The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets

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THE VALUATION OF RISK ASSETS AND THE SELECTION OF RISKY INVESTMENTS IN STOCK PORTFOLIOS AND CAPITAL BUDGETS*

John Lintner

Introduction and Preview of Some Conclusions

THE effects of risk and uncertainty upon asset prices, upon rational decision rules for individuals and institutions to use in selecting security portfolios, and upon the proper selection of projects to include in corporate capital budgets, have increasingly engaged the attention of professional economists and other students of the capital markets and of business finance in recent years. The essential purpose of the present paper is to push back the frontiers of our knowledge of the logical structure of these related issues, albeit under idealized conditions. The immediately following text describes the contents of the paper and summarizes some of the principal results.

The first two sections of this paper deal with the *problem of selecting* optimal security portfolios by risk-averse investors who have the alternative of investing in risk-free securities with a positive return (or borrowing at the same rate of interest) and who can sell short if they wish. The first gives alternative and hopefully more transparent proofs (under these more general market conditions) for Tobin's important "separation theorem" that ". . . the proportionate composition of the non-cash assets is independent of their aggregate share of the investment balance . . . " (and hence of the optimal holding of cash) for risk averters in purely competitive markets when utility functions are quadratic *or* rates of return are multivariate normal.[1] We then note that the same conclusion follows from an earlier theorem of Roy's [19] without dependence on quadratic utilities or normality. The second section shows that *if short sales are permitted*, the best portfolio-mix of risk assets can be determined by the solution of a single simple set of simultaneous equations without recourse to programming methods, and when covariances are zero, a still simpler ratio scheme gives the optimum, whether or not short sales are permitted. When covariances are not all zero and short sales are excluded, a single quadratic programming solution is required, but sufficient.

Following these extensions of Tobin's classic work, we concentrate on the set of risk assets held in risk averters' portfolios. In section III we develop various significant *equilibrium properties within* the risk asset portfolio. In particular, we establish conditions under which stocks will be held long (short) in optimal portfolios even when "risk premiums" are negative (positive). We also develop expressions for different combinations of expected rate of return on a given security, and its standard deviation, variance, and/or covariances which will result in the same relative holding of a stock, *ceteris paribus*. These "indifference functions" provide direct evidence on the moot issue of the appropriate functional relationships between "required rates of return" and relevant risk parameter(s) — and on the related issue of how "risk classes" of securities may best be delineated (if they are to be used).[2]

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[Professor Sharpe's paper, "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk" (*Journal of Finance*, September 1964) appeared after this paper was in final form and on its way to the printers. My first section, which parallels the first half of his paper (with corresponding conclusions), sets the algebraic framework for sections II, III and VI, (which have no counterpart in his paper) and for section IV on the equilibrium prices of risk assets, concerning which our results differ significantly for reasons which will be explored elsewhere. Sharpe does not take up the capital budgeting problem developed in section V below.]

[1] Tobin [21, especially pp. 82–85]. Tobin assumed that funds are to be a allocated only over "monetary assets" (risk-free cash and default-free bonds of uncertain resale price) and allowed no short sales or borrowing. See also footnote 24 below. Other approaches are reviewed in Farrar [38].

[2] It should be noted that the classic paper by Modigliani and Miller [16] was silent on these issues. Corporations were assumed to be divided into homogeneous classes having the property that all shares of all corporations in any given class differed (at most) by a "scale factor," and hence (*a*) were perfectly correlated with each other and (*b*) were perfect substitutes for each other in perfect markets (p. 266). No comment was made on the measure of risk or uncertainty (or other attributes) relevant to the identification of different "equiva-

There seems to be a general presumption among economists that relative risks are best measured by the standard deviation (or coefficient of variation) of the rate of return,[3] but in the simplest cases considered — specifically when all covariances are considered to be invariant (or zero) — the indifference functions are shown to be linear between expected rates of return and their *variance*, not standard deviation.[4] (With variances fixed, the indifference function between the i^{th} expected rate of return and its pooled covariance with other stocks is hyperbolic.) There is no simple relation between the expected rate of return required to maintain an investor's relative holding of a stock and its standard deviation. Specifically, when covariances are non-zero and variable, the indifference functions are complex and non-linear *even if* it is assumed that the *correlations* between rates of return on different securities are invariant.

To this point we follow Tobin [21] and Markowitz [14] in assuming that current security prices are given, and that each investor acts on his own (perhaps unique) probability distribution over rates of return given these market prices. In the rest of the paper, we assume that investors' joint probability distributions pertain to dollar returns rather than rates of return[5], and for simplicity we assume that all investors assign identical sets of means, variances, and covariances to the distribution of these dollar returns. However unrealisic the latter assumption may be, it enables us, in section IV, to derive a set of (stable) equilibrium market prices which at least fully and explicitly reflect the presence of uncertainty *per se* (as distinct from the effects of diverse expectations), and to derive further implications of such uncertainty. In particular, the aggregate market value of any company's equity is equal to the capitalization at the risk-free interest rate of a uniquely defined *certainty-equivalent* of the probability distribution of the aggregate dollar returns to all holders of its stock. For each company, this certainty equivalent is the expected value of these uncertain returns less an adjustment term which is proportional to their aggregate risk. The factor of proportionality is the *same for all companies* in equilibirum, and may be regarded as a *market price of dollar risk.* The relevant risk of each company's stock is measured, moreover, not by the standard deviation of its dollar returns, but by the *sum* of the *variance* of its own aggregate dollar returns *and* their *total covariance* with those of all other stocks.

The next section considers some of the implications of these results for the normative aspects of the capital budgeting decisions of a company whose stock is traded in the market. For simplicity, we impose further assumptions required to make capital budgeting decisions independent of decisions on how the budget is financed.[6] The capital budgeting problem becomes a quadratic programming problem analogous to that introduced earlier for the individual investor. This capital budgeting-portfolio problem is formulated, its solution is given and some of its more important properties examined. Specifically, the minimum expected return (in dollars of expected present value) required to justify the allocation of funds to a given risky project is shown to be an increasing function of each of the following factors: (*i*) the risk-free rate of return; (*ii*) the "market price of (dollar) risk"; (*iii*) the variance in the project's own present value return; (*iv*) the project's aggregate present value return-*co*variance with assets already held by the company, and (*v*) its total covariance with other projects concurrently included in the capital budget. *All five* factors are involved explicitly in the corresponding (derived) formula for the minimum acceptable *expected rate* of return on an investment project. In this model, all means

lent return" classes. Both Propositions I (market value of *firm* independent of capital structure) and II (the linear relation between the expected return on equity shares and the debt-equity ratio for firms within a given class) are derived from the above assumptions (and the further assumption that corporate bonds are riskless securities); they involve no inter-class comparisons, ". . . nor do they involve any assertion as to what is an adequate compensation to investors for assuming a given degree of risk. . . ." (p. 279).

[3] This is, for instance, the presumption of Hirschleifer [8, p. 113], although he was careful not to commit himself to this measure alone in a paper primarily focussed on other issues. For an inductive argument in favor of the standard deviation of the rate of return as the best measure of risk, see Gordon [5, especially pp. 69 and 761. See also Dorfman in [3, p. 129 ff.] and Baumol [2].

[4] Except in dominantly "short" portfolios, the constant term will be larger, and the slope lower, the higher the (fixed) level of covariances of the given stocks with other stocks.

[5] The dollar return in the period is the sum of the cash dividend and the increase in market price during the period.

[6] We also assume that common stock portfolios are not "inferior goods," that the value of *all other* common stocks is invariant, and any effect of changes in capital budgets on the *covariances* between the values of different companies' *stocks* is ignored.

and (co)variances of present values must be calculated at the riskless rate r^*. We also show that *there can be no "risk-discount" rate* to be used in computing present values to accept or reject individual projects. In particular, the "*cost of capital*" as defined (for uncertainty) anywhere in the literature *is not the appropriate rate* to use in these decisions *even if* all new projects have the same "risk" as existing assets.

The final section of the paper briefly examines the complications introduced by institutional limits on amounts which either individuals or corporations may borrow at given rates, by rising costs of borrowed funds, and certain other "real world" complications. It is emphasized that the results of this paper are not being presented as directly applicable to practical decisions, because many of the factors which matter very siginificantly in practice have had to be ignored or assumed away. The function of these simplifying assumptions has been to permit a rigorous development of theoretical relationships and theorems which reorient much current theory (especially on capital budgeting) and provide a basis for further work.[7] More detailed conclusions will be found emphasized at numerous points in the text.

I — Portfolio Selection for an Individual Investor: The Separation Theorem

Market Assumptions

We assume that (1) *each individual* investor can invest any part of his capital in certain *risk-free assets* (e. g. deposits in insured savings accounts[8]) all of which pay interest at a common positive rate, exogeneously determined; and that (2) he can invest *any fraction* of his capital *in any* or all of a given finite set of *risky* securities which are (3) traded in a single *purely competitive market*, free of transactions costs and taxes, at given market prices,[9] which consequently do not depend on his investments or transactions. We also assume that (4) any investor may, if he wishes, borrow funds to invest in risk assets. Except in the final section, we assume that the *interest rate paid* on such loans is the same as he would have received had he invested in risk-free savings accounts, and that there is *no limit* on the amount he can borrow at this rate. Finally (5) he makes all purchases and sales of securities and all deposits and loans at discrete points in time, so that in selecting his portfolio at any "transaction point," each investor will consider only (*i*) the cash throw-off (typically interest payments and dividends received) within the period to the next transaction point and (*ii*) changes in the market prices of stocks during this same period. The *return* on any common stock is defined to be the sum of the cash dividends received plus the change in its market price. The return on any portfolio is measured in exactly the same way, including interest received or paid.

Assumptions Regarding Investors

(1) Since we posit the existence of assets yielding *positive risk-free* returns, we assume that each investor has already decided the fraction of his total capital he wishes to hold in cash and non-interest bearing deposits for reasons of liquidity or transactions requirements.[10] Henceforth, we will speak of *an investor's capital* as the stock of funds he has available for profitable investment *after* optimal cash holdings have been deducted. We also assume that (2) each investor will have assigned a *joint probability distribution* incorporating his best judgments regarding the returns on all *individual stocks*, or at least will have specified an expected value and variance to every return and a covariance or correlation to every pair of returns. All expected values of returns are finite, all variances are non-zero and finite, and all correlations of returns are less than one in absolute value (i. e. the covariance matrix is positive-definite). The investor computes the expected value and variance of the total return on any possible *portfolio*, or mix of any specified amounts of any or all of the individual stocks, by forming the appropriately weighted average or sum of these components expected returns, variances and covariances.

[7] The relation between the results of this paper and the models which were used in [11] and [12] is indicated at the end of section V.

[8] Government bonds of appropriate maturity provide another important example when their "yield" is substituted for the word "interest."

[9] Solely for convenience, we shall usually refer to all these investments as common stocks, although the analysis is of course quite general.

[10] These latter decisions are independent of the decisions regarding the allocation of remaining funds between risk-free assets with positive return and risky stocks, which are of direct concern in this paper, because the risk-free assets with positive returns clearly dominate those with no return once liquidity and transactions requirements are satisfied at the margin.

With respect to an investor's *criterion for choices* among different attainable combinations of assets, we assume that (3) if any two mixtures of assets have the *same expected return*, the investor will prefer the one having the *smaller variance* of return, and if any two mixtures of assets have the *same variance* of returns, he will prefer the one having the *greater expected value*. Tobin [21, pp. 75-76] has shown that such preferences are implied by maximization of the expected value of a von Neumann-Morgenstern utility function if *either* (*a*) the investor's *utility* function is *concave* and *quadratic or* (*b*) the investor's *utility* function is *concave, and* he has assigned probability distributions such that the *returns* on *all possible portfolios differ at most by a location and scale parameter*, (which will be the case if the joint distribution of all individual stocks is multivariate normal).

Alternative Proofs of the Separation Theorem

Since the interest rates on riskless savings bank deposits ("loans to the bank") and on borrowed funds are being assumed to be the same, we can treat borrowing as negative lending. Any portfolio can then be described in terms of (i) the *gross* amount invested in stocks, (ii) the fraction of this amount invested in each individual stock, and (iii) the *net* amount invested in loans (a negative value showing that the investor has borrowed rather than lent). But since the *total net* investment (the algebraic sum of stocks plus loans) is a given amount, the problem simply requires finding the jointly optimal values for (1) the ratio of the gross investment in stocks to the total net investment, and (2) the ratio of the gross investment in each individual stock to the total gross investment in stocks. It turns out that although the solution of (1) depends upon that of (2), in our context the latter is independent of the former. Specifically, the *separation theorem* asserts that:

Given the assumptions about borrowing, lending, and investor preferences stated earlier in this section, *the optimal proportionate composition of the stock (risk-asset) portfolio* (i.e. the solution to sub-problem 2 above) *is independent of the ratio of the gross investment in stocks to the total net investment.*

Tobin proved this important separation theorem by deriving the detailed solution for the optimal mix of risk assets *conditional* on a given gross investment in this portfolio, and then formally proving the critical invariance property stated in the theorem. Tobin used more restrictive assumptions that we do regarding the available investment opportunities and he permitted no borrowing.[11] Under our somewhat broadened assumptions in these respects, the problem fits neatly into a traditional Fisher framework, with different available combinations of expected values and standard deviations of return on alternative *stock portfolios* taking the place of the original "production opportunity" set and with the alternative investment choices being concurrent rather than between time periods. Within this framework, alternative and more transparent proofs of the separation theorem are available which do not involve the actual calculation of the best allocation in stocks over individual stock issues. As did Fisher, we shall present a simple algebraic proof[12], set out the logic of the argument leading to the theorem, and depict the essential geometry of the problem.[13]

As a preliminary step, we need to establish the relation between the investor's total investment in *any* arbitrary mixture or portfolio of individual stocks, his total net return from all his investments (including riskless assets and any borrowing), and the risk parameters of his investment position. Let the *interest rate* on riskless assets or borrowing be r^*, and the *uncertain return* (dividends plus price appreciation) *per dollar invested in the given portfolio of stocks* be r. Let w represent the *ratio* of gross investment in stocks to

[11] Tobin considered the special case where cash with no return was the only riskless asset available. While he formally required that all assets be held in non-negative quantities (thereby ruling out short sales), and that the total value of risk assets held not be greater than the investment balance available without borrowing, these non-negativity and maximum value constraints were not introduced into his formal solution of the optimal investment mix, which in turn was used in proving the invariance property stated in the theorem. Our proof of the theorem is independent of the programming constraints neglected in Tobin's proof. Later in this section we show that when short sales are properly and explicitly introduced into the set of possible portfolios, the resulting equations for the optimum portfolio mix are identical to those derived by Tobin, but that insistence on no short sales results in a somewhat more complex programming problem (when covariances are non-zero), which may however, be readily handled with computer programs now available.

[12] An alternative algebraic proof using utility functions explicitly is presented in the appendix, note I.

[13] Lockwood Rainhard, Jr. has also independently developed and presented a similar proof of the theorem in an unpublished seminar paper.

total *net* investment (stock plus riskless assets minus borrowing). Then the investor's net return per dollar of total net investment will be

(1) $\bar{y}=(1-w)r^*+w\tilde{r}=r^*+w(\bar{r}-r^*)$; $0 \leqq w < \infty$,

where a value of $w<1$ indicates that the investor holds some of his capital in riskless assets and receives interest amounting to $(1-w)r^*$; while $w>1$ indicates that the investor borrows to buy stocks on margin and pays interest amounting to the absolute value of $(1-w)r^*$. From (1) we determine the mean and variance of the net return per dollar of total net investment to be:

(2*a*) $\bar{y}=r^*+w(\bar{r}-r^*)$, and

(2*b*) $\sigma^2_y=w^2\sigma^2_r$.

Finally, after eliminating w between these two equations, we find that the direct relation between the expected value of the investor's net return per dollar of his total net investment and the risk parameters of his investment position is:

(3*a*) $\bar{y}=r^*+\theta\sigma_y$, where

(3*b*) $\theta=(\bar{r}-r^*)/\sigma_r$.

In terms of *any* arbitrarily selected *stock* portfolio, therefore, the investor's *net* expected rate of return on his total net investment is related *linearly* to the *risk* of return on his total net investment as *measured* by *the standard deviation* of his return. Given *any* selected stock portfolio, this linear function corresponds to Fisher's "market opportunity line"; its intercept is the risk-free rate r^* and its slope is given by θ, which is determined by the parameters $\bar{r}$ and σ_r of the particular stock portfolio being considered. We also see from (2*a*) that, by a suitable choice of w, the investor can use *any* stock mix (and *its* associated "market opportunity line") to obtain an expected return, $\bar{y}$, as high as he likes; but that, because of (2*b*)and (3*b*), as he increases his investment w in the (tentatively chosen) mix, the standard deviation σ_y (and hence the variance σ^2_y) of the return on his total investment also becomes proportionately greater.

Now consider all possible stock portfolios. Those portfolios having the same θ value will lie on the same "market opportunity line," but those having different θ values *will offer different "market opportunity lines"* (between expected return and risk) for the investor to use. The investor's problem is to choose which stock portfolio-mix (or market opportunity line *or* θ value) to use *and* how intensively to use it (the proper value of w). Since *any* expected return $\bar{y}$ can be obtained from *any* stock mix, an investor adhering to our choice criterion will minimize the variance of his over-all return σ^2_y associated with *any* expected return he may choose *by confining all his investment in stocks to the mix with the largest* θ *value. This portfolio minimizes the variance associated with any* $\bar{y}$ *(and hence any* w *value)* the investor may prefer, and *consequently, is independent* of $\bar{y}$ and w. This establishes the separation theorem[14], once we note that our assumptions regarding available portfolios[15] insure the existence of a maximum θ.

It is equally apparent that *after* determining the optimal stock portfolio (mix) by maximizing θ, the investor can complete his choice of an over-all investment position by substituting the θ of this optimal mix in (3) and decide which over-all investment position by substituting of the available $(\bar{y}, \sigma_y)$ pairs he prefers by referring to his own utility function. Substitution of this best $\bar{y}$ value in (2*a*) determines a unique best value of the ratio w of gross investment in the optimal stock portfolio to his total net investment, and hence, the optimal amount of investments in riskless savings deposits or the optimal amount of borrowing as well.

This separation theorem thus has four immediate *corrolaries* which can be stated:

(*i*) *Given* the assumptions about borrowing and lending stated above, any investor whose choices maximize the expectation of any particular utility function consistent with these conditions will make *identical decisions regarding the proportionate composition of his stock* (risk-asset) *portfolio. This is true regardless of the particular utility function*[16] whose expectation he maximizes.

(*ii*) Under these assumptions, only a *single point* on the Markowitz "Efficient Frontier" *is relevant* to the investor's decision regarding his investments in *risk* assets.[17] (The next section

[14] See also the appendix, note I for a different form of proof.

[15] Specifically, that the amount invested in any stock in any stock mix is infinitely divisible, that all expected returns on individual stocks are finite, that all variances are positive and finite, and that the variance-covariance matrix is positive-definite.

[16] When probability assessments are multivariate normal, the utility function may be polynomial, exponential, etc. Even in the "non-normal" case when utility functions *are* quadratic, they may vary in its parameters. See also the reference to Roy's work in the text below.

[17] When the above conditions hold (see also final para-

shows this point can be obtained directly without calculating the remainder of the efficient set.)

Given the same assumptions, (*iii*) the parameters of the investor's particular utility within the relevant set determine *only* the ratio of his total gross investment in stocks to his total *net* investment (including riskless assets and borrowing); and (*iv*) the investor's wealth is also, consequently, relevant to determining the *absolute size* of his investment in individual stocks, but *not* to the *relative distribution* of his gross investment in stocks *among individual issues*.

The Geometry of the Separation Theorem and Its Corrolaries

The algebraic derivations given above can be represented graphically as in chart 1. Any given available stock portfolio is characterized by a pair of values $(\sigma_r, \bar{r})$ which can be represented as a point in a plane with axes σ_y and $\bar{y}$. Our assumptions insure that the points representing all available stock mixes lie in a finite region, all parts of which lie to the right of the vertical axis, and that this region is bounded by a closed curve.[18] The contours of the investor's utility function are concave upward, and any movement in a north and or west direction denotes contours of greater utility. Equation (3) shows that all the (σ_y, y) pairs attainable by combining, borrowing, or lending with *any* particular stock portfolio lie on a ray from the point $(0, r^*)$ though the point corresponding to the stock mix in question. Each possible stock portfolio thus determines a unique "market opportunity line". Given the properties of the utility function, it is obvious that shifts from one possible mix to another which *rotate* the associated market opportunity line *counter colckwise* will *move the investor to preferred positions regardless of the point on the line* he had tentatively chosen. The slope of this market-opportunity line given by (3) is θ, and the limit of the favorable rotation is given by the maximum attainable θ, which identifies the optimal mix M.[19] Once this best mix, M, has been determined, the investor completes the optimization of his total investment position by selecting the point on the ray through M which is tangent to a utility contour in the standard manner. If his utility contours are as in the U_i set in chart 1, he uses savings accounts and does not borrow. If his utility contours are as in U_j set, he borrows in order to have a gross investment in his best stock mix greater than his net investment balance.

Risk Aversion, Normality and the Separation Theorem

The above analysis has been based on the assumptions regarding markets and investors stated at the beginning of this section. One crucial premise was investor *risk-aversion* in the form of *preference for expected return* and *preference against return-variance*, *ceteris paribus*. We noted that Tobin has shown that *either* concave-*quadratic* utility functions *or* multivariate *normality* (of probability assessments) *and any concave* utility were *sufficient* conditions to validate this premise, but they were *not* shown (or alleged) to be *necessary* conditions. This is probably fortunate because the quadratic utility of income (or wealth!) function, in spite of its popularity in theoretical work, has several undesirably restrictive and implausible properties,[20] and, despite

graph of this section), the modest narrowing of the relevant range of Markowitz' Efficient Set suggested by Baumol [2] is still larger than needed by a factor strictly proportionate to the number of portfolios he retains in his truncated set! This is true since the relevant set is a single portfolio under these conditions.

18 See Markowitz [14] as cited in the appendix, note I.

19 The analogy with the standard Fisher two-period production-opportunity case in perfect markets with equal borrowing and lending rates is clear. The optimal set of production opportunities available is found by moving along the envelope function of efficient combinations of projects onto ever higher present value lines to the highest attainable. This best set of production opportunities is independent of the investor's particular utility function which determines only whether he then lends or borrows in the market (and by how much in either case) to reach hi best over-all position. The only differences between this case and ours lie in the concurrent nature of the comparisons (instead of inter-period), and the rotation of the market opportunity lines around the common pivot of the riskless return (instead of parallel shifts in present value lines). See Fisher [4] and also Hirschlaifer [7], figure 1 and section 1*a*.

20 In brief, not only does the quadratic function imply negative marginal utilities of income or wealth much "too soon" in empirical work unless the risk-aversion parameter is very small — in which case it cannot account for the degree of risk-aversion empirically found,— it also implies that, over a major part of the range of empirical data, common stocks, like potatoes in Ireland, are "inferior" goods. Offering more return at the same risk would so sate investors that they would reduce their risk-investments *because* they were more attractive. (Thereby, as Tobin [21] noted, denying the negatively sloped demand curves for *riskless* assets which are standard doctrine in "liquidity preference theory" — a conclusion which cannot, incidentally, be avoided by "limit arguments" on quadratic utilities such as he used, once borrowing and leverage are admitted.)

its mathematical convenience, multivariate normality is doubtless also suspect, especially perhaps in considering common stocks.

It is, consequently, very relevant to note that by using the Bienaymé-Tchebycheff inequality, Roy [19] has shown that investors operating on his "Safety First" principle (i.e. make risky investments so as to minimize the upper bound of the probability that the realized outcome will fall below a pre-assigned "disaster level") should maximize the ratio of the *excess* expected portfolio return (over the disaster level) to the standard deviation of the return on the portfolio[21] — which is precisely our criterion of max θ when his disaster level is equated to the risk-free rate r^*. This result, of course, does not depend on multivariate normality, and uses a different argument and form of utility function.

The *Separation Theorem*, and its Corrolaries (*i*) and (*ii*) above — and all the rest of our following analysis which depends on the maximization of θ — is thus rigorously appropriate in the non-multivariate normal case for Safety-Firsters who minimax the stated upper bound of the chance of doing less well on portfolios including risk assets than they can do on riskless investments, just as it is for concave-expected utility maximizers in the "normal" case. On the basis of the same probability judgments, these Safety-Firsters will use the same proximate criterion function (max θ) and will choose proportionately the same risk asset portfolios as the more orothodox "utility maximizers" we have hitherto considered.

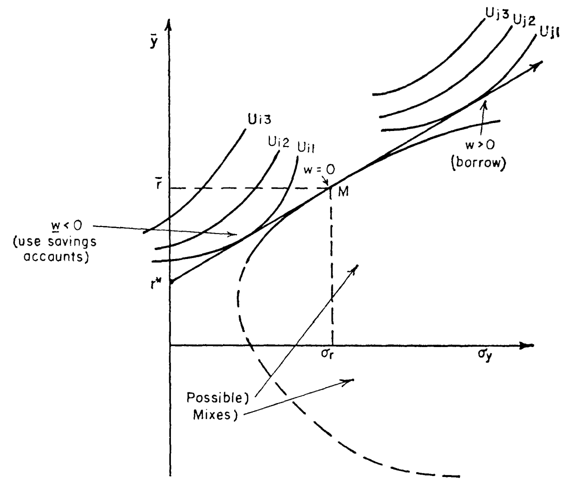

FIGURE I

This function also implausibly implies, as Pratt [17] and Arrow [1] have noted, that the insurance premiums which people would be willing to pay to hedge *given* risks *rise* progressively with wealth or income. For a related result, see Hicks [6, p. 802].

[21] Roy also notes that when judgmental distributions *are* multivariate normal, maximization of this criterion *minimizes* the probability of "disaster" (failure to do better in stocks than savings deposits or government bonds held to maturity). It should be noted, however, minimization of the probability of short falls from "disaster" levels in this "normal" case is strictly *equivalent* to expected utility maximization under *all* risk-averters' utility functions. The equivalence is *not* restricted to the utility function of the form (0, 1) (zero if "disaster" occurs, one if it doesn't), as claimed by Roy [19, p. 432] and Markowitz [14, p. 293 and following.].

II — Portfolio Selection: The Optimal Stock Mix

Before finding the optimal stock mix — the mix which maximizes θ in (3*b*) above — it is necessary to express the return on any arbitrary mix in terms of the returns on individual stocks included in the portfolio. Although short sales are excluded by assumption in most of the writings on portfolio optimization, this restrictive assumption is arbitrary for some purposes at least, and we therefore broaden the analysis in this paper to include short sales whenever they are permitted.

Computation of Returns on a Stock Mix, When Short Sales are Permitted

We assume that there are m different stocks in the market, denoted by $i = 1, 2, \ldots, m$, and treat short sales as negative purchases. We shall use the following basic notation:

- $|h_i|$ – The ratio of the gross investment in the i^{th} stock (the market value of the amount bought *or* sold) to the gross investment in all stocks. A positive value of h_i indicates a *purchase*, while a negative value indicates a *short sale.*
- $\tilde{r}_i$ – The return per dollar invested in a *purchase* of the i^{th} stock (cash dividends plus price appreciation)
- $\tilde{r}$ – As above, the return per dollar invested in a particular *mix* or *portfolio* of stocks.

Consider now a gross investment in the entire mix, so that the actual investment in the i^{th} stock is equal to $|h_i|$. The returns on purchases and short sales need to be considered separately.

First, we see that if $|h_i|$ is invested in a *pur-*

chase ($h_i > 0$), the return will be simply $h_i\tilde{r}_i$. For reasons which will be clear immediately however, we write this in the form:

(4a) $h_i\tilde{r}_i = h_i(\tilde{r}_i - r^*) + |h_i|\, r^*$.

Now suppose that $|h_i|$ is invested in a *short sale* ($h_i < 0$), this gross investment being equal to the price received for the stock. (The price received must be deposited in escrow, and in addition, an amount equal to margin requirements on the current price of the stock sold must be remitted or loaned to the actual owner of the securities borrowed to effect the short sale.) In computing the *return* on a short sale, we know that the short seller must pay to the person who lends him the stock any dividends which accrue while the stock is sold short (and hence borrowed), and his capital gain (or loss) is the negative of any price appreciation during this period. In addition, the short seller will receive interest at the riskless rate r^* on the sales price placed in escrow, and he may or may not *also* receive interest at the same rate on his cash remittance to the lender of the stock. To facilitate the formal analysis, we *assume* that *both interest components* are *always received* by the short seller, and that margin requirements are 100%. In this case, the short seller's *return* per dollar of his gross investment will be ($2r^* - r_i$), and if he invests $|h_i|$ in the short sale ($h_i < 0$), its contribution to his portfolio return will be:

(4b) $|h_i|\,(2r^* - \tilde{r}_i) = h_i(\tilde{r}_i - r^*) + |h_i|\, r^*$.

Since the right-hand sides of (4a) and (4b) are identical, the total return per dollar invested in *any* stock mix can be written as:

[22]In recent years, it has become increasingly common for the short seller to waive interest on his deposit with the lender of the security — in market parlance, for the borrowers of stock to obtain it "flat" — and when the demand for borrowing stock is large relative to the supply available for this purpose, the borrower may pay a cash premium to the lender of the stock. See Sidney M. Robbins, [18, pp. 58–59]. It will be noted that these practices reduce the expected return of short sales without changing the variance. The formal procedures developed below permit the identification of the appropriate stocks for short sale assuming the expected return is ($2r^* - \tilde{r}_i$). If these stocks were to be borrowed "flat" or a premium paid, it would be *simply necessary to iterate the solution after replacing* ($\tilde{r}_i - r^*$) in (4b) and (5) *for these stocks* with the value ($\tilde{r}_i$) — and *if*, in addition, a premium p_i is paid, the term ($\tilde{r}_i + p_i$) should be substituted (where $p_i \geqq 0$ is the premium (if any) per dollar of sales price of the stock to be paid to lender of the stock). With equal lending and borrowing rates, changes in margin requirements will not affect the calculations. (I am indebted to Prof. Schlaifer for suggesting the use of absolute values in analyzing short sales.)

(5) $\tilde{r} = \Sigma_i\, [h_i(\tilde{r}_i - r^*) + |h_i|\, r^*]$
$= r^* + \Sigma_i h_i(\tilde{r}_i - r^*)$

because $\Sigma_i\, |h_i| = 1$ by the definition of $|h_i|$.

The expectation and variance of the return on any stock mix is consequently

(6a) $\bar{r} = r^* + \Sigma_i h_i(\bar{r}_i - r^*) = r^* + \Sigma_i h_i \bar{x}_i$,

(6b) $\check{r} = \Sigma_{ij} h_i h_j \check{r}_{ij} = \Sigma_{ij} h_i h_j \check{x}_{ij}$

where $\check{r}_{ij}$ represents the variance $\sigma_{rij}{}^2$ when $i = j$, and covariances when $i \neq j$. The notation has been further simplified in the right-hand expressions by defining:

(7) $\tilde{x}_i = \tilde{r}_i - r^*$,

and making appropriate substitutions in the middle expressions. The quantity θ defined in (3b) can thus be written:

(8) $$\theta = \frac{\bar{r} - r^*}{(\check{r})^{1/2}} = \frac{\bar{x}}{(\check{x})^{1/2}} = \frac{\Sigma_i h_i \bar{x}_i}{(\Sigma_{ij} h_i h_j \check{x}_{ij})^{1/2}}.$$

Since h_i may be either positive or negative, equation (6a) shows that a portfolio with $\bar{r} \gtrless r^*$ and hence with $\theta > 0$ exists if there is one or more stocks with $\bar{r}_i$ not exactly equal to r^*. We assume throughout the rest of the paper that such a portfolio exists.

Determination of the Optimal Stock Portfolio

As shown in the proof of the Separation Theorem above, the optimal stock portfolio is the one which maximizes θ as defined in equation (8). We, of course, wish to maximize this value subject to the constraint

(9) $\Sigma_i\, |h_i| = 1$,

which follows from the definition of $|h_i|$. But we observe from equation (8) that θ is a *homogeneous function of order zero* in the h_i: the value of θ is *unchanged by any proportionate* change in all h_i. Our problem thus reduces to the simpler one of finding a vector of values yielding the *unconstrained* maximum of θ in equation (8), after which we may scale these initial solution values to satisfy the constraint.

The Optimum Portfolio When Short Sales are Permitted

We first examine the partial derivatives of (8) with respect to the h_i and find:

(10) $$\frac{\partial\theta}{\partial h_i} = (\sigma_x)^{-1}\,[\bar{x}_i - \lambda(h_i \check{x}_{ii} + \Sigma_j h_j \check{x}_{ij})\,],$$

where,

(11) $\lambda = \bar{x}/\sigma_x{}^2 = \Sigma_i h_i \bar{x}_i / \Sigma_i \Sigma_j h_i h_j \check{x}_{ij}$.

The *necessary and sufficient conditions* on the *relative* values of the h_i for a stationary *and the unique* (*global*) *maximum*[23] are obtained by setting the derivatives in (10) equal to zero, which give the set of equations

$$(12) \quad z_i\check{x}_{ii} + \Sigma_j z_j \check{x}_{ij} = \bar{x}_i, \quad i = 1, 2, \ldots, m;$$

where we write

$$(13) \quad z_i = \lambda h_i.$$

It will be noted the set of equations (12) — which are identical to those Tobin derived by a different route[24] — are *linear* in the own-*variances*, pooled *covariances*, and *excess returns* of the respective securities; and since the covariance matrix $\underline{\underline{\check{x}}}$ is positive definite and hence non-singular, this system of equations has a unique solution

$$(14) \quad z_i^0 = \Sigma_j \check{x}^{ij}\bar{x}_j$$

where $\check{x}^{ij}$ represents the ij^{th} element of $(\underline{\underline{\check{x}}})^{-1}$, the inverse of the covariance matrix. Using (13), (7), and (6*b*), this solution may also be written in terms of the primary variables of the problem in the form

$$(15) \quad h_i^0 = (\lambda^0)^{-1}\Sigma_j \check{r}^{ij}(\bar{r}_j - r^*), \quad \text{all } i.$$

Moreover, since (13) implies

$$(16) \quad \Sigma_i |z_i| = \lambda \Sigma_i |h_i|,$$

λ^0 may readily be evaluated, after introducing the constraint (9) as

$$(17) \quad \Sigma_i |z_i^0| = \lambda^0 \Sigma_i |h_i^0| = \lambda^0$$

The optimal *relative* investments z_i^0 can consequently be scaled to the optimal proportions of the stock portfolio h_i^0, by dividing each z_i^0 by the sum of their absolute values. A comparison of equations (16) and (11) shows further that:

$$(18) \quad \Sigma_i |z_i^0| = \lambda^0 = \bar{x}^0/\sigma_{x^0}{}^2;$$

i.e. the sum of the absolute values of the z_i^0 yields, as a byproduct, the value of the ratio of the expected excess rate of return on the optimal portfolio to the variance of the return on this best portfolio.

It is also of interest to note that if we form the corresponding λ-ratio of the expected excess return to its variance for each i^{th} stock, we have at the optimum:

$$(19) \quad h_i^0 = (\lambda_i/\lambda^0) - \Sigma_{j \neq i} h_j^0 \check{x}_{ij}/\check{x}_{ii} \text{ where } \lambda_i = \bar{x}_i/\check{x}_{ii}.$$

The optimal fraction of each security in the best portfolio is equal to the ratio of *its* λ_i to that of the entire portfolio, *less* the ratio of its pooled covariance with other securities to its own variance. Consequently, *if* the investor were to act on the assumption that all covariances were zero, he could pick his optimal portfolio mix very simply by determining the λ_i ratio of the expected excess return $\bar{x}_i = \bar{r}_i - r^*$ of each stock to its variance $\check{x}_{ii} = \check{r}_{ii}$, and setting each $h_i = \lambda_i/\Sigma\lambda_i$; for with no covariances,[25] $\Sigma\lambda_i = \lambda^0 = \bar{x}^0/\sigma_{x^0}{}^2$. With this simplifying assumption, the λ_i ratios of each stock suffice to determine the optimal mix by simple arithmetic;[26] in the more general case with non-zero covariances, a single set[27] of linear equations must be solved in the usual way, but no (linear or non-linear) programming is required and no more than one point on the "efficient frontier" need ever be computed, given the assumptions under which we are working.

The Optimum Portfolio When Short Sales are not Permitted

The exclusion of short sales does not complicate the above analysis *if* the investor is willing to act on an assumption of no correlations between the returns on different stocks. In this case, he finds his best portfolio of "long" holding by merely eliminating all securities whose λ_i-

[23]It is clear from a comparison of equations (8) and (11), showing that sgn θ = sgn λ, that only the vectors of h_i values corresponding to $\lambda > 0$ are relevant to the maximization of θ. Moreover, since θ as given in (8) and all its first partials shown in (10) are continuous functions of the h_i, it follows that when short sales are permitted, any maximum of θ must be a stationary value, and any stationary value is a maximum (rather than a minimum) when $\lambda > 0$ because θ is a convex function with a positive-definite quadratic form in its denominator. For the same reason, any maximum of θ is a unique (global) maximum.

[24]See Tobin, [21], equation (3.22), p. 83. Tobin had, however, formally required no short selling or borrowing, implying that this set of equations is valid under these constraints [so long as there is a single riskless asset (pp. 84–85)]; but the constraints were ignored in his derivation. We have shown that this set of equations *is* valid *when short sales* are properly included in the portfolio *and borrowing* is available in perfect markets in unlimited amounts. The alternative set of equilibrium conditions required when short sales are ruled out is given immediately below. The complications introduced by borrowing restrictions are examined in the final section of the paper.

[25] With no covariances, the set of equations (12) reduces to $\lambda h_i = \bar{x}_i/\check{x}_{ii} = \lambda_i$, and after summing over all $i = 1, 2 \ldots m$, and using the constraint (9), we have immediately that $|\lambda^0| = \Sigma_i |\lambda_i|$, and $\lambda^0 > 0$ for max θ (instead of min θ).

[26] Using a more restricted market setting, Hicks [6, p. 801] has also reached an equivalent result when covariances are zero (as he assumed throughout).

[27] See, however, footnote 22, above.

ratio is negative, and investing in the remaining issues in the proportions $h_i = \lambda_i/\Sigma\lambda_i$ in accordance with the preceding paragraph.

But in the more generally realistic cases when covariances are nonzero *and* short sales are not admitted, the solution of a single bilinear or quadratic programming problem is required to determine the optimal portfolio. (All other points on the "efficient frontier," of course, continue to be irrelevant so long as there is a riskless asset and a "perfect" borrowing market.) The optimal portfolio mix is now given by the set of h_i^0 which maximize θ in equation (8) subject to the constraint that all $h_i \geqq 0$. As before, the (further) constraint that the sum of the h_i be unity (equation 9) may be ignored in the initial solution for the *relative* values of the h_i [because θ in (8) is homogeneous of order zero]. To find this optimum, we form the Lagrangian function

$$(20) \qquad \phi(\underline{h}, \underline{u}) = \theta + \Sigma_i u_i h_i$$

which is to be maximized subject to $h_i \geqq 0$ and $u_i \geqq 0$. Using (11), we have immediately

$$(21) \qquad \frac{\partial\phi}{\partial h_i} \gtreqless 0 \leftrightarrow \bar{x}_i - \lambda(h_i \check{x}_{ii} + \Sigma_j h_j \check{x}_{ij}) + \alpha u_i \gtreqless 0.$$

As in the previous cases, we also must have $\lambda > 0$ for a maximum (rather than a minimum) of ϕ, and we shall write $z_i = \lambda h_i$ and $v_i = \alpha u_i$. The necssary and sufficient conditions for the vector of *relative* holdings z_i^0 which maximizes θ in (20) are consequently,[28] using the Kuhn-Tucker theorem [9],

$$(22a) \quad z_i^0 \check{x}_{ii} + \Sigma_j z_j^0 \check{x}_{ij} - v_i^0 = \bar{x}_i, \; i = 1, 2, \ldots m;$$

where

$$(22b\text{-}d) \quad z_i^0 \geqq 0, \; v_i^0 \geqq 0, \; z_i^0 v_i^0 = 0.$$

This system of equations can be expeditiously solved by the Wilson Simplicial Algorithm [23].

Now let m' denote the number of stocks with strictly positive holdings $z_i^0 > 0$ in (22*b*), and renumber the entire set of stocks so that the subset satisfying this strict inequality [and, hence also, by (22*d*) $v_i^0 = 0$] are denoted 1, 2, . . . , m'. *Within this m' subset of stocks found to belong in the optimal portfolio with positive holdings*, we consequently have, using the constraint (19),

$$(17a) \quad \Sigma_{i=1}^{m'} z_i^0 = \lambda^0 \Sigma_{i=1}^{m'} h_i^0 = \lambda^0$$

so that the *fraction* of the optimal portfolio invested in the i^{th} stock (where $i = 1, 2 \ldots m'$) is

$$(23) \qquad h_i^0 = z_i^0/\lambda^0 = z_i^0/\Sigma_{i=1}^{m'} z_i^0 .$$

Once again, using (17*a*) and (11), the sum of the z_i^0 within this set of stocks held yields as a by-product the ratio of the expected excess rate of return on the optimal *portfolio* to the variance of the return on this best portfolio:

$$(18a) \quad \Sigma_{i=1}^{m'} z_i^0 = \lambda^0 = \bar{x}^0/\sigma^2_{x^0}.$$

Moreover, since $z_i^0 > 0$ in (22*a* and 22*b*) strictly implies $v_i^0 = 0$ by virtue of (22*c*), equation (22*a*) *for the subset of positively held stocks $i = 1, 2 \ldots m'$* is formally identical to equation (12). We can, consequently, use these equations to bring out certain significant properties of the security portfolios which will be held by risk-averse investors trading in perfect markets.[29] *In the rest of this paper, all statements with respect to "other stocks" will refer to other stocks included within the portfolio.*

III Risk Premiums and Other Properties of Stocks Held Long or Short in Optimal Portfolios

Since the covariances between most pairs of stocks will be positive, it is clear from equation (19) that stocks held long ($h_i^0 > 0$) in a portfolio will generally be those whose expected

[28] Equation (22*a*–22*d*) can readily be shown to satisfy the six necessary and two further sufficient conditions of the Kuhn-Tucker theorem. Apart from the constraints $\underline{h} \geqq 0$ and $\underline{u} \geqq 0$ which are automatically satisfied by the computing algorithm [conditions (22*b* and 22*c*)] the four *necessary* conditions are:

1) $\left[\frac{\partial\phi}{\partial h_i}\right]^0 \leqq 0$. This condition is satisfied *as a strict equality* in our solutions by virtue of equation (22*a*) [See equation (21)]. This strict equality also shows that,

2) $h_i^0 \left[\frac{\partial\phi}{\partial h_i}\right]^0 = 0$, the first complementary slackness condition is also satisfied.

3) $\left[\frac{\partial\phi}{\partial u_i}\right]^0 \geqq 0$. This condition is satisfied because from equation (20), $\left[\frac{\partial\phi}{\partial u_i}\right]^0 = h_i^0 \geqq 0$ by virtue of equation (22*b*). This same equation shows that the second complementary slackness condition,

4) $u_i^0 \left[\frac{\partial\phi}{\partial u_i}\right]^0 = 0$, may be written $u_i^0 h_i^0 = 0$ which is also satisfied because of equation (22*c*) since $\alpha \neq 0$.

The two additional *sufficiency* conditions are of course satisfied because the variance-covariance matrix $\underline{\underline{x}}$ is positive definite, making $\phi(\underline{h}, \underline{u}^0)$ a concave function on $\underline{h}$ and $\phi(\underline{h}^0, \underline{u})$ a convex function of $\underline{u}$.

[29] More precisely, the properties of portfolios when both the investors and the markets satisfy the conditions stated at the outset of section I or, alternatively, when investors satisfy Roy's premises as noted previously.

return is enough greater than the risk-free rate to offset the disutility, so to speak, of the contribution of their variance and pooled covariance to the risk of the entire portfolio. This much is standard doctrine. Positive covariances with other securities held long in the portfolio raise the minimum level of $\bar{x}_i > 0$ which will lead to the inclusion of the i^{th} stock as a positive holding in the optimal portfolio. But equation (19) shows that stocks whose expected returns are *less* than the riskless rate (i.e. $\bar{x}_i < 0$ or $\bar{r}_i < r^*$) will *also* be held *long* ($h_i^0 > 0$) *provided* that *either* (*a*) they are *negatively correlated* in sufficient degree with other important stocks *held long* in the portfolio, or (*b*) that they are *positively correlated* in sufficient degree with other important stocks *held short* in the portfolio. The precise condition for $h_i^0 > 0$ when $\bar{x}_i < 0$ is that the weighted sum of the i^{th} covariances be sufficiently negative to satisfy

(19*a*) $h_i^0 > 0 \leftrightarrow |\Sigma_{j \neq i} h_j^0 \check{x}_{ij}| > |\bar{x}_i/\lambda^0|$,

which follows from (19) since $\check{x}_{ii} > 0$.

Since our $\bar{x}_i$ is precisely what is usually called the "risk premium" in the literature, we have just shown that *the "risk premiums" required on risky securities* (i.e. those with σ_i and $\sigma_i^2 > 0$) for them *to be held long by optimizing risk-averse investors in perfect markets need not always be positive*, as generally presumed. *They will in fact be negative* under either of the conditions stated in (*a*) and (*b*) above, summarized in (19*a*). The explanation is, of course, that a long holding of a security which is negatively correlated with other long holdings tends to reduce the variance of the whole portfolio by offsetting some of the variance contributed by the other securities in the portfolio, and this "variance-offsetting" effect may dominate the security's own-variance and even a negative expected excess return $\bar{x}_i < 0$.

Positive correlations with other securities held short in the portfolio have a similar variance-offsetting effect.[30]

Correspondingly, it is apparent from (19) itself that any stock with *positive* excess returns or risk premiums ($\bar{x}_i > 0$) will be held *short* ($h_i^0 < 0$) in the portfolio *provided that either* (*a*) it is *positively correlated* in sufficient degree with other stocks *held long* in the portfolio, or (*b*) it is *negatively correlated* in sufficient degree with other stocks *held short* in the portfolio. *Positive* (negative) *risk premiums are neither a sufficient nor a necessary condition for a stock to be held long* (short).

Indifference Contours

Equation (12) (and the equivalent set (22*a*) restricted to stocks actually held in portfolios) also enables us to examine the *indifference contours* between expected excess returns, variances, or standard deviations and covariances of securities which will result in the *same fraction* h_i^0 of the investor's portfolio being held in a given security. The general presumption in the literature, as noted in our introduction,[31] is that the market values of risk assets are adjusted in perfect markets to maintain a *linear* relation between expected rates of return (our $\bar{r}_i = \bar{x}_i + r^*$) *and risk* as measured by the *standard deviation* of return σ_i on the security in question. This presumption probably arises from the fact that this relation *is* valid for trade offs *between* a riskless security *and* a single risk asset (or a *given mix* of risk assets to be held in fixed proportions). But it can *not* be validly attributed to indifferent trade offs *between* risk assets *within* optimizing risk-asset portfolios. In point of fact, it can easily be shown that there is a *strictly linear indifference contour* between the *expected return* $\bar{r}_i$ (or the expected excess return $\bar{x}_i$) *and the variance* σ_i^2 (*not* the standard deviation σ_i) of the individual security, and this linear function has very straightforward properties. The assumption made in this derivation that the covariances σ_{ij} with other securities are invariant is a more reasonable one than is perhaps readily apparent.[32] Subject to the acceptability

[30] Stocks with negative expected excess returns or "risk premiums" ($\bar{x}_i < 0$) will, of course, enter into portfolios only as short sales (provided these are permitted) when the inequality in (19*a*) is reversed, i.e.

$h_i^0 < 0 \leftrightarrow \Sigma_{j \neq i} h_j^0 \check{x}_{ij} + \bar{x}_i/\lambda^0 < 0$. When short sales are not permitted, and (19*a*) is not satisfied, stocks with $\bar{x}_i < 0$ simply do not appear in the portfolio at all.

[31] See footnote 3 for references and quotations.

[32] Fixed covariances are directly implied by the assumption that every pair of i^{th} and j^{th} stocks are related by a one-common-factor model (e.g. the general state of the economy or the general level of the stock market), so that, letting $\tilde{\mu}$ represent the general exogenous factor and $\tilde{\omega}$ the random outcome of endogenous factors under management's control, we have

$\tilde{x}_i = a_i + b_i \tilde{\mu} + \tilde{\omega}_i$

$\tilde{x}_j = a_j + b_j \tilde{\mu} + \tilde{\omega}_j$

with $\tilde{\mu}$, $\tilde{\omega}_i$, and $\tilde{\omega}_j$ mutually independent. This model implies

$\sigma_i^2 = b_i^2 \sigma_\mu^2 + \sigma_\omega^2$, and $\sigma_{ij} = b_i b_j \sigma_\mu^2$,

so that if management, say, varies the part under its control, of this latter assumption, it follows that *risk classes of securities should be scaled in terms of variances* of returns rather than standard deviations (with the level of covariances reflected in the parameters of the linear function). The complexities involved when indifference contours are scaled on covariances or standard deviations are indicated below.

The conclusion that the indifference contour between $\bar{x}_i$ and the variance σ_i^2 is *linear* in the general case when all covariances σ_{ij} are held constant is established in the appendix, note **II**, by totally differentiating the equilibrium conditions (12) [or the equivalent set (22*a*) restricted to the m' stocks held in the portfolio]. But *all* pairs of values of $\bar{x}_i$ and σ_i^2 along the linear indifference coutour which holds h_i^0 fixed at some given level also rigorously imply that the proportionate mix of *all other* stocks in the portfolio is *also unchanged.* Consequently, we may proceed to derive other properties of this indifference contour by examining a simple "two security" portfolio. (The i^{th} security is renumbered "1," and "all other" securities are called the second security.) If we then solve the equilibrium conditions[33] (12) in this two-stock case and hold $K = h_1^0/h_2^0$ constant, we have

$$(24) \quad K = h_1^0/h_2^0 = \text{constant} = (\bar{x}_1\sigma_2^2 - \bar{x}_2\sigma_{12})/(\bar{x}_2\sigma_1^2 - \bar{x}_1\sigma_{12})$$

which leads to the desired explicit expression, using $\bar{r}_1 = \bar{x}_1 + r^*$,

$$(25) \quad \bar{r}_1 = r^* + W\sigma_{12} + WK\sigma_1^2,$$

where

$$(25a) \quad W = \bar{x}_2/(\sigma_2^2 + K\sigma_{12}).$$

Since[34] $WK = \lambda^0 h_1^0$ and $\lambda^0 > 0$, the *slope* of this indifference contour between $\bar{x}_1$ and σ_1^2 will always be positive when $h^0_1 > 0$ (as would be expected, because when σ_{12} is held constant, increased variance requires added return to justify any given positive holding[35]); but when the first stock is held short, its expected (or excess) return and its variance along the contour vary inversely (as they should since "shorts" profit from price declines). Moreover, if we regard σ_{12} as an exogenous "shift" parameter, the *constant term* (or intercept) of this indifference contour varies directly[36] with σ_{12}, and the slope of $\bar{x}_1$ on σ_1^2 varies inversely[37] with σ_{12} in the usual case, when $\bar{x}_2 > 0$.

Now note that (25) and (25*a*) can be written

$$(25b) \quad \bar{r}_1 = r^* + \bar{x}_2(\sigma_{12} + K\sigma_1^2)/(\sigma_2^2 + K\sigma_{12}),$$

which clearly depicts a hyperbolic (rather than linear) indifference contour on σ_{12} if σ_1^2 is regarded as fixed, and a more complex function between $\bar{r}_1$ (or x_1) and the standard deviation σ_1, which may be written (using $\sigma_{12} = \sigma_1\sigma_2\rho$),

$$(25b') \quad \bar{x}_1 = \frac{\bar{x}_2 K\sigma_1^2\,[1 + \rho(K\sigma_1/\sigma_2)^{-1}]}{\sigma_2^2\,(1 + \rho K\sigma_1/\sigma_2)}$$

The *slope* of the indifference contour between $\bar{x}_1$ and σ_1 is a still more involved function, which may be written most simply as

$$(25c) \quad \frac{\partial \bar{x}_1}{\partial \sigma_1} = \frac{\bar{x}_2\,[2K\sigma_1\sigma_2^2 + (K^2\sigma_1^2\sigma_2 + \sigma_2^3)\rho]}{(\sigma_2^2 + K\sigma_{12})^2} = 2K\sigma_1\bar{x}_2\frac{1 + (\rho/2)\,[(K\sigma_1/\sigma_2) + (\sigma_2/K\sigma_1)].}{\sigma^2(1 + \rho K\sigma_1/\sigma_2)^2}$$

It is true, in the usual situation with $K > 0$, $\bar{x}_2 > 0$, and $\rho > 0$, that $\bar{x}_1$ $(= \bar{r}_1 - r^*)$ and $\partial x_1/\partial\sigma_1$ are necessarily positive as common doctrine presumes, *but* the complex non-linearity is evident even in this "normal case" restricted to two stocks — and the *positive risk premium* $\bar{x}_1$ *and positive slope* on σ_1, of course, *cannot be generalized.* For instance, in the admittedly less usual but important case with $\bar{x}_2 > 0$ and the intercorrelation $\rho < 0$, *both* $\bar{x}_1$ and $\partial\bar{x}_1/\partial\sigma_1$ are *alternatively negative and positive* over different ranges [38] of σ_1 for any fixed h_1^0 or $K > 0$.

$\bar{\omega}$ and σ_ω^2, the covariance will be unchanged. (This single-common-factor model is essentially the same as what Sharpe [20] calls the "diagonal" model.)

[33] The explicit solution is $z_1^0 = \lambda^0 h_1^0 = (\bar{x}_1\sigma_2^2 - \bar{x}_2\sigma_{12})/(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)$; and $z_2^0 = \lambda^0 h_2^0 = (\bar{x}_2\sigma_1^2 - \bar{x}_1\sigma_{12})/(\sigma_1^2\sigma_1^2 - \sigma_{12}^2)$; where $\lambda^0 = z_1^0 + z_2^0$.

[34] Upon substituting (24) in (25) and using the preceding footnote, we have $W = \lambda^0 h_2^0 = z_2^0$, from which it follows that $WK = \lambda^0 h_2^0 h_1^0/h_2^0 = \lambda^0 h_1^0$.
As noted earlier, we have $\lambda^0 > 0$ (because the investor maximizes and does not minimize θ). [It may be noted that W is used instead of z_2^0 in (25) in order to incorporate the restriction on the indifference contours that K is constant, and thereby to obtain an expression (25*a*) which does not contain $\bar{x}_1$ and σ_1^2 (as does z_2^0 without the constraint of constant K).]

[35] Note that this is true whether the "other security" is held long or short.

[36] Let the constant term in (25) be $C = r^* + W\sigma_{12}$. Then

$$\frac{\partial C}{\partial \sigma_{12}} = \frac{(\sigma_2^2 + K\sigma_{12})\,\bar{x}_2 - \bar{x}_2\sigma_{12}K}{(\sigma_2^2 + K\sigma_{12})^2} = \frac{\bar{x}_2\,\sigma_2^2}{(\sigma_2^2 + K\sigma_{12})^2}$$

which has the same sign as $\bar{x}_2$, independent of the sign of K, σ_{12}, or $\bar{x}_1$.

[37] We have $\partial WK/\partial\sigma_{12} = -K^2\bar{x}_2/(\sigma_2^2 + K\sigma_{12})^2$, which has a sign opposite to that of $\bar{x}_2$.

[38] With $K > 0$, $\bar{x}_2 > 0$, and $\rho < 0$, we have from (25*b'*)

$\bar{x}_1 < 0$ if $0 < K\sigma_1/\sigma_2 < |\rho|$, and

$\bar{x}_1 > 0$ if $|\rho| < K\sigma_1/\sigma_2 < |\rho^{-1}|$.

On the other hand, from (25*c*) we have

Moreover, *in contrast* to the $\bar{x}_i - \sigma_1^2$ contour examined above, the pairs of values along the $\bar{x}_i - \sigma_i$ contour which hold h_i^0 constant do *not* imply an unchanged mix[39] of the other stocks in the optimizing portfolio when $m' > 2$; nor is λ^0 invariant along an $\bar{x}_i - \sigma_i$ contour, as it is along the $\bar{x}_1 - \sigma_1^2$ contour with covariances constant. For both reasons, the indifference contour between $\bar{x}_1$ and σ_1 for portfolios of $m' > 2$ stocks is very much more complex than for the two-stock case, whereas the "two-stock" contour (3) between $\bar{x}_1$ and σ_1^2 is exact for any number of stocks (when "all other" stocks are pooled in fixed proportions, as we have seen they can validly be). We should also observe that there does not seem to be an easy set of economically interesting assumptions which lead to *fixed correlations* as σ_1 varies (as assumed in deriving $\bar{x}_1 - \sigma_1$ indifference contours) in marked contrast to the quite interesting and plausible "single-factor" model (see footnote 32 above) which directly validates the assumption of fixed covariances used in deriving the $\bar{x}_1 - \sigma_1^2$ indifference contours.

In sum, we conclude that — however natural or plausible it may have seemed to relate risk premiums to standard deviations of return *within* portfolios of risk assets, and to scale risk classes of securities on this same basis — risk premiums can most simply *and* plausibly be related directly to *variances* of returns (with the level of covariances reflected in the parameters of the linear function). Since the principal function of the concept of "risk class" has been to delineate a required level of risk premium, we conclude further that risk classes should also be delineated in the same units (variances) if, indeed, the concept of risk class should be used at all.[40]

IV — Market Prices of Shares Implied by Shareholder Optimization in Purely Competitive Markets Under Idealized Uncertainty

Our analysis to this point has followed Tobin [21] and Markowitz [14] in assuming that current security prices are *exogenous data*, and that each investor acts on his own (doubtless unique) probability distribution over rates of return, *given* these market prices. I shall continue to make the same assumptions concerning markets and investors introduced in section I. In particular, it is assumed that security markets are purely competitive, transactions costs and taxes are zero, and *all* investors prefer a greater mean rate of return for a given variance and a lesser rate of return variance for any given mean return rate. But in this and the following section, I shall *assume* (1) that investors' joint probability distributions pertain to *dollar returns rather than rates* of return — the dollar return in the period being the sum of the cash dividend and the increase of market price during the period. Also, for simplicity, assume that (2) for *any* given set of market prices for all stocks, *all* investors assign *identical* sets of means, variances, and convariances to the joint distribution of these dollar returns (and hence for *any* set of prices, to the vector of means and the variance-covariances matrix of the rates of return $\bar{r}_i$ of all stocks), and that all correlations between stocks are < 1.

This assumption of identical probability beliefs or judgments by all investors in the market restricts the applicability of the analysis of this and the following section to what I have elsewhere characterized as *idealized uncertainty* [10, pp. 246–247]. But however unrealistic this latter assumption may be, it does enable us to derive a set of (stable) equilibrium market prices — and an important theorem concerning the properties of these prices — which at least fully and explicitly reflect the presence of uncertainty *per se* (as distinct from the effects of diverse judgmental distributions among investors).

Note first that the assumption of identical probability judgments means that (1) *the same stock mix will be optimal for every investor* (although the actual dollar gross investment in this mix — and the ratio, w, of gross investment in this mix to his net investment balance — will vary from one investor to the next). It consequently follows that, when the market is in equilibrium, (2) the h_i^0 given by equation (15) or (12) can be interpreted as the ratio of the *aggregate market* value of the i^{th} stock to the total aggregate market value of all stocks, and hence, (3) *all h_i will be strictly positive.*

In order to develop further results, define

$$\partial\bar{x}_1/\partial\sigma_1 < 0 \text{ if } 0 < K\sigma_1/\sigma_2 < |\rho^{-1}| - \sqrt{\rho^{-2} - 1}\ ,$$

and

$$\partial\bar{x}_1/\partial\sigma_1 > 0 \text{ if } |\rho^{-1}| - \sqrt{\rho^{-2} - 1} < K\sigma_1/\sigma_2 < |\rho^{-1}|.$$

[39] See appendix, note 11(*b*).

[40] However, see below, especially the "fifth" through "seventh" points enumerated near the end of Section V.

V_{0i} — the aggregate market value of the i^{th} stock at time zero,

$\bar{R}_i$ — the aggregate return on the i^{th} stock (the sum of aggregate cash dividends paid and appreciation in aggregate market value over the transaction period); and

$T \equiv \Sigma_i V_{0i}$, the aggregate market value of *all* stock in the market at time zero.

The original economic definitions of the variables in the portfolio optimization problem give

(26*a*) $h_i = V_{0i}/T,$

(26*b*) $\bar{r}_i = \bar{R}_i/V_{0i},$

(26*c*) $\bar{x}_i = \bar{r}_i - r^* = (\bar{R}_i - r^* V_{0i})/V_{0i},$

(26*d*) $\check{x}_{ij} = \check{r}_{ij} = \check{R}_{ij}/V_{0i} V_{0j},$

where $\check{R}_{ij}$ is the covariance of the aggregate dollar returns of the i^{th} and j^{th} stocks (and $\check{R}_{ii}$ is the i^{th} stock's aggregate return variance). The equilibrium conditions (12) may now be written

$$(12a) \quad \frac{\bar{R}_i - r^* V_i^0}{V_{0i}} = \lambda \frac{V_i^0}{T} \frac{\check{R}_{ii}}{(V_{0i})^2} + \lambda \Sigma_{j \neq i} \frac{V_{0j}}{T} \frac{\check{R}_{ij}}{V_{0i} V_{0j}},$$

which reduces to

$$(27) \quad \bar{R}_i - r^* V_{0i} = (\lambda/T)\,[\check{R}_{ii} + \Sigma_{j \neq i} \check{R}_{ij}] = (\lambda/T)\,\Sigma_j \check{R}_{ij}.$$

Now $\bar{R}_i - r^* V_{0i}$ represents the *expected* excess of the aggregate dollar return on the i^{th} security over earnings at the riskless rate on its aggregate market value, and $\Sigma_j \check{R}_{ij}$ represents the aggregate risk (direct dollar return variance and total covariance) entailed in holding the stock. Equation (27) consequently establishes the following:

Theorem: Under Idealized Uncertainty, equilibrium in purely competitive markets of risk-averse investors requires that the values of all stocks will have adjusted themselves so that the *ratio* of the expected excess aggregate dollar returns of each stock to the aggregate dollar risk of holding the stock will be *the same for all* stocks (and equal to λ/T), when the risk of each stock is measured by the variance of its own dollar return and its combined covariance with that of all other stocks.

But we seek an explicit equation[41] for V_{0i}, and to this end we note that partial summation of equation (27) over *all other* stocks gives us

$$(28) \quad \Sigma_{k \neq i} (\bar{R}_k - r^* V_{0k}) = (\lambda/T) \Sigma_{k \neq i} \Sigma_j \check{R}_{kj}.$$

After dividing each side of (27) by the corresponding side of (28), and solving for V_{0i}, we then find that the aggregate market value of the i^{th} stock is related to the concurrent market values of the *other* $(m - 1)$ stocks by

$$(29) \quad V_{0i} = (\bar{R}_i - W_i)/r^*$$

where

$$(29\text{a}) \quad W_i = \gamma_i \Sigma_j \check{R}_{ij} = \gamma_i (\check{R}_{ii} + \Sigma_{j \neq i} \check{R}_{ij})$$

and

$$(29\text{b}) \quad \gamma_i = \frac{\Sigma_{k \neq i} (\bar{R}_k - r^* V_{0k})}{\Sigma_{k \neq i} \Sigma_j \check{R}_{kj}} = \frac{\Sigma_{k \neq i} (\bar{R}_k - r^* V_{0k})}{\Sigma_{k \neq i} \Sigma_{j \neq i} \check{R}_{kj} + \Sigma_{j \neq i} \check{R}_{ij}}.$$

Since (29b) *appears* to make the slope coefficient γ_i unique to each company, we must note immediately that dividing each side of (27) by its summation over *all* stocks shows that the aggregate market value of the i^{th} stock is *also* related to the concurrent market values of *all* (m) stocks[42] by equation (29) when W_i is written as

$$(29c) \quad W_i = (\lambda/T)\, \Sigma_j \check{R}_{ij},$$

and

$$(29d) \quad \lambda/T = \frac{\Sigma_i (\bar{R}_i - r^* V_{0i})}{\Sigma_i \Sigma_j \check{R}_{ij}}.$$

But from equations (28) and (29*b*), we see that

$$(29e) \quad \gamma_i = \gamma_j = \lambda/T,$$

a *common value* for *all companies in the market.* The values of W_i given by (29*a*) and (29*c*) are consequently *identical*, and *the subscripts on* γ *should henceforth be ignored.*

In words, equations (29) establish the following further

Theorem: Under Idealized Uncertainty, in purely competitive markets of risk-averse investors,

A) the total market value of any stock in equilibrium is equal to the *capitalization* at the *risk-free interest rate* r^* of the *certainty equivalent* $(R_i - W_i)$ of its uncertain *aggregate dollar return* $\bar{R}_i$;

B) *the difference* W_i between the expected value $\bar{R}_i$ of these returns and their certainty

[41] I do not simply rearrange equation (27) at this point since (λ/T) includes V_{0i} as one of its terms (see equation (29*d*) below).

[42] Alternatively, equations (29) and (29*c*) follow directly from (27), and (29*d*) may be established by substituting (26*a–d*) in (11).

equivalent is *proportional* for *each* company to *its aggregate risk* represented by the *sum* $(\Sigma_j \check{R}_{ij})$ of the *variance* of these returns and their total covariance with those of all other stocks; and

C) the factor of proportionality $(\gamma = \lambda/T)$ is the *same* for *all* companies in the market.

Certain corrolaries are immediately apparent:

Corrolary I: Market values of securities are related to standard deviations of dollar returns by way of variances and covariances, *not directly* and *not linearly*.

Corrolary II: The aggregate risk $(\Sigma_j R_{ij})$ of the i^{th} stock which is directly relevant to its aggregate market value V_{0i} is simply *its contribution* to the aggregate *variance* of the dollar returns (for *all* holders together) of *all* stocks (which is $\Sigma_i \Sigma_j \check{R}_{ij}$).

Corrolary III: The *ratio* $(\bar{R}_i - W_i)/\bar{R}_i$ of the *certainty-equivalent* of aggregate dollar returns to their expected value is, in general, *different for each* i^{th} company when the market is in equilibrium;[43] but for all companies, this certainty-equivalent to expected-dollar-return ratio is the *same linear function* $\{1-\gamma [\Sigma_j \check{R}_{ij}/\bar{R}_i]\}$ of total dollar risk $(\Sigma_j \check{R}_{ij})$ attributable to the i^{th} stock deflated by its expected dollar return $\bar{R}_i$.

Several further implications also follow immediately. First, note that equation (29) can be written

$$(29') \quad V_{0i} = (\bar{R}_i - W_i)/r^* = (V_{0i} + \bar{R}_i - W_i)/(1 + r^*) = (\bar{H}_i - W_i)/(1 + r^*).$$

Since $\bar{R}_i$ was defined as the sum of the aggregate cash dividend and increase in value in the equity during the period, the *sum* $V_{0i} + \bar{R}_i$ is equal to the expected value of the sum (denoted $\bar{H}_i$) of the cash dividend and end-of-period aggregate market value of the equity, and the elements of the covariance matrix $\underline{\check{H}}$ are identical to those in $\underline{\check{R}}$. *All* equations (29) can consequently be validly rewritten substituting H for R throughout [and $(1 + r^*)$ for r^*], *thus explicitly determining all current values* V_{0i} *directly by the joint probability distributions over the end-of-period realizations*[44] $\tilde{H}_i$. (The value of W_i, incidentally, is not affected by these substitutions.) Our assumption that investors hold joint probability distributions over dollar returns $\tilde{R}_i$ is consequently *equivalent* to an assumption that they hold distributions over end-of-period realizations, and *our analysis applies equally under either assumption.*

Moreover, after the indicated substitutions, equation (29′) *shows that the current aggregate value of any equity is equal to the certainty-equivalent of the sum of its prospective cash receipts (to shareholders) and total market value at the end of the period, discounted at the riskless rate* r^*. Similarly, by an extension of the same lines of analysis, the certainty equivalent of the cash dividend and market value at the end of the first period clearly may be regarded as the then-present-values using riskless discount rates of the certainty-equivalents of random receipts still further in the future. *The analysis thus justifies viewing market values as riskless-rate present values of certainty-equivalents of random future receipts,* where certainty-equivalents are related to expected values by way of variances and covariances weighted by adjustment factors γ_{it}, which may or may not be the same for each future period t.

Still another implication of equation (29) is of a more negative character. Those who like (or hope) to find a "risk" discount rate k_r with which to discount expected values under uncertainty will find from (29) that, using a subscript i for the individual firm

$$(29'') \quad V_{0i} = \frac{\bar{R}_i}{k_{ri}} = \frac{\bar{R}_i}{r^* (1 - W_i/\bar{R}_i)^{-1}} = \frac{\bar{R}_i}{r^* (1 - \gamma \Sigma_j \check{R}_{ij}/\bar{R}_i)^{-1}}$$

so that

$$(30) \quad k_{ri} = r^* (1 - \gamma \Sigma_j \check{R}_{ij}/\bar{R}_i)^{-1}.$$

It is apparent that (*i*) the *appropriate "risk" discount rate* k_{ri} *is unique to each individual company in a competitive equilibrium* (because of the first half of corrolary III above); (*ii*) that efforts to derive it complicate rather than simplify the analysis, since (*iii*) it is a *derived* rather than a primary variable; and that (*iv*) it explicitly involves all the elements required for the determination of V_{0i} itself, and, (*v*) does so in a more

[43] From equations (27), (29), (29*a*), and (29*e*), this statement is true for all pairs of stocks having different aggregate market values, $V_{0i} \neq V_{0j}$.

[44] Because we are assuming only "idealized" uncertainty, the distribution of these end-of-period realizations will be independent of judgments regarding the dividend receipt and end-of-period market value separately. See Lintner [10] and Modigliani-Miller [16].

complex and non-linear fashion.[45] Having established these points, the rest of our analysis returns to the more direct and simpler relation of equation (29).

V — Corporate Capital Budgeting Under Idealized Uncertainty

Capital budgeting decisions within a corporation affect both the expected value and variances — and hence, the certainty-equivalents — of its prospective aggregate dollar returns to its owners. When the requisite conditions are satisfied, equation (29) thus provides a normative criterion for these decisions, derived from a competitive equilibrium in the securities market.

In developing these important implications of the results of the last section, I of course maintain the assumptions of idealized uncertainty in purely competitive markets of risk-averse investors with identical probability distributions, and I continue to assume, for simplicity, that there are no transactions costs or taxes. The identity of probability distributions over outcomes now covers corporate management as well as investors, and includes potential corporate investments in the capital budget as well as assets currently held by the company. Every corporate management, *ex ante*, assigns probability zero to default on its debt, and all investors also treat corporate debt as a riskless asset. I thus extend the riskless investment (or borrowing) alternative from individual investors to corporations. Each company can invest any amount of its capital budget in a perfectly safe security (savings deposit or certificate of deposit) at the riskless rate r^*, or it may borrow unlimited amounts at the *same* rate in the current or any future period.[46] I also assume that the investment opportunities available to the company in any time period are regarded as independent of the size and composition of the capital budget in any other time period.[47] I also assume there is no limited liability to corporate stock, nor any institutional or legal restriction on the investment purview of any investor, and that the riskless rate r^* is expected by everyone to remain constant over time.

Note that this set of assumptions is sufficient to validate the famous (taxless) Propositions I and II of Modigliani and Miller [15]. In particular, under these severely idealized conditions, for any given size and composition of corporate assets (investments), investors will be indifferent to the *financing* decisions of the company. Subject to these conditions, we can, consequently, derive valid decision rules for capital budgets which do not explicitly depend upon concurrent financing decisions. Moreover, these conditions make the present values of the cash flows *to any* company from its real (and financial) assets and operations equal to the total market value of investors' *claims* to these flows, i.e., to the sum of the aggregate market value of its common (and preferred) stock outstanding and its borrowings (debt)[48]. They also make any change in shareholders claims equal to the change in the present values of flows (before interest deductions) to the company less any change in debt service. The *changes* in the market value of the equity V_{0i} induced by capital budgeting decisions will consequently be precisely equal to

$$
\begin{aligned}
(31) \qquad \Delta V_{0i} &= \Delta\,(\bar{R}_i - W_i)/(1 + r^*) \\
&= \Delta\,(\bar{H}_i - W_i)/(1 + r^*),
\end{aligned}
$$

where $\Delta\bar{H}_i$ is the net change induced in the *expected* present value at the end of the first period of the cash inflows (net of interest charges) to the i^{th} company attributable to *its assets*[49] when all present values are computed at the riskless rate r^*.

These relationships may be further simplified in a useful way by making three additional assumptions: that (*i*) the *aggregate market value*

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[45] It may also be noted that even when *covariances* between stocks are constant, the elasticity of k_{ri} with respect to the variance $\check{R}_{ii}$ (and *a fortiori*) to the standard deviation of return) is a unique (to the company) multiple of a hyperbolic relation of a variance-expected-return ratio:

$$
(30a) \quad \frac{\check{R}_{ii}}{k_{ri}} \; \frac{\partial k_{ri}}{\partial \check{R}_{ii}} = \frac{\gamma\, \check{R}_{ii}/\bar{R}_i}{1 - \gamma\,(\Sigma_{j\neq i}\check{R}_{ij}/\bar{R}_i) - \gamma\,\check{R}_{ii}/\bar{R}_i}.
$$

[46] The effects of removing the latter assumption are considered briefly in the final section.

[47] This simplifying assumption specifies a (stochastic) comparative static framework which rules out the complications introduced by making investor expectations of future growth in a company's investment opportunities conditional on current investment decisions. I examine the latter complications in other papers [11], and [12].

[48] See Lintner [10]. Note that in [10, especially p. 265, top 1st column] I argued that additional assumptions were needed to validate the "entity theory" under uncertainty — the last sentence of the preceding paragraph, and the stipulation that corporate bonds are riskless meet the requirement. See, however, Modigliani-Miller [16].

[49] By definition, ΔH_i is the change in the expected sum of dividend payment and market value of the equity at the end of the period. This is made equal to the statement in the text by the assumptions under which we are operating.

of *all other* stocks—and (*ii*) the *covariances* $\check{R}_{ij}$ *with* all other *stocks* are invariant to the capital budgeting decisions of the i^{th} company; while (*iii*) the (optimal) *portfolio* of risk assets is not an "inferior good" (in the classic Slutsky-Hicks sense) *vis a vis* riskless assets. The reasonableness of (*iii*) is obvious (especially in the context of a universe of risk-averse investors!), and given (*iii*), assumption (*i*) is a convenience which only involves ignoring (generally small) second-order feedback effects (which will not reverse signs); while the plausibility of (*ii*) as a good working first approximation was indicated above (footnote 32).[50]

In this context, we now show that capital budgeting decisions by the i^{th} firm will raise the aggregate market value of its equity V_{0i}—and hence by common agreement be in the interest of its shareholders—so long as the induced change in expected dollar return is greater than the product of the market price γ of risk and the induced variance of dollar returns, i.e.,

$$(32)\quad \Delta\bar{R}_i - \gamma\Delta\check{R}_{ii} = \Delta\bar{H}_i - \gamma\Delta\check{H}_{ii} > 0.$$

This assertion (or theorem) can be proved as follows. The total differential of (29) is

$$(29f)\quad r^*\Delta V_{0i} - \Delta\bar{R}_i + \gamma\Delta\check{R}_{ii} + (\Sigma_j\check{R}_{ij})\Delta\gamma = 0$$

so that under the above assumptions

$$(29g)\quad \Delta\bar{R}_i \geqq \gamma\,\Delta\check{R}_{ii} + (\Sigma_j\check{R}_{ij})d\,\gamma \rightarrow \Delta V_{0i} \geqq 0 \rightarrow \Delta T \geqq 0.$$

But using (29*e*) and (29*d*), we have

$$(29h)\quad \Delta\gamma = (\Delta\bar{R}_i - \gamma\Delta\check{R}_{ii})/\Sigma_i\Sigma_j\check{R}_{ij}$$

so that

$$(29i)\quad \Delta\bar{R}_i = \gamma\Delta\check{R}_{ii} \rightarrow \Delta\gamma = 0 \rightarrow \Delta V_{0i} = 0 \rightarrow \Delta T = 0,$$

and the first equality in (29*i*) defines the relevant *indifference function*.[51] Moreover, using (29*h*) and the fact that $\Sigma_j\check{R}_{ij} < \Sigma_i\Sigma_j\check{R}_{ij}$, we have from (29*g*):

$$(29j)\quad \Delta\bar{R}_i \geqq \gamma\Delta\check{R}_{ii} \rightarrow \Delta\bar{R}_i \geqq \gamma\Delta\check{R}_{ii} + (\Sigma_j\check{R}_{ij})\,\Delta\gamma,$$

and consequently

$$(29k)\quad \Delta\bar{R}_i \geqq \gamma\,\Delta\check{R}_{ii} \rightarrow \Delta V_{0i} \geqq 0 \rightarrow \Delta T \geqq 0,$$

from which (32) follows immediately.

In order to explore the implications of (32) further, it will now be convenient to consider in more detail the capital budgeting decisions of a company whose *existing assets* have a present value computed at the rate r^* (and measured at the *end* of the first period) of $\tilde{H}_0^{(1)}$, a random variable with expected value $\bar{H}_0^{(1)}$ and variance $\check{H}_{00}$.

The company may be provisionally holding any fraction of $\bar{H}_0$ in savings deposits or CD's yielding r^*, and it may use any such funds (or borrow unlimited amounts at the *same* rate) to make new "real" investments. We assume that the company has available a set of new projects $1, 2 \ldots j \ldots n$ which respectively involve *current* investment outlays of $H_j^{(0)}$, and which have present values of the relevant incremental cash flows (valued at the *end* of the first period) of $\tilde{H}_j^{(1)}$. Since any diversion (or borrowing) of funds to invest in any project involves an opportunity cost of $r^* H_j^{(0)}$, we also have the "excess" dollar end-of-period present value return

$$(33)\quad \tilde{X}_j^{(1)} = \tilde{H}_j^{(1)} - r^* H_{j0}^{(0)}.$$

Finally, we shall denote the $(n+1)$ 'th order covariance matrix (including the existing assets $\tilde{H}_0$) by $\underline{\underline{\check{H}}}$ or $\underline{\underline{\check{X}}}$ whose corresponding elements $\check{H}_{jk} = \check{X}_{jk}$.

Determination of the Optimal Corporate Capital-Budget-Portfolio

In this simplified context, it is entirely reasonable to expect that the corporation will seek to maximize the left side of equation[52] (32) as its capital budgeting criterion. At first blush, a very complex *integer* quadratic-programming solution would seem to be required, but fortunately we can break the problem down inductively and find a valid formulation which can be solved in essentially the same manner as an individual investor's portfolio decision.

[50] It is, however, necessary in general to redefine the variables in terms of *dollar* returns (rather than rates of return), but this seems equally reasonable.

[51] Note that this indifference function can also be derived by substituting equations (26*a–d*) directly into that found in section III above (equation 6*b*) in appendix Note II or equation (25) in the text) for the relevant case where covariances are invariant.

[52] Under our assumption that stock portfolios are not inferior goods, sgn ΔT = s gn $[\Delta\bar{R}_i - \gamma\Delta\check{R}_{ii}]$ so that (although generally small in terms of percentages) the induced change in aggregate values of all stocks will reinforce the induced change in the *relative* value of the i^{th} stock; the fact that $\Delta\gamma$ also has the same sign introduces a countervailing feedback, but as shown above [note especially (29*g*)], this latter effect is of second order and cannot reverse the sign of the criterion we use. In view of the overwhelming informational requirements of determining the maximum of a fully inclusive criterion function which allowed formula induced adjustments external to the firm, *and* the fact our criterion is a monotone rising function of this ultimate ideal, the position in the text follows.

First, we note that if a single project j is added to an existing body of assets $\bar{H}_0{}^0$, we have

$$(34a)\quad \Delta\bar{R}_i - \gamma\Delta\check{R}_{ii} = \bar{H}_j{}^{(1)} - r^*H_j{}^0 - \gamma\,[H_{jj} + 2\check{H}_{j0}] = X_j{}^{(1)} - \gamma\,[\check{X}_{jj} + 2\check{X}_{j0}].$$

Now suppose a project k is also added. The total change from j and k *together* is

$$(34b)\quad (\Delta R_i - \gamma\Delta\check{R}_{ii}) = X_j{}^{(1)} + X_k{}^{(1)} - \gamma\,[\check{X}_{jj} + \check{X}_{kk} + 2\check{X}_{j0} + 2\check{X}_{k0} + 2\check{X}_{jk}],$$

while the *increment* due to adding *k with j already in the budget* is

$$(34c)\quad (\Delta R_i - \gamma\check{R}_{ii}) = \bar{X}_k{}^{(1)} - \gamma\,[\check{X}_{kk} + 2\check{X}_{k0} + 2\check{X}_{jk}].$$

Given the goal of maximizing the left side of (32), the k^{th} project should be added to the budget (already provisionally containing j) *if and only if* the *right* side of (34c) is $>$ o— and if this condition is satisfied, the same test expression written for j, given inclusion of k, will show whether j should stay in. Equation (34c) appropriately generalized to any number of projects, is thus a *necessary condition* to be satisfied by *each project in an optimal budget*, given the inclusion of all other projects simultaneously satisfying this condition.

The unstructured iterative or search procedure suggested by our two-project development can obviously be short-circuited by programming methods, and the integer aspect of the programming (in this situation) can conveniently be by-passed by assuming that the company may accept all or any fractional part a_j, $0 \leqq a_j \leqq 1$, of any project (since it turns out that all a_j in the final solution will take on *only* limiting values). Finally, thanks to this latter fact, the objective of maximizing the left side of (32) is equivalent[53] to maximizing

$$(32')\quad Z = H_0{}^{(1)} + \Sigma_j\, a_j \bar{H}_j{}^{(1)} - r^*\Sigma_j a_j H_j{}^{(0)} - \gamma[\Sigma_j a_j \check{H}_{jj} + 2\Sigma_j a_j \check{H}_{j0} + 2\Sigma_{j\neq k\neq 0} a_j a_k \check{H}_{jk}] = \bar{H}_0{}^{(1)} + \Sigma_j a_j \bar{X}_j - \gamma[\Sigma_j a_j \check{X}_{jj} + 2\Sigma_j a_j \check{X}_{j0} + 2\Sigma_{j\neq k\neq 0}\, a_j a_k \check{X}_{jk}],$$

subject to the constraints that $0 \leqq a_j \leqq 1$ for all a_j, $j = 1, 2 \ldots n$. Not only will all a_j be binary variables in the solution, but the generalized form of the necessary condition (34c) will be given by the solution [see equation (37) below].

In order to maximize Z in (32′) subject to the constraints on a_j, we let $q_j = 1 - a_j$ for convenience, and form the Lagrangian function

$$(35)\quad \psi\,(a, \mu, \eta) = Z + \Sigma_j\, \mu_j\, a_j + \Sigma_j\, \eta_j\, q_j$$

which is to be maximized subject to $a_j \geqq 0$, $q_j \geqq 0$, $\mu_j \geqq 0$, and $\eta_j \geqq 0$, where μ_j and η_j are the Lagrangian multipliers associated with the respective constraints $a_j \geqq 0$ and $q_j \geqq 0$. Using (33), we have immediately

$$(35')\quad \frac{\partial\psi}{\partial a_j} \gtreqless 0 \leftrightarrow \bar{X}_j - \gamma\,[a_j\check{X}_{jj} + 2\Sigma_j a_j \check{X}_{j0} + 2\Sigma_{k\neq 0}\, a_k \check{X}_{jk}] + \mu_j - \eta_j \gtreqless 0.$$

Using the Kuhn-Tucker Theorem [9], the necessary and sufficient conditions for the optimal vector of investments $a_j{}^0$ which maximize ψ in (35) are consequently[54]

$$(36a)\quad \gamma\,[a_j{}^0\, \check{X}_{jj} + 2a_j{}^0 \check{X}_{j0} + 2\Sigma_{k\neq j\neq 0}\, a_k{}^0 \check{X}_{jk}] - \mu_j{}^0 + \eta_j{}^0 = \bar{X}_j$$

when

$$(36b, c, d, e)\quad a_j{}^0 \geqq 0,\ q_j{}^0 \geqq 0,\ \mu_j{}^0 \geqq 0,\ \eta_j{}^0 \geqq 0$$

and

$$(36f, g)\quad \mu_j{}^0\, a_j{}^0 = 0,\ \eta_j{}^0\, q_j{}^0 = 0,$$

where

$$j = 1, 2 \ldots n$$

in each set (36a) − (36g).

Once again, these equations can be readily solved by the Wilson Simplicial Algorithm [23] on modern computing equipment. It may be observed that this formulation in terms of independent investment projects can readily be generalized to cover mutually exclusive, contingent, and compound projects[55] with no difficulty. It is also apparent that the absence of a financing constraint (due principally to our assumption

[53] For the reason given, the maximum of (32′) is the same as it would be if (32′) had been written in the more natural way using $a_i{}^2$ instead of a_j as the coefficient of $\check{H}_{jj}$; the use of a_j is required to make the form of (35′) and (37) satisfy the requirement of (34c).

[54] The proof that the indicated solution satisfies the Kuhn-Tucker conditions with respect to the variables $a_j{}^0$ and $\mu_j{}^0$ is identical to that given above footnote 28 upon the substitution of $\underline{\underline{X}}$ for $\underline{\underline{x}}$, a_j for h_i, and μ_j for u_i, and need not be repeated. The two additional *necessary* conditions are

(3′) $\left[\frac{\partial\psi}{\partial\eta_j}\right]^0 \geqq 0$, which is satisfied, since from (35) we have $\left[\frac{\partial\psi}{\partial\eta_j}\right]^0 = q_j{}^0 \geqq 0$ by virtue of (36c); and this latter relation shows that the corresponding complementary slackness condition,

(4′) $\mu^0{}_j\ \left[\frac{\partial\psi}{\partial\mu_j}\right]^0 = 0$, may be written $\mu_j{}^0\, q_j{}^0 = 0$, and is therefore satisfied because of (36g).

All three *sufficiency* conditions are also satisfied because the variance-covariance matrix $\underline{\underline{X}}$ is positive definite, making $\psi\ (\underline{a}, \underline{u}^0, \underline{n}^0)$ a concave function on $\underline{a}$ and $\psi\ (\underline{a}, \underline{u}^0, \underline{n}^0)$ a convex function on both $\underline{u}$ and $\underline{n}$.

[55] See Weingartner [22], 11 and 32–34.

that new riskless debt is available in unlimited amounts at a fixed rate r^*) insures that all projects will either be accepted or rejected *in toto*. All a_j^0 will be either 0 or 1, and the troublesome problems associated with fractional projects or recourse to integer (non-linear) programming do not arise.

Consider now the set of *accepted* projects, and denote this subset with asterisks. We then have all $a_{j*}^0 = a_{k*}^0 = 1$; the corresponding $\mu_{j*}^0 = \mu_{k*}^0 = 0$; and for any project j^*, the corresponding $\eta_{j*}^0 > 0$ (i.e. *strictly positive*),[56] and the number η_{j*}^0 is the "dual evaluator" or "shadow price" registering the *net gain* to the company *and* its shareholders of accepting the project. Rewriting the corresponding equation from (36*a*), we have[57]

$$(37) \qquad \eta_{j*}^0 = \bar{H}_{j*}^{(1)} - r^* H_{j*}^{(0)} - \gamma \, [\check{H}_{j*j*} + 2\check{H}_{j*0} + 2\Sigma_{k* \neq j* \neq 0} \check{H}_{j*k*}] > 0.$$

Several important features and implications of these results should be emphasized. First of all, note that we have shown that *even* when uncertainty is admitted in only this highly simplified way, and when any effect of changes in capital budgets on the *covariances* between returns on different companies' *stocks* is ignored, the minimum expected return (in dollars of expected present value $\bar{H}_{j*}^{(1)}$) required to justify the allocation of funds to a given risky project costing a given sum $H_{j*}^{(0)}$ is an increasing function of each of the following factors: (*i*) the risk-free rate of return r^*; (*ii*) the "market price of dollar risk", γ; (*iii*) the variance $\check{H}_{j*j*}$ in the project's own present value return; (*iv*) the project's aggregate present value return-*co*variance $\check{H}_{j*0}$ with assets already held by the company, and (*v*) its total covariance $\Sigma_{k* \neq j* \neq 0} \check{H}_{j*k*}$ with other projects concurrently included in the capital budget.

Second, it follows from this analysis that, if uncertainty is recognized to be an important fact of life, and risk-aversion is a significant property of relevant utility functions, appropriate *risk-variables* must be introduced *explicitly* into the analytical framework used in analysis, and that these risk-variables will be *essential components* of any optimal decision rules developed. Important insights can be, and have been, derived from "certainty" models, including some *qualitative* notions of the *conditional* effects of changes in availability of funds due to fund-suppliers' reactions to uncertainty,[58] but such models ignore the decision-maker's problem of optimizing *his* investment decisions in the face of the stochastic character of the outcomes among which *he* must choose.

Third, it is clear that *stochastic considerations are a primary source of interdependencies among projects*, and these must *also enter explicitly* into optimal decision rules. In particular, note that, although own-variances are necessarily positive and subtracted in equation (37), the net gain n_{j*}^0 may still be positive and justify acceptance *even if* the expected end-of-period "excess" present-value return $(\check{X}_{j*}^{(1)} = \bar{H}_{j*}^{(1)} - r^* H_{j*}^{(0)})$ is negative[59]— so long as its total present-value-covariances $(\check{H}_{j*0} + \Sigma_{k* \neq j* \neq 0} \check{H}_{j*k*})$ are also negative and sufficiently large. *Sufficiently risk-reducing investments rationally belong in corporate capital budgets even at the expense of lowering expected present value returns*— an important (and realistic) feature of rational capital budgeting procedure not covered (nor even implied) in traditional analyses.

Fourth, note that, as would by now be expected, for any fixed r^* and γ, the net gain from a project is a *linear* function of its (present value) *variance* and *covariances* with existing company assets and concurrent projects. Standard deviations are not involved except as a component of (co)variances.

Fifth, the fact that the risk of a project involves all the elements in the bracketed term in (37), including covariances with other concurrent projects, indicates that in practice it will often be extremely difficult, if not impossible, to classify *projects* into respectively homogeneous "risk classes." The practice is convenient (and desirable where it does not introduce significant bias) but our analysis shows it is *not essential*, and the considerations which follow show it to be *a*

[56] We are of course here ignoring the very exceptional and coincidental case in which $\eta_{j*}^0 = 0$ which implies that a_{j*}^0 is indeterminate in the range $0 \leqq a_{0j} \leqq 1$, the company being *totally indifferent* whether or not all (or any part) of a project is undertaken.

[57] We use $\check{H}_{j*k*}$ to denote elements the original covariance matrix $\check{\underline{H}}$ *after* all rows and columns associated with rejected projects have been removed.

[58] See Weingartner [22] and works there cited. Weingartner would of course agree with the conclusion stated here, see pp. 193–194.

[59] Indeed, in extreme cases, a project should be accepted even if the expected end-of-period present value $\bar{H}_{j*}^{(1)}$ is less than cost $H_{j*}^{(0)}$, provided negative correlations with existing assets and other concurrent investments are sufficiently strong and negative.

dangerous expedient which is positively misleading as generally employed in the literature.

Sixth, it must be emphasized that — following the requirements of the market equilibrium conditions (29) from which equations (36), (37), and (38) were derived — *all means and (co)variances of present values have been calculated using the riskless rate* r^*. In this connection, recall the non-linear effect on present values of varying the discount rate used in their computation. Also remember the further facts that (*i*) the means and variances of the distributions of present values computed at different discount rates do not vary in proportion to each other when different discount rates are applied to the same set of future stochastic cash flow data, and that (*ii*) the changes induced in the means and variances of the present values of different projects having different patterns and durations of future cash flows will also differ greatly as discount rates are altered. From these considerations alone, it necessarily follows that *there can be no single "risk discount rate"* to use in computing present values for the purpose of deciding on the acceptance or rejection of *different individual projects* out of a subset of projects *even if all projects in the subset have the same degree of "risk."*[60] The same conclusion follows *a fortiori* among projects with different risks.

Seventh, the preceding considerations, again *a fortiori*, insure that *even if all new projects have the same degree of "risk" as existing assets, the "cost of capital"* (as defined for uncertainty *anywhere* in the literature) *is not the appropriate discount rate to use in accept-reject decisions on individual projects* for capital budgeting.[61] This is true whether the "cost of capital" is to be used as a "hurdle rate" (which the "expected return" must exceed) *or* as a discount rate in obtaining present values of net cash inflows and outflows.

Perhaps at this point the reader should be reminded of the rather heroic set of simplifying assumptions which were made at the beginning of this section. One consequence of the unreality of these assumptions is, clearly, that the results are not being presented as directly applicable to practical decisions at this stage. Too many factors that matter very significantly have been left out (or assumed away). But the very simplicity of the assumptions has enabled us to develop rigorous proofs of the above propositions which do differ substantially from current treatments of "capital budgeting under uncertainty." A little reflection should convince the reader that *all the above conclusions will still hold under more realistic (complex) conditions.*

Since we have shown that selection of individual projects to go in a capital budget under uncertainty by means of "risk-discount" rates (or by the so-called "cost of capital") is fundamentally in error, we should probably note that the decision criteria given by the solutions of equation (36) [and the acceptance condition (37)] — which directly involve the means and variances of present values computed at the riskless rate — do have a valid counterpart in the form of a "required expected rate of return." Specifically, if we let $[\Sigma \check{H}_{j^*}]$ represent the entire bracket in equation (37), and divide through by the original cost of the project $H_{j^*}^{(0)}$, we have

$$(38) \quad \bar{H}_{j^*}^{(1)}/H_{j^*}^{(0)} = r_{j^*} > r^* + \gamma\,[\Sigma \check{H}_{j^*}]/H_{j^*}^{(0)}.$$

Now the ratio of the expected *end-of-period* present value $\bar{H}_{j^*}^{(1)}$ to the initial cost $H_{j^*}^{(0)}$ — i.e. the left side of (38), which we write r_{j^*} — is precisely (the expected value of) what Lutz called the *net* short term marginal efficiency of the investment [13 p. 159]. We can thus say that the *minimum acceptable expected rate of return* on a project is a (positively sloped) linear function of the ratio of the project's *aggregate incremental* present-value-variance-covariance $(\Sigma \check{H}_{j^*})$ to its cost $H_{j^*}^{(0)}$. The slope coefficient is still the "market price of dollar risk", γ, and the intercept is the risk-free rate r^*. (It will be observed that our "accept-reject" rule for individual projects under uncertainty thus reduces to Lutz' rule under certainty — as it should — since with certainty the right-hand ratio term is zero.) To

[60] Note, as a corollary, it also follows that *even if* the world were simple enough that a single "as if" risk-discount rate could in principle be found, the same considerations insure that *there can be no simple function relating the appropriate "risk-discount" rate to the riskless rate r* and "degree of risk," however measured.* But especially in this context, it must be emphasized that a single risk discount rate *would produce non-optimal choices* among projects *even if* (*i*) all projects could be assigned to meaningful risk-classes, *unless* it were also true that (*ii*) all projects had the same (actual) time-pattern of net cash flows and the same life (which is a condition having probability measure zero under uncertainty!).

[61] Note particularly that, even though we are operating under assumptions which validate Modigliani and Miller's propositions I and II, and the form of finance is *not* relevant to the choice of projects, we nevertheless cannot accept their use of their ρ_k — their cost of capital — as the relevant discount rate.

avoid misunderstanding and misuse of this relation, however, several further observations must be emphasized.

a) Equation (38) — like equation (37) from which it was derived — states a necessary condition of the (Kuhn-Tucker) optimum with respect to the projects selected. It may validly be *used to choose* the desirable projects out of the larger set of *possible* projects *if the covariances among potential projects* $\breve{H}_{j \neq k \neq 0}$ *are all zero.*[62] *Otherwise, a programming solution of equation set* (36) *is required*[63] *to find which subset of projects* H_{j*} satisfy either (37) or (38), essentially because the total variance of any project $[\Sigma \breve{H}_j]$ is dependent on *which* other projects are *concurrently included* in the budget.

b) Although the risk-free rate r^* enters equation (38) *explicitly* only as the intercept [or constant in the linear (in)equation form], it must be emphasized again that it *also enters implicitly as the discount rate used in computing the means and variances of all present values which appear* in the (in)equation. *In consequence*, (*i*) *any shift* in the value of r^* *changes every term* in the function. (*ii*) The changes in $\bar{H}^{(1)}{}_{j*}$ and $\Sigma \breve{H}_{j*}$ are *non-linear and non-proportional* to each other.[64] Since (*iii*) any shift in the value of r^* changes *every covariance* in equation (36*a*) *non-proportionately*, (*iv*) the *optimal subset of projects* j^* is *not invariant* to a change in the risk-free rate r^*. Therefore (*v*), *in principal, any shift in the value of* r^* *requires a new programming solution of the entire set of equations* (36).

c) Even for a predetermined and fixed r^*, and even with respect only to *included* projects, the condition expressed in (38) is rigorously *valid only under the full set* of simplifying assumptions stated at the beginning of this section. In addition, the programming solution of equation (36), and its derivative property (38), *simultaneously determines both the optimal composition and the optimal size* of the capital budget *only under this full set* of simplifying assumptions. Indeed, even if the twin assumptions of a fixed riskless rate r^* and of formally unlimited borrowing opportunities at this rate are retained[65], *but* other assumptions are (realistically) generalized — specifically to permit expected returns on new investments at any time to depend in part on investments made in prior periods, and to make the "entity value" in part a function of the finance mix used — *then* the (set of) programming solutions merely determines the optimal *mix or composition* of the capital budget *conditional* on each possible aggregate budget size and risk.[66] Given the resulting "investment opportunity function" — which is the three-dimensional Markowitz-type envelope of efficient sets of projects — the optimal capital budget size and risk can be determined directly by market criteria (as developed in [11] and [12])[67] but will depend explicitly on concurrent financing decisions (e.g. retentions and leverage).[68]

VI — Some Implications of More Relaxed Assumptions

We have come a fairly long way under a progressively larger set of restrictive assumptions. The purpose of the exercise has not been to provide results *directly* applicable to practical decisions at this stage — too much (other than uncertainty *per se*) that matters greatly in prac-

[62] Note that covariances $\breve{H}_{j0}$ with *existing* assets need not be zero since they are independent of other projects and may be combined with the own-variance $\breve{H}_{jj}$.

[63] In strict theory, an iterative *exhaustive* search over *all possible* combinations *could* obviate the programming procedure, but the number of combinations would be very large in practical problems, and economy dictates programming methods.

[64] This statement is true *even if* the set of projects j^* were invariant to a change in r^* which in general will not be the case, as noted in the following text statement.

[65] If these assumptions are not retained, the position *and* composition of the investment opportunity function (defined immediately below in the text) are themselves dependent on the relevant discount rate, for the reasons given in the "sixth" point above and the preceding paragraph. (See also Lutz [13 p. 160].) Optimization then requires the solution of a much different and more complex set of (in)equations, simultaneously encompassing finance-mix *and* investment mix.

[66] This stage of the analysis corresponds, in the standard "theory of the firm," to the determination of the optimal mix of factors for each possible scale.

[67] I should note here, however, that on the basis of the above analysis, the correct *marginal expected* rate of return for the investment opportunity function should be the value of r_j^* [See left side equation (38) above] for the marginally included project at each budget size, i.e. the ratio of end-of-period present value computed at the riskless rate r^* to the project cost — rather than the different rate (generally used by other authors) stated in [12, p. 54 top]. Correspondingly, the relevant *average* expected return is the same ratio computed for the budget as a whole. Correspondingly, the relevant *variance* is the variance of this ratio. None of the subsequent analysis or results of [12] are affected by this corrected specification of the inputs to the investment opportunity function.

[68] This latter solution determines the optimal point on the investment opportunity function at which to operate. The optimal *mix* of projects to include in the capital budget is that which corresponds to the optimal point on the investment opportunity function.

tice has been assumed away — but rather to develop rigorously some of the fundamental implications of uncertainty *as such* for an important class of decisions about which there has been much confusion in the theoretical literature. The more negative conclusions reached — such as, for instance, the serious distortions inherently involved in the prevalent use of a "risk-discount rate" or a "company-risk-class" "cost-of-capital" for project selection in capital budgeting — clearly will hold under more general conditions, as will the primary role under uncertainty of the *risk-free rate* (whether used to calculate *distributions* of present values *or* to form *present values of certainty-equivalents*). But others of our more affirmative results, and especially the particular equations developed, are just as clearly inherently conditional on the simplifying assumptions which have been made. While it would be out of place to undertake any exhaustive inventory here, we should nevertheless note the impact of relaxing certain key assumptions upon some of these other conclusions.

The particular formulas in sections II–V depend *inter-alia* on the *Separation Theorem* and each investor's consequent preference for the stock *mix* which maximizes θ. Recall that in proving the Separation Theorem in section I we assumed that the investor could borrow unlimited amounts at the rate r^* equal to the rate on savings deposits. Four alternatives to this assumption may be considered briefly. (1) *Borrowing Limits*: The Theorem (and the subsequent development) holds *provided* that the margin requirements turn out *not* to be binding; but if the investor's utility function is such that, given the portfolio which maximizes θ, he prefers a w greater than is permitted, *then* the Theorem does not hold and the utility function must be used explicitly to determine the optimal stock mix.[69] (2) *Borrowing rate* r^{**} *greater* than "lending rate" r^*: (*a*) If the max θ using r^* implies a $w < 1$, the theorem holds in original form; (*b*) if the max θ using r^* implies $w > 1$ *and* (upon recomputation) the max θ using r^{**} in equations (3*b*), (7) and (8) implies $w > 1$, the theorem also holds but r^{**} (rather than r^*) *must be used* in sections II–V; (*c*) if max θ using r^* implies $w > 1$ *and* max θ using r^{**} implies $w < 1$, *then* there will be no borrowing *and* the utility function must be used explicitly to determine the optimal stock mix.[70] (3) *Borrowing rate an increasing function of leverage* $(w - 1)$: The theorem still holds under condition (2*a*) above, but if max θ using r^* implies $w > 1$ *then* the *optimal mix and the optimal financing* must *be determined simultaneously using the utility function* explicitly.[71] (4) The latter conclusion also follows immediately *if the borrowing rate is not independent of the stock mix.*

[69] See appendix, note III.

The *qualitative* conclusions of sections II and III hold even if the Separation Theorem does not, but the formulas would be much more complex. Similarly, the stock market equilibrium in section IV — and the parameters used for capital budgeting decisions in section V — will be altered if different investors in the market are affected differently by the "real world" considerations in the preceding paragraph (because of different utility functions, or probability assessments), or by differential tax rates. Note also that even if all our original assumptions through section IV are accepted for investors, the results in section V would have to be modified to allow for all real world complications in the cost and availability of debt and the tax treatment of debt interest versus other operating income. Finally, although explicitly ruled out in section V, it must be recalled that "limited liability," legal or other institutional restrictions or premiums, or the presence of "market risk" (as distinct from default risk) on corporate debt, *are sufficient both* to make the optimal *project mix* in the capital budget *conditional* on the finance mix (notably retentions and leverage), *and* the finance mix itself *also* something to be optimized.

Obviously, the need for further work on all these topics is great. The present paper will have succeeded in its essential purpose if it has rigorously pushed back the frontiers of theoretical understanding, and opened the doors to more fruitful theoretical and applied work.

[70] See appendix, note IV.

[71] See appendix, note V.

APPENDIX

Note I — Alternative Proof of Separation Theorem and Its Corrolaries

In this note, I present an alternative proof of the *Separation Theorem* and its corrolaries using utility functions explicity. Some readers may prefer this form, since it follows traditional theory more closely.

Let $\bar{y}$ and σ_y be the expected value and variance of the rate of return on any asset mixture and A_0 be the amount of the investor's total net investment. Given the assumptions regarding the market and the investor, stated in the text, the investor will seek to maximize the expected utility of a function which can be written in general form as

$$(1') \quad E\,[U(A_0\bar{y},\, A_0\sigma_y)] = \bar{U}(A_0\bar{y},\, A_0\sigma_y),$$

subject to his investment opportunities characterized by the risk-free rate r^*, at which he can invest in savings deposits or borrow any amount he desires, and by the set of all stock mixes available to him, each of which in turn is represented by a pair of values $(\bar{r},\ \sigma_r)$. Our assumptions establish the following properties[72] of the utility function in (1′):

$$(1a') \quad \begin{cases} \partial\bar{U}/\partial\bar{y} = A_0\bar{U}_1 > 0; \ \partial\bar{U}/\partial\sigma_y = A_0\bar{U}_2 < 0; \\ \left.\dfrac{d\bar{y}}{d\sigma_y}\right|_{\bar{U}} = -\bar{U}_2/\bar{U}_1 > 0; \ \left.\dfrac{d^2\bar{y}}{d\sigma_y^2}\right|_{\bar{U}} > 0. \end{cases}$$

Also, with the assumptions we have made,[73] all available stock mixes will lie *in a finite region* all parts of which are strictly to the right of the vertical axis in the σ_r, r plane since all available mixes will have positive variance. The boundary of this region will be a closed curve[74] and the region is convex.[75] Moreover, since $\bar{U}_1 > 0$ and $\bar{U}_2 < 0$ in (1a′), all mixes within this region are dominated by those whose $(\sigma_r,\ r)$ values lie on the part of the boundary associated with values of $\bar{r} > 0$, *and* for which changes in σ_r and $\bar{r}$ are positively associated. This is Markowitz' Efficient Set or "E–V" Frontier. We may write its equation[76] as

$$(2') \quad \bar{r} = f(\sigma_r),\ f'(\sigma_r) > 0,\ f''(\sigma_r) < 0.$$

Substituting (2′) in (2) and (3) in the text, we find the first order conditions for the maximization of (1) subject to (2), (3), and (2′) to be given by the equalities in

$$(3a') \quad \partial\bar{U}/\partial w = \bar{U}_1(\bar{r} - r^*) + \bar{U}_2\sigma_r \geqq 0.$$

$$(3b') \quad \partial\bar{U}/\partial\sigma_r = \bar{U}_1 w f'(\sigma) + \bar{U}_2 w \geqq 0.$$

which immediately reduce to the two equations [using (3a) from the text]

$$(4') \quad \theta = -\bar{U}_2/\bar{U}_1 = f'(\sigma_r).$$

Second order conditions for a maximum are satisfied because of the concavity of (1′) and (2′). The separation theorem follows immediately from (4′) when we note that the equation of the first and third members $\theta = f'(\sigma)$ *is precisely the condition for the maximization*[77] *of* θ, since

$$(5a') \quad \frac{\partial\theta}{\partial\sigma_r} = \frac{\sigma_r[f'(\sigma_r)] - [\bar{r} - r^*]}{\sigma_r^2} = \frac{f'(\sigma_r) - \theta}{\sigma_r}$$

$$(5b') \quad \frac{\partial^2\theta}{\partial(\sigma_r)^2} = \frac{\sigma_r f''(\sigma_r) + [f'(\sigma_r) - \theta] - [f'(\sigma_r) - \theta]}{\sigma_r^2} = f''(\sigma_r)/\sigma_r < 0 \text{ for all } \sigma_r > 0.$$

A necessary condition for the maximization of (1′) is consequently the maximization of θ (as asserted), which is independent of w. The value of $(-\bar{U}_2/\bar{U}_1)$, however, directly depends on w (for *any* given value of θ), and a second necessary condition for the maximization of $\bar{U}$ is that w be adjusted to bring this value $(-\bar{U}_2/\bar{U}_1)$ into equality with θ, thereby satisfying the usual tangency condition between utility contours and the market opportunity function (3) in the text. These two necessary conditions are also *sufficient* because of the concavity of (1′) and the positive-definite property of the matrix of risk-investment opportunities. Q.E.D.

Note II

a) Indifference Contours Between x_i and σ^2_i When all σ_{ij} are Constant

The conclusion that the indifference contour between $\bar{x}_i$ and the variance σ_i^2 is *linear* in the general case when all covariances σ_{ij} are held constant can best be established by totally differentiating the equilibrium conditions (12) in the text [or the equivalent set (22a) restricted to the m' stocks held in the portfolio] which yields the set of equations

[72] For formal proof of these properties, see Tobin, [21], pp. 72–77.

[73] Specifically, that the amount invested in any stock in any stock mix is infinitely divisible, that all expected returns on individual stocks are finite, that all variances are positive and finite, and that the variance-covariance matrix is positive-definite.

[74] Markowitz [14] has shown that, in general, this closed curve will be made up of successive hyperbolic segments which are strictly tangent at points of overlap.

[75] Harry Markowitz, [14], chapter VII. The shape of the boundary follows from the fact that the point corresponding to any mix (in positive proportions summing to one) of any two points on the boundary lies to the left of the straight line joining those two points; and all points on and within the boundary belong to the set of available $(\sigma_r, \bar{r})$ pairs because any such point corresponds to an appropriate combination in positive proportions of at least one pair of points on the boundary.

[76] Note that the stated conditions on *the* derivatives in (2′) hold even in the exceptional cases of discontinuity. Markowitz [14], p. 153.

[77] This conclusion clearly holds even in the exceptional cases (noted in the preceding footnote) in which the derivatives of $r = f(\sigma_r)$ are not continuous. Equation (3a′) will hold as an exact equality because of the continuity of the utility function, giving $\theta = -\bar{U}_2/\bar{U}_1$. By equation (3b′), expected utility $\bar{U}$ increases with σ_r for all $f'(\sigma) \geqq -\bar{U}_2/\bar{U}_1 = \theta$, and the max σ_r consistent with $f'(\sigma) \geqq \theta$ maximizes θ by equation (5a′).

$$\lambda^0\,\sigma_1^2\,dh_1^0 + \lambda^0\,\sigma_{12}\,dh_2^0 + \ldots + \lambda^0\,\sigma_{1i}\,dh_i^0 + \ldots + \lambda^0\,\sigma_{1m'}\,dh_{m'}^0 + \frac{\bar{x}_1}{\lambda^0}\,d\lambda^0 = 0$$

$$\vdots$$

$$\lambda^0\sigma_{i1}{}^0 dh_1^0 + \lambda^0\sigma_{i2} dh_2^0 + \ldots + \lambda^0\sigma_i^2 dh_i^0 + \ldots + \lambda^0\sigma_{im'} dh_{m'}^0 + \frac{\bar{x}_i}{\lambda^0}\,d\lambda^0 = dx_i - \lambda^0 h_i d\sigma_i^2 \tag{6}$$

$$\vdots$$

$$\lambda^0\sigma_{m'1} dh_1^0 + \lambda^0\sigma_{m'2} dh_2^0 + \ldots + \lambda^0\sigma_{m'i} dh_i^0 + \ldots + \lambda^0\sigma^2_{m'} dh_{m'}^0 + \frac{\bar{x}_{m'}}{\lambda^0}\,d\lambda^0 = 0$$

$$dh_1^0 + dh_2^0 + \ldots + dh_i^0 + \ldots + dh_{m'}^0 = 0$$

Denoting the coefficient matrix on the left by $\underline{\underline{H}}$, and the i, j^{th} element of its inverse by H^{ij}, we have by Cramer's rule,

$(6a')$ $dh_i^0 = (d\bar{x}_i - \lambda^0 h_i^0\, d\sigma_i^2)\, H^{ii}$.

Since $\underline{\underline{H}}$ is non-singular, h_i^0 will be constant along an indifference contour if and only if

$(6b')$ $d\bar{x}_i = \lambda^0 h_i^0\, d\sigma_i^2$.

The indifference contour is strictly linear because the slope coefficient $\lambda^0 h_i^0$ is invariant to the absolute levels of $\bar{x}_i$ and σ_i^2 when h_i^0 is constant, as may be seen by noting that

$(6c')$ $d\lambda_0{}^0 = (d\bar{x}_i - \lambda^0\, h_i^0\, d\sigma_i^2)\, H^{i\lambda^0}$

so that

$(6d')$ $dh_i^0 = 0 \longrightarrow d\lambda_i{}^0 = 0$,

when only $\bar{x}_i$ and σ_i^2 are varied. Moreover, any pair of changes $d\bar{x}_i$ and $d\sigma_i^2$ which hold $dh_i^0 = 0$ by ($6a'$ and b') imply *no change* in the relative holding h_j^0 of *any other* security, since $dh_j^0 = (d\bar{x}_i - \lambda^0\, h_i^0\, d\sigma_i^2)\, H^{ij} = 0$ for all $j \neq i$ when $dh_i^0 = 0$. Consequently, *all* pairs of values of $\bar{x}_i$ and σ_i^2 along the linear indifference contour which holds h_i^0 fixed at some given level rigorously imply that the proportionate mix of *all other* stocks in the portfolio is *also unchanged* — as was also to be shown.

b) Indifference Contours Between x_i and σ_i When ρ Constant

If the equilibrium conditions (12) are differentiated totally to determine the indifference contours between $\bar{x}_i$ and σ_i, the left-hand side of equations ($6'$) above will be unaffected, but the right side will be changed as follows: In the i^{th} equation

$$d\bar{x}_i - \lambda^0\,[2h_i^0\sigma_i - \Sigma_{j\neq i} h_j^0\sigma_j\rho_{ij}]\, d\sigma_i = d\bar{x}_i - \lambda^0\,(h_i^0\sigma_i - \bar{x}_i/\sigma_i) d\sigma_i$$

replaces $d\bar{x}_i - \lambda^0 h_i^0 d\sigma_i^2$; the last equation is unchanged; and in all other equations $-\lambda^0 h_i^0 \sigma_j \rho_{ij} d\sigma_i$ replaces 0. We then have

$(7a')$ $dh_i^0 = [d\bar{x}_i - \lambda^0\,(h_i^0\sigma_i - \bar{x}_i/\sigma_i)\, d\sigma_i]\, H^{ii} - \lambda^0\, h_i^0\, \Sigma_{j\neq i}\sigma_j\, \rho_{ij}\, H^{ij}\, d\sigma_i$;

$(7b')$ $dh_j^0 = [d\bar{x}_i - \lambda^0\,(h_i^0\sigma_i - \bar{x}_i/\sigma_i)\, d\sigma_i]\, H^{ij} - \lambda^0 h_i^0\, \Sigma_{K\neq i}\, \sigma_K\, \rho_{iK}\, H^{jK}\, d\sigma_i$;

$(7c')$ $d\lambda^0 = [d\bar{x}_i - \lambda^0\,(h_i^0\sigma_i - \bar{x}_i/\sigma_i)\, d\sigma_i]\, H^{i\lambda^0} - \lambda^0\, h_i\, \Sigma_{K\neq i}\, \sigma_K\, \rho_{iK}\, H^{\lambda^0 K}\, d\sigma_i$.

Clearly, in this case, $dh_i^0 = 0$ does *not* imply $dh_j^0 = 0$, *nor* does it imply $d\lambda^0 = 0$.

Note III — Borrowing Limits Effective

In principle, in this case the investor must compute all the Markowitz efficient boundary segment joining M (which maximizes θ in figure 1) to the point N corresponding to the greatest attainable $\bar{r}$. Given the fixed margin w, he must then project all points on this original (unlevered) efficient set (see equation $2'$ above) to determine the new (levered) efficient set of (σ_y, $\bar{y}$) pairs attainable by using equations ($2a$, b) in the text; and he will then choose the (σ_y, $\bar{y}$) pair from this latter set which maximizes utility. With concave utility functions this optimum (σ_y, $\bar{y}$) pair will satisfy the standard optimizing tangency conditions between the (recomputed) efficient set and the utility function. The situation is illustrated in figure 2.

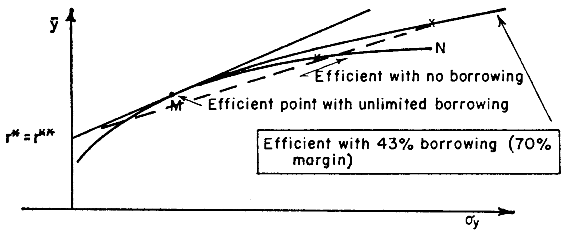

FIGURE II

Note IV — Borrowing Rate r^{**} is Higher than Lending Rate r^*

The conclusions stated in the text are obvious from the graph of this case (which incidentally is *formally* identical to Hirschleifer's treatment of the same case under certainty in [7].)

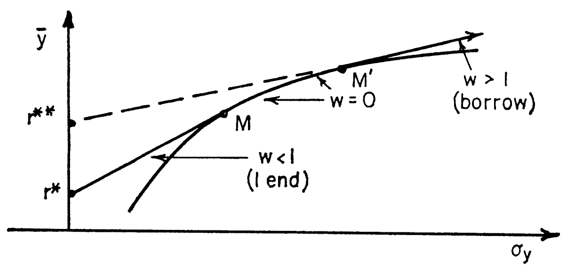

FIGURE III

The optimum depends uniquely upon the utility function if it is tangent to the efficient set with no borrowing in the range MM'.

Note V — Borrowing Rate is Dependent on Leverage

With $r^{**} = g(w)$, $g'(w) > 0$, and when the optimum $w > 1$ so that borrowing is undertaken, θ itself from equation (3) in the text becomes a function of w, which we will write $\theta\,(w)$. The optimizing equations, corresponding to ($3'a$, b) above in note I, then become

(6a') $\partial \bar{U}/\partial w = \bar{U}_1[(r - r^{**}) - wg'(w)] + \bar{U}_2\,\sigma_r \geqq 0$

(6b') $\partial \bar{U}/\partial \sigma_r = \bar{U}_1 wf'(\sigma) + \bar{U}_2 w \geqq 0$

which reduce to the two equations

(7') $\theta(w) - wg'(w)/\sigma_r = -\bar{U}_2/\bar{U}_1 = f'(\sigma)$.

The equation of the first and third members $\theta(w) - wg'(w)/\sigma_r = f'(\sigma)$ is *no longer* equal to the *maximization* of θ itself, *nor* is the solution of this equation *independent* of w which is required for the validity of the Separation Theorem. It follows that the selection of the optimal stock mix (indexed by θ) and of w *simultaneously depend upon the parameters of the utility function* (and, with normal distribution, *also* upon its *form*). Q.E.D.

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