1: Vector Valued Functions

In the previous chapter, we learned about vectors and were introduced to the power of vectors within mathematics. In this chapter, we'll build on this foundation to define functions whose input is a real number and whose output is a vector. We'll see how to graph these functions and apply calculus techniques to analyze their behavior. Most importantly, we'll see *why* we are interested in doing this: we'll see beautiful applications to the study of moving objects.

1.1 Vector–Valued Functions

We are very familiar with **real valued functions**, that is, functions whose output is a real number. This section introduces **vector–valued functions** – functions whose output is a vector.

Definition 1 Vector–Valued Functions

A **vector–valued function** is a function of the form

$$\vec{r}(t) = \langle\, f(t), g(t)\, \rangle \quad \text{or} \quad \vec{r}(t) = \langle\, f(t), g(t), h(t)\, \rangle,$$

where f, g and h are real valued functions.

The **domain** of $\vec{r}$ is the set of all values of t for which $\vec{r}(t)$ is defined. The **range** of $\vec{r}$ is the set of all possible output vectors $\vec{r}(t)$.

Evaluating and Graphing Vector–Valued Functions

Evaluating a vector–valued function at a specific value of t is straightforward; simply evaluate each component function at that value of t. For instance, if $\vec{r}(t) = \langle t^2, t^2 + t - 1\rangle$, then $\vec{r}(-2) = \langle 4, 1\rangle$. We can sketch this vector, as is done in Figure 1.1(a). Plotting lots of vectors is cumbersome, though, so generally we do not sketch the whole vector but just the terminal point. The **graph** of a vector–valued function is the set of all terminal points of $\vec{r}(t)$, where the initial point of each vector is always the origin. In Figure 1.1(b) we sketch the graph of $\vec{r}$; we can indicate individual points on the graph with their respective vector, as shown.

Vector–valued functions are closely related to parametric equations of graphs. While in both methods we plot points $\big(x(t), y(t)\big)$ or $\big(x(t), y(t), z(t)\big)$ to produce a graph, in the context of vector–valued functions each such point represents a vector. The implications of this will be more fully realized in the next section as we apply calculus ideas to these functions.

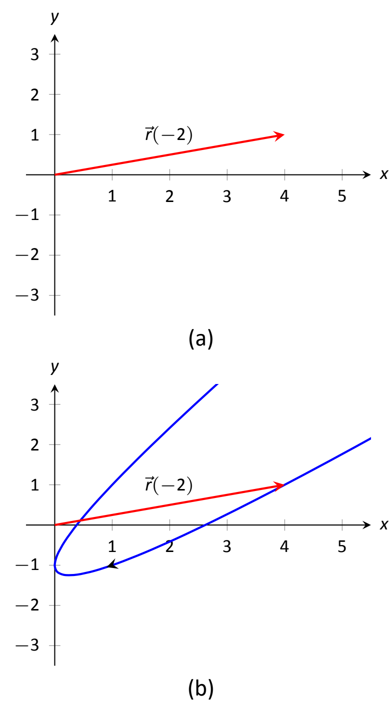
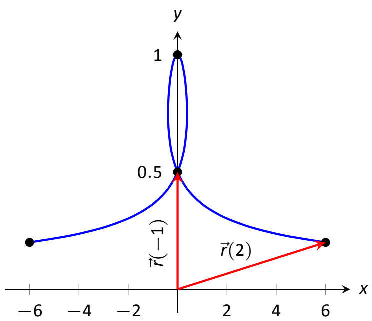


Figure 1.1: Sketching the graph of a vector–valued function.

t	$t^3 - t$	$\frac{1}{t^2+1}$
-2	-6	1/5
-1	0	1/2
0	0	1
1	0	1/2
2	6	1/5

(a)



(b)

Figure 1.2: Sketching the vector–valued function of Example 1.

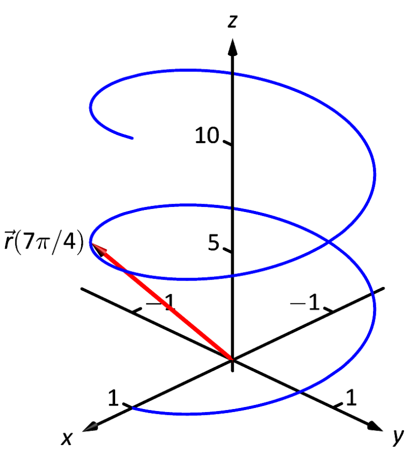


Figure 1.3: The graph of $\vec{r}(t)$ in Example 2.

Example 1 Graphing vector–valued functions

Graph $\vec{r}(t) = \left\langle t^3 - t, \frac{1}{t^2+1} \right\rangle$, for $-2 \leq t \leq 2$. Sketch $\vec{r}(-1)$ and $\vec{r}(2)$.

Solution We start by making a table of t, x and y values as shown in Figure 1.2(a). Plotting these points gives an indication of what the graph looks like. In Figure 1.2(b), we indicate these points and sketch the full graph. We also highlight $\vec{r}(-1)$ and $\vec{r}(2)$ on the graph.

Example 2 Graphing vector–valued functions.

Graph $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 4\pi$.

Solution We can again plot points, but careful consideration of this function is very revealing. Momentarily ignoring the third component, we see the x and y components trace out a circle of radius 1 centered at the origin. Noticing that the z component is t, we see that as the graph winds around the z-axis, it is also increasing at a constant rate in the positive z direction, forming a spiral. This is graphed in Figure 1.3. In the graph $\vec{r}(7\pi/4) \approx (0.707, -0.707, 5.498)$ is highlighted to help us understand the graph.

Algebra of Vector–Valued Functions

Definition 2 Operations on Vector–Valued Functions

Let $\vec{r}_1(t) = \langle f_1(t), g_1(t) \rangle$ and $\vec{r}_2(t) = \langle f_2(t), g_2(t) \rangle$ be vector–valued functions in $\mathbb{R}^2$ and let c be a scalar. Then:

1. $\vec{r}_1(t) \pm \vec{r}_2(t) = \langle\, f_1(t) \pm f_2(t), g_1(t) \pm g_2(t) \,\rangle$.
2. $c\vec{r}_1(t) = \langle\, cf_1(t), cg_1(t) \,\rangle$.

A similar definition holds for vector–valued functions in $\mathbb{R}^3$.

This definition states that we add, subtract and scale vector-valued functions component–wise. Combining vector–valued functions in this way can be very useful (as well as create interesting graphs).

Example 3 Adding and scaling vector–valued functions.

Let $\vec{r}_1(t) = \langle\, 0.2t, 0.3t \,\rangle$, $\vec{r}_2(t) = \langle\, \cos t, \sin t \,\rangle$ and $\vec{r}(t) = \vec{r}_1(t) + \vec{r}_2(t)$. Graph $\vec{r}_1(t)$, $\vec{r}_2(t)$, $\vec{r}(t)$ and $5\vec{r}(t)$ on $-10 \leq t \leq 10$.

Notes:

Solution We can graph $\vec{r}_1$ and $\vec{r}_2$ easily by plotting points (or just using technology). Let's think about each for a moment to better understand how vector–valued functions work.

We can rewrite $\vec{r}_1(t) = \langle 0.2t, 0.3t \rangle$ as $\vec{r}_1(t) = t\,\langle 0.2, 0.3\rangle$. That is, the function $\vec{r}_1$ scales the vector $\langle 0.2, 0.3\rangle$ by t. This scaling of a vector produces a line in the direction of $\langle 0.2, 0.3\rangle$.

We are familiar with $\vec{r}_2(t) = \langle \cos t, \sin t \rangle$; it traces out a circle, centered at the origin, of radius 1. Figure 1.4(a) graphs $\vec{r}_1(t)$ and $\vec{r}_2(t)$.

Adding $\vec{r}_1(t)$ to $\vec{r}_2(t)$ produces $\vec{r}(t) = \langle \cos t + 0.2t, \sin t + 0.3t \rangle$, graphed in Figure 1.4(b). The linear movement of the line combines with the circle to create loops that move in the direction of $\langle 0.2, 0.3\rangle$. (We encourage the reader to experiment by changing $\vec{r}_1(t)$ to $\langle 2t, 3t\rangle$, etc., and observe the effects on the loops.)

Multiplying $\vec{r}(t)$ by 5 scales the function by 5, producing $5\vec{r}(t) = \langle 5\cos t + 1, 5\sin t + 1.5\rangle$, which is graphed in Figure 1.4(c) along with $\vec{r}(t)$. The new function is "5 times bigger" than $\vec{r}(t)$. Note how the graph of $5\vec{r}(t)$ in (c) looks identical to the graph of $\vec{r}(t)$ in (b). This is due to the fact that the x and y bounds of the plot in (c) are exactly 5 times larger than the bounds in (b).

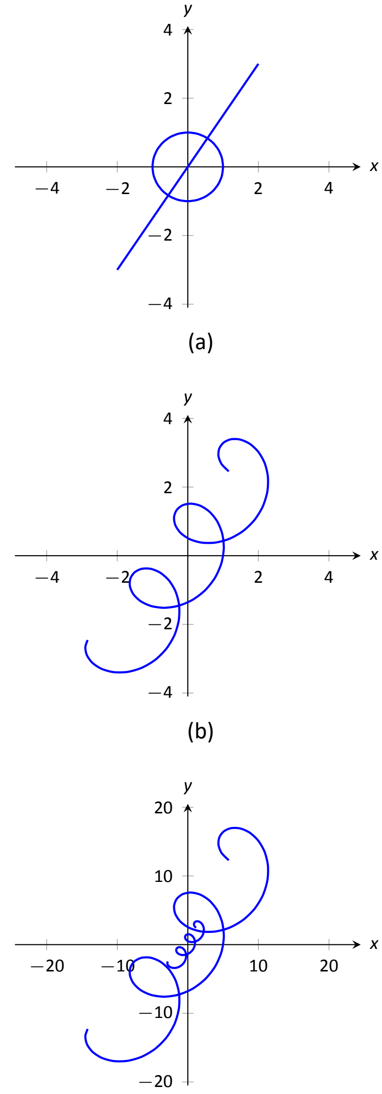


Figure 1.4: Graphing the functions in Example 3.

Example 4 **Adding and scaling vector–valued functions.**

A **cycloid** is a graph traced by a point p on a rolling circle, as shown in Figure 1.5. Find an equation describing the cycloid, where the circle has radius 1.

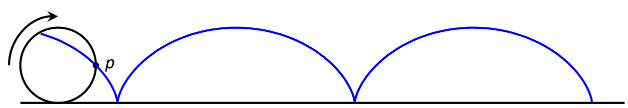


Figure 1.5: Tracing a cycloid.

Solution This problem is not very difficult if we approach it in a clever way. We start by letting $\vec{p}(t)$ describe the position of the point p on the circle, where the circle is centered at the origin and only rotates clockwise (i.e., it does not roll). This is relatively simple given our previous experiences with parametric equations; $\vec{p}(t) = \langle \cos t, -\sin t\rangle$.

We now want the circle to roll. We represent this by letting $\vec{c}(t)$ represent the location of the center of the circle. It should be clear that the y component of $\vec{c}(t)$ should be 1; the center of the circle is always going to be 1 if it rolls on a horizontal surface.

The x component of $\vec{c}(t)$ is a linear function of t: $f(t) = mt$ for some scalar m. When $t = 0$, $f(t) = 0$ (the circle starts centered on the y-axis). When $t = 2\pi$, the circle has made one complete revolution, traveling a distance equal to its

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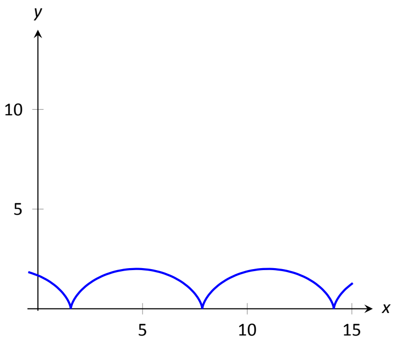


Figure 1.6: The cycloid in Example 4.

circumference, which is also 2π. This gives us a point on our line $f(t) = mt$, the point $(2\pi, 2\pi)$. It should be clear that $m = 1$ and $f(t) = t$. So $\vec{c}(t) = \langle t, 1\rangle$.

We now combine $\vec{p}$ and $\vec{c}$ together to form the equation of the cycloid: $\vec{r}(t) = \vec{p}(t) + \vec{c}(t) = \langle \cos t + t, -\sin t + 1\rangle$, which is graphed in Figure 1.6.

Displacement

A vector–valued function $\vec{r}(t)$ is often used to describe the position of a moving object at time t. At $t = t_0$, the object is at $\vec{r}(t_0)$; at $t = t_1$, the object is at $\vec{r}(t_1)$. Knowing the locations $\vec{r}(t_0)$ and $\vec{r}(t_1)$ give no indication of the path taken between them, but often we only care about the difference of the locations, $\vec{r}(t_1) - \vec{r}(t_0)$, the **displacement**.

Definition 3 Displacement

Let $\vec{r}(t)$ be a vector–valued function and let $t_0 < t_1$ be values in the domain. The **displacement** $\vec{d}$ of $\vec{r}$, from $t = t_0$ to $t = t_1$, is

$$\vec{d} = \vec{r}(t_1) - \vec{r}(t_0).$$

When the displacement vector is drawn with initial point at $\vec{r}(t_0)$, its terminal point is $\vec{r}(t_1)$. We think of it as the vector which points from a starting position to an ending position.

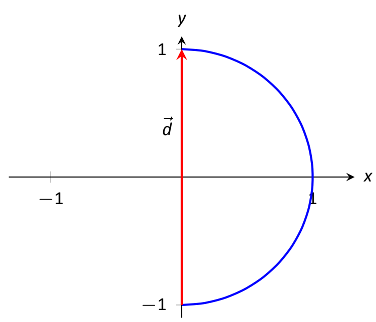


Figure 1.7: Graphing the displacement of a position function in Example 5.

Example 5 Finding and graphing displacement vectors

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t)\right\rangle$. Graph $\vec{r}(t)$ on $-1 \leq t \leq 1$, and find the displacement of $\vec{r}(t)$ on this interval.

Solution The function $\vec{r}(t)$ traces out the unit circle, though at a different rate than the "usual" $\langle \cos t, \sin t\rangle$ parametrization. At $t_0 = -1$, we have $\vec{r}(t_0) = \langle 0, -1\rangle$; at $t_1 = 1$, we have $\vec{r}(t_1) = \langle 0, 1\rangle$. The displacement of $\vec{r}(t)$ on $[-1, 1]$ is thus $\vec{d} = \langle 0, 1\rangle - \langle 0, -1\rangle = \langle 0, 2\rangle$.

A graph of $\vec{r}(t)$ on $[-1, 1]$ is given in Figure 1.7, along with the displacement vector $\vec{d}$ on this interval.

Measuring displacement makes us contemplate related, yet very different, concepts. Considering the semi–circular path the object in Example 5 took, we can quickly verify that the object ended up a distance of 2 units from its initial location. That is, we can compute $||\,\vec{d}\,|| = 2$. However, measuring *distance from the starting point* is different from measuring *distance traveled*. Being a semi–

Notes:

circle, we can measure the distance traveled by this object as $\pi \approx 3.14$ units. Knowing *distance from the starting point* allows us to compute **average rate of change.**

Definition 4 Average Rate of Change

Let $\vec{r}(t)$ be a vector–valued function, where each of its component functions is continuous on its domain, and let $t_0 < t_1$. The **average rate of change** of $\vec{r}(t)$ on $[t_0, t_1]$ is

$$\text{average rate of change} = \frac{\vec{r}(t_1) - \vec{r}(t_0)}{t_1 - t_0}.$$

Example 6 Average rate of change

Let $\vec{r}(t) = \left\langle \cos(\frac{\pi}{2}t), \sin(\frac{\pi}{2}t) \right\rangle$ as in Example 5. Find the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ and on $[-1, 5]$.

Solution We computed in Example 5 that the displacement of $\vec{r}(t)$ on $[-1, 1]$ was $\vec{d} = \langle 0, 2 \rangle$. Thus the average rate of change of $\vec{r}(t)$ on $[-1, 1]$ is:

$$\frac{\vec{r}(1) - \vec{r}(-1)}{1 - (-1)} = \frac{\langle 0, 2 \rangle}{2} = \langle 0, 1 \rangle .$$

We interpret this as follows: the object followed a semi–circular path, meaning it moved towards the right then moved back to the left, while climbing slowly, then quickly, then slowly again. *On average*, however, it progressed straight up at a constant rate of $\langle 0, 1 \rangle$ per unit of time.

We can quickly see that the displacement on $[-1, 5]$ is the same as on $[-1, 1]$, so $\vec{d} = \langle 0, 2 \rangle$. The average rate of change is different, though:

$$\frac{\vec{r}(5) - \vec{r}(-1)}{5 - (-1)} = \frac{\langle 0, 2 \rangle}{6} = \langle 0, 1/3 \rangle .$$

As it took "3 times as long" to arrive at the same place, this average rate of change on $[-1, 5]$ is $1/3$ the average rate of change on $[-1, 1]$.

We considered average rates of change in Sections 1.1 and 2.1 as we studied limits and derivatives. The same is true here; in the following section we apply calculus concepts to vector–valued functions as we find limits, derivatives, and integrals. Understanding the average rate of change will give us an understanding of the derivative; displacement gives us one application of integration.

Notes:

Exercises 1.1

Terms and Concepts

11 01 ex 01 1. Vector–valued functions are closely related to ________ of graphs.

11 01 ex 02 2. When sketching vector–valued functions, technically one isn't graphing points, but rather ________.

11 01 ex 31 3. It can be useful to think of ________ as a vector that points from a starting position to an ending position.

Problems

11 01 exset 01 **In Exercises 4 – 11, sketch the vector–valued function on the given interval.**

11 01 ex 03 4. $\vec{r}(t) = \langle t^2, t^2 - 1 \rangle$, for $-2 \le t \le 2$.

11 01 ex 04 5. $\vec{r}(t) = \langle t^2, t^3 \rangle$, for $-2 \le t \le 2$.

11 01 ex 05 6. $\vec{r}(t) = \langle 1/t, 1/t^2 \rangle$, for $-2 \le t \le 2$.

11 01 ex 06 7. $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t \rangle$, for $-2\pi \le t \le 2\pi$.

11 01 ex 07 8. $\vec{r}(t) = \langle \frac{1}{10}t^2, \sin t \rangle$, for $-2\pi \le t \le 2\pi$.

11 01 ex 08 9. $\vec{r}(t) = \langle 3\sin(\pi t), 2\cos(\pi t) \rangle$, on $[0, 2]$.

11 01 ex 09 10. $\vec{r}(t) = \langle 3\cos t, 2\sin(2t) \rangle$, on $[0, 2\pi]$.

11 01 ex 10 11. $\vec{r}(t) = \langle 2\sec t, \tan t \rangle$, on $[-\pi, \pi]$.

11 01 exset 02 **In Exercises 12 – 15, sketch the vector–valued function on the given interval in $\mathbb{R}^3$. Technology may be useful in creating the sketch.**

11 01 ex 11 12. $\vec{r}(t) = \langle 2\cos t, t, 2\sin t \rangle$, on $[0, 2\pi]$.

11 01 ex 12 13. $\vec{r}(t) = \langle 3\cos t, \sin t, t/\pi \rangle$ on $[0, 2\pi]$.

11 01 ex 13 14. $\vec{r}(t) = \langle \cos t, \sin t, \sin t \rangle$ on $[0, 2\pi]$.

11 01 ex 14 15. $\vec{r}(t) = \langle \cos t, \sin t, \sin(2t) \rangle$ on $[0, 2\pi]$.

11 01 exset 03 **In Exercises 16 – 19, find $|| \vec{r}(t) ||$.**

11 01 ex 15 16. $\vec{r}(t) = \langle t, t^2 \rangle$.

11 01 ex 16 17. $\vec{r}(t) = \langle 5\cos t, 3\sin t \rangle$.

11 01 ex 17 18. $\vec{r}(t) = \langle 2\cos t, 2\sin t, t \rangle$.

11 01 ex 18 19. $\vec{r}(t) = \langle \cos t, t, t^2 \rangle$.

11 01 exset 04 **In Exercises 20 – 27, create a vector–valued function whose graph matches the given description.**

11 01 ex 19 20. A circle of radius 2, centered at $(1, 2)$, traced counter–clockwise once on $[0, 2\pi]$.

11 01 ex 20 21. A circle of radius 3, centered at $(5, 5)$, traced clockwise once on $[0, 2\pi]$.

11 01 ex 21 22. An ellipse, centered at $(0, 0)$ with vertical major axis of length 10 and minor axis of length 3, traced once counter–clockwise on $[0, 2\pi]$.

11 01 ex 22 23. An ellipse, centered at $(3, -2)$ with horizontal major axis of length 6 and minor axis of length 4, traced once clockwise on $[0, 2\pi]$.

11 01 ex 23 24. A line through $(2, 3)$ with a slope of 5.

11 01 ex 24 25. A line through $(1, 5)$ with a slope of $-1/2$.

11 01 ex 25 26. A vertically oriented helix with radius of 2 that starts at $(2, 0, 0)$ and ends at $(2, 0, 4\pi)$ after 1 revolution on $[0, 2\pi]$.

11 01 ex 26 27. A vertically oriented helix with radius of 3 that starts at $(3, 0, 0)$ and ends at $(3, 0, 3)$ after 2 revolutions on $[0, 1]$.

11 01 exset 05 **In Exercises 28 – 31, find the average rate of change of $\vec{r}(t)$ on the given interval.**

11 01 ex 27 28. $\vec{r}(t) = \langle t, t^2 \rangle$ on $[-2, 2]$.

11 01 ex 28 29. $\vec{r}(t) = \langle t, t + \sin t \rangle$ on $[0, 2\pi]$.

11 01 ex 29 30. $\vec{r}(t) = \langle 3\cos t, 2\sin t, t \rangle$ on $[0, 2\pi]$.

11 01 ex 30 31. $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ on $[-1, 3]$.

1.2 Calculus and Vector–Valued Functions

The previous section introduced us to a new mathematical object, the vector–valued function. We now apply calculus concepts to these functions. We start with the limit, then work our way through derivatives to integrals.

Limits of Vector–Valued Functions

The initial definition of the limit of a vector–valued function is a bit intimidating, as was the definition of the limit in Definition 1. The theorem following the definition shows that in practice, taking limits of vector–valued functions is no more difficult than taking limits of real–valued functions.

Definition 5 Limits of Vector–Valued Functions

Let I be an open interval containing c, and let $\vec{r}(t)$ be a vector–valued function defined on I, except possibly at c. The **limit of $\vec{r}(t)$, as t approaches c, is $\vec{L}$**, expressed as

$$\lim_{t\to c} \vec{r}(t) = \vec{L},$$

means that given any $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t \neq c$, if $|t - c| < \delta$, we have $||\,\vec{r}(t) - \vec{L}\,|| < \varepsilon$.

Note how the measurement of distance between real numbers is the absolute value of their difference; the measure of distance between vectors is the vector norm, or magnitude, of their difference.

Theorem 1 states that we can compute limits of vector–valued functions component–wise.

Theorem 1 Limits of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle\, f(t), g(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^2$ defined on an open interval I containing c. Then
$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\ \lim_{t\to c} g(t) \right\rangle.$$
2. Let $\vec{r}(t) = \langle\, f(t), g(t), h(t)\,\rangle$ be a vector–valued function in $\mathbb{R}^3$ defined on an open interval I containing c. Then
$$\lim_{t\to c} \vec{r}(t) = \left\langle \lim_{t\to c} f(t)\,,\ \lim_{t\to c} g(t)\,,\ \lim_{t\to c} h(t) \right\rangle$$

Notes:

Example 7 **Finding limits of vector–valued functions**

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Find $\lim_{t\to 0} \vec{r}(t)$.

Solution We apply the theorem and compute limits component–wise.

$$\begin{aligned}\lim_{t\to 0} \vec{r}(t) &= \left\langle \lim_{t\to 0} \frac{\sin t}{t},\ \lim_{t\to 0} t^2 - 3t + 3,\ \lim_{t\to 0} \cos t \right\rangle \\ &= \langle 1, 3, 1 \rangle .\end{aligned}$$

Continuity

Definition 6 **Continuity of Vector–Valued Functions**

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c.

1. $\vec{r}(t)$ is **continuous at** c if $\lim_{t\to c} \vec{r}(t) = \vec{r}(c)$.
2. If $\vec{r}(t)$ is continuous at all c in I, then $\vec{r}(t)$ is **continuous on** I**.**

We again have a theorem that lets us evaluate continuity component–wise.

Theorem 2 **Continuity of Vector–Valued Functions**

Let $\vec{r}(t)$ be a vector–valued function defined on an open interval I containing c. Then $\vec{r}(t)$ is continuous at c if, and only if, each of its component functions is continuous at c.

Example 8 **Evaluating continuity of vector–valued functions**

Let $\vec{r}(t) = \left\langle \frac{\sin t}{t},\ t^2 - 3t + 3,\ \cos t \right\rangle$. Determine whether $\vec{r}$ is continuous at $t = 0$ and $t = 1$.

Solution While the second and third components of $\vec{r}(t)$ are defined at $t = 0$, the first component, $(\sin t)/t$, is not. Since the first component is not even defined at $t = 0$, $\vec{r}(t)$ is not defined at $t = 0$, and hence it is not continuous at $t = 0$.

Notes:

At $t = 1$ each of the component functions is continuous. Therefore $\vec{r}(t)$ is continuous at $t = 1$.

Derivatives

Consider a vector–valued function $\vec{r}$ defined on an open interval I containing t_0 and t_1. We can compute the displacement of $\vec{r}$ on $[t_0, t_1]$, as shown in Figure 1.8(a). Recall that dividing the displacement vector by $t_1 - t_0$ gives the average rate of change on $[t_0, t_1]$, as shown in (b).

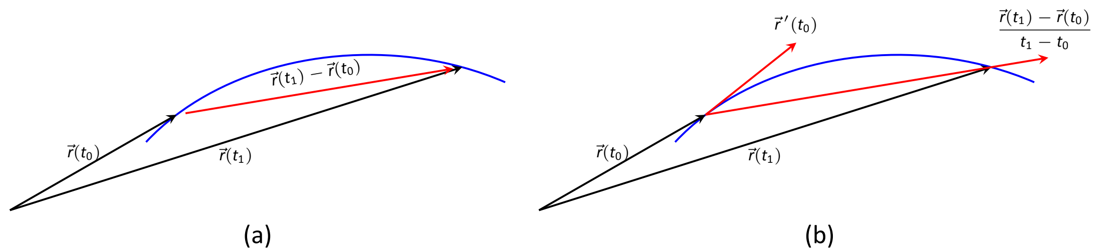


Figure 1.8: Illustrating displacement, leading to an understanding of the derivative of vector–valued functions.

The **derivative** of a vector–valued function is a measure of the *instantaneous* rate of change, measured by taking the limit as the length of $[t_0, t_1]$ goes to 0. Instead of thinking of an interval as $[t_0, t_1]$, we think of it as $[c, c + h]$ for some value of h (hence the interval has length h). The *average* rate of change is

$$\frac{\vec{r}(c+h) - \vec{r}(c)}{h}$$

for any value of $h \neq 0$. We take the limit as $h \to 0$ to measure the instantaneous rate of change; this is the derivative of $\vec{r}$.

Definition 7 Derivative of a Vector–Valued Function

Let $\vec{r}(t)$ be continuous on an open interval I containing c.

1. The derivative of $\vec{r}$ at $t = c$ is
$$\vec{r}\,'(c) = \lim_{h\to 0} \frac{\vec{r}(c+h) - \vec{r}(c)}{h}.$$

2. The derivative of $\vec{r}$ is
$$\vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}.$$

Alternate notations for the derivative of $\vec{r}$ include:

$$\vec{r}\,'(t) = \frac{d}{dt}\big(\vec{r}(t)\big) = \frac{d\vec{r}}{dt}.$$

Notes:

If a vector–valued function has a derivative for all c in an open interval I, we say that $\vec{r}(t)$ is **differentiable** on I.

Once again we might view this definition as intimidating, but recall that we can evaluate limits component–wise. The following theorem verifies that this means we can compute derivatives component–wise as well, making the task not too difficult.

Theorem 3 Derivatives of Vector–Valued Functions

1. Let $\vec{r}(t) = \langle f(t), g(t) \rangle$. Then

$$\vec{r}\,'(t) = \langle f'(t), g'(t) \rangle .$$

2. Let $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$. Then

$$\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle .$$

Example 9 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle t^2, t \rangle$.

1. Sketch $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes.
2. Compute $\vec{r}\,'(1)$ and sketch this vector with its initial point at the origin and at $\vec{r}(1)$.

Solution

1. Theorem 3 allows us to compute derivatives component–wise, so

$$\vec{r}\,'(t) = \langle 2t, 1 \rangle .$$

$\vec{r}(t)$ and $\vec{r}\,'(t)$ are graphed together in Figure 1.9(a). Note how plotting the two of these together, in this way, is not very illuminating. When dealing with real–valued functions, plotting $f(x)$ with $f'(x)$ gave us useful information as we were able to compare f and f' at the same x-values. When dealing with vector–valued functions, it is hard to tell which points on the graph of $\vec{r}\,'$ correspond to which points on the graph of $\vec{r}$.

2. We easily compute $\vec{r}\,'(1) = \langle 2, 1 \rangle$, which is drawn in Figure 1.9 with its initial point at the origin, as well as at $\vec{r}(1) = \langle 1, 1 \rangle$. These are sketched in Figure 1.9(b).

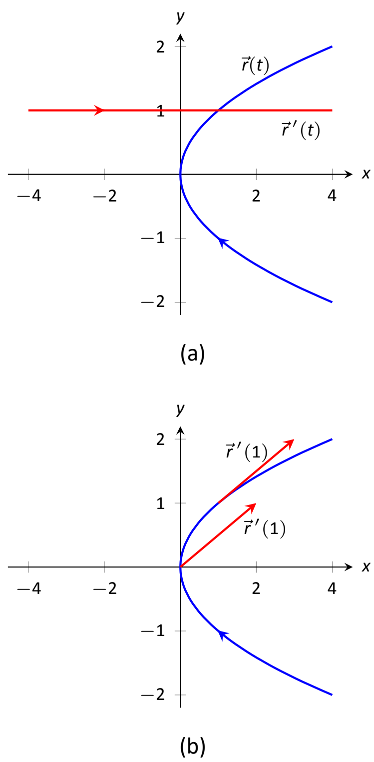


Figure 1.9: Graphing the derivative of a vector–valued function in Example 9.

Notes:

Example 10 Derivatives of vector–valued functions

Let $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$. Compute $\vec{r}\,'(t)$ and $\vec{r}\,'(\pi/2)$. Sketch $\vec{r}\,'(\pi/2)$ with its initial point at the origin and at $\vec{r}(\pi/2)$.

Solution We compute $\vec{r}\,'$ as $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle$. At $t = \pi/2$, we have $\vec{r}\,'(\pi/2) = \langle -1, 0, 1 \rangle$. Figure 1.10 shows a graph of $\vec{r}(t)$, with $\vec{r}\,'(\pi/2)$ plotted with its initial point at the origin and at $\vec{r}(\pi/2)$.

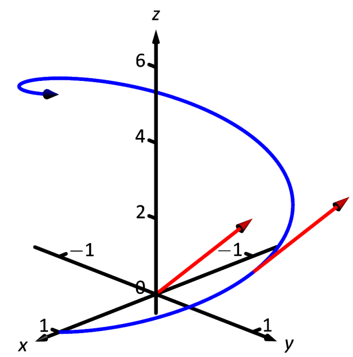


Figure 1.10: Viewing a vector–valued function and its derivative at one point.

In Examples 9 and 10, sketching a particular derivative with its initial point at the origin did not seem to reveal anything significant. However, when we sketched the vector with its initial point on the corresponding point on the graph, we did see something significant: the vector appeared to be *tangent* to the graph. We have not yet defined what "tangent" means in terms of curves in space; in fact, we use the derivative to define this term.

Definition 8 Tangent Vector, Tangent Line

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I containing c, where $\vec{r}\,'(c) \neq \vec{0}$.

1. A vector $\vec{v}$ is **tangent to the graph of** $\vec{r}(t)$ **at** $t = c$ if $\vec{v}$ is parallel to $\vec{r}\,'(c)$.
2. The **tangent line** to the graph of $\vec{r}(t)$ at $t = c$ is the line through $\vec{r}(c)$ with direction parallel to $\vec{r}\,'(c)$. An equation of the tangent line is
$$\vec{\ell}(t) = \vec{r}(c) + t\,\vec{r}\,'(c).$$

Example 11 Finding tangent lines to curves in space

Let $\vec{r}(t) = \left\langle t, t^2, t^3 \right\rangle$ on $[-1.5, 1.5]$. Find the vector equation of the line tangent to the graph of $\vec{r}$ at $t = -1$.

Solution To find the equation of a line, we need a point on the line and the line's direction. The point is given by $\vec{r}(-1) = \langle -1, 1, -1 \rangle$. (To be clear, $\langle -1, 1, -1 \rangle$ is a *vector*, not a point, but we use the point "pointed to" by this vector.)

The direction comes from $\vec{r}\,'(-1)$. We compute, component–wise, $\vec{r}\,'(t) = \left\langle 1, 2t, 3t^2 \right\rangle$. Thus $\vec{r}\,'(-1) = \langle 1, -2, 3 \rangle$.

The vector equation of the line is $\ell(t) = \langle -1, 1, -1 \rangle + t \langle 1, -2, 3 \rangle$. This line and $\vec{r}(t)$ are sketched in Figure 1.11.

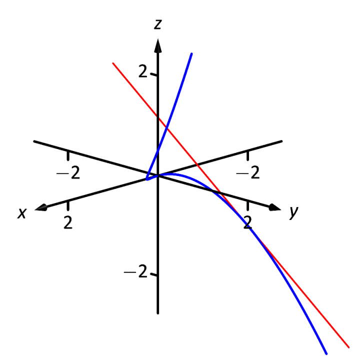


Figure 1.11: Graphing a curve in space with its tangent line.

Notes:

Example 12 Finding tangent lines to curves
Find the equations of the lines tangent to $\vec{r}(t) = \langle t^3, t^2 \rangle$ at $t = -1$ and $t = 0$.

SOLUTION We find that $\vec{r}\,'(t) = \langle 3t^2, 2t \rangle$. At $t = -1$, we have

$$\vec{r}(-1) = \langle -1, 1 \rangle \quad \text{and} \quad \vec{r}\,'(-1) = \langle 3, -2 \rangle,$$

so the equation of the line tangent to the graph of $\vec{r}(t)$ at $t = -1$ is

$$\ell(t) = \langle -1, 1 \rangle + t \langle 3, -2 \rangle .$$

This line is graphed with $\vec{r}(t)$ in Figure 1.12.

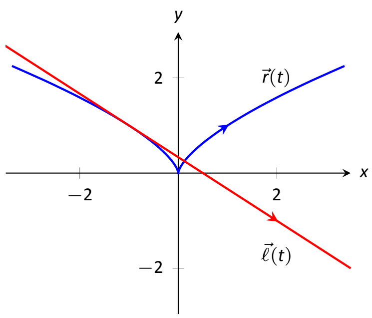


Figure 1.12: Graphing $\vec{r}(t)$ and its tangent line in Example 12.

At $t = 0$, we have $\vec{r}\,'(0) = \langle 0, 0 \rangle = \vec{0}$! This implies that the tangent line "has no direction." We cannot apply Definition 8, hence cannot find the equation of the tangent line.

We were unable to compute the equation of the tangent line to $\vec{r}(t) = \langle t^3, t^2 \rangle$ at $t = 0$ because $\vec{r}\,'(0) = \vec{0}$. The graph in Figure 1.12 shows that there is a cusp at this point. This leads us to another definition of **smooth**, previously defined by Definition 46 in Section 9.2.

Definition 9 Smooth Vector–Valued Functions

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I where $\vec{r}\,'(t)$ is continuous on I. $\vec{r}(t)$ is **smooth** on I if $\vec{r}\,'(t) \neq \vec{0}$ on I.

Having established derivatives of vector–valued functions, we now explore the relationships between the derivative and other vector operations. The following theorem states how the derivative interacts with vector addition and the various vector products.

Notes:

Theorem 4 Properies of Derivatives of Vector–Valued Functions

Let $\vec{r}$ and $\vec{s}$ be differentiable vector–valued functions, let f be a differentiable real–valued function, and let c be a real number.

1. $\dfrac{d}{dt}\Big(\vec{r}(t) \pm \vec{s}(t)\Big) = \vec{r}\,'(t) \pm \vec{s}\,'(t)$
2. $\dfrac{d}{dt}\Big(c\vec{r}(t)\Big) = c\vec{r}\,'(t)$
3. $\dfrac{d}{dt}\Big(f(t)\vec{r}(t)\Big) = f\,'(t)\vec{r}(t) + f(t)\vec{r}\,'(t)$ **Product Rule**
4. $\dfrac{d}{dt}\Big(\vec{r}(t)\cdot\vec{s}(t)\Big) = \vec{r}\,'(t)\cdot\vec{s}(t) + \vec{r}(t)\cdot\vec{s}\,'(t)$ **Product Rule**
5. $\dfrac{d}{dt}\Big(\vec{r}(t)\times\vec{s}(t)\Big) = \vec{r}\,'(t)\times\vec{s}(t) + \vec{r}(t)\times\vec{s}\,'(t)$ **Product Rule**
6. $\dfrac{d}{dt}\Big(\vec{r}\big(f(t)\big)\Big) = \vec{r}\,'\big(f(t)\big)f\,'(t)$ **Chain Rule**

Example 13 Using derivative properties of vector–valued functions

Let $\vec{r}(t) = \langle t, t^2-1\rangle$ and let $\vec{u}(t)$ be the unit vector that points in the direction of $\vec{r}(t)$.

1. Graph $\vec{r}(t)$ and $\vec{u}(t)$ on the same axes, on $[-2, 2]$.
2. Find $\vec{u}\,'(t)$ and sketch $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$. Sketch each with initial point the corresponding point on the graph of $\vec{u}$.

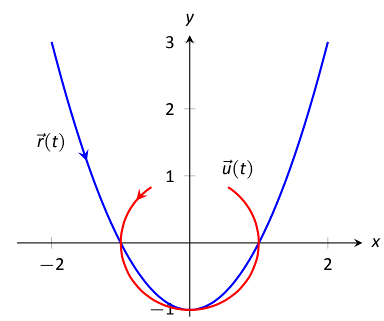


Figure 1.13: Graphing $\vec{r}(t)$ and $\vec{u}(t)$ in Example 13.

Solution

1. To form the unit vector that points in the direction of $\vec{r}$, we need to divide $\vec{r}(t)$ by its magnitude.

$$||\,\vec{r}(t)\,|| = \sqrt{t^2+(t^2-1)^2} \quad\Rightarrow\quad \vec{u}(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}}\,\langle t, t^2-1\rangle\,.$$

$\vec{r}(t)$ and $\vec{u}(t)$ are graphed in Figure 1.13. Note how the graph of $\vec{u}(t)$ forms part of a circle; this must be the case, as the length of $\vec{u}(t)$ is 1 for all t.

2. To compute $\vec{u}\,'(t)$, we use Theorem 4, writing

$$\vec{u}(t) = f(t)\vec{r}(t), \quad \text{where} \quad f(t) = \frac{1}{\sqrt{t^2+(t^2-1)^2}} = \big(t^2+(t^2-1)^2\big)^{-1/2}.$$

Notes:

(We *could* write

$$\vec{u}(t) = \left\langle \frac{t}{\sqrt{t^2+(t^2-1)^2}}, \frac{t^2-1}{\sqrt{t^2+(t^2-1)^2}} \right\rangle$$

and then take the derivative. It is a matter of preference; this latter method requires two applications of the Quotient Rule where our method uses the Product and Chain Rules.)

We find $f'(t)$ using the Chain Rule:

$$\begin{aligned} f'(t) &= -\frac{1}{2}\big(t^2+(t^2-1)^2\big)^{-3/2}\big(2t+2(t^2-1)(2t)\big) \\ &= -\frac{2t(2t^2-1)}{2\big(\sqrt{t^2+(t^2-1)^2}\,\big)^3} \end{aligned}$$

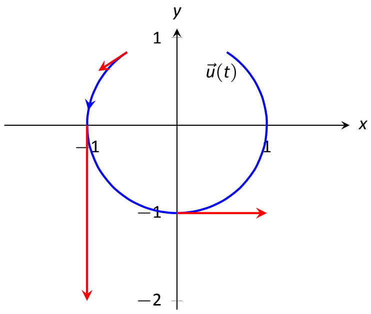


Figure 1.14: Graphing some of the derivatives of $\vec{u}(t)$ in Example 13.

We now find $\vec{u}\,'(t)$ using part 3 of Theorem 4:

$$\begin{aligned} \vec{u}\,'(t) &= f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \\ &= -\frac{2t(2t^2-1)}{2\big(\sqrt{t^2+(t^2-1)^2}\,\big)^3}\left\langle t, t^2-1\right\rangle + \frac{1}{\sqrt{t^2+(t^2-1)^2}}\left\langle 1, 2t\right\rangle . \end{aligned}$$

This is admittedly very "messy;" such is usually the case when we deal with unit vectors. We can use this formula to compute $\vec{u}\,'(-2)$, $\vec{u}\,'(-1)$ and $\vec{u}\,'(0)$:

$$\begin{aligned} \vec{u}\,'(-2) &= \left\langle -\frac{15}{13\sqrt{13}}, -\frac{10}{13\sqrt{13}} \right\rangle \approx \langle -0.320, -0.213\rangle \\ \vec{u}\,'(-1) &= \langle 0, -2\rangle \\ \vec{u}\,'(0) &= \langle 1, 0\rangle \end{aligned}$$

Each of these is sketched in Figure 1.14. Note how the length of the vector gives an indication of how quickly the circle is being traced at that point. When $t=-2$, the circle is being drawn relatively slow; when $t=-1$, the circle is being traced much more quickly.

It is a basic geometric fact that a line tangent to a circle at a point P is perpendicular to the line passing through the center of the circle and P. This is illustrated in Figure 1.14; each tangent vector is perpendicular to the line that passes through its initial point and the center of the circle. Since the center of the circle is the origin, we can state this another way: $\vec{u}\,'(t)$ is orthogonal to $\vec{u}(t)$.

Recall that the dot product serves as a test for orthogonality: if $\vec{u}\cdot\vec{v}=0$, then $\vec{u}$ is orthogonal to $\vec{v}$. Thus in the above example, $\vec{u}(t)\cdot\vec{u}\,'(t)=0$.

Notes:

This is true of any vector–valued function that has a constant length, that is, that traces out part of a circle. It has important implications later on, so we state it as a theorem (and leave its formal proof as an Exercise.)

Theorem 5 Vector–Valued Functions of Constant Length

Let $\vec{r}(t)$ be a differentiable vector–valued function on an open interval I of constant length. That is, $||\,\vec{r}(t)\,|| = c$ for all t in I (equivalently, $\vec{r}(t)\cdot\vec{r}(t) = c^2$ for all t in I). Then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$ for all t in I.

Integration

Before formally defining integrals of vector–valued functions, consider the following equation that our calculus experience tells us *should* be true:

$$\int_a^b \vec{r}\,'(t)\,dt = \vec{r}(b) - \vec{r}(a).$$

That is, the integral of a rate of change function should give total change. In the context of vector–valued functions, this total change is displacement. The above equation *is* true; we now develop the theory to show why.

We can define antiderivatives and the indefinite integral of vector–valued functions in the same manner we defined indefinite integrals in Definition 19. However, we cannot define the definite integral of a vector–valued function as we did in Definition 20. That definition was based on the signed area between a function $y = f(x)$ and the x-axis. An area–based definition will not be useful in the context of vector–valued functions. Instead, we define the definite integral of a vector–valued function in a manner similar to that of Theorem 38, utilizing Riemann sums.

Notes:

Definition 10 Antiderivatives, Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t)$ be a continuous vector–valued function on $[a, b]$. An **antiderivative** of $\vec{r}(t)$ is a function $\vec{R}(t)$ such that $\vec{R}'(t) = \vec{r}(t)$.

The set of all antiderivatives of $\vec{r}(t)$ is the **indefinite integral of** $\vec{r}(t)$, denoted by

$$\int \vec{r}(t)\ dt.$$

The definite integral of $\vec{r}(t)$ on $[a, b]$ is

$$\int_a^b \vec{r}(t)\ dt = \lim_{||\Delta t||\to 0} \sum_{i=1}^{n} \vec{r}(c_i)\Delta t_i,$$

where Δt_i is the length of the i^{th} subinterval of a partition of $[a, b]$, $||\Delta t||$ is the length of the largest subinterval in the partition, and c_i is any value in the i^{th} subinterval of the partition.

It is probably difficult to infer meaning from the definition of the definite integral. The important thing to realize from the definition is that it is built upon limits, which we can evaluate component–wise.

The following theorem simplifies the computation of definite integrals; the rest of this section and the following section will give meaning and application to these integrals.

Theorem 6 Indefinite and Definite Integrals of Vector–Valued Functions

Let $\vec{r}(t) = \langle f(t), g(t)\rangle$ be a vector–valued function in $\mathbb{R}^2$.

1. $\displaystyle\int \vec{r}(t)\ dt = \left\langle \int f(t)\ dt, \int g(t)\ dt \right\rangle$

2. $\displaystyle\int_a^b \vec{r}(t)\ dt = \left\langle \int_a^b f(t)\ dt, \int_a^b g(t)\ dt \right\rangle$

A similar statement holds for vector–valued functions in $\mathbb{R}^3$.

Example 14 Evaluating a definite integral of a vector–valued function

Notes:

Let $\vec{r}(t) = \left\langle e^{2t}, \sin t \right\rangle$. Evaluate $\displaystyle\int_0^1 \vec{r}(t)\,dt$.

Solution We follow Theorem 6.

$$\begin{aligned}
\int_0^1 \vec{r}(t)\,dt &= \int_0^1 \left\langle e^{2t}, \sin t \right\rangle\,dt \\
&= \left\langle \int_0^1 e^{2t}\,dt\,, \int_0^1 \sin t\,dt \right\rangle \\
&= \left\langle \frac{1}{2}e^{2t}\Big|_0^1\,, -\cos t\Big|_0^1 \right\rangle \\
&= \left\langle \frac{1}{2}(e^2-1)\,, -\cos(1)+1 \right\rangle \\
&\approx \langle 3.19, 0.460 \rangle\,.
\end{aligned}$$

Example 15 Solving an initial value problem

Let $\vec{r}\,''(t) = \langle 2, \cos t, 12t \rangle$. Find $\vec{r}(t)$ where:

- $\vec{r}(0) = \langle -7, -1, 2 \rangle$ and
- $\vec{r}\,'(0) = \langle 5, 3, 0 \rangle$.

Solution Knowing $\vec{r}\,''(t) = \langle 2, \cos t, 12t \rangle$, we find $\vec{r}\,'(t)$ by evaluating the indefinite integral.

$$\begin{aligned}
\int \vec{r}\,''(t)\,dt &= \left\langle \int 2\,dt\,, \int \cos t\,dt\,, \int 12t\,dt \right\rangle \\
&= \left\langle 2t + C_1, \sin t + C_2, 6t^2 + C_3 \right\rangle \\
&= \left\langle 2t, \sin t, 6t^2 \right\rangle + \langle C_1, C_2, C_3 \rangle \\
&= \left\langle 2t, \sin t, 6t^2 \right\rangle + \vec{C}.
\end{aligned}$$

Note how each indefinite integral creates its own constant which we collect as one constant vector $\vec{C}$. Knowing $\vec{r}\,'(0) = \langle 5, 3, 0 \rangle$ allows us to solve for $\vec{C}$:

$$\begin{aligned}
\vec{r}\,'(t) &= \left\langle 2t, \sin t, 6t^2 \right\rangle + \vec{C} \\
\vec{r}\,'(0) &= \langle 0, 0, 0 \rangle + \vec{C} \\
\langle 5, 3, 0 \rangle &= \vec{C}.
\end{aligned}$$

So $\vec{r}\,'(t) = \left\langle 2t, \sin t, 6t^2 \right\rangle + \langle 5, 3, 0 \rangle = \left\langle 2t + 5, \sin t + 3, 6t^2 \right\rangle$. To find $\vec{r}(t)$, we integrate once more.

Notes:

$$\int \vec{r}\,'(t)\;dt = \left\langle \int 2t+5\;dt, \int \sin t + 3\;dt, \int 6t^2\;dt \right\rangle$$
$$= \left\langle t^2+5t, -\cos t + 3t, 2t^3 \right\rangle + \vec{C}.$$

With $\vec{r}(0) = \langle -7,-1,2\rangle$, we solve for $\vec{C}$:

$$\begin{aligned} \vec{r}(t) &= \left\langle t^2+5t, -\cos t+3t, 2t^3\right\rangle + \vec{C} \\ \vec{r}(0) &= \langle 0,-1,0\rangle + \vec{C} \\ \langle -7,-1,2\rangle &= \langle 0,-1,0\rangle + \vec{C} \\ \langle -7,0,2\rangle &= \vec{C}. \end{aligned}$$

So $\vec{r}(t) = \left\langle t^2+5t, -\cos t+3t, 2t^3\right\rangle + \langle -7,0,2\rangle = \left\langle t^2+5t-7, -\cos t+3t, 2t^3+2\right\rangle$.

What does the integration of a vector–valued function *mean*? There are many applications, but none as direct as "the area under the curve" that we used in understanding the integral of a real–valued function.

A key understanding for us comes from considering the integral of a derivative:

$$\int_a^b \vec{r}\,'(t)\;dt = \vec{r}(t)\Big|_a^b = \vec{r}(b) - \vec{r}(a).$$

Integrating a *rate of change* function gives *displacement*.

Noting that vector–valued functions are closely related to parametric equations, we can describe the arc length of the graph of a vector–valued function as an integral. Given parametric equations $x = f(t)$, $y = g(t)$, the arc length on $[a,b]$ of the graph is

$$\text{Arc Length} = \int_a^b \sqrt{f\,'(t)^2 + g\,'(t)^2}\;dt,$$

as stated in Theorem 82. If $\vec{r}(t) = \langle f(t), g(t)\rangle$, note that $\sqrt{f\,'(t)^2 + g\,'(t)^2} = ||\,\vec{r}\,'(t)\,||$. Therefore we can express the arc length of the graph of a vector–valued function as an integral of the magnitude of its derivative.

Theorem 7 Arc Length of a Vector–Valued Function

Let $\vec{r}(t)$ be a vector–valued function where $\vec{r}\,'(t)$ is continuous on $[a,b]$. The arc length L of the graph of $\vec{r}(t)$ is

$$L = \int_a^b ||\,\vec{r}\,'(t)\,||\;dt.$$

Notes:

Note that we are actually integrating a scalar–function here, not a vector–valued function.

The next section takes what we have established thus far and applies it to objects in motion. We will let $\vec{r}(t)$ describe the path of an object in the plane or in space and will discover the information provided by $\vec{r}\,'(t)$ and $\vec{r}\,''(t)$.

Notes:

Exercises 1.2

Terms and Concepts

11 02 ex 01 1. Limits, derivatives and integrals of vector–valued functions are all evaluated ________–wise.

11 02 ex 02 2. The definite integral of a rate of change function gives ________.

11 02 ex 03 3. Why is it generally not useful to graph both $\vec{r}(t)$ and $\vec{r}\,'(t)$ on the same axes?

Problems

11 02 exset 01 **In Exercises 4 – 7, evaluate the given limit.**

11 02 ex 04 4. $\lim\limits_{t\to 5} \left\langle 2t+1, 3t^2-1, \sin t \right\rangle$

11 02 ex 05 5. $\lim\limits_{t\to 3} \left\langle e^t, \dfrac{t^2-9}{t+3} \right\rangle$

11 02 ex 06 6. $\lim\limits_{t\to 0} \left\langle \dfrac{t}{\sin t}, (1+t)^{\frac{1}{t}} \right\rangle$

11 02 ex 07 7. $\lim\limits_{h\to 0} \dfrac{\vec{r}(t+h)-\vec{r}(t)}{h}$, where $\vec{r}(t) = \left\langle t^2, t, 1 \right\rangle$.

11 02 exset 02 **In Exercises 8 – 9, identify the interval(s) on which $\vec{r}(t)$ is continuous.**

11 02 ex 08 8. $\vec{r}(t) = \left\langle t^2, 1/t \right\rangle$

11 02 ex 09 9. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t \right\rangle$

11 02 exset 03 **In Exercises 10 – 14, find the derivative of the given function.**

11 02 ex 10 10. $\vec{r}(t) = \left\langle \cos t, e^t, \ln t \right\rangle$

11 02 ex 11 11. $\vec{r}(t) = \left\langle \dfrac{1}{t}, \dfrac{2t-1}{3t+1}, \tan t \right\rangle$

11 02 ex 12 12. $\vec{r}(t) = (t^2)\left\langle \sin t, 2t+5 \right\rangle$

11 02 ex 13 13. $r(t) = \left\langle t^2+1, t-1 \right\rangle \cdot \left\langle \sin t, 2t+5 \right\rangle$

11 02 ex 14 14. $\vec{r}(t) = \left\langle t^2+1, t-1, 1 \right\rangle \times \left\langle \sin t, 2t+5, 1 \right\rangle$

11 02 exset 04 **In Exercises 15 – 18, find $\vec{r}\,'(t)$. Sketch $\vec{r}(t)$ and $\vec{r}\,'(1)$, with the initial point of $\vec{r}\,'(1)$ at $\vec{r}(1)$.**

11 02 ex 15 15. $\vec{r}(t) = \left\langle t^2+t, t^2-t \right\rangle$

11 02 ex 16 16. $\vec{r}(t) = \left\langle t^2-2t+2, t^3-3t^2+2t \right\rangle$

11 02 ex 17 17. $\vec{r}(t) = \left\langle t^2+1, t^3-t \right\rangle$

11 02 ex 18 18. $\vec{r}(t) = \left\langle t^2-4t+5, t^3-6t^2+11t-6 \right\rangle$

11 02 exset 05 **In Exercises 19 – 22, give the equation of the line tangent to the graph of $\vec{r}(t)$ at the given t value.**

11 02 ex 19 19. $\vec{r}(t) = \left\langle t^2+t, t^2-t \right\rangle$ at $t=1$.

11 02 ex 20 20. $\vec{r}(t) = \left\langle 3\cos t, \sin t \right\rangle$ at $t=\pi/4$.

11 02 ex 21 21. $\vec{r}(t) = \left\langle 3\cos t, 3\sin t, t \right\rangle$ at $t=\pi$.

11 02 ex 22 22. $\vec{r}(t) = \left\langle e^t, \tan t, t \right\rangle$ at $t=0$.

11 02 exset 06 **In Exercises 23 – 26, find the value(s) of t for which $\vec{r}(t)$ is not smooth.**

11 02 ex 23 23. $\vec{r}(t) = \left\langle \cos t, \sin t - t \right\rangle$

11 02 ex 24 24. $\vec{r}(t) = \left\langle t^2-2t+1, t^3+t^2-5t+3 \right\rangle$

11 02 ex 25 25. $\vec{r}(t) = \left\langle \cos t - \sin t, \sin t - \cos t, \cos(4t) \right\rangle$

11 02 ex 26 26. $\vec{r}(t) = \left\langle t^3-3t+2, -\cos(\pi t), \sin^2(\pi t) \right\rangle$

11 02 exset 07 **Exercises 27 – 29 ask you to verify parts of Theorem 4. In each let $f(t) = t^3$, $\vec{r}(t) = \left\langle t^2, t-1, 1 \right\rangle$ and $\vec{s}(t) = \left\langle \sin t, e^t, t \right\rangle$. Compute the various derivatives as indicated.**

11 02 ex 27 27. Simplify $f(t)\vec{r}(t)$, then find its derivative; show this is the same as $f\,'(t)\vec{r}(t)+f(t)\vec{r}\,'(t)$.

11 02 ex 28 28. Simplify $\vec{r}(t)\cdot\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\cdot\vec{s}(t)+\vec{r}(t)\cdot\vec{s}\,'(t)$.

11 02 ex 29 29. Simplify $\vec{r}(t)\times\vec{s}(t)$, then find its derivative; show this is the same as $\vec{r}\,'(t)\times\vec{s}(t)+\vec{r}(t)\times\vec{s}\,'(t)$.

11 02 exset 08 **In Exercises 30 – 33, evaluate the given definite or indefinite integral.**

11 02 ex 30 30. $\displaystyle\int \left\langle t^3, \cos t, te^t \right\rangle\, dt$

11 02 ex 31 31. $\displaystyle\int \left\langle \frac{1}{1+t^2}, \sec^2 t \right\rangle\, dt$

11 02 ex 32 32. $\displaystyle\int_0^{\pi} \left\langle -\sin t, \cos t \right\rangle\, dt$

11 02 ex 33 33. $\displaystyle\int_{-2}^{2} \left\langle 2t+1, 2t-1 \right\rangle\, dt$

11 02 exset 09 **In Exercises 34 – 37, solve the given initial value problems.**

11 02 ex 34 34. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \left\langle t, \sin t \right\rangle$ and $\vec{r}(0) = \left\langle 2, 2 \right\rangle$.

11 02 ex 35 35. Find $\vec{r}(t)$, given that $\vec{r}\,'(t) = \langle 1/(t+1), \tan t\rangle$ and $\vec{r}(0) = \langle 1, 2\rangle$.

11 02 ex 36 36. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle t^2, t, 1\right\rangle$, $\vec{r}\,'(0) = \langle 1, 2, 3\rangle$ and $\vec{r}(0) = \langle 4, 5, 6\rangle$.

11 02 ex 37 37. Find $\vec{r}(t)$, given that $\vec{r}\,''(t) = \left\langle \cos t, \sin t, e^t\right\rangle$, $\vec{r}\,'(0) = \langle 0, 0, 0\rangle$ and $\vec{r}(0) = \langle 0, 0, 0\rangle$.

11 02 exset 10 **In Exercises 38 – 41 , find the arc length of $\vec{r}(t)$ on the indicated interval.**

11 02 ex 38 38. $\vec{r}(t) = \langle 2\cos t, 2\sin t, 3t\rangle$ on $[0, 2\pi]$.

11 02 ex 39 39. $\vec{r}(t) = \langle 5\cos t, 3\sin t, 4\sin t\rangle$ on $[0, 2\pi]$.

11 02 ex 40 40. $\vec{r}(t) = \left\langle t^3, t^2, t^3\right\rangle$ on $[0, 1]$.

11 02 ex 41 41. $\vec{r}(t) = \left\langle e^{-t}\cos t, e^{-t}\sin t\right\rangle$ on $[0, 1]$.

11 02 ex 42 42. Prove Theorem 5; that is, show if $\vec{r}(t)$ has constant length and is differentiable, then $\vec{r}(t)\cdot\vec{r}\,'(t) = 0$. (Hint: use the Product Rule to compute $\frac{d}{dt}\big(\vec{r}(t)\cdot\vec{r}(t)\big)$.)

A: Solutions To Selected Problems

Chapter 1

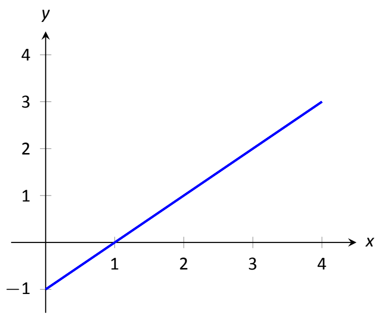
Section 1.1

11 01 ex 01 1. parametric equations

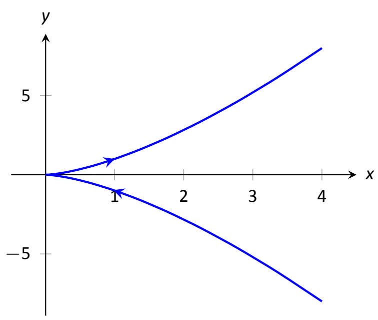
11 01 ex 02 2. vectors

11 01 ex 31 3. displacement

11 01 ex 03 4.



11 01 ex 04 5.



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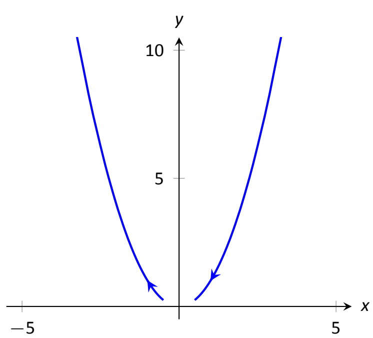

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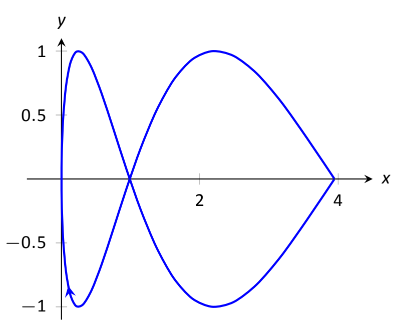

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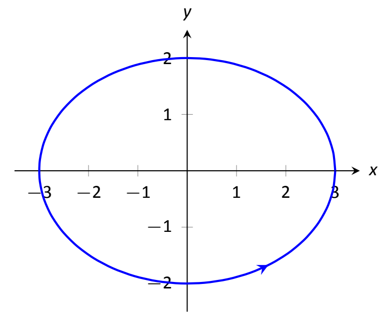
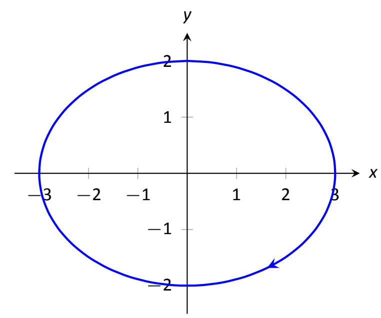


11 01 ex 08 9.



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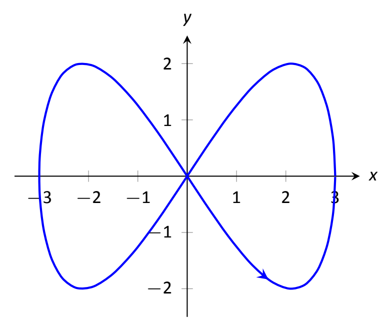
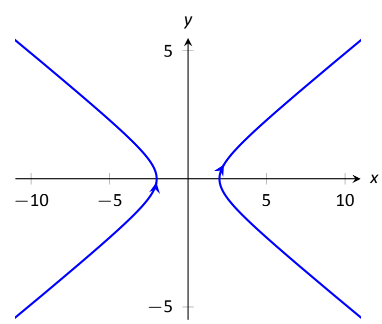
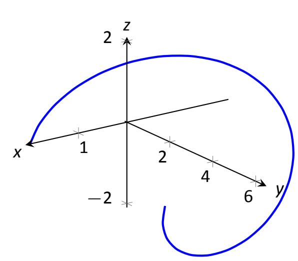


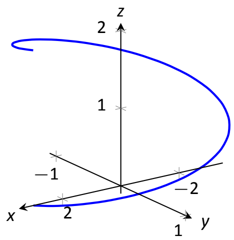
11 01 ex 10 11.



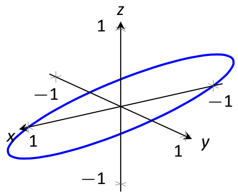
11 01 ex 11 12.



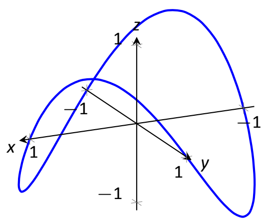
11 01 ex 12 13.



11 01 ex 13 14.



11 01 ex 14 15.



11 01 ex 15 16. $\|\vec{r}(t)\| = \sqrt{t^2 + t^4} = |t|\sqrt{t^2 + 1}$.

11 01 ex 16 17. $\|\vec{r}(t)\| = \sqrt{25\cos^2 t + 9\sin^2 t}$.

11 01 ex 17 18. $\|\vec{r}(t)\| = \sqrt{4\cos^2 t + 4\sin^2 t + t^2} = \sqrt{t^2 + 4}$.

11 01 ex 18 19. $\|\vec{r}(t)\| = \sqrt{\cos^2 t + t^2 + t^4}$.

11 01 ex 19 20. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 2\cos t + 1, 2\sin t + 2\rangle$.

11 01 ex 20 21. Answers may vary; three solutions are $\vec{r}(t) = \langle 3\sin t + 5, 3\cos t + 5\rangle$, $\vec{r}(t) = \langle -3\cos t + 5, 3\sin t + 5\rangle$ and $\vec{r}(t) = \langle 3\cos t + 5, -3\sin t + 5\rangle$.

11 01 ex 21 22. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 1.5\cos t, 5\sin t\rangle$.

11 01 ex 22 23. Answers may vary, though most direct solutions are $\vec{r}(t) = \langle -3\cos t + 3, 2\sin t - 2\rangle$, $\vec{r}(t) = \langle 3\cos t + 3, -2\sin t - 2\rangle$ and $\vec{r}(t) = \langle 3\sin t + 3, 2\cos t - 2\rangle$.

11 01 ex 23 24. Answers may vary, though most direct solutions are $\vec{r}(t) = \langle t, 5(t-2) + 3\rangle$ and $\vec{r}(t) = \langle t + 2, 5t + 3\rangle$.

11 01 ex 24 25. Answers may vary, though most direct solutions are $\vec{r}(t) = \langle t, -1/2(t-1) + 5\rangle$, $\vec{r}(t) = \langle t + 1, -1/2t + 5\rangle$, $\vec{r}(t) = \langle -2t + 1, t + 5\rangle$ and $\vec{r}(t) = \langle 2t + 1, -t + 5\rangle$.

11 01 ex 25 26. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 2\cos t, 2\sin t, 2t\rangle$.

11 01 ex 26 27. Answers may vary, though most direct solution is $\vec{r}(t) = \langle 3\cos(4\pi t), 3\sin(4\pi t), 3t\rangle$.

11 01 ex 27 28. $\langle 1, 0\rangle$

11 01 ex 28 29. $\langle 1, 1\rangle$

11 01 ex 29 30. $\langle 0, 0, 1\rangle$

11 01 ex 30 31. $\langle 1, 2, 7\rangle$

Section 1.2

11 02 ex 01 1. component

11 02 ex 02 2. displacement

11 02 ex 03 3. It is difficult to identify the points on the graphs of $\vec{r}(t)$ and $\vec{r}\,'(t)$ that correspond to each other.

11 02 ex 04 4. $\langle 11, 74, \sin 5\rangle$

11 02 ex 05 5. $\langle e^3, 0\rangle$

11 02 ex 06 6. $\langle 1, e\rangle$

11 02 ex 07 7. $\langle 2t, 1, 0\rangle$

11 02 ex 08 8. $(-\infty, 0) \bigcup (0, \infty)$

11 02 ex 09 9. $(0, \infty)$

11 02 ex 10 10. $\vec{r}\,'(t) = \langle -\sin t, e^t, 1/t\rangle$

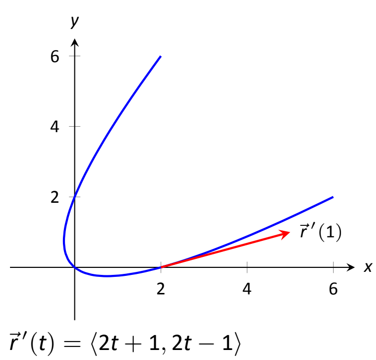
11 02 ex 11 11. $\vec{r}\,'(t) = \langle -1/t^2, 5/(3t+1)^2, \sec^2 t\rangle$

11 02 ex 12 12. $\vec{r}\,'(t) = (2t)\langle \sin t, 2t + 5\rangle + (t^2)\langle \cos t, 2\rangle = \langle 2t\sin t + t^2\cos t, 6t^2 + 10t\rangle$

11 02 ex 13 13.
$\vec{r}\,'(t) = \langle 2t, 1\rangle \cdot \langle \sin t, 2t + 5\rangle + \langle t^2 + 1, t - 1\rangle \cdot \langle \cos t, 2\rangle = (t^2 + 1)\cos t + 2t\sin t + 4t + 3$

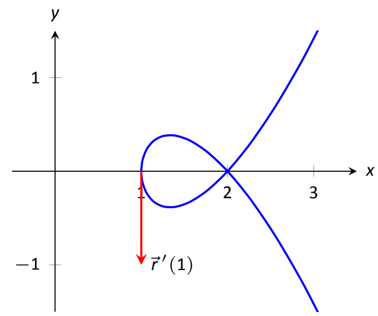
11 02 ex 14 14. $\vec{r}\,'(t) = \langle 2t, 1, 0\rangle \times \langle \sin t, 2t + 5, 1\rangle + \langle t^2 + 1, t - 1, 1\rangle \times \langle \cos t, 2, 0\rangle = \langle -1, \cos t - 2t, 6t^2 + 10t + 2 + \cos t - \sin t - t\cos t\rangle$

11 02 ex 15 15.



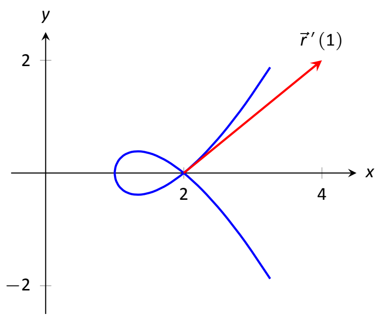
$\vec{r}\,'(t) = \langle 2t + 1, 2t - 1\rangle$

11 02 ex 16 16.



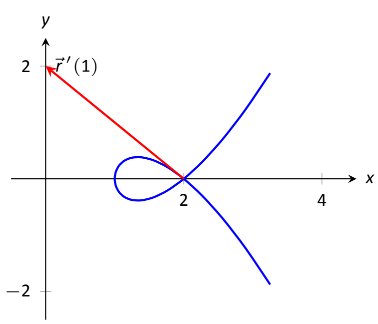
$\vec{r}\,'(t) = \left\langle 2t-2, 3t^2-6t+2\right\rangle$

11 02 ex 17 17.



$\vec{r}\,'(t) = \left\langle 2t, 3t^2-1\right\rangle$

11 02 ex 18 18.



$\vec{r}\,'(t) = \left\langle 2t-4, 3t^2-12t+11\right\rangle$

11 02 ex 19 19. $\ell(t) = \langle 2,0\rangle + t\,\langle 3,1\rangle$

11 02 ex 20 20. $\ell(t) = \left\langle 3\sqrt{2}/2, \sqrt{2}/2\right\rangle + t\left\langle -3\sqrt{2}/2, \sqrt{2}/2\right\rangle$

11 02 ex 21 21. $\ell(t) = \langle -3,0,\pi\rangle + t\,\langle 0,-3,1\rangle$

11 02 ex 22 22. $\ell(t) = \langle 1,0,0\rangle + t\,\langle 1,1,1\rangle$

11 02 ex 23 23. $t = 2n\pi$, where n is an integer; so $t = \ldots -4\pi, -2\pi, 0, 2\pi, 4\pi, \ldots$

11 02 ex 24 24. $t = 1$

11 02 ex 25 25. $\vec{r}(t)$ is not smooth at $t = 3\pi/4 + n\pi$, where n is an integer

11 02 ex 26 26. $t = \pm 1$

11 02 ex 27 27. Both derivatives return $\left\langle 5t^4, 4t^3-3t^2, 3t^2\right\rangle$.

11 02 ex 28 28. Both derivatives return $2\sin t + t^2\cos t + te^t + 1$.

11 02 ex 29 29. Both derivatives return $\left\langle 2t - e^t - 1, \cos t - 3t^2, (t^2+2t)e^t - (t-1)\cos t - \sin t\right\rangle$.

11 02 ex 30 30. $\left\langle \frac{1}{4}t^4, \sin t, te^t - e^t\right\rangle + \vec{C}$

11 02 ex 31 31. $\left\langle \tan^{-1} t, \tan t\right\rangle + \vec{C}$

11 02 ex 32 32. $\langle -2, 0\rangle$

11 02 ex 33 33. $\langle 4, -4\rangle$

11 02 ex 34 34. $\vec{r}(t) = \left\langle \frac{1}{2}t^2 + 2, -\cos t + 3\right\rangle$

11 02 ex 35 35. $\vec{r}(t) = \langle \ln|t+1| + 1, -\ln|\cos t| + 2\rangle$

11 02 ex 36 36. $\vec{r}(t) = \left\langle t^4/12 + t + 4,\ t^3/6 + 2t + 5,\ t^2/2 + 3t + 6\right\rangle$

11 02 ex 37 37. $\vec{r}(t) = \left\langle -\cos t + 1, t - \sin t, e^t - t - 1\right\rangle$

11 02 ex 38 38. $2\sqrt{13}\pi$

11 02 ex 39 39. 10π

11 02 ex 40 40. $\frac{1}{54}\left((22)^{3/2} - 8\right)$

11 02 ex 41 41. $\sqrt{2}(1 - e^{-1})$

11 02 ex 42 42. As $\vec{r}(t)$ has constant length, $\vec{r}(t)\cdot\vec{r}(t) = c^2$ for some constant c. Thus

$$\begin{aligned}
\vec{r}(t)\cdot\vec{r}(t) &= c^2\\
\frac{d}{dt}\big(\vec{r}(t)\cdot\vec{r}(t)\big) &= \frac{d}{dt}\big(c^2\big)\\
\vec{r}\,'(t)\cdot\vec{r}(t) + \vec{r}(t)\cdot\vec{r}\,'(t) &= 0\\
2\vec{r}(t)\cdot\vec{r}\,'(t) &= 0\\
\vec{r}(t)\cdot\vec{r}\,'(t) &= 0.
\end{aligned}$$