

# Advanced Probabilistic Machine Learning and Applications

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## Tutorial 6: Bethe approximation and BP

### Exercise 1: graph coloring problem (part I)

Coloring is a classical problem of graph theory. Given a (unweighted, undirected) graph  $G(V, E)$ , a coloring  $M \subseteq V$  is an assignment of labels, called colors, to the vertices of a graph such that no two adjacent vertices share the same color.

- (i) Write a probability distribution for the coloring problem.
- (ii) Consider a “soft” constraint instead which relaxes the “hard” one and write the corresponding interaction function of the factor node.

*Hint:* the soft constraint allows two neighboring nodes to have the same color, but penalized this a lot.

- (iii) Draw a factor graph corresponding to it.

## Exercise 2: representing models using factor graphs

Write **both** the following problems (i) in terms of a probability distribution and (ii) in terms of a graphical model by drawing an example of the corresponding factor graph.

### (a) p-spin model

One model that is commonly studied in physics is the so-called Ising 3-spin model. The Hamiltonian of this model is written as

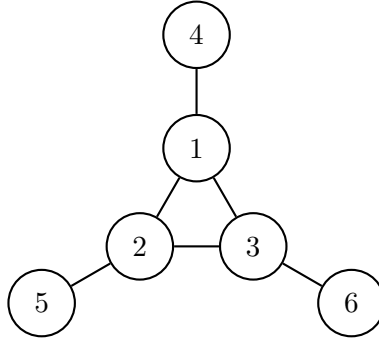
$$H(\mathbf{s}) = - \sum_{(ijk) \in E} J_{ijk} s_i s_j s_k - \sum_{i=1}^N h_i s_i \quad (1)$$

where  $E$  is a given set of (unordered) triplets  $i \neq j \neq k$ ,  $J_{ijk}$  is the interaction strength for the triplet  $(ijk) \in E$ , and  $h_i$  is a magnetic field on spin  $i$ . The spins are Ising, which in physics means  $s_i \in \{+1, -1\}$ .

### (b) Independent set problem

Independent set is a problem defined and studied in combinatorics and graph theory. Given a (unweighted, undirected) graph  $G(V, E)$ , an independent set  $S \subseteq V$  is defined as a subset of nodes such that if  $i \in S$  then for all  $j \in \partial i$  we have  $j \notin S$ . In other words in for all  $(ij) \in E$  only  $i$  or  $j$  can belong to the independent set.

For example, suppose we have the following graph:



Moreover, for problem (b):

(iii) Write a probability distribution that is uniform over all independent sets on a given graph.

(iv) Write a probability distribution that gives a larger weight to larger independent sets, where the size of an independent set is simply  $|S|$ .