Discrete Conditional Probability

For: Probability Theory 1

Conditional Probability

This section answers questions about dependent events. We want to find the Probability that given E what about F?

> This is denoted by
>
> $$P(F|E)$$
>
> It reads, probability of F given E
> $|$ is known as *given*

Formula & Context

$$P(F|E) = \frac{P(F \cap E)}{P(E)}$$

The top part $F \cap E$ will represent the Event Space where Both F and E have occurred.

See that this boils down to

> What is the probability that both E and F occur divided by the probability of E. As E has been presumed to occur already.

Examples

Example A

Roll a die once. Let X be its outcome. Let the event

$$F = \{X = 6\}$$

and let

$$E = \{X > 4\}$$

The distribution function $m(\omega)$ outputs $\frac{1}{6}$ for all $\omega = 1, 2, \ldots, 6$
Then $P(F) = \frac{1}{6}$

Suppose we roll the dice and learn E has occurred. This means the dice rolled has been either a 5 or 6.

See that because we know E has occurred.

$$\begin{aligned}P(F|E) &= \frac{P(F \cap E)}{P(E)} \\ &= \frac{P(X=6)}{P(X \in \{5,6\})} \\ &= \frac{\frac{1}{6}}{\frac{2}{6}} \\ &= \frac{1}{2}\end{aligned}$$

Example B

A survey found that 89.935% of women will live to 60 while 57.062% will love to 80. Given that a woman is **already** 60, what are the chances she will live to 80? The sample space was $100\,000$ women.

Let E be the subset of women alive at 60. $|E| = 89,935$
F is the subset of women alive at 80, $|F| = 57,062$

See that this woman has already lived to 60. This means that she has passed the 'First Filter'.

$$P(F|E) = \frac{|F|}{|E|} = .6352$$

We divide by E because we know she didn't die.

Example C

Recall the election example from Chapter 1:
It reads

> three candidates A, B, and C are running for office. We decided that A and B have an equal chance of winning and C is only 1/2 as likely to win as A.

We can assign probabilities as

$$
\begin{aligned}
P(A) = P(B) &= \frac{2}{5} \\
P(C) = &= \frac{1}{5}
\end{aligned}
$$

Suppose A drops the race.
If we follow the example rules we define

$$
\begin{aligned}
P(B|A) &= \frac{2}{3} \\
P(C|A) &= \frac{1}{3}
\end{aligned}
$$

This of course does **not** apply to real life. We cannot just assume all of A's supporters will follow this $2:1$ ratio.

Example D

Let there be 2 bags. Bag$_1$ has 2 black balls and 3 white, while Bag$_2$ has 1 back and 1 white. A bag and ball is chosen at random.

Let B be the event that a *Black* ball has been chosen and I be the event Bag$_1$ is chosen. Find $P(I|B)$

We need to find the chance of picking Bag$_1$ if we know a back ball came out.

$$P(I|B) = \frac{P(I \cap B)}{P(B)}$$

$P(B)$ is the probability that a black ball is chosen. This can happen from any Bag.

$$P(B) = P(B \cap I) + P(B \cap II)$$

We now need to find these probabilities.
$P(B \cap I) = \frac{1}{2}\frac{2}{5}$ since $\frac{1}{2}$ chance of choosing bag and then $\frac{2}{5}$ of choosing the black ball.

Likewise $P(B \cap II) = \frac{1}{2}\frac{1}{2}$

$$\implies P(B) = \frac{1}{5} + \frac{1}{4}$$

$$\text{Answer: } \frac{1/5}{1/5 + 1/4} = \boxed{\frac{4}{9}}$$

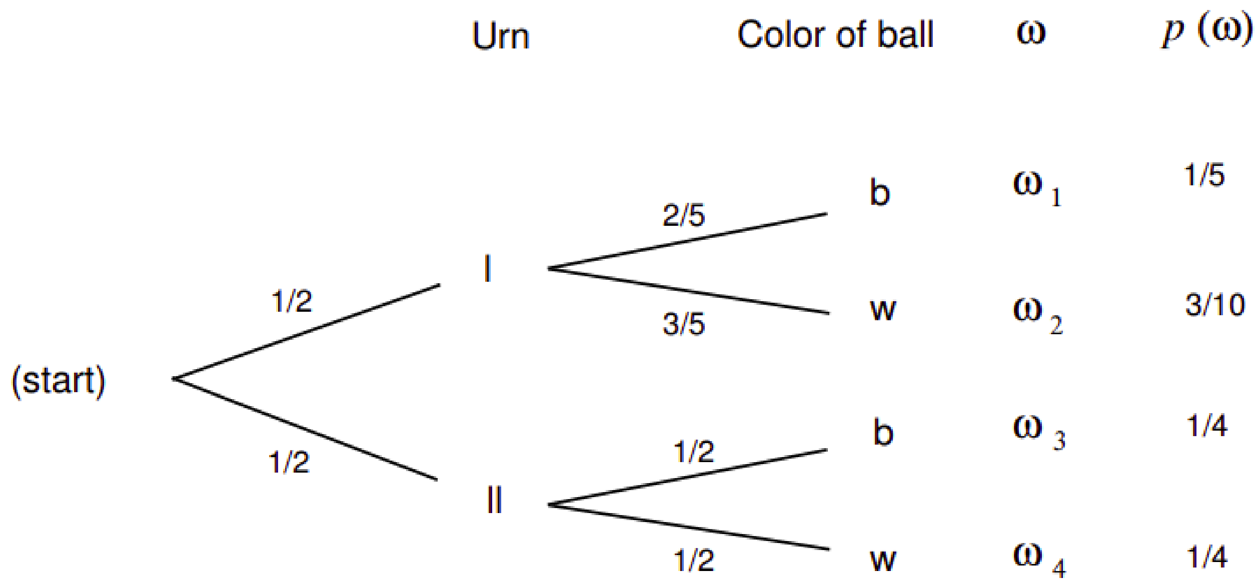

Figure 4.1: Tree diagram.