1: Functions of Several Variables

1.1 The Multivariable Chain Rule

Consider driving an off-road vehicle along a dirt road. As you drive, your elevation likely changes. What factors determine how quickly your elevation rises and falls? After some thought, generally one recognizes that one's *velocity* (speed and direction) and the *terrain* influence your rise and fall.

One can represent the terrain as the surface defined by a multivariable function $z = f(x, y)$; one can represent the path of the off-road vehicle, as seen from above, with a vector–valued function $\vec{r}(t) = \langle x(t), y(t)\rangle$; the velocity of the vehicle is thus $\vec{r}\,'(t) = \langle x'(t), y'(t)\rangle$.

Consider Figure 1.1.1 in which a surface $z = f(x, y)$ is drawn, along with a dashed curve in the x-y plane. Restricting f to just the points on this circle gives the curve shown on the surface (i.e., "the path of the off-road vehicle.") The derivative $\frac{df}{dt}$ gives the instantaneous rate of change of f with respect to t. If we consider an object traveling along this path, $\frac{df}{dt} = \frac{dz}{dt}$ gives the rate at which the object rises/falls (i.e., "the rate of elevation change" of the vehicle.) Conceptually, the Multivariable Chain Rule combines terrain and velocity information properly to compute this rate of elevation change.

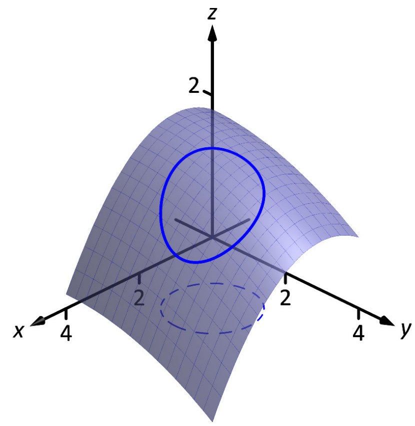

Figure 1.1.1: Understanding the application of the Multivariable Chain Rule.

Abstractly, let z be a function of x and y; that is, $z = f(x, y)$ for some function f, and let x and y each be functions of t. By choosing a t-value, x- and y-values are determined, which in turn determine z: this defines z as a function of t. The Multivariable Chain Rule gives a method of computing $\frac{dz}{dt}$.

Theorem 1.1.1 Multivariable Chain Rule, Part I

Let $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, where f, g and h are differentiable functions. Then $z = f(x, y) = f\big(g(t), h(t)\big)$ is a function of t, and

$$\begin{aligned}\frac{dz}{dt} = \frac{df}{dt} &= f_x(x,y)\frac{dx}{dt} + f_y(x,y)\frac{dy}{dt}\\ &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\\ &= \langle f_x, f_y\rangle \cdot \langle x', y'\rangle.\end{aligned}$$

The Chain Rule of Section 2.5 states that $\dfrac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)g'(x)$. If

$t = g(x)$, we can express the Chain Rule as

$$\frac{df}{dx} = \frac{df}{dt}\frac{dt}{dx};$$

recall that the derivative notation is deliberately chosen to reflect their fraction–like properties. A similar effect is seen in Theorem 1.1.1. In the second line of equations, one can think of the dx and ∂x as "sort of" canceling out, and likewise with dy and ∂y.

Notice, too, the third line of equations in Theorem 1.1.1. The vector $\langle f_x, f_y\rangle$ contains information about the surface (terrain); the vector $\langle x', y'\rangle$ can represent velocity. In the context measuring the rate of elevation change of the off-road vehicle, the Multivariable Chain Rule states it can be found through a product of terrain and velocity information.

We now practice applying the Multivariable Chain Rule.

Example 1.1.1 Using the Multivariable Chain Rule

Let $z = x^2y + x$, where $x = \sin t$ and $y = e^{5t}$. Find $\dfrac{dz}{dt}$ using the Chain Rule.

Solution Following Theorem 1.1.1, we find

$$f_x(x, y) = 2xy + 1, \qquad f_y(x, y) = x^2, \qquad \frac{dx}{dt} = \cos t, \qquad \frac{dy}{dt} = 5e^{5t}.$$

Applying the theorem, we have

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t}.$$

This may look odd, as it seems that $\frac{dz}{dt}$ is a function of x, y and t. Since x and y are functions of t, $\frac{dz}{dt}$ is really just a function of t, and we can replace x with $\sin t$ and y with e^{5t}:

$$\frac{dz}{dt} = (2xy + 1)\cos t + 5x^2e^{5t} = (2\sin(t)e^{5t} + 1)\cos t + 5e^{5t}\sin^2 t.$$

The previous example can make us wonder: if we substituted for x and y at the end to show that $\frac{dz}{dt}$ is really just a function of t, why not substitute before differentiating, showing clearly that z is a function of t?

That is, $z = x^2y + x = (\sin t)^2e^{5t} + \sin t$. Applying the Chain and Product Rules, we have

$$\frac{dz}{dt} = 2\sin t\cos t\, e^{5t} + 5\sin^2 t\, e^{5t} + \cos t,$$

which matches the result from the example.

Notes:

This may now make one wonder "What's the point? If we could already find the derivative, why learn another way of finding it?" In some cases, applying this rule makes deriving simpler, but this is hardly the power of the Chain Rule. Rather, in the case where $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, the Chain Rule is extremely powerful when *we do not know what f, g and/or h are*. It may be hard to believe, but often in "the real world" we know rate–of–change information (i.e., information about derivatives) without explicitly knowing the underlying functions. The Chain Rule allows us to combine several rates of change to find another rate of change. The Chain Rule also has theoretic use, giving us insight into the behavior of certain constructions (as we'll see in the next section).

We demonstrate this in the next example.

Example 1.1.2 Applying the Multivarible Chain Rule
An object travels along a path on a surface. The exact path and surface are not known, but at time $t = t_0$ it is known that :

$$\frac{\partial z}{\partial x} = 5, \qquad \frac{\partial z}{\partial y} = -2, \qquad \frac{dx}{dt} = 3 \qquad \text{and} \qquad \frac{dy}{dt} = 7.$$

Find $\frac{dz}{dt}$ at time t_0.

SOLUTION The Multivariable Chain Rule states that

$$\begin{aligned} \frac{dz}{dt} &= \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} \\ &= 5(3) + (-2)(7) \\ &= 1. \end{aligned}$$

By knowing certain rates–of–change information about the surface and about the path of the particle in the x-y plane, we can determine how quickly the object is rising/falling.

We next apply the Chain Rule to solve a max/min problem.

Example 1.1.3 Applying the Multivariable Chain Rule
Consider the surface $z = x^2 + y^2 - xy$, a paraboloid, on which a particle moves with x and y coordinates given by $x = \cos t$ and $y = \sin t$. Find $\frac{dz}{dt}$ when $t = 0$, and find where the particle reaches its maximum/minimum z-values.

SOLUTION It is straightforward to compute

$$f_x(x, y) = 2x - y, \qquad f_y(x, y) = 2y - x, \qquad \frac{dx}{dt} = -\sin t, \qquad \frac{dy}{dt} = \cos t.$$

Notes:

Combining these according to the Chain Rule gives:

$$\frac{dz}{dt} = -(2x-y)\sin t + (2y-x)\cos t.$$

When $t = 0$, $x = 1$ and $y = 0$. Thus $\frac{dz}{dt} = -(2)(0) + (-1)(1) = -1$. When $t = 0$, the particle is moving down, as shown in Figure 1.1.2.

To find where z-value is maximized/minimized on the particle's path, we set $\frac{dz}{dt} = 0$ and solve for t:

$$\begin{aligned} \frac{dz}{dt} = 0 &= -(2x-y)\sin t + (2y-x)\cos t \\ 0 &= -(2\cos t - \sin t)\sin t + (2\sin t - \cos t)\cos t \\ 0 &= \sin^2 t - \cos^2 t \\ \cos^2 t &= \sin^2 t \\ t &= n\frac{\pi}{4} \quad \text{(for odd } n\text{)} \end{aligned}$$

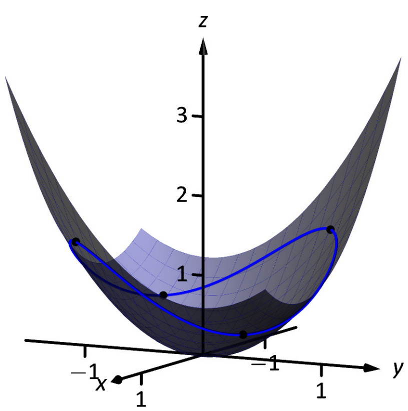

Figure 1.1.2: Plotting the path of a particle on a surface in Example 1.1.3.

We can use the First Derivative Test to find that on $[0, 2\pi]$, z has reaches its absolute minimum at $t = \pi/4$ and $5\pi/4$; it reaches its absolute maximum at $t = 3\pi/4$ and $7\pi/4$, as shown in Figure 1.1.2.

We can extend the Chain Rule to include the situation where z is a function of more than one variable, and each of these variables is also a function of more than one variable. The basic case of this is where $z = f(x, y)$, and x and y are functions of two variables, say s and t.

Notes:

Theorem 1.1.2 Multivariable Chain Rule, Part II

1. Let $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, where f, g and h are differentiable functions. Then z is a function of s and t, and

   - $\dfrac{\partial z}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$, and
   - $\dfrac{\partial z}{\partial t} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t}$.

2. Let $z = f(x_1, x_2, \ldots, x_m)$ be a differentiable function of m variables, where each of the x_i is a differentiable function of the variables $t_1, t_2, \ldots, t_n$. Then z is a function of the t_i, and

$$\frac{\partial z}{\partial t_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial f}{\partial x_m}\frac{\partial x_m}{\partial t_i}.$$

Example 1.1.4 Using the Multivarible Chain Rule, Part II

Let $z = x^2y + x$, $x = s^2 + 3t$ and $y = 2s - t$. Find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, and evaluate each when $s = 1$ and $t = 2$.

Solution Following Theorem 1.1.2, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = 2xy + 1 \qquad \frac{\partial f}{\partial y} = x^2,$$

$$\frac{\partial x}{\partial s} = 2s \qquad \frac{\partial x}{\partial t} = 3 \qquad \frac{\partial y}{\partial s} = 2 \qquad \frac{\partial y}{\partial t} = -1.$$

Thus

$$\frac{\partial z}{\partial s} = (2xy+1)(2s) + (x^2)(2) = 4xys + 2s + 2x^2, \quad \text{and}$$

$$\frac{\partial z}{\partial t} = (2xy+1)(3) + (x^2)(-1) = 6xy - x^2 + 3.$$

When $s = 1$ and $t = 2$, $x = 7$ and $y = 0$, so

$$\frac{\partial z}{\partial s} = 100 \qquad \text{and} \qquad \frac{\partial z}{\partial t} = -46.$$

Example 1.1.5 Using the Multivarible Chain Rule, Part II

Let $w = xy + z^2$, where $x = t^2e^s$, $y = t\cos s$, and $z = s\sin t$. Find $\frac{\partial w}{\partial t}$ when $s = 0$ and $t = \pi$.

Notes:

Solution Following Theorem 1.1.2, we compute the following partial derivatives:

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x \qquad \frac{\partial f}{\partial z} = 2z,$$

$$\frac{\partial x}{\partial t} = 2te^s \qquad \frac{\partial y}{\partial t} = \cos s \qquad \frac{\partial z}{\partial t} = s\cos t.$$

Thus

$$\frac{\partial w}{\partial t} = y(2te^s) + x(\cos s) + 2z(s\cos t).$$

When $s = 0$ and $t = \pi$, we have $x = \pi^2$, $y = \pi$ and $z = 0$. Thus

$$\frac{\partial w}{\partial t} = \pi(2\pi) + \pi^2 = 3\pi^2.$$

Implicit Differentiation

We studied finding $\frac{dy}{dx}$ when y is given as an implicit function of x in detail in Section 2.6. We find here that the Multivariable Chain Rule gives a simpler method of finding $\frac{dy}{dx}$.

For instance, consider the implicit function $x^2y - xy^3 = 3$. We learned to use the following steps to find $\frac{dy}{dx}$:

$$\begin{aligned}\frac{d}{dx}\left(x^2y - xy^3\right) &= \frac{d}{dx}\left(3\right)\\ 2xy + x^2\frac{dy}{dx} - y^3 - 3xy^2\frac{dy}{dx} &= 0\\ \frac{dy}{dx} &= -\frac{2xy - y^3}{x^2 - 3xy^2}. \end{aligned} \tag{1.1}$$

Instead of using this method, consider $z = x^2y - xy^3$. The implicit function above describes the level curve $z = 3$. Considering x and y as functions of x, the Multivariable Chain Rule states that

$$\frac{dz}{dx} = \frac{\partial z}{\partial x}\frac{dx}{dx} + \frac{\partial z}{\partial y}\frac{dy}{dx}. \tag{1.2}$$

Since z is constant (in our example, $z = 3$), $\frac{dz}{dx} = 0$. We also know $\frac{dx}{dx} = 1$. Equation (1.2) becomes

$$\begin{aligned} 0 &= \frac{\partial z}{\partial x}(1) + \frac{\partial z}{\partial y}\frac{dy}{dx} \quad \Rightarrow\\ \frac{dy}{dx} &= -\frac{\partial z}{\partial x}\bigg/\frac{\partial z}{\partial y}\\ &= -\frac{f_x}{f_y}. \end{aligned}$$

Notes:

Note how our solution for $\frac{dy}{dx}$ in Equation (1.1) is just the partial derivative of z with respect to x, divided by the partial derivative of z with respect to y.

We state the above as a theorem.

Theorem 1.1.3 Implicit Differentiation

Let f be a differentiable function of x and y, where $f(x, y) = c$ defines y as an implicit function of x, for some constant c. Then

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

We practice using Theorem 1.1.3 by applying it to a problem from Section 2.6.

Example 1.1.6 Implicit Differentiation

Given the implicitly defined function $\sin(x^2y^2) + y^3 = x + y$, find y'. Note: this is the same problem as given in Example 2.6.4 of Section 2.6, where the solution took about a full page to find.

Solution Let $f(x, y) = \sin(x^2y^2) + y^3 - x - y$; the implicitly defined function above is equivalent to $f(x, y) = 0$. We find $\frac{dy}{dx}$ by applying Theorem 1.1.3. We find

$$f_x(x, y) = 2xy^2\cos(x^2y^2) - 1 \qquad \text{and} \qquad f_y(x, y) = 2x^2y\cos(x^2y^2) + 3y^2 - 1,$$

so

$$\frac{dy}{dx} = -\frac{2xy^2\cos(x^2y^2) - 1}{2x^2y\cos(x^2y^2) + 3y^2 - 1},$$

which matches our solution from Example 2.6.4.

Notes:

Exercises 1.1

Terms and Concepts

12 08 ex 01 1. Let a level curve of $z = f(x, y)$ be described by $x = g(t)$, $y = h(t)$. Explain why $\frac{dz}{dt} = 0$.

12 08 ex 02 2. Fill in the blank: The single variable Chain Rule states $\frac{d}{dx}\Big(f\big(g(x)\big)\Big) = f'\big(g(x)\big)\cdot$________.

12 08 ex 03 3. Fill in the blank: The Multivariable Chain Rule states $\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot$ ________ $+$ ________ $\cdot\frac{dy}{dt}$.

12 08 ex 04 4. If $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$, we can substitute and write z as an explicit function of t.
T/F: Using the Multivariable Chain Rule to find $\frac{dz}{dt}$ is sometimes easier than first substituting and then taking the derivative.

12 08 ex 05 5. T/F: The Multivariable Chain Rule is only useful when all the related functions are known explicitly.

12 08 ex 06 6. The Multivariable Chain Rule allows us to compute implicit derivatives easily by just computing two ________ derivatives.

Problems

12 08 exset 01 **In Exercises 7 – 12, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given.**

(a) Use the Multivariable Chain Rule to compute $\frac{dz}{dt}$.

(b) Evaluate $\frac{dz}{dt}$ at the indicated t-value.

12 08 ex 07 7. $z = 3x + 4y, \quad x = t^2, \quad y = 2t; \quad t = 1$

12 08 ex 19 8. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1; \quad t = 1$

12 08 ex 20 9. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3; \quad t = \pi/4$

12 08 ex 09 10. $z = \frac{x}{y^2 + 1}, \quad x = \cos t, \quad y = \sin t; \quad t = \pi/2$

12 08 ex 10 11. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t; \quad t = \pi/4$

12 08 ex 08 12. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2; \quad t = 3$

12 08 exset 02 **In Exercises 13 – 18, functions $z = f(x, y)$, $x = g(t)$ and $y = h(t)$ are given. Find the values of t where $\frac{dz}{dt} = 0$. Note: these are the same surfaces/curves as found in Exercises 7 – 12.**

12 08 ex 11 13. $z = 3x + 4y, \quad x = t^2, \quad y = 2t$

12 08 ex 21 14. $z = x^2 - y^2, \quad x = t, \quad y = t^2 - 1$

12 08 ex 22 15. $z = 5x + 2y, \quad x = 2\cos t + 1, \quad y = \sin t - 3$

12 08 ex 13 16. $z = \frac{x}{y^2 + 1}, \quad x = \cos t, \quad y = \sin t$

12 08 ex 14 17. $z = x^2 + 2y^2, \quad x = \sin t, \quad y = 3\sin t$

12 08 ex 12 18. $z = \cos x \sin y, \quad x = \pi t, \quad y = 2\pi t + \pi/2$

12 08 exset 03 **In Exercises 19 – 22, functions $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$ are given.**

(a) Use the Multivariable Chain Rule to compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

(b) Evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at the indicated s and t values.

12 08 ex 15 19. $z = x^2 y, \quad x = s - t, \quad y = 2s + 4t; \quad s = 1, t = 0$

12 08 ex 16 20. $z = \cos\left(\pi x + \frac{\pi}{2}y\right), \quad x = st^2, \quad y = s^2 t; \quad s = 1, t = 1$

12 08 ex 17 21. $z = x^2 + y^2, \quad x = s\cos t, \quad y = s\sin t; \quad s = 2, t = \pi/4$

12 08 ex 18 22. $z = e^{-(x^2+y^2)}, \quad x = t, \quad y = st^2; \quad s = 1, t = 1$

12 08 exset 04 **In Exercises 23 – 26, find $\frac{dy}{dx}$ using Implicit Differentiation and Theorem 1.1.3.**

12 08 ex 23 23. $x^2 \tan y = 50$

12 08 ex 24 24. $(3x^2 + 2y^3)^4 = 2$

12 08 ex 25 25. $\frac{x^2 + y}{x + y^2} = 17$

12 08 ex 26 26. $\ln(x^2 + xy + y^2) = 1$

12 08 exset 05 **In Exercises 27 – 30, find $\frac{dz}{dt}$, or $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$, using the supplied information.**

12 08 ex 27 27. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1, \quad \frac{dx}{dt} = 4, \quad \frac{dy}{dt} = -5$

12 08 ex 28 28. $\frac{\partial z}{\partial x} = 1, \quad \frac{\partial z}{\partial y} = -3, \quad \frac{dx}{dt} = 6, \quad \frac{dy}{dt} = 2$

12 08 ex 29 29. $\frac{\partial z}{\partial x} = -4, \quad \frac{\partial z}{\partial y} = 9,$
$\frac{\partial x}{\partial s} = 5, \quad \frac{\partial x}{\partial t} = 7, \quad \frac{\partial y}{\partial s} = -2, \quad \frac{\partial y}{\partial t} = 6$

12 08 ex 30 30. $\frac{\partial z}{\partial x} = 2, \quad \frac{\partial z}{\partial y} = 1,$
$\frac{\partial x}{\partial s} = -2, \quad \frac{\partial x}{\partial t} = 3, \quad \frac{\partial y}{\partial s} = 2, \quad \frac{\partial y}{\partial t} = -1$

A: Solutions To Selected Problems

Chapter 1

Section 1.1

12 08 ex 01 1. Because the parametric equations describe a level curve, z is constant for all t. Therefore $\frac{dz}{dt} = 0$.

12 08 ex 02 2. $g'(x)$

12 08 ex 03 3. $\frac{dx}{dt}$, and $\frac{\partial f}{\partial y}$

12 08 ex 04 4. T

12 08 ex 05 5. F

12 08 ex 06 6. partial

12 08 ex 07 7.
   (a) $\frac{dz}{dt} = 3(2t) + 4(2) = 6t + 8$.
   (b) At $t = 1$, $\frac{dz}{dt} = 14$.

12 08 ex 19 8.
   (a) $\frac{dz}{dt} = 2x(1) - 2y(2t) = 2x - 4yt$
   (b) At $t = 1$, $x = 1$, $y = 0$ and $\frac{dz}{dt} = 2$.

12 08 ex 20 9.
   (a) $\frac{dz}{dt} = 5(-2\sin t) + 2(\cos t) = -10\sin t + 2\cos t$
   (b) At $t = \pi/4$, $\frac{dz}{dt} = -4\sqrt{2}$.

12 08 ex 09 10.
   (a) $\displaystyle\frac{dz}{dt} = \frac{1}{1+y^2}(-\sin t) - \frac{2xy}{(y^2+1)^2}(\cos t)$.
   (b) At $t = \pi/2$, $x = 0$, $y = 1$, and $\frac{dz}{dt} = -1/2$.

12 08 ex 10 11.
   (a) $\displaystyle\frac{dz}{dt} = 2x(\cos t) + 4y(3\cos t)$.
   (b) At $t = \pi/4$, $x = \sqrt{2}/2$, $y = 3\sqrt{2}/2$, and $\frac{dz}{dt} = 19$.

12 08 ex 08 12.
   (a) $\frac{dz}{dt} = -\sin x \sin y(\pi) + \cos x \cos y(2\pi)$.
   (b) At $t = 3$, $x = 3\pi$, $y = 13\pi/2$, and $\frac{dz}{dt} = 0$.

12 08 ex 11 13. $t = -4/3$; this corresponds to a minimum

12 08 ex 21 14. $t = 0, \pm\sqrt{3/2}$

12 08 ex 22 15. $t = \tan^{-1}(1/5) + n\pi$, where n is an integer

12 08 ex 13 16. We find that

$$\frac{dz}{dt} = -\frac{2\cos^2 t \sin t}{(1+\sin^2 t)^2} - \frac{\sin t}{1+\sin^2 t}.$$

Setting this equal to 0, finding a common denominator and factoring out $\sin t$, we get

$$\sin t\left(\frac{2\cos^2 t + \sin^2 t + 1}{(1+\sin^2 t)^2}\right) = 0.$$

We have $\sin t = 0$ when $t = \pi n$, where n is an integer. The expression in the parenthesis above is always positive, and hence never equal 0. So all solutions are $t = \pi n$, n is an integer.

12 08 ex 14 17. We find that

$$\frac{dz}{dt} = 38\cos t \sin t.$$

Thus $\frac{dz}{dt} = 0$ when $t = \pi n$ or $\pi n + \pi/2$, where n is any integer.

12 08 ex 12 18. We find that

$$\frac{dz}{dt} = -\pi\sin(\pi t)\sin(2\pi t + \pi/2) + 2\pi\cos(\pi t)\cos(2\pi t + \pi/2).$$

One can "easily" see that when t is an integer, $\sin(\pi t) = 0$ and $\cos(2\pi t + \pi/2) = 0$, hence $\frac{dz}{dt} = 0$ when t is an integer. There are other places where z has a relative max/min that require more work. First, verify that $\sin(2\pi t + \pi/2) = \cos(2\pi t)$, and $\cos(2\pi t + \pi/2) = -\sin(2\pi t)$. This lets us rewrite $\frac{dz}{dt} = 0$ as

$$-\sin(\pi t)\cos(2\pi t) - 2\cos(\pi t)\sin(2\pi t) = 0.$$

By bringing one term to the other side of the equality then dividing, we can get

$$2\tan(2\pi t) = -\tan(\pi t).$$

Using the angle sum/difference formulas found in the back of the book, we know

$$\tan(2\pi t) = \tan(\pi t) + \tan(\pi t) = \frac{\tan(\pi t) + \tan(\pi t)}{1 - \tan^2(\pi t)}.$$

Thus we write

$$2\frac{\tan(\pi t) + \tan(\pi t)}{1 - \tan^2(\pi t)} = -\tan(\pi t).$$

Solving for $\tan^2(\pi t)$, we find

$$\tan^2(\pi t) = 5 \quad \Rightarrow \quad \tan(\pi t) = \pm\sqrt{5},$$

and so

$$\pi t = \tan^{-1}(\pm\sqrt{5}) = \pm\tan^{-1}(\sqrt{5}).$$

Since the period of tangent is π, we can adjust our answer to be

$$\pi t = \pm\tan^{-1}(\sqrt{5}) + n\pi, \text{ where } n \text{ is an integer.}$$

Dividing by π, we find

$$t = \pm\frac{1}{\pi}\tan^{-1}(\sqrt{5}) + n, \text{ where } n \text{ is an integer.}$$

12 08 ex 15 19.
   (a) $\frac{\partial z}{\partial s} = 2xy(1) + x^2(2) = 2xy + 2x^2$;
   $\frac{\partial z}{\partial t} = 2xy(-1) + x^2(4) = -2xy + 4x^2$
   (b) With $s = 1$, $t = 0$, $x = 1$ and $y = 2$. Thus $\frac{\partial z}{\partial s} = 6$ and $\frac{\partial z}{\partial t} = 0$

12 08 ex 16 20.

(a) $\frac{\partial z}{\partial s} =$
$-\pi \sin(\pi x+\pi y/2)(t^2) - \frac{1}{2}\pi\sin(\pi x+\pi y/2)(2st) =$
$-\pi\Big(t^2\sin(\pi x+\pi y/2) + st\sin(\pi x+\pi y/2)\Big);$
$\frac{\partial z}{\partial t} =$
$-\pi \sin(\pi x+\pi y/2)(2st) - \frac{1}{2}\pi\sin(\pi x+\pi y/2)(s^2) =$
$-\pi\Big(2st\sin(\pi x+\pi y/2) + \frac{1}{2}s^2\sin(\pi x+\pi y/2)\Big)$

(b) With $s=1$, $t=1$, $x=1$ and $y=1$. Thus $\frac{\partial z}{\partial s} = 2\pi$ and $\frac{\partial z}{\partial t} = 5\pi/2$

12 08 ex 17 21.

(a) $\frac{\partial z}{\partial s} = 2x(\cos t) + 2y(\sin t) = 2x\cos t + 2y\sin t$;
$\frac{\partial z}{\partial t} = 2x(-s\sin t) + 2y(s\cos t) =$
$-2xs\sin t + 2ys\cos t$

(b) With $s=2$, $t=\pi/4$, $x=\sqrt{2}$ and $y=\sqrt{2}$. Thus $\frac{\partial z}{\partial s} = 4$ and $\frac{\partial z}{\partial t} = 0$

12 08 ex 18 22.

(a) $\frac{\partial z}{\partial s} = -2xe^{-(x^2+y^2)}(0) - 2ye^{-(x^2+y^2)}(t^2) =$
$-2yt^2e^{-(x^2+y^2)}$;
$\frac{\partial z}{\partial t} = -2xe^{-(x^2+y^2)}(1) - 2ye^{-(x^2+y^2)}(2st) =$
$-2xe^{-(x^2+y^2)} - 4stye^{-(x^2+y^2)}$

(b) With $s=1$, $t=1$, $x=1$ and $y=1$. Thus $\frac{\partial z}{\partial s} = -2/e^2$ and $\frac{\partial z}{\partial t} = -6/e^2$

12 08 ex 23 23. $f_x = 2x\tan y$, $f_y = x^2\sec^2 y$;
$$\frac{dy}{dx} = -\frac{2\tan y}{x\sec^2 y}$$

12 08 ex 24 24. $f_x = 4(3x^2+2y^3)^3(6x)$, $f_y = 4(3x^2+2y^3)^3(6y^2)$;
$$\frac{dy}{dx} = -\frac{x}{y^2}$$

12 08 ex 25 25. $f_x = \dfrac{(x+y^2)(2x) - (x^2+y)(1)}{(x+y^2)^2}$,
$$f_y = \frac{(x+y^2)(1) - (x^2+y)(2y)}{(x+y^2)^2};$$
$$\frac{dy}{dx} = -\frac{2x(x+y^2) - (x^2+y)}{x+y^2-2y(x^2+y)}$$

12 08 ex 26 26. $f_x = \dfrac{2x+y}{x^2+xy+y^2}$, $f_y = \dfrac{x+2y}{x^2+xy+y^2}$;
$$\frac{dy}{dx} = -\frac{2x+y}{2y+x}$$

12 08 ex 27 27. $\frac{dz}{dt} = 2(4) + 1(-5) = 3$.

12 08 ex 28 28. $\frac{dz}{dt} = 1(6) + (-3)(2) = 0$.

12 08 ex 29 29. $\frac{\partial z}{\partial s} = -4(5) + 9(-2) = -38$,
$\frac{\partial z}{\partial t} = -4(7) + 9(6) = 26$.

12 08 ex 30 30. $\frac{\partial z}{\partial s} = 2(-2) + 1(2) = -2$,
$\frac{\partial z}{\partial t} = 2(3) + 1(-1) = 5$.