Dynamic Simulation of a Hot Air Balloon

Thomas A. Badgwell
Corporate Strategic Research
ExxonMobil Research & Engineering

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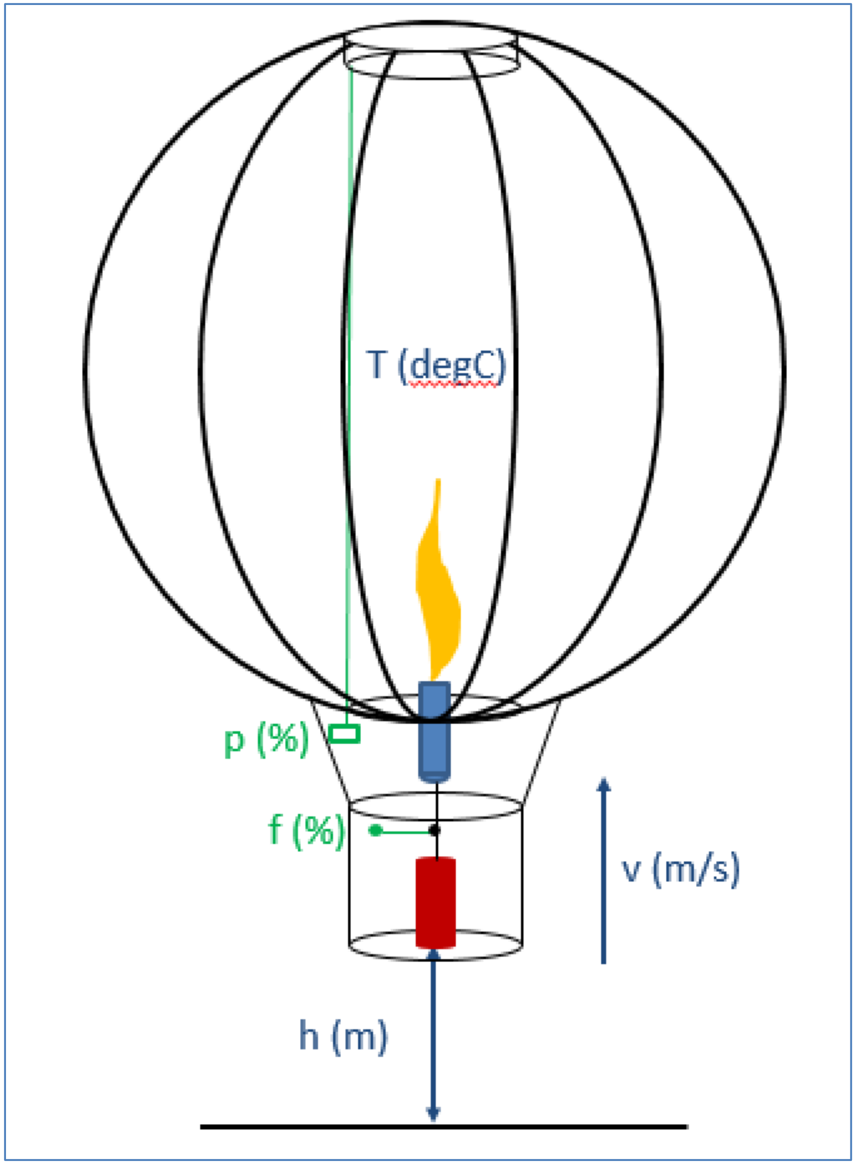


Outline

Choose the following scaled variables:

$$\xi = h/h_r \quad \theta_i = T_i/T_r \quad \theta_s = T_s/T_r \quad \tau = t/t_r \quad \Gamma = f/f_r \quad \Lambda = p/p_r$$

$$h_r = 1000m \quad T_r = T_0 \quad t_r = (h_r/g)^{0.5} \quad f_r = \frac{\rho_0 V C_{p0} T_r}{\in \Delta H_c t_r} \quad p_r = \left(C_{p0} k_v t_r\right)^{-1}$$

Then the full hot air balloon model can be written in dimensionless form:

$$\frac{d^2\xi}{d\tau^2} = \alpha\mu\theta_s^{\gamma-1}\left[1 - \frac{\theta_s}{\theta_i}\right] - \mu - \omega\frac{d\xi}{d\tau}\left|\frac{d\xi}{d\tau}\right| \qquad \theta_s = 1 - \delta\xi$$

$$\frac{d\theta_i}{d\tau} = -(\theta_i - \theta_s)(\beta + \Lambda) + \Gamma$$

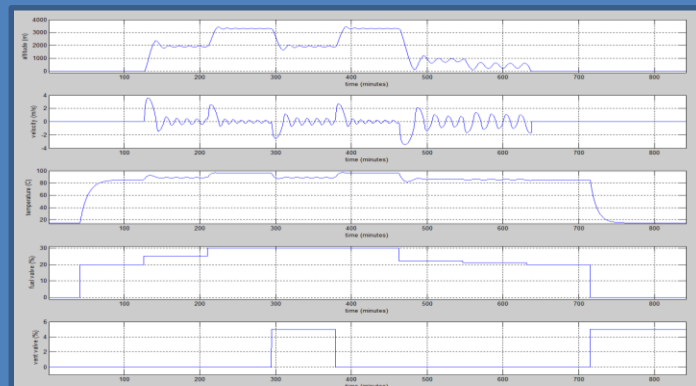
$\alpha = \frac{\rho_0 V}{m_p}$	Balloon number	$\omega = \frac{k_v h_r}{m_T}$	Drag number
$\gamma = \frac{Mg}{aR}$	Atmosphere number	$\delta = \frac{a h_r}{T_0}$	Thermal drop number
$\mu = \frac{m_p}{m_T}$	Ratio payload weight to total weight	$\beta = \frac{U A t_r}{\rho_0 V C_{p0}}$	Heat loss number

Model development

```
function dxdt = hab(t,x,parms)
%
% This function evaluates the ode rhs for the hot air balloon simulation.
%
% Tom Badgwell 06/07/17

% Get parameters

alpha = parms.alpha;
gamma = parms.gamma;
mu    = parms.mu;
omega = parms.omega;
delta = parms.delta;
beta  = parms.beta;

% Get inputs

u = parms.u;

% Calculate derivatives

dxdt = zeros(3,1);
Ths = 1 - delta*x(1);
dxdt(1) = x(2);
dxdt(2) = alpha*mu*(Ths^(gamma-1))*(1-(Ths/x(3))) - mu -omega*x(2)*abs(x(2));
dxdt(3) = -(x(3) - Ths)*(beta + u(2)) + u(1);

end
```

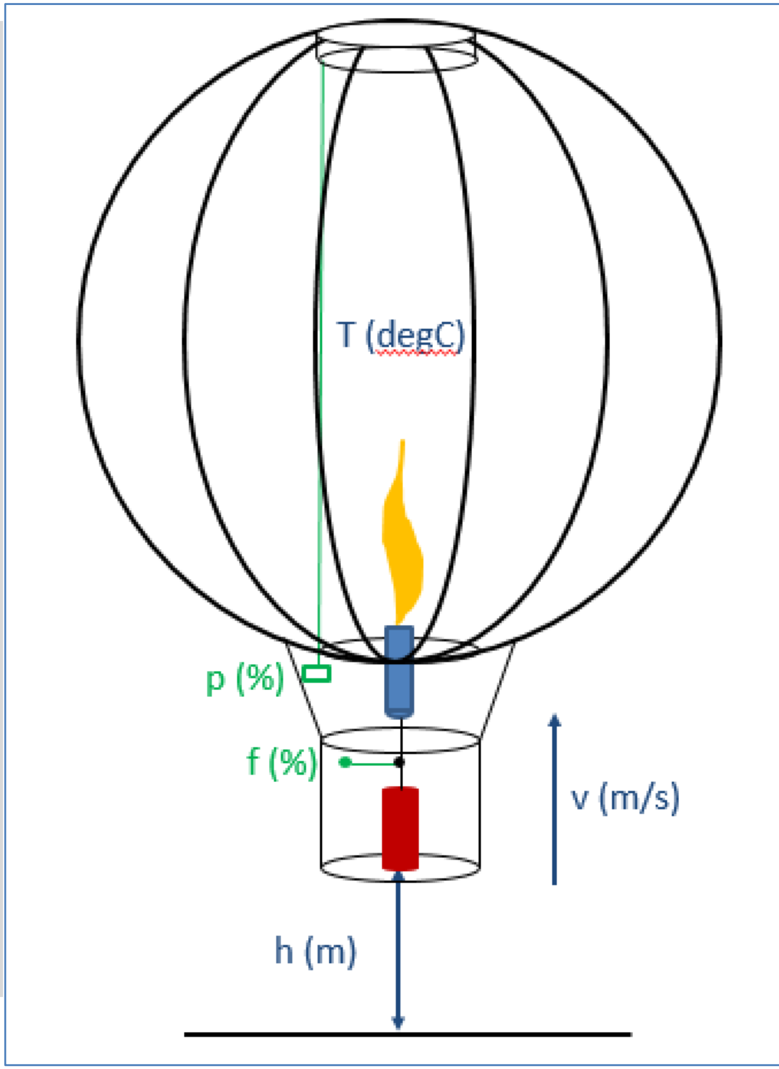
Matlab code

Simulation results



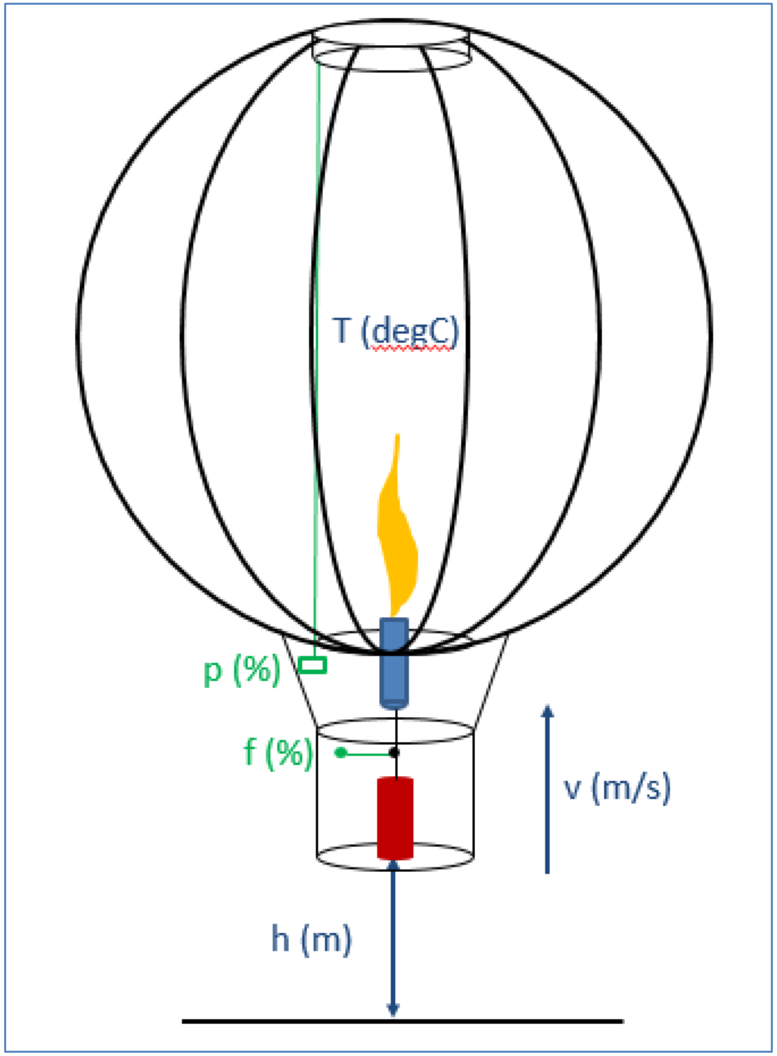
Model development

- List key model assumptions
- Derive model equations (from assumptions)
- Scale the variables to develop a dimensionless model
  - Facilitates analysis
  - Simplifies numerical solution
- Analyze the model
  - Determine operating region
  - Verify good behavior over the operating region (no singularities, exploding exponentials, etc.
- Work through an example
  - Provides a sanity check on model assumptions and analysis
- Iterate

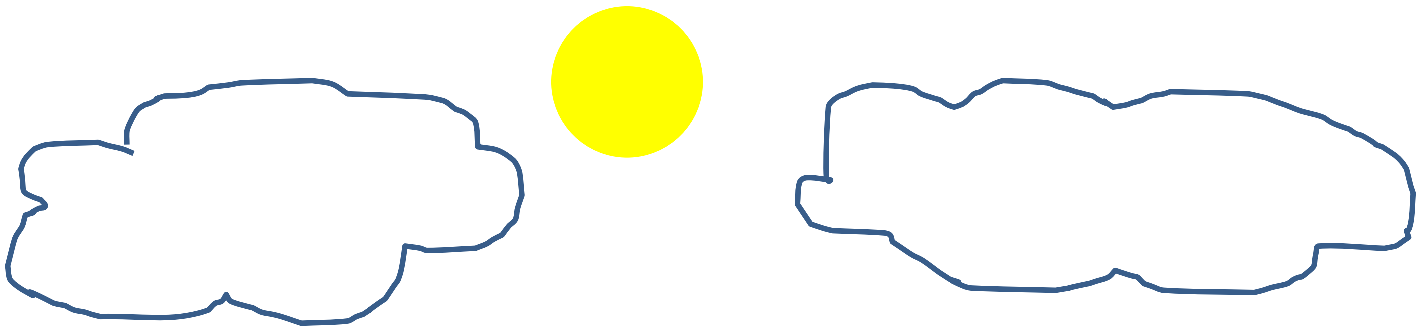


Model development: Modeling assumptions

1. We focus on vertical movement of the balloon, which can be controlled by the fuel valve position f and vent valve position p.
2. Vertical velocity v depends on lift, weight, and drag forces.
3. Lift force is equal to the weight of the air displaced by the envelope (Archimedes' principle).
4. Drag force is proportional to the square of velocity.
5. Air is an ideal gas, and atmospheric temperature falls off linearly with altitude [1].
6. The envelope interior is isothermal at temperature T_i.
7. Envelope volume V is constant, and pressure inside and outside the envelope is the same.
8. The envelope interior temperature T_i changes based on fuel valve position f, vent valve position p, and heat loss to the atmosphere.
9. The internal energy of the gas in the envelope dominates (over its kinetic and potential energy).
10. Changes in the internal energy of the gas in the envelope due to pressure changes can be ignored.



Model components: Standard Atmosphere [1]



- Air is an ideal gas
- Atmospheric temperature falls off linearly with altitude, up to 11km

$$P_s = \frac{\rho_s R T_s}{M} \qquad M = 28.97\ \mathrm{g/mol}$$

$$T_s = T_o - ah \qquad a = 6.5 \times 10^{-3}$$

$$\frac{\boldsymbol{\rho}_s}{\boldsymbol{\rho}_0} = \left(\frac{T_s}{T_0}\right)^{\gamma-1} \qquad \gamma = \frac{gM}{aR} = \frac{(.02897\ kg/mol)(9.807\ m/s^2)}{(6.5\ x\ 10^{-3} K/m)(8.314\ J/molK\ )} = 5.257$$

$$\frac{P_s}{P_0} = \left(\frac{P_s}{P_0}\right)^{\gamma}$$

Model components: Vertical Force Balance

The vertical force balance considers lift, weight, and drag:

$$m_T \frac{d^2 h}{dt^2} = L - m_p g - k \frac{dh}{dt}\left|\frac{dh}{dt}\right|$$

Lift force is given by Archimedes' principle: The lift force is equal to the weight of the air displaced by the envelope:

$$L = (\rho_s - \rho_i) V g$$

Requirement for air pressure to be the same on the inside and outside of the envelope gives:

$$\rho_i = \rho_s \left(\frac{T_s}{T_i}\right)$$

Combining these with the standard atmosphere model gives the following vertical force balance :

$$m_T \frac{d^2 h}{dt^2} = \rho_0 V g \left(\frac{T_s}{T_0}\right)^{\gamma - 1} \left[1 - \left(\frac{T_s}{T_0}\right)\left(\frac{T_0}{T_i}\right)\right] - m_p g - k \frac{dh}{dt}\left|\frac{dh}{dt}\right|$$

Model development: Envelope energy balance

The envelope energy balance considers energy input from burning fuel, energy output from venting gas, and heat loss to the atmosphere:

$$\frac{d}{dt}(E_i) = -UA(T_s - T_i) - C_{pi}(T_i - T_s)k_v p$$

Now we make some simplifying assumptions regarding energy changes of the gas in the envelope:

- The internal energy of the gas in the envelope dominates (over its kinetic and potential energy).
- Changes in the internal energy of the gas in the envelope due to pressure changes can be ignored.

These assumptions allow us to approximate the envelope energy balance as follows:

$$\rho_0 V C_{p0} \frac{d}{dt} T_i = -UA(T_i - T_s) + \in \Delta H_c f \; - C_{p0}(T_i - T_s)k_v p$$

Hot Air Balloon Model: Dimensional Equations

Standard model of the atmosphere:

$$P_s = \frac{\rho_s R T_s}{M} \qquad M = 28.97 \text{ g/mol} \qquad T_s = T_o - ah \qquad a = 6.5 \text{ x } 10^{-3}$$

Vertical force balance:

$$m_T \frac{d^2 h}{dt^2} = (\rho_s - \rho_i) V g - m_p g \; - k \frac{dh}{dt} \left| \frac{dh}{dt} \right|$$

Energy balance:

$$\rho_0 V C_{p0} \frac{d}{dt} T_i = -UA(T_i - T_s) + \in \Delta H_c f \; - C_{p0}(T_i - T_s) k_v p$$

Dimensionless Hot Air Balloon Model

Choose the following scaled variables:

$$\xi = h/h_r \quad \theta_i = T_i/T_r \quad \theta_s = T_s/T_r \quad \tau = t/t_r \quad \Gamma = f/f_r \quad \Lambda = p/p_r$$

$$h_r = 1000m \quad T_r = T_0 \quad t_r = (h_r/g)^{0.5} \quad f_r = \frac{\rho_0 V C_{p0} T_r}{\in \Delta H_c t_r} \quad p_r = \left(C_{p0} k_v t_r\right)^{-1}$$

Then the full hot air balloon model can be written in dimensionless form:

$$\frac{d^2\xi}{d\tau^2} = \alpha\mu\theta_s^{\gamma-1}\left[1 - \frac{\theta_s}{\theta_i}\right] - \mu - \omega\frac{d\xi}{d\tau}\left|\frac{d\xi}{d\tau}\right| \qquad \theta_s = 1 - \delta\xi$$

$$\frac{d\theta_i}{d\tau} = -(\theta_i - \theta_s)(\beta + \Lambda) + \Gamma$$

$\alpha = \dfrac{\rho_0 V}{m_p}$ Balloon number

$\omega = \dfrac{k_v h_r}{m_T}$ Drag number

$\gamma = \dfrac{Mg}{aR}$ Atmosphere number

$\delta = \dfrac{a h_r}{T_0}$ Thermal drop number

$\mu = \dfrac{m_p}{m_T}$ Ratio payload weight to total weight

$\beta = \dfrac{UAt_r}{\rho_0 V C_{p0}}$ Heat loss number

Dimensionless Hot Air Balloon Model: Takeoff Condition

From the vertical force balance, the condition for takeoff is:

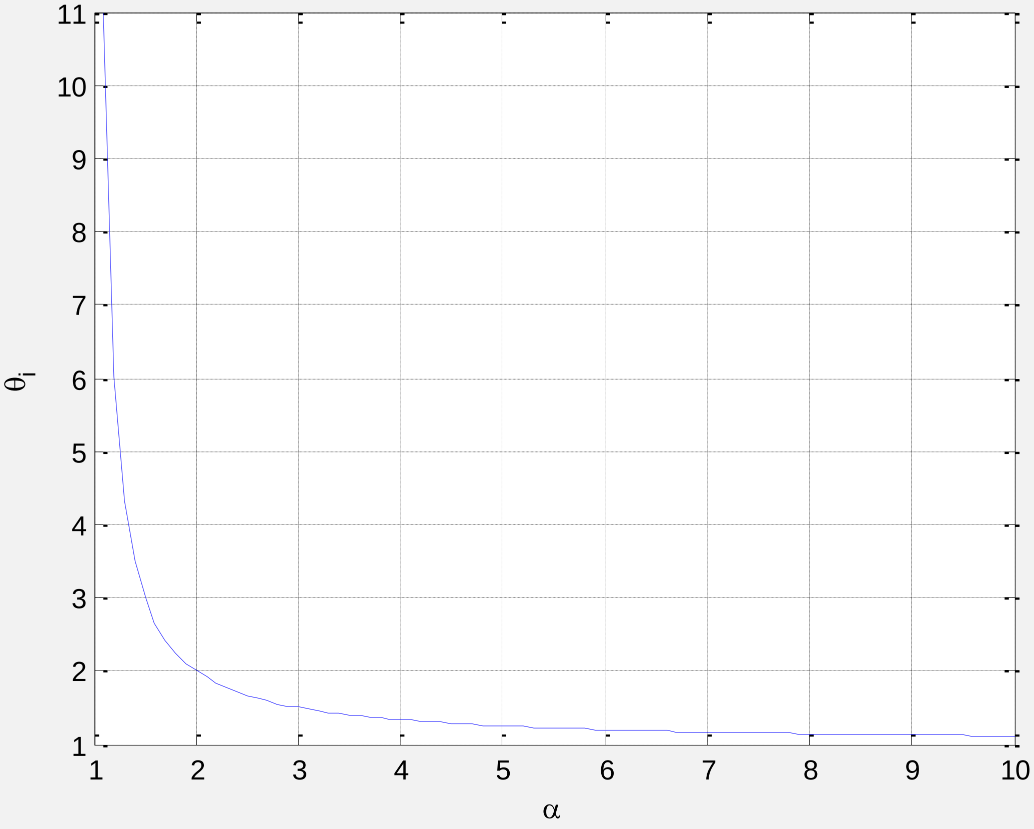
$$\frac{d^2\xi}{d\tau^2} = \frac{d\xi}{d\tau} = 0 \qquad \theta_s = 1$$

$$\alpha\mu\left[1 - \frac{1}{\theta_i}\right] \geq \mu$$

$$\theta_i \geq \frac{\propto}{\propto - 1}$$ for takeoff

Implication is:

$$\propto \geq 1 \quad \rightarrow \quad \rho_0 V \geq m_p$$



Example: AX7-77 Balloon Model Parameters

The takeoff takes place at STP (15 degC and 1 atm):

$$\rho_0 = \frac{MP_0}{RT_0} = 1.225\ kg/m^3$$

$$\propto = 5.098$$

The conditions at takeoff are:

$$\theta_i \geq \frac{\propto}{\propto - 1} = 1.244 \quad \rightarrow \quad T_i = 3.58.5K = 85.3\ degC$$

$$\rho_i = \rho_0 \frac{T_i}{T_0} = 0.9848\ kg/m^3$$

$$m_i = \rho_i V = 2140\ kg \quad \rightarrow \quad m_T = m_p + m_i = 2671\ kg$$

$$\mu = m_p/m_T = 0.1961$$

We choose $h_r = 1000m$: $\delta = \frac{ah_r}{T_0} = 0.02255$

Example: AX7-77 Balloon Model Parameters

To calculate the parameter ω, assume a free-fall velocity of -15 m/s:

$$\omega = -\frac{\mu}{\frac{d\xi}{d\tau}\left|\frac{d\xi}{d\tau}\right|} = 8.544$$

To calculate the parameter β, assume that it takes 10 minutes for the envelope to cool off 95% of the way to ambient temperature:

$$\beta = -\frac{1}{\frac{\tau_c}{t_r}} = 0.01683$$

To calculate the scale factor f_r, assume that the fuel valve is set to 20% at takeoff:

$$0 = -(\theta_i - \theta_s)\beta + \Gamma \quad \rightarrow \quad f_r = 4870\%$$

To calculate the scale factor p_r, assume that with the fuel valve stuck wide open ($f = 100\%$) the balloon can still stay on the ground if the vent valve is full open ($\mathrm{p} = 100\%$):

$$0 = -(\theta_i - \theta_s)(\beta + \Lambda) + \Gamma \quad \rightarrow \quad p_r = 1485\%$$

Example: AX7-77 Balloon

As an example, let's simulate the flight of a AX7-77 balloon from Head Balloons, Inc. [2]:

- Carries pilot and two passengers
- Envelope volume is 77,000 ft^3 or 2180 m^3
- Maximum envelope temperature is 120 degC
- Assume takeoff at STP (15 degC, 1 atm)
- Sample loading from FAA flight manual supplement:

	Envelope	225 lb
	Basket	190 lb
	4x10 gallon fuel tanks	280 lb
	Pilot	170 lb
	Passenger #1	125 lb
	Passenger #2	165 lb

Total		1155 lb or 523.8 kg

Matlab Code: AX7-77 Balloon Model Numerical Solution

This example was implemented in Matlab (R2012b):

- The ode is implemented in the function hab.m:

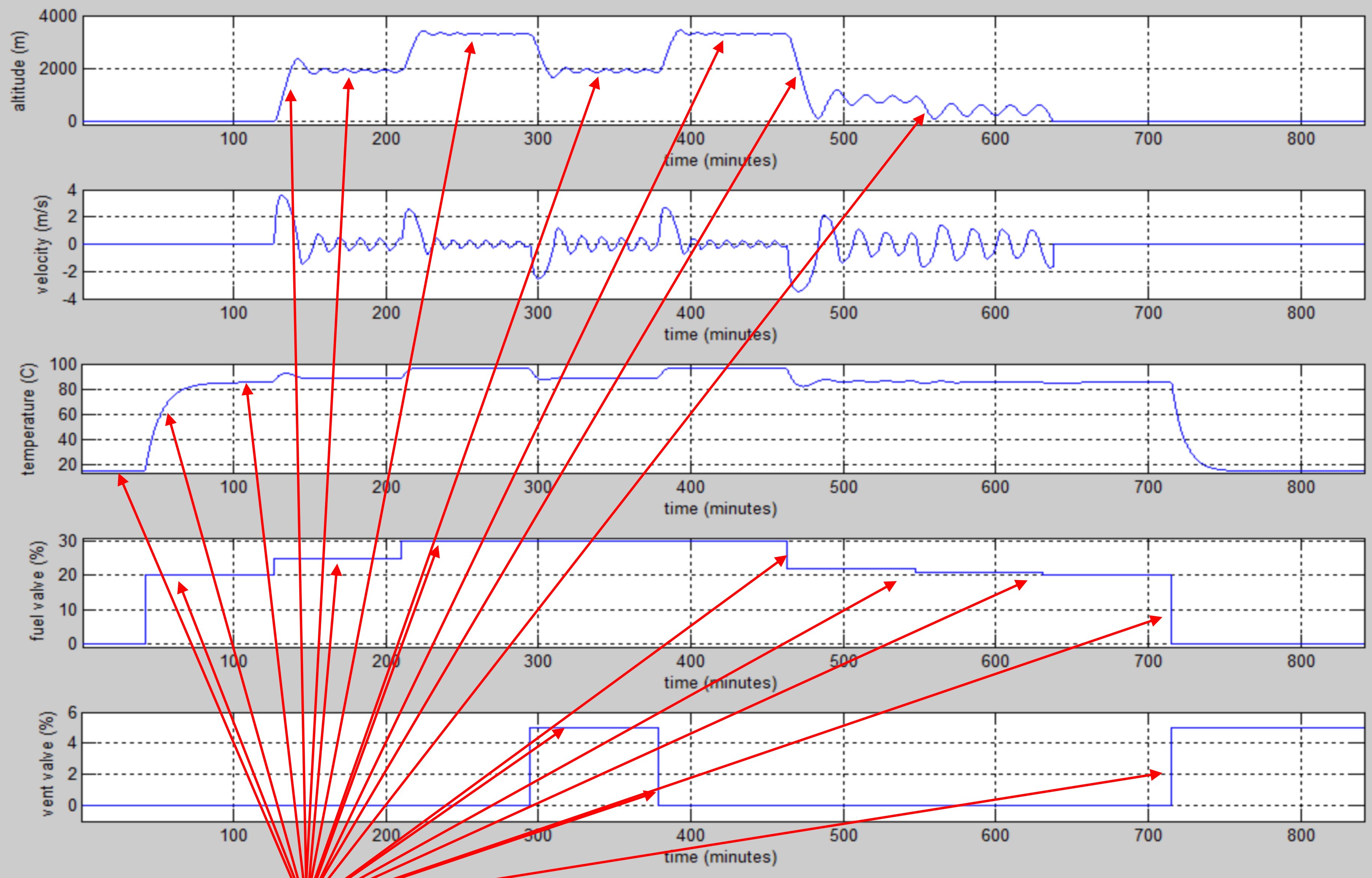
```
dxdt = zeros(3,1);
Ths = 1 - delta*x(1);
dxdt(1) = x(2);
dxdt(2) = alpha*mu*(Ths^(gamma-1))*(1-(Ths/x(3))) - mu -omega*x(2)*abs(x(2));
dxdt(3) = -(x(3) - Ths)*(beta + u(2)) + u(1);
```

- The simulation is performed in the script HotAirBalloon.m:

```
% Integrate the model for this time step

parms.u = uk(k,:);
tstop   = tstart + dt;
tspan   = [tstart tstop];
[t,x]   = ode45(@(t,x) hab(t,x,parms), tspan, xstart);
```

`ode45` is based on an explicit Runge-Kutta (4,5) formula, the Dormand-Prince pair. It is a *one-step* solver – in computing `y(`t_n`)`, it needs only the solution at the immediately preceding time point, `y(`t_{n-1}`)`. In general, `ode45` is the best function to apply as a *first try* for most problems. [3]

Simulation Results: AX7-77 Balloon Flight



On ground; shutoff fuel, open vent

References

[1] John D. Anderson, Introduction to Flight, McGraw-Hill, (1978).
[2] FAA Approved Balloon Flight Manual Supplement for Head Balloons, Inc. Model AX7-77 w/Parachute Top (1984).