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Fuzzy possibilistic portfolio selection model with *VaR* constraint and risk-free investment

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We propose a possibilistic portfolio model with *VaR* constraint and risk-free investment based on the possibilistic mean and variance, while assuming that the expected rate of returns is a fuzzy number. The model shows more clearly that, in the financial market affected by several non-probabilistic factors, risk-averse investors wish not only to reach the expected rate of returns in their actual investment, but also to assure that the maximum of their possible future risk is lower than an expected loss. Under the condition that the expected rate of returns is a normal distribution fuzzy variable, we proposed a theorem as the solution, and derive a crisp equivalent form of the possibilistic portfolio under constraints of *VaR* and risk-free investment. This model is an expansion of the fuzzy possibilistic mean–variance model by Zhang (2007). Finally, an empirical study is carried out using the data concerning some stocks of various industries listed at the Shanghai Stock Exchange. A conclusion is reached that the investors are able to choose a portfolio more suitable to them under the *VaR* constraint.

1. Introduction

Markowitz (1952, 1959) proposed a mean–variance model for portfolio selection in the earliest stage. Over the last 50 years, the theory has played an important role in the development of modern portfolio selection theory. In Markowitz's portfolio theory it is assumed that all the investors disgust the risk and asset returns are random variables. The expected return of a portfolio and the variance of returns are referred to as the investment return and the risk of the investment, respectively. It combines probability theory with optimization techniques to model the investment behavior under uncertainty. Researches on the mean–variance portfolio selection problem typically include Sharpe (1970), Merton (1972), Perold (1984), Pang (1980) and Elliott et al. (2010) etc.

Another standard benchmark for measuring firm-wide risk is value at risk (*VaR*) (see Duffie and Pan (1997)). For a given time horizon t and the confidence level β, the *VaR* of a portfolio is the loss in the portfolio's market value over the time horizon $\triangle t$ that is exceeded with probability $1-\beta$. For example, a 99% *VaR* for a 10-day holding period, implies that the maximum loss incurred over the next 10 days should only exceed the limit once in every 100 cases. Therefore it reflects the potential downside risk faced on investments in terms of nominal losses. The topic was described in monographs (see e.g., Gourieroux et al. (2000), Jorion (1997) and Longin (2000)). In particular, Alexander and Baptista (2002, 2004) and Xu et al. (2010a, 2010b) analyzed the economic implications of using a mean-*VaR* model for portfolio selection and the portfolio selection implications arising from imposing a value at risk constraint on the mean–variance model.

Those above-mentioned notions are of the crisp-stochastic category. It means that the input data and parameters (goals, objectives, and constraints) are determined as crisp (real) numbers or unique distribution functions. Thus, it is supposed that a decision maker is able to determine exactly the decision parameters and the unique input data (distribution functions are known). These assumptions are fulfilled in many situations and the traditional *VaR* method is a useful methodology for predicting financial risks. However, because of the complexity of financial systems, there are several situations where the input data are not precise but only fuzzy. Therefore the decision maker should not consider parameters (goals and constraints) using crisp numbers or unique distribution functions, but instead he/she should use fuzzy numbers or fuzzy probability distribution functions (see e.g., Zdeněk and Zmeskal (2005)). Thus, the fuzzy set (number) is characterized as an important approach of fuzzy information. In recent years, researchers investigated many fuzzy portfolio selection problems (see e.g., Watada (1997), Ramaswamy (1998), León et al. (2002), Wang and Zhu (2002), Tanaka et al. (2000), Xu et al. (2010a, 2010b) and Zhang et al. (2010)). Dubois and Prade (1987) defined an interval-valued expectation of fuzzy numbers, viewing them as consonant random sets. They also showed that this expectation

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remained additive in the sense of addition of fuzzy numbers. Carlsson and Fullér (2001) introduced the notations of lower and upper possibilistic mean values, and introduced the notation of crisp possibilistic mean value and crisp possibilistic variance of continuous possibility distributions. Zhang and Nie (2003) extended the concepts of possibilistic mean and possibilistic variance proposed by Carlsson and Fullér, and presented the notions of upper and lower possibilistic variances and covariances of fuzzy numbers. Based on this, Zhang et al. (2007) proposed two kinds of portfolio selection models based on upper and lower possibilistic means and possibilistic variances and introduced the notions of upper and lower possibilistic efficient portfolios. Furthermore, they presented an algorithm which can be used to derive the explicit expressions of the possibilistic efficient frontiers for the possibilistic mean-variance portfolio selection problem dealing with lower bounds on asset holdings. However, Zhang et al. (2007) only considered the case that the investors hope that the return (rate) of their portfolio to reach an expected value. In real financial markets, risk-averse investors wish not only to reach the expected rate of returns in their actual investment, but also to assure the maximum possible loss should not exceed an expected loss. In this paper, we propose a possibilistic portfolio model under the constraints of *VaR* and risk-free investment. The model shows that risk-aversion investors wish not only to reach the expected rate of returns in their actual investment, but also to assure that the maximum of their possible future risk be lower than the *VaR*. With the assumption that the returns of asset are normal distribution fuzzy variables, we proposed a theorem as the solution, and derive a crisp equivalent form of the possibilistic portfolio under constraints of *VaR* and risk-free investment. We further developed the possibilistic mean–variance model.

The rest of the paper is organized as follows. In Section 2, we introduce the basic concepts of the possibilistic mean and variance of a fuzzy number. In Section 3, we propose a possibilistic portfolio model under constraints of *VaR* and risk-free investment. Suppose the expected rate of returns is a normal distribution fuzzy variable, we proposed a theorem as the solution, and derive a crisp equivalent form of the possibilistic portfolio under constraints of *VaR* and risk-free investment. In Section 4, a numerical example is given to illustrate our proposed effective approaches. Finally, Section 5 presents conclusions.

2. Possibilistic mean value and variance

Let us introduce some definitions which we need in the following sections. A fuzzy number $\tilde{A}$ is a fuzzy set of the real line $\mathbb{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by $\mathscr{F}$.

Carlsson and Fullér (2001) defined the upper and lower possibilistic means of a fuzzy number $\tilde{A}$ with γ-level set $\left[\tilde{A}\right]^\gamma = [a_1(\gamma), a_2(\gamma)]$ $(\gamma > 0)$ as

$$M_U\left(\tilde{A}\right) = \frac{\int_0^1 pos\left[\tilde{A} \geq a_2(\gamma)\right] a_2(\gamma) d\gamma}{\int_0^1 pos\left[\tilde{A} \geq a_2(\gamma)\right] d\gamma}, \tag{1}$$

$$M_L\left(\tilde{A}\right) = \frac{\int_0^1 pos\left[\tilde{A} \leq a_1(\gamma)\right] a_1(\gamma) d\gamma}{\int_0^1 pos\left[\tilde{A} \leq a_1(\gamma)\right] d\gamma}, \tag{2}$$

respectively, where *pos* denotes possibility measure.

$$pos\left[\tilde{A} \geq a_2(\gamma)\right] = \Pi([a_2(\gamma), +\infty)) = \sup_{t \geq a_2(\gamma)} A(t) = \gamma, \tag{3}$$

$$pos\left[\tilde{A} \leq a_1(\gamma)\right] = \Pi((-\infty, a_1(\gamma)]) = \sup_{t \leq a_1(\gamma)} A(t) = \gamma. \tag{4}$$

Therefore, according to Eqs. (3) and (4), $M_U\left(\tilde{A}\right)$ and $M_L\left(\tilde{A}\right)$ can be written in the following form:

$$M_U\left(\tilde{A}\right) = 2\int_0^1 \gamma a_2(\gamma) d\gamma, \tag{5}$$

$$M_L\left(\tilde{A}\right) = 2\int_0^1 \gamma a_1(\gamma) d\gamma. \tag{6}$$

The possibilistic mean value of $\tilde{A}$ is the arithmetic mean of its lower possibilistic and upper possibilistic mean values as follows:

$$\overline{M}\left(\tilde{A}\right) = \frac{M_U\left(\tilde{A}\right) + M_L\left(\tilde{A}\right)}{2}. \tag{7}$$

Corresponding to the upper and lower possibilistic means, Zhang and Nie (2003) introduced the upper and lower possibilistic variances and covariances of fuzzy numbers. The upper and lower possibilistic variances of a fuzzy number $\tilde{A}$ with γ-level set $\left[\tilde{A}\right]^\gamma = [a_1(\gamma), a_2(\gamma)]$ $(\gamma > 0)$ are defined as

$$\begin{aligned}\sigma_U^2\left(\tilde{A}\right) &= \frac{\int_0^1 pos\left[\tilde{A} \geq a_2(\gamma)\right]\left(M_U\left(\tilde{A}\right) - a_2(\gamma)\right)^2 d\gamma}{\int_0^1 pos\left[\tilde{A} \geq a_2(\gamma)\right] d\gamma} \\ &= 2\int_0^1 \gamma\left(M_U\left(\tilde{A}\right) - a_2(\gamma)\right)^2 d\gamma\end{aligned} \tag{8}$$

and

$$\begin{aligned}\sigma_L^2\left(\tilde{A}\right) &= \frac{\int_0^1 pos\left[\tilde{A} \leq a_1(\gamma)\right]\left(M_L\left(\tilde{A}\right) - a_1(\gamma)\right)^2 d\gamma}{\int_0^1 pos\left[\tilde{A} \leq a_1(\gamma)\right] d\gamma} \\ &= 2\int_0^1 \gamma\left(M_L\left(\tilde{A}\right) - a_1(\gamma)\right)^2 d\gamma,\end{aligned} \tag{9}$$

respectively.

The upper and lower possibilistic covariances between fuzzy numbers $\tilde{A}$ with γ-level set $\left[\tilde{A}\right]^\gamma = [a_1(\gamma), a_2(\gamma)]$ and $\tilde{B}$ with γ-level set $\left[\tilde{B}\right]^\gamma = [b_1(\gamma), b_2(\gamma)]$ $(\gamma \in [0,1])$ are defined as

$$\mathrm{Cov}_U\left(\tilde{A}, \tilde{B}\right) = 2\int_0^1 \gamma\left(M_U\left(\tilde{A}\right) - a_2(\gamma)\right)\left(M_U\left(\tilde{B}\right) - b_2(\gamma)\right) d\gamma \tag{10}$$

and

$$\mathrm{Cov}_L\left(\tilde{A}, \tilde{B}\right) = 2\int_0^1 \gamma\left(M_L\left(\tilde{A}\right) - a_1(\gamma)\right)\left(M_L\left(\tilde{B}\right) - b_1(\gamma)\right) d\gamma. \tag{11}$$

The possibilistic variance of fuzzy number $\tilde{A}$ is defined as

$$\overline{\sigma^2}\left(\tilde{A}\right) = \frac{\sigma_U^2\left(\tilde{A}\right) + \sigma_L^2\left(\tilde{A}\right)}{2}. \tag{12}$$

The possibilistic covariance between fuzzy numbers $\tilde{A}$ and $\tilde{B}$ is defined as

$$\overline{\mathrm{Cov}}\left(\tilde{A}, \tilde{B}\right) = \frac{\mathrm{Cov}_U\left(\tilde{A}, \tilde{B}\right) + \mathrm{Cov}_L\left(\tilde{A}, \tilde{B}\right)}{2}. \tag{13}$$

Lemma 1. *Let $\lambda_1, \lambda_2 \in R$ and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers, then*

$$\overline{M}\left(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}\right) = \lambda_1 \overline{M}\left(\tilde{A}\right) + \lambda_2 \overline{M}\left(\tilde{B}\right).$$

Lemma 2. *Let $\lambda_1, \lambda_2 \in R$ and let $\tilde{A}$ and $\tilde{B}$ be fuzzy numbers, then*

$$\begin{aligned}\overline{\sigma^2}\left(\lambda_1 \tilde{A} + \lambda_2 \tilde{B}\right) = {} & \lambda_1^2 \overline{\sigma^2}\left(\tilde{A}\right) + \lambda_2^2 \overline{\sigma^2}\left(\tilde{B}\right) \\ & + 2|\lambda_1 \lambda_2| \overline{\mathrm{Cov}}\left(\phi(\lambda_1)\tilde{A}, \phi(\lambda_2)\tilde{B}\right),\end{aligned}$$

where $\phi(x)$ is a sign function of $x \in R$.

3. Possibilistic portfolio model under constraints of *VaR* and risk-free investment

3.1. Mathematics modeling

Suppose that there are n risky assets and one risk-free asset available for investment. Let $\tilde{\xi}_i$ be the return rate of asset i, $i=1,2,\cdots,n$, which is a fuzzy number. Let x_i represent the proportion invested in asset i, and r_f is the risk-free asset return.

As a preparation for establishing the new model, we need to determine the following values. First, the return $\tilde{r}_p$ on the portfolio is given by

$$\tilde{r}_p=\sum_{i=1}^{n}x_i\tilde{\xi}_i+r_f\left(1-\sum_{i=1}^{n}x_i\right), \tag{14}$$

where $\tilde{r}_p$ is a fuzzy number.

The possibilistic mean of the portfolio return $\tilde{r}_p$ is written as

$$\overline{M}\left(\tilde{r}_p\right)=\sum_{i=1}^{n}x_i\frac{M_U\left(\tilde{\xi}_i\right)+M_L\left(\tilde{\xi}_i\right)}{2}+r_f\left(1-\sum_{i=1}^{n}x_i\right). \tag{15}$$

According to Lemma 2 and $x_i\geq 0$, $i=1,2,\cdots,n$, it is known that the possibilistic variance of $\tilde{r}_p$ is given by

$$\begin{aligned}\overline{\sigma^2}&=\sum_{i=1}^{n}x_i^2\overline{\sigma^2_{\tilde{\xi}_i}}+2\sum_{i>j=1}^{n}\left|x_ix_j\right|\overline{\mathrm{Cov}}\left(\tilde{\xi}_i,\tilde{\xi}_j\right)\\&=\sum_{i=1}^{n}x_i^2\overline{\sigma^2_{\tilde{\xi}_i}}+2\sum_{i>j=1}^{n}x_ix_j\overline{\mathrm{Cov}}\left(\tilde{\xi}_i,\tilde{\xi}_j\right).\end{aligned} \tag{16}$$

Analogous to the possibilistic mean–variance model (2007), in this paper, the possibilistic mean is used to describe the portfolio return, and the possibilistic variance is used to describe the portfolio risk. Meanwhile, a value at risk (*VaR*) constraint is imposed on our portfolio model. Under this structure, the possibilistic portfolio model under constraints of *VaR* and risk-free investment can be formulated as

$$\begin{cases}min\quad \overline{\sigma^2}=\sum_{i=1}^{n}x_i^2\overline{\sigma^2_{\tilde{\xi}_i}}+2\sum_{i>j=1}^{n}x_ix_j\overline{\mathrm{Cov}}\left(\tilde{\xi}_i\tilde{\xi}_j\right)\\ s.t.\sum_{i=1}^{n}x_i\frac{M_U\left(\tilde{\xi}_i\right)+M_L\left(\tilde{\xi}_i\right)}{2}+r_f\left(1-\sum_{i=1}^{n}x_i\right)\geq\bar{r},\\ pos\left(\tilde{\xi}_ix_i\leq VaR\right)\leq 1-\beta,\\ \sum_{i=1}^{n}x_i\leq 1,\\ 0\leq l_i\leq x_i\leq u_i, i=1,2,\cdots,n,\end{cases} \tag{17}$$

where pos denotes possibility measure, $\bar{r}$ is the underestimated expected return rate. l_i and u_i represent the lower bound and the upper bound on investment in asset i, respectively. *VaR* is defined as value at risk by the β–confidence level.

The model shows that risk-averse investors wish not only to reach the expected rate of returns in their actual investment, but also to ensure that the maximum of their possible future risk is lower than an expected loss.

Definition 1. *A portfolio* $X^*=(x_1^*,x_2^*,\cdots,x_n^*)'$ *in (17) is called the possibilistic efficient portfolio if no portfolio X exists such that* $\overline{M}\left(\tilde{r}_X\right)\geq\overline{M}\left(\tilde{r}_{X^*}\right)$ *and* $\overline{\sigma^2}\left(\tilde{r}_X\right)\leq\overline{\sigma^2}\left(\tilde{r}_{X^*}\right)$, *where at least one of the inequalities is strict.*

The set of all the possibilistic efficient portfolios forms the possibilistic efficient frontier, the possibilistic efficient frontier can be traced out by solving the portfolio selection problem (17) for all possible values of $\bar{r}$.

3.2. Fuzzy normal distribution

Suppose that the return rate of asset i is a normal distribution fuzzy variable expressed as $\tilde{\xi}_i\sim FN(\mu_i,\sigma_i)$, and its membership function is

$$A_{\tilde{\xi}_i}(t)=exp\left\{-\left[(t-\mu_i)/\sigma_i\right]^2\right\}.$$

The level set of $\tilde{\xi}_i$ is defined as

$$\left[\tilde{\xi}_i\right]^\gamma=\left[\mu_i-\sigma_i\sqrt{ln\gamma^{-1}},\mu_i+\sigma_i\sqrt{ln\gamma^{-1}}\right],\gamma\in(0,1),i=1,2,\cdots,n.$$

Theorem 1. *Assume that the return rates of assets are normal distribution fuzzy variables expressed as* $\tilde{\xi}_i\sim FN(\mu_i,\sigma_i)$, $i=1,2,\cdots,n$, *then*

$$\sum_{i=1}^{n}x_i\tilde{\xi}_i\sim FN\left(\sum_{i=1}^{n}x_i\mu_i,\left(\frac{1}{2}-\frac{\pi}{8}\right)\left(\sum_{i=1}^{n}x_i\sigma_i\right)^2\right), \tag{18}$$

where $x_i\geq 0$, $i=1,2,\cdots,n$.

Proof. According to Lemma 1, the possibilistic mean value of $\sum_{i=1}^{n}x_i\tilde{\xi}_i$ can be calculated by

$$\overline{M}\left(\sum_{i=1}^{n}x_i\tilde{\xi}_i\right)=\sum_{i=1}^{n}x_i\overline{M}\left(\tilde{\xi}_i\right)=\sum_{i=1}^{n}x_i\mu_i.$$

From Eqs. (5) and (6), we can deduce that

$$\begin{aligned}M_U\left(\tilde{\xi}_i\right)&=2\int_0^1\gamma\left(\mu_i+\sigma_i\sqrt{ln\gamma^{-1}}\right)d\gamma\\&=\mu_i+2\sigma_i\int_0^1\gamma\sqrt{ln\gamma^{-1}}d\gamma\\&=\mu_i+\sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}}\end{aligned} \tag{19}$$

and

$$\begin{aligned}M_L\left(\tilde{\xi}_i\right)&=2\int_0^1\gamma\left(\mu_i-\sigma_i\sqrt{ln\gamma^{-1}}\right)d\gamma\\&=\mu_i-\sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}}.\end{aligned} \tag{20}$$

Using the same way as above, we can obtain the following results:

$$\begin{aligned}\sigma_U^2&=2\int_0^1\gamma\left[M_U\left(\tilde{\xi}_i\right)-a_2(\gamma)\right]^2d\gamma\\&=2\int_0^1\gamma\left(\mu_i+\sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}}-\mu_i-\sigma_i\sqrt{ln\gamma^{-1}}\right)^2d\gamma=\left(\frac{1}{2}-\frac{\pi}{8}\right)\sigma_i^2\end{aligned}$$

and

$$\sigma_L^2=2\int_0^1\gamma\left[M_L\left(\tilde{\xi}_i\right)-a_1(\gamma)\right]^2d\gamma=\left(\frac{1}{2}-\frac{\pi}{8}\right)\sigma_i^2.$$

Thus, the possibilistic variance can be written as

$$\overline{\sigma^2_{\tilde{\xi}_i}}=\frac{\sigma_U^2+\sigma_L^2}{2}=\left(\frac{1}{2}-\frac{\pi}{8}\right)\sigma_i^2. \tag{21}$$

Table 1
The possibilistic distributions of returns of five stocks.

Stock	Stock1	Stock2	Stock3	Stock4	Stock5
μ_i	0.05	0.10	0.18	0.26	0.35
σ_i	0.118	0.167	0.223	0.268	0.322

Moreover, according to Eqs. (10) and (11), the upper and lower possibilistic covariances are given by

$$\mathrm{Cov}_U\left(\tilde{\xi}_i, \tilde{\xi}_j\right) = 2\int_0^1 \gamma\left[M_U\left(\tilde{\xi}_i\right) - a_2(\gamma)\right]\left[M_U\left(\tilde{\xi}_j\right) - b_2(\gamma)\right]d\gamma = \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i\sigma_j,$$

$$\mathrm{Cov}_L\left(\tilde{\xi}_i, \tilde{\xi}_j\right) = 2\int_0^1 \gamma\left[M_L\left(\tilde{\xi}_i\right) - a_1(\gamma)\right]\left[M_L\left(\tilde{\xi}_j\right) - b_1(\gamma)\right]d\gamma = \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i\sigma_j.$$

The possibilistic covariance is

$$\overline{\mathrm{Cov}}\left(\tilde{\xi}_i, \tilde{\xi}_j\right) = \frac{\mathrm{Cov}_U\left(\tilde{\xi}_i, \tilde{\xi}_j\right) + \mathrm{Cov}_L\left(\tilde{\xi}_i, \tilde{\xi}_j\right)}{2} = \left(\frac{1}{2} - \frac{\pi}{8}\right)\sigma_i\sigma_j. \tag{22}$$

According to Lemma 2 and $x_i \geq 0$, we can compute the possibilistic variance of $\sum_{i=1}^n x_i\tilde{\xi}_i$

$$\begin{aligned}\overline{\sigma^2}_{\sum_{i=1}^n x_i\tilde{\xi}_i} &= \sum_{i=1}^n x_i^2\overline{\sigma^2_{\tilde{\xi}_i}} + 2\sum_{i>j=1}^n x_i x_j \overline{\mathrm{Cov}}\left(\tilde{\xi}_i, \tilde{\xi}_j\right) \\ &= \sum_{i=1}^n\left(\frac{1}{2} - \frac{\pi}{8}\right)x_i^2\sigma_i^2 + 2\sum_{i>j=1}^n\left(\frac{1}{2} - \frac{\pi}{8}\right)x_i x_j\sigma_i\sigma_j \\ &= \left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^n x_i\sigma_i\right)^2,\end{aligned}$$

so, the proof of the theorem is completed. □

According to Theorem 1, the membership function of $\sum_{i=1}^n \tilde{\xi}_i x_i$ is defined by

$$A(t) = exp\left\{\frac{-\left(t - \sum_{i=1}^n x_i\mu_i\right)^2}{\left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^n x_i\sigma_i\right)^2}\right\},$$

then

$$\begin{aligned} pos\left(\sum_{i=1}^n \tilde{\xi}_i x_i \leq VaR\right) &= sup_{t\leq VaR}\left\{exp\left\{\frac{-\left(t - \sum_{i=1}^n x_i\mu_i\right)^2}{\left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^n x_i\sigma_i\right)^2}\right\}\right\} \\ &= \begin{cases} exp\left\{\frac{-\left(VaR - \sum_{i=1}^n x_i\mu_i\right)^2}{\left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^n x_i\sigma_i\right)^2}\right\} & \text{for } \sum_{i=1}^n x_i\mu_i > VaR, \\ 1 & \text{for } \sum_{i=1}^n x_i\mu_i \leq VaR. \end{cases}\end{aligned} \tag{23}$$

According to Eqs. (17) and (23), we can obtain

$$-\left(VaR - \sum_{i=1}^n x_i\mu_i\right)^2 \leq \left(\frac{1}{2} - \frac{\pi}{8}\right)ln(1-\beta)\left(\sum_{i=1}^n x_i\sigma_i\right)^2. \tag{24}$$

It is known from Eqs. (14), (19) and (20), that when the return rates of assets are normal distribution fuzzy variables, the upper and lower possibilistic means of $\tilde{r}_p$ are given by

$$M_U\left(\tilde{r}_p\right) = \sum_{i=1}^n\left(\mu_i + \sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}} - r_f\right)x_i + r_f,$$

$$M_L\left(\tilde{r}_p\right) = \sum_{i=1}^n\left(\mu_i - \sigma_i\frac{\sqrt{\pi}}{2\sqrt{2}} - r_f\right)x_i + r_f.$$

Thus, the possibilistic mean of $\tilde{r}_p$ is written as

$$\overline{M} = \frac{M_U\left(\tilde{r}_p\right) + M_L\left(\tilde{r}_p\right)}{2} = \sum_{i=1}^n\left(\mu_i - r_f\right)x_i + r_f.$$

Therefore, for a normal distribution fuzzy variable, the possibilistic portfolio model under constraints of *VaR* and risk-free investment can be formulated as

$$\begin{cases} min \quad \overline{\sigma^2} = \left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^n x_i^2\sigma_i^2 + 2\sum_{i>j=1}^n x_i x_j\sigma_i\sigma_j\right) \\ s.t. \sum_{i=1}^n x_i\left(\mu_i - r_f\right) + r_f \geq \bar{r}, \\ -\left(VaR - \sum_{i=1}^n x_i\mu_i\right)^2 \leq \left(\frac{1}{2} - \frac{\pi}{8}\right)ln(1-\beta)\left(\sum_{i=1}^n x_i\sigma_i\right)^2, \\ \sum_{i=1}^n x_i \leq 1, \\ 0 \leq l_i \leq x_i \leq u_i, i = 1, 2, \cdots, n. \end{cases} \tag{25}$$

4. Numerical example

In order to illustrate our proposed effective approaches for the portfolios in this paper, we give a real portfolio adjusting example. In this example, we select five stocks from Shanghai Stock Exchange. Original data come from these securities' monthly closed prices from January 2008 to December 2008. Based on the historical data, the corporations' financial reports and the future information, we can estimate their returns with the following possibilistic distributions.

For the return rate of each asset $\tilde{\xi}_i \sim FN(\mu_i, \sigma_i)$, Table 1 gives the frequency distribution of monthly returns of five stocks.

Therefore, the $\gamma-$ level set of $\tilde{\xi}_i$ $(i = 1, \cdots 5)$ is given by

$$\left[\tilde{\xi}_1\right]^\gamma = \left[0.05 - 0.118\sqrt{ln\gamma^{-1}}, 0.05 + 0.118\sqrt{ln\gamma^{-1}}\right]$$

Table 2
Some possibilistic efficient portfolios with $\beta = 0.90$, $VaR = -0.4\%$ and $r_f = 0.72\%$.

$\bar{r}$ (%)	0.7200	3.2100	8.1000	9.2800	10.1400	12.0300	13.8700	14.9900	21.2300	21.9500
$\overline{\sigma^2}$ (%)	3.9145	3.9145	6.7442	8.7618	10.5577	15.2536	20.6530	24.3393	56.6103	63.1772
x_1	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500	0.0500
x_2	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.000	0.0724	0.1500
x_3	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.3000	0.3000
x_4	0.0000	0.0000	0.0000	0.0000	0.0000	0.0994	0.1722	0.2165	0.3000	0.3000
x_5	0.1000	0.1000	0.1586	0.1931	0.2000	0.2000	0.2000	0.2000	0.2000	0.2000
$\sum_{i=1}^5 x_i$	0.2500	0.2500	0.3086	0.3431	0.3746	0.3431	0.5222	0.5665	0.9224	1

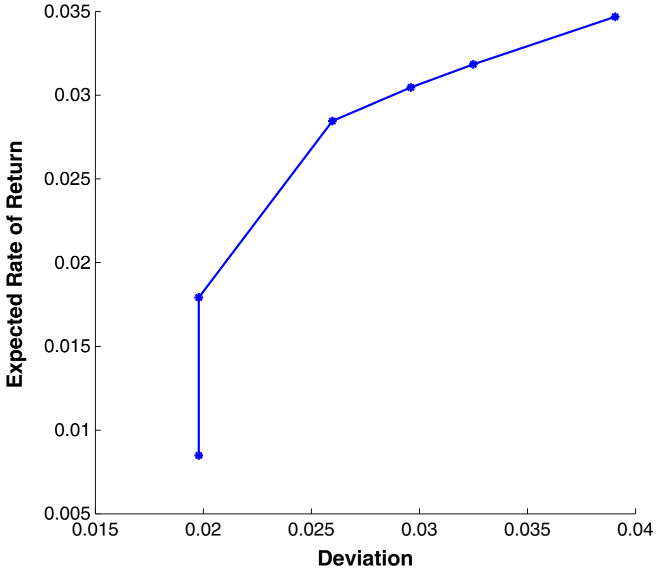

Fig. 1. Some possibilistic efficient portfolios $\beta = 0.90$, $VaR = -0.4\%$ and $r_f = 0.72\%$.

$$\left[\tilde{\xi}_2\right]^\gamma = \left[0.10 - 0.167\sqrt{ln\gamma^{-1}}, 0.10 + 0.167\sqrt{ln\gamma^{-1}}\right]$$

$$\left[\tilde{\xi}_3\right]^\gamma = \left[0.18 - 0.223\sqrt{ln\gamma^{-1}}, 0.18 + 0.223\sqrt{ln\gamma^{-1}}\right]$$

$$\left[\tilde{\xi}_4\right]^\gamma = \left[0.26 - 0.268\sqrt{ln\gamma^{-1}}, 0.26 + 0.268\sqrt{ln\gamma^{-1}}\right]$$

$$\left[\tilde{\xi}_5\right]^\gamma = \left[0.35 - 0.322\sqrt{ln\gamma^{-1}}, 0.35 + 0.322\sqrt{ln\gamma^{-1}}\right].$$

We let the risk-free asset as the bank account. According to the current savings deposit rate of the year, the return of risk-free asset r_f is equal to 0.72%. The lower bound of investment ratio x_i is $l = [0.05,0,0.1,0,0.1]$, and the upper bound is $u = [0.3,0.4,0.3,0.3,0.2]$.

Suppose $\beta = 0.9$ and $VaR = -0.4\%$, by solving the model (25), the possibilistic efficient portfolios for the different objective value $\bar{r}$ are obtained as shown in Table 2. We find that, when the objective

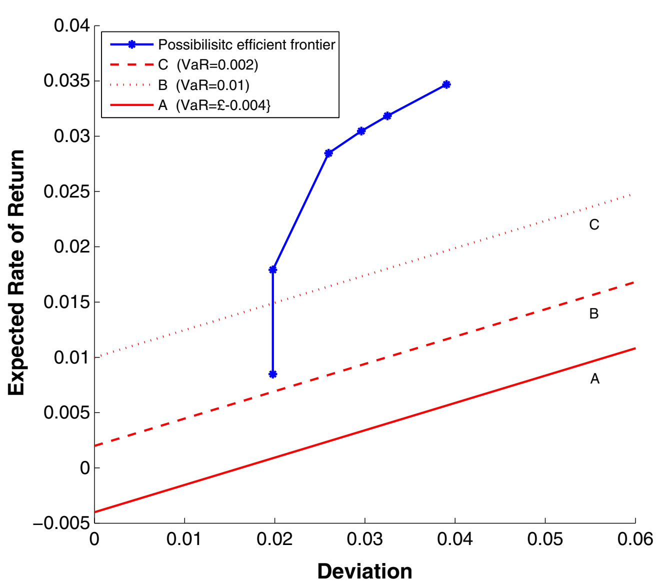

Fig. 2. Some possibilistic efficient portfolios with different *VaR*.

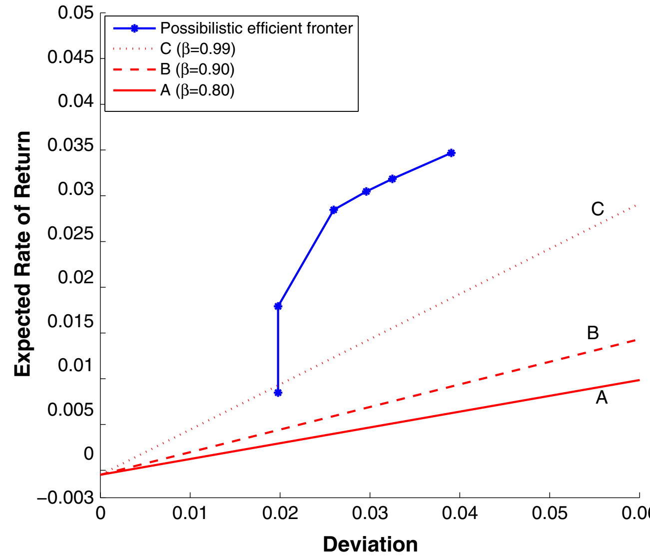

Fig. 3. Some possibilistic efficient portfolios with different β.

value $\bar{r}$ is small (e.g. $\bar{r} = 3.21\%$), the investment proportion of assets is exactly the lower bound value given in advance, and the sum of the proportion is 0.25. It means that 25% of capitals are invested in risky assets and the remaining 75% are invested in risk-free assets. As the objective value $\bar{r}$ increases, the proportion of risky assets is increased consequently. For example, if $\bar{r} = 21.95\%$, the proportion of portfolio is 0.05, 0.15, 0.3, 03, 0.2, respectively. The sum of the proportions is 1, which means all capitals are invested on risky assets.

On the basis of Table 2, we get the efficient frontier of the possibilistic portfolio as shown in Fig. 1. Because the lower bound of investment proportion is given in advance, such as $l = [0.05,0,0.1,0,0.1]$, it means that the investor would bear the risk of 3.9145%, therefore it is impossible for the efficient frontier to intersect the vertical axis in Fig. 1.

In addition, the effect of *VaR* constraint on portfolio is determined by the slope and intercept of *VaR* line, and the location of the efficient frontier. As shown in Fig. 2, when the location of the efficient frontier remains unmoved, and the slope of *VaR* line is constant, the intercept changes of *VaR* would have an effect on the optimal solution of the model. As *VaR* constraint moves from point A to B, *VaR* changes wouldn't have any effects on the optimal solution of the model. However, as *VaR* constraint moves from point A to C, the optimal solution of the model shall be different.

In Fig. 3, when the location of the efficient frontier remains unmoved, and the intercept of VaR line is constant, the slope changes of *VaR* line would have an effect on the optimal solution of the model. As *VaR* constraint moves from point A to B, *VaR* changes wouldn't have any effects on the optimal solution of the model. However, as *VaR* constraint moves from point A to C, the optimal solution of the model shall be different.

5. Conclusion

In this paper, we propose the possibilistic portfolio model under constraints of *VaR* and risk-free investment by utilizing the possibilistic mean and variance of a fuzzy number. When the returns of assets are normal distribution fuzzy variables, we also obtain its specific possibilistic portfolio model. In order to illustrate our proposed effective approaches for the portfolios, we select five stocks from Shanghai Stock Exchange. Based on these data, we analyze an approximately optimal result, and obtain its efficient frontier. A conclusion is reached that our method not only extends the possibilistic mean–variance model, but also can be used to help some risk-averse investors select the appropriate proportion of investments under complex market situations.

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