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# PORTFOLIO SELECTION: THE EFFECTS OF UNCERTAIN MEANS, VARIANCES, AND COVARIANCES

*George M. Frankfurter, Herbert E. Phillips, and John P. Seagle**

Present models for selecting portfolios according to the mean-variance criteria do not account for the simultaneous effect of error in estimating means, variances, and covariances of security returns. This paper describes an experiment in which the impact of estimation error is so strong that the usefulness of present mean-variance approaches to portfolio selection is brought into question.

## I. Introduction

The literature dealing with portfolio theory has grown considerably since Markowitz [6], Tobin [11], Sharpe [10], and others introduced the mean-variance approach to portfolio selection. This approach is based on a theory of preference ordering that supposes the investor is made happy by anticipation of gain but is vexed by his uncertainties. Accordingly, a collection of securities is called "efficient," if it (1) maximizes expected return for a given level of variance of return, and (2) minimizes variance of return for a given level of expected return. Such a preference ordering system would apply in general only under a quadratic utility function, which has unrealistic properties [9]. For this study it is sufficient to assume that the investor would not purposely select the riskier of two portfolios offering the same expected return.

To set the stage for the experiment, we consider well-behaved securities. Suppose that the returns on securities have a multivariate normal distribution [7, Chapter 9]

$$f(R) = f(R_1, R_2, \ldots, R_s) \tag{1}$$

$$= |H|^{1/2} \cdot (2\pi)^{-s/2} \exp(-1/2(R-\mu)H(R-\mu)')$$

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where

$$\mu = (\mu_1, \mu_2, \ldots, \mu_s) \tag{2}$$

is a 1 x s vector and H is an s x s positive definite symmetric matrix with inverse

$$\Sigma = ||\sigma_{ij}|| = ||\rho_{ij}\sigma_i\sigma_j|| \tag{3}$$

which is the variance-covariance matrix. Let

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_s] \tag{4}$$

be an s x 1 vector of security weights, where $\lambda_i$ denotes the portion of portfolio "p" that is invested in security "i". The mean and variance of portfolio "p" are obtained from

$$E = \mu\lambda \tag{5}$$

and

$$V = \lambda'\Sigma\lambda \quad . \tag{6}$$

The parameters given by (2) and (3) completely determine the multivariate normal density set out in (1). Given the required parameters, computational techniques exist for computing the efficient E-V combinations [6, 10]. In practice, however, these parameters are never known. For more general distributions and criteria, more parameters would be needed.

With the exception of papers by Mao and Särndal [5], Fried [2], and Kalymon [3], questions having to do with the consequence of using estimations in place of known parameters in portfolio selection models have largely been ignored. Mao and Särndal [5] recognize that the parameters of a distribution of security returns, such as that given by equation (1), would, in practice, depend upon unknown future values of state variables. A procedure is presented for maximizing expected utility given a prior probability distribution on the state variables and a likelihood function on the parameters which is conditional on the state variables. Fried [2] presents a model for obtaining the needed priors, using standard forecasting techniques. Kalymon [3] considers the effect of uncertainty about means and shows that standard measures of risk fail to account for error in mean estimation. He recognizes, but chooses to disregard, the additional impact of error in estimating variances and covariances. We include this source of error in an experiment and demonstrate the effect on portfolio selections of error in estimating the parameters of distributions of security returns.

In the experimental design presented below it is recognized that estimates

$$\tilde{\mu} = (\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_s), \text{ and} \tag{7}$$

$$\tilde{\Sigma} = ||\tilde{\sigma}_{ij}|| = ||\tilde{\rho}_{ij}\tilde{\sigma}_i\tilde{\sigma}_j||, \tag{8}$$

which may depend on both sample and subjective information, are used in practice to parameterize f(R) [See equation (1)], thereby causing the computed E-V combinations.

$$\tilde{E} = \tilde{\mu}\lambda, \tag{9}$$

and

$$\tilde{V} = \lambda'\tilde{\Sigma}\lambda \tag{10}$$

to be estimates also.[1]

Uncertainty regarding the means and the variance-covariance matrix [see equations (2) and (3)] must be considered simultaneously, because estimates [see equations (7) and (8)] are typically based on the same sample data or forecasts. The experimental results demonstrate that estimation error in $\tilde{\mu}$ and $\tilde{\Sigma}$ can cause the analyst to select substantially inferior portfolios a large proportion of the time.

The design of the experiment is set out in Section II. Section III describes the execution, and the results are given in Section IV.

## II. Design of the Experiment

The analyst may have information (subjective or time series data) regarding security returns for any number of accounting periods. This information constitutes, at best, just a single sample of some multidimensional process (perhaps a multivariate normal process) that will generate the future security returns that are actually relevant in portfolio analyses. With such data, however, one can make but one prediction (or estimation) of future security returns.

[1] Estimates [equations (7) through (10)] are marked with the tilde to indicate that they are random variables.

It would be advantageous for this study to conduct an empirical analysis on estimators that have entered into actual portfolio selection decisions, but this is not possible -- firms change with time, as do other factors that affect security returns. This point has been a source of serious confusion in some recent papers [4]. In this paper, the desired ends are achieved, but by different means and in a necessarily abstract setting.

If it can be assumed that security returns are, in general, distributed at least roughly in accordance with a known and analytically tractable distribution, then sample observations for any set of "s" securities can be generated by Monte Carlo methods. In this paper, the pretense is made that security returns are distributed in accordance with the multivariate normal density given by equation (1); such a density is completely defined by parameters $\mu$ and $\Sigma$, as given by equations (2) and (3), respectively. Given these parameters, s-tuples of security returns

$$R = [R_1, R_2, \ldots, R_s] \tag{11}$$

can be generated by means of a series of regression structures, as described below.

Suppose that "s" securities are selected and ordered arbitrarily 1, 2, ..., s. It follows from equation (1) that

$$f(R_1) = (\sigma_1^2 \cdot 2\pi)^{-1/2} \cdot \exp\,[-1/2(R_1-\mu)\sigma_1^{-2}] \tag{12}$$

which is a univariate normal distribution. Thus, any return on security "1" may be written

$$R_1 = \mu_1 + \varepsilon_1 \tag{13}$$

where

$$\varepsilon_1 \sim N(0,\ \sigma_1^2). \tag{14}$$

Returns on securities 1 and 2 are bivariate normal.

$$f(R_1,\ R_2) = |\Sigma_{12}|^{-1/2} \cdot (2\pi)^{-1} \cdot \exp\,[-1/2(R_1-\mu_1,\ R_2-\mu_2) \cdot \Sigma_{12}^{-1} \cdot (R_1-\mu_1,\ R_2-\mu_2)'] \tag{15}$$

where $\Sigma_{12}$ is the variance-covariance matrix for securities 1 and 2, which is a partition of $\Sigma$ [7, p. 213]. Once $R_1$ has been generated in accord with (13), $R_2$ can be obtained from the univariate normal distribution with mean

$$\mu_2 + (\sigma_{12}/\sigma_{11})\cdot(R_1-\mu_1)$$

and variance

$$\sigma_{22.1} \equiv \sigma_{22}-(\sigma_{12})^2/\sigma_{11}$$

$R_2$ can also be expressed in the form of a regression equation

$$R_2 = \mu_2 + \rho_{12}\frac{\sigma_2}{\sigma_1}(R_1-\mu_1) + \varepsilon_{2.1} \tag{16}$$

where $\varepsilon_{2.1}$ is the error term of a regression of $R_2$ against $R_1$.

$$\varepsilon_{2.1} \sim N(0, \sigma_{22.1}) \tag{17}$$

Returns $R_3$, $R_4$, ..., $R_s$ can be sequentially generated in an analogous manner, using a series of multiple regression equations.

## III. The Experiment

Assuming a world where just three securities exist, a 1 x 3 vector of means [see equation (2)] and a 3 x 3 variance-covariance matrix [see equation (3)] were obtained for three actual securities by statistical methods. These values, shown in Table 1, were used to parameterize a multivariate normal density [see equation (1)]; this density was then modeled in the data generating algorithm described in the previous section.

TABLE 1

SECURITIES USED IN THE SIMULATION

| Security Number | 1 | 2 | 3 |
|---|---|---|---|
| Mean Returns Percent | 16.64 | 6.64 | 21.35 |
| Covariances: | | | |
| with Sec. 1 | 2102 | -115 | 1115 |
| with Sec. 2 | -115 | 1664 | -37 |
| with Sec. 2 | 1115 | -37 | 2223 |
| Firm | Chrysler | N.Y. Shipping | Bulova |

Sample data consisting of three-tuples of returns were generated and collected in trial groups representing from 5 to 50 accounting periods. Each group thus represents the totality of information that a security analyst might have in a real decision-making situation. For each trial group, a vector of means and a variance-covariance matrix were estimated [see equations (7) and (8)] by standard correlation methods. For each of 100 trial groups of returns, estimates of the portfolio mean and variance were obtained by equations (9) and (10), respectively, for different security combinations. By varing the $\lambda_i$ [see equation (4)], any number of portfolios can be defined for the security space shown in Table 1. In the experiment, the $\lambda_i$ were varied from 0 to 1 in steps of 0.1, defining 66 different portfolios. These are shown in Table 2, ordered according to E and V, as calculated from the "true" parameters of Table 1, using equations (5) and (6). In Table 2 an asterisk is used to identify a portfolio that is "truly" efficient according to the mean-variance criterion; we say that such a portfolio is "efficient in parameter space."

A security analyst would not know the true parameter values; in practice, these must be estimated. Sample data were generated according to the multivariate normal data generating algorithm, represented in part by equations (12) through (17) above. For each of four different lengths of run representing, respectively, historical data for 5, 10, 25, and 50 independent accounting periods, 100 different trial groups of data were generated. For each trial group a vector of means was obtained in accordance with equation (7) and a variance-covariance matrix was obtained in accordance with equation (8). By substituting (4), (7), and (8) into equations (9) and (10), a mean and variance was calculated for each of the 66 portfolios. Portfolios that appeared efficient according to the sample based mean-variance criterion were identified for use in preparing Figure I.

## IV. Results

In Figure I we plot the relative frequency with which portfolios appeared on the efficient frontier in the sample trials. The figure is presented in four parts, one for each length of run, as labeled. Portfolios are identified according to the mean and variance, based on the parameter values of Table 2.[2]

[2]These are rounded to the nearest values shown on the axes. Each portfolio of Table 2 is unique to a position in Figure I, but each such position may, and many do, contain more than one portfolio. For positions that contain more than one portfolio, the relative frequency efficient for one or more of these portfolios is shown.

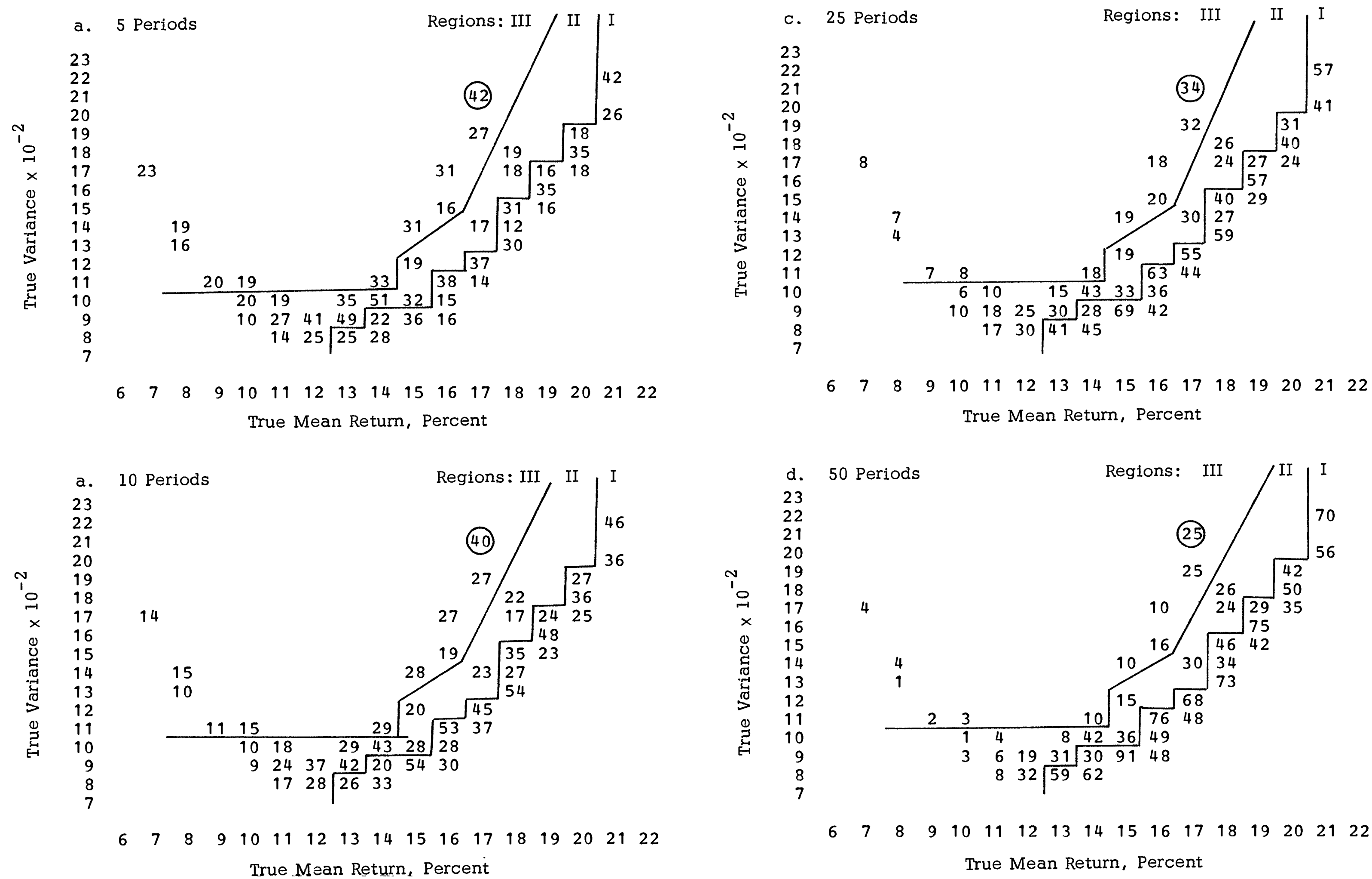


Figure I
Relative Frequency (percent in Which Portfolios
Appear Efficient in Sample Trials

Table 2

PORTFOLIO SUMMARY REPORT

| Portfolio Number | Portfolio Weights | | | Mean Return | Portfolio Variance | Portfolio Number | Portfolio Weights | | | Mean Return | Portfolio Variance |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1* | 0 | 0 | 10 | 21.35 | 22.22 | 34 | 7 | 2 | 1 | 15.11 | 12.41 |
| 2* | 1 | 0 | 9 | 20.88 | 20.22 | 35* | 4 | 3 | 3 | 15.05 | 09.19 |
| 3* | 2 | 0 | 8 | 20.41 | 18.63 | 36 | 1 | 4 | 5 | 15.00 | 09.30 |
| 4* | 3 | 0 | 7 | 19.94 | 17.46 | 37 | 8 | 2 | 0 | 14.64 | 13.75 |
| 5 | 0 | 1 | 9 | 19.88 | 18.10 | 38 | 5 | 3 | 2 | 14.58 | 09.48 |
| 6* | 4 | 0 | 6 | 19.46 | 16.72 | 39* | 2 | 4 | 4 | 14.52 | 08.54 |
| 7* | 1 | 1 | 8 | 19.41 | 16.30 | 40 | 6 | 3 | 1 | 14.11 | 10.19 |
| 8 | 5 | 0 | 5 | 18.99 | 16.38 | 41* | 3 | 4 | 3 | 14.05 | 08.20 |
| 9* | 2 | 1 | 7 | 18.94 | 14.92 | 42 | 0 | 5 | 5 | 14.00 | 09.53 |
| 10 | 6 | 0 | 4 | 18.52 | 16.47 | 43 | 7 | 3 | 0 | 13.64 | 11.31 |
| 11* | 3 | 1 | 6 | 18.47 | 13.96 | 44 | 4 | 4 | 2 | 13.58 | 08.27 |
| 12 | 0 | 2 | 8 | 18.41 | 14.77 | 45 | 1 | 5 | 4 | 13.53 | 08.56 |
| 13 | 7 | 0 | 3 | 18.05 | 16.98 | 46 | 5 | 4 | 1 | 13.11 | 08.76 |
| 14* | 4 | 1 | 5 | 17.99 | 13.42 | 47* | 2 | 5 | 3 | 13.05 | 08.00 |
| 15* | 1 | 2 | 7 | 17.94 | 13.18 | 48 | 6 | 4 | 0 | 12.64 | 09.68 |
| 16 | 8 | 0 | 2 | 17.58 | 17.91 | 49* | 3 | 5 | 2 | 12.58 | 07.86 |
| 17 | 5 | 1 | 4 | 17.52 | 13.29 | 50 | 0 | 6 | 4 | 12.53 | 09.37 |
| 18* | 2 | 2 | 6 | 17.47 | 12.00 | 51 | 4 | 5 | 1 | 12.11 | 08.14 |
| 19 | 9 | 0 | 1 | 17.11 | 19.25 | 52 | 1 | 6 | 3 | 12.06 | 08.60 |
| 20 | 6 | 1 | 3 | 17.05 | 13.59 | 53 | 5 | 5 | 0 | 11.64 | 08.84 |
| 21* | 3 | 2 | 5 | 17.00 | 11.25 | 54 | 2 | 6 | 2 | 11.58 | 08.25 |
| 22 | 0 | 3 | 7 | 16.94 | 12.23 | 55 | 3 | 6 | 1 | 11.11 | 08.31 |
| 23 | 10 | 0 | 0 | 16.64 | 21.02 | 56 | 0 | 7 | 3 | 11.06 | 10.00 |
| 24 | 7 | 1 | 2 | 16.58 | 14.30 | 57 | 4 | 6 | 0 | 10.64 | 08.80 |
| 25* | 4 | 2 | 4 | 16.52 | 10.91 | 58 | 1 | 7 | 2 | 10.58 | 09.44 |
| 26* | 1 | 3 | 6 | 16.47 | 10.84 | 59 | 2 | 7 | 1 | 10.11 | 09.29 |
| 27 | 8 | 1 | 1 | 16.11 | 15.43 | 60 | 3 | 7 | 0 | 09.64 | 09.56 |
| 28 | 5 | 2 | 3 | 16.05 | 10.99 | 61 | 0 | 8 | 2 | 09.58 | 11.42 |
| 29* | 2 | 3 | 5 | 16.00 | 09.88 | 62 | 1 | 8 | 1 | 09.11 | 11.06 |
| 30 | 9 | 1 | 0 | 15.64 | 16.98 | 63 | 2 | 8 | 0 | 08.64 | 11.12 |
| 31 | 6 | 2 | 2 | 15.58 | 11.49 | 64 | 0 | 9 | 1 | 08.12 | 13.64 |
| 32* | 3 | 3 | 4 | 15.52 | 09.32 | 65 | 1 | 9 | 0 | 07.64 | 13.48 |
| 33 | 0 | 4 | 6 | 15.47 | 10.49 | 66 | 0 | 10 | 0 | 06.64 | 16.64 |

Note: Mean returns are given in percentages, variances in $(\text{percent})^2 \times 10^{-2}$
Weights are fractions 10. Efficient portfolios are indicated by an *.

It is clear from the figure that the makeup of the efficient set of portfolios varies substantially among sample trials, even though generation of all sample data was based upon the assumed parameter values. Moreover, portfolios that are extremely inefficient in terms of the true parameters may appear efficient a large proportion of the time. For example, the circled portfolio has a true mean (in parameter space) of 17, and a true variance of 21. This portfolio is dominated by any portfolio that lies below or to its right. Notice that the circled portfolio appeared on the efficient frontier 42 percent of the time for a length of run of five, and compare this value with the relative frequency associated with other portfolios that clearly dominate it. In samples corresponding to 10, 25, and 50 accounting periods, the relative frequency of the circled portfolio declined.

In general, the efficient and inefficient portfolios seem to appear on the efficient frontier with relative frequencies that would *not* distinguish them. The magnitude of the percentages shown in Figure I is not very sensitive to sampling error, as the standard deviations of these percentages range from 3 percentage points at a relative frequency of 10 percent to 10 percentage points at a relative frequency of 50 percent.

The figure is broken down into three regions. Portfolios, such as the circled portfolio, are in Region III; this region contains portfolios that are most inefficient in the sense that they are dominated by a large number of portfolios by substantial margins. Region I contains portfolios that are, in a loose sense, close to or on the efficient frontier (in parameter space), and Region II contains the remaining portfolios.

As the number of accounting periods in a trial group rises, mean and variance-covariance estimators become more peaked -- this follows from well-known postulates in statistics. In the case at hand, however, the relative frequency with which inferior portfolios (Region III) appear on the efficient frontier in sample trials declines as the sample period rises, but not sharply -- there are important exceptions. In general, these relative frequencies are nowhere negligible, not even for a run of 50 accounting periods. By comparison with real world situations, 50 accounting periods of information about a security represent a considerable amount of information. For a five-period run, the average relative frequency with which the inferior portfolios of Region III appear on the efficient frontier is actually larger than for the portfolios of Region I.[3] This surprising result is quite consistent with the results displayed in Table 3.

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[3]This is not obvious from the figure because some positions in Region I contain several portfolios, while Region III intervals generally contain just one portfolio.

## TABLE 3

### Relative Frequency (percent) in Which Efficient Portfolios are Dominated in Sample Trials for a Length of Run of 5 Accounting Periods

| Dominating Portfolios | | | Efficient Portfolios | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Port. | | | 1 | 2 | 3 | 4 | 6 | 7 | 9 | 11 | 14 | 15 | 18 | 21 | 25 | 26 | 29 | 32 | 35 | 39 | 41 | 47 | 49 |
| | Mean | | 21 | 21 | 20 | 20 | 19 | 19 | 19 | 18 | 18 | 18 | 17 | 17 | 17 | 16 | 16 | 16 | 15 | 15 | 14 | 13 | 13 |
| | | Var. | 22 | 20 | 19 | 17 | 17 | 16 | 15 | 14 | 13 | 13 | 12 | 11 | 11 | 11 | 10 | 9 | 9 | 9 | 8 | 8 | 8 |
| 1* | 21 | 22 | 0 | 8 | 8 | 11 | 14 | 1 | 2 | 4 | 5 | 1 | 1 | 1 | 4 | 2 | 1 | 1 | 2 | 3 | 3 | 4 | 4 |
| 2* | 21 | 20 | 1 | 0 | 3 | 5 | 5 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 2 | 3 | 3 |
| 3* | 20 | 19 | 3 | 2 | 0 | 1 | 1 | 4 | 1 | 2 | 3 | 2 | 2 | 2 | 2 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 3 |
| 4* | 20 | 17 | 4 | 4 | 2 | 0 | 1 | 5 | 2 | 1 | 2 | 5 | 2 | 2 | 2 | 4 | 2 | 2 | 2 | 2 | 2 | 3 | 3 |
| 5 | 20 | 18 | 2 | 11 | 17 | 16 | 16 | 8 | 10 | 13 | 14 | 0 | 1 | 4 | 6 | 1 | 0 | 2 | 6 | 3 | 3 | 6 | 5 |
| 6* | 19 | 17 | 5 | 3 | 2 | 1 | 0 | 3 | 3 | 2 | 2 | 7 | 3 | 2 | 3 | 6 | 2 | 3 | 3 | 5 | 3 | 6 | 5 |
| 7* | 19 | 16 | 0 | 1 | 1 | 8 | 7 | 0 | 4 | 5 | 6 | 0 | 0 | 1 | 1 | 1 | 2 | 1 | 2 | 3 | 3 | 4 | 4 |
| 8 | 19 | 16 | 5 | 4 | 3 | 2 | 1 | 3 | 2 | 4 | 2 | 5 | 6 | 3 | 2 | 6 | 4 | 3 | 4 | 4 | 5 | 5 | 5 |
| 9* | 19 | 15 | 1 | 1 | 2 | 1 | 3 | 3 | 0 | 2 | 3 | 4 | 2 | 2 | 3 | 3 | 2 | 3 | 2 | 2 | 3 | 3 | 4 |
| 10 | 19 | 16 | 6 | 4 | 3 | 2 | 2 | 3 | 3 | 2 | 3 | 5 | 5 | 4 | 2 | 5 | 4 | 2 | 3 | 4 | 3 | 4 | 3 |
| 11* | 18 | 14 | 1 | 2 | 2 | 1 | 0 | 2 | 1 | 0 | 1 | 2 | 4 | 1 | 0 | 4 | 1 | 1 | 1 | 2 | 2 | 4 | 3 |
| 12 | 18 | 15 | 0 | 6 | 10 | 14 | 19 | 10 | 18 | 15 | 15 | 9 | 11 | 16 | 17 | 1 | 2 | 8 | 12 | 5 | 8 | 6 | 9 |
| 13 | 18 | 17 | 6 | 5 | 4 | 4 | 3 | 3 | 3 | 2 | 2 | 3 | 3 | 2 | 2 | 4 | 2 | 2 | 2 | 3 | 2 | 3 | 3 |
| 14* | 18 | 13 | 3 | 2 | 2 | 1 | 1 | 3 | 3 | 2 | 0 | 2 | 2 | 1 | 1 | 5 | 2 | 2 | 2 | 3 | 2 | 4 | 2 |
| 15* | 18 | 13 | 1 | 2 | 1 | 2 | 8 | 1 | 1 | 7 | 8 | 0 | 4 | 6 | 6 | 2 | 2 | 5 | 7 | 4 | 5 | 5 | 6 |
| 16 | 18 | 18 | 5 | 4 | 4 | 3 | 2 | 3 | 2 | 2 | 1 | 2 | 2 | 2 | 1 | 3 | 2 | 1 | 1 | 1 | 1 | 1 | 1 |
| 17 | 18 | 13 | 3 | 3 | 2 | 2 | 0 | 3 | 3 | 1 | 1 | 3 | 2 | 4 | 1 | 5 | 4 | 2 | 1 | 3 | 1 | 1 | 1 |
| 18* | 17 | 12 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 4 | 1 | 0 | 1 | 2 | 2 | 1 | 2 | 2 | 1 | 2 | 1 | 3 |
| 19** | 17 | 19 | 10 | 10 | 9 | 8 | 7 | 7 | 6 | 4 | 4 | 5 | 5 | 3 | 4 | 4 | 4 | 4 | 2 | 4 | 1 | 3 | 2 |
| 20 | 17 | 14 | 4 | 4 | 4 | 4 | 3 | 6 | 4 | 4 | 3 | 4 | 3 | 2 | 2 | 5 | 5 | 3 | 1 | 3 | 2 | 2 | 1 |
| 21* | 17 | 11 | 1 | 1 | 1 | 2 | 0 | 1 | 0 | 2 | 0 | 2 | 1 | 0 | 1 | 0 | 2 | 1 | 3 | 3 | 1 | 3 | 2 |
| 22 | 17 | 12 | 3 | 3 | 9 | 12 | 13 | 6 | 11 | 15 | 19 | 10 | 17 | 15 | 16 | 10 | 16 | 18 | 18 | 5 | 10 | 6 | 5 |
| 23** | 17 | 21 | 24 | 23 | 22 | 20 | 19 | 21 | 19 | 14 | 14 | 17 | 15 | 11 | 9 | 12 | 9 | 8 | 4 | 7 | 5 | 7 | 4 |
| 24 | 17 | 14 | 2 | 2 | 2 | 1 | 2 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 2 | 3 | 2 | 0 | 3 | 0 | 2 | 0 |
| 25* | 17 | 11 | 1 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 2 | 2 | 1 | 1 | 0 | 1 | 2 | 0 | 0 | 2 | 0 | 3 | 2 |
| 26* | 16 | 11 | 3 | 3 | 3 | 3 | 6 | 3 | 3 | 6 | 9 | 2 | 5 | 9 | 8 | 0 | 6 | 6 | 6 | 3 | 5 | 3 | 4 |
| 27** | 16 | 15 | 5 | 4 | 4 | 4 | 4 | 6 | 6 | 6 | 5 | 4 | 3 | 2 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 3 | 2 |
| 28 | 16 | 11 | 4 | 4 | 1 | 0 | 0 | 4 | 4 | 3 | 0 | 4 | 4 | 4 | 3 | 3 | 3 | 4 | 1 | 6 | 1 | 6 | 1 |
| 29* | 16 | 10 | 2 | 2 | 2 | 2 | 1 | 2 | 2 | 1 | 4 | 1 | 2 | 4 | 3 | 2 | 0 | 0 | 1 | 1 | 2 | 4 | 3 |
| 30** | 16 | 17 | 20 | 19 | 19 | 18 | 17 | 21 | 21 | 20 | 16 | 17 | 15 | 14 | 12 | 15 | 13 | 9 | 7 | 9 | 8 | 8 | 5 |
| 31 | 16 | 11 | 5 | 4 | 4 | 2 | 2 | 4 | 4 | 4 | 4 | 6 | 6 | 5 | 5 | 5 | 4 | 3 | 4 | 5 | 4 | 6 | 3 |
| 32* | 16 | 9 | 3 | 3 | 2 | 3 | 2 | 2 | 2 | 2 | 1 | 2 | 1 | 0 | 3 | 3 | 1 | 0 | 1 | 3 | 2 | 6 | 5 |
| 33 | 15 | 10 | 6 | 6 | 7 | 11 | 13 | 5 | 9 | 12 | 13 | 7 | 10 | 11 | 18 | 8 | 15 | 15 | 14 | 13 | 17 | 5 | 12 |
| 34 | 15 | 12 | 7 | 7 | 8 | 8 | 5 | 7 | 6 | 6 | 6 | 8 | 8 | 8 | 4 | 6 | 4 | 3 | 1 | 6 | 5 | 5 | 4 |
| 35* | 15 | 9 | 2 | 1 | 1 | 1 | 3 | 1 | 0 | 0 | 1 | 2 | 2 | 0 | 0 | 2 | 2 | 2 | 0 | 2 | 3 | 4 | 5 |
| 36 | 15 | 9 | 3 | 3 | 4 | 3 | 4 | 3 | 3 | 3 | 6 | 3 | 2 | 4 | 6 | 1 | 2 | 7 | 7 | 4 | 5 | 2 | 3 |
| 37** | 15 | 14 | 16 | 16 | 15 | 14 | 15 | 17 | 17 | 16 | 14 | 20 | 20 | 16 | 16 | 15 | 13 | 13 | 10 | 10 | 8 | 7 | 6 |
| 38 | 15 | 9 | 4 | 2 | 1 | 1 | 1 | 4 | 2 | 1 | 1 | 4 | 4 | 4 | 1 | 5 | 5 | 5 | 2 | 4 | 6 | 6 | 5 |
| 39* | 15 | 9 | 3 | 3 | 4 | 4 | 3 | 2 | 3 | 4 | 3 | 2 | 2 | 3 | 6 | 0 | 0 | 3 | 7 | 0 | 3 | 2 | 3 |
| 40 | 14 | 10 | 8 | 6 | 5 | 5 | 4 | 8 | 8 | 5 | 5 | 7 | 8 | 7 | 8 | 10 | 10 | 8 | 6 | 6 | 4 | 6 | 6 |
| 41* | 14 | 8 | 3 | 3 | 2 | 3 | 3 | 3 | 1 | 1 | 3 | 1 | 2 | 1 | 2 | 3 | 1 | 1 | 3 | 1 | 0 | 3 | 1 |
| 42 | 14 | 10 | 8 | 7 | 7 | 11 | 11 | 7 | 9 | 10 | 12 | 5 | 10 | 10 | 13 | 7 | 9 | 11 | 15 | 13 | 12 | 12 | 14 |
| 43** | 14 | 11 | 16 | 16 | 14 | 13 | 13 | 16 | 15 | 16 | 13 | 16 | 16 | 14 | 14 | 20 | 18 | 17 | 16 | 13 | 11 | 10 | 7 |
| 44 | 14 | 8 | 4 | 4 | 3 | 1 | 3 | 3 | 3 | 1 | 2 | 4 | 3 | 3 | 2 | 6 | 7 | 3 | 2 | 6 | 5 | 5 | 9 |
| 45 | 14 | 9 | 3 | 5 | 5 | 6 | 6 | 4 | 5 | 4 | 5 | 2 | 4 | 4 | 5 | 1 | 2 | 3 | 8 | 2 | 5 | 3 | 6 |
| 46 | 13 | 9 | 6 | 5 | 5 | 4 | 2 | 5 | 5 | 4 | 2 | 8 | 5 | 5 | 4 | 7 | 6 | 8 | 4 | 9 | 7 | 6 | 5 |
| 47* | 13 | 8 | 4 | 3 | 5 | 6 | 6 | 3 | 5 | 6 | 6 | 1 | 3 | 4 | 6 | 1 | 2 | 2 | 5 | 2 | 2 | 0 | 5 |
| 48 | 13 | 10 | 13 | 13 | 12 | 10 | 9 | 13 | 12 | 12 | 9 | 13 | 14 | 12 | 11 | 15 | 13 | 11 | 12 | 17 | 16 | 14 | 11 |
| 49* | 13 | 8 | 5 | 4 | 3 | 2 | 3 | 4 | 2 | 2 | 3 | 3 | 1 | 2 | 2 | 3 | 3 | 1 | 2 | 3 | 1 | 2 | 0 |
| 50 | 13 | 9 | 8 | 8 | 8 | 11 | 11 | 6 | 8 | 11 | 11 | 5 | 9 | 10 | 10 | 3 | 7 | 8 | 10 | 5 | 10 | 11 | 12 |
| 51 | 12 | 8 | 6 | 4 | 3 | 2 | 2 | 4 | 4 | 2 | 2 | 5 | 3 | 1 | 2 | 5 | 5 | 3 | 1 | 5 | 4 | 3 | 2 |
| 52 | 12 | 9 | 1 | 2 | 3 | 3 | 5 | 1 | 2 | 3 | 5 | 1 | 1 | 2 | 3 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 3 |
| 53 | 12 | 9 | 13 | 11 | 10 | 9 | 8 | 12 | 10 | 10 | 8 | 12 | 12 | 9 | 8 | 11 | 11 | 11 | 7 | 12 | 10 | 14 | 14 |
| 54 | 12 | 8 | 1 | 1 | 2 | 3 | 3 | 1 | 2 | 3 | 3 | 0 | 1 | 2 | 3 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| 55 | 11 | 8 | 1 | 1 | 1 | 1 | 3 | 0 | 0 | 1 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| 56 | 11 | 10 | 6 | 6 | 7 | 7 | 8 | 5 | 4 | 5 | 8 | 4 | 3 | 4 | 5 | 3 | 4 | 4 | 5 | 3 | 3 | 3 | 7 |
| 57 | 11 | 9 | 6 | 7 | 7 | 7 | 7 | 6 | 6 | 7 | 7 | 6 | 6 | 6 | 5 | 7 | 5 | 5 | 5 | 6 | 5 | 6 | 5 |
| 58 | 11 | 9 | 0 | 1 | 2 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| 59 | 10 | 9 | 1 | 2 | 2 | 3 | 4 | 1 | 1 | 2 | 2 | 1 | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 |
| 60 | 10 | 10 | 7 | 7 | 7 | 8 | 9 | 7 | 7 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 6 |
| 61** | 10 | 11 | 7 | 7 | 7 | 6 | 6 | 6 | 6 | 5 | 6 | 5 | 5 | 6 | 6 | 5 | 3 | 4 | 5 | 4 | 5 | 4 | 4 |
| 62** | 9 | 11 | 2 | 3 | 3 | 3 | 2 | 2 | 3 | 3 | 2 | 2 | 3 | 2 | 2 | 2 | 3 | 2 | 2 | 3 | 2 | 2 | 1 |
| 63** | 9 | 11 | 7 | 7 | 7 | 7 | 7 | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 3 | 6 | 6 | 6 | 3 | 6 | 5 | 6 | 3 |
| 64** | 8 | 14 | 10 | 10 | 10 | 9 | 9 | 9 | 10 | 9 | 8 | 9 | 9 | 7 | 7 | 8 | 7 | 7 | 7 | 6 | 6 | 5 | 5 |
| 65** | 8 | 13 | 5 | 5 | 6 | 5 | 2 | 5 | 6 | 5 | 2 | 5 | 5 | 4 | 2 | 5 | 5 | 2 | 2 | 4 | 1 | 3 | 1 |
| 66** | 7 | 17 | 12 | 12 | 12 | 11 | 8 | 12 | 11 | 7 | 7 | 11 | 9 | 7 | 7 | 9 | 9 | 6 | 6 | 5 | 6 | 4 | 3 |

* Efficient in parameter space; ** Region III as defined in Figure I

Table 3 shows the relative frequency of sample trials in which each truly efficient portfolio (in parameter space) is dominated[4] by each of the remaining 65 portfolios. In this table, portfolio A is said to dominate Portfolio B in a sample trial if the following conditions are met:

(1) Portfolio A appears on the efficient frontier

and

(2) $\tilde{E}_A \geqq \tilde{E}_B$ and $\tilde{V}_A < \tilde{V}_B$

or

(3) $\tilde{E}_A > \tilde{E}_B$ and $\tilde{V}_A \leqq \tilde{V}_B$ .

In the table, rows that correspond to the inferior portfolios of Region III are marked with a double asterisk; these rows show that Region III portfolios dominate the truly efficient portfolios in a high proportion of trials. Investors with different risk preferences would choose different points on the efficient frontier. Table 3 shows that, in sample trials, each inferior portfolio may dominate any efficient portfolio on the true efficient frontier. Moreover, for each portfolio in Region III, the proportion of sample trials for which it dominates a truly efficient portfolio tends to be high for any point on the efficient frontier.

Most Region III portfolios involve fairly high concentrations of security 1, Chrysler, as shown by the weights column of Table 2. Chrysler offers a respectable return but has a high variance. Because of this high variance, both mean and variance estimates of portfolios concentrated in Chrysler are subject to relatively more error than other portfolios. It is not surprising, therefore, that in sample trials, portfolios such as these are found throughout mean-variance space, including the entire length of the efficient frontier.

## V. Conclusions

Present models for selecting portfolios according to the mean-variance criteria do not account for error in estimating the required parameters. The experimental results reported in this paper demonstrate that this error is, potentially, of sufficient importance to bring into question the usefulness of models that ignore it.

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[4]Each column denotes a truly efficient portfolio. Rows that denote truly efficient portfolios are marked with a single asterisk, as in Table 2. Rows that denote Region III portfolios, from Figure I, are marked with a double asterisk.

In the experiment, only sampling error for a multivariate normal process, which is invariant to time, is allowed. This error can only be magnified when more realistic conditions, such as estimation on the basis of judgment [8], and time dependency, are taken into account. Even in the grossly oversimplified world of this experiment, however, a mean-variance approach performed poorly. One obvious question to be asked is: Under realistic conditions, are portfolios selected according to mean-variance criteria any more likely to be efficient (in the sense of the model) than portfolios that are selected at random? The experimental results reported in this paper, while not conclusive, strongly suggest that the answer is "no."

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