \sqrt{x^2+1}\,dx$

05 05 ex 14 16. $\displaystyle\int_{0}^{\pi} x\sin x\,dx$

05 05 ex 15 17. $\displaystyle\int_{0}^{\pi/2} \sqrt{\cos x}\,dx$

05 05 ex 16 18. $\displaystyle\int_{1}^{4} \ln x\,dx$

05 05 ex 17 19. $\displaystyle\int_{-1}^{1} \frac{1}{\sin x + 2}\,dx$

05 05 ex 18 20. $\displaystyle\int_{0}^{6} \frac{1}{\sin x + 2}\,dx$

05 05 exset 04 **In Exercises 21 – 24, find n such that the error in approximating the given definite integral is less than 0.0001 when using:**

(a) the Trapezoidal Rule

(b) Simpson's Rule

05 05 ex 19 21. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

05 05 ex 20 22. $\displaystyle\int_{1}^{4} \frac{1}{\sqrt{x}}\,dx$

05 05 ex 21 23. $\displaystyle\int_{0}^{\pi} \cos\left(x^2\right)\,dx$

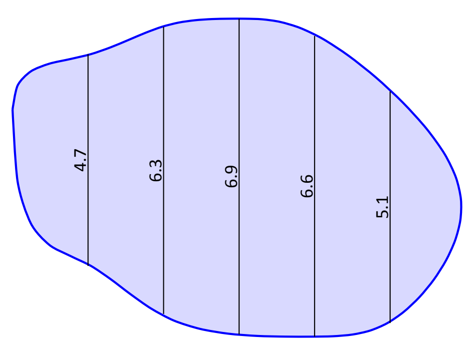
05 05 ex 22 24. $\displaystyle\int_{0}^{5} x^4\,dx$

05 05 exset 01 **In Exercises 25 – 26, a region is given. Find the area of the region using Simpson's Rule:**

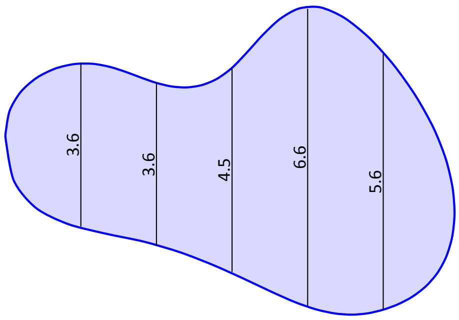
(a) where the measurements are in centimeters, taken in 1 cm increments, and

(b) where the measurements are in hundreds of yards, taken in 100 yd increments.

05 05 ex 01 25.



05 05 ex 02

26. 

A: Solutions To Selected Problems

Chapter 1

Section 1.1

05 01 ex 01 1. Answers will vary.

05 01 ex 02 2. “an”

05 01 ex 03 3. Answers will vary.

05 01 ex 04 4. opposite; opposite

05 01 ex 05 5. Answers will vary.

05 01 ex 06 6. velocity

05 01 ex 07 7. velocity

05 01 ex 42 8. $F(x) + G(x)$

05 01 ex 08 9. $3/4x^4 + C$

05 01 ex 09 10. $1/9x^9 + C$

05 01 ex 10 11. $10/3x^3 - 2x + C$

05 01 ex 11 12. $t + C$

05 01 ex 12 13. $s + C$

05 01 ex 13 14. $-1/(3t) + C$

05 01 ex 14 15. $-3/(t) + C$

05 01 ex 15 16. $2\sqrt{x} + C$

05 01 ex 16 17. $\tan\theta + C$

05 01 ex 17 18. $-\cos\theta + C$

05 01 ex 18 19. $\sec x - \csc x + C$

05 01 ex 19 20. $5e^\theta + C$

05 01 ex 20 21. $3^t/\ln 3 + C$

05 01 ex 21 22. $\frac{5^t}{2\ln 5} + C$

05 01 ex 22 23. $4/3t^3 + 6t^2 + 9t + C$

05 01 ex 23 24. $t^6/6 + t^4/4 - 3t^2 + C$

05 01 ex 24 25. $x^6/6 + C$

05 01 ex 25 26. $e^\pi x + C$

05 01 ex 26 27. $ax + C$

05 01 ex 39 28.

(a) $x > 0$

(b) $1/x$

(c) $x < 0$

(d) $1/x$

(e) $\ln|x| + C$. Explanations will vary.

05 01 ex 28 29. $-\cos x + 3$

05 01 ex 29 30. $5e^x + 5$

05 01 ex 30 31. $x^4 - x^3 + 7$

05 01 ex 31 32. $\tan x + 4$

05 01 ex 32 33. $7^x/\ln 7 + 1 - 49/\ln 7$

05 01 ex 33 34. $5/2x^2 + 7x + 3$

05 01 ex 34 35. $\frac{7x^3}{6} - \frac{9x}{2} + \frac{40}{3}$

05 01 ex 35 36. $5e^x - 2x$

05 01 ex 36 37. $\theta - \sin(\theta) - \pi + 4$

05 01 ex 37 38. $2x^4 + \cos x + \frac{2^x}{(\ln 2)^2} + (5 - \frac{1}{\ln 2})x - 1 - \frac{1}{(\ln 2)^2}$

05 01 ex 38 39. $3x - 2$

05 01 ex 40 40. No answer provided.

05 01 ex 41 41. $dy = (2xe^x\cos x + x^2e^x\cos x - x^2e^x\sin x)dx$

Section 1.2

05 05 ex 23 1. F

05 05 ex 24 2. When the antiderivative cannot be computed and when the integrand is unknown.

05 05 ex 25 3. They are superseded by the Trapezoidal Rule; it takes an equal amount of work and is generally more accurate.

05 05 ex 26 4. A quadratic function (i.e., parabola)

05 05 ex 03 5.

(a) $3/4$

(b) $2/3$

(c) $2/3$

05 05 ex 04 6.

(a) 250

(b) 250

(c) 250

05 05 ex 05 7.

(a) $\frac{1}{4}(1 + \sqrt{2})\pi \approx 1.896$

(b) $\frac{1}{6}(1 + 2\sqrt{2})\pi \approx 2.005$

(c) 2

05 05 ex 06 8.

(a) $2 + \sqrt{2} + \sqrt{3} \approx 5.15$

(b) $2/3(3 + \sqrt{2} + 2\sqrt{3}) \approx 5.25$

(c) $16/3 \approx 5.33$

05 05 ex 07 9.

(a) 38.5781

(b) $147/4 \approx 36.75$

(c) $147/4 \approx 36.75$

05 05 ex 08 10.

(a) 0.2207

(b) 0.2005

(c) $1/5$

05 05 ex 09 11.

(a) 0

(b) 0

(c) 0

05 05 ex 10 12.

(a) $9/2(1+\sqrt{3}) \approx 12.294$

(b) $3+6\sqrt{3} \approx 13.392$

(c) $9\pi/2 \approx 14.137$

05 05 ex 11 13. Trapezoidal Rule: 0.9006
Simpson's Rule: 0.90452

05 05 ex 12 14. Trapezoidal Rule: 3.0241
Simpson's Rule: 2.9315

05 05 ex 13 15. Trapezoidal Rule: 13.9604
Simpson's Rule: 13.9066

05 05 ex 14 16. Trapezoidal Rule: 3.0695
Simpson's Rule: 3.14295

05 05 ex 15 17. Trapezoidal Rule: 1.1703
Simpson's Rule: 1.1873

05 05 ex 16 18. Trapezoidal Rule: 2.52971
Simpson's Rule: 2.5447

05 05 ex 17 19. Trapezoidal Rule: 1.0803
Simpson's Rule: 1.077

05 05 ex 18 20. Trapezoidal Rule: 3.5472
Simpson's Rule: 3.6133

05 05 ex 19 21.

(a) $n = 161$ (using max $\big(f''(x)\big) = 1$)

(b) $n = 12$ (using max $\big(f^{(4)}(x)\big) = 1$)

05 05 ex 20 22.

(a) $n = 150$ (using max $\big(f''(x)\big) = 1$)

(b) $n = 18$ (using max $\big(f^{(4)}(x)\big) = 7$)

05 05 ex 21 23.

(a) $n = 1004$ (using max $\big(f''(x)\big) = 39$)

(b) $n = 62$ (using max $\big(f^{(4)}(x)\big) = 800$)

05 05 ex 22 24.

(a) $n = 5591$ (using max $\big(f''(x)\big) = 300$)

(b) $n = 46$ (using max $\big(f^{(4)}(x)\big) = 24$)

05 05 ex 01 25.

(a) Area is 30.8667 cm^2.

(b) Area is $308,667$ yd^2.

05 05 ex 02 26.

(a) Area is 25.0667 cm^2

(b) Area is 250,667 yd^2