1: VECTORS

This chapter introduces a new mathematical object, the **vector**. Defined in Section 1.2, we will see that vectors provide a powerful language for describing quantities that have magnitude and direction aspects. A simple example of such a quantity is force: when applying a force, one is generally interested in how much force is applied (i.e., the magnitude of the force) and the direction in which the force was applied. Vectors will play an important role in many of the subsequent chapters in this text.

This chapter begins with moving our mathematics out of the plane and into "space." That is, we begin to think mathematically not only in two dimensions, but in three. With this foundation, we can explore vectors both in the plane and in space.

1.1 Introduction to Cartesian Coordinates in Space

Up to this point in this text we have considered mathematics in a 2–dimensional world. We have plotted graphs on the x-y plane using rectangular and polar coordinates and found the area of regions in the plane. We have considered properties of *solid* objects, such as volume and surface area, but only by first defining a curve in the plane and then rotating it out of the plane.

While there is wonderful mathematics to explore in "2D," we live in a "3D" world and eventually we will want to apply mathematics involving this third dimension. In this section we introduce Cartesian coordinates in space and explore basic surfaces. This will lay a foundation for much of what we do in the remainder of the text.

Each point P in space can be represented with an ordered triple, $P = (a, b, c)$, where a, b and c represent the relative position of P along the x-, y- and z-axes, respectively. Each axis is perpendicular to the other two.

Visualizing points in space on paper can be problematic, as we are trying to represent a 3-dimensional concept on a 2–dimensional medium. We cannot draw three lines representing the three axes in which each line is perpendicular to the other two. Despite this issue, standard conventions exist for plotting shapes in space that we will discuss that are more than adequate.

One convention is that the axes must conform to the **right hand rule**. This rule states that when the index finger of the right hand is extended in the direction of the positive x-axis, and the middle finger (bent "inward" so it is perpendicular to the palm) points along the positive y-axis, then the extended thumb will point in the direction of the positive z-axis. (It may take some thought to

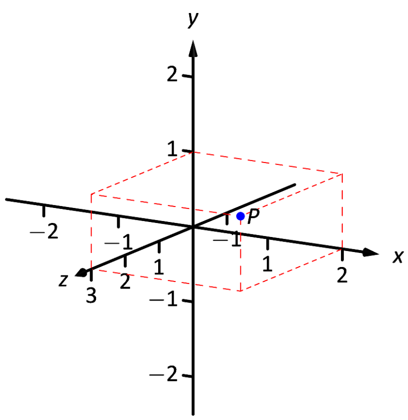

Figure 1.1.1: Plotting the point $P = (2, 1, 3)$ in space.

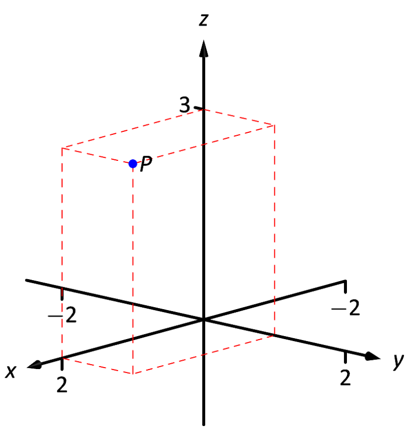

Figure 1.1.2: Plotting the point $P = (2, 1, 3)$ in space with a perspective used in this text.

verify this, but this system is inherently different from the one created by using the "left hand rule.")

As long as the coordinate axes are positioned so that they follow this rule, it does not matter how the axes are drawn on paper. There are two popular methods that we briefly discuss.

In Figure 1.1.1 we see the point $P = (2, 1, 3)$ plotted on a set of axes. The basic convention here is that the x-y plane is drawn in its standard way, with the z-axis down to the left. The perspective is that the paper represents the x-y plane and the positive z axis is coming up, off the page. This method is preferred by many engineers. Because it can be hard to tell where a single point lies in relation to all the axes, dashed lines have been added to let one see how far along each axis the point lies.

One can also consider the x-y plane as being a horizontal plane in, say, a room, where the positive z-axis is pointing up. When one steps back and looks at this room, one might draw the axes as shown in Figure 1.1.2. The same point P is drawn, again with dashed lines. This point of view is preferred by most mathematicians, and is the convention adopted by this text.

Measuring Distances

It is of critical importance to know how to measure distances between points in space. The formula for doing so is based on measuring distance in the plane, and is known (in both contexts) as the Euclidean measure of distance.

Definition 1.1.1 Distance In Space

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in space. The distance D between P and Q is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

We refer to the line segment that connects points P and Q in space as $\overline{PQ}$, and refer to the length of this segment as $||\overline{PQ}||$. The above distance formula allows us to compute the length of this segment.

Example 1.1.1 Length of a line segment

Let $P = (1, 4, -1)$ and let $Q = (2, 1, 1)$. Draw the line segment $\overline{PQ}$ and find its length.

Solution The points P and Q are plotted in Figure 1.1.3; no special consideration need be made to draw the line segment connecting these two

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points; simply connect them with a straight line. One *cannot* actually measure this line on the page and deduce anything meaningful; its true length must be measured analytically. Applying Definition 1.1.1, we have

$$||\overline{PQ}|| = \sqrt{(2-1)^2 + (1-4)^2 + (1-(-1))^2} = \sqrt{14} \approx 3.74.$$

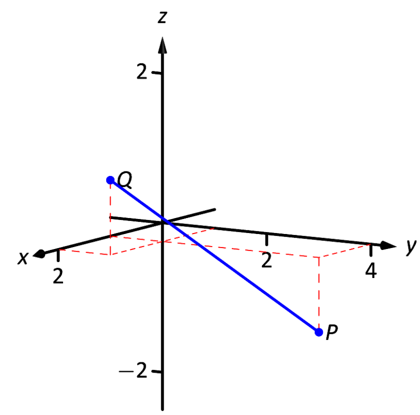

Figure 1.1.3: Plotting points P and Q in Example 1.1.1.

Spheres

Just as a circle is the set of all points in the *plane* equidistant from a given point (its center), a sphere is the set of all points in *space* that are equidistant from a given point. Definition 1.1.1 allows us to write an equation of the sphere.

We start with a point $C = (a, b, c)$ which is to be the center of a sphere with radius r. If a point $P = (x, y, z)$ lies on the sphere, then P is r units from C; that is,

$$||\overline{PC}|| = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2} = r.$$

Squaring both sides, we get the standard equation of a sphere in space with center at $C = (a, b, c)$ with radius r, as given in the following Key Idea.

Key Idea 1.1.1 Standard Equation of a Sphere in Space

The standard equation of the sphere with radius r, centered at $C = (a, b, c)$, is

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2.$$

Example 1.1.2 Equation of a sphere

Find the center and radius of the sphere defined by $x^2+2x+y^2-4y+z^2-6z = 2$.

Solution To determine the center and radius, we must put the equation in standard form. This requires us to complete the square (three times).

$$\begin{aligned} x^2 + 2x + y^2 - 4y + z^2 - 6z &= 2 \\ (x^2 + 2x + 1) + (y^2 - 4y + 4) + (z^2 - 6z + 9) - 14 &= 2 \\ (x+1)^2 + (y-2)^2 + (z-3)^2 &= 16 \end{aligned}$$

The sphere is centered at $(-1, 2, 3)$ and has a radius of 4.

The equation of a sphere is an example of an implicit function defining a surface in space. In the case of a sphere, the variables x, y and z are all used. We now consider situations where surfaces are defined where one or two of these

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variables are absent.

Introduction to Planes in Space

The coordinate axes naturally define three planes (shown in Figure 1.1.4), the **coordinate planes**: the x-y plane, the y-z plane and the x-z plane. The x-y plane is characterized as the set of all points in space where the z-value is 0. This, in fact, gives us an equation that describes this plane: $z = 0$. Likewise, the x-z plane is all points where the y-value is 0, characterized by $y = 0$.

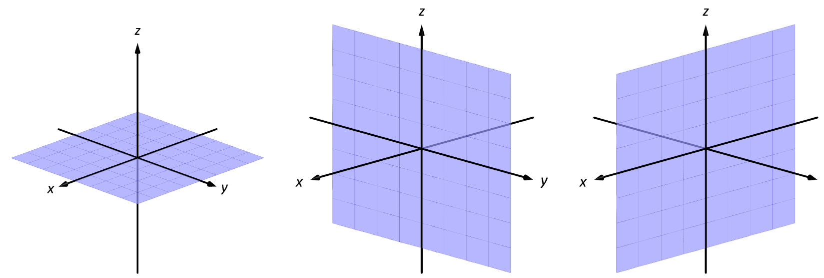

Figure 1.1.4: The coordinate planes.

The equation $x = 2$ describes all points in space where the x-value is 2. This is a plane, parallel to the y-z coordinate plane, shown in Figure 1.1.5.

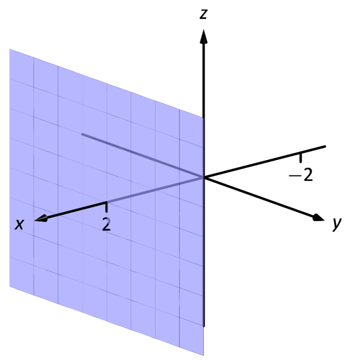

Figure 1.1.5: The plane $x = 2$.

Example 1.1.3 Regions defined by planes

Sketch the region defined by the inequalities $-1 \leq y \leq 2$.

SOLUTION The region is all points between the planes $y = -1$ and $y = 2$. These planes are sketched in Figure 1.1.6, which are parallel to the x-z plane. Thus the region extends infinitely in the x and z directions, and is bounded by planes in the y direction.

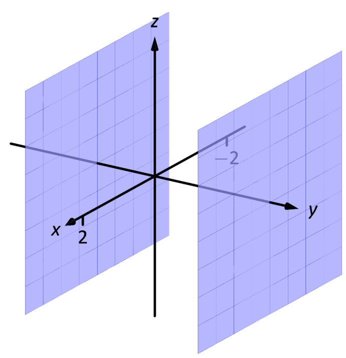

Figure 1.1.6: Sketching the boundaries of a region in Example 1.1.3.

Cylinders

The equation $x = 1$ obviously lacks the y and z variables, meaning it defines points where the y and z coordinates can take on any value. Now consider the equation $x^2 + y^2 = 1$ *in space.* In *the plane*, this equation describes a circle of radius 1, centered at the origin. In space, the z coordinate is not specified, meaning it can take on any value. In Figure 1.1.8 (a), we show part of the graph of the equation $x^2 + y^2 = 1$ by sketching 3 circles: the bottom one has a constant z-value of -1.5, the middle one has a z-value of 0 and the top circle has a

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z-value of 1. By plotting *all* possible z-values, we get the surface shown in Figure 1.1.8 (b). This surface looks like a "tube," or a "cylinder"; mathematicians call this surface a **cylinder** for an entirely different reason.

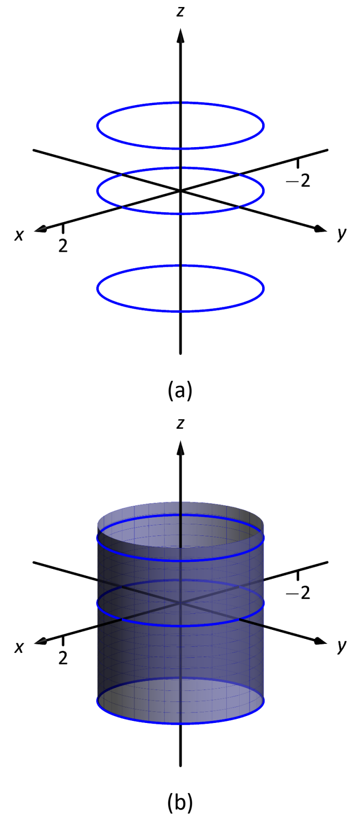

Figure 1.1.8: Sketching $x^2 + y^2 = 1$.

Definition 1.1.2 Cylinder

Let C be a curve in a plane and let L be a line not parallel to C. A **cylinder** is the set of all lines parallel to L that pass through C. The curve C is the **directrix** of the cylinder, and the lines are the **rulings**.

In this text, we consider curves C that lie in planes parallel to one of the coordinate planes, and lines L that are perpendicular to these planes, forming **right cylinders**. Thus the directrix can be defined using equations involving 2 variables, and the rulings will be parallel to the axis of the 3$^{\text{rd}}$ variable.

In the example preceding the definition, the curve $x^2 + y^2 = 1$ in the x-y plane is the directrix and the rulings are lines parallel to the z-axis. (Any circle shown in Figure 1.1.8 can be considered a directrix; we simply choose the one where $z = 0$.) Sample rulings can also be viewed in part (b) of the figure. More examples will help us understand this definition.

Example 1.1.4 Graphing cylinders

Graph the cylinder following cylinders.

1. $z = y^2$ 2. $x = \sin z$

Solution

1. We can view the equation $z = y^2$ as a parabola in the y-z plane, as illustrated in Figure 1.1.7 (a). As x does not appear in the equation, the rulings are lines through this parabola parallel to the x-axis, shown in (b). These rulings give a general idea as to what the surface looks like, drawn in (c).

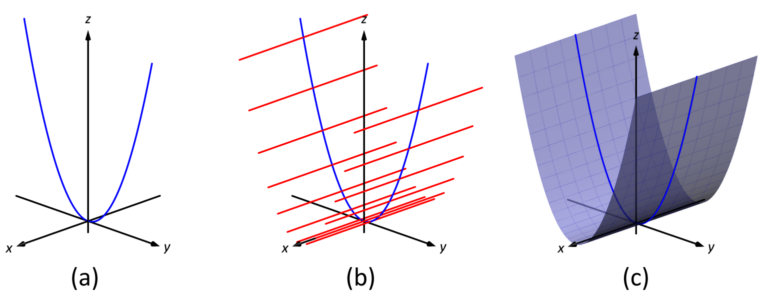

Figure 1.1.7: Sketching the cylinder defined by $z = y^2$.

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2. We can view the equation $x = \sin z$ as a sine curve that exists in the x-z plane, as shown in Figure 1.1.9 (a). The rules are parallel to the y axis as the variable y does not appear in the equation $x = \sin z$; some of these are shown in part (b). The surface is shown in part (c) of the figure.

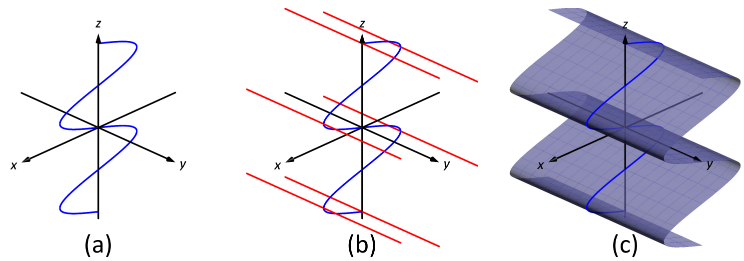

Figure 1.1.9: Sketching the cylinder defined by $x = \sin z$.

Surfaces of Revolution

One of the applications of integration we learned previously was to find the volume of solids of revolution – solids formed by revolving a curve about a horizontal or vertical axis. We now consider how to find the equation of the surface of such a solid.

Consider the surface formed by revolving $y = \sqrt{x}$ about the x-axis. Cross–sections of this surface parallel to the y-z plane are circles, as shown in Figure 1.1.10(a). Each circle has equation of the form $y^2 + z^2 = r^2$ for some radius r. The radius is a function of x; in fact, it is $r(x) = \sqrt{x}$. Thus the equation of the surface shown in Figure 1.1.10b is $y^2 + z^2 = (\sqrt{x})^2$.

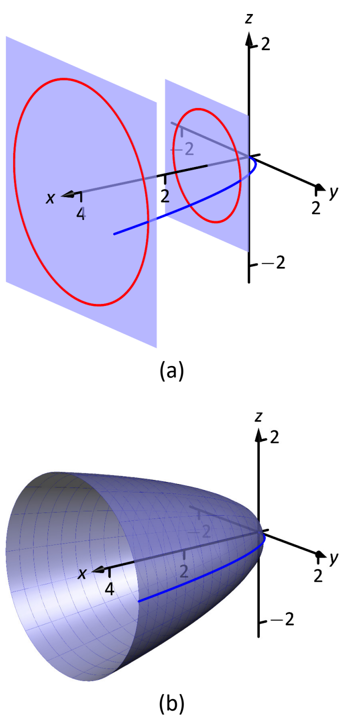

Figure 1.1.10: Introducing surfaces of revolution.

We generalize the above principles to give the equations of surfaces formed by revolving curves about the coordinate axes.

Key Idea 1.1.2 Surfaces of Revolution, Part 1

Let r be a radius function.

1. The equation of the surface formed by revolving $y = r(x)$ or $z = r(x)$ about the x-axis is $y^2 + z^2 = r(x)^2$.
2. The equation of the surface formed by revolving $x = r(y)$ or $z = r(y)$ about the y-axis is $x^2 + z^2 = r(y)^2$.
3. The equation of the surface formed by revolving $x = r(z)$ or $y = r(z)$ about the z-axis is $x^2 + y^2 = r(z)^2$.

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Example 1.1.5 Finding equation of a surface of revolution

Let $y = \sin z$ on $[0, \pi]$. Find the equation of the surface of revolution formed by revolving $y = \sin z$ about the z-axis.

Solution Using Key Idea 1.1.2, we find the surface has equation $x^2 + y^2 = \sin^2 z$. The curve is sketched in Figure 1.1.11(a) and the surface is drawn in Figure 1.1.11(b).

Note how the surface (and hence the resulting equation) is the same if we began with the curve $x = \sin z$, which is also drawn in Figure 1.1.11(a).

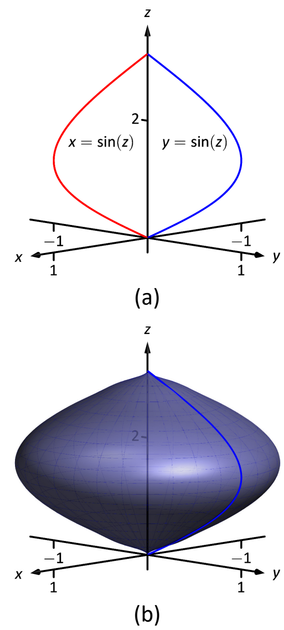

Figure 1.1.11: Revolving $y = \sin z$ about the z-axis in Example 1.1.5.

This particular method of creating surfaces of revolution is limited. For instance, in Example 7.3.4 of Section 7.3 we found the volume of the solid formed by revolving $y = \sin x$ about the y-axis. Our current method of forming surfaces can only rotate $y = \sin x$ about the x-axis. Trying to rewrite $y = \sin x$ as a function of y is not trivial, as simply writing $x = \sin^{-1} y$ only gives part of the region we desire.

What we desire is a way of writing the surface of revolution formed by rotating $y = f(x)$ about the y-axis. We start by first recognizing this surface is the same as revolving $z = f(x)$ about the z-axis. This will give us a more natural way of viewing the surface.

A value of x is a measurement of distance from the z-axis. At the distance r, we plot a z-height of $f(r)$. When rotating $f(x)$ about the z-axis, we want all points a distance of r from the z-axis in the x-y plane to have a z-height of $f(r)$. All such points satisfy the equation $r^2 = x^2 + y^2$; hence $r = \sqrt{x^2 + y^2}$. Replacing r with $\sqrt{x^2 + y^2}$ in $f(r)$ gives $z = f(\sqrt{x^2 + y^2})$. This is the equation of the surface.

> **Key Idea 1.1.3 Surfaces of Revolution, Part 2**
>
> Let $z = f(x)$, $x \geq 0$, be a curve in the x-z plane. The surface formed by revolving this curve about the z-axis has equation $z = f\big(\sqrt{x^2 + y^2}\big)$.

Example 1.1.6 Finding equation of surface of revolution

Find the equation of the surface found by revolving $z = \sin x$ about the z-axis.

Solution Using Key Idea 1.1.3, the surface has equation $z = \sin\left(\sqrt{x^2 + y^2}\right)$. The curve and surface are graphed in Figure 1.1.12.

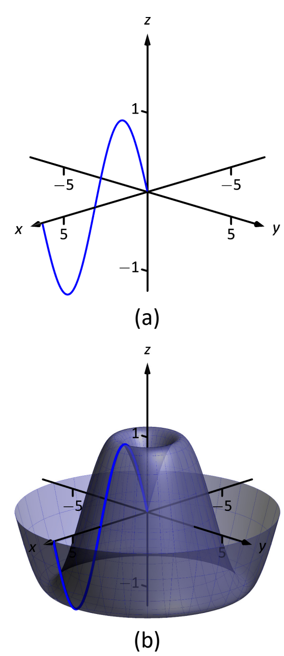

Figure 1.1.12: Revolving $z = \sin x$ about the z-axis in Example 1.1.6.

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Quadric Surfaces

Spheres, planes and cylinders are important surfaces to understand. We now consider one last type of surface, a **quadric surface**. The definition may look intimidating, but we will show how to analyze these surfaces in an illuminating way.

> **Definition 1.1.3 Quadric Surface**
>
> A **quadric surface** is the graph of the general second–degree equation in three variables:
>
> $$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

When the coefficients D, E or F are not zero, the basic shapes of the quadric surfaces are rotated in space. We will focus on quadric surfaces where these coefficients are 0; we will not consider rotations. There are six basic quadric surfaces: the elliptic paraboloid, elliptic cone, ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, and the hyperbolic paraboloid.

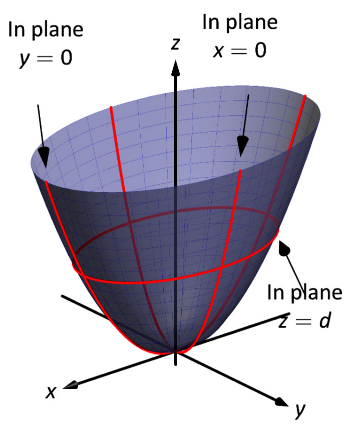

Figure 1.1.13: The elliptic paraboloid $z = x^2/4 + y^2$.

We study each shape by considering **traces**, that is, intersections of each surface with a plane parallel to a coordinate plane. For instance, consider the elliptic paraboloid $z = x^2/4 + y^2$, shown in Figure 1.1.13. If we intersect this shape with the plane $z = d$ (i.e., replace z with d), we have the equation:

$$d = \frac{x^2}{4} + y^2.$$

Divide both sides by d:

$$1 = \frac{x^2}{4d} + \frac{y^2}{d}.$$

This describes an ellipse – so cross sections parallel to the x-y coordinate plane are ellipses. This ellipse is drawn in the figure.

Now consider cross sections parallel to the x-z plane. For instance, letting $y = 0$ gives the equation $z = x^2/4$, clearly a parabola. Intersecting with the plane $x = 0$ gives a cross section defined by $z = y^2$, another parabola. These parabolas are also sketched in the figure.

Thus we see where the elliptic paraboloid gets its name: some cross sections are ellipses, and others are parabolas.

Such an analysis can be made with each of the quadric surfaces. We give a sample equation of each, provide a sketch with representative traces, and describe these traces.

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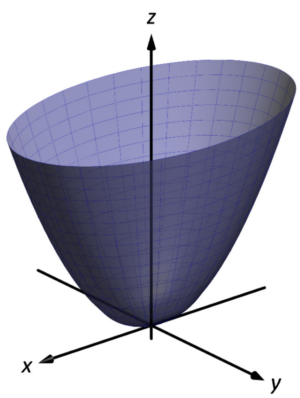
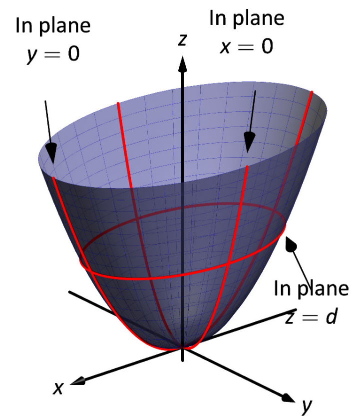
Elliptic Paraboloid, $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



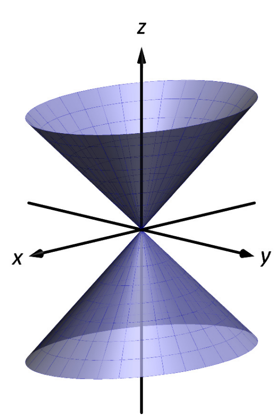
Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Ellipse



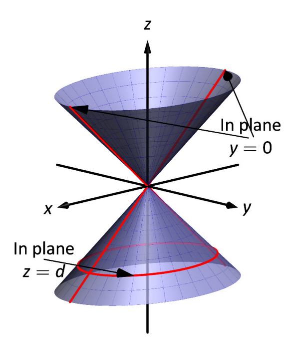
One variable in the equation of the elliptic paraboloid will be raised to the first power; above, this is the z variable. The paraboloid will "open" in the direction of this variable's axis. Thus $x = y^2/a^2 + z^2/b^2$ is an elliptic paraboloid that opens along the x-axis.

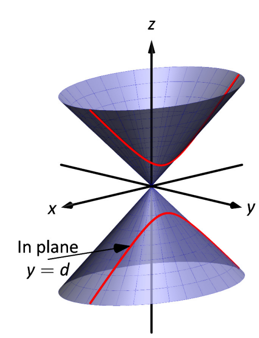
Multiplying the right hand side by (-1) defines an elliptic paraboloid that "opens" in the opposite direction.

Elliptic Cone, $z^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$



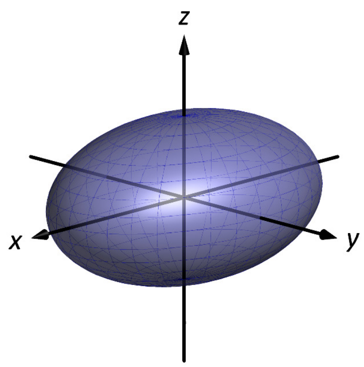
Plane	Trace
$x = 0$	Crossed Lines
$y = 0$	Crossed Lines
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse



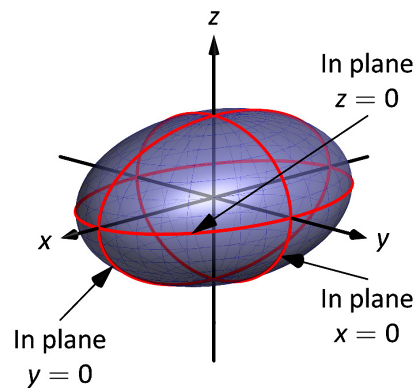


One can rewrite the equation as $z^2 - x^2/a^2 - y^2/b^2 = 0$. The one variable with a positive coefficient corresponds to the axis that the cones "open" along.

Ellipsoid, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

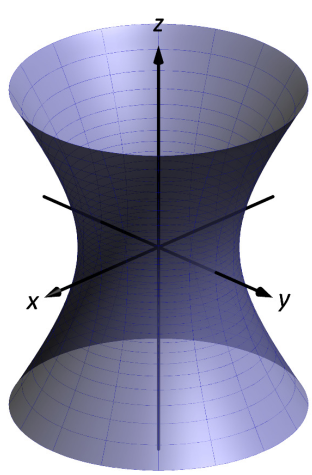


Plane	Trace
$x = d$	Ellipse
$y = d$	Ellipse
$z = d$	Ellipse

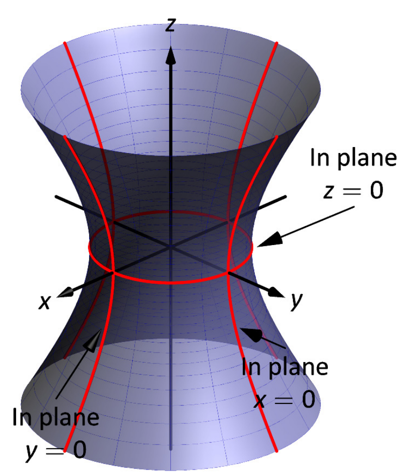


If $a = b = c \neq 0$, the ellipsoid is a sphere with radius a; compare to Key Idea 1.1.1.

Hyperboloid of One Sheet, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$

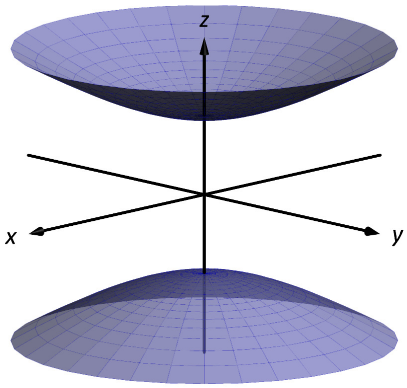


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

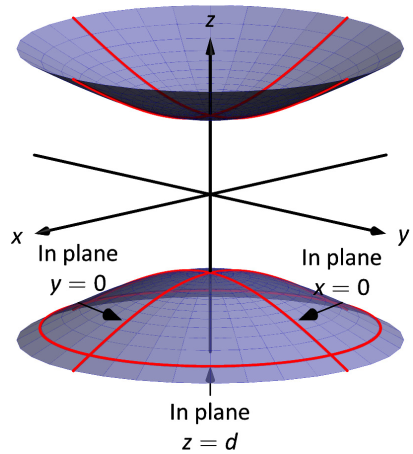


The one variable with a negative coefficient corresponds to the axis that the hyperboloid "opens" along.

Hyperboloid of Two Sheets, $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

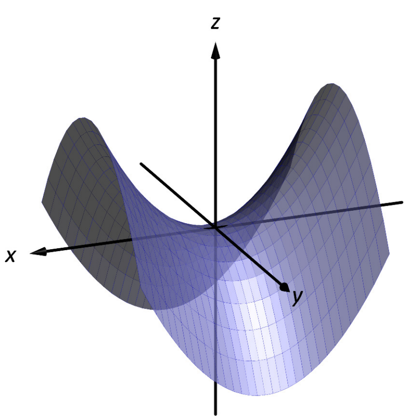


Plane	Trace
$x = d$	Hyperbola
$y = d$	Hyperbola
$z = d$	Ellipse

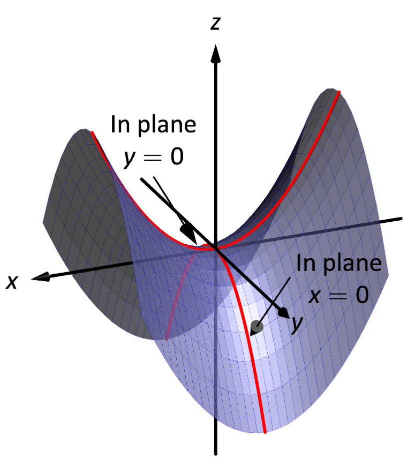


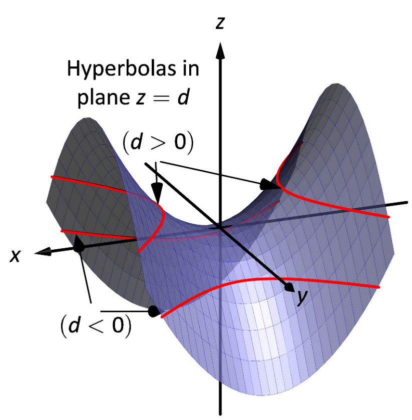
The one variable with a positive coefficient corresponds to the axis that the hyperboloid "opens" along. In the case illustrated, when $|d| < |c|$, there is no trace.

Hyperbolic Paraboloid, $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$



Plane	Trace
$x = d$	Parabola
$y = d$	Parabola
$z = d$	Hyperbola





The parabolic traces will open along the axis of the one variable that is raised to the first power.

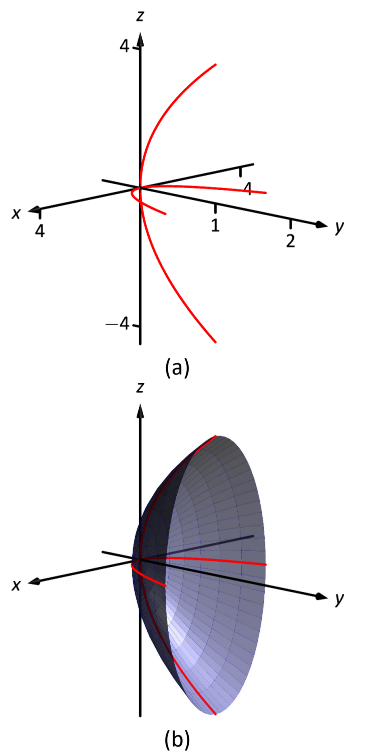


Figure 1.1.14: Sketching an elliptic paraboloid.

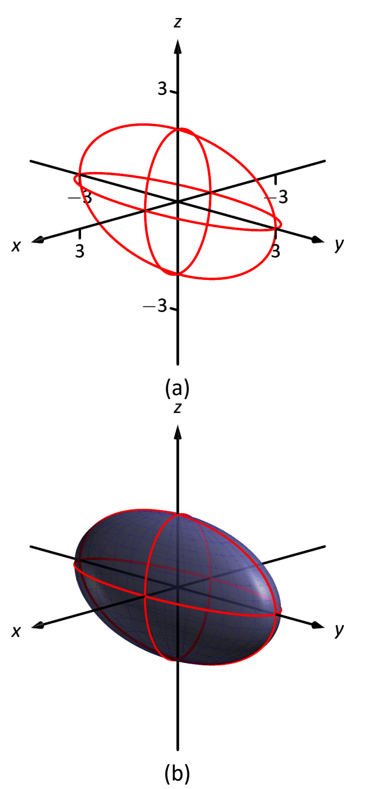


Figure 1.1.15: Sketching an ellipsoid.

Example 1.1.7 Sketching quadric surfaces

Sketch the quadric surface defined by the given equation.

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$ 2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1.$ 3. $z = y^2 - x^2.$

Solution

1. $y = \dfrac{x^2}{4} + \dfrac{z^2}{16}$:

   We first identify the quadric by pattern–matching with the equations given previously. Only two surfaces have equations where one variable is raised to the first power, the elliptic paraboloid and the hyperbolic paraboloid. In the latter case, the other variables have different signs, so we conclude that this describes a hyperbolic paraboloid. As the variable with the first power is y, we note the paraboloid opens along the y-axis.

   To make a decent sketch by hand, we need only draw a few traces. In this case, the traces $x = 0$ and $z = 0$ form parabolas that outline the shape.

   $x = 0$: The trace is the parabola $y = z^2/16$

   $z = 0$: The trace is the parabola $y = x^2/4$.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 1.1.14(a). This is enough to give an idea of what the paraboloid looks like. The surface is filled in in (b).

2. $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$:

   This is an ellipsoid. We can get a good idea of its shape by drawing the traces in the coordinate planes.

   $x = 0$: The trace is the ellipse $\dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$. The major axis is along the y–axis with length 6 (as $b = 3$, the length of the axis is 6); the minor axis is along the z-axis with length 4.

   $y = 0$: The trace is the ellipse $x^2 + \dfrac{z^2}{4} = 1$. The major axis is along the z-axis, and the minor axis has length 2 along the x-axis.

   $z = 0$: The trace is the ellipse $x^2 + \dfrac{y^2}{9} = 1$, with major axis along the y-axis.

   Graphing each trace in the respective plane creates a sketch as shown in Figure 1.1.15(a). Filling in the surface gives Figure 1.1.15(b).

3. $z = y^2 - x^2$:

Notes:

This defines a hyperbolic paraboloid, very similar to the one shown in the gallery of quadric sections. Consider the traces in the $y-z$ and $x-z$ planes:

$x = 0$: The trace is $z = y^2$, a parabola opening up in the $y - z$ plane.

$y = 0$: The trace is $z = -x^2$, a parabola opening down in the $x - z$ plane.

Sketching these two parabolas gives a sketch like that in Figure 1.1.16 (a), and filling in the surface gives a sketch like (b).

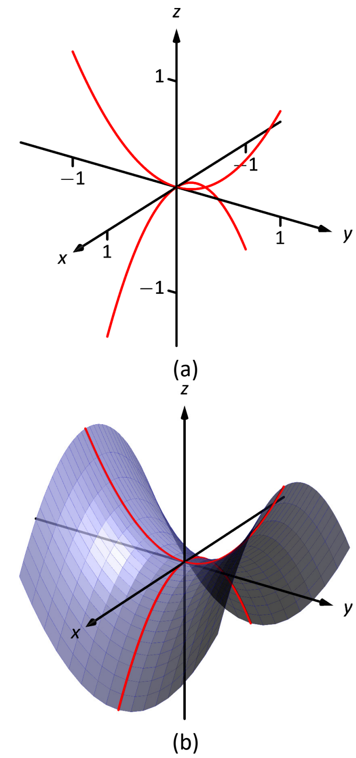


Figure 1.1.16: Sketching a hyperbolic paraboloid.

Example 1.1.8 Identifying quadric surfaces

Consider the quadric surface shown in Figure 1.1.17. Which of the following equations best fits this surface?

(a) $x^2 - y^2 - \frac{z^2}{9} = 0$ (c) $z^2 - x^2 - y^2 = 1$

(b) $x^2 - y^2 - z^2 = 1$ (d) $4x^2 - y^2 - \frac{z^2}{9} = 1$

Solution The image clearly displays a hyperboloid of two sheets. The gallery informs us that the equation will have a form similar to $\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

We can immediately eliminate option (a), as the constant in that equation is not 1.

The hyperboloid "opens" along the x-axis, meaning x must be the only variable with a positive coefficient, eliminating (c).

The hyperboloid is wider in the z-direction than in the y-direction, so we need an equation where $c > b$. This eliminates (b), leaving us with (d). We should verify that the equation given in (d), $4x^2 - y^2 - \frac{z^2}{9} = 1$, fits.

We already established that this equation describes a hyperboloid of two sheets that opens in the x-direction and is wider in the z-direction than in the y. Now note the coefficient of the x-term. Rewriting $4x^2$ in standard form, we have: $4x^2 = \dfrac{x^2}{(1/2)^2}$. Thus when $y = 0$ and $z = 0$, x must be $1/2$; i.e., each hyperboloid "starts" at $x = 1/2$. This matches our figure.

We conclude that $4x^2 - y^2 - \dfrac{z^2}{9} = 1$ best fits the graph.

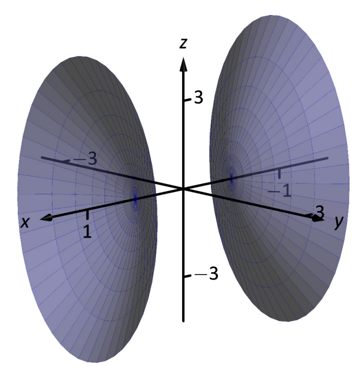


Figure 1.1.17: A possible equation of this quadric surface is found in Example 1.1.8.

This section has introduced points in space and shown how equations can describe surfaces. The next sections explore *vectors*, an important mathematical object that we'll use to explore curves in space.

Notes:

Exercises 1.1

Terms and Concepts

10 01 ex 08 1. Axes drawn in space must conform to the ________ ________ rule.

10 01 ex 01 2. In the plane, the equation $x = 2$ defines a ________; in space, $x = 2$ defines a ________.

10 01 ex 02 3. In the plane, the equation $y = x^2$ defines a ________; in space, $y = x^2$ defines a ________.

10 01 ex 03 4. Which quadric surface looks like a Pringles® chip?

10 01 ex 04 5. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the x-axis, what quadric surface is formed?

10 01 ex 05 6. Consider the hyperbola $x^2 - y^2 = 1$ in the plane. If this hyperbola is rotated about the y-axis, what quadric surface is formed?

Problems

10 01 ex 06 7. The points $A = (1, 4, 2)$, $B = (2, 6, 3)$ and $C = (4, 3, 1)$ form a triangle in space. Find the distances between each pair of points and determine if the triangle is a right triangle.

10 01 ex 07 8. The points $A = (1, 1, 3)$, $B = (3, 2, 7)$, $C = (2, 0, 8)$ and $D = (0, -1, 4)$ form a quadrilateral $ABCD$ in space. Is this a parallelogram?

10 01 ex 09 9. Find the center and radius of the sphere defined by
$x^2 - 8x + y^2 + 2y + z^2 + 8 = 0$.

10 01 ex 10 10. Find the center and radius of the sphere defined by
$x^2 + y^2 + z^2 + 4x - 2y - 4z + 4 = 0$.

10 01 exset 01 **In Exercises 11 – 14, describe the region in space defined by the inequalities.**

10 01 ex 11 11. $x^2 + y^2 + z^2 < 1$

10 01 ex 12 12. $0 \le x \le 3$

10 01 ex 13 13. $x \ge 0,\ y \ge 0,\ z \ge 0$

10 01 ex 14 14. $y \ge 3$

10 01 exset 02 **In Exercises 15 – 18, sketch the cylinder in space.**

10 01 ex 15 15. $z = x^3$

10 01 ex 16 16. $y = \cos z$

10 01 ex 17 17. $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$

10 01 ex 18 18. $y = \dfrac{1}{x}$

10 01 exset 05 **In Exercises 19 – 22, give the equation of the surface of revolution described.**

10 01 ex 29 19. Revolve $z = \dfrac{1}{1+y^2}$ about the y-axis.

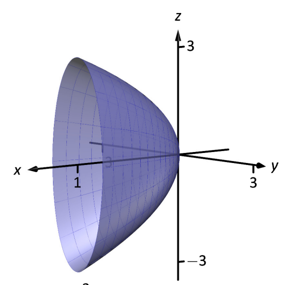
10 01 ex 30 20. Revolve $y = x^2$ about the x-axis.

10 01 ex 31 21. Revolve $z = x^2$ about the z-axis.

10 01 ex 32 22. Revolve $z = 1/x$ about the z-axis.

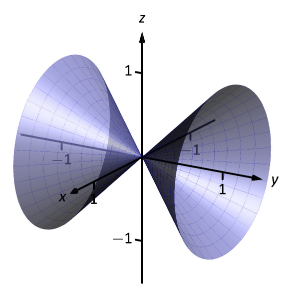
10 01 exset 03 **In Exercises 23 – 26, a quadric surface is sketched. Determine which of the given equations best fits the graph.**

10 01 ex 19 23.



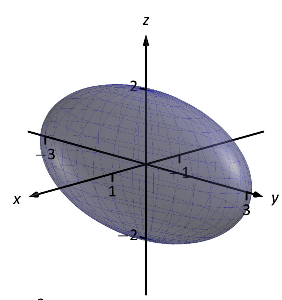
(a) $x = y^2 + \dfrac{z^2}{9}$ (b) $x = y^2 + \dfrac{z^2}{3}$

10 01 ex 20 24.



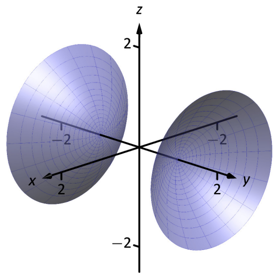
(a) $x^2 - y^2 - z^2 = 0$ (b) $x^2 - y^2 + z^2 = 0$

10 01 ex 21 25.



(a) $x^2 + \dfrac{y^2}{3} + \dfrac{z^2}{2} = 1$ (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

10 01 ex 22 26.



(a) $y^2 - x^2 - z^2 = 1$ (b) $y^2 + x^2 - z^2 = 1$

10 01 exset 04 **In Exercises 27 – 32, sketch the quadric surface.**

10 01 ex 28 27. $z - y^2 + x^2 = 0$

10 01 ex 24 28. $z^2 = x^2 + \dfrac{y^2}{4}$

10 01 ex 23 29. $x = -y^2 - z^2$

10 01 ex 27 30. $16x^2 - 16y^2 - 16z^2 = 1$

10 01 ex 26 31. $\dfrac{x^2}{9} - y^2 + \dfrac{z^2}{25} = 1$

10 01 ex 25 32. $4x^2 + 2y^2 + z^2 = 4$

1.2 An Introduction to Vectors

Many quantities we think about daily can be described by a single number: temperature, speed, cost, weight and height. There are also many other concepts we encounter daily that cannot be described with just one number. For instance, a weather forecaster often describes wind with its speed and its direction (". . . with winds from the southeast gusting up to 30 mph . . ."). When applying a force, we are concerned with both the magnitude and direction of that force. In both of these examples, *direction* is important. Because of this, we study *vectors*, mathematical objects that convey both magnitude and direction information.

One "bare–bones" definition of a vector is based on what we wrote above: "a vector is a mathematical object with magnitude and direction parameters." This definition leaves much to be desired, as it gives no indication as to how such an object is to be used. Several other definitions exist; we choose here a definition rooted in a geometric visualization of vectors. It is very simplistic but readily permits further investigation.

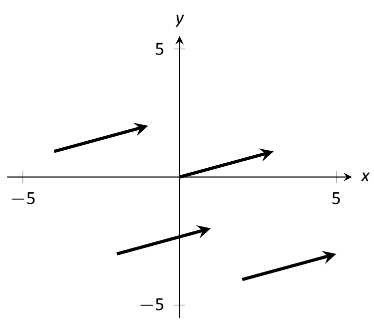


Figure 1.2.1: Drawing the same vector with different initial points.

Definition 1.2.1 Vector

A **vector** is a directed line segment.

Given points P and Q (either in the plane or in space), we denote with $\vec{PQ}$ the vector *from P to Q*. The point P is said to be the **initial point** of the vector, and the point Q is the **terminal point**.

The **magnitude**, **length** or **norm** of $\vec{PQ}$ is the length of the line segment $\overline{PQ}$: $||\,\vec{PQ}\,|| = ||\,\overline{PQ}\,||$.

Two vectors are **equal** if they have the same magnitude and direction.

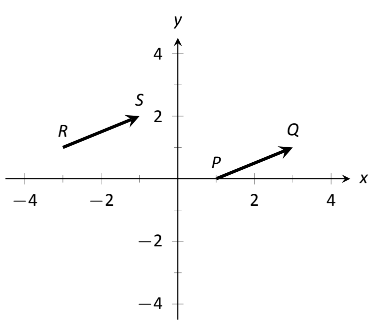


Figure 1.2.2: Illustrating how equal vectors have the same displacement.

Figure 1.2.1 shows multiple instances of the same vector. Each directed line segment has the same direction and length (magnitude), hence each is the same vector.

We use $\mathbb{R}^2$ (pronounced "r two") to represent all the vectors in the plane, and use $\mathbb{R}^3$ (pronounced "r three") to represent all the vectors in space.

Consider the vectors $\vec{PQ}$ and $\vec{RS}$ as shown in Figure 1.2.2. The vectors look to be equal; that is, they seem to have the same length and direction. Indeed, they are. Both vectors move 2 units to the right and 1 unit up from the initial point to reach the terminal point. One can analyze this movement to measure the

Notes:

magnitude of the vector, and the movement itself gives direction information (one could also measure the slope of the line passing through P and Q or R and S). Since they have the same length and direction, these two vectors are equal.

This demonstrates that inherently all we care about is *displacement*; that is, how far in the x, y and possibly z directions the terminal point is from the initial point. Both the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ in Figure 1.2.2 have an x-displacement of 2 and a y-displacement of 1. This suggests a standard way of describing vectors in the plane. A vector whose x-displacement is a and whose y-displacement is b will have terminal point (a, b) when the initial point is the origin, $(0, 0)$. This leads us to a definition of a standard and concise way of referring to vectors.

Definition 1.2.2 Component Form of a Vector

1. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^2$, whose terminal point is (a, b) when its initial point is $(0, 0)$, is $\langle a, b\rangle$.
2. The **component form** of a vector $\vec{v}$ in $\mathbb{R}^3$, whose terminal point is (a, b, c) when its initial point is $(0, 0, 0)$, is $\langle a, b, c\rangle$.

The numbers a, b (and c, respectively) are the **components** of $\vec{v}$.

It follows from the definition that the component form of the vector $\overrightarrow{PQ}$, where $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ is

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1\rangle ;$$

in space, where $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$, the component form of $\overrightarrow{PQ}$ is

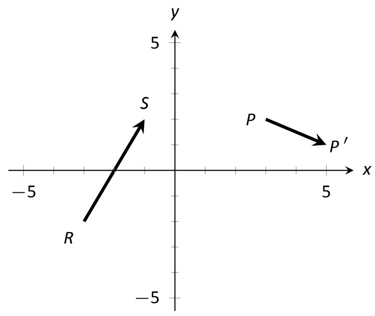
$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle .$$

We practice using this notation in the following example.

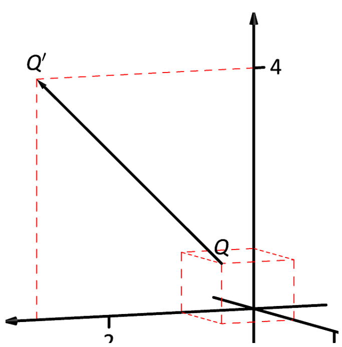
Example 1.2.1 Using component form notation for vectors

1. Sketch the vector $\vec{v} = \langle 2, -1\rangle$ starting at $P = (3, 2)$ and find its magnitude.
2. Find the component form of the vector $\vec{w}$ whose initial point is $R = (-3, -2)$ and whose terminal point is $S = (-1, 2)$.
3. Sketch the vector $\vec{u} = \langle 2, -1, 3\rangle$ starting at the point $Q = (1, 1, 1)$ and find its magnitude.

Notes:



(a)



(b)

Figure 1.2.3: Graphing vectors in Example 1.2.1.

Solution

1. Using P as the initial point, we move 2 units in the positive x-direction and -1 units in the positive y-direction to arrive at the terminal point $P' = (5, 1)$, as drawn in Figure 1.2.3(a).

   The magnitude of $\vec{v}$ is determined directly from the component form:

   $$||\,\vec{v}\,|| = \sqrt{2^2 + (-1)^2} = \sqrt{5}.$$

2. Using the note following Definition 1.2.2, we have

   $$\overrightarrow{RS} = \langle -1 - (-3), 2 - (-2) \rangle = \langle 2, 4 \rangle .$$

   One can readily see from Figure 1.2.3(a) that the x- and y-displacement of $\overrightarrow{RS}$ is 2 and 4, respectively, as the component form suggests.

3. Using Q as the initial point, we move 2 units in the positive x-direction, -1 unit in the positive y-direction, and 3 units in the positive z-direction to arrive at the terminal point $Q' = (3, 0, 4)$, illustrated in Figure 1.2.3(b).

   The magnitude of $\vec{u}$ is:

   $$||\,\vec{u}\,|| = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}.$$

Now that we have defined vectors, and have created a nice notation by which to describe them, we start considering how vectors interact with each other. That is, we define an *algebra* on vectors.

Notes:

Definition 1.2.3 Vector Algebra

1. Let $\vec{u} = \langle u_1, u_2 \rangle$ and $\vec{v} = \langle v_1, v_2 \rangle$ be vectors in $\mathbb{R}^2$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector

   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector

   $$c\vec{v} = c \langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle .$$

2. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ be vectors in $\mathbb{R}^3$, and let c be a scalar.

   (a) The addition, or sum, of the vectors $\vec{u}$ and $\vec{v}$ is the vector

   $$\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle .$$

   (b) The scalar product of c and $\vec{v}$ is the vector

   $$c\vec{v} = c \langle v_1, v_2, v_3 \rangle = \langle cv_1, cv_2, cv_3 \rangle .$$

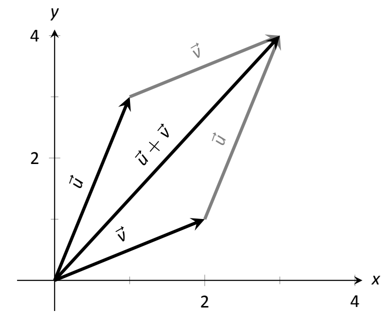


Figure 1.2.5: Illustrating how to add vectors using the Head to Tail Rule and Parallelogram Law.

In short, we say addition and scalar multiplication are computed "component–wise."

Example 1.2.2 Adding vectors

Sketch the vectors $\vec{u} = \langle 1, 3 \rangle$, $\vec{v} = \langle 2, 1 \rangle$ and $\vec{u} + \vec{v}$ all with initial point at the origin.

Solution We first compute $\vec{u} + \vec{v}$.

$$\begin{aligned} \vec{u} + \vec{v} &= \langle 1, 3 \rangle + \langle 2, 1 \rangle \\ &= \langle 3, 4 \rangle . \end{aligned}$$

These are all sketched in Figure 1.2.4.

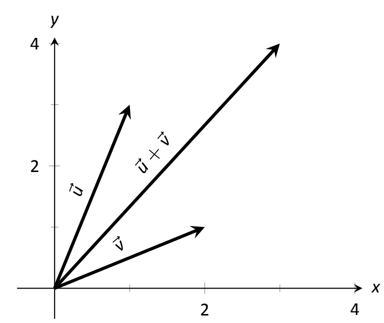


Figure 1.2.4: Graphing the sum of vectors in Example 1.2.2.

As vectors convey magnitude and direction information, the sum of vectors also convey length and magnitude information. Adding $\vec{u} + \vec{v}$ suggests the following idea:

"Starting at an initial point, go out $\vec{u}$, then go out $\vec{v}$."

Notes:

This idea is sketched in Figure 1.2.5, where the initial point of $\vec{v}$ is the terminal point of $\vec{u}$. This is known as the "Head to Tail Rule" of adding vectors. Vector addition is very important. For instance, if the vectors $\vec{u}$ and $\vec{v}$ represent forces acting on a body, the sum $\vec{u}+\vec{v}$ gives the resulting force. Because of various physical applications of vector addition, the sum $\vec{u}+\vec{v}$ is often referred to as the **resultant vector**, or just the "resultant."

Analytically, it is easy to see that $\vec{u}+\vec{v}=\vec{v}+\vec{u}$. Figure 1.2.5 also gives a graphical representation of this, using gray vectors. Note that the vectors $\vec{u}$ and $\vec{v}$, when arranged as in the figure, form a parallelogram. Because of this, the Head to Tail Rule is also known as the Parallelogram Law: the vector $\vec{u}+\vec{v}$ is defined by forming the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$; the initial point of $\vec{u}+\vec{v}$ is the common initial point of parallelogram, and the terminal point of the sum is the common terminal point of the parallelogram.

While not illustrated here, the Head to Tail Rule and Parallelogram Law hold for vectors in $\mathbb{R}^3$ as well.

It follows from the properties of the real numbers and Definition 1.2.3 that

$$\vec{u}-\vec{v}=\vec{u}+(-1)\vec{v}.$$

The Parallelogram Law gives us a good way to visualize this subtraction. We demonstrate this in the following example.

Example 1.2.3 Vector Subtraction

Let $\vec{u}=\langle 3,1\rangle$ and $\vec{v}=\langle 1,2\rangle$. Compute and sketch $\vec{u}-\vec{v}$.

Solution The computation of $\vec{u}-\vec{v}$ is straightforward, and we show all steps below. Usually the formal step of multiplying by (-1) is omitted and we "just subtract."

$$\begin{aligned}\vec{u}-\vec{v}&=\vec{u}+(-1)\vec{v}\\&=\langle 3,1\rangle+\langle -1,-2\rangle\\&=\langle 2,-1\rangle .\end{aligned}$$

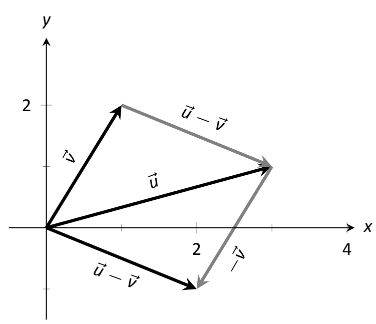


Figure 1.2.6: Illustrating how to subtract vectors graphically.

Figure 1.2.6 illustrates, using the Head to Tail Rule, how the subtraction can be viewed as the sum $\vec{u}+(-\vec{v})$. The figure also illustrates how $\vec{u}-\vec{v}$ can be obtained by looking only at the terminal points of $\vec{u}$ and $\vec{v}$ (when their initial points are the same).

Example 1.2.4 Scaling vectors

1. Sketch the vectors $\vec{v}=\langle 2,1\rangle$ and $2\vec{v}$ with initial point at the origin.
2. Compute the magnitudes of $\vec{v}$ and $2\vec{v}$.

Notes:

Solution

1. We compute $2\vec{v}$:

$$\begin{aligned} 2\vec{v} &= 2\langle 2, 1\rangle \\ &= \langle 4, 2\rangle . \end{aligned}$$

   Both $\vec{v}$ and $2\vec{v}$ are sketched in Figure 1.2.7. Make note that $2\vec{v}$ does not start at the terminal point of $\vec{v}$; rather, its initial point is also the origin.

2. The figure suggests that $2\vec{v}$ is twice as long as $\vec{v}$. We compute their magnitudes to confirm this.

$$\begin{aligned} ||\,\vec{v}\,|| &= \sqrt{2^2+1^2} \\ &= \sqrt{5}. \\ ||\,2\vec{v}\,|| &= \sqrt{4^2+2^2} \\ &= \sqrt{20} \\ &= \sqrt{4\cdot 5} = 2\sqrt{5}. \end{aligned}$$

   As we suspected, $2\vec{v}$ is twice as long as $\vec{v}$.

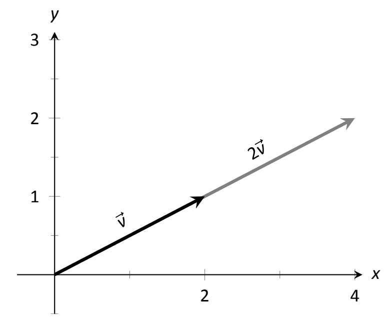


Figure 1.2.7: Graphing vectors $\vec{v}$ and $2\vec{v}$ in Example 1.2.4.

The **zero vector** is the vector whose initial point is also its terminal point. It is denoted by $\vec{0}$. Its component form, in $\mathbb{R}^2$, is $\langle 0, 0\rangle$; in $\mathbb{R}^3$, it is $\langle 0, 0, 0\rangle$. Usually the context makes is clear whether $\vec{0}$ is referring to a vector in the plane or in space.

Our examples have illustrated key principles in vector algebra: how to add and subtract vectors and how to multiply vectors by a scalar. The following theorem states formally the properties of these operations.

Notes:

Theorem 1.2.1 Properties of Vector Operations

The following are true for all scalars c and d, and for all vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^2$ or where $\vec{u}$, $\vec{v}$ and $\vec{w}$ are all in $\mathbb{R}^3$:

1. $\vec{u}+\vec{v}=\vec{v}+\vec{u}$ Commutative Property
2. $(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$ Associative Property
3. $\vec{v}+\vec{0}=\vec{v}$ Additive Identity
4. $(cd)\vec{v}=c(d\vec{v})$
5. $c(\vec{u}+\vec{v})=c\vec{u}+c\vec{v}$ Distributive Property
6. $(c+d)\vec{v}=c\vec{v}+d\vec{v}$ Distributive Property
7. $0\vec{v}=\vec{0}$
8. $||\,c\vec{v}\,||=|c|\cdot||\,\vec{v}\,||$
9. $||\,\vec{u}\,||=0$ if, and only if, $\vec{u}=\vec{0}$.

As stated before, each vector $\vec{v}$ conveys magnitude and direction information. We have a method of extracting the magnitude, which we write as $||\,\vec{v}\,||$. *Unit vectors* are a way of extracting just the direction information from a vector.

Definition 1.2.4 Unit Vector

A **unit vector** is a vector $\vec{v}$ with a magnitude of 1; that is,

$$||\,\vec{v}\,||=1.$$

Consider this scenario: you are given a vector $\vec{v}$ and are told to create a vector of length 10 in the direction of $\vec{v}$. How does one do that? If we knew that $\vec{u}$ was the unit vector in the direction of $\vec{v}$, the answer would be easy: $10\vec{u}$. So how do we find $\vec{u}$?

Property 8 of Theorem 1.2.1 holds the key. If we divide $\vec{v}$ by its magnitude, it becomes a vector of length 1. Consider:

$$\begin{aligned}\left|\left|\frac{1}{||\,\vec{v}\,||}\vec{v}\right|\right| &= \frac{1}{||\,\vec{v}\,||}||\,\vec{v}\,|| \qquad \text{(we can pull out } \frac{1}{||\,\vec{v}\,||} \text{ as it is a scalar)}\\ &= 1.\end{aligned}$$

Notes:

So the vector of length 10 in the direction of $\vec{v}$ is $10\dfrac{1}{||\,\vec{v}\,||}\vec{v}$. An example will make this more clear.

Example 1.2.5 Using Unit Vectors

Let $\vec{v} = \langle 3, 1\rangle$ and let $\vec{w} = \langle 1, 2, 2\rangle$.

1. Find the unit vector in the direction of $\vec{v}$.
2. Find the unit vector in the direction of $\vec{w}$.
3. Find the vector in the direction of $\vec{v}$ with magnitude 5.

Solution

1. We find $||\,\vec{v}\,|| = \sqrt{10}$. So the unit vector $\vec{u}$ in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{\sqrt{10}}\vec{v} = \left\langle \frac{3}{\sqrt{10}}, \frac{1}{\sqrt{10}}\right\rangle.$$
2. We find $||\,\vec{w}\,|| = 3$, so the unit vector $\vec{z}$ in the direction of $\vec{w}$ is
$$\vec{u} = \frac{1}{3}\vec{w} = \left\langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right\rangle.$$
3. To create a vector with magnitude 5 in the direction of $\vec{v}$, we multiply the unit vector $\vec{u}$ by 5. Thus $5\vec{u} = \left\langle 15/\sqrt{10}, 5/\sqrt{10}\right\rangle$ is the vector we seek. This is sketched in Figure 1.2.8.

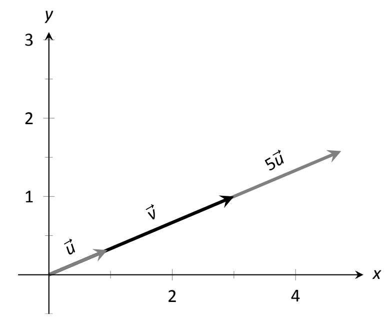


Figure 1.2.8: Graphing vectors in Example 1.2.5. All vectors shown have their initial point at the origin.

The basic formation of the unit vector $\vec{u}$ in the direction of a vector $\vec{v}$ leads to a interesting equation. It is:

$$\vec{v} = ||\,\vec{v}\,||\frac{1}{||\,\vec{v}\,||}\vec{v}.$$

We rewrite the equation with parentheses to make a point:

$$\vec{v} = \underbrace{||\,\vec{v}\,||}_{\text{magnitude}} \cdot \underbrace{\left(\frac{1}{||\,\vec{v}\,||}\vec{v}\right)}_{\text{direction}}.$$

This equation illustrates the fact that a vector has both magnitude and direction, where we view a unit vector as supplying *only* direction information. Identifying unit vectors with direction allows us to define **parallel vectors**.

Notes:

Definition 1.2.5 Parallel Vectors

1. Unit vectors $\vec{u}_1$ and $\vec{u}_2$ are **parallel** if $\vec{u}_1 = \pm\vec{u}_2$.
2. Nonzero vectors $\vec{v}_1$ and $\vec{v}_2$ are **parallel** if their respective unit vectors are parallel.

Note: $\vec{0}$ is directionless; because $||\,\vec{0}\,|| = 0$, there is no unit vector in the "direction" of $\vec{0}$.
Some texts define two vectors as being parallel if one is a scalar multiple of the other. By this definition, $\vec{0}$ is parallel to all vectors as $\vec{0} = 0\vec{v}$ for all $\vec{v}$.
We prefer the given definition of parallel as it is grounded in the fact that unit vectors provide direction information. One may adopt the convention that $\vec{0}$ is parallel to all vectors if they desire. (See also the marginal note on page 604.)

It is equivalent to say that vectors $\vec{v}_1$ and $\vec{v}_2$ are parallel if there is a scalar $c \neq 0$ such that $\vec{v}_1 = c\vec{v}_2$ (see marginal note).

If one graphed all unit vectors in $\mathbb{R}^2$ with the initial point at the origin, then the terminal points would all lie on the unit circle. Based on what we know from trigonometry, we can then say that the component form of all unit vectors in $\mathbb{R}^2$ is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.

A similar construction in $\mathbb{R}^3$ shows that the terminal points all lie on the unit sphere. These vectors also have a particular component form, but its derivation is not as straightforward as the one for unit vectors in $\mathbb{R}^2$. Important concepts about unit vectors are given in the following Key Idea.

Key Idea 1.2.1 Unit Vectors

1. The unit vector in the direction of $\vec{v}$ is
$$\vec{u} = \frac{1}{||\,\vec{v}\,||}\vec{v}.$$
2. A vector $\vec{u}$ in $\mathbb{R}^2$ is a unit vector if, and only if, its component form is $\langle \cos\theta, \sin\theta \rangle$ for some angle θ.
3. A vector $\vec{u}$ in $\mathbb{R}^3$ is a unit vector if, and only if, its component form is $\langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta \rangle$ for some angles θ and φ.

These formulas can come in handy in a variety of situations, especially the formula for unit vectors in the plane.

Example 1.2.6 Finding Component Forces

Consider a weight of 50lb hanging from two chains, as shown in Figure 1.2.9. One chain makes an angle of 30° with the vertical, and the other an angle of 45°. Find the force applied to each chain.

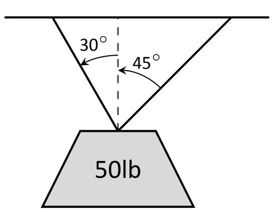


Figure 1.2.9: A diagram of a weight hanging from 2 chains in Example 1.2.6.

Solution Knowing that gravity is pulling the 50lb weight straight down,

Notes:

we can create a vector $\vec{F}$ to represent this force.

$$\vec{F} = 50\langle 0, -1\rangle = \langle 0, -50\rangle .$$

We can view each chain as "pulling" the weight up, preventing it from falling. We can represent the force from each chain with a vector. Let $\vec{F}_1$ represent the force from the chain making an angle of 30° with the vertical, and let $\vec{F}_2$ represent the force form the other chain. Convert all angles to be measured from the horizontal (as shown in Figure 1.2.10), and apply Key Idea 1.2.1. As we do not yet know the magnitudes of these vectors, (that is the problem at hand), we use m_1 and m_2 to represent them.

$$\vec{F}_1 = m_1\langle \cos 120^\circ, \sin 120^\circ\rangle$$

$$\vec{F}_2 = m_2\langle \cos 45^\circ, \sin 45^\circ\rangle$$

As the weight is not moving, we know the sum of the forces is $\vec{0}$. This gives:

$$\vec{F} + \vec{F}_1 + \vec{F}_2 = \vec{0}$$

$$\langle 0, -50\rangle + m_1\langle \cos 120^\circ, \sin 120^\circ\rangle + m_2\langle \cos 45^\circ, \sin 45^\circ\rangle = \vec{0}$$

The sum of the entries in the first component is 0, and the sum of the entries in the second component is also 0. This leads us to the following two equations:

$$m_1 \cos 120^\circ + m_2 \cos 45^\circ = 0$$
$$m_1 \sin 120^\circ + m_2 \sin 45^\circ = 50$$

This is a simple 2-equation, 2-unkown system of linear equations. We leave it to the reader to verify that the solution is

$$m_1 = 50(\sqrt{3} - 1) \approx 36.6; \qquad m_2 = \frac{50\sqrt{2}}{1 + \sqrt{3}} \approx 25.88.$$

It might seem odd that the sum of the forces applied to the chains is more than 50lb. We leave it to a physics class to discuss the full details, but offer this short explanation. Our equations were established so that the *vertical* components of each force sums to 50lb, thus supporting the weight. Since the chains are at an angle, they also pull against each other, creating an "additional" horizontal force while holding the weight in place.

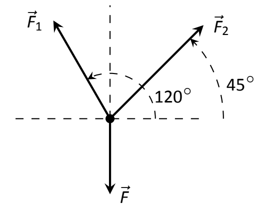


Figure 1.2.10: A diagram of the force vectors from Example 1.2.6.

Unit vectors were very important in the previous calculation; they allowed us to define a vector in the proper direction but with an unknown magnitude. Our computations were then computed component–wise. Because such calculations are often necessary, the *standard unit vectors* can be useful.

Notes:

Definition 1.2.6 Standard Unit Vectors

1. In $\mathbb{R}^2$, the standard unit vectors are

$$\vec{i} = \langle 1, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1 \rangle .$$

2. In $\mathbb{R}^3$, the standard unit vectors are

$$\vec{i} = \langle 1, 0, 0 \rangle \quad \text{and} \quad \vec{j} = \langle 0, 1, 0 \rangle \quad \text{and} \quad \vec{k} = \langle 0, 0, 1 \rangle .$$

Example 1.2.7 Using standard unit vectors

1. Rewrite $\vec{v} = \langle 2, -3 \rangle$ using the standard unit vectors.
2. Rewrite $\vec{w} = 4\vec{i} - 5\vec{j} + 2\vec{k}$ in component form.

Solution

1.
$$\begin{aligned} \vec{v} &= \langle 2, -3 \rangle \\ &= \langle 2, 0 \rangle + \langle 0, -3 \rangle \\ &= 2 \langle 1, 0 \rangle - 3 \langle 0, 1 \rangle \\ &= 2\vec{i} - 3\vec{j} \end{aligned}$$

2.
$$\begin{aligned} \vec{w} &= 4\vec{i} - 5\vec{j} + 2\vec{k} \\ &= \langle 4, 0, 0 \rangle + \langle 0, -5, 0 \rangle + \langle 0, 0, 2 \rangle \\ &= \langle 4, -5, 2 \rangle \end{aligned}$$

These two examples demonstrate that converting between component form and the standard unit vectors is rather straightforward. Many mathematicians prefer component form, and it is the preferred notation in this text. Many engineers prefer using the standard unit vectors, and many engineering text use that notation.

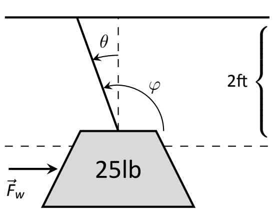


Figure 1.2.11: A figure of a weight being pushed by the wind in Example 1.2.8.

Example 1.2.8 Finding Component Force

A weight of 25lb is suspended from a chain of length 2ft while a wind pushes the weight to the right with constant force of 5lb as shown in Figure 1.2.11. What angle will the chain make with the vertical as a result of the wind's pushing? How much higher will the weight be?

Notes:

Solution The force of the wind is represented by the vector $\vec{F}_w = 5\vec{i}$. The force of gravity on the weight is represented by $\vec{F}_g = -25\vec{j}$. The direction and magnitude of the vector representing the force on the chain are both unknown. We represent this force with

$$\vec{F}_c = m\,\langle \cos\varphi, \sin\varphi\rangle = m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j}$$

for some magnitude m and some angle with the horizontal φ. (Note: θ is the angle the chain makes with the *vertical*; φ is the angle with the *horizontal*.)

As the weight is at equilibrium, the sum of the forces is $\vec{0}$:

$$\begin{aligned} \vec{F}_c + \vec{F}_w + \vec{F}_g &= \vec{0} \\ m\cos\varphi\,\vec{i} + m\sin\varphi\,\vec{j} + 5\vec{i} - 25\vec{j} &= \vec{0} \end{aligned}$$

Thus the sum of the $\vec{i}$ and $\vec{j}$ components are 0, leading us to the following system of equations:

$$\begin{aligned} 5 + m\cos\varphi &= 0 \\ -25 + m\sin\varphi &= 0 \end{aligned} \tag{1.1}$$

This is enough to determine $\vec{F}_c$ already, as we know $m\cos\varphi = -5$ and $m\sin\varphi = 25$. Thus $F_c = \langle -5, 25\rangle$. We can use this to find the magnitude m:

$$m = \sqrt{(-5)^2 + 25^2} = 5\sqrt{26} \approx 25.5\text{lb}.$$

We can then use either equality from Equation (1.1) to solve for φ. We choose the first equality as using arccosine will return an angle in the 2$^{\text{nd}}$ quadrant:

$$5 + 5\sqrt{26}\cos\varphi = 0 \quad \Rightarrow \quad \varphi = \cos^{-1}\left(\frac{-5}{5\sqrt{26}}\right) \approx 1.7682 \approx 101.31^\circ.$$

Subtracting 90° from this angle gives us an angle of 11.31° with the vertical.

We can now use trigonometry to find out how high the weight is lifted. The diagram shows that a right triangle is formed with the 2ft chain as the hypotenuse with an interior angle of 11.31°. The length of the adjacent side (in the diagram, the dashed vertical line) is $2\cos 11.31^\circ \approx 1.96$ft. Thus the weight is lifted by about 0.04ft, almost 1/2in.

The algebra we have applied to vectors is already demonstrating itself to be very useful. There are two more fundamental operations we can perform with vectors, the *dot product* and the *cross product*. The next two sections explore each in turn.

Notes:

Exercises 1.2

Terms and Concepts

10 02 ex 01 1. Name two different things that cannot be described with just one number, but rather need 2 or more numbers to fully describe them.

10 02 ex 02 2. What is the difference between $(1, 2)$ and $\langle 1, 2\rangle$?

10 02 ex 03 3. What is a unit vector?

10 02 ex 36 4. Unit vectors can be thought of as conveying what type of information?

10 02 ex 04 5. What does it mean for two vectors to be parallel?

10 02 ex 05 6. What effect does multiplying a vector by -2 have?

Problems

10 02 exset 01 **In Exercises 7 – 10, points P and Q are given. Write the vector $\overrightarrow{PQ}$ in component form and using the standard unit vectors.**

10 02 ex 06 7. $P = (2, -1), \quad Q = (3, 5)$

10 02 ex 07 8. $P = (3, 2), \quad Q = (7, -2)$

10 02 ex 08 9. $P = (0, 3, -1), \quad Q = (6, 2, 5)$

10 02 ex 09 10. $P = (2, 1, 2), \quad Q = (4, 3, 2)$

10 02 ex 10 11. Let $\vec{u} = \langle 1, -2\rangle$ and $\vec{v} = \langle 1, 1\rangle$.

   (a) Find $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $2\vec{u}-3\vec{v}$.

   (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.

   (c) Find $\vec{x}$ where $\vec{u}+\vec{x} = 2\vec{v}-\vec{x}$.

10 02 ex 11 12. Let $\vec{u} = \langle 1, 1, -1\rangle$ and $\vec{v} = \langle 2, 1, 2\rangle$.

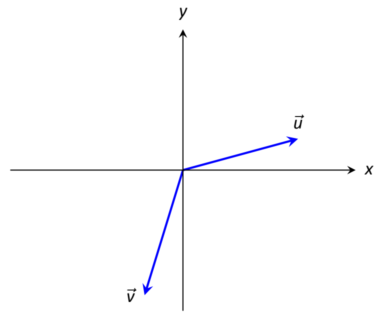
   (a) Find $\vec{u}+\vec{v}$, $\vec{u}-\vec{v}$, $\pi\vec{u}-\sqrt{2}\vec{v}$.

   (b) Sketch the above vectors on the same axes, along with $\vec{u}$ and $\vec{v}$.

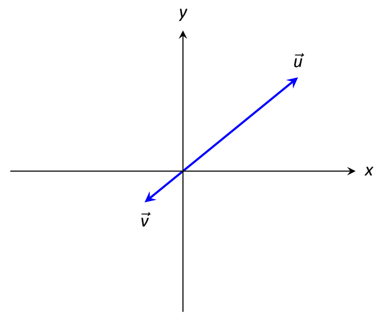
   (c) Find $\vec{x}$ where $\vec{u}+\vec{x} = \vec{v}+2\vec{x}$.

10 02 exset 02 **In Exercises 13 – 16, sketch $\vec{u}$, $\vec{v}$, $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ on the same axes.**

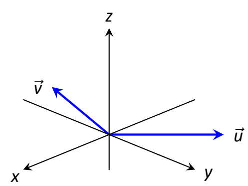
10 02 ex 12 13.



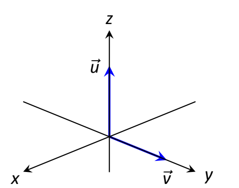
10 02 ex 13 14.



10 02 ex 14 15.



10 02 ex 15 16.



10 02 exset 03 **In Exercises 17 – 20, find $||\,\vec{u}\,||$, $||\,\vec{v}\,||$, $||\,\vec{u}+\vec{v}\,||$ and $||\,\vec{u}-\vec{v}\,||$.**

10 02 ex 16 17. $\vec{u} = \langle 2, 1\rangle, \quad \vec{v} = \langle 3, -2\rangle$

10 02 ex 17 18. $\vec{u} = \langle -3, 2, 2\rangle, \quad \vec{v} = \langle 1, -1, 1\rangle$

10 02 ex 18 19. $\vec{u} = \langle 1, 2\rangle, \quad \vec{v} = \langle -3, -6\rangle$

10 02 ex 19 20. $\vec{u} = \langle 2, -3, 6\rangle, \quad \vec{v} = \langle 10, -15, 30\rangle$

10 02 ex 20 21. Under what conditions is $||\,\vec{u}\,|| + ||\,\vec{v}\,|| = ||\,\vec{u}+\vec{v}\,||$?

10 02 exset 04 **In Exercises 22 – 25, find the unit vector $\vec{u}$ in the direction of $\vec{v}$.**

10 02 ex 21 22. $\vec{v} = \langle 3, 7\rangle$

10 02 ex 22 23. $\vec{v} = \langle 6, 8\rangle$

10 02 ex 23 24. $\vec{v} = \langle 1, -2, 2\rangle$

10 02 ex 24 25. $\vec{v} = \langle 2, -2, 2\rangle$

10 02 ex 25 26. Find the unit vector in the first quadrant of $\mathbb{R}^2$ that makes a 50° angle with the x-axis.

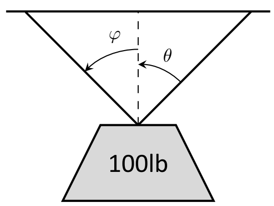
10 02 ex 26 27. Find the unit vector in the second quadrant of $\mathbb{R}^2$ that makes a 30° angle with the y-axis.

10 02 ex 27 28. Verify, from Key Idea 1.2.1, that

$$\vec{u} = \langle \sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\rangle$$

is a unit vector for all angles θ and φ.

10 02 exset 05 **A weight of 100lb is suspended from two chains, making angles with the vertical of θ and φ as shown in the figure below.**



In Exercises 29 – 32, angles θ and φ are given. Find the force applied to each chain.

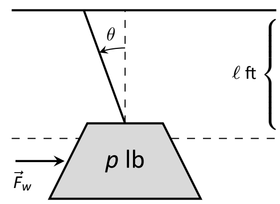
10 02 ex 28 29. $\theta = 30^\circ, \quad \varphi = 30^\circ$

10 02 ex 29 30. $\theta = 60^\circ, \quad \varphi = 60^\circ$

10 02 ex 30 31. $\theta = 20^\circ, \quad \varphi = 15^\circ$

10 02 ex 31 32. $\theta = 0^\circ, \quad \varphi = 0^\circ$

10 02 exset 06 **A weight of $p$lb is suspended from a chain of length ℓ while a constant force of $\vec{F}_w$ pushes the weight to the right, making an angle of θ with the vertical, as shown in the figure below.**



In Exercises 33 – 36, a force $\vec{F}_w$ and length ℓ are given. Find the angle θ and the height the weight is lifted as it moves to the right.

10 02 ex 32 33. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 1\text{lb}$

10 02 ex 33 34. $\vec{F}_w = 1\text{lb}, \quad \ell = 1\text{ft}, \quad p = 10\text{lb}$

10 02 ex 34 35. $\vec{F}_w = 1\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

10 02 ex 35 36. $\vec{F}_w = 10\text{lb}, \quad \ell = 10\text{ft}, \quad p = 1\text{lb}$

A: Solutions To Selected Problems

Chapter 1

Section 1.1

10 01 ex 08 1. right hand

10 01 ex 01 2. line; plane

10 01 ex 02 3. curve (a parabola); surface (a cylinder)

10 01 ex 03 4. a hyperbolic paraboloid

10 01 ex 04 5. a hyperbolid of two sheets

10 01 ex 05 6. a hyperboloid of one sheet

10 01 ex 06 7. $\| \overline{AB} \| = \sqrt{6}$; $\| \overline{BC} \| = \sqrt{17}$; $\| \overline{AC} \| = \sqrt{11}$. Yes, it is a right triangle as $\| \overline{AB} \|^2 + \| \overline{AC} \|^2 = \| \overline{BC} \|^2$.

10 01 ex 07 8. Yes, as opposite sides have equal length. $\| \overline{AB} \| = \sqrt{21} = \| \overline{CD} \|$; $\| \overline{BC} \| = \sqrt{6} = \| \overline{AD} \|$.

10 01 ex 09 9. Center at $(4, -1, 0)$; radius = 3

10 01 ex 10 10. Center at $(-2, 1, 2)$; radius = $\sqrt{5}$

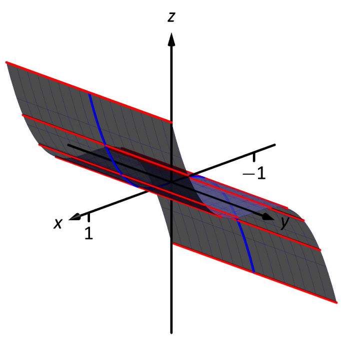
10 01 ex 11 11. Interior of a sphere with radius 1 centered at the origin.

10 01 ex 12 12. Region bounded between the planes $x = 0$ (the $y - z$ coordinate plane) and $x = 3$.

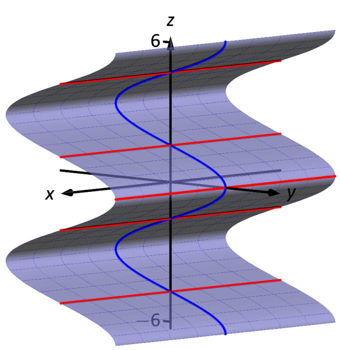
10 01 ex 13 13. The first octant of space; all points (x, y, z) where each of x, y and z are non-negative. (Analogous to the first quadrant in the plane.)

10 01 ex 14 14. All points in space where the y value is greater than 3; viewing space as often depicted in this text, this is the region "to the right" of the plane $y = 3$ (which is parallel to the $x - z$ coordinate plane.)

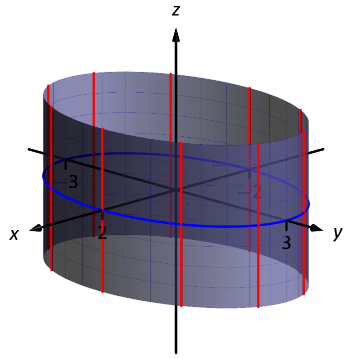
10 01 ex 15 15.



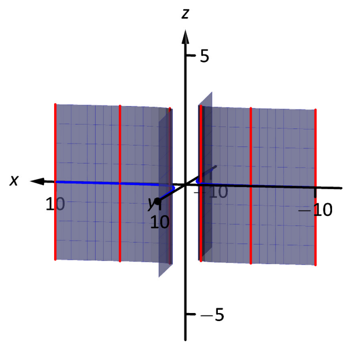
10 01 ex 16 16.



10 01 ex 17 17.



10 01 ex 18 18.



10 01 ex 29 19. $x^2 + z^2 = \frac{1}{(1+y^2)^2}$

10 01 ex 30 20. $y^2 + z^2 = x^4$

10 01 ex 31 21. $z = (\sqrt{x^2 + y^2})^2 = x^2 + y^2$

10 01 ex 32 22. $z = \frac{1}{\sqrt{x^2+y^2}}$

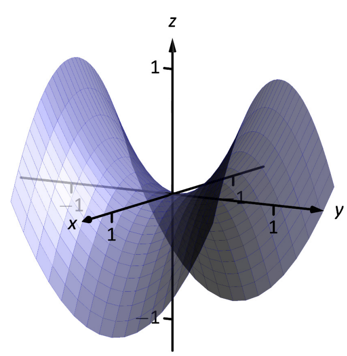
10 01 ex 19 23. (a) $x = y^2 + \dfrac{z^2}{9}$

10 01 ex 20 24. (b) $x^2 - y^2 + z^2 = 0$

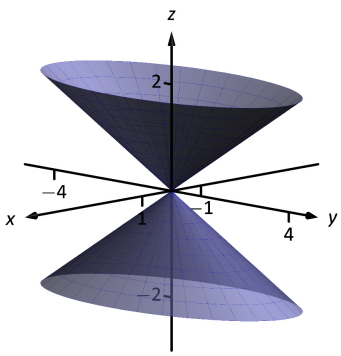
10 01 ex 21 25. (b) $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$

10 01 ex 22 26. (a) $y^2 - x^2 - z^2 = 1$

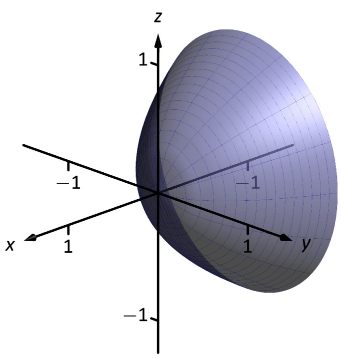
10 01 ex 28 27.



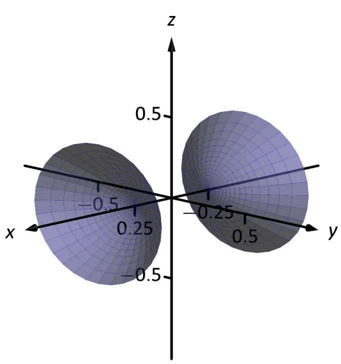
10 01 ex 24 28.



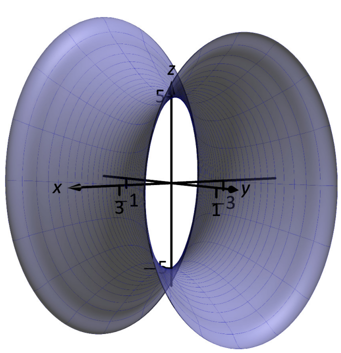
10 01 ex 23 29.



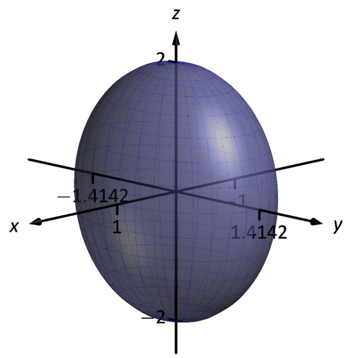
10 01 ex 27 30.



10 01 ex 26 31.



10 01 ex 25 32.



Section 1.2

10 02 ex 01 1. Answers will vary.

10 02 ex 02 2. $(1, 2)$ is a point; $\langle 1, 2\rangle$ is a vector that describes a displacement of 1 unit in the x-direction and 2 units in the y-direction.

10 02 ex 03 3. A vector with magnitude 1.

10 02 ex 36 4. Direction

10 02 ex 04 5. Their respective unit vectors are parallel; unit vectors $\vec{u}_1$ and $\vec{u}_2$ are parallel if $\vec{u}_1 = \pm\vec{u}_2$.

10 02 ex 05 6. It stretches the vector by a factor of 2, and points it in the opposite direction.

10 02 ex 06 7. $\overrightarrow{PQ} = \langle 1, 6\rangle = 1\vec{i} + 6\vec{j}$

10 02 ex 07 8. $\overrightarrow{PQ} = \langle 4, -4\rangle = 4\vec{i} - 4\vec{j}$

10 02 ex 08 9. $\overrightarrow{PQ} = \langle 6, -1, 6\rangle = 6\vec{i} - \vec{j} + 6\vec{k}$

10 02 ex 09 10. $\overrightarrow{PQ} = \langle 2, 2, 0\rangle = 2\vec{i} + 2\vec{j}$

10 02 ex 10 11.

(a) $\vec{u} + \vec{v} = \langle 2, -1\rangle$; $\vec{u} - \vec{v} = \langle 0, -3\rangle$; $2\vec{u} - 3\vec{v} = \langle -1, -7\rangle$.

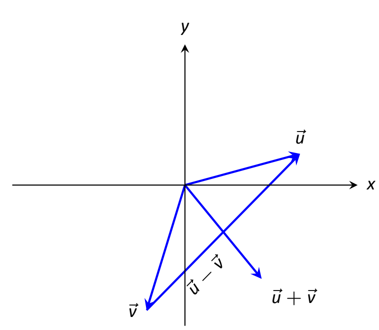
(c) $\vec{x} = \langle 1/2, 2\rangle$.

10 02 ex 11 12.

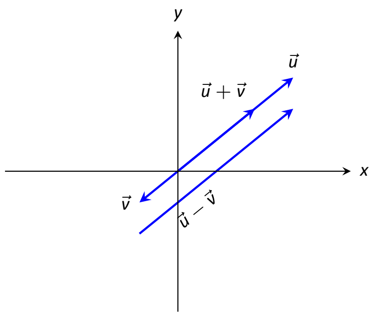
(a) $\vec{u} + \vec{v} = \langle 3, 2, 1\rangle$; $\vec{u} - \vec{v} = \langle -1, 0, -3\rangle$; $\pi\vec{u} - \sqrt{2}\vec{v} = \langle \pi - 2\sqrt{2}, \pi - \sqrt{2}, -\pi - 2\sqrt{2}\rangle$.

(c) $\vec{x} = \langle -1, 0, -3\rangle$.

10 02 ex 12 13.

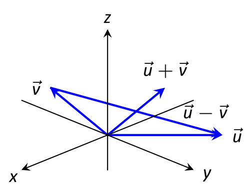


10 02 ex 13 14.

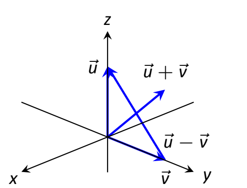


Sketch of $\vec u-\vec v$ shifted for clarity.

10 02 ex 14 15.



10 02 ex 15 16.



10 02 ex 16 17. $||\,\vec u\,|| = \sqrt{5}$, $||\,\vec v\,|| = \sqrt{13}$, $||\,\vec u+\vec v\,|| = \sqrt{26}$, $||\,\vec u-\vec v\,|| = \sqrt{10}$

10 02 ex 17 18. $||\,\vec u\,|| = \sqrt{17}$, $||\,\vec v\,|| = \sqrt{3}$, $||\,\vec u+\vec v\,|| = \sqrt{14}$, $||\,\vec u-\vec v\,|| = \sqrt{26}$

10 02 ex 18 19. $||\,\vec u\,|| = \sqrt{5}$, $||\,\vec v\,|| = 3\sqrt{5}$, $||\,\vec u+\vec v\,|| = 2\sqrt{5}$, $||\,\vec u-\vec v\,|| = 4\sqrt{5}$

10 02 ex 19 20. $||\,\vec u\,|| = 7$, $||\,\vec v\,|| = 35$, $||\,\vec u+\vec v\,|| = 42$, $||\,\vec u-\vec v\,|| = 28$

10 02 ex 20 21. When $\vec u$ and $\vec v$ have the same direction. (Note: parallel is not enough.)

10 02 ex 21 22. $\vec u = \langle 3/\sqrt{58}, 7/\sqrt{58}\rangle$

10 02 ex 22 23. $\vec u = \langle 0.6, 0.8\rangle$

10 02 ex 23 24. $\vec u = \langle 1/3, -2/3, 2/3\rangle$

10 02 ex 24 25. $\vec u = \langle 1/\sqrt{3}, -1/\sqrt{3}, 1/\sqrt{3}\rangle$

10 02 ex 25 26. $\vec u = \langle \cos 50^\circ, \sin 50^\circ\rangle \approx \langle 0.643, 0.766\rangle$.

10 02 ex 26 27. $\vec u = \langle \cos 120^\circ, \sin 120^\circ\rangle = \langle -1/2, \sqrt{3}/2\rangle$.

10 02 ex 27 28.

$$\begin{aligned} ||\,\vec u\,|| &= \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} \\ &= \sqrt{\sin^2\theta(\cos^2\varphi + \sin^2\varphi) + \cos^2\theta} \\ &= \sqrt{\sin^2\theta + \cos^2\theta} \\ &= 1. \end{aligned}$$

10 02 ex 28 29. The force on each chain is $100/\sqrt{3} \approx 57.735$lb.

10 02 ex 29 30. The force on each chain is 100lb.

10 02 ex 30 31. The force on the chain with angle θ is approx. 45.124lb; the force on the chain with angle φ is approx. 59.629lb.

10 02 ex 31 32. The force on each chain is 50lb.

10 02 ex 32 33. $\theta = 45^\circ$; the weight is lifted 0.29 ft (about 3.5in).

10 02 ex 33 34. $\theta = 5.71^\circ$; the weight is lifted 0.005 ft (about 1/16th of an inch).

10 02 ex 34 35. $\theta = 45^\circ$; the weight is lifted 2.93 ft.

10 02 ex 35 36. $\theta = 84.29^\circ$; the weight is lifted 9 ft.