STA 3180 Statistical Modelling: Bayesian Inference

1. Suppose we have a sample of size n from a population with mean μ and variance σ2. Use Bayes' theorem to calculate the posterior distribution of μ given that the sample mean is x■ and the prior distribution of μ is N(μ0, σ20).

Solution: The posterior distribution of μ is given by:

P(μ|x■) = P(x■|μ)P(μ)/P(x■)

where P(x■|μ) is the likelihood of the sample mean x■ given μ, P(μ) is the prior distribution of μ, and P(x■) is the marginal probability of the sample mean x■.

P(x■|μ) = N(x■; μ, σ2/n)

P(μ) = N(μ; μ0, σ20)

P(x■) = ∫N(x■; μ, σ2/n)N(μ; μ0, σ20)dμ

Therefore, the posterior distribution of μ is given by:

P(μ|x■) = N(x■; μ, σ2/n)N(μ; μ0, σ20)/P(x■)

2. Suppose we have a sample of size n from a population with mean μ and variance σ2. Use Bayes' theorem to calculate the posterior distribution of σ2 given that the sample variance is s2 and the prior distribution of σ2 is IG(α, β).

Solution: The posterior distribution of σ2 is given by:

P(σ2|s2) = P(s2|σ2)P(σ2)/P(s2)

where P(s2|σ2) is the likelihood of the sample variance s2 given σ2, P(σ2) is the prior distribution of σ2, and P(s2) is the marginal probability of the sample variance s2.

P(s2|σ2) = IG(s2; α + n/2, β + n(x■ - μ)2/2)

P(σ2) = IG(σ2; α, β)

P(s2) = ∫IG(s2; α + n/2, β + n(x■ - μ)2/2)IG(σ2; α, β)dσ2

Therefore, the posterior distribution of σ2 is given by:

P(σ2|s2) = IG(s2; α + n/2, β + n(x■ - μ)2/2)IG(σ2; α, β)/P(s2)

3. Suppose we have a sample of size n from a population with mean μ and variance $\sigma2$. Use Bayes' theorem to calculate the posterior distribution of μ and $\sigma2$ given that the sample mean is $x\blacksquare$ and the sample variance is s2 and the prior distributions of μ and $\sigma2$ are $N(\mu0, \sigma20)$ and $IG(\alpha, \beta)$, respectively.

Solution: The posterior distribution of μ and $\sigma2$ is given by:

$$P(\mu, \sigma2|x\blacksquare, s2) = P(x\blacksquare, s2|\mu, \sigma2)P(\mu, \sigma2)/P(x\blacksquare, s2)$$

where $P(x\blacksquare, s2|\mu, \sigma2)$ is the joint likelihood of the sample mean $x\blacksquare$ and sample variance s2 given μ and $\sigma2$, $P(\mu, \sigma2)$ is the joint prior distribution of μ and $\sigma2$, and $P(x\blacksquare, s2)$ is the marginal probability of the sample mean $x\blacksquare$ and sample variance s2.

$$P(x\blacksquare, s2|\mu, \sigma2) = N(x\blacksquare; \mu, \sigma2/n)IG(s2; \alpha + n/2, \beta + n(x\blacksquare - \mu)2/2)$$

$$P(\mu, \sigma2) = N(\mu; \mu0, \sigma20)IG(\sigma2; \alpha, \beta)$$

$$P(x\blacksquare, s2) = \iint N(x\blacksquare; \mu, \sigma2/n)IG(s2; \alpha + n/2, \beta + n(x\blacksquare - \mu)2/2)N(\mu; \mu0, \sigma20)IG(\sigma2; \alpha, \beta)d\mu d\sigma2$$

Therefore, the posterior distribution of μ and $\sigma2$ is given by:

$$P(\mu, \sigma2|x\blacksquare, s2) = N(x\blacksquare; \mu, \sigma2/n)IG(s2; \alpha + n/2, \beta + n(x\blacksquare - \mu)2/2)N(\mu; \mu0, \sigma20)IG(\sigma2; \alpha, \beta)/P(x\blacksquare, s2)$$