Contents

1 Tutorial 10: Poisson Matrix Factorization

In this tutorial we will implement the EM and VI algorithms for the Poisson Matrix Factorization model, seen in Lecture 10. We will analyse the *football* dataset, that you can find in the folder **football/football_net.gml**, and in the jupyter notebook it has been already pre-processed. You will also find a **functions.py** script, where you can find some useful functions.

1.1 Exercise 1: implementing PMF with EM

We first implement the EM algorithm.

(a) Complete the functions of the class **PMF_EM**.

(b) Plot the log-likelihood values.

(c) Plot the obtained partition, with both overlapping and hard communities. Compare with the ground truth.

Solution in the jupyter file *L10_tutorial_solution.ipynb*.

1.2 Exercise 2: implementing PMF with VI

We now implement the VI algorithm.

(a) Complete the functions of the class **PMF_VI**. This implies to compute analytically the ELBO before implementing it.

(b) Plot the elbo values.

(c) Plot the obtained partition, with both overlapping and hard communities. Compare with the ground truth.

The implementation is in the jupyter file *L10_tutorial_solution.ipynb*.

(a) The Evidence Lower Bound (ELBO) is defined as:

$$\begin{aligned}\mathrm{ELBO}(q) &= \mathbb{E}_q\left[\log P(A,u,v)\right] - \mathbb{E}_q\left[\log q(u,v)\right] && (1)\\ &= \mathbb{E}_q\left[\log P(A|u,v)\right] + \mathbb{E}_q\left[\log P(u,v)\right] - \mathbb{E}_q\left[\log q(u,v)\right] && (2)\end{aligned}$$

where we assume $P(u,v) = P(u)P(v)$, $q(u,v) = q(u)q(v)$, and Gamma distributions. Thus,

$$\begin{aligned}\log P(A|u,v) &= \sum_{ij}\left[A_{ij}\log\sum_k u_{ik}v_{jk} - \sum_k u_{ik}v_{jk}\right] && (3)\\ \log P(u) &= \sum_{ik}\left[a_{ik}\log b_{ik} - \log\Gamma(a_{ik}) + (a_{ik}-1)\log u_{ik} - b_{ik}u_{ik}\right] && (4)\\ \log P(v) &= \sum_{jk}\left[c_{jk}\log d_{jk} - \log\Gamma(c_{jk}) + (c_{jk}-1)\log v_{jk} - d_{jk}v_{jk}\right] && (5)\\ \log q(u) &= \sum_{ik}\left[\alpha_{ik}^{shp}\log\alpha_{ik}^{rte} - \log\Gamma(\alpha_{ik}^{shp}) + (\alpha_{ik}^{shp}-1)\log u_{ik} - \alpha_{ik}^{rte}u_{ik}\right] && (6)\\ \log q(v) &= \sum_{jk}\left[\beta_{jk}^{shp}\log\beta_{jk}^{rte} - \log\Gamma(\beta_{jk}^{shp}) + (\beta_{jk}^{shp}-1)\log v_{jk} - \beta_{jk}^{rte}v_{jk}\right]\ . && (7)\end{aligned}$$

The ELBO is then,

$$
\begin{aligned}
\mathrm{ELBO}(q) \quad \propto \quad & \sum_{ij}\left[A_{ij}\mathbb{E}_q\left(\log\sum_k u_{ik}v_{jk}\right) - \sum_k \frac{\alpha_{ik}^{shp}}{\alpha_{ik}^{rte}}\frac{\beta_{ik}^{shp}}{\beta_{ik}^{rte}}\right] + \\
& \sum_{ik}\left[a_{ik}\left[\psi(\alpha_{ik}^{shp}) - \log\alpha_{ik}^{rte}\right] - b_{ik}\frac{\alpha_{ik}^{shp}}{\alpha_{ik}^{rte}}\right] + \sum_{jk}\left[c_{ik}\left[\psi(\beta_{jk}^{shp}) - \log\beta_{jk}^{rte}\right] - d_{jk}\frac{\beta_{jk}^{shp}}{\beta_{jk}^{rte}}\right] - \\
& \sum_{ik}\left[\alpha_{ik}^{shp}\log\alpha_{ik}^{rte} - \log\Gamma(\alpha_{ik}^{shp}) + \alpha_{ik}^{shp}\left[\psi(\alpha_{ik}^{shp}) - \log\alpha_{ik}^{rte}\right] - \alpha_{ik}^{shp}\right] - \\
& \sum_{jk}\left[\beta_{jk}^{shp}\log\beta_{jk}^{rte} - \log\Gamma(\beta_{jk}^{shp}) + \beta_{jk}^{shp}\left[\psi(\beta_{jk}^{shp}) - \log\beta_{jk}^{rte}\right] - \beta_{jk}^{shp}\right] \qquad (8)
\end{aligned}
$$

where the proportionality is because we are omitting constant terms. Moreover, we used the following result:

$$
X \sim \mathrm{Gam}(\alpha,\beta) \implies \mathbb{E}[X] = \frac{\alpha}{\beta} \quad , \quad \mathbb{E}[\log X] = \psi(\alpha) - \log\beta\ . \qquad (9)
$$

Additionally, we use the approximation:

$$
\mathbb{E}_q\left(\log\sum_k u_{ik}v_{jk}\right) \approx \sum_k \mathbb{E}_q\left[\log u_{ik}v_{jk}\right]\ . \qquad (10)
$$

Thus,

$$
\begin{aligned}
\mathrm{ELBO}(q) \quad \propto \quad & \sum_{ijk}\left[A_{ij}\left[\psi(\alpha_{ik}^{shp}) - \log\alpha_{ik}^{rte} + \psi(\beta_{jk}^{shp}) - \log\beta_{jk}^{rte}\right] - \frac{\alpha_{ik}^{shp}}{\alpha_{ik}^{rte}}\frac{\beta_{ik}^{shp}}{\beta_{ik}^{rte}}\right] + \\
& \sum_{ik}\left[(a_{ik} - \alpha_{ik}^{shp})\psi(\alpha_{ik}^{shp}) - a_{ik}\log\alpha_{ik}^{rte} - \alpha_{ik}^{shp}\left(1 - \frac{b_{ik}}{\alpha_{ik}^{rte}}\right) + \log\Gamma(\alpha_{ik}^{shp})\right] + \\
& \sum_{jk}\left[(c_{jk} - \beta_{jk}^{shp})\psi(\beta_{jk}^{shp}) - c_{jk}\log\beta_{jk}^{rte} - \beta_{jk}^{shp}\left(1 - \frac{d_{jk}}{\beta_{jk}^{rte}}\right) + \log\Gamma(\beta_{jk}^{shp})\right] . \qquad (11)
\end{aligned}
$$