Improved Estimation for Markowitz Portfolios using James-Stein Estimators, 1979

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IMPROVED ESTIMATION FOR MARKOWITZ PORTFOLIOS

USING JAMES-STEIN TYPE ESTIMATORS

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ABSTRACT

Given a set of N assets a portfolio is determined by a set of weights X_i, i = 1, 2, ..., N; $\sum_{i=1}^{N} X_i = 1$ indicating the proportion of the value of the portfolio devoted to each asset. A Markowitz Efficient Portfolio is the vector of weights $\underset{\sim}{X}_m$ that minimizes the variance σ_m^2 of the total return from the portfolio, subject to the condition that the portfolio mean premium return μ_m has a certain value.

The estimators for the Nx1 vector $\underset{\sim}{X}_m$, the return premium μ_m and the variance σ_m^2 are ratios of functions of estimators of the mean premium return vector $\underset{\sim}{\mu}$ and the inverse of the covariance matrix $\underset{\sim}{\Lambda}$. The performance of these ratios as estimators depends on the magnitude of the coefficient of variation of the ratio denominators. A variety of James-Stein type estimators for $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$ are used to reduce the mean square error. Under the assumption that returns are normally distributed a Monte Carlo simulation is used to compare the various estimators for a sample of N=20 stocks, at sample sizes T=60 and T=100.

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1. INTRODUCTION

The theory of portfolio analysis involves the determination of sets of assets that are efficient in a risk-return space. Efficient portfolios are those combinations of assets which have, maximum return for a given level of risk or alternatively, minimum risk for a given level of return. In the Markowitz formulation of portfolio analysis, the measures of return and risk are the mean and variance of the portfolios' returns. The objects of choice are therefore, mean and variance because investors possess quadratic preference functions for return or alternatively the distributions of asset returns are completely specified by their first two moments.

The number of assets available to investors is nearly limitless. The available assets range from riskless zero variance securities (Federal Government bills and zero coupon bonds held to maturity), to various other financial securities and real assets. The majority of financial research has restricted itself to financial assets, and more often common stocks.

An efficient portfolio (allowing unrestricted short sales of assets) is determined by minimizing portfolio variance, subject to a mean portfolio premium return and the additional constraint that investment proportions in risky assets sum to one. The Lagrangian is

$$\min_{\underset{\sim}{X}} L = \underset{\sim}{X}' \underset{\sim}{\Sigma} \underset{\sim}{X} - \lambda_1\{\underset{\sim}{X}'\underset{\sim}{\mu} - \mu_m\} - \lambda_2\{\underset{\sim}{X}'\underset{\sim}{e}-1\} \tag{1.1}$$

where $\underset{\sim}{X}$ is an N x 1 vector representing the unknown proportions invested in risky positive variance assets.

$\underset{\sim}{\Sigma}$ is the N x N covariance matrix of risky assets with rank r = N, the number of assets.

$\underset{\sim}{\mu} = (\underset{\sim}{\mu}^* - \underset{\sim}{e}\, E_z)$ is the N x 1 vector of asset mean return premiums, where $\underset{\sim}{\mu}^*$ is the mean return vector, $\underset{\sim}{e}$ is the unit vector and E_z is the fixed return on the riskless asset z.

μ_m is the desired mean premium return = (mean portfolio return $- E_z$).

Solution of (1.1) gives the investment proportions or weight vector $\underset{\sim}{X}_m$, mean premium return μ_m and variance σ_m^2 of the optimal portfolio m. (See Merton, 1972 and Roll, 1977 for a summary.)

$$\underset{\sim}{X}_m = \frac{\underset{\sim}{\Lambda}\,\underset{\sim}{\mu}}{b} = \frac{\underset{\sim}{Y}_m}{b}$$

$$\mu_m = \underset{\sim}{\mu}'\,\underset{\sim}{X}_m = \frac{a}{b} \tag{1.2}$$

$$\sigma_m^2 = \underset{\sim}{X}_m'\,\underset{\sim}{\Sigma}\,\underset{\sim}{X}_m = \frac{a}{b^2}$$

where $\underset{\sim}{\Lambda}$ is the inverse of Σ, $\underset{\sim}{Y}_m = \underset{\sim}{\Lambda}\,\underset{\sim}{\mu}$, is the nonstandardized weight vector, $a = \underset{\sim}{\mu}'\,\underset{\sim}{\Lambda}\,\underset{\sim}{\mu}$ and $b = \underset{\sim}{e}'\,\underset{\sim}{\Lambda}\,\underset{\sim}{\mu}$.

The portfolio m, for given E_z, is a unique Markowitz portfolio, which could be combined with investment in the riskless asset z to produce portfolios which are termed Sharpe (1964) efficient. However, the solutions (1.2) to this optimization problem do not depend upon the existence of a zero variance asset, with return E_z. In the absence of a zero variance asset, E_z can be thought of as the mean return from a positive variance portfolio, whose return is orthogonal to the return on the unique portfolio m. In the parlance of financial economics, E_z is referred to as the return on the zero beta portfolio (see Black, 1972 for a discussion).

A major problem, that belies the implementation of this normative theory of portfolio analysis, is the formation of rational expectations regarding the mean return premium vector $\underset{\sim}{\mu}$ and the covariance matrix $\underset{\sim}{\Sigma}$, that is appropriate for the investors' holding period. In this paper, we assume that the returns from the N stocks are stationary random time series, which are distributed as a multivariate normal with mean $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$. While the assumption of multivariate normality is suspect for daily and weekly returns, the distribution does not seem to be significantly different from a normal for monthly returns. (See Fama, 1976 for a review of the evidence on normality and stationarity.) Given that the time series of monthly returns is stationary,sample estimates of the mean return vector and covariance matrix may be obtained from past returns of the N assets under consideration.

Using a simulation approach for three stocks, Frankfurter et al. (1971) conclude that, since sampling error is so large, portfolios selected according to the Markowitz criterion are likely not more efficient than an equally weighted portfolio. For the two asset case, Dickenson (1974a and 1974b) indicates poor reliability of the estimators of the proportions vector and variance of the global minimum risk portfolio. In addition, for the two asset case a preliminary indication of the bias in estimating the weight vector of Markowitz is provided. A number of authors, including Barry (1974, 1978) and Klein and Bawa (1976, 1977) have addressed the effect of estimation risk on efficient portfolios. However, little research has been produced which deals with the sampling distributions of the relevant estimators.

Jobson and Korkie (1978a, 1978b) obtained the asymptotic distributions of conventional estimators of μ_m, σ_m^2 and $\underset{\sim}{X}_m$. In addition, Taylor series approximations to the expectations and variances of the estimators were derived. A Monte Carlo simulation showed that, for sample sizes T=60 and T=100, the simulation moments of the estimators substantially exceeded the approximate moments.

This paper examines the behavior of a class of estimators based on James-Stein estimators of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$. Section 2 provides a summary of the estimation problem and discusses the implementation of James-Stein estimators. The results of a Monte Carlo experiment designed to examine the sampling behavior of these estimators for sample sizes T=60 and T=100 are discussed in Section 3. Section 4 is the conclusion.

2. ESTIMATION OF THE PARAMETERS

2.1 The Estimation Problem

Assume a random sample of T return premium observations on each of N stocks which is denoted by the N x 1 vector $\underset{\sim}{r}_t$, t = 1, 2, ..., T. Now assume that r_t is multivariate normal with mean vector $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$ and denote the sample mean vector and covariance matrix by

$$\underset{\sim}{\bar{r}} = \frac{1}{T} \sum_{t=1}^{T} \underset{\sim}{r}_t \quad \text{and} \quad \underset{\sim}{V} = \frac{1}{(T-1)} \sum_{t=1}^{T} (\underset{\sim}{r}_t - \underset{\sim}{\bar{r}})(\underset{\sim}{r}_t - \underset{\sim}{\bar{r}})' \,.$$

The parameters μ_m, σ_m^2 and $\underset{\sim}{X}_m$ are functions of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$. Best unbiased estimators of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$ are provided by $\underset{\sim}{\bar{r}}$ and $\underset{\sim}{W} = \frac{(T-N-2)}{(T-1)} \underset{\sim}{V}^{-1}$.

Estimators of μ_m, σ_m^2 and $\underset{\sim}{X}_m$ may be obtained by replacing $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$ in (1.2) by $\underset{\sim}{\bar{r}}$ and $\underset{\sim}{W}$.

The asymptotic distributions of these ratio estimators are presented in Jobson and Korkie (1978a). The sampling distributions of the estimators for sample sizes T=60, 100, 300, 500 and 1000 were also studied using a Monte Carlo experiment. A sample size of 300 was found to be necessary before the simulation means and variances for $\underset{\sim}{X}_m$ were comparable to their asymptotic counterparts. For μ_m a sample of 500 was required, while for σ_m^2 a sample of size 1000 was required. The lack of agreement with the asymptotic results was shown to be consistent with results obtained by Hayya et al. (1975) for the sampling behavior of ratios of random variables.

In Jobson and Korkie (1978b), approximate expressions using Taylor series are obtained for the bias and variance for the estimators of μ_m, σ_m^2 and $\underset{\sim}{X}_m$ including terms of $0(\frac{1}{T^2})$. A Monte Carlo experiment showed that a sample size of 300 was required for the Monte Carlo means and variances to be comparable to the approximate expressions. For samples of size 60 and 100 the simulation means and variances far exceeded the approximate expressions. The Taylor series approximations for the means and variances are shown to require sample sizes T of at least 140 for convergence.

The degree of instability in the three estimators was shown to be related to the magnitudes of the coefficient of variation of the estimators of the denominators b and b^2 of μ_m, σ_m^2 and $\underset{\sim}{X}_m$ given by (1.2). Estimators which bring about a reduction in the coefficient of variation of the estimators of b and b^2 might therefore be preferable. In the next section we discuss the use of James-Stein type estimators of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$.

2.2 James-Stein Type Estimators

The sample quantities $\bar{r}$ and W are the best unbiased estimators of μ and Λ respectively, while the maximum likelihood estimators are given by $\bar{r}$ and $(\frac{T}{T-1})V^{-1}$. For the special case $\Sigma = \sigma^2 I$, σ^2 known, and $N > 2$, Stein (1955) showed that the maximum likelihood estimator $\bar{r}$ of μ is inadmissable for the squared error loss function $L = (\bar{r} - \mu)'(\bar{r} - \mu)$. James and Stein (1960) showed that for this loss function the estimator

$\delta = (1 - \frac{(N-2)\sigma^2}{\bar{r}'\bar{r}})\,\bar{r}$ dominates the estimator $\bar{r}$. The estimator δ has a risk of 2 at $\mu=0$ and increases gradually with $\mu'\mu$ towards the value of the risk of $\bar{r}$ as $\mu'\mu$ approaches ∞. For the estimator

$\delta_o = \mu_o + (1 - \frac{(N-2)\sigma^2}{\bar{r}'\bar{r}})\,(\bar{r} - \mu_o)$ the risk is minimum at some arbitrary μ

If σ^2 is unknown it may be replaced by an independent estimate s^2 and the result extends to

$\delta_s = \mu_o + (1 - \frac{(\frac{N-2}{T+2})s^2}{\bar{r}'\bar{r}})\,(\bar{r} - \mu_o)$. For the case of the general covariance matrix Σ, the loss function is changed to $L = (\bar{r} - \mu)'\ \Lambda\ (\bar{r} - \mu)$. If a Wishart matrix S is an independent estimate of Σ with $(T-1)$ degrees of freedom and expectation $(T-1)\Sigma$ the estimator becomes

$\delta_s^* = \mu_o + (1 - \frac{(\frac{N-2}{T-N+3})}{\bar{r}'S^{-1}\bar{r}})\,(\bar{r} - \mu_o)$. Once again δ_s^* is admissable while $\bar{r}$ is not.

Since it is possible for $\bar{r}'S^{-1}\bar{r} < (\frac{N-2}{T-N-3})$ in the expression for δ^*_S, an improved estimator employing only the positive part of $(1 - \frac{(\frac{N-2}{T-N-3})}{\bar{r}'S^{-1}\bar{r}})$ was suggested by Stein (1962). The revised estimator is given by

$$\delta^{*+}_S = \mu_0 + (1 - \frac{(\frac{N-2}{T-N-3})}{\bar{r}'S^{-1}\bar{r}})^+ (\bar{r} - \mu_0) \text{ where } (1 - \frac{(\frac{N-2}{T-N-3})}{\bar{r}'S^{-1}\bar{r}})^+ = \max\left\{0,\ (1 - \frac{(\frac{N-2}{T-N-3})}{\bar{r}'S^{-1}\bar{r}})\right\}$$

The estimator may also be written

$$\delta^{*+}_S = \alpha\, \mu_0 + (1 - \alpha)\, \bar{r} \qquad (2.1)$$

where $\alpha = \min\left\{1,\ \frac{(\frac{N-2}{T-N-3})}{\bar{r}'S^{-1}\bar{r}}\right\}$.

The connection between the J-S type estimator and the Bayes estimator was observed by Lindley (1962). Bayesian derivations of J-S estimators were presented by Efron and Morris (1973) and Lin and Tsai (1973). Refinements of the J-S type estimators have been discussed recently by Bock (1975), Efron and Morris (1976), Berger (1976), Berger and Bock (1976), Berger, Bock, Brown, Cosella and Glesser (1977) and Alam (1977). In some cases, a more arbitrary loss function $L = (\bar{r} - \mu)'\, Q\, (\bar{r} - \mu)$ is employed where Q is a known positive definite matrix.

A demonstration of the application of J-S type estimators was given by Efron and Morris (1975). In place of μ_0 in (2.1) they employed the average of the elements of $\bar{r}$, $\left[\frac{e'\bar{r}}{N}\right]$. The estimator (2.1) becomes

$$\delta_\alpha = \alpha \left[\frac{e'\bar{r}}{N}\right] + (1 - \alpha)\, \bar{r} \qquad (2.2)$$

Estimation of the inverse of the covariance matrix, $\underset{\sim}{\Lambda}$, may also be improved by using estimators analagous to the J-S estimators for $\underset{\sim}{\mu}$. Efron and Morris (1976) employing the loss function

$$L = \frac{\mathrm{tr}\ (\hat{\underset{\sim}{\Lambda}} - \underset{\sim}{\Lambda})\ \underset{\sim}{S}}{(T-1)\,\mathrm{tr}(\underset{\sim}{\Lambda})}$$

showed that the estimator $\hat{\underset{\sim}{\Lambda}}_0 = \underset{\sim}{W} + \dfrac{(N^2+N-2)\ \underset{\sim}{I}}{(T-1)\,\mathrm{tr}(\underset{\sim}{V})}$ is uniformly better than the conventional unbiased estimator $\underset{\sim}{W}$, where I is the identity matrix. The second term of the expression for $\hat{\underset{\sim}{\Lambda}}_0$ is a multiple of the best unbiased estimator of $\underset{\sim}{\Lambda}$ when $\underset{\sim}{\Sigma}$ is known to be proportional to the identity matrix. This estimator is given by

$\hat{\underset{\sim}{\Lambda}}_1 = \dfrac{(N(T-1)-2)}{(T-1)\,\mathrm{tr}(\underset{\sim}{V})}\ \underset{\sim}{I}$. A more general form therefore suggested by Efron and Morris (1976) is given by

$$\hat{\underset{\sim}{\Lambda}} = (1-\beta)\ \hat{\underset{\sim}{\Lambda}}_0 + \beta\ \hat{\underset{\sim}{\Lambda}}_1 \tag{2.3}$$

where $0 \le \beta \le 1$.

The parameters μ_m, σ_m^2 and $\underset{\sim}{X}_m$ are functions of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$. Although the estimators given by (2.2) and (2.3) with

$\alpha = \min \left\{ 1, \dfrac{\left(\frac{N-2}{T-N-3}\right)}{\bar{\underset{\sim}{r}}'\underset{\sim}{S}^{-1}\bar{\underset{\sim}{r}}} \right\}$ and $\beta = 0$ are preferred as estimators of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$, their performance in the estimation of μ_m, σ_m^2 and $\underset{\sim}{X}_m$ is unknown. In the next section we employ the estimators given by (2.2) and (2.3) allowing α and β to vary in the closed interval [0,1]. The various forms of the estimators are compared in a Monte Carlo experiment.

3. RESULTS OF A SAMPLING EXPERIMENT

3.1 Design of the Experiment

A Monte Carlo simulation was conducted to compare the performances of various forms of the J-S type estimators in the estimation of μ_m, σ_m^2 and $\underset{\sim}{X}_m$. The same asset population of N = 20 stocks used in Jobson and Korkie (1978a, 1978b) was used in this experiment. From the chosen multivariate normal population 500 random samples of size T=60 and T=100 were selected. All computations in the simulation were performed in double precision Fortran. The multivariate normal deviates were generated from subroutine GGNRM of the ISML Subroutine Library.

For each repetition, estimates of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$ were obtained using expressions (2.2) and (2.3). Both α and β were allowed to vary in increments of 0.10 over the closed interval [0,1] yielding a total of 121 cases. In addition for all 11 values of α the case employing the conventional estimator $\underset{\sim}{W}$ of $\underset{\sim}{\Lambda}$ was also used and will be referred to as the case β=*. As a result a total of 132 estimators of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Lambda}$ were employed. For each case the estimators of μ_m, σ_m^2 and $\underset{\sim}{X}_m$ were obtained using the expressions given by (1.2). The mean square error and average bias were determined for each case over the 500 trials.

3.2 Discussion of Results

Tables 1 through 6 summarize the results of the simulation experiment. Tables 1 and 2 show the squared bias (BSQ) and the mean square error (MSE) for the estimators of μ_m. For both sample sizes, and for each value of β, the values of BSQ decline monotonically to a minimum and then increase monotonically, as α varies from 0.0 to 1.0. The value of α for which BSQ = 0.0 depends on β. For T=60, the value of MSE when BSQ = 0.0, increases from a minimum of 0.18 at $\alpha=0.7$ and $\beta=0.0$ or $\beta=0.1$, while for T=100 MSE is smallest at $\alpha=0.7$, $\beta=*$ or $\alpha=0.6$ and $\beta=0.0$.

The performance of the estimators of σ_m^2 are shown in Tables 3 and 4 for T=60 and T=100. For each β, the values of MSE also decline monotonically to a minimum and then increase monotonically as α moves from 0.0 to 1.0. The smallest values of MSE correspond to $\alpha=1.0$, for T=60 and $.6 \leq \alpha \leq 1.0$ for T=100. For BSQ, the minimum α value is $0.0 \leq \alpha \leq 0.9$. The MSE values are smallest when $\beta=*$ and $\alpha=1.0$. For the case, $\beta=*$ and $\alpha=0.9$ relatively small values of BSQ and MSE occur simultaneously.

Tables 5 and 6 display the total squared bias (BSQ) and the total mean square error (MSE) for the estimators of the 20 elements of $\underset{\sim}{X}_m$. With the exception of $\beta=*$ and $\beta=1.0$, the minimum values of BSQ occur at $\alpha=0.0$. For $\beta=*$, the minimum values of BSQ occur when $0.2 \leq \alpha \leq 0.4$. For all values of β, the magnitude of MSE declines approximately monotonically as α moves from $\alpha=0.0$ to $\alpha=1.0$.

In Tables 5 and 6 BSQ values vary from 0.00 to 0.30. For a given level of BSQ, the smallest values of MSE are at small values of β or $\beta=*$. For values of BSQ near 0.0, minimum MSE values occur at $\beta=*$ and α near 0.3. If overall minimum MSE values are required these occur near $\beta=0.0$ and $\alpha=1.0$.

From the examination of the tables, we conclude that there is little to be gained by using an estimator of $\underset{\sim}{\Lambda}$ other than $\underset{\sim}{W}$. The ideal value of however, varies depending on which parameters μ_m, σ_m^2 and $\underset{\sim}{X}_m$ are being estimated and the desired objectives regarding BSQ relative to MSE. For $\underset{\sim}{X}_m$, values of α near $\alpha=0.30$ are desirable if BSQ is to be minimized while $\alpha=1.0$ yields the smallest values of MSE. For μ_m, a value of BSQ=0.0 and a minimum MSE are obtainable for α near 0.7. For σ_m^2, values of $\alpha=0.90$ or 1.0 appear to simultaneously keep BSQ and MSE minimal.

In comparison to the conventional estimators of μ_m, σ_m^2 and $\underset{\sim}{X}_m$ employed by Jobson and Korkie (1978a, 1978b) the Tables 1 through 6 show that substantial improvements in BSQ and MSE are obtainable through the use of James-Stein type estimators. The estimators corresponding to $\alpha=0.0$ and $\beta=*$ in each table gives the conventional estimator. For estimators of μ_m and σ_m^2 the values of BSQ and MSE for the conventional estimators are substantially larger than the minimum values in other parts of the tables. Similarly, for the elements of $\underset{\sim}{X}_m$ the MSE values are also much larger. For T=100, the BSQ for $\underset{\sim}{X}_m$ at $\beta=*$, $\alpha=0.0$ was close to 0.0; however, for T=60 the BSQ value is relatively large.

4. CONCLUSION

On the basis of this simulation experiment, it would appear that substantial reductions in bias and mean square error may be realized through the use of J-S type estimators for $\underset{\sim}{\mu}$. The J-S type estimators of $\underset{\sim}{\Lambda}$, however, did not seem to be advantageous. The particular form of the J-S estimator for $\underset{\sim}{\mu}$, which is most suitable, depends on whether one is estimating μ_m, σ_m^2 or $\underset{\sim}{X}_m$ and also in some cases on the tradeoff between bias and mean square error.

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1. Estimators of μ_m, T=60

β \ α		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
*	BSQ	52.27	18.49	9.12	4.49	2.02	0.77	0.21	0.02	0.00	0.04	0.05
	MSE	357.00	49.50	21.40	10.10	4.60	1.91	0.68	0.24	0.19	0.27	0.34
0.0	BSQ	5.11	3.17	1.82	0.96	0.44	0.15	0.03	0.00	0.02	0.04	0.05
	MSE	9.61	6.03	3.57	1.97	0.99	0.46	0.22	0.18	0.22	0.29	0.34
0.1	BSQ	4.08	2.53	1.46	0.76	0.34	0.12	0.02	0.00	0.02	0.04	0.05
	MSE	7.62	4.81	2.86	1.59	0.81	0.38	0.21	0.18	0.23	0.30	0.34
0.2	BSQ	3.31	2.04	1.17	0.61	0.26	0.08	0.01	0.00	0.02	0.04	0.05
	MSE	6.15	3.90	2.33	1.30	0.67	0.33	0.20	0.19	0.24	0.30	0.34
0.3	BSQ	2.69	1.66	0.94	0.49	0.21	0.06	0.01	0.00	0.02	0.04	0.05
	MSE	5.06	3.22	1.94	1.09	0.57	0.30	0.20	0.19	0.24	0.30	0.34
0.4	BSQ	2.19	1.35	0.77	0.38	0.16	0.04	0.00	0.00	0.03	0.04	0.05
	MSE	4.25	2.73	1.65	0.94	0.51	0.28	0.20	0.20	0.25	0.31	0.34
0.5	BSQ	1.82	1.10	0.62	0.31	0.13	0.03	0.00	0.01	0.03	0.04	0.05
	MSE	3.68	2.38	1.46	0.85	0.48	0.28	0.21	0.21	0.26	0.31	0.34
0.6	BSQ	1.48	0.92	0.52	0.25	0.10	0.02	0.00	0.01	0.03	0.05	0.05
	MSE	3.34	2.19	1.37	0.83	0.48	0.30	0.23	0.23	0.26	0.31	0.34
0.7	BSQ	1.28	0.77	0.44	0.21	0.08	0.02	0.00	0.01	0.03	0.05	0.05
	MSE	3.30	2.22	1.44	0.90	0.55	0.35	0.27	0.25	0.27	0.31	0.34
0.8	BSQ	1.10	0.67	0.37	0.18	0.07	0.01	0.00	0.01	0.03	0.05	0.05
	MSE	3.91	2.72	1.84	1.20	0.77	0.50	0.36	0.30	0.29	0.32	0.34
0.9	BSQ	1.06	0.66	0.36	0.18	0.07	0.01	0.00	0.01	0.03	0.04	0.05
	MSE	7.45	5.48	3.92	2.72	1.82	1.18	0.75	0.49	0.37	0.33	0.34
1.0	BSQ	24.01	15.37	9.30	3.46	2.59	1.10	0.35	0.05	0.00	0.03	0.05
	MSE	7610.00	4990.00	3110.00	1820.00	984.00	474.00	193.00	61.00	12.00	0.97	0.34

2. Estimators of μ_m, T=100

β \ α		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
*	BSQ	8.94	4.67	2.50	1.21	0.50	0.16	0.02	0.00	0.03	0.06	0.07
	MSE	38.30	10.20	4.96	2.40	1.06	0.40	0.14	0.10	0.16	0.23	0.27
0.0	BSQ	2.46	1.46	0.79	0.37	0.14	0.03	0.00	0.01	0.04	0.07	0.07
	MSE	4.24	2.54	1.42	0.72	0.33	0.15	0.10	0.13	0.19	0.24	0.27
0.1	BSQ	1.74	1.02	0.55	0.25	0.08	0.01	0.00	0.02	0.04	0.07	0.07
	MSE	2.99	1.80	1.01	0.51	0.24	0.13	0.11	0.14	0.20	0.25	0.27
0.2	BSQ	1.28	0.74	0.38	0.17	0.05	0.00	0.00	0.03	0.05	0.07	0.07
	MSE	2.19	1.32	0.74	0.38	0.19	0.11	0.11	0.15	0.21	0.25	0.27
0.3	BSQ	0.94	0.53	0.27	0.11	0.03	0.00	0.01	0.03	0.05	0.07	0.07
	MSE	1.65	0.99	0.56	0.29	0.16	0.11	0.12	0.16	0.21	0.25	0.27
0.4	BSQ	0.71	0.40	0.19	0.07	0.02	0.00	0.01	0.03	0.05	0.07	0.07
	MSE	1.27	0.76	0.43	0.23	0.14	0.11	0.13	0.17	0.22	0.25	0.27
0.5	BSQ	0.52	0.28	0.13	0.05	0.01	0.00	0.01	0.03	0.05	0.07	0.07
	MSE	1.00	0.60	0.34	0.19	0.13	0.12	0.14	0.18	0.22	0.26	0.27
0.6	BSQ	0.38	0.20	0.09	0.03	0.00	0.00	0.02	0.04	0.06	0.07	0.07
	MSE	0.80	0.49	0.28	0.17	0.13	0.12	0.15	0.19	0.23	0.26	0.27
0.7	BSQ	0.29	0.14	0.06	0.02	0.00	0.00	0.02	0.04	0.06	0.07	0.07
	MSE	0.66	0.41	0.24	0.16	0.13	0.13	0.16	0.20	0.23	0.26	0.27
0.8	BSQ	0.21	0.10	0.04	0.01	0.00	0.01	0.02	0.04	0.06	0.07	0.07
	MSE	0.57	0.35	0.22	0.15	0.13	0.14	0.17	0.21	0.24	0.26	0.27
0.9	BSQ	0.15	0.07	0.02	0.00	0.00	0.01	0.03	0.04	0.06	0.07	0.07
	MSE	0.50	0.32	0.21	0.15	0.14	0.15	0.18	0.21	0.24	0.26	0.27
1.0	BSQ	0.11	0.05	0.01	0.00	0.00	0.01	0.03	0.05	0.06	0.07	0.07
	MSE	0.47	0.30	0.20	0.16	0.15	0.16	0.19	0.22	0.24	0.26	0.27

3. Estimators of σ_m^2, T=60

β \ α		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
*	BSQ	150×10^4	29800	9330	3860	1670	670	222	157	46	9	23
	MSE	115×10^6	592000	247000	145000	84300	45300	21000	7640	1790	173	30
0.0	BSQ	205	90	29	3	2	17	41	68	95	116	132
	MSE	24000	16500	10900	6750	3890	2040	961	413	197	146	132
0.1	BSQ	106	41	9	0	8	27	51	78	102	122	139
	MSE	17300	11900	7790	4830	2790	1490	728	346	195	159	139
0.2	BSQ	52	15	1	3	16	37	61	85	11	13	145
	MSE	12600	8630	5660	3520	2060	1130	586	313	204	177	145
0.3	BSQ	24	4	0	8	24	45	68	92	113	131	150
	MSE	9260	6360	4190	2640	1580	904	511	310	228	205	150
0.4	BSQ	10	0	3	14	30	51	74	96	116	134	155
	MSE	6950	4810	3210	2070	1290	795	500	345	276	252	155
0.5	BSQ	4	0	6	18	35	55	76	97	117	134	160
	MSE	5440	3840	2650	1790	1200	813	577	446	380	346	160
0.6	BSQ	2	1	7	19	36	54	75	95	114	131	164
	MSE	4710	3480	2550	1870	1390	1060	850	719	639	574	164
0.7	BSQ	3	0	4	14	29	46	65	84	103	121	168
	MSE	5310	4260	3450	2840	2370	2040	1790	1620	1470	1290	168
0.8	BSQ	18	5	0	2	10	21	36	53	72	93	171
	MSE	11200	10100	9100	8310	7660	7110	6640	6200	5690	4700	171
0.9	BSQ	31	222	157	104	63	34	14	3	0	11	175
	MSE	99500	96100	92900	89800	86600	83100	79000	73500	64500	45100	175
1.0	BSQ	**	**	**	**	**	**	**	**	**	**	178
	MSE	**	**	**	**	**	**	**	**	**	**	178

** very large numbers

4. Estimators of σ_m^2, T=100

β \ α		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
*	BSQ	20100	3710	1617	730	303	102	21	0	6	19	25
	MSE	1960000	117000	48700	22100	9880	4190	1590	501	115	30	28
0.0	BSQ	132	52	12	0	5	21	42	64	83	96	101
	MSE	7240	4540	2760	1610	894	468	237	131	98	98	101
0.1	BSQ	35	8	0	6	20	40	61	81	98	110	114
	MSE	3990	2600	1640	996	577	324	187	125	108	111	114
0.2	BSQ	4	0	7	20	38	58	78	96	111	121	125
	MSE	2480	1660	1080	672	409	249	163	126	118	122	125
0.3	BSQ	1	7	20	37	56	75	94	110	123	132	135
	MSE	1670	1140	754	488	313	200	152	130	127	132	135
0.4	BSQ	8	20	36	53	72	90	107	121	133	141	143
	MSE	1190	825	561	377	257	186	149	136	136	141	143
0.5	BSQ	20	35	51	69	86	103	119	132	142	148	151
	MSE	892	631	442	310	225	175	150	143	144	148	151
0.6	BSQ	34	49	66	83	99	115	129	141	150	155	157
	MSE	705	509	367	269	206	170	153	149	151	155	157
0.7	BSQ	48	63	80	96	111	125	138	149	157	161	163
	MSE	589	434	322	246	197	169	157	155	158	161	163
0.8	BSQ	61	76	92	107	122	135	146	156	163	167	168
	MSE	525	393	298	234	193	171	162	161	164	167	168
0.9	BSQ	72	88	103	118	131	143	154	162	168	172	173
	MSE	503	378	289	230	193	174	167	166	169	172	173
1.0	BSQ	83	98	113	127	139	151	160	168	173	177	178
	MSE	522	386	293	232	196	178	171	171	174	177	178

5. Estimators of $\underset{\sim}{X}m$, T=60

β \ α		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
*	BSQ	0.31	0.06	0.02	0.01	0.01	0.02	0.03	0.05	0.07	0.09	0.12
	MSE	60.20	10.70	6.41	4.39	3.12	2.21	1.53	1.02	0.65	0.42	0.33
0.0	BSQ	0.07	0.08	0.08	0.09	0.09	0.10	0.12	0.13	0.14	0.16	0.18
	MSE	1.72	1.43	1.18	0.96	0.77	0.61	0.47	0.37	0.29	0.24	0.22
0.1	BSQ	0.09	0.09	0.10	0.11	0.11	0.12	0.13	0.14	0.16	0.17	0.19
	MSE	1.36	1.14	0.95	0.79	0.64	0.52	0.42	0.34	0.28	0.24	0.22
0.2	BSQ	0.11	0.11	0.12	0.12	0.13	0.14	0.15	0.16	0.17	0.19	0.21
	MSE	1.09	0.93	0.79	0.66	0.55	0.46	0.38	0.32	0.27	0.24	0.23
0.3	BSQ	0.13	0.13	0.14	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.22
	MSE	0.89	0.77	0.66	0.57	0.48	0.41	0.35	0.30	0.27	0.25	0.23
0.4	BSQ	0.14	0.15	0.15	0.16	0.17	0.17	0.18	0.19	0.20	0.21	0.23
	MSE	0.75	0.65	0.57	0.50	0.44	0.38	0.33	0.30	0.27	0.25	0.24
0.5	BSQ	0.16	0.16	0.17	0.18	0.18	0.19	0.20	0.21	0.22	0.23	0.24
	MSE	0.64	0.57	0.51	0.45	0.40	0.36	0.33	0.30	0.28	0.26	0.25
0.6	BSQ	0.18	0.18	0.19	0.19	0.20	0.21	0.21	0.22	0.24	0.25	0.25
	MSE	0.57	0.52	0.47	0.43	0.39	0.35	0.33	0.30	0.29	0.28	0.26
0.7	BSQ	0.19	0.20	0.20	0.21	0.21	0.22	0.23	0.23	0.25	0.26	0.27
	MSE	0.53	0.49	0.45	0.42	0.39	0.36	0.34	0.32	0.30	0.29	0.27
0.8	BSQ	0.21	0.21	0.22	0.22	0.23	0.23	0.24	0.25	0.26	0.27	0.28
	MSE	0.54	0.51	0.48	0.45	0.42	0.39	0.37	0.35	0.34	0.33	0.28
0.9	BSQ	0.22	0.22	0.23	0.23	0.24	0.24	0.25	0.26	0.26	0.27	0.29
	MSE	0.73	0.69	0.65	0.62	0.58	0.55	0.52	0.50	0.47	0.43	0.29
1.0	BSQ	0.80	0.67	0.56	0.47	0.39	0.33	0.29	0.27	0.26	0.27	0.30
	MSE	418.00	338.00	267.00	205.00	151.00	105.00	67.00	37.80	16.90	4.44	0.30

6. Estimators of $\underset{\sim}{\chi}_m$, T=100

β \ α		0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
*	BSQ	0.03	0.01	0.00	0.00	0.01	0.02	0.03	0.05	0.07	0.09	0.12
	MSE	6.09	2.87	2.02	1.47	1.07	0.77	0.55	0.39	0.28	0.22	0.23
0.0	BSQ	0.04	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.12	0.14	0.16
	MSE	1.02	0.83	0.68	0.55	0.44	0.35	0.28	0.23	0.19	0.18	0.19
0.1	BSQ	0.07	0.07	0.08	0.08	0.09	0.10	0.11	0.13	0.14	0.16	0.17
	MSE	0.75	0.63	0.53	0.44	0.36	0.30	0.25	0.22	0.19	0.19	0.20
0.2	BSQ	0.09	0.10	0.10	0.11	0.12	0.13	0.14	0.15	0.16	0.18	0.19
	MSE	0.59	0.50	0.43	0.37	0.32	0.27	0.24	0.21	0.20	0.20	0.21
0.3	BSQ	0.12	0.12	0.13	0.14	0.14	0.15	0.16	0.17	0.18	0.20	0.21
	MSE	0.48	0.42	0.37	0.33	0.29	0.26	0.23	0.22	0.21	0.21	0.22
0.4	BSQ	0.14	0.15	0.15	0.16	0.17	0.17	0.18	0.19	0.20	0.21	0.22
	MSE	0.41	0.37	0.33	0.30	0.27	0.25	0.24	0.23	0.22	0.22	0.23
0.5	BSQ	0.16	0.17	0.17	0.18	0.19	0.19	0.20	0.21	0.22	0.23	0.24
	MSE	0.37	0.34	0.31	0.29	0.27	0.25	0.24	0.23	0.23	0.23	0.24
0.6	BSQ	0.18	0.19	0.19	0.20	0.21	0.21	0.22	0.23	0.24	0.24	0.25
	MSE	0.34	0.32	0.30	0.28	0.27	0.26	0.25	0.25	0.24	0.25	0.25
0.7	BSQ	0.20	0.21	0.21	0.22	0.22	0.23	0.24	0.24	0.25	0.26	0.26
	MSE	0.32	0.31	0.29	0.28	0.27	0.26	0.26	0.26	0.26	0.26	0.27
0.8	BSQ	0.22	0.23	0.23	0.24	0.24	0.25	0.25	0.26	0.26	0.27	0.28
	MSE	0.32	0.30	0.29	0.28	0.28	0.27	0.27	0.27	0.27	0.27	0.28
0.9	BSQ	0.24	0.24	0.25	0.25	0.26	0.26	0.27	0.27	0.28	0.28	0.29
	MSE	0.32	0.31	0.30	0.29	0.29	0.28	0.28	0.28	0.28	0.28	0.29
1.0	BSQ	0.26	0.26	0.26	0.27	0.27	0.28	0.28	0.28	0.29	0.29	0.30
	MSE	0.32	0.31	0.31	0.30	0.30	0.29	0.29	0.29	0.29	0.29	0.30