5: INTEGRATION

We have spent considerable time considering the derivatives of a function and their applications. In the following chapters, we are going to starting thinking in "the other direction." That is, given a function $f(x)$, we are going to consider functions $F(x)$ such that $F'(x) = f(x)$. There are numerous reasons this will prove to be useful: these functions will help us compute area, volume, mass, force, pressure, work, and much more.

5.1 Antiderivatives and Indefinite Integration

Given a function $y = f(x)$, a *differential equation* is one that incorporates y, x, and the derivatives of y. For instance, a simple differential equation is:

$$y' = 2x.$$

Solving a differential equation amounts to finding a function y that satisfies the given equation. Take a moment and consider that equation; can you find a function y such that $y' = 2x$?

Can you find another?

And yet another?

Hopefully one was able to come up with at least one solution: $y = x^2$. "Finding another" may have seemed impossible until one realizes that a function like $y = x^2 + 1$ also has a derivative of $2x$. Once that discovery is made, finding "yet another" is not difficult; the function $y = x^2 + 123,456,789$ also has a derivative of $2x$. The differential equation $y' = 2x$ has many solutions. This leads us to some definitions.

Definition 5.1.1   Antiderivatives and Indefinite Integrals

Let a function $f(x)$ be given. An **antiderivative** of $f(x)$ is a function $F(x)$ such that $F'(x) = f(x)$.

The set of all antiderivatives of $f(x)$ is the **indefinite integral of f**, denoted by

$$\int f(x)\, dx.$$

Make a note about our definition: we refer to *an* antiderivative of f, as opposed to *the* antiderivative of f, since there is *always* an infinite number of them.

We often use upper-case letters to denote antiderivatives.

Knowing one antiderivative of f allows us to find infinitely more, simply by adding a constant. Not only does this give us *more* antiderivatives, it gives us *all* of them.

Theorem 5.1.1   Antiderivative Forms

Let $F(x)$ and $G(x)$ be antiderivatives of $f(x)$ on an interval I. Then there exists a constant C such that, on I,

$$G(x) = F(x) + C.$$

Given a function f defined on an interval I and one of its antiderivatives F, we know *all* antiderivatives of f on I have the form $F(x) + C$ for some constant C. Using Definition 5.1.1, we can say that

$$\int f(x)\, dx = F(x) + C.$$
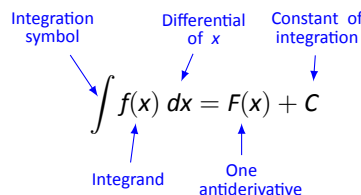
Let's analyze this indefinite integral notation.

Figure 5.1.1: Understanding the indefinite integral notation.

Figure 5.1.1 shows the typical notation of the indefinite integral. The integration symbol, $\int$, is in reality an "elongated S," representing "take the sum." We will later see how *sums* and *antiderivatives* are related.

The function we want to find an antiderivative of is called the *integrand*. It contains the differential of the variable we are integrating with respect to. The $\int$ symbol and the differential dx are not "bookends" with a function sandwiched in between; rather, the symbol $\int$ means "find all antiderivatives of what follows," and the function $f(x)$ and dx are multiplied together; the dx does not "just sit there."

Let's practice using this notation.

Notes:

Example 5.1.1   Evaluating indefinite integrals

Evaluate $\int \sin x\, dx$.

SOLUTION   We are asked to find all functions $F(x)$ such that $F'(x) = \sin x$. Some thought will lead us to one solution: $F(x) = -\cos x$, because $\frac{d}{dx}(-\cos x) = \sin x$.

The indefinite integral of $\sin x$ is thus $-\cos x$, plus a constant of integration. So:

$$\int \sin x\, dx = -\cos x + C.$$

A commonly asked question is "What happened to the dx?" The unenlightened response is "Don't worry about it. It just goes away." A full understanding includes the following.

This process of *antidifferentiation* is really solving a *differential* question. The integral

$$\int \sin x\, dx$$

presents us with a differential, $dy = \sin x\, dx$. It is asking: "What is y?" We found lots of solutions, all of the form $y = -\cos x + C$.

Letting $dy = \sin x\, dx$, rewrite

$$\int \sin x\, dx \quad \text{as} \quad \int dy.$$

This is asking: "What functions have a differential of the form dy?" The answer is "Functions of the form $y + C$, where C is a constant." What is y? We have lots of choices, all differing by a constant; the simplest choice is $y = -\cos x$.

Understanding all of this is more important later as we try to find antiderivatives of more complicated functions. In this section, we will simply explore the rules of indefinite integration, and one can succeed for now with answering "What happened to the dx?" with "It went away."

Let's practice once more before stating integration rules.

Example 5.1.2   Evaluating indefinite integrals

Evaluate $\int (3x^2 + 4x + 5)\, dx$.

SOLUTION   We seek a function $F(x)$ whose derivative is $3x^2 + 4x + 5$. When taking derivatives, we can consider functions term–by–term, so we can likely do that here.

What functions have a derivative of $3x^2$? Some thought will lead us to a cubic, specifically $x^3 + C_1$, where C_1 is a constant.

Notes:

What functions have a derivative of $4x$? Here the x term is raised to the first power, so we likely seek a quadratic. Some thought should lead us to $2x^2 + C_2$, where C_2 is a constant.

Finally, what functions have a derivative of 5? Functions of the form $5x + C_3$, where C_3 is a constant.

Our answer appears to be

$$\int (3x^2 + 4x + 5)\, dx = x^3 + C_1 + 2x^2 + C_2 + 5x + C_3.$$

We do not need three separate constants of integration; combine them as one constant, giving the final answer of

$$\int (3x^2 + 4x + 5)\, dx = x^3 + 2x^2 + 5x + C.$$

It is easy to verify our answer; take the derivative of $x^3 + 2x^3 + 5x + C$ and see we indeed get $3x^2 + 4x + 5$.

This final step of "verifying our answer" is important both practically and theoretically. In general, taking derivatives is easier than finding antiderivatives so checking our work is easy and vital as we learn.

We also see that taking the derivative of our answer returns the function in the integrand. Thus we can say that:

$$\frac{d}{dx}\left(\int f(x)\, dx \right) = f(x).$$

Differentiation "undoes" the work done by antidifferentiation.

Theorem 2.7.3 gave a list of the derivatives of common functions we had learned at that point. We restate part of that list here to stress the relationship between derivatives and antiderivatives. This list will also be useful as a glossary of common antiderivatives as we learn.

Notes:

Theorem 5.1.2   Derivatives and Antiderivatives

Common Differentiation Rules   Common Indefinite Integral Rules

1. $\frac{d}{dx}\left(cf(x)\right) = c \cdot f'(x)$ 1. $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$

2. $\frac{d}{dx}\left(f(x) \pm g(x)\right) =$ 2. $\int \left(f(x) \pm g(x)\right)\,dx =$
   $f'(x) \pm g'(x)$ $\int f(x)\,dx \pm \int g(x)\,dx$

3. $\frac{d}{dx}\left(C\right) = 0$ 3. $\int 0\,dx = C$

4. $\frac{d}{dx}\left(x\right) = 1$ 4. $\int 1\,dx = \int dx = x + C$

5. $\frac{d}{dx}\left(x^n\right) = n \cdot x^{n-1}$ 5. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$

6. $\frac{d}{dx}\left(\sin x\right) = \cos x$ 6. $\int \cos x\,dx = \sin x + C$

7. $\frac{d}{dx}\left(\cos x\right) = -\sin x$ 7. $\int \sin x\,dx = -\cos x + C$

8. $\frac{d}{dx}\left(\tan x\right) = \sec^2 x$ 8. $\int \sec^2 x\,dx = \tan x + C$

9. $\frac{d}{dx}\left(\csc x\right) = -\csc x \cot x$ 9. $\int \csc x \cot x\,dx = -\csc x + C$

10. $\frac{d}{dx}\left(\sec x\right) = \sec x \tan x$ 10. $\int \sec x \tan x\,dx = \sec x + C$

11. $\frac{d}{dx}\left(\cot x\right) = -\csc^2 x$ 11. $\int \csc^2 x\,dx = -\cot x + C$

12. $\frac{d}{dx}\left(e^x\right) = e^x$ 12. $\int e^x\,dx = e^x + C$

13. $\frac{d}{dx}\left(a^x\right) = \ln a \cdot a^x$ 13. $\int a^x\,dx = \frac{1}{\ln a} \cdot a^x + C$

14. $\frac{d}{dx}\left(\ln x\right) = \frac{1}{x}$ 14. $\int \frac{1}{x}\,dx = \ln |x| + C$

We highlight a few important points from Theorem 5.1.2:

- Rule #1 states $\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$. This is the Constant Multiple Rule: we can temporarily ignore constants when finding antiderivatives, just as we did when computing derivatives (i.e., $\frac{d}{dx}\left(3x^2\right)$ is just as easy to compute as $\frac{d}{dx}\left(x^2\right)$). An example:

$$\int 5 \cos x\,dx = 5 \cdot \int \cos x\,dx = 5 \cdot (\sin x + C) = 5 \sin x + C.$$

In the last step we can consider the constant as also being multiplied by

Notes:

5, but "5 times a constant" is still a constant, so we just write "C".

- Rule #2 is the Sum/Difference Rule: we can split integrals apart when the integrand contains terms that are added/subtracted, as we did in Example 5.1.2. So:

$$\int (3x^2 + 4x + 5)\, dx = \int 3x^2\, dx + \int 4x\, dx + \int 5\, dx$$
$$= 3\int x^2\, dx + 4\int x\, dx + \int 5\, dx$$
$$= 3 \cdot \frac{1}{3}x^3 + 4 \cdot \frac{1}{2}x^2 + 5x + C$$
$$= x^3 + 2x^2 + 5x + C$$

In practice we generally do not write out all these steps, but we demonstrate them here for completeness.

- Rule #5 is the Power Rule of indefinite integration. There are two important things to keep in mind:

  1. Notice the restriction that $n \neq -1$. This is important: $\int \frac{1}{x}\, dx \neq$ "$\frac{1}{0}x^0 + C$"; rather, see Rule #14.

  2. We are presenting antidifferentiation as the "inverse operation" of differentiation. Here is a useful quote to remember:

     "Inverse operations do the opposite things in the opposite order."

     When taking a derivative using the Power Rule, we **first** *multiply* by the power, then **second** *subtract* 1 from the power. To find the antiderivative, do the opposite things in the opposite order: **first** *add* one to the power, then **second** *divide* by the power.

- Note that Rule #14 incorporates the absolute value of x. The exercises will work the reader through why this is the case; for now, know the absolute value is important and cannot be ignored.

Initial Value Problems

In Section 2.3 we saw that the derivative of a position function gave a velocity function, and the derivative of a velocity function describes acceleration. We can now go "the other way:" the antiderivative of an acceleration function gives a velocity function, etc. While there is just one derivative of a given function, there are infinitely many antiderivatives. Therefore we cannot ask "What is *the* velocity of an object whose acceleration is -32ft/s^2?", since there is more than one answer.

Notes:

We can find *the* answer if we provide more information with the question, as done in the following example. Often the additional information comes in the form of an *initial value*, a value of the function that one knows beforehand.

Example 5.1.3   Solving initial value problems
The acceleration due to gravity of a falling object is -32 ft/s^2. At time $t = 3$, a falling object had a velocity of -10 ft/s. Find the equation of the object's velocity.

    SOLUTION   We want to know a velocity function, $v(t)$. We know two things:

- The acceleration, i.e., $v'(t) = -32$, and

- the velocity at a specific time, i.e., $v(3) = -10$.

Using the first piece of information, we know that $v(t)$ is an antiderivative of $v'(t) = -32$. So we begin by finding the indefinite integral of -32:

$$\int (-32)\, dt = -32t + C = v(t).$$

Now we use the fact that $v(3) = -10$ to find C:

$$v(t) = -32t + C$$
$$v(3) = -10$$
$$-32(3) + C = -10$$
$$C = 86$$

    Thus $v(t) = -32t + 86$. We can use this equation to understand the motion of the object: when $t = 0$, the object had a velocity of $v(0) = 86$ ft/s. Since the velocity is positive, the object was moving upward.
    When did the object begin moving down? Immediately after $v(t) = 0$:

$$-32t + 86 = 0 \quad \Rightarrow \quad t = \frac{43}{16} \approx 2.69\text{s}.$$

Recognize that we are able to determine quite a bit about the path of the object knowing just its acceleration and its velocity at a single point in time.

Example 5.1.4   Solving initial value problems
Find $f(t)$, given that $f''(t) = \cos t$, $f'(0) = 3$ and $f(0) = 5$.

    SOLUTION   We start by finding $f'(t)$, which is an antiderivative of $f''(t)$:

$$\int f''(t)\, dt = \int \cos t\, dt = \sin t + C = f'(t).$$

Notes:

So $f'(t) = \sin t + C$ for the correct value of C. We are given that $f'(0) = 3$, so:

$$f'(0) = 3 \quad \Rightarrow \quad \sin 0 + C = 3 \quad \Rightarrow \quad C = 3.$$

Using the initial value, we have found $f'(t) = \sin t + 3$.

We now find $f(t)$ by integrating again.

$$f(t) = \int f'(t)\, dt = \int (\sin t + 3)\, dt = -\cos t + 3t + C.$$

We are given that $f(0) = 5$, so

$$-\cos 0 + 3(0) + C = 5$$
$$-1 + C = 5$$
$$C = 6$$

Thus $f(t) = -\cos t + 3t + 6$.

This section introduced antiderivatives and the indefinite integral. We found they are needed when finding a function given information about its derivative(s). For instance, we found a velocity function given an acceleration function.

In the next section, we will see how position and velocity are unexpectedly related by the areas of certain regions on a graph of the velocity function. Then, in Section 5.4, we will see how areas and antiderivatives are closely tied together. This connection is incredibly important, as indicated by the name of the theorem that describes it: The Fundamental Theorem of Calculus.

Notes:

Exercises 5.1

Terms and Concepts

1. Define the term "antiderivative" in your own words.

2. Is it more accurate to refer to "the" antiderivative of $f(x)$ or "an" antiderivative of $f(x)$?

3. Use your own words to define the indefinite integral of $f(x)$.

4. Fill in the blanks: "Inverse operations do the _____ things in the _____ order."

5. What is an "initial value problem"?

6. The derivative of a position function is a _____ function.

7. The antiderivative of an acceleration function is a _____ function.

8. If $F(x)$ is an antiderivative of $f(x)$, and $G(x)$ is an antiderivative of $g(x)$, give an antiderivative of $f(x) + g(x)$.

Problems

In Exercises 9 – 27, evaluate the given indefinite integral.

9. $\int 3x^3 \, dx$

10. $\int x^8 \, dx$

11. $\int (10x^2 - 2) \, dx$

12. $\int dt$

13. $\int 1 \, ds$

14. $\int \frac{1}{3t^2} \, dt$

15. $\int \frac{3}{t^2} \, dt$

16. $\int \frac{1}{\sqrt{x}} \, dx$

17. $\int \sec^2 \theta \, d\theta$

18. $\int \sin \theta \, d\theta$

19. $\int (\sec x \tan x + \csc x \cot x) \, dx$

20. $\int 5e^\theta \, d\theta$

21. $\int 3^t \, dt$

22. $\int \frac{5^t}{2} \, dt$

23. $\int (2t + 3)^2 \, dt$

24. $\int (t^2 + 3)(t^3 - 2t) \, dt$

25. $\int x^2 x^3 \, dx$

26. $\int e^\pi \, dx$

27. $\int a \, dx$

28. This problem investigates why Theorem 5.1.2 states that $\int \frac{1}{x} \, dx = \ln |x| + C$.

    (a) What is the domain of $y = \ln x$?
    (b) Find $\frac{d}{dx}\left(\ln x \right)$.
    (c) What is the domain of $y = \ln(-x)$?
    (d) Find $\frac{d}{dx}\left(\ln(-x) \right)$.
    (e) You should find that $1/x$ has two types of antiderivatives, depending on whether $x > 0$ or $x < 0$. In one expression, give a formula for $\int \frac{1}{x} \, dx$ that takes these different domains into account, and explain your answer.

In Exercises 29 – 39, find $f(x)$ described by the given initial value problem.

29. $f'(x) = \sin x$ and $f(0) = 2$

30. $f'(x) = 5e^x$ and $f(0) = 10$

31. $f'(x) = 4x^3 - 3x^2$ and $f(-1) = 9$

32. $f'(x) = \sec^2 x$ and $f(\pi/4) = 5$

33. $f'(x) = 7^x$ and $f(2) = 1$

34. $f''(x) = 5$ and $f'(0) = 7, f(0) = 3$

35. $f''(x) = 7x$ and $f'(1) = -1, f(1) = 10$

36. $f''(x) = 5e^x$ and $f'(0) = 3, f(0) = 5$

37. $f''(\theta) = \sin\theta$ and $f'(\pi) = 2, f(\pi) = 4$

38. $f''(x) = 24x^2 + 2^x - \cos x$ and $f'(0) = 5, f(0) = 0$

39. $f''(x) = 0$ and $f'(1) = 3, f(1) = 1$

Review

40. Use information gained from the first and second derivatives to sketch $f(x) = \dfrac{1}{e^x + 1}$.

41. Given $y = x^2 e^x \cos x$, find dy.

5.2 The Definite Integral

We start with an easy problem. An object travels in a straight line at a constant velocity of 5 ft/s for 10 seconds. How far away from its starting point is the object?

We approach this problem with the familiar "Distance = Rate × Time" equation. In this case, Distance = 5ft/s × 10s = 50 feet.

It is interesting to note that this solution of 50 feet can be represented graphically. Consider Figure 5.2.1, where the constant velocity of 5ft/s is graphed on the axes. Shading the area under the line from $t = 0$ to $t = 10$ gives a rectangle with an area of 50 square units; when one considers the units of the axes, we can say this area represents 50 ft.

Now consider a slightly harder situation (and not particularly realistic): an object travels in a straight line with a constant velocity of 5ft/s for 10 seconds, then instantly reverses course at a rate of 2ft/s for 4 seconds. (Since the object is traveling in the opposite direction when reversing course, we say the velocity is a constant -2ft/s.) How far away from the starting point is the object – what is its *displacement*?

Here we use "Distance = Rate$_1$ × Time$_1$ + Rate$_2$ × Time$_2$," which is

$$\text{Distance} = 5 \cdot 10 + (-2) \cdot 4 = 42 \text{ ft.}$$

Hence the object is 42 feet from its starting location.

We can again depict this situation graphically. In Figure 5.2.2 we have the velocities graphed as straight lines on $[0, 10]$ and $[10, 14]$, respectively. The displacement of the object is

"Area above the t–axis — Area below the t–axis,"

which is easy to calculate as $50 - 8 = 42$ feet.

Now consider a more difficult problem.

Example 5.2.1   Finding position using velocity

The velocity of an object moving straight up/down under the acceleration of gravity is given as $v(t) = -32t + 48$, where time t is given in seconds and velocity is in ft/s. When $t = 0$, the object had a height of 0 ft.

1. What was the initial velocity of the object?

2. What was the maximum height of the object?

3. What was the height of the object at time $t = 2$?

SOLUTION   It is straightforward to find the initial velocity; at time $t = 0$, $v(0) = -32 \cdot 0 + 48 = 48$ ft/s.
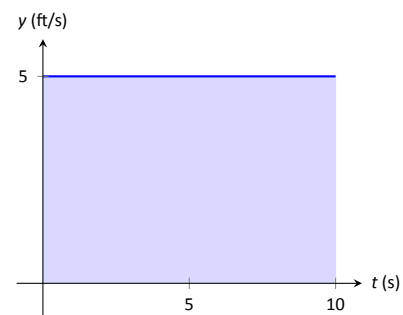

y (ft/s)

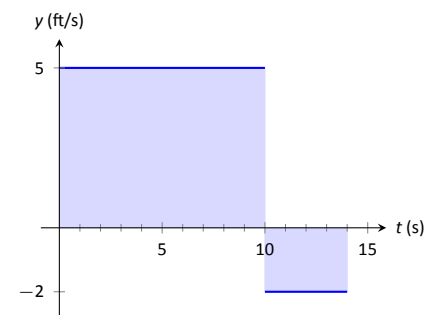Figure 5.2.1: The area under a constant velocity function corresponds to distance traveled.

y (ft/s)

Figure 5.2.2: The total displacement is the area above the t–axis minus the area below the t–axis.
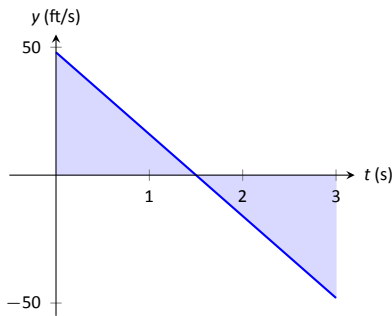

Notes:

To answer questions about the height of the object, we need to find the object's position function $s(t)$. This is an initial value problem, which we studied in the previous section. We are told the initial height is 0, i.e., $s(0) = 0$. We know $s'(t) = v(t) = -32t + 48$. To find s, we find the indefinite integral of $v(t)$:

$$\int v(t)\,dt = \int (-32t + 48)\,dt = -16t^2 + 48t + C = s(t).$$

Since $s(0) = 0$, we conclude that $C = 0$ and $s(t) = -16t^2 + 48t$.

To find the maximum height of the object, we need to find the maximum of s. Recalling our work finding extreme values, we find the critical points of s by setting its derivative equal to 0 and solving for t:

$$s'(t) = -32t + 48 = 0 \quad \Rightarrow \quad t = 48/32 = 1.5\text{s}.$$

(Notice how we ended up just finding when the velocity was 0ft/s!) The first derivative test shows this is a maximum, so the maximum height of the object is found at

$$s(1.5) = -16(1.5)^2 + 48(1.5) = 36\text{ft}.$$

The height at time $t = 2$ is now straightforward to compute: it is $s(2) = 32\text{ft}$.

While we have answered all three questions, let's look at them again graphically, using the concepts of area that we explored earlier.

Figure 5.2.3 shows a graph of $v(t)$ on axes from $t = 0$ to $t = 3$. It is again straightforward to find $v(0)$. How can we use the graph to find the maximum height of the object?

Recall how in our previous work that the displacement of the object (in this case, its height) was found as the area under the velocity curve, as shaded in the figure. Moreover, the area between the curve and the t–axis that is below the t–axis counted as "negative" area. That is, it represents the object coming back toward its starting position. So to find the maximum distance from the starting point – the maximum height – we find the area under the velocity line that is above the t–axis, i.e., from $t = 0$ to $t = 1.5$. This region is a triangle; its area is

$$\text{Area } = \frac{1}{2}\text{Base} \times \text{Height} = \frac{1}{2} \times 1.5\text{s} \times 48\text{ft/s} = 36\text{ft},$$

which matches our previous calculation of the maximum height.

Finally, to find the height of the object at time $t = 2$ we calculate the total signed area under the velocity function from $t = 0$ to $t = 2$. This signed area is equal to $s(2)$, the displacement (i.e., signed distance) from the starting position at $t = 0$ to the position at time $t = 2$. That is,

Displacement = Area above the t–axis $-$ Area below t–axis.

Figure 5.2.3: A graph of $v(t) = -32t + 48$; the shaded areas help determine displacement.

Notes:

The regions are triangles, and we find

$$\text{Displacement} = \frac{1}{2}(1.5\text{s})(48\text{ft/s}) - \frac{1}{2}(.5\text{s})(16\text{ft/s}) = 32\text{ft}.$$

This also matches our previous calculation of the height at $t = 2$.

Notice how we answered each question in this example in two ways. Our first method was to manipulate equations using our understanding of antiderivatives and derivatives. Our second method was geometric: we answered questions looking at a graph and finding the areas of certain regions of this graph.

The above example does not *prove* a relationship between area under a velocity function and displacement, but it does imply a relationship exists. Section 5.4 will fully establish fact that the area under a velocity function is displacement.

Given a graph of a function $y = f(x)$, we will find that there is great use in computing the area between the curve $y = f(x)$ and the x-axis. Because of this, we need to define some terms.

Definition 5.2.1   The Definite Integral, Total Signed Area

Let $y = f(x)$ be defined on a closed interval $[a, b]$. The **total signed area from $x = a$ to $x = b$ under f** is:
(area under f and above the x–axis on $[a, b]$) − (area above f and under the x–axis on $[a, b]$).

The **definite integral of f on $[a, b]$** is the total signed area of f on $[a, b]$, denoted

$$\int_a^b f(x)\, dx,$$

where a and b are the **bounds of integration.**

By our definition, the definite integral gives the "signed area under f." We usually drop the word "signed" when talking about the definite integral, and simply say the definite integral gives "the area under f" or, more commonly, "the area under the curve."

The previous section introduced the indefinite integral, which related to antiderivatives. We have now defined the definite integral, which relates to areas under a function. The two are very much related, as we'll see when we learn the Fundamental Theorem of Calculus in Section 5.4. Recall that earlier we said that the "$\int$" symbol was an "elongated S" that represented finding a "sum." In the context of the definite integral, this notation makes a bit more sense, as we are adding up areas under the function f.
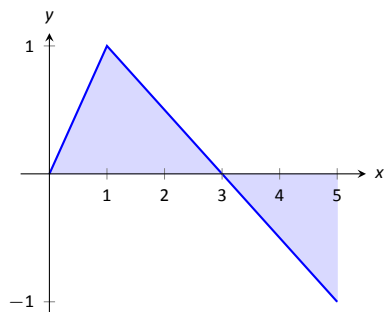

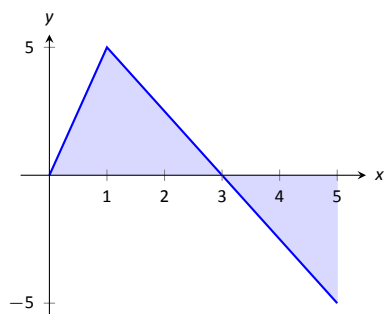Notes:

Figure 5.2.4: A graph of $f(x)$ in Example 5.2.2.

Figure 5.2.5: A graph of $5f$ in Example 5.2.2. (Yes, it looks just like the graph of f in Figure 5.2.4, just with a different y-scale.)

We practice using this notation.

Example 5.2.2   Evaluating definite integrals
Consider the function f given in Figure 5.2.4.

Find:

1. $\displaystyle\int_0^3 f(x)\,dx$

2. $\displaystyle\int_3^5 f(x)\,dx$

3. $\displaystyle\int_0^5 f(x)\,dx$

4. $\displaystyle\int_0^3 5f(x)\,dx$

5. $\displaystyle\int_1^1 f(x)\,dx$

SOLUTION

1. $\int_0^3 f(x)\,dx$ is the area under f on the interval $[0,3]$. This region is a triangle, so the area is $\int_0^3 f(x)\,dx = \frac{1}{2}(3)(1) = 1.5$.

2. $\int_3^5 f(x)\,dx$ represents the area of the triangle found under the x–axis on $[3,5]$. The area is $\frac{1}{2}(2)(1) = 1$; since it is found *under* the x–axis, this is "negative area." Therefore $\int_3^5 f(x)\,dx = -1$.

3. $\int_0^5 f(x)\,dx$ is the total signed area under f on $[0,5]$. This is $1.5+(-1) = 0.5$.

4. $\int_0^3 5f(x)\,dx$ is the area under $5f$ on $[0,3]$. This is sketched in Figure 5.2.5. Again, the region is a triangle, with height 5 times that of the height of the original triangle. Thus the area is $\int_0^3 5f(x)\,dx = 15/2 = 7.5$.

5. $\int_1^1 f(x)\,dx$ is the area under f on the "interval" $[1,1]$. This describes a line segment, not a region; it has no width. Therefore the area is 0.

This example illustrates some of the properties of the definite integral, given here.

Notes:

Theorem 5.2.1   Properties of the Definite Integral

Let f and g be defined on a closed interval I that contains the values a, b and c, and let k be a constant. The following hold:

1. $\displaystyle\int_a^a f(x)\, dx = 0$

2. $\displaystyle\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx$

3. $\displaystyle\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

4. $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$

5. $\displaystyle\int_a^b k \cdot f(x)\, dx = k \cdot \int_a^b f(x)\, dx$

We give a brief justification of Theorem 5.2.1 here.

1. As demonstrated in Example 5.2.2, there is no "area under the curve" when the region has no width; hence this definite integral is 0.

2. This states that total area is the sum of the areas of subregions. It is easily considered when we let $a < b < c$. We can break the interval $[a, c]$ into two subintervals, $[a, b]$ and $[b, c]$. The total area over $[a, c]$ is the area over $[a, b]$ plus the area over $[b, c]$.

   It is important to note that this still holds true even if $a < b < c$ is not true. We discuss this in the next point.

3. This property can be viewed a merely a convention to make other properties work well. (Later we will see how this property has a justification all its own, not necessarily in support of other properties.) Suppose $b < a < c$. The discussion from the previous point clearly justifies

$$\int_b^a f(x)\, dx + \int_a^c f(x)\, dx = \int_b^c f(x)\, dx. \tag{5.1}$$

   However, we still claim that, as originally stated,

$$\int_a^b f(x)\, dx + \int_b^c f(x)\, dx = \int_a^c f(x)\, dx. \tag{5.2}$$

Notes:

How do Equations (5.1) and (5.2) relate? Start with Equation (5.1):

$$\int_b^a f(x)\, dx + \int_a^c f(x)\, dx = \int_b^c f(x)\, dx$$

$$\int_a^c f(x)\, dx = -\int_b^a f(x)\, dx + \int_b^c f(x)\, dx$$

Property (3) justifies changing the sign and switching the bounds of integration on the $-\int_b^a f(x)\, dx$ term; when this is done, Equations (5.1) and (5.2) are equivalent.

The conclusion is this: by adopting the convention of Property (3), Property (2) holds no matter the order of a, b and c. Again, in the next section we will see another justification for this property.

4,5. Each of these may be non–intuitive. Property (5) states that when one scales a function by, for instance, 7, the area of the enclosed region also is scaled by a factor of 7. Both Properties (4) and (5) can be proved using geometry. The details are not complicated but are not discussed here.

Example 5.2.3   **Evaluating definite integrals using Theorem 5.2.1.**
Consider the graph of a function $f(x)$ shown in Figure 5.2.6. Answer the following:
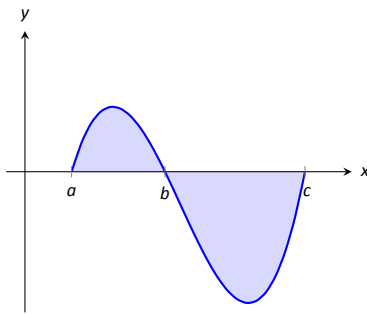
1. Which value is greater: $\int_a^b f(x)\, dx$ or $\int_b^c f(x)\, dx$?

2. Is $\int_a^c f(x)\, dx$ greater or less than 0?

3. Which value is greater: $\int_a^b f(x)\, dx$ or $\int_c^b f(x)\, dx$?

SOLUTION

1. $\int_a^b f(x)\, dx$ has a positive value (since the area is above the x–axis) whereas $\int_b^c f(x)\, dx$ has a negative value. Hence $\int_a^b f(x)\, dx$ is bigger.

2. $\int_a^c f(x)\, dx$ is the total signed area under f between $x = a$ and $x = c$. Since the region below the x–axis looks to be larger than the region above, we conclude that the definite integral has a value less than 0.

3. Note how the second integral has the bounds "reversed." Therefore $\int_c^b f(x)\, dx$ represents a positive number, greater than the area described by the first definite integral. Hence $\int_c^b f(x)\, dx$ is greater.

Figure 5.2.6: A graph of a function in Example 5.2.3.

Notes:

The area definition of the definite integral allows us to use geometry to compute the definite integral of some simple functions.

Example 5.2.4 Evaluating definite integrals using geometry

Evaluate the following definite integrals:

$$1.\ \int_{-2}^{5} (2x - 4)\, dx \qquad 2.\ \int_{-3}^{3} \sqrt{9 - x^2}\, dx.$$

SOLUTION

1. It is useful to sketch the function in the integrand, as shown in Figure 5.2.7(a). We see we need to compute the areas of two regions, which we have labeled R_1 and R_2. Both are triangles, so the area computation is straightforward:

$$R_1 : \frac{1}{2}(4)(8) = 16 \qquad R_2 : \frac{1}{2}(3)6 = 9.$$

Region R_1 lies under the x–axis, hence it is counted as negative area (we can think of the triangle's height as being "-8"), so

$$\int_{-2}^{5} (2x - 4)\, dx = -16 + 9 = -7.$$

2. Recognize that the integrand of this definite integral describes a half circle, as sketched in Figure 5.2.7(b), with radius 3. Thus the area is:

$$\int_{-3}^{3} \sqrt{9 - x^2}\, dx = \frac{1}{2}\pi r^2 = \frac{9}{2}\pi.$$

### Example 5.2.5 Understanding motion given velocity

Consider the graph of a velocity function of an object moving in a straight line, given in Figure 5.2.8, where the numbers in the given regions gives the area of that region. Assume that the definite integral of a velocity function gives displacement. Find the maximum speed of the object and its maximum displacement from its starting position.

SOLUTION Since the graph gives velocity, finding the maximum speed is simple: it looks to be 15ft/s.

At time $t = 0$, the displacement is 0; the object is at its starting position. At time $t = a$, the object has moved backward 11 feet. Between times $t = a$ and
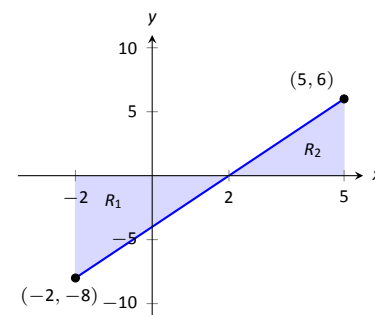
(a)
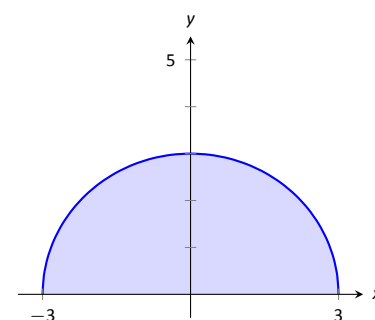
(b)

Figure 5.2.7: A graph of $f(x) = 2x - 4$ in (a) and $f(x) = \sqrt{9 - x^2}$ in (b), from Example 5.2.4.
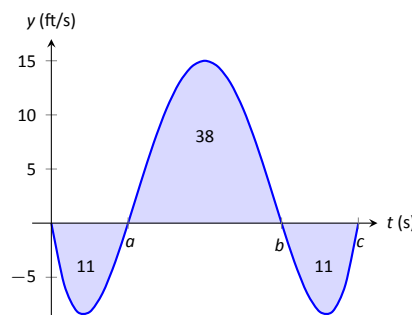
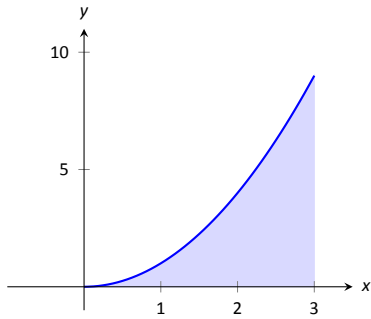Figure 5.2.8: A graph of a velocity in Example 5.2.5.

Notes:

Figure 5.2.9: What is the area below $y = x^2$ on $[0, 3]$? The region is not a usual geometric shape.

$t = b$, the object moves forward 38 feet, bringing it into a position 27 feet forward of its starting position. From $t = b$ to $t = c$ the object is moving backwards again, hence its maximum displacement is 27 feet from its starting position.

In our examples, we have either found the areas of regions that have nice geometric shapes (such as rectangles, triangles and circles) or the areas were given to us. Consider Figure 5.2.9, where a region below $y = x^2$ is shaded. What is its area? The function $y = x^2$ is relatively simple, yet the shape it defines has an area that is not simple to find geometrically.

In the next section we will explore how to find the areas of such regions.

Notes:

Exercises 5.2

Terms and Concepts

05 02 ex 01

1. What is "total signed area"?

05 02 ex 02

2. What is "displacement"?

05 02 ex 03

3. What is $\int_3^3 \sin x\, dx$?
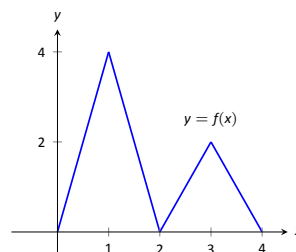
05 02 ex 04

4. Give a single definite integral that has the same value as
$$\int_0^1 (2x+3)\, dx + \int_1^2 (2x+3)\, dx.$$

05 02 ex 07

7.

(a) $\int_0^2 f(x)\, dx$

(d) $\int_0^1 4x\, dx$
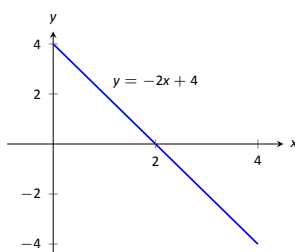
(b) $\int_2^4 f(x)\, dx$

(e) $\int_2^3 (2x-4)\, dx$

(c) $\int_2^4 2f(x)\, dx$

(f) $\int_2^3 (4x-8)\, dx$

Problems

05 02 exset 01

In Exercises 5 – 10, a graph of a function $f(x)$ is given. Using the geometry of the graph, evaluate the definite integrals.

05 02 ex 05
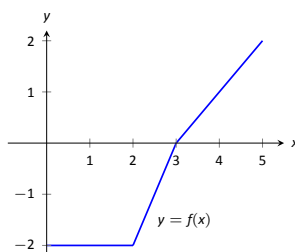
5.

(a) $\int_0^1 (-2x+4)\, dx$

(d) $\int_1^3 (-2x+4)\, dx$

(b) $\int_0^2 (-2x+4)\, dx$

(e) $\int_2^4 (-2x+4)\, dx$

(c) $\int_0^3 (-2x+4)\, dx$

(f) $\int_0^1 (-6x+12)\, dx$

05 02 ex 08

8.

(a) $\int_0^1 (x-1)\, dx$

(d) $\int_2^3 (x-1)\, dx$

(b) $\int_0^2 (x-1)\, dx$

(e) $\int_1^4 (x-1)\, dx$

(c) $\int_0^3 (x-1)\, dx$

(f) $\int_1^4 \big((x-1)+1\big)\, dx$

05 02 ex 06

6.

(a) $\int_0^2 f(x)\, dx$

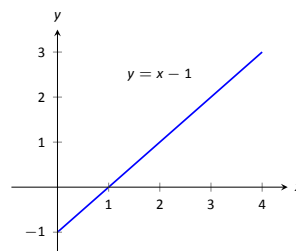(d) $\int_2^5 f(x)\, dx$

(b) $\int_0^3 f(x)\, dx$

(e) $\int_5^3 f(x)\, dx$

(c) $\int_0^5 f(x)\, dx$

(f) $\int_0^3 -2f(x)\, dx$

05 02 ex 09

9.

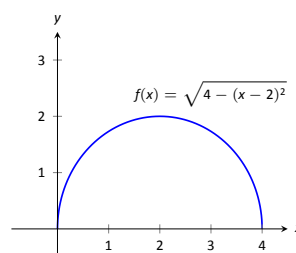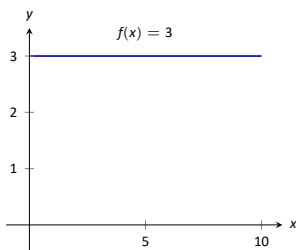
(a) $\int_0^2 f(x)\, dx$

(c) $\int_0^4 f(x)\, dx$

(b) $\int_2^4 f(x)\, dx$

(d) $\int_0^4 5f(x)\, dx$

10.

$f(x) = 3$

(a) $\displaystyle\int_0^5 f(x)\, dx$

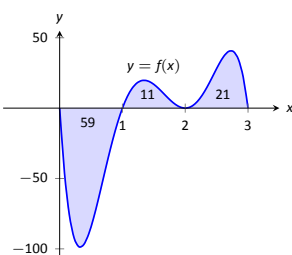(c) $\displaystyle\int_0^0 f(x)\, dx$

(b) $\displaystyle\int_3^7 f(x)\, dx$

(d) $\displaystyle\int_a^b f(x)\, dx$, where $0 \le a \le b \le 10$

In Exercises 11 – 14, a graph of a function $f(x)$ is given; the numbers inside the shaded regions give the area of that region. Evaluate the definite integrals using this area information.
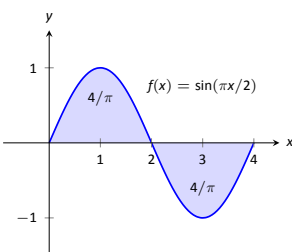
11.

$y = f(x)$

(a) $\displaystyle\int_0^1 f(x)\, dx$

(c) $\displaystyle\int_0^3 f(x)\, dx$

(b) $\displaystyle\int_0^2 f(x)\, dx$

(d) $\displaystyle\int_1^2 -3f(x)\, dx$

12.

$f(x) = \sin(\pi x/2)$

(a) $\displaystyle\int_0^2 f(x)\, dx$

(c) $\displaystyle\int_0^4 f(x)\, dx$

(b) $\displaystyle\int_2^4 f(x)\, dx$

(d) $\displaystyle\int_0^1 f(x)\, dx$

13.

$f(x) = 3x^2 - 3$

(a) $\displaystyle\int_{-2}^{-1} f(x)\, dx$

(c) $\displaystyle\int_{-1}^1 f(x)\, dx$

(b) $\displaystyle\int_1^2 f(x)\, dx$

(d) $\displaystyle\int_0^1 f(x)\, dx$

14.

$f(x) = x^2$
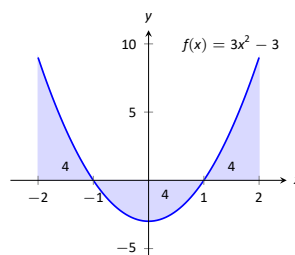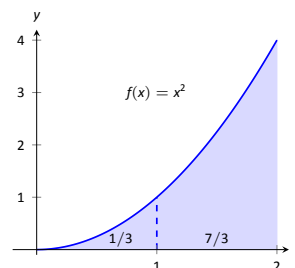
(a) $\displaystyle\int_0^2 5x^2\, dx$

(c) $\displaystyle\int_1^3 (x-1)^2\, dx$

(b) $\displaystyle\int_0^2 (x^2 + 3)\, dx$

(d) $\displaystyle\int_2^4 \left((x-2)^2 + 5\right) dx$

In Exercises 15 – 16, a graph of the velocity function of an object moving in a straight line is given. Answer the questions based on that graph.

15.
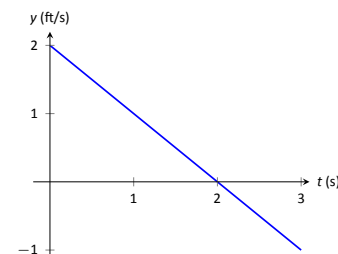
(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0, 3]$?

16.
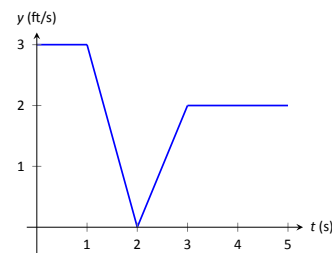
(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) What is the object's total displacement on $[0, 5]$?

17. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 64$, where t is in seconds, from a height of 48 feet.

(a) What is the object's maximum velocity?

(b) What is the object's maximum displacement?

(c) When does the maximum displacement occur?

(d) When will the object reach a height of 0? (Hint: find when the displacement is -48ft.)

18. An object is thrown straight up with a velocity, in ft/s, given by $v(t) = -32t + 96$, where t is in seconds, from a height of 64 feet.

(a) What is the object's initial velocity?

(b) When is the object's displacement 0?

(c) How long does it take for the object to return to its initial height?

(d) When will the object reach a height of 210 feet?

In Exercises 19 – 22, let

- $\int_0^2 f(x)\, dx = 5,$

- $\int_0^3 f(x)\, dx = 7,$

- $\int_0^2 g(x)\, dx = -3,$ **and**

- $\int_2^3 g(x)\, dx = 5.$

Use these values to evaluate the given definite integrals.

19. $\int_0^2 \big(f(x) + g(x)\big)\, dx$

20. $\int_0^3 \big(f(x) - g(x)\big)\, dx$

21. $\int_2^3 \big(3f(x) + 2g(x)\big)\, dx$

22. Find nonzero values for a and b such that
$$\int_0^3 \big(af(x) + bg(x)\big)\, dx = 0$$

In Exercises 23 – 26, let

- $\int_0^3 s(t)\, dt = 10,$

- $\int_3^5 s(t)\, dt = 8,$

- $\int_3^5 r(t)\, dt = -1,$ **and**

- $\int_0^5 r(t)\, dt = 11.$

Use these values to evaluate the given definite integrals.

23. $\int_0^3 \big(s(t) + r(t)\big)\, dt$

24. $\int_5^0 \big(s(t) - r(t)\big)\, dt$

25. $\int_3^3 \big(\pi s(t) - 7r(t)\big)\, dt$

26. Find nonzero values for a and b such that
$$\int_0^5 \big(ar(t) + bs(t)\big)\, dt = 0$$

Review

In Exercises 27 – 30, evaluate the given indefinite integral.

27. $\int \big(x^3 - 2x^2 + 7x - 9\big)\, dx$

28. $\int \big(\sin x - \cos x + \sec^2 x\big)\, dx$

29. $\int \Big(\sqrt[3]{t} + \dfrac{1}{t^2} + 2^t\Big)\, dt$

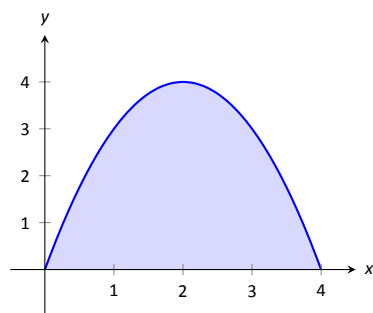30. $\int \Big(\dfrac{1}{x} - \csc x \cot x\Big)\, dx$

Figure 5.3.1: A graph of $f(x) = 4x - x^2$. What is the area of the shaded region?
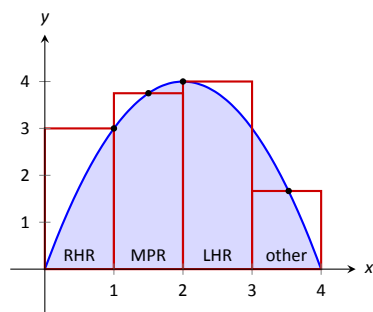

Figure 5.3.2: Approximating $\int_0^4 (4x - x^2)\, dx$ using rectangles. The heights of the rectangles are determined using different rules.

5.3   Riemann Sums

In the previous section we defined the definite integral of a function on $[a, b]$ to be the signed area between the curve and the x–axis. Some areas were simple to compute; we ended the section with a region whose area was not simple to compute. In this section we develop a technique to find such areas.

A fundamental calculus technique is to first answer a given problem with an approximation, then refine that approximation to make it better, then use limits in the refining process to find the exact answer. That is exactly what we will do here.

Consider the region given in Figure 5.3.1, which is the area under $y = 4x - x^2$ on $[0, 4]$. What is the signed area of this region – i.e., what is $\int_0^4 (4x - x^2)\, dx$?

We start by approximating. We can surround the region with a rectangle with height and width of 4 and find the area is approximately 16 square units. This is obviously an *over–approximation*; we are including area in the rectangle that is not under the parabola.

We have an approximation of the area, using one rectangle. How can we refine our approximation to make it better? The key to this section is this answer: *use more rectangles.*

Let's use 4 rectangles with an equal width of 1. This *partitions* the interval $[0, 4]$ into 4 *subintervals*, $[0, 1]$, $[1, 2]$, $[2, 3]$ and $[3, 4]$. On each subinterval we will draw a rectangle.

There are three common ways to determine the height of these rectangles: the **Left Hand Rule**, the **Right Hand Rule**, and the **Midpoint Rule**. The **Left Hand Rule** says to evaluate the function at the left–hand endpoint of the subinterval and make the rectangle that height. In Figure 5.3.2, the rectangle drawn on the interval $[2, 3]$ has height determined by the Left Hand Rule; it has a height of $f(2)$. (The rectangle is labeled "LHR.")

The **Right Hand Rule** says the opposite: on each subinterval, evaluate the function at the right endpoint and make the rectangle that height. In the figure, the rectangle drawn on $[0, 1]$ is drawn using $f(1)$ as its height; this rectangle is labeled "RHR.".

The **Midpoint Rule** says that on each subinterval, evaluate the function at the midpoint and make the rectangle that height. The rectangle drawn on $[1, 2]$ was made using the Midpoint Rule, with a height of $f(1.5)$. That rectangle is labeled "MPR."

These are the three most common rules for determining the heights of approximating rectangles, but one is not forced to use one of these three methods. The rectangle on $[3, 4]$ has a height of approximately $f(3.53)$, very close to the Midpoint Rule. It was chosen so that the area of the rectangle is *exactly* the area of the region under f on $[3, 4]$. (Later you'll be able to figure how to do this, too.)

The following example will approximate the value of $\int_0^4 (4x - x^2)\, dx$ using

Notes:

these rules.

Example 5.3.1   **Using the Left Hand, Right Hand and Midpoint Rules**
Approximate the value of $\int_0^4 (4x - x^2)\, dx$ using the Left Hand Rule, the Right Hand Rule, and the Midpoint Rule, using 4 equally spaced subintervals.

SOLUTION   We break the interval $[0, 4]$ into four subintervals as before. In Figure 5.3.3 we see 4 rectangles drawn on $f(x) = 4x - x^2$ using the Left Hand Rule. (The areas of the rectangles are given in each figure.)
Note how in the first subinterval, $[0, 1]$, the rectangle has height $f(0) = 0$. We add up the areas of each rectangle (height$\times$ width) for our Left Hand Rule approximation:

$$f(0) \cdot 1 + f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 =$$
$$0 + 3 + 4 + 3 = 10.$$

Figure 5.3.4 shows 4 rectangles drawn under f using the Right Hand Rule; note how the $[3, 4]$ subinterval has a rectangle of height 0.
In this example, these rectangle seem to be the mirror image of those found in Figure 5.3.3. (This is because of the symmetry of our shaded region.) Our approximation gives the same answer as before, though calculated a different way:

$$f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 =$$
$$3 + 4 + 3 + 0 = 10.$$

Figure 5.3.5 shows 4 rectangles drawn under f using the Midpoint Rule. This gives an approximation of $\int_0^4 (4x - x^2)\, dx$ as:

$$f(0.5) \cdot 1 + f(1.5) \cdot 1 + f(2.5) \cdot 1 + f(3.5) \cdot 1 =$$
$$1.75 + 3.75 + 3.75 + 1.75 = 11.$$

Our three methods provide two approximations of $\int_0^4 (4x - x^2)\, dx$: 10 and 11.

Summation Notation

It is hard to tell at this moment which is a better approximation: 10 or 11? We can continue to refine our approximation by using more rectangles. The notation can become unwieldy, though, as we add up longer and longer lists of numbers. We introduce **summation notation** to ameliorate this problem.
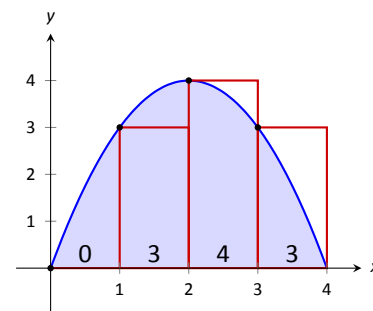

Figure 5.3.3: Approximating $\int_0^4 (4x - x^2)\, dx$ using the Left Hand Rule in Example 5.3.1.
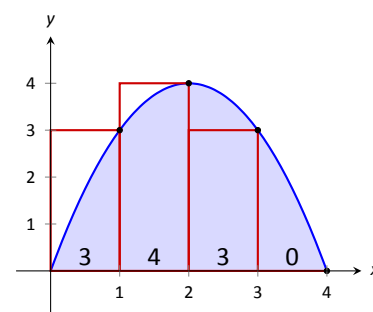

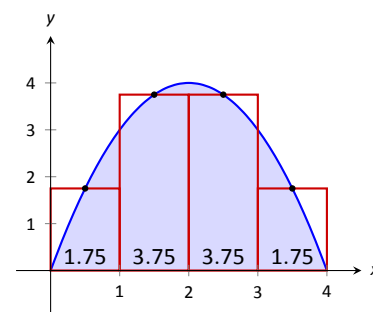Figure 5.3.4: Approximating $\int_0^4 (4x - x^2)\, dx$ using the Right Hand Rule in Example 5.3.1.

Figure 5.3.5: Approximating $\int_0^4 (4x - x^2)\, dx$ using the Midpoint Rule in Example 5.3.1.

Notes:

Suppose we wish to add up a list of numbers a_1, a_2, a_3, ..., a_9. Instead of writing

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9,$$

we use summation notation and write

Figure 5.3.6: Understanding summation notation.

The upper case sigma represents the term "sum." The index of summation in this example is i; any symbol can be used. By convention, the index takes on only the integer values between (and including) the lower and upper bounds.

Let's practice using this notation.

Example 5.3.2   Using summation notation
Let the numbers $\{a_i\}$ be defined as $a_i = 2i - 1$ for integers i, where $i \geq 1$. So $a_1 = 1$, $a_2 = 3$, $a_3 = 5$, etc. (The output is the positive odd integers). Evaluate the following summations:

$$1. \sum_{i=1}^{6} a_i \qquad\qquad 2. \sum_{i=3}^{7}(3a_i - 4) \qquad\qquad 3. \sum_{i=1}^{4}(a_i)^2$$

Solution

1.
$$\sum_{i=1}^{6} a_i = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$$
$$= 1 + 3 + 5 + 7 + 9 + 11$$
$$= 36.$$

2. Note the starting value is different than 1:

$$\sum_{i=3}^{7}(3a_i - 4) = (3a_3 - 4) + (3a_4 - 4) + (3a_5 - 4) + (3a_6 - 4) + (3a_7 - 4)$$
$$= 11 + 17 + 23 + 29 + 35$$
$$= 115.$$

Notes:

3.

$$\sum_{i=1}^{4}(a_i)^2 = (a_1)^2 + (a_2)^2 + (a_3)^2 + (a_4)^2$$
$$= 1^2 + 3^2 + 5^2 + 7^2$$
$$= 84$$

It might seem odd to stress a new, concise way of writing summations only to write each term out as we add them up. It is. The following theorem gives some of the properties of summations that allow us to work with them without writing individual terms. Examples will follow.

Theorem 5.3.1   Properties of Summations

1. $\displaystyle\sum_{i=1}^{n} c = c \cdot n$, where c is a constant.

2. $\displaystyle\sum_{i=m}^{n}(a_i \pm b_i) = \sum_{i=m}^{n} a_i \pm \sum_{i=m}^{n} b_i$

3. $\displaystyle\sum_{i=m}^{n} c \cdot a_i = c \cdot \sum_{i=m}^{n} a_i$

4. $\displaystyle\sum_{i=m}^{j} a_i + \sum_{i=j+1}^{n} a_i = \sum_{i=m}^{n} a_i$

5. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

6. $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

7. $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

Example 5.3.3   Evaluating summations using Theorem 5.3.1
Revisit Example 5.3.2 and, using Theorem 5.3.1, evaluate

$$\sum_{i=1}^{6} a_i = \sum_{i=1}^{6}(2i - 1).$$

Notes:

SOLUTION

$$\sum_{i=1}^{6}(2i-1) = \sum_{i=1}^{6}2i - \sum_{i=1}^{6}(1)$$

$$= \left(2\sum_{i=1}^{6}i\right) - 6$$

$$= 2\frac{6(6+1)}{2} - 6$$

$$= 42 - 6 = 36$$

We obtained the same answer without writing out all six terms. When dealing with small sizes of n, it may be faster to write the terms out by hand. However, Theorem 5.3.1 is incredibly important when dealing with large sums as we'll soon see.

Riemann Sums

Consider again $\int_0^4 (4x - x^2)\,dx$. We will approximate this definite integral using 16 equally spaced subintervals and the Right Hand Rule in Example 5.3.4. Before doing so, it will pay to do some careful preparation.

Figure 5.3.7 shows a number line of $[0,4]$ divided, or *partitioned*, into 16 equally spaced subintervals. We denote 0 as x_1; we have marked the values of x_5, x_9, x_{13} and x_{17}. We could mark them all, but the figure would get crowded. While it is easy to figure that $x_{10} = 2.25$, in general, we want a method of determining the value of x_i without consulting the figure. Consider:

$$x_i = x_1 + (i-1)\Delta x$$

starting value · subinterval size

number of subintervals between x_1 and x_i

So $x_{10} = x_1 + 9(4/16) = 2.25$.

If we had partitioned $[0,4]$ into 100 equally spaced subintervals, each subinterval would have length $\Delta x = 4/100 = 0.04$. We could compute x_{32} as

$$x_{32} = x_1 + 31(4/100) = 1.24.$$

(That was far faster than creating a sketch first.)

Notes:
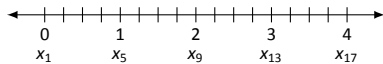
Figure 5.3.7:   Dividing $[0,4]$ into 16 equally spaced subintervals.

Given any subdivision of $[0, 4]$, the first subinterval is $[x_1, x_2]$; the second is $[x_2, x_3]$; the ith subinterval is $[x_i, x_{i+1}]$.

When using the Left Hand Rule, the height of the ith rectangle will be $f(x_i)$.

When using the Right Hand Rule, the height of the ith rectangle will be $f(x_{i+1})$.

When using the Midpoint Rule, the height of the ith rectangle will be $f\left(\dfrac{x_i + x_{i+1}}{2}\right)$.

Thus approximating $\int_0^4 (4x - x^2)\, dx$ with 16 equally spaced subintervals can be expressed as follows, where $\Delta x = 4/16 = 1/4$:

Left Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_i) \Delta x$

Right Hand Rule: $\displaystyle\sum_{i=1}^{16} f(x_{i+1}) \Delta x$

Midpoint Rule: $\displaystyle\sum_{i=1}^{16} f\left(\dfrac{x_i + x_{i+1}}{2}\right) \Delta x$

We use these formulas in the next two examples. The following example lets us practice using the Right Hand Rule and the summation formulas introduced in Theorem 5.3.1.

Example 5.3.4   Approximating definite integrals using sums
Approximate $\int_0^4 (4x - x^2)\, dx$ using the Right Hand Rule and summation formulas with 16 and 1000 equally spaced intervals.

SOLUTION   Using the formula derived before, using 16 equally spaced intervals and the Right Hand Rule, we can approximate the definite integral as

$$\sum_{i=1}^{16} f(x_{i+1}) \Delta x.$$

We have $\Delta x = 4/16 = 0.25$. Since $x_i = 0 + (i-1)\Delta x$, we have

$$\begin{aligned}
x_{i+1} &= 0 + \big((i+1) - 1\big)\Delta x \\
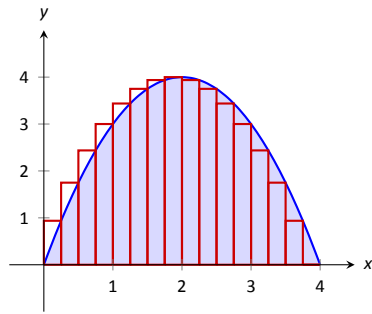&= i \Delta x
\end{aligned}$$

Notes:

Using the summation formulas, consider:

$$\int_0^4 (4x - x^2)\,dx \approx \sum_{i=1}^{16} f(x_{i+1})\Delta x$$

$$= \sum_{i=1}^{16} f(i\Delta x)\Delta x$$

$$= \sum_{i=1}^{16} \left(4i\Delta x - (i\Delta x)^2\right)\Delta x$$

$$= \sum_{i=1}^{16} (4i\Delta x^2 - i^2\Delta x^3)$$

$$= (4\Delta x^2)\sum_{i=1}^{16} i - \Delta x^3\sum_{i=1}^{16} i^2 \qquad (5.3)$$

$$= (4\Delta x^2)\frac{16\cdot 17}{2} - \Delta x^3\frac{16(17)(33)}{6} \quad (\Delta x = 0.25)$$

$$= 10.625$$

We were able to sum up the areas of 16 rectangles with very little computation. In Figure 5.3.8 the function and the 16 rectangles are graphed. While some rectangles over–approximate the area, other under–approximate the area (by about the same amount). Thus our approximate area of 10.625 is likely a fairly good approximation.

Notice Equation (5.3); by changing the 16's to 1,000's (and appropriately changing the value of Δx), we can use that equation to sum up 1000 rectangles! We do so here, skipping from the original summand to the equivalent of Equation (5.3) to save space. Note that $\Delta x = 4/1000 = 0.004$.

$$\int_0^4 (4x - x^2)\,dx \approx \sum_{i=1}^{1000} f(x_{i+1})\Delta x$$

$$= (4\Delta x^2)\sum_{i=1}^{1000} i - \Delta x^3\sum_{i=1}^{1000} i^2$$

$$= (4\Delta x^2)\frac{1000\cdot 1001}{2} - \Delta x^3\frac{1000(1001)(2001)}{6}$$

$$= 10.666656$$

Using many, many rectangles, we have a likely good approximation of $\int_0^4 (4x - x^2)\Delta x$. That is,

$$\int_0^4 (4x - x^2)\,dx \approx 10.666656.$$

Figure 5.3.8:   Approximating $\int_0^4 (4x - x^2)\,dx$ with the Right Hand Rule and 16 evenly spaced subintervals.

Notes:

Before the above example, we stated what the summations for the Left Hand, Right Hand and Midpoint Rules looked like. Each had the same basic structure, which was:

1. each rectangle has the same width, which we referred to as Δx, and

2. each rectangle's height is determined by evaluating f at a particular point in each subinterval. For instance, the Left Hand Rule states that each rectangle's height is determined by evaluating f at the left hand endpoint of the subinterval the rectangle lives on.

One could partition an interval $[a, b]$ with subintervals that do not have the same size. We refer to the length of the ith subinterval as Δx_i. Also, one could determine each rectangle's height by evaluating f at *any* point c_i in the ith subinterval. Thus the height of the ith subinterval would be $f(c_i)$, and the area of the ith rectangle would be $f(c_i)\Delta x_i$. These ideas are formally defined below.

Definition 5.3.1   Partition

A **partition** Δx of a closed interval $[a, b]$ is a set of numbers $x_1, x_2, \ldots x_{n+1}$ where
$$a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b.$$

The length of the ith subinterval, $[x_i, x_{i+1}]$, is $\Delta x_i = x_{i+1} - x_i$. If $[a, b]$ is partitioned into subintervals of equal length, we let Δx represent the length of each subinterval.

The **size of the partition**, denoted $||\Delta x||$, is the length of the largest subinterval of the partition.

Summations of rectangles with area $f(c_i)\Delta x_i$ are named after mathematician Georg Friedrich Bernhard Riemann, as given in the following definition.

Definition 5.3.2   Riemann Sum

Let f be defined on a closed interval $[a, b]$ and let Δx be a partition of $[a, b]$, and let c_i denote any value in the ith subinterval.
The sum
$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

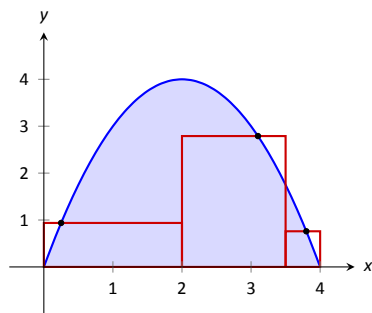is a **Riemann sum** of f on $[a, b]$.

Notes:

Figure 5.3.9: An example of a general Riemann sum to approximate $\int_0^4 (4x - x^2)\, dx$.

Figure 5.3.9 shows the approximating rectangles of a Riemann sum of $\int_0^4 (4x - x^2)\, dx$. While the rectangles in this example do not approximate well the shaded area, they demonstrate that the subinterval widths may vary and the heights of the rectangles can be determined without following a particular rule.

"Usually" Riemann sums are calculated using one of the three methods we have introduced. The uniformity of construction makes computations easier. Before working another example, let's summarize some of what we have learned in a convenient way.

Key Idea 5.3.1   Riemann Sum Concepts

Consider $\displaystyle\int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(c_i)\Delta x_i.$

1. When the n subintervals have equal length, $\Delta x_i = \Delta x = \dfrac{b - a}{n}.$

2. The ith term of an equally spaced partition is $x_i = a + (i - 1)\Delta x.$ (This makes $x_{n+1} = b.$)

3. The Left Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_i)\Delta x.$

4. The Right Hand Rule summation is: $\displaystyle\sum_{i=1}^{n} f(x_{i+1})\Delta x.$

5. The Midpoint Rule summation is: $\displaystyle\sum_{i=1}^{n} f\left(\dfrac{x_i + x_{i+1}}{2}\right)\Delta x.$

Let's do another example.

Example 5.3.5   Approximating definite integrals with sums
Approximate $\int_{-2}^{3} (5x + 2)\, dx$ using the Midpoint Rule and 10 equally spaced intervals.

SOLUTION   Following Key Idea 5.3.1, we have

$$\Delta x = \frac{3 - (-2)}{10} = 1/2 \quad \text{and} \quad x_i = (-2) + (1/2)(i - 1) = i/2 - 5/2.$$

Notes:

As we are using the Midpoint Rule, we will also need x_{i+1} and $\dfrac{x_i + x_{i+1}}{2}$. Since $x_i = i/2 - 5/2$, $x_{i+1} = (i+1)/2 - 5/2 = i/2 - 2$. This gives

$$\frac{x_i + x_{i+1}}{2} = \frac{(i/2 - 5/2) + (i/2 - 2)}{2} = \frac{i - 9/2}{2} = i/2 - 9/4.$$

We now construct the Riemann sum and compute its value using summation formulas.

$$\int_{-2}^{3} (5x + 2)\, dx \approx \sum_{i=1}^{10} f\left(\frac{x_i + x_{i+1}}{2}\right) \Delta x$$

$$= \sum_{i=1}^{10} f(i/2 - 9/4)\Delta x$$

$$= \sum_{i=1}^{10} \left(5(i/2 - 9/4) + 2\right)\Delta x$$

$$= \Delta x \sum_{i=1}^{10} \left[\left(\frac{5}{2}\right)i - \frac{37}{4}\right]$$

$$= \Delta x \left(\frac{5}{2}\sum_{i=1}^{10}(i) - \sum_{i=1}^{10}\left(\frac{37}{4}\right)\right)$$

$$= \frac{1}{2}\left(\frac{5}{2} \cdot \frac{10(11)}{2} - 10 \cdot \frac{37}{4}\right)$$

$$= \frac{45}{2} = 22.5$$

Figure 5.3.10: Approximating $\int_{-2}^{3}(5x + 2)\, dx$ using the Midpoint Rule and 10 evenly spaced subintervals in Example 5.3.5.

Note the graph of $f(x) = 5x + 2$ in Figure 5.3.10. The regions whose area is computed by the definite integral are triangles, meaning we can find the exact answer without summation techniques. We find that the exact answer is indeed 22.5. One of the strengths of the Midpoint Rule is that often each rectangle includes area that should not be counted, but misses other area that should. When the partition size is small, these two amounts are about equal and these errors almost "cancel each other out." In this example, since our function is a line, these errors are exactly equal and they do cancel each other out, giving us the exact answer.

Note too that when the function is negative, the rectangles have a "negative" height. When we compute the area of the rectangle, we use $f(c_i)\Delta x$; when f is negative, the area is counted as negative.

Notice in the previous example that while we used 10 equally spaced intervals, the number "10" didn't play a big role in the calculations until the very end.
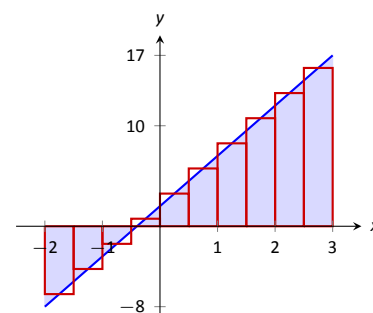
Notes:

Mathematicians love to abstract ideas; let's approximate the area of another region using n subintervals, where we do not specify a value of n until the very end.

Example 5.3.6   **Approximating definite integrals with a formula, using sums**
Revisit $\int_0^4 (4x-x^2)\,dx$ yet again. Approximate this definite integral using the Right Hand Rule with n equally spaced subintervals.

   SOLUTION   Using Key Idea 5.3.1, we know $\Delta x = \frac{4-0}{n} = 4/n$. We also find $x_i = 0 + \Delta x(i-1) = 4(i-1)/n$. The Right Hand Rule uses x_{i+1}, which is $x_{i+1} = 4i/n$.

   We construct the Right Hand Rule Riemann sum as follows. Be sure to follow each step carefully. If you get stuck, and do not understand how one line proceeds to the next, you may skip to the result and consider how this result is used. You should come back, though, and work through each step for full understanding.

$$\int_0^4 (4x - x^2)\,dx \approx \sum_{i=1}^{n} f(x_{i+1})\Delta x$$

$$= \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\Delta x$$

$$= \sum_{i=1}^{n} \left[4\frac{4i}{n} - \left(\frac{4i}{n}\right)^2\right]\Delta x$$

$$= \sum_{i=1}^{n} \left(\frac{16\Delta x}{n}\right)i - \sum_{i=1}^{n} \left(\frac{16\Delta x}{n^2}\right)i^2$$

$$= \left(\frac{16\Delta x}{n}\right)\sum_{i=1}^{n} i - \left(\frac{16\Delta x}{n^2}\right)\sum_{i=1}^{n} i^2$$

$$= \left(\frac{16\Delta x}{n}\right)\cdot\frac{n(n+1)}{2} - \left(\frac{16\Delta x}{n^2}\right)\frac{n(n+1)(2n+1)}{6} \quad \left(\begin{smallmatrix}\text{recall}\\ \Delta x = 4/n\end{smallmatrix}\right)$$

$$= \frac{32(n+1)}{n} - \frac{32(n+1)(2n+1)}{3n^2} \quad \text{(now simplify)}$$

$$= \frac{32}{3}\left(1 - \frac{1}{n^2}\right)$$

   The result is an amazing, easy to use formula. To approximate the definite integral with 10 equally spaced subintervals and the Right Hand Rule, set $n = 10$ and compute

$$\int_0^4 (4x - x^2)\,dx \approx \frac{32}{3}\left(1 - \frac{1}{10^2}\right) = 10.56.$$

Notes:

Recall how earlier we approximated the definite integral with 4 subintervals; with $n = 4$, the formula gives 10, our answer as before.

It is now easy to approximate the integral with 1,000,000 subintervals! Hand-held calculators will round off the answer a bit prematurely giving an answer of 10.66666667. (The actual answer is 10.666666666656.)

We now take an important leap. Up to this point, our mathematics has been limited to geometry and algebra (finding areas and manipulating expressions). Now we apply *calculus*. For any *finite n*, we know that

$$\int_0^4 (4x - x^2)\, dx \approx \frac{32}{3}\left(1 - \frac{1}{n^2}\right).$$

Both common sense and high–level mathematics tell us that as *n* gets large, the approximation gets better. In fact, if we take the *limit* as $n \to \infty$, we get the *exact area* described by $\int_0^4 (4x - x^2)\, dx$. That is,

$$\int_0^4 (4x - x^2)\, dx = \lim_{n\to\infty} \frac{32}{3}\left(1 - \frac{1}{n^2}\right)$$
$$= \frac{32}{3}(1 - 0)$$
$$= \frac{32}{3} = 10.\overline{6}$$

This is a fantastic result. By considering *n* equally–spaced subintervals, we obtained a formula for an approximation of the definite integral that involved our variable *n*. As *n* grows large – without bound – the error shrinks to zero and we obtain the exact area.

This section started with a fundamental calculus technique: make an approximation, refine the approximation to make it better, then use limits in the refining process to get an exact answer. That is precisely what we just did.

Let's practice this again.

Example 5.3.7   **Approximating definite integrals with a formula, using sums**
Find a formula that approximates $\int_{-1}^5 x^3\, dx$ using the Right Hand Rule and *n* equally spaced subintervals, then take the limit as $n \to \infty$ to find the exact area.

SOLUTION   Following Key Idea 5.3.1, we have $\Delta x = \frac{5-(-1)}{n} = 6/n$. We have $x_i = (-1) + (i-1)\Delta x$; as the Right Hand Rule uses x_{i+1}, we have $x_{i+1} = (-1) + i\Delta x$.

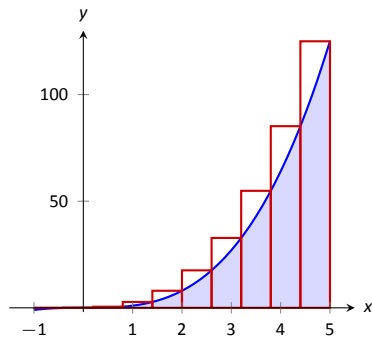The Riemann sum corresponding to the Right Hand Rule is (followed by sim-

Notes:

plifications):

$$\int_{-1}^{5} x^3 \, dx \approx \sum_{i=1}^{n} f(x_{i+1}) \Delta x$$

$$= \sum_{i=1}^{n} f(-1 + i\Delta x)\Delta x$$

$$= \sum_{i=1}^{n} (-1 + i\Delta x)^3 \Delta x$$

$$= \sum_{i=1}^{n} \left((i\Delta x)^3 - 3(i\Delta x)^2 + 3i\Delta x - 1 \right) \Delta x \quad \text{(now distribute } \Delta x)$$

$$= \sum_{i=1}^{n} \left(i^3 \Delta x^4 - 3i^2 \Delta x^3 + 3i\Delta x^2 - \Delta x \right) \quad \text{(now split up summation)}$$

$$= \Delta x^4 \sum_{i=1}^{n} i^3 - 3\Delta x^3 \sum_{i=1}^{n} i^2 + 3\Delta x^2 \sum_{i=1}^{n} i - \sum_{i=1}^{n} \Delta x$$

$$= \Delta x^4 \left(\frac{n(n+1)}{2} \right)^2 - 3\Delta x^3 \frac{n(n+1)(2n+1)}{6} + 3\Delta x^2 \frac{n(n+1)}{2} - n\Delta x$$

(use $\Delta x = 6/n$)

$$= \frac{1296}{n^4} \cdot \frac{n^2(n+1)^2}{4} - 3\frac{216}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} + 3\frac{36}{n^2} \frac{n(n+1)}{2} - 6$$

(now do a sizable amount of algebra to simplify)

$$= 156 + \frac{378}{n} + \frac{216}{n^2}$$

Once again, we have found a compact formula for approximating the definite integral with n equally spaced subintervals and the Right Hand Rule. Using 10 subintervals, we have an approximation of 195.96 (these rectangles are shown in Figure 5.3.11). Using $n = 100$ gives an approximation of 159.802.

Now find the exact answer using a limit:

$$\int_{-1}^{5} x^3 \, dx = \lim_{n \to \infty} \left(156 + \frac{378}{n} + \frac{216}{n^2} \right) = 156.$$

Limits of Riemann Sums

We have used limits to evaluate given definite integrals. Will this always work? We will show, given not–very–restrictive conditions, that yes, it will always work.

Figure 5.3.11: Approximating $\int_{-1}^{5} x^3 \, dx$ using the Right Hand Rule and 10 evenly spaced subintervals.

Notes:

The previous two examples demonstrated how an expression such as

$$\sum_{i=1}^{n} f(x_{i+1})\Delta x$$

can be rewritten as an expression explicitly involving n, such as $32/3(1 - 1/n^2)$.

Viewed in this manner, we can think of the summation as a function of n. An n value is given (where n is a positive integer), and the sum of areas of n equally spaced rectangles is returned, using the Left Hand, Right Hand, or Midpoint Rules.

Given a definite integral $\int_a^b f(x)\, dx$, let:

- $S_L(n) = \sum_{i=1}^{n} f(x_i)\Delta x$, the sum of equally spaced rectangles formed using the Left Hand Rule,

- $S_R(n) = \sum_{i=1}^{n} f(x_{i+1})\Delta x$, the sum of equally spaced rectangles formed using the Right Hand Rule, and

- $S_M(n) = \sum_{i=1}^{n} f\left(\dfrac{x_i + x_{i+1}}{2}\right)\Delta x$, the sum of equally spaced rectangles formed using the Midpoint Rule.

Recall the definition of a limit as $n \to \infty$: $\lim\limits_{n\to\infty} S_L(n) = K$ if, given any $\varepsilon > 0$, there exists $N > 0$ such that

$$|S_L(n) - K| < \varepsilon \quad \text{when} \quad n \geq N.$$

The following theorem states that we can use any of our three rules to find the exact value of a definite integral $\int_a^b f(x)\, dx$. It also goes two steps further. The theorem states that the height of each rectangle doesn't have to be determined following a specific rule, but could be $f(c_i)$, where c_i is any point in the ith subinterval, as discussed before Riemann Sums where defined in Definition 5.3.2.

The theorem goes on to state that the rectangles do not need to be of the same width. Using the notation of Definition 5.3.1, let Δx_i denote the length of the ith subinterval in a partition of $[a, b]$ and let $||\Delta x||$ represent the length of the largest subinterval in the partition: that is, $||\Delta x||$ is the largest of all the Δx_i's. If $||\Delta x||$ is small, then $[a, b]$ must be partitioned into many subintervals, since all subintervals must have small lengths. "Taking the limit as $||\Delta x||$ goes to zero" implies that the number n of subintervals in the partition is growing to

Notes:

infinity, as the largest subinterval length is becoming arbitrarily small. We then interpret the expression

$$\lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i$$

as "the limit of the sum of the areas of rectangles, where the width of each rectangle can be different but getting small, and the height of each rectangle is not necessarily determined by a particular rule." The theorem states that this Riemann Sum also gives the value of the definite integral of f over $[a, b]$.

Theorem 5.3.2   Definite Integrals and the Limit of Riemann Sums

Let f be continuous on the closed interval $[a, b]$ and let $S_L(n)$, $S_R(n)$, $S_M(n)$, Δx, Δx_i and c_i be defined as before. Then:

1. $\displaystyle \lim_{n \to \infty} S_L(n) = \lim_{n \to \infty} S_R(n) = \lim_{n \to \infty} S_M(n) = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x$,

2. $\displaystyle \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x = \int_a^b f(x)\, dx$, and

3. $\displaystyle \lim_{||\Delta x|| \to 0} \sum_{i=1}^{n} f(c_i) \Delta x_i = \int_a^b f(x)\, dx$.

We summarize what we have learned over the past few sections here.

- Knowing the "area under the curve" can be useful. One common example: the area under a velocity curve is displacement.

- We have defined the definite integral, $\int_a^b f(x)\, dx$, to be the signed area under f on the interval $[a, b]$.

- While we can approximate a definite integral many ways, we have focused on using rectangles whose heights can be determined using the Left Hand Rule, the Right Hand Rule and the Midpoint Rule.

- Sums of rectangles of this type are called Riemann sums.

- The exact value of the definite integral can be computed using the limit of a Riemann sum. We generally use one of the above methods as it makes the algebra simpler.

Notes:

We first learned of derivatives through limits then learned rules that made the process simpler. We know of a way to evaluate a definite integral using limits; in the next section we will see how the Fundamental Theorem of Calculus makes the process simpler. The key feature of this theorem is its connection between the indefinite integral and the definite integral.

Notes:

Exercises 5.3

Terms and Concepts

1. A fundamental calculus technique is to use _____ to re-fine approximations to get an exact answer.

2. What is the upper bound in the summation

$$\sum_{i=7}^{14}(48i - 201)?$$

3. This section approximates definite integrals using what geometric shape?

4. T/F: A sum using the Right Hand Rule is an example of a Riemann Sum.

Problems

In Exercises 5 – 12, write out each term of the summation and compute the sum.

5. $\displaystyle\sum_{i=2}^{4} i^2$

6. $\displaystyle\sum_{i=-1}^{3} (4i - 2)$

7. $\displaystyle\sum_{i=-2}^{2} \sin(\pi i/2)$

8. $\displaystyle\sum_{i=1}^{10} 5$

9. $\displaystyle\sum_{i=1}^{5} \frac{1}{i}$

10. $\displaystyle\sum_{i=1}^{6} (-1)^i i$

11. $\displaystyle\sum_{i=1}^{4} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

12. $\displaystyle\sum_{i=0}^{5} (-1)^i \cos(\pi i)$

In Exercises 13 – 16, write each sum in summation notation.

13. $3 + 6 + 9 + 12 + 15$

14. $-1 + 0 + 3 + 8 + 15 + 24 + 35 + 48 + 63$

15. $\dfrac{1}{2} + \dfrac{2}{3} + \dfrac{3}{4} + \dfrac{4}{5}$

16. $1 - e + e^2 - e^3 + e^4$

In Exercises 17 – 24, evaluate the summation using Theorem 5.3.1.

17. $\displaystyle\sum_{i=1}^{10} 5$

18. $\displaystyle\sum_{i=1}^{25} i$

19. $\displaystyle\sum_{i=1}^{10} (3i^2 - 2i)$

20. $\displaystyle\sum_{i=1}^{15} (2i^3 - 10)$

21. $\displaystyle\sum_{i=1}^{10} (-4i^3 + 10i^2 - 7i + 11)$

22. $\displaystyle\sum_{i=1}^{10} (i^3 - 3i^2 + 2i + 7)$

23. $1 + 2 + 3 + \ldots + 99 + 100$

24. $1 + 4 + 9 + \ldots + 361 + 400$

Theorem 5.3.1 states

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{k} a_i + \sum_{i=k+1}^{n} a_i \text{, so}$$

$$\sum_{i=k+1}^{n} a_i = \sum_{i=1}^{n} a_i - \sum_{i=1}^{k} a_i .$$

Use this fact, along with other parts of Theorem 5.3.1, to evaluate the summations given in Exercises 25 – 28.

25. $\displaystyle\sum_{i=11}^{20} i$

26. $\displaystyle\sum_{i=16}^{25} i^3$

27. $\displaystyle\sum_{i=7}^{12} 4$

28. $\displaystyle\sum_{i=5}^{10} 4i^3$

In Exercises 29 – 34, a definite integral

$$\int_a^b f(x)\, dx \text{ is given.}$$

   (a) Graph $f(x)$ **on** $[a, b]$.

   (b) Add to the sketch rectangles using the provided rule.

   (c) Approximate $\int_a^b f(x)\, dx$ **by summing the areas of the rectangles.**

29. $\int_{-3}^{3} x^2\, dx$, with 6 rectangles using the Left Hand Rule.

30. $\int_0^2 (5 - x^2)\, dx$, with 4 rectangles using the Midpoint Rule.

31. $\int_0^{\pi} \sin x\, dx$, with 6 rectangles using the Right Hand Rule.

32. $\int_0^3 2^x\, dx$, with 5 rectangles using the Left Hand Rule.

33. $\int_1^2 \ln x\, dx$, with 3 rectangles using the Midpoint Rule.

34. $\int_1^9 \frac{1}{x}\, dx$, with 4 rectangles using the Right Hand Rule.

In Exercises 35 – 40, a definite integral

$$\int_a^b f(x)\, dx \text{ is given. As demonstrated in Examples 5.3.6}$$

and 5.3.7, do the following.

   (a) Find a formula to approximate $\int_a^b f(x)\, dx$ **using** n **subintervals and the provided rule.**

   (b) Evaluate the formula using $n = 10$, 100 **and** 1,000.

   (c) Find the limit of the formula, as $n \to \infty$, **to find the exact value of** $\int_a^b f(x)\, dx$.

35. $\int_0^1 x^3\, dx$, using the Right Hand Rule.

36. $\int_{-1}^1 3x^2\, dx$, using the Left Hand Rule.

37. $\int_{-1}^3 (3x - 1)\, dx$, using the Midpoint Rule.

38. $\int_1^4 (2x^2 - 3)\, dx$, using the Left Hand Rule.

39. $\int_{-10}^{10} (5 - x)\, dx$, using the Right Hand Rule.

40. $\int_0^1 (x^3 - x^2)\, dx$, using the Right Hand Rule.

Review

In Exercises 41 – 46, find an antiderivative of the given function.

41. $f(x) = 5 \sec^2 x$

42. $f(x) = \dfrac{7}{x}$

43. $g(t) = 4t^5 - 5t^3 + 8$

44. $g(t) = 5 \cdot 8^t$

45. $g(t) = \cos t + \sin t$
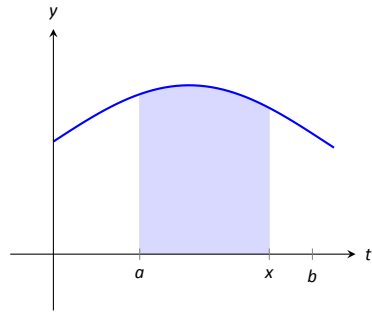
46. $f(x) = \dfrac{1}{\sqrt{x}}$

5.4   The Fundamental Theorem of Calculus

Figure 5.4.1: The area of the shaded region is $F(x) = \int_a^x f(t)\, dt$.

Let $f(t)$ be a continuous function defined on $[a, b]$. The definite integral $\int_a^b f(x)\, dx$ is the "area under f" on $[a, b]$. We can turn this concept into a function by letting the upper (or lower) bound vary.

Let $F(x) = \int_a^x f(t)\, dt$. It computes the area under f on $[a, x]$ as illustrated in Figure 5.4.1. We can study this function using our knowledge of the definite integral. For instance, $F(a) = 0$ since $\int_a^a f(t)\, dt = 0$.

We can also apply calculus ideas to $F(x)$; in particular, we can compute its derivative. While this may seem like an innocuous thing to do, it has far–reaching implications, as demonstrated by the fact that the result is given as an important theorem.

Theorem 5.4.1   **The Fundamental Theorem of Calculus, Part 1**

Let f be continuous on $[a, b]$ and let $F(x) = \int_a^x f(t)\, dt$. Then F is a differentiable function on (a, b), and

$$F'(x) = f(x).$$

Initially this seems simple, as demonstrated in the following example.

Example 5.4.1   **Using the Fundamental Theorem of Calculus, Part 1**

Let $F(x) = \displaystyle\int_{-5}^x (t^2 + \sin t)\, dt$. What is $F'(x)$?

SOLUTION   Using the Fundamental Theorem of Calculus, we have $F'(x) = x^2 + \sin x$.

This simple example reveals something incredible: $F(x)$ is an antiderivative of $x^2 + \sin x$! Therefore, $F(x) = \frac{1}{3}x^3 - \cos x + C$ for some value of C. (We can find C, but generally we do not care. We know that $F(-5) = 0$, which allows us to compute C. In this case, $C = \cos(-5) + \frac{125}{3}$.)

We have done more than found a complicated way of computing an antiderivative. Consider a function f defined on an open interval containing a, b and c. Suppose we want to compute $\int_a^b f(t)\, dt$. First, let $F(x) = \int_c^x f(t)\, dt$. Using

---

Notes:

the properties of the definite integral found in Theorem 5.2.1, we know

$$\int_a^b f(t)\, dt = \int_a^c f(t)\, dt + \int_c^b f(t)\, dt$$

$$= -\int_c^a f(t)\, dt + \int_c^b f(t)\, dt$$

$$= -F(a) + F(b)$$

$$= F(b) - F(a).$$

We now see how indefinite integrals and definite integrals are related: we can evaluate a definite integral using antiderivatives! This is the second part of the Fundamental Theorem of Calculus.

Theorem 5.4.2   The Fundamental Theorem of Calculus, Part 2

Let f be continuous on $[a, b]$ and let F be *any* antiderivative of f. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Example 5.4.2   Using the Fundamental Theorem of Calculus, Part 2
We spent a great deal of time in the previous section studying $\int_0^4 (4x - x^2)\, dx$. Using the Fundamental Theorem of Calculus, evaluate this definite integral.

SOLUTION   We need an antiderivative of $f(x) = 4x - x^2$. All antiderivatives of f have the form $F(x) = 2x^2 - \frac{1}{3}x^3 + C$; for simplicity, choose $C = 0$.
   The Fundamental Theorem of Calculus states

$$\int_0^4 (4x - x^2)\, dx = F(4) - F(0) = \left(2(4)^2 - \frac{1}{3}4^3\right) - \left(0 - 0\right) = 32 - \frac{64}{3} = 32/3.$$

This is the same answer we obtained using limits in the previous section, just with much less work.

Notation: A special notation is often used in the process of evaluating definite integrals using the Fundamental Theorem of Calculus. Instead of explicitly writing $F(b) - F(a)$, the notation $F(x)\Big|_a^b$ is used. Thus the solution to Example 5.4.2 would be written as:

$$\int_0^4 (4x - x^2)\, dx = \left(2x^2 - \frac{1}{3}x^3\right)\Big|_0^4 = \left(2(4)^2 - \frac{1}{3}4^3\right) - \left(0 - 0\right) = 32/3.$$

Notes:

The Constant *C*: *Any* antiderivative $F(x)$ can be chosen when using the Fundamental Theorem of Calculus to evaluate a definite integral, meaning any value of C can be picked. The constant *always* cancels out of the expression when evaluating $F(b) - F(a)$, so it does not matter what value is picked. This being the case, we might as well let $C = 0$.

Example 5.4.3   Using the Fundamental Theorem of Calculus, Part 2
Evaluate the following definite integrals.

1. $\displaystyle\int_{-2}^{2} x^3\, dx$ 2. $\displaystyle\int_{0}^{\pi} \sin x\, dx$ 3. $\displaystyle\int_{0}^{5} e^t\, dt$ 4. $\displaystyle\int_{4}^{9} \sqrt{u}\, du$ 5. $\displaystyle\int_{1}^{5} 2\, dx$

   SOLUTION

1. $\displaystyle\int_{-2}^{2} x^3\, dx = \left.\frac{1}{4}x^4\right|_{-2}^{2} = \left(\frac{1}{4}2^4\right) - \left(\frac{1}{4}(-2)^4\right) = 0.$

2. $\displaystyle\int_{0}^{\pi} \sin x\, dx = \left.-\cos x\right|_{0}^{\pi} = -\cos\pi - (-\cos 0) = 1 + 1 = 2.$

   (This is interesting; it says that the area under one "hump" of a sine curve is 2.)

3. $\displaystyle\int_{0}^{5} e^t\, dt = \left.e^t\right|_{0}^{5} = e^5 - e^0 = e^5 - 1 \approx 147.41.$

4. $\displaystyle\int_{4}^{9} \sqrt{u}\, du = \int_{4}^{9} u^{\frac{1}{2}}\, du = \left.\frac{2}{3}u^{\frac{3}{2}}\right|_{4}^{9} = \frac{2}{3}\left(9^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = \frac{2}{3}(27 - 8) = \frac{38}{3}.$

5. $\displaystyle\int_{1}^{5} 2\, dx = \left.2x\right|_{1}^{5} = 2(5) - 2 = 2(5 - 1) = 8.$

   This integral is interesting; the integrand is a constant function, hence we are finding the area of a rectangle with width $(5 - 1) = 4$ and height 2. Notice how the evaluation of the definite integral led to $2(4) = 8$.

   In general, if c is a constant, then $\int_{a}^{b} c\, dx = c(b - a)$.

Understanding Motion with the Fundamental Theorem of Calculus

We established, starting with Key Idea 2.2.1, that the derivative of a position function is a velocity function, and the derivative of a velocity function is an acceleration function. Now consider definite integrals of velocity and acceleration functions. Specifically, if $v(t)$ is a velocity function, what does $\displaystyle\int_{a}^{b} v(t)\, dt$ mean?

Notes:

The Fundamental Theorem of Calculus states that

$$\int_a^b v(t)\,dt = V(b) - V(a),$$

where $V(t)$ is any antiderivative of $v(t)$. Since $v(t)$ is a velocity function, $V(t)$ must be a position function, and $V(b) - V(a)$ measures a change in position, or **displacement**.

Example 5.4.4   Finding displacement
A ball is thrown straight up with velocity given by $v(t) = -32t + 20$ft/s, where t is measured in seconds. Find, and interpret, $\int_0^1 v(t)\,dt$.

SOLUTION   Using the Fundamental Theorem of Calculus, we have

$$\int_0^1 v(t)\,dt = \int_0^1 (-32t + 20)\,dt$$

$$= -16t^2 + 20t\Big|_0^1$$

$$= 4.$$

Thus if a ball is thrown straight up into the air with velocity $v(t) = -32t + 20$, the height of the ball, 1 second later, will be 4 feet above the initial height. (Note that the ball has *traveled* much farther. It has gone up to its peak and is falling down, but the difference between its height at $t = 0$ and $t = 1$ is 4ft.)

Integrating a rate of change function gives total change. Velocity is the rate of position change; integrating velocity gives the total change of position, i.e., displacement.

Integrating a speed function gives a similar, though different, result. Speed is also the rate of position change, but does not account for direction. So integrating a speed function gives total change of position, without the possibility of "negative position change." Hence the integral of a speed function gives *distance traveled.*

As acceleration is the rate of velocity change, integrating an acceleration function gives total change in velocity. We do not have a simple term for this analogous to displacement. If $a(t) = 5$miles/h^2 and t is measured in hours, then

$$\int_0^3 a(t)\,dt = 15$$

means the velocity has increased by 15m/h from $t = 0$ to $t = 3$.

Notes:

The Fundamental Theorem of Calculus and the Chain Rule

Part 1 of the Fundamental Theorem of Calculus (FTC) states that given $F(x) = \int_a^x f(t)\, dt$, $F'(x) = f(x)$. Using other notation, $\dfrac{d}{dx}\big(F(x)\big) = f(x)$. While we have just practiced evaluating definite integrals, sometimes finding antiderivatives is impossible and we need to rely on other techniques to approximate the value of a definite integral. Functions written as $F(x) = \int_a^x f(t)\, dt$ are useful in such situations.

It may be of further use to compose such a function with another. As an example, we may compose $F(x)$ with $g(x)$ to get

$$F\big(g(x)\big) = \int_a^{g(x)} f(t)\, dt.$$

What is the derivative of such a function? The Chain Rule can be employed to state

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'\big(g(x)\big)g'(x) = f\big(g(x)\big)g'(x).$$

An example will help us understand this.

Example 5.4.5   **The FTC, Part 1, and the Chain Rule**

Find the derivative of $F(x) = \displaystyle\int_2^{x^2} \ln t\, dt$.

SOLUTION   We can view $F(x)$ as being the function $G(x) = \displaystyle\int_2^x \ln t\, dt$ composed with $g(x) = x^2$; that is, $F(x) = G\big(g(x)\big)$. The Fundamental Theorem of Calculus states that $G'(x) = \ln x$. The Chain Rule gives us

$$\begin{aligned}
F'(x) &= G'\big(g(x)\big)g'(x) \\
&= \ln(g(x))g'(x) \\
&= \ln(x^2)2x \\
&= 2x\ln x^2
\end{aligned}$$

Normally, the steps defining $G(x)$ and $g(x)$ are skipped.

Practice this once more.

Example 5.4.6   **The FTC, Part 1, and the Chain Rule**

Find the derivative of $F(x) = \displaystyle\int_{\cos x}^5 t^3\, dt$.

Notes:

SOLUTION   Note that $F(x) = -\int_{5}^{\cos x} t^3 \, dt$. Viewed this way, the derivative of F is straightforward:

$$F'(x) = \sin x \cos^3 x.$$

Area Between Curves

Consider continuous functions $f(x)$ and $g(x)$ defined on $[a, b]$, where $f(x) \geq g(x)$ for all x in $[a, b]$, as demonstrated in Figure 5.4.2. What is the area of the shaded region bounded by the two curves over $[a, b]$?

The area can be found by recognizing that this area is "the area under $f -$ the area under g." Using mathematical notation, the area is

$$\int_{a}^{b} f(x) \, dx - \int_{a}^{b} g(x) \, dx.$$

Properties of the definite integral allow us to simplify this expression to

$$\int_{a}^{b} \big(f(x) - g(x)\big) \, dx.$$
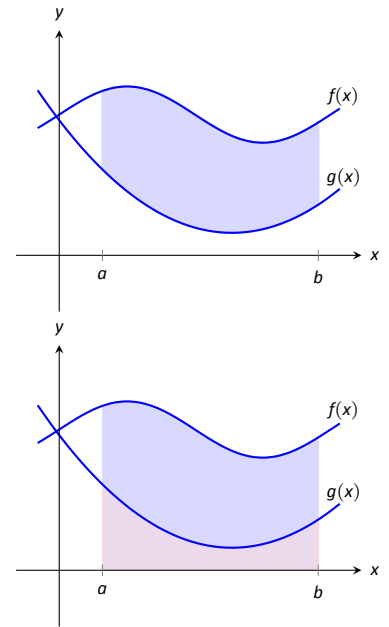

Figure 5.4.2: Finding the area bounded by two functions on an interval; it is found by subtracting the area under g from the area under f.
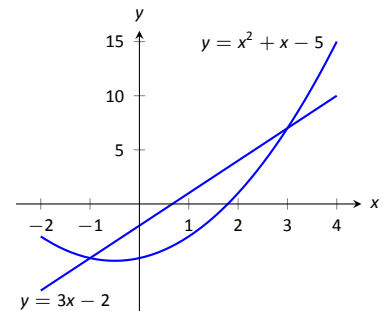

Theorem 5.4.3   Area Between Curves

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_{a}^{b} \big(f(x) - g(x)\big) \, dx.$$

Example 5.4.7   Finding area between curves
Find the area of the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$.

SOLUTION   It will help to sketch these two functions, as done in Figure 5.4.3. The region whose area we seek is completely bounded by these two functions; they seem to intersect at $x = -1$ and $x = 3$. To check, set $x^2 + x - 5 =$

Figure 5.4.3: Sketching the region enclosed by $y = x^2 + x - 5$ and $y = 3x - 2$ in Example 5.4.7.
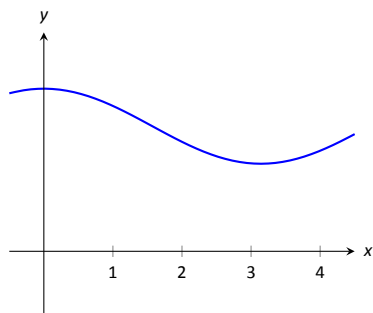

Notes:

Figure 5.4.4: A graph of a function f to introduce the Mean Value Theorem.
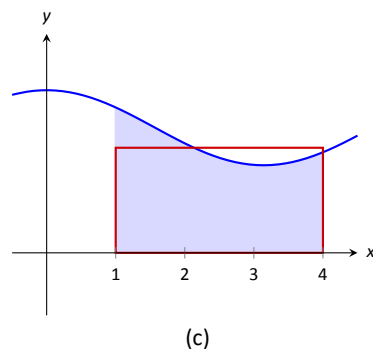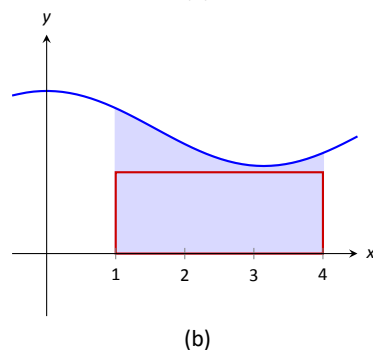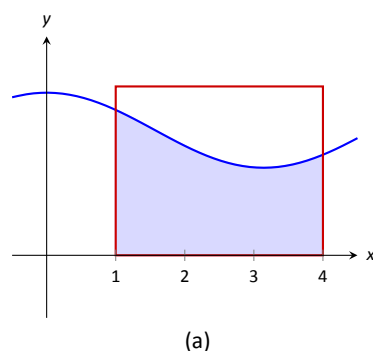

(a)

(b)

(c)

Figure 5.4.5: Differently sized rectangles give upper and lower bounds on $\int_1^4 f(x)\, dx$; the last rectangle matches the area exactly.

$3x - 2$ and solve for x:

$$x^2 + x - 5 = 3x - 2$$
$$(x^2 + x - 5) - (3x - 2) = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$
$$x = -1,\ 3.$$

Following Theorem 5.4.3, the area is

$$\int_{-1}^{3} \left(3x - 2 - (x^2 + x - 5)\right) dx = \int_{-1}^{3} (-x^2 + 2x + 3)\, dx$$
$$= \left(-\frac{1}{3}x^3 + x^2 + 3x\right)\Big|_{-1}^{3}$$
$$= -\frac{1}{3}(27) + 9 + 9 - \left(\frac{1}{3} + 1 - 3\right)$$
$$= 10\frac{2}{3} = 10.\overline{6}$$

The Mean Value Theorem and Average Value

Consider the graph of a function f in Figure 5.4.4 and the area defined by $\int_1^4 f(x)\, dx$. Three rectangles are drawn in Figure 5.4.5; in (a), the height of the rectangle is greater than f on $[1, 4]$, hence the area of this rectangle is is greater than $\int_0^4 f(x)\, dx$.

In (b), the height of the rectangle is smaller than f on $[1, 4]$, hence the area of this rectangle is less than $\int_1^4 f(x)\, dx$.

Finally, in (c) the height of the rectangle is such that the area of the rectangle is *exactly* that of $\int_0^4 f(x)\, dx$. Since rectangles that are "too big", as in (a), and rectangles that are "too little," as in (b), give areas greater/lesser than $\int_1^4 f(x)\, dx$, it makes sense that there is a rectangle, whose top intersects $f(x)$ somewhere on $[1, 4]$, whose area is *exactly* that of the definite integral.

We state this idea formally in a theorem.

Notes:

Theorem 5.4.4   **The Mean Value Theorem of Integration**

Let f be continuous on $[a, b]$. There exists a value c in $[a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a).$$

Figure 5.4.6: A graph of $y = \sin x$ on $[0, \pi]$ and the rectangle guaranteed by the Mean Value Theorem.

This is an *existential* statement; c exists, but we do not provide a method of finding it. Theorem 5.4.4 is directly connected to the Mean Value Theorem of Differentiation, given as Theorem 3.2.1; we leave it to the reader to see how.

We demonstrate the principles involved in this version of the Mean Value Theorem in the following example.

Example 5.4.8   **Using the Mean Value Theorem**
Consider $\int_0^\pi \sin x\,dx$. Find a value c guaranteed by the Mean Value Theorem.

SOLUTION   We first need to evaluate $\int_0^\pi \sin x\,dx$. (This was previously done in Example 5.4.3.)

$$\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = 2.$$

Thus we seek a value c in $[0, \pi]$ such that $\pi \sin c = 2$.

$$\pi \sin c = 2 \;\Rightarrow\; \sin c = 2/\pi \;\Rightarrow\; c = \arcsin(2/\pi) \approx 0.69.$$

In Figure 5.4.6 $\sin x$ is sketched along with a rectangle with height $\sin(0.69)$. The area of the rectangle is the same as the area under $\sin x$ on $[0, \pi]$.

Let f be a function on $[a, b]$ with c such that $f(c)(b-a) = \int_a^b f(x)\,dx$. Consider $\int_a^b \big(f(x) - f(c)\big)\,dx$:

$$\int_a^b \big(f(x) - f(c)\big)\,dx = \int_a^b f(x) - \int_a^b f(c)\,dx$$
$$= f(c)(b - a) - f(c)(b - a)$$
$$= 0.$$

When $f(x)$ is shifted by $-f(c)$, the amount of area under f above the x–axis on $[a, b]$ is the same as the amount of area below the x–axis above f; see Figure 5.4.7 for an illustration of this. In this sense, we can say that $f(c)$ is the *average value* of f on $[a, b]$.
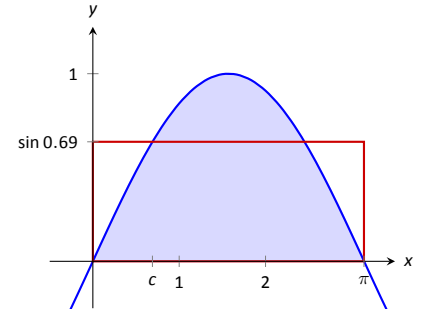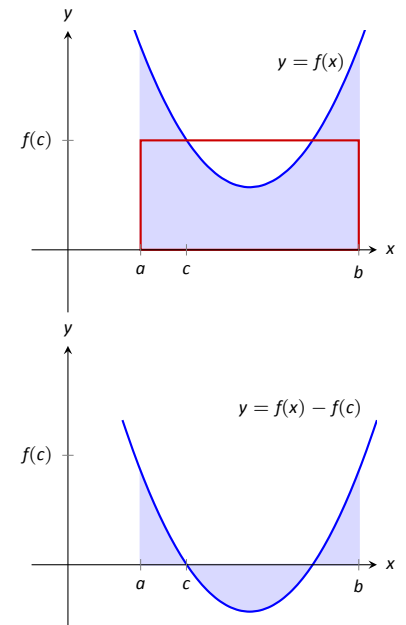

Figure 5.4.7: On top, a graph of $y = f(x)$ and the rectangle guaranteed by the Mean Value Theorem. Below, $y = f(x)$ is shifted down by $f(c)$; the resulting "area under the curve" is 0.

Notes:

The value $f(c)$ is the average value in another sense. First, recognize that the Mean Value Theorem can be rewritten as

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx,$$

for some value of c in $[a, b]$. Next, partition the interval $[a, b]$ into n equally spaced subintervals, $a = x_1 < x_2 < \ldots < x_{n+1} = b$ and choose any c_i in $[x_i, x_{i+1}]$. The average of the numbers $f(c_1), f(c_2), ..., f(c_n)$ is:

$$\frac{1}{n}\Big(f(c_1) + f(c_2) + \ldots + f(c_n)\Big) = \frac{1}{n}\sum_{i=1}^n f(c_i).$$

Multiply this last expression by 1 in the form of $\frac{(b-a)}{(b-a)}$:

$$\frac{1}{n}\sum_{i=1}^n f(c_i) = \sum_{i=1}^n f(c_i)\frac{1}{n}$$

$$= \sum_{i=1}^n f(c_i)\frac{1}{n}\frac{(b-a)}{(b-a)}$$

$$= \frac{1}{b-a}\sum_{i=1}^n f(c_i)\frac{b-a}{n}$$

$$= \frac{1}{b-a}\sum_{i=1}^n f(c_i)\Delta x \quad \text{(where } \Delta x = (b-a)/n\text{)}$$

Now take the limit as $n \to \infty$:

$$\lim_{n\to\infty} \frac{1}{b-a}\sum_{i=1}^n f(c_i)\Delta x \quad = \quad \frac{1}{b-a}\int_a^b f(x)\, dx \quad = \quad f(c).$$

This tells us this: when we evaluate f at n (somewhat) equally spaced points in $[a, b]$, the average value of these samples is $f(c)$ as $n \to \infty$.

This leads us to a definition.

Definition 5.4.1   **The Average Value of f on $[a, b]$**

Let f be continuous on $[a, b]$. The **average value of f on $[a, b]$** is $f(c)$, where c is a value in $[a, b]$ guaranteed by the Mean Value Theorem. I.e.,

$$\text{Average Value of } f \text{ on } [a, b] = \frac{1}{b-a}\int_a^b f(x)\, dx.$$

Notes:

An application of this definition is given in the following example.

Example 5.4.9   Finding the average value of a function

An object moves back and forth along a straight line with a velocity given by $v(t) = (t - 1)^2$ on $[0, 3]$, where t is measured in seconds and $v(t)$ is measured in ft/s.

What is the average velocity of the object?

Solution   By our definition, the average velocity is:

$$\frac{1}{3 - 0} \int_0^3 (t - 1)^2 \, dt = \frac{1}{3} \int_0^3 \left(t^2 - 2t + 1\right) dt = \frac{1}{3} \left(\frac{1}{3}t^3 - t^2 + t\right)\Big|_0^3 = 1 \text{ ft/s.}$$

We can understand the above example through a simpler situation. Suppose you drove 100 miles in 2 hours. What was your average speed? The answer is simple: displacement/time = 100 miles/2 hours = 50 mph.

What was the displacement of the object in Example 5.4.9? We calculate this by integrating its velocity function: $\int_0^3 (t - 1)^2 \, dt = 3$ ft. Its final position was 3 feet from its initial position after 3 seconds: its average velocity was 1 ft/s.

This section has laid the groundwork for a lot of great mathematics to follow. The most important lesson is this: definite integrals can be evaluated using antiderivatives. Since the previous section established that definite integrals are the limit of Riemann sums, we can later create Riemann sums to approximate values other than "area under the curve," convert the sums to definite integrals, then evaluate these using the Fundamental Theorem of Calculus. This will allow us to compute the work done by a variable force, the volume of certain solids, the arc length of curves, and more.

The downside is this: generally speaking, computing antiderivatives is much more difficult than computing derivatives. The next chapter is devoted to techniques of finding antiderivatives so that a wide variety of definite integrals can be evaluated. Before that, the next section explores techniques of approximating the value of definite integrals beyond using the Left Hand, Right Hand and Midpoint Rules.

Notes:

Exercises 5.4

Terms and Concepts

1. How are definite and indefinite integrals related?

2. What constant of integration is most commonly used when evaluating definite integrals?

3. T/F: If f is a continuous function, then $F(x) = \int_a^x f(t)\,dt$ is also a continuous function.

4. The definite integral can be used to find "the area under a curve." Give two other uses for definite integrals.

Problems

In Exercises 5 – 28, evaluate the definite integral.

5. $\int_1^3 (3x^2 - 2x + 1)\,dx$

6. $\int_0^4 (x-1)^2\,dx$

7. $\int_{-1}^1 (x^3 - x^5)\,dx$

8. $\int_{\pi/2}^{\pi} \cos x\,dx$

9. $\int_0^{\pi/4} \sec^2 x\,dx$

10. $\int_1^e \frac{1}{x}\,dx$

11. $\int_{-1}^1 5^x\,dx$

12. $\int_{-2}^{-1} (4 - 2x^3)\,dx$

13. $\int_0^{\pi} (2\cos x - 2\sin x)\,dx$

14. $\int_1^3 e^x\,dx$

15. $\int_0^4 \sqrt{t}\,dt$

16. $\int_9^{25} \frac{1}{\sqrt{t}}\,dt$

17. $\int_1^8 \sqrt[3]{x}\,dx$

18. $\int_1^2 \frac{1}{x}\,dx$

19. $\int_1^2 \frac{1}{x^2}\,dx$

20. $\int_1^2 \frac{1}{x^3}\,dx$

21. $\int_0^1 x\,dx$

22. $\int_0^1 x^2\,dx$

23. $\int_0^1 x^3\,dx$

24. $\int_0^1 x^{100}\,dx$

25. $\int_{-4}^4 dx$

26. $\int_{-10}^{-5} 3\,dx$

27. $\int_{-2}^2 0\,dx$

28. $\int_{\pi/6}^{\pi/3} \csc x \cot x\,dx$

29. Explain why:

(a) $\int_{-1}^1 x^n\,dx = 0$, when n is a positive, odd integer, and

(b) $\int_{-1}^1 x^n\,dx = 2\int_0^1 x^n\,dx$ when n is a positive, even integer.

30. Explain why $\int_a^{a+2\pi} \sin t\,dt = 0$ for all values of a.

In Exercises 31 – 34, find a value c guaranteed by the Mean Value Theorem.

31. $\int_0^2 x^2\, dx$

32. $\int_{-2}^2 x^2\, dx$

33. $\int_0^1 e^x\, dx$

34. $\int_0^{16} \sqrt{x}\, dx$

In Exercises 35 – 40, find the average value of the function on the given interval.

35. $f(x) = \sin x$ on $[0, \pi/2]$

36. $y = \sin x$ on $[0, \pi]$

37. $y = x$ on $[0, 4]$

38. $y = x^2$ on $[0, 4]$

39. $y = x^3$ on $[0, 4]$

40. $g(t) = 1/t$ on $[1, e]$

In Exercises 41 – 46, a velocity function of an object moving along a straight line is given. Find the displacement of the object over the given time interval.

41. $v(t) = -32t + 20$ft/s on $[0, 5]$

42. $v(t) = -32t + 200$ft/s on $[0, 10]$

43. $v(t) = 10$ft/s on $[0, 3]$.

44. $v(t) = 2^t$mph on $[-1, 1]$

45. $v(t) = \cos t$ ft/s on $[0, 3\pi/2]$

46. $v(t) = \sqrt[4]{t}$ ft/s on $[0, 16]$

In Exercises 47 – 50, an acceleration function of an object moving along a straight line is given. Find the change of the object's velocity over the given time interval.

47. $a(t) = -32$ft/s^2 on $[0, 2]$

48. $a(t) = 10$ft/s^2 on $[0, 5]$

49. $a(t) = t$ ft/s^2 on $[0, 2]$

50. $a(t) = \cos t$ ft/s^2 on $[0, \pi]$

In Exercises 51 – 54, sketch the given functions and find the area of the enclosed region.

51. $y = 2x$, $y = 5x$, and $x = 3$.

52. $y = -x + 1$, $y = 3x + 6$, $x = 2$ and $x = -1$.

53. $y = x^2 - 2x + 5$, $y = 5x - 5$.

54. $y = 2x^2 + 2x - 5$, $y = x^2 + 3x + 7$.

In Exercises 55 – 58, find $F'(x)$.

55. $F(x) = \int_2^{x^3+x} \frac{1}{t}\, dt$

56. $F(x) = \int_{x^3}^0 t^3\, dt$

57. $F(x) = \int_x^{x^2} (t + 2)\, dt$

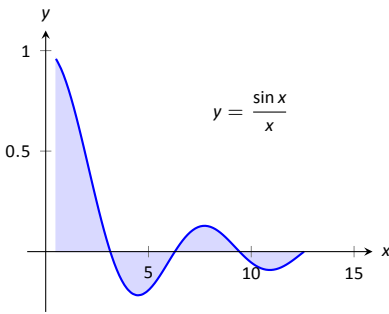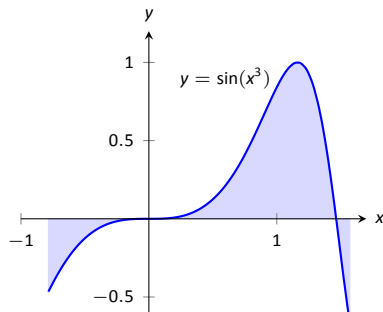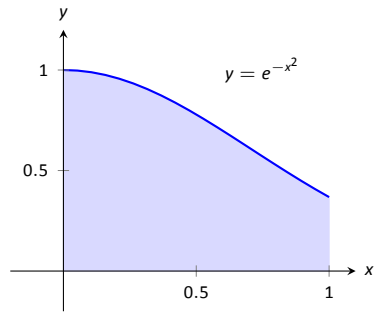58. $F(x) = \int_{\ln x}^{e^x} \sin t\, dt$

Figure 5.5.1: Graphically representing three definite integrals that cannot be evaluated using antiderivatives.

5.5 Numerical Integration

The Fundamental Theorem of Calculus gives a concrete technique for finding the exact value of a definite integral. That technique is based on computing antiderivatives. Despite the power of this theorem, there are still situations where we must *approximate* the value of the definite integral instead of finding its exact value. The first situation we explore is where we *cannot* compute the antiderivative of the integrand. The second case is when we actually do not know the function in the integrand, but only its value when evaluated at certain points.

An **elementary function** is any function that is a combination of polynomials, n^{th} roots, rational, exponential, logarithmic and trigonometric functions. We can compute the derivative of any elementary function, but there are many elementary functions of which we cannot compute an antiderivative. For example, the following functions do not have antiderivatives that we can express with elementary functions:

$$e^{-x^2}, \quad \sin(x^3) \quad \text{and} \quad \frac{\sin x}{x}.$$

The simplest way to refer to the antiderivatives of e^{-x^2} is to simply write $\int e^{-x^2}\,dx$.

This section outlines three common methods of approximating the value of definite integrals. We describe each as a systematic method of approximating area under a curve. By approximating this area accurately, we find an accurate approximation of the corresponding definite integral.

We will apply the methods we learn in this section to the following definite integrals:

$$\int_0^1 e^{-x^2}\,dx, \quad \int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx, \quad \text{and} \quad \int_{0.5}^{4\pi} \frac{\sin(x)}{x}\,dx,$$

as pictured in Figure 5.5.1.

The Left and Right Hand Rule Methods

In Section 5.3 we addressed the problem of evaluating definite integrals by approximating the area under the curve using rectangles. We revisit those ideas here before introducing other methods of approximating definite integrals.

We start with a review of notation. Let f be a continuous function on the interval $[a,b]$. We wish to approximate $\int_a^b f(x)\,dx$. We partition $[a,b]$ into n equally spaced subintervals, each of length $\Delta x = \dfrac{b-a}{n}$. The endpoints of these

Notes:

subintervals are labeled as

$$x_1 = a, \ x_2 = a + \Delta x, \ x_3 = a + 2\Delta x, \ \ldots, \ x_i = a + (i-1)\Delta x, \ \ldots, \ x_{n+1} = b.$$

Key Idea 5.3.1 states that to use the Left Hand Rule we use the summation $\sum_{i=1}^{n} f(x_i)\Delta x$ and to use the Right Hand Rule we use $\sum_{i=1}^{n} f(x_{i+1})\Delta x$. We review the use of these rules in the context of examples.

Example 5.5.1 Approximating definite integrals with rectangles

Approximate $\displaystyle\int_0^1 e^{-x^2} \, dx$ using the Left and Right Hand Rules with 5 equally spaced subintervals.

SOLUTION We begin by partitioning the interval $[0, 1]$ into 5 equally spaced intervals. We have $\Delta x = \frac{1-0}{5} = 1/5 = 0.2$, so

$$x_1 = 0, \ x_2 = 0.2, \ x_3 = 0.4, \ x_4 = 0.6, \ x_5 = 0.8, \ \text{and} \ x_6 = 1.$$

Using the Left Hand Rule, we have:

$$\begin{aligned}
\sum_{i=1}^{n} f(x_i)\Delta x &= \big(f(x_1) + f(x_2) + f(x_3) + f(x_4) + f(x_5)\big)\Delta x \\
&= \big(f(0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)\big)\Delta x \\
&\approx \big(1 + 0.961 + 0.852 + 0.698 + 0.527\big)(0.2) \\
&\approx 0.808.
\end{aligned}$$

Using the Right Hand Rule, we have:

$$\begin{aligned}
\sum_{i=1}^{n} f(x_{i+1})\Delta x &= \big(f(x_2) + f(x_3) + f(x_4) + f(x_5) + f(x_6)\big)\Delta x \\
&= \big(f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)\big)\Delta x \\
&\approx \big(0.961 + 0.852 + 0.698 + 0.527 + 0.368\big)(0.2) \\
&\approx 0.681.
\end{aligned}$$

Figure 5.5.2 shows the rectangles used in each method to approximate the definite integral. These graphs show that in this particular case, the Left Hand Rule is an over approximation and the Right Hand Rule is an under approximation. To get a better approximation, we could use more rectangles, as we did in
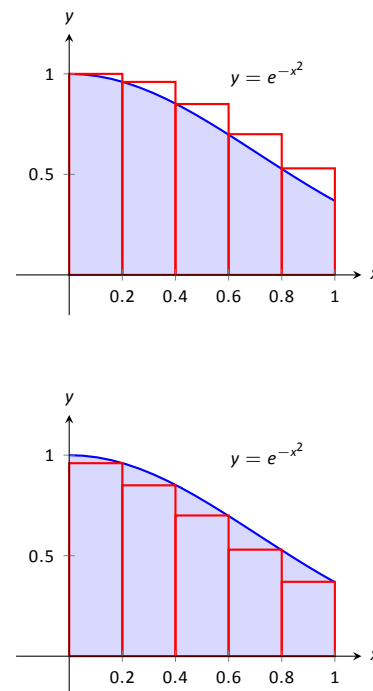

Figure 5.5.2: Approximating $\int_0^1 e^{-x^2} \, dx$ in Example 5.5.1.

Notes:

x_i	Exact	Approx.	$\sin(x_i^3)$
x_1	$-\pi/4$	-0.785	-0.466
x_2	$-7\pi/40$	-0.550	-0.165
x_3	$-\pi/10$	-0.314	-0.031
x_4	$-\pi/40$	-0.0785	0
x_5	$\pi/20$	0.157	0.004
x_6	$\pi/8$	0.393	0.061
x_7	$\pi/5$	0.628	0.246
x_8	$11\pi/40$	0.864	0.601
x_9	$7\pi/20$	1.10	0.971
x_{10}	$17\pi/40$	1.34	0.690
x_{11}	$\pi/2$	1.57	-0.670

Figure 5.5.3: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.2.
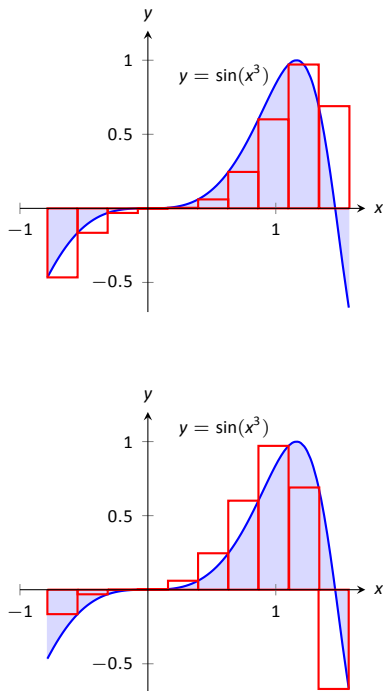

Figure 5.5.4: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ in Example 5.5.2.

Section 5.3. We could also average the Left and Right Hand Rule results together, giving

$$\frac{0.808 + 0.681}{2} = 0.7445.$$

The actual answer, accurate to 4 places after the decimal, is 0.7468, showing our average is a good approximation.

Example 5.5.2   Approximating definite integrals with rectangles

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Left and Right Hand Rules with 10 equally spaced subintervals.

SOLUTION   We begin by finding Δx:

$$\frac{b-a}{n} = \frac{\pi/2 - (-\pi/4)}{10} = \frac{3\pi}{40} \approx 0.236.$$

It is useful to write out the endpoints of the subintervals in a table; in Figure 5.5.3, we give the exact values of the endpoints, their decimal approximations, and decimal approximations of $\sin(x^3)$ evaluated at these points.

Once this table is created, it is straightforward to approximate the definite integral using the Left and Right Hand Rules. (Note: the table itself is easy to create, especially with a standard spreadsheet program on a computer. The last two columns are all that are needed.) The Left Hand Rule sums the first 10 values of $\sin(x_i^3)$ and multiplies the sum by Δx; the Right Hand Rule sums the last 10 values of $\sin(x_i^3)$ and multiplies by Δx. Therefore we have:

Left Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.91)(0.236) = 0.451.$

Right Hand Rule: $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx (1.71)(0.236) = 0.404.$

Average of the Left and Right Hand Rules: 0.4275.

The actual answer, accurate to 3 places after the decimal, is 0.460. Our approximations were once again fairly good. The rectangles used in each approximation are shown in Figure 5.5.4. It is clear from the graphs that using more rectangles (and hence, narrower rectangles) should result in a more accurate approximation.

The Trapezoidal Rule

In Example 5.5.1 we approximated the value of $\int_0^1 e^{-x^2}\,dx$ with 5 rectangles of equal width. Figure 5.5.2 shows the rectangles used in the Left and Right

Notes:

Hand Rules. These graphs clearly show that rectangles do not match the shape of the graph all that well, and that accurate approximations will only come by using lots of rectangles.

Instead of using rectangles to approximate the area, we can instead use *trapezoids.* In Figure 5.5.5, we show the region under $f(x) = e^{-x^2}$ on $[0,1]$ approximated with 5 trapezoids of equal width; the top "corners" of each trapezoid lies on the graph of $f(x)$. It is clear from this figure that these trapezoids more accurately approximate the area under f and hence should give a better approximation of $\int_0^1 e^{-x^2}\, dx$. (In fact, these trapezoids seem to give a *great* approximation of the area!)

The formula for the area of a trapezoid is given in Figure 5.5.6. We approximate $\int_0^1 e^{-x^2}\, dx$ with these trapezoids in the following example.

Example 5.5.3   Approximating definite integrals using trapezoids

Use 5 trapezoids of equal width to approximate $\displaystyle\int_0^1 e^{-x^2}\, dx$.

SOLUTION   To compute the areas of the 5 trapezoids in Figure 5.5.5, it will again be useful to create a table of values as shown in Figure 5.5.7.

The leftmost trapezoid has legs of length 1 and 0.961 and a height of 0.2. Thus, by our formula, the area of the leftmost trapezoid is:

$$\frac{1+0.961}{2}(0.2) = 0.1961.$$

Moving right, the next trapezoid has legs of length 0.961 and 0.852 and a height of 0.2. Thus its area is:

$$\frac{0.961+0.852}{2}(0.2) = 0.1813.$$

The sum of the areas of all 5 trapezoids is:

$$\frac{1+0.961}{2}(0.2) + \frac{0.961+0.852}{2}(0.2) + \frac{0.852+0.698}{2}(0.2)+$$
$$\frac{0.698+0.527}{2}(0.2) + \frac{0.527+0.368}{2}(0.2) = 0.7445.$$

We approximate $\displaystyle\int_0^1 e^{-x^2}\, dx \approx 0.7445.$

There are many things to observe in this example. Note how each term in the final summation was multiplied by both 1/2 and by $\Delta x = 0.2$. We can factor these coefficients out, leaving a more concise summation as:

$$\frac{1}{2}(0.2)\Big[(1+0.961)+(0.961+0.852)+(0.852+0.698)+(0.698+0.527)+(0.527+0.368)\Big].$$
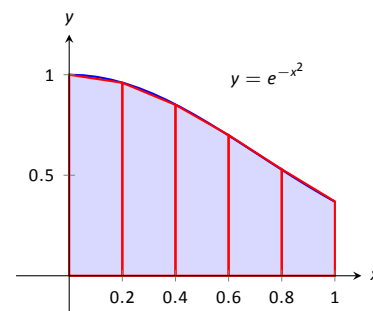

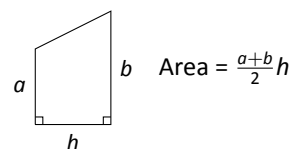Figure 5.5.5: Approximating $\int_0^1 e^{-x^2}\, dx$ using 5 trapezoids of equal widths.

Figure 5.5.6: The area of a trapezoid.

x_i	$e^{-x_i^2}$
0	1
0.2	0.961
0.4	0.852
0.6	0.698
0.8	0.527
1	0.368

Figure 5.5.7: A table of values of e^{-x^2}.

Notes:

Now notice that all numbers except for the first and the last are added twice. Therefore we can write the summation even more concisely as

$$\frac{0.2}{2}\Big[1 + 2(0.961 + 0.852 + 0.698 + 0.527) + 0.368\Big].$$

This is the heart of the **Trapezoidal Rule**, wherein a definite integral $\int_a^b f(x)\,dx$ is approximated by using trapezoids of equal widths to approximate the corresponding area under f. Using n equally spaced subintervals with endpoints x_1, $x_2, \ldots, x_{n+1}$, we again have $\Delta x = \dfrac{b-a}{n}$. Thus:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^n \frac{f(x_i) + f(x_{i+1})}{2}\Delta x$$

$$= \frac{\Delta x}{2}\sum_{i=1}^n \big(f(x_i) + f(x_{i+1})\big)$$

$$= \frac{\Delta x}{2}\Big[f(x_1) + 2\sum_{i=2}^n f(x_i) + f(x_{n+1})\Big].$$

Example 5.5.4   Using the Trapezoidal Rule

Revisit Example 5.5.2 and approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx$ using the Trapezoidal Rule and 10 equally spaced subintervals.

SOLUTION   We refer back to Figure 5.5.3 for the table of values of $\sin(x^3)$. Recall that $\Delta x = 3\pi/40 \approx 0.236$. Thus we have:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\,dx \approx \frac{0.236}{2}\Big[-0.466 + 2\big(-0.165 + (-0.031) + \ldots + 0.69\big) + (-0.67)\Big]$$

$$= 0.4275.$$

Notice how "quickly" the Trapezoidal Rule can be implemented once the table of values is created. This is true for all the methods explored in this section; the real work is creating a table of x_i and $f(x_i)$ values. Once this is completed, approximating the definite integral is not difficult. Again, using technology is wise. Spreadsheets can make quick work of these computations and make using lots of subintervals easy.

Also notice the approximations the Trapezoidal Rule gives. It is the average of the approximations given by the Left and Right Hand Rules! This effectively

Notes:

renders the Left and Right Hand Rules obsolete. They are useful when first learning about definite integrals, but if a real approximation is needed, one is generally better off using the Trapezoidal Rule instead of either the Left or Right Hand Rule.

How can we improve on the Trapezoidal Rule, apart from using more and more trapezoids? The answer is clear once we look back and consider what we have *really* done so far. The Left Hand Rule is not *really* about using rectangles to approximate area. Instead, it approximates a function f with constant functions on small subintervals and then computes the definite integral of these constant functions. The Trapezoidal Rule is really approximating a function f with a linear function on a small subinterval, then computes the definite integral of this linear function. In both of these cases the definite integrals are easy to compute in geometric terms.

So we have a progression: we start by approximating f with a constant function and then with a linear function. What is next? A quadratic function. By approximating the curve of a function with lots of parabolas, we generally get an even better approximation of the definite integral. We call this process **Simpson's Rule**, named after Thomas Simpson (1710-1761), even though others had used this rule as much as 100 years prior.

Simpson's Rule

Given one point, we can create a constant function that goes through that point. Given two points, we can create a linear function that goes through those points. Given three points, we can create a quadratic function that goes through those three points (given that no two have the same x–value).

Consider three points (x_1, y_1), (x_2, y_2) and (x_3, y_3) whose x–values are equally spaced and $x_1 < x_2 < x_3$. Let f be the quadratic function that goes through these three points. It is not hard to show that

$$\int_{x_1}^{x_3} f(x)\, dx = \frac{x_3 - x_1}{6}\left(y_1 + 4y_2 + y_3\right). \tag{5.4}$$

Consider Figure 5.5.8. A function f goes through the 3 points shown and the parabola g that also goes through those points is graphed with a dashed line. Using our equation from above, we know exactly that

$$\int_{1}^{3} g(x)\, dx = \frac{3 - 1}{6}\left(3 + 4(1) + 2\right) = 3.$$

Since g is a good approximation for f on $[1, 3]$, we can state that
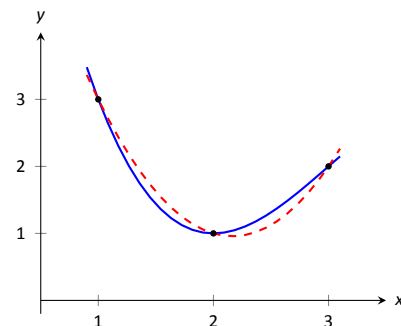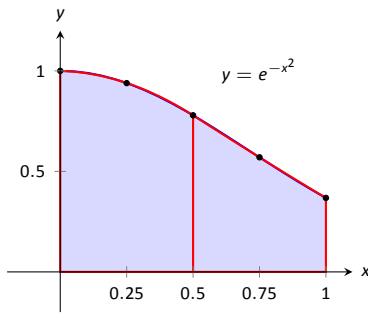
$$\int_{1}^{3} f(x)\, dx \approx 3.$$

Figure 5.5.8: A graph of a function f and a parabola that approximates it well on $[1, 3]$.

Notes:

x_i	$e^{-x_i^2}$
0	1
0.25	0.939
0.5	0.779
0.75	0.570
1	0.368

(a)

(b)

Figure 5.5.9: A table of values to approximate $\int_0^1 e^{-x^2}\, dx$, along with a graph of the function.

x_i	$\sin(x_i^3)$
−0.785	−0.466
−0.550	−0.165
−0.314	−0.031
−0.0785	0
0.157	0.004
0.393	0.061
0.628	0.246
0.864	0.601
1.10	0.971
1.34	0.690
1.57	−0.670

Figure 5.5.10: Table of values used to approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ in Example 5.5.6.

Notice how the interval $[1, 3]$ was split into two subintervals as we needed 3 points. Because of this, whenever we use Simpson's Rule, we need to break the interval into an even number of subintervals.

In general, to approximate $\int_a^b f(x)\, dx$ using Simpson's Rule, subdivide $[a, b]$ into n subintervals, where n is even and each subinterval has width $\Delta x = (b - a)/n$. We approximate f with $n/2$ parabolic curves, using Equation (5.4) to compute the area under these parabolas. Adding up these areas gives the formula:

$$\int_a^b f(x)\, dx \approx \frac{\Delta x}{3}\Big[f(x_1)+4f(x_2)+2f(x_3)+4f(x_4)+\ldots+2f(x_{n-1})+4f(x_n)+f(x_{n+1})\Big].$$

Note how the coefficients of the terms in the summation have the pattern 1, 4, 2, 4, 2, 4, . . ., 2, 4, 1.

Let's demonstrate Simpson's Rule with a concrete example.

Example 5.5.5   Using Simpson's Rule

Approximate $\int_0^1 e^{-x^2}\, dx$ using Simpson's Rule and 4 equally spaced subintervals.

SOLUTION   We begin by making a table of values as we have in the past, as shown in Figure 5.5.9(a). Simpson's Rule states that

$$\int_0^1 e^{-x^2}\, dx \approx \frac{0.25}{3}\Big[1 + 4(0.939) + 2(0.779) + 4(0.570) + 0.368\Big] = 0.74683\overline{3}.$$

Recall in Example 5.5.1 we stated that the correct answer, accurate to 4 places after the decimal, was 0.7468. Our approximation with Simpson's Rule, with 4 subintervals, is better than our approximation with the Trapezoidal Rule using 5!

Figure 5.5.9(b) shows $f(x) = e^{-x^2}$ along with its approximating parabolas, demonstrating how good our approximation is. The approximating curves are nearly indistinguishable from the actual function.

Example 5.5.6   Using Simpson's Rule

Approximate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ using Simpson's Rule and 10 equally spaced intervals.

SOLUTION   Figure 5.5.10 shows the table of values that we used in the past for this problem, shown here again for convenience. Again, $\Delta x = (\pi/2 + \pi/4)/10 \approx 0.236$.

Notes:

Simpson's Rule states that

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx \approx \frac{0.236}{3}\Big[(-0.466) + 4(-0.165) + 2(-0.031) + \ldots$$

$$\ldots + 2(0.971) + 4(0.69) + (-0.67)\Big]$$

$$= 0.4701$$

Recall that the actual value, accurate to 3 decimal places, is 0.460. Our approximation is within one $1/100^{\text{th}}$ of the correct value. The graph in Figure 5.5.11 shows how closely the parabolas match the shape of the graph.
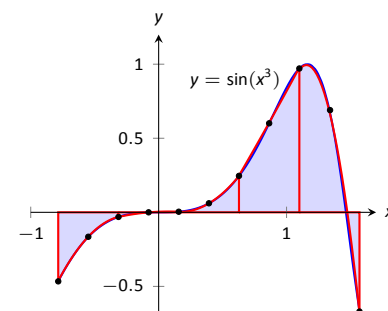
Summary and Error Analysis

We summarize the key concepts of this section thus far in the following Key Idea.

Figure 5.5.11: Approximating $\int_{-\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x^3)\, dx$ in Example 5.5.6 with Simpson's Rule and 10 equally spaced intervals.

Key Idea 5.5.1   Numerical Integration

Let f be a continuous function on $[a, b]$, let n be a positive integer, and let $\Delta x = \dfrac{b - a}{n}$.

Set $x_1 = a$, $x_2 = a + \Delta x$, $\ldots$, $x_i = a + (i - 1)\Delta x$, $x_{n+1} = b$.

Consider $\displaystyle\int_a^b f(x)\, dx$.

Left Hand Rule: $\displaystyle\int_a^b f(x)\, dx \approx \Delta x\Big[f(x_1) + f(x_2) + \ldots + f(x_n)\Big]$.

Right Hand Rule: $\displaystyle\int_a^b f(x)\, dx \approx \Delta x\Big[f(x_2) + f(x_3) + \ldots + f(x_{n+1})\Big]$.

Trapezoidal Rule: $\displaystyle\int_a^b f(x)\, dx \approx \frac{\Delta x}{2}\Big[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\Big]$.

Simpson's Rule: $\displaystyle\int_a^b f(x)\, dx \approx \frac{\Delta x}{3}\Big[f(x_1) + 4f(x_2) + 2f(x_3) + \ldots + 4f(x_n) + f(x_{n+1})\Big]$ (n even).

In our examples, we approximated the value of a definite integral using a given method then compared it to the "right" answer. This should have raised several questions in the reader's mind, such as:

1. How was the "right" answer computed?

2. If the right answer can be found, what is the point of approximating?

3. If there is value to approximating, how are we supposed to know if the approximation is any good?

Notes:

These are good questions, and their answers are educational. In the examples, *the* right answer was never computed. Rather, an approximation accurate to a certain number of places after the decimal was given. In Example 5.5.1, we do not know the *exact* answer, but we know it starts with 0.7468. These more accurate approximations were computed using numerical integration but with more precision (i.e., more subintervals and the help of a computer).

Since the exact answer cannot be found, approximation still has its place. How are we to tell if the approximation is any good?

"Trial and error" provides one way. Using technology, make an approximation with, say, 10, 100, and 200 subintervals. This likely will not take much time at all, and a trend should emerge. If a trend does not emerge, try using yet more subintervals. Keep in mind that trial and error is never foolproof; you might stumble upon a problem in which a trend will not emerge.

A second method is to use Error Analysis. While the details are beyond the scope of this text, there are some formulas that give *bounds* for how good your approximation will be. For instance, the formula might state that the approximation is within 0.1 of the correct answer. If the approximation is 1.58, then one knows that the correct answer is between 1.48 and 1.68. By using lots of subintervals, one can get an approximation as accurate as one likes. Theorem 5.5.1 states what these bounds are.

> **Theorem 5.5.1   Error Bounds in the Trapezoidal and Simpson's Rules**
>
> 1. Let E_T be the error in approximating $\int_a^b f(x)\, dx$ using the Trapezoidal Rule.
>
>   If f has a continuous 2nd derivative on $[a, b]$ and M is any upper bound of $\left|f''(x)\right|$ on $[a, b]$, then
>
> $$E_T \leq \frac{(b-a)^3}{12n^2} M.$$
>
> 2. Let E_S be the error in approximating $\int_a^b f(x)\, dx$ using Simpson's Rule.
>
>   If f has a continuous 4th derivative on $[a, b]$ and M is any upper bound of $\left|f^{(4)}\right|$ on $[a, b]$, then
>
> $$E_S \leq \frac{(b-a)^5}{180n^4} M.$$

There are some key things to note about this theorem.

Notes:

1. The larger the interval, the larger the error. This should make sense intuitively.

2. The error shrinks as more subintervals are used (i.e., as n gets larger).

3. The error in Simpson's Rule has a term relating to the 4th derivative of f. Consider a cubic polynomial: it's 4th derivative is 0. Therefore, the error in approximating the definite integral of a cubic polynomial with Simpson's Rule is 0 – Simpson's Rule computes the exact answer!

We revisit Examples 5.5.3 and 5.5.5 and compute the error bounds using Theorem 5.5.1 in the following example.

Example 5.5.7   Computing error bounds

Find the error bounds when approximating $\int_0^1 e^{-x^2}\, dx$ using the Trapezoidal Rule and 5 subintervals, and using Simpson's Rule with 4 subintervals.

SOLUTION
Trapezoidal Rule with $n = 5$:

We start by computing the 2nd derivative of $f(x) = e^{-x^2}$:

$$f''(x) = e^{-x^2}(4x^2 - 2).$$

Figure 5.5.12 shows a graph of $f''(x)$ on $[0, 1]$. It is clear that the largest value of f'', in absolute value, is 2. Thus we let $M = 2$ and apply the error formula from Theorem 5.5.1.

$$E_T = \frac{(1-0)^3}{12 \cdot 5^2} \cdot 2 = 0.00\overline{6}.$$

Our error estimation formula states that our approximation of 0.7445 found in Example 5.5.3 is within 0.0067 of the correct answer, hence we know that

$$0.7445 - 0.0067 = .7378 \leq \int_0^1 e^{-x^2}\, dx \leq 0.7512 = 0.7445 + 0.0067.$$

We had earlier computed the exact answer, correct to 4 decimal places, to be 0.7468, affirming the validity of Theorem 5.5.1.

Simpson's Rule with $n = 4$:

We start by computing the 4th derivative of $f(x) = e^{-x^2}$:

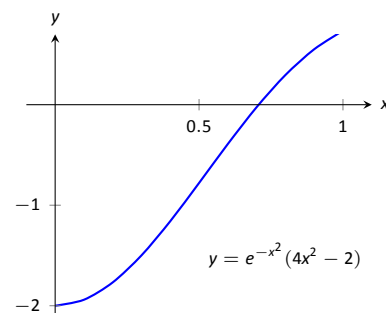$$f^{(4)}(x) = e^{-x^2}(16x^4 - 48x^2 + 12).$$

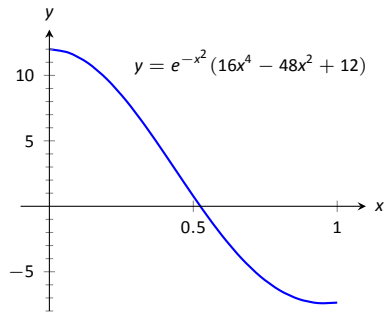Figure 5.5.12: Graphing $f''(x)$ in Example 5.5.7 to help establish error bounds.

In the figure: $y = e^{-x^2}(4x^2 - 2)$

Notes:

Figure 5.5.13: Graphing $f^{(4)}(x)$ in Example 5.5.7 to help establish error bounds.

Time	Speed (mph)
0	0
1	25
2	22
3	19
4	39
5	0
6	43
7	59
8	54
9	51
10	43
11	35
12	40
13	43
14	30
15	0
16	0
17	28
18	40
19	42
20	40
21	39
22	40
23	23
24	0

Figure 5.5.14: Speed data collected at 30 second intervals for Example 5.5.8.

Figure 5.5.13 shows a graph of $f^{(4)}(x)$ on $[0, 1]$. It is clear that the largest value of $f^{(4)}$, in absolute value, is 12. Thus we let $M = 12$ and apply the error formula from Theorem 5.5.1.

$$E_s = \frac{(1-0)^5}{180 \cdot 4^4} \cdot 12 = 0.00026.$$

Our error estimation formula states that our approximation of $0.7468\overline{3}$ found in Example 5.5.5 is within 0.00026 of the correct answer, hence we know that

$$0.74683 - 0.00026 = .74657 \leq \int_0^1 e^{-x^2} \, dx \leq 0.74709 = 0.74683 + 0.00026.$$

Once again we affirm the validity of Theorem 5.5.1.

At the beginning of this section we mentioned two main situations where numerical integration was desirable. We have considered the case where an antiderivative of the integrand cannot be computed. We now investigate the situation where the integrand is not known. This is, in fact, the most widely used application of Numerical Integration methods. "Most of the time" we observe behavior but do not know "the" function that describes it. We instead collect data about the behavior and make approximations based off of this data. We demonstrate this in an example.

Example 5.5.8 Approximating distance traveled
One of the authors drove his daughter home from school while she recorded their speed every 30 seconds. The data is given in Figure 5.5.14. Approximate the distance they traveled.

SOLUTION Recall that by integrating a speed function we get distance traveled. We have information about $v(t)$; we will use Simpson's Rule to approximate $\int_a^b v(t) \, dt$.

The most difficult aspect of this problem is converting the given data into the form we need it to be in. The speed is measured in miles per hour, whereas the time is measured in 30 second increments.

We need to compute $\Delta x = (b - a)/n$. Clearly, $n = 24$. What are a and b? Since we start at time $t = 0$, we have that $a = 0$. The final recorded time came after 24 periods of 30 seconds, which is 12 minutes or 1/5 of an hour. Thus we have

$$\Delta x = \frac{b - a}{n} = \frac{1/5 - 0}{24} = \frac{1}{120}; \quad \frac{\Delta x}{3} = \frac{1}{360}.$$

Notes:

Thus the distance traveled is approximately:

$$\int_0^{0.2} v(t) \, dt \approx \frac{1}{360} \Big[f(x_1) + 4f(x_2) + 2f(x_3) + \cdots + 4f(x_n) + f(x_{n+1}) \Big]$$
$$= \frac{1}{360} \Big[0 + 4 \cdot 25 + 2 \cdot 22 + \cdots + 2 \cdot 40 + 4 \cdot 23 + 0 \Big]$$
$$\approx 6.2167 \text{ miles.}$$

We approximate the author drove 6.2 miles. (Because we are sure the reader wants to know, the author's odometer recorded the distance as about 6.05 miles.)

We started this chapter learning about antiderivatives and indefinite integrals. We then seemed to change focus by looking at areas between the graph of a function and the *x*-axis. We defined these areas as the definite integral of the function, using a notation very similar to the notation of the indefinite integral. The Fundamental Theorem of Calculus tied these two seemingly separate concepts together: we can find areas under a curve, i.e., we can evaluate a definite integral, using antiderivatives.

We ended the chapter by noting that antiderivatives are sometimes more than difficult to find: they are impossible. Therefore we developed numerical techniques that gave us good approximations of definite integrals.

We used the definite integral to compute areas, and also to compute displacements and distances traveled. There is far more we can do than that. In Chapter 7 we'll see more applications of the definite integral. Before that, in Chapter 6 we'll learn advanced techniques of integration, analogous to learning rules like the Product, Quotient and Chain Rules of differentiation.

Notes:

Exercises 5.5

Terms and Concepts

1. T/F: Simpson's Rule is a method of approximating an-
   tiderivatives.

2. What are the two basic situations where approximating the
   value of a definite integral is necessary?

3. Why are the Left and Right Hand Rules rarely used?

4. Simpson's Rule is based on approximating portions of a
   function with what type of function?

Problems

In Exercises 5 – 12, a definite integral is given.

  (a) Approximate the definite integral with the Trapezoidal
     Rule and $n = 4$.

  (b) Approximate the definite integral with Simpson's Rule
     and $n = 4$.

  (c) Find the exact value of the integral.

5. $\int_{-1}^{1} x^2 \, dx$

6. $\int_{0}^{10} 5x \, dx$

7. $\int_{0}^{\pi} \sin x \, dx$

8. $\int_{0}^{4} \sqrt{x} \, dx$

9. $\int_{0}^{3} (x^3 + 2x^2 - 5x + 7) \, dx$

10. $\int_{0}^{1} x^4 \, dx$

11. $\int_{0}^{2\pi} \cos x \, dx$

12. $\int_{-3}^{3} \sqrt{9 - x^2} \, dx$

In Exercises 13 – 20, approximate the definite integral with
the Trapezoidal Rule and Simpson's Rule, with $n = 6$.

13. $\int_{0}^{1} \cos(x^2) \, dx$

14. $\int_{-1}^{1} e^{x^2} \, dx$

15. $\int_{0}^{5} \sqrt{x^2 + 1} \, dx$

16. $\int_{0}^{\pi} x \sin x \, dx$

17. $\int_{0}^{\pi/2} \sqrt{\cos x} \, dx$

18. $\int_{1}^{4} \ln x \, dx$

19. $\int_{-1}^{1} \frac{1}{\sin x + 2} \, dx$

20. $\int_{0}^{6} \frac{1}{\sin x + 2} \, dx$

In Exercises 21 – 24, find n such that the error in approximat-
ing the given definite integral is less than 0.0001 when using:

  (a) the Trapezoidal Rule

  (b) Simpson's Rule

21. $\int_{0}^{\pi} \sin x \, dx$

22. $\int_{1}^{4} \frac{1}{\sqrt{x}} \, dx$

23. $\int_{0}^{\pi} \cos(x^2) \, dx$

24. $\int_{0}^{5} x^4 \, dx$

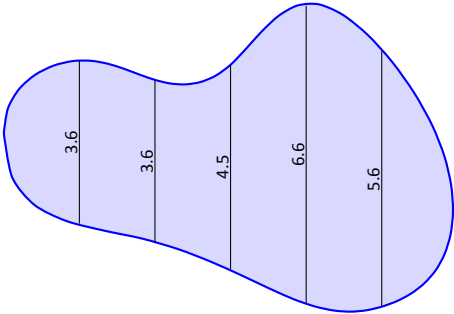In Exercises 25 – 26, a region is given. Find the area of the
region using Simpson's Rule:

  (a) where the measurements are in centimeters, taken in
     1 cm increments, and

  (b) where the measurements are in hundreds of yards,
     taken in 100 yd increments.

25.

26.
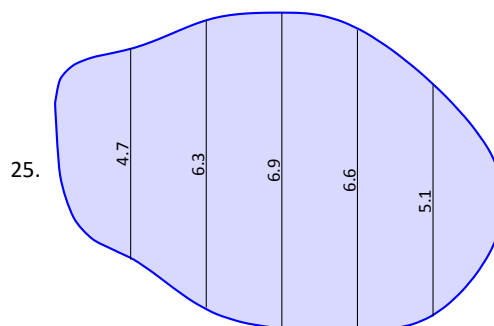
3.6 3.6 4.5 6.6 5.6

A: Solutions To Selected Problems

Chapter 5

Section 5.1

1. Answers will vary.

2. "an"

3. Answers will vary.

4. opposite; opposite

5. Answers will vary.

6. velocity

7. velocity

8. $F(x) + G(x)$

9. $3/4x^4 + C$

10. $1/9x^9 + C$

11. $10/3x^3 - 2x + C$

12. $t + C$

13. $s + C$

14. $-1/(3t) + C$

15. $-3/(t) + C$

16. $2\sqrt{x} + C$

17. $\tan\theta + C$

18. $-\cos\theta + C$

19. $\sec x - \csc x + C$

20. $5e^\theta + C$

21. $3^t / \ln 3 + C$

22. $\frac{5^t}{2\ln 5} + C$

23. $4/3t^3 + 6t^2 + 9t + C$

24. $t^6/6 + t^4/4 - 3t^2 + C$

25. $x^6/6 + C$

26. $e^\pi x + C$

27. $ax + C$

28.
   (a) $x > 0$
   (b) $1/x$
   (c) $x < 0$
   (d) $1/x$
   (e) $\ln|x| + C$. Explanations will vary.

29. $-\cos x + 3$

30. $5e^x + 5$

31. $x^4 - x^3 + 7$

32. $\tan x + 4$

33. $7^x / \ln 7 + 1 - 49/\ln 7$

34. $5/2x^2 + 7x + 3$

35. $\frac{7x^3}{6} - \frac{9x}{2} + \frac{40}{3}$

36. $5e^x - 2x$

37. $\theta - \sin(\theta) - \pi + 4$

38. $2x^4 + \cos x + \frac{2^x}{(\ln 2)^2} + (5 - \frac{1}{\ln 2})x - 1 - \frac{1}{(\ln 2)^2}$

39. $3x - 2$

40. No answer provided.

41. $dy = (2xe^x \cos x + x^2 e^x \cos x - x^2 e^x \sin x)dx$

Section 5.2

1. Answers will vary.

2. Answers will vary.

3. 0

4. $\int 0^2 (2x + 3)\, dx$

5.
   (a) 3
   (b) 4
   (c) 3
   (d) 0
   (e) -4
   (f) 9

6.
   (a) -4
   (b) -5
   (c) -3
   (d) 1
   (e) -2
   (f) 10

7.
   (a) 4
   (b) 2
   (c) 4
   (d) 2
   (e) 1
   (f) 2

8.
   (a) $-1/2$
   (b) 0
   (c) $3/2$
   (d) $3/2$
   (e) $9/2$
   (f) $15/2$

9.

    (a) π

    (b) π

    (c) 2π

    (d) 10π

10.

    (a) 15

    (b) 12

    (c) 0

    (d) $3(b-a)$

11.

    (a) -59

    (b) -48

    (c) -27

    (d) -33

12.

    (a) $4/\pi$

    (b) $-4/\pi$

    (c) 0

    (d) $2/\pi$

13.

    (a) 4

    (b) 4

    (c) -4

    (d) -2

14.

    (a) $40/3$

    (b) $26/3$

    (c) $8/3$

    (d) $38/3$

15.

    (a) 2ft/s

    (b) 2ft

    (c) 1.5ft

16.

    (a) 3ft/s

    (b) 9.5ft

    (c) 9.5ft

17.

    (a) 64ft/s

    (b) 64ft

    (c) $t = 2$

    (d) $t = 2 + \sqrt{7} \approx 4.65$ seconds

18.

    (a) 96ft/s

    (b) 6 seconds

    (c) 6 seconds

    (d) Never; the maximum height is 208ft.

19. 2

20. 5

21. 16

22. Answers can vary; one solution is $a = -2, b = 7$

23. 24

24. -7

25. -7

26. Answers can vary; one solution is $a = -18, b = 11$

27. $1/4x^4 - 2/3x^3 + 7/2x^2 - 9x + C$

28. $-\cos x - \sin x + \tan x + C$

29. $3/4t^{4/3} - 1/t + 2^t/\ln 2 + C$

30. $\ln|x| + \csc x + C$

Section 5.3

1. limits

2. 14

3. Rectangles.

4. T

5. $2^2 + 3^2 + 4^2 = 29$

6. $-6 - 2 + 2 + 6 + 10 = 10$

7. $0 - 1 + 0 + 1 + 0 = 0$

8. $5 + 5 + \ldots + 5 = 50$

9. $1 + 1/2 + 1/3 + 1/4 + 1/5 = 137/60$

10. $-1 + 2 - 3 + 4 - 5 + 6 = 3$

11. $1/2 + 1/6 + 1/12 + 1/20 = 4/5$

12. $1 + 1 + 1 + 1 + 1 + 1 = 6$

13. Answers may vary; $\sum_{i=1}^{5} 3i$

14. Answers may vary; $\sum_{i=0}^{8}(i^2 - 1)$

15. Answers may vary; $\sum_{i=1}^{4} \frac{i}{i+1}$

16. Answers may vary; $\sum_{i=0}^{4}(-1)^i e^i$

17. $5 \cdot 10 = 50$

18. 325

19. 1045

20. 28,650

21. -8525

22. 2050

23. 5050

24. 2870

25. 155

26. $91, 225$

27. 24

28. 11, 700

29. 19

30. 59/8

31. $\pi/3 + \pi/(2\sqrt{3}) \approx 1.954$

32. 8.144

33. 0.388584

34. $496/315 \approx 1.5746$

35.

   (a) Exact expressions will vary; $\frac{(1+n)^2}{4n^2}$.

   (b) 121/400, 10201/40000, 1002001/4000000

   (c) 1/4

36.

   (a) Exact expressions will vary; $2 + 4/n^2$.

   (b) 51/25, 5001/2500, 500001/250000

   (c) 2

37.

   (a) 8.

   (b) 8, 8, 8

   (c) 8

38.

   (a) Exact expressions will vary;
       $20/3 - 96/(3n) + 64/(3n^2)$.

   (b) 92/25, 3968/625, 103667/15625

   (c) 20/3

39.

   (a) Exact expressions will vary; $100 - 200/n$.

   (b) 80, 98, 499/5

   (c) 100

40.

   (a) Exact expressions will vary; $-1/12(1 - 1/n^2)$.

   (b) $-33/400, -3333/40000, -333333/4000000$

   (c) $-1/12$

41. $F(x) = 5\tan x + 4$

42. $F(x) = 7\ln|x| + 14$

43. $G(t) = 4/6t^6 - 5/4t^4 + 8t + 9$

44. $G(t) = 5 \cdot 8^t/\ln 8 + 900$

45. $G(t) = \sin t - \cos t - 78$

46. $F(x) = 2\sqrt{x} - \pi$

Section 5.4

1. Answers will vary.

2. 0

3. T

4. Answers will vary.

5. 20

6. 28/3

7. 0

8. 1

9. 1

10. 1

11. $(5 - 1/5)/\ln 5$

12. 23/2

13. -4

14. $e^3 - e$

15. 16/3

16. 4

17. 45/4

18. ln 2

19. 1/2

20. 3/8

21. 1/2

22. 1/3

23. 1/4

24. 1/101

25. 8

26. 15

27. 0

28. $2 - 2/\sqrt{3}$

29. Explanations will vary. A sketch will help.

30. $\int_a^{a+2\pi} \sin t\, dt = \cos(a + 2\pi) - \cos(a)$. Since cosine is periodic with period 2π, $\cos(a + 2\pi) = \cos(a)$, and hence the integral is 0.

31. $c = 2/\sqrt{3}$

32. $c = \pm 2/\sqrt{3}$

33. $c = \ln(e - 1) \approx 0.54$

34. $c = 64/9 \approx 7.1$

35. $2/\pi$

36. $2/pi$

37. 2

38. 16/3

39. 16

40. $1/(e - 1)$

41. -300ft

42. 400ft

43. 30ft

44. $1.5/\ln(2) \approx 2.164$miles

45. -1ft

46. $128/5$ft

47. -64ft/s

48. 50ft/s

49. 2ft/s

50. 0ft/s

51. $27/2$

52. 21

53. $9/2$

54. $343/6$

55. $F'(x) = (3x^2 + 1)\frac{1}{x^3+x}$

56. $F'(x) = -3x^{11}$

57. $F'(x) = 2x(x^2 + 2) - (x + 2)$

58. $F'(x) = e^x \sin(e^x) - 1/x \sin(\ln x)$

Section 5.5

1. F

2. When the antiderivative cannot be computed and when the integrand is unknown.

3. They are superseded by the Trapezoidal Rule; it takes an equal amount of work and is generally more accurate.

4. A quadratic function (i.e., parabola)

5.
   (a) $3/4$
   (b) $2/3$
   (c) $2/3$

6.
   (a) 250
   (b) 250
   (c) 250

7.
   (a) $\frac{1}{4}(1 + \sqrt{2})\pi \approx 1.896$
   (b) $\frac{1}{6}(1 + 2\sqrt{2})\pi \approx 2.005$
   (c) 2

8.
   (a) $2 + \sqrt{2} + \sqrt{3} \approx 5.15$
   (b) $2/3(3 + \sqrt{2} + 2\sqrt{3}) \approx 5.25$
   (c) $16/3 \approx 5.33$

9.
   (a) 38.5781
   (b) $147/4 \approx 36.75$
   (c) $147/4 \approx 36.75$

10.
   (a) 0.2207
   (b) 0.2005

(c) $1/5$

11.
   (a) 0
   (b) 0
   (c) 0

12.
   (a) $9/2(1 + \sqrt{3}) \approx 12.294$
   (b) $3 + 6\sqrt{3} \approx 13.392$
   (c) $9\pi/2 \approx 14.137$

13. Trapezoidal Rule: 0.9006
   Simpson's Rule: 0.90452

14. Trapezoidal Rule: 3.0241
   Simpson's Rule: 2.9315

15. Trapezoidal Rule: 13.9604
   Simpson's Rule: 13.9066

16. Trapezoidal Rule: 3.0695
   Simpson's Rule: 3.14295

17. Trapezoidal Rule: 1.1703
   Simpson's Rule: 1.1873

18. Trapezoidal Rule: 2.52971
   Simpson's Rule: 2.5447

19. Trapezoidal Rule: 1.0803
   Simpson's Rule: 1.077

20. Trapezoidal Rule: 3.5472
   Simpson's Rule: 3.6133

21.
   (a) $n = 161$ (using max $(f''(x)) = 1$)
   (b) $n = 12$ (using max $(f^{(4)}(x)) = 1$)

22.
   (a) $n = 150$ (using max $(f''(x)) = 1$)
   (b) $n = 18$ (using max $(f^{(4)}(x)) = 7$)

23.
   (a) $n = 1004$ (using max $(f''(x)) = 39$)
   (b) $n = 62$ (using max $(f^{(4)}(x)) = 800$)

24.
   (a) $n = 5591$ (using max $(f''(x)) = 300$)
   (b) $n = 46$ (using max $(f^{(4)}(x)) = 24$)

25.
   (a) Area is 30.8667 cm^2.
   (b) Area is 308, 667 yd^2.

26.
   (a) Area is 25.0667 cm^2
   (b) Area is 250,667 yd^2

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