Advanced Probabilistic Machine Learning and Applications

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Tutorial 4: Mean field approach

Exercise 1: Curie-Weiss model

The Hamiltonian of the Curie-Weiss model can be written as:

$$H(\mathbf{s}) = -\frac{J}{2N}\sum_{i,j} s_i s_j - h\sum_i s_i. \tag{1}$$

From the lecture, we know that the Mean Field free variational energy is:

$$F[Q] = -\frac{J}{2}m^2 - hm + \frac{1}{\beta}\left[\frac{1+m}{2}\log\left(\frac{1+m}{2}\right) + \frac{1-m}{2}\log\left(\frac{1-m}{2}\right)\right]$$

where $m := \mathbb{E}_Q\left[\sum_i s_i/N\right]$ is the expected total magnetisation.

Perform the following tasks.

1. Plot the function $F[Q]$ as a function of m in the two following cases:
   (i) at $h = 0$, for different values of J larger and lower than 1;
   (ii) at a value of J larger than 1 for different values (positive and negative) of h.

   Describe what you see in both cases.
2. The minimizer m^* is also the solution of the self-consitency equation:

   $$m = \tanh\left[\frac{J}{2}m + h\right].$$

   Compute the value of m^* in the three following cases:
   (i) $h = 10^{-6}$ and J between 0 and 2;
   (ii) $h = -10^{-6}$ and J between 0 and 2;
   (iii) $J = 1.5$ and h between -1 and 1.
3. Focusing on the case $J = 1.5$. With $h = 0.1, 0.2$, how many solutions to the self-consistency equations are there? Which one is the correct one? (Plot the function $F[Q]$ to answer to these questions).

Exercise 2: Sampling from the Curie-Weiss model (MCMC method)

Consider again the Curie-Weiss Hamiltonian in Equation (1). A practical way to sample configurations of N spins from the Gibbs probability distribution:

$$P(\mathbf{s}) = \frac{e^{-\beta H(\mathbf{s})}}{Z},$$

is the Monte-Carlo-Markov-Chain (MCMC) method, and in particular the Metropolis-Hastings algorithm. This works as follows:

1. Choose a starting configuration for the N spins values $s_i = \pm 1$, for $i = 1, \ldots, N$
2. Choose a spin i at random. Compute the current value of the energy $H_{\rm now}$ and the value of the energy $H_{\rm flip}$ if the spins i is flipped (that is if $s_i^{\rm new} = -s_i^{\rm old}$)
3. Sample a number $r \sim U(0,1)$
4. **if** $r < \exp[\beta(H_{\rm now} - H_{\rm flip})]$ **then:** perform the flip (i.e. $s_i^{\rm new} = -s_i^{\rm old}$)
5. **if** $r \geq \exp[\beta(H_{\rm now} - H_{\rm flip})]$ **then:** leave s_i as it is
6. Go back to step 2

If one is running this program long enough, it is guaranteed that the final configuration of spins will have been chosen with the correct probability.

Perform the following tasks.

1. Write a code to implement the MCMC dynamics.
2. Run the code starting from a configuration where all spins are equal to 1. Take $h = 0$, $J = 1$, $\beta = 1.2$ and run your dynamics for a time that is long enough (say, with $t_{\rm max} = 100N$ attempts to flips spins). Monitor the value of the magnetization per spin $m = \sum_i s_i/N$ as a function of time. Make a plot for $N = 10, 50, 100, 200, 1000$ spins.
   Compare with the exact solution at $N = \infty$.
   Comment.
3. Start by a configuration where all spins are equal to 1 and take $h = -0.1$, $J = 1$, $\beta = 1.2$. Monitor again the value of the magnetization per spin $m = \sum_i s_i/N$ as a function of time. Make a plot for $N = 10, 50, 100, 200, 1000$ spins.
   Compare with the exact solution at $N = \infty$.
   Comment.