

# Tutorial 4: Mean Field approach

TBD and Caterina De Bacco

## Exercise 1: Curie-Weiss model again

The Hamiltonian of the Curie-Weiss model can be written as:

$$H(\mathbf{s}) = -\frac{J}{2N} \sum_{i,j} s_i s_j - h \sum_i s_i.$$

From the lecture, we know that the Mean Field free variational energy is:

$$F[Q] = -\frac{J}{2} m^2 - h m + \frac{1}{\beta} \left[ \frac{1+m}{2} \log \left( \frac{1+m}{2} \right) + \frac{1-m}{2} \log \left( \frac{1-m}{2} \right) \right] \quad (1)$$

where  $m$  is the expected total magnetisation  $\mathbb{E}_Q[\sum_i s_i / N]$ .

- (a) Plot the function  $F[Q]$  as a function of  $m$  in the two following cases: (i) at  $h = 0$  for different values of  $J$  larger and lower than 1 and (ii) at a value of  $J$  larger than 1 for different values (positive and negative) of  $h$ . Describe what you see in both cases.
- (b) The minimizer  $m^*$  is also the solution of the “self-consistent equation”:

$$m = \tanh \left[ \frac{J}{2} m + h \right].$$

Compute the value of  $m^*$  in the three following cases:

- (i)  $h = 10^{-6}$  and  $J$  between 0 and 2;
  - (ii)  $h = -10^{-6}$  and  $J$  between 0 and 2; and
  - (iii)  $J = 1.5$  and  $h$  between  $-1$  and  $1$ .
- (c) Focusing on the case  $J = 1.5$ . With  $h = 0.1, 0.2$ , how many solutions to the self-consistency equations are there? Which one is the correct one? Plot the function  $F[Q]$  to answer to these questions.

## Exercise 2: sampling from the Curie-Weiss model

Consider again the Hamiltonian of the Curie-Weiss model.

A practical way to sample configurations of  $N$  spins from the Gibbs probability distribution:

$$P(\mathbf{s}) = \frac{e^{-\beta H(\mathbf{s})}}{Z} \quad ,$$

with

$$H(\mathbf{s}) = -\frac{J}{2N} \sum_{i,j} s_i s_j - h \sum_i s_i \quad ,$$

is the Monte-Carlo-Markov-Chain (MCMC) method, and in particular the Metropolis-Hastings algorithm.

It works as follows:

1. Choose a starting configuration for the  $N$  spins values  $s_i = \pm 1$  for  $i = 1, \dots, N$ .
2. Choose a spin  $i$  at random. Compute the current value of the energy  $H_{\text{now}}$  and the value of the energy  $H_{\text{flip}}$  if the spins  $i$  is flipped (that is if  $s_i^{\text{new}} = -s_i^{\text{old}}$ ).
3. Sample a number  $r$  uniformly in  $[0, 1]$  and, if  $r < e^{\beta(H_{\text{now}} - H_{\text{flip}})}$  perform the flip (i.e.  $s_i^{\text{new}} = -s_i^{\text{old}}$ ) otherwise leave it as it is.
4. Goto step 2.

If one is performing this program long enough, it is guarantied that the final configuration ( $\mathbf{s}$ ) will have been chosen with the correct probability.

- (a) Write a code to perform the MCMC dynamics, and start by a configuration where all spins are equal to 1. Take  $h = 0$ ,  $J = 1$ ,  $\beta = 1.2$  and try your dynamics for a long enough time (say, with  $t_{\text{max}} = 100N$  attempts to flips spins) and monitor the value of the magnetization per spin  $m = \sum_i s_i / N$  as a function of time. Make a plot for  $N = 10, 50, 100, 200, 1000$  spins. Compare with the exact solution at  $N = \infty$ . Comment.
- (b) Start by a configuration where all spins are equal to 1 and take  $h = -0.1$ ,  $J = 1$ ,  $\beta = 1.2$ . Monitor again the value of the magnetization per spin  $m = \sum_i s_i / N$  as a function of time. Make a plot for  $N = 10, 50, 100, 200, 1000$  spins. Compare with the exact solution at  $N = \infty$ . Comment.