Management Science

Publication details, including instructions for authors and subscription information:
http://pubsonline.informs.org

On the Use of Mean-Variance and Quadratic Approximations in Implementing Dynamic Investment Strategies: A Comparison of Returns and Investment Policies

Robert R. Grauer, Nils H. Hakansson,

On the Use of Mean-variance and Quadratic Approximations in Implementing Dynamic Investment Strategies: A Comparison of Returns and Investment Policies*

Robert R. Grauer • Nils H. Hakansson
Department of Economics and Faculty of Business Administration, Simon Fraser University, Burnaby, British Columbia, Canada V5A 1S6
Haas School of Business, University of California, Berkeley, 350 *Barrows Hall, Berkeley, California* 94720

This paper compares two approximation schemes for calculating the optimal portfolios in the discrete-time dynamic investment model, specifically, the mean-variance (MV) and the quadratic approximations, to the exact power function method. Future returns are estimated via the empirical probability assessment approach. The results show that (i) with quarterly revision, the MV model approximates the dynamic model very well; (ii) with annual revision, there are often sharp differences between the power function model and the MV approximation; and (iii) these differences become even larger when the quadratic approximation is used.
(*Dynamic Investment; Mean-variance Analysis; Asset Allocation; Investment Management*)

1. Introduction

In several previous studies, discrete-time dynamic portfolio theory (see Mossin (1968), Hakansson (1971, 1974), Leland (1972), Ross (1974), and Huberman and Ross (1983)) was applied to the asset allocation problem in conjunction with the empirical probability assessment approach (EPAA) to implement a set of "active" investment strategies. In a domestic setting, Grauer and Hakansson (1982, 1985, 1986) employed the model to construct and rebalance portfolios composed of U.S. stocks, corporate bonds, government bonds, and a risk-free asset. In a global environment, Grauer and Hakansson (1987) explored the performance of the model by including the four principal U.S. asset categories and up to fourteen non-U.S. equity and bond categories in the universe. In an industry-rotation application, Grauer, Hakansson, and Shen (1990) examined the model where the investment universe consisted of twelve value- or equal-weighted industry groups. Finally, Grauer and Hakansson (1991a) refined the EPAA to generating probability assessments by including an inflation adapter and applied it to the problem of "timing the market," i.e., to the problem of choosing between the value-weighted (CRSP) market portfolio and cash or borrowing. The approach is noteworthy for two reasons. First, the only information provided to the model was realized, past (joint) returns. Second, it sometimes generated statistically significant positive abnormal returns. This raises the question: Is the (abnormal) performance attributable to the input to the optimizer or the model itself? As to the first possibility, Grauer and Hakansson (1991b) and Grauer and Shen (1992) have explored ways of controlling for estimation error with mixed results.

The primary purpose of this paper is to compare the performance of mean-variance (MV) and quadratic approximations to the (exact) power function formulation that springs from the dynamic reinvestment model, where the approximations are chosen so that, for each power function, the quadratic and the MV investors

* Accepted for the focused issue on financial modeling, *Management Science*, Vol 38, No. 11, November 1992

0025-1909/93/3907/0856$01.25

have similar levels of risk aversion. A number of authors have argued that, in the single period case, power function policies can be well approximated by MV policies, e.g., Levy and Markowitz (1979), Pulley (1981, 1983), Kallberg and Ziemba (1983), and Kroll, Levy, and Markowitz (1984). However, there is an opposing intuition which suggests that the power functions' strong aversion to low returns and bankruptcy will lead them to select portfolios that are not MV-efficient, e.g., Hakansson (1971) and Grauer (1981a, 1981b, 1986). Therefore, a second and main purpose of the paper is to shed further light on whether the power policies differ from the corresponding MV and quadratic policies when returns are compounded over many periods.

Our principal findings, based on domestic and global portfolios selected from up to ten stock and bond categories and cash or borrowing, are as follows. First, with quarterly revision, the MV model approximates the exact power function model very well. Second, with annual revision, the portfolio compositions and returns earned by the more risk averse power function strategies bear little resemblance to those of the corresponding MV approximations. Third, the quadratic approximations to the more risk averse power function strategies are even less satisfactory.

The paper proceeds as follows. Section 2 outlines the discrete-time dynamic investment model and the method employed to make it operational. Section III formulates the MV and quadratic utility approximations to the power function models and summarizes the principal findings drawn from earlier studies that have compared these approaches. Section 4 describes the data, §5 reports the results, and §6 contains a summary and concluding comments.

2. The Dynamic Investment Model

The discrete-time dynamic investment model used is the same as the one employed in Grauer and Hakansson (1986) and the reader is therefore referred to that paper (specifically pp. 288–291) for details. It is based on the pure reinvestment version of dynamic investment theory.[1] In particular, if $U_n(w_n)$ is the *induced* utility of wealth w with n periods to go (to the horizon) and r is the single-period return, the important convergence result (see Hakansson 1974),

$$U_n(w_n) \to \frac{1}{\gamma} w^{\gamma}, \quad \text{for some } \gamma < 1,$$

holds for a very broad class of terminal utility functions $U_0(w_0)$ when returns are independent (but nonstationary) from period to period. Convergence implies that use of the stationary, myopic decision rule

$$\max E\left[\frac{1}{\gamma}(1+r)^{\gamma}\right], \quad \text{for some } \gamma < 1, \tag{1}$$

in each period is optimal. Consequently, the family (1) encompasses a broad variety of different goal formulations for investors with intermediate- to long-term investment horizons.[2] Since the relative risk aversion function $(-wU''(w)/U'(w))$ for (1) is $1-\gamma$, the family (1) incorporates the full range of risk attitudes from zero to infinity.

More specifically, at the beginning of each period t, the investor chooses a portfolio, x_t, on the basis of some member, γ, of the family of objective functions for returns r given by

$$V(1+r) = \frac{1}{\gamma}(1+r)^{\gamma}.$$

This is equivalent to solving the following problem in each period t:

$$\max_{x_t} E\left[\frac{1}{\gamma}(1+r_t(x_t))^{\gamma}\right] = \max_{x_t} \sum_s \pi_{ts} \frac{1}{\gamma}(1+r_{ts}(x_t))^{\gamma} \tag{2}$$

subject to:

$$x_{it} \geq 0, \quad x_{Lt} \geq 0, \quad x_{Bt} \leq 0, \quad \text{all } i, \tag{3}$$

$$\sum_i x_{it} + x_{Lt} + x_{Bt} = 1, \tag{4}$$

$$\sum_i m_{it} x_{it} \leq 1, \tag{5}$$

$$\Pr(1 + r_t(x_t) \geq 0) = 1, \tag{6}$$

[1] The simple reinvestment formulation does ignore consumption of course

[2] A plot of the functions $(1/\gamma)(1+r)^{\gamma}$ for several values of γ was given in Grauer and Hakansson (1982, p. 42)

where

$r_{ts}(x_t) = \sum_i x_{it}r_{its} + x_{Lt}r_{Lt} + x_{Bt}r^d_{Bt}$ is the (ex ante) return on the portfolio in period t if state s occurs,

$\gamma \leq 1$ = a parameter that remains fixed over time,

x_{it} = the amount invested in risky asset category i in period t as a fraction of own capital,

x_{Lt} = the amount lent in period t as a fraction of own capital,

x_{Bt} = the amount borrowed in period t as a fraction of own capital,

$x_t = (x_{1t}, \ldots, x_{nt}, x_{Lt}, x_{Bt})$,

r_{it} = the anticipated total return (dividend yield plus capital gains or losses) on asset category i in period t,

r_{Lt} = the return on the riskfree asset in period t,

r^d_{Bt} = the interest rate on borrowing at the time of the decision at the beginning of period t,

m_{it} = the initial margin requirement for asset category i in period t expressed as a fraction,

π_{ts} = the probability of state s at the end of period t, in which case the random return r_{it} will assume the value r_{its}.

Constraint (3) rules out short sales and (4) is the budget constraint. Constraint (5) serves to limit borrowing (when desired) to the maximum permissible under the margin requirements that apply to the various asset categories. Finally, constraint (6) rules out any (ex ante) probability of bankruptcy.

The inputs to the model are based on the empirical probability assessment approach. Suppose quarterly revision is used. Then, at the beginning of quarter t, the portfolio problem (2)–(6) for that quarter uses the following inputs: the (observable) riskfree return for quarter t, the (observable) call money rate + 1% at the beginning of quarter t, and the (observable) realized returns for the risky assets for the previous n quarters. Each joint realization in quarters $t - n$ through $t - 1$ is given probability $1/n$ of occurring in quarter t. Thus, estimates are updated each period on a moving basis and used in raw form without adjustment of any kind. On the other hand, since the whole joint distribution is specified and used, there is no information loss; all moments and correlations are implicitly taken into account. It may be noted that the empirical distribution of the past n periods is optimal if the investor has no information about the form and parameters of the true distribution, but believes that this distribution went into effect n periods ago.[3]

With these inputs in place, the portfolio weights for the various asset categories and the proportion of assets borrowed are calculated by solving system (2)–(6) via nonlinear programming methods.[4] At the end of quarter t, the realized returns on the risky assets are observed, along with the realized borrowing rate r^r_{Bt} (which may differ from the decision borrowing rate r^d_{Bt}).[5] Then, using the weights selected at the beginning of the quarter, the realized return on the portfolio chosen for quarter t is recorded. The cycle is then repeated in all subsequent quarters.[6]

All reported returns are gross of transaction costs and taxes and assume that the investor in question had no influence on prices. There are several reasons for this approach. First, since this is one of the first studies in this area, we wish to keep the complications to a minimum. Second, the return series used as inputs and for comparisons also exclude transaction costs (for reinvestment of interest and dividends) and taxes. Third, many investors are tax-exempt and various techniques are available for keeping transaction costs low. Finally,

[3] For a comprehensive overview of these issues and the problems associated with the estimation of return distributions, see Bawa, Brown, and Klein (1979)

[4] The solvency constraints (6) are not binding for the power functions, with $\gamma < 1$, and discrete probability distributions with a finite number of outcomes because of the functions' strong aversion to low levels of wealth or returns Nonetheless, it is convenient to explicitly consider (6) so that the nonlinear programming algorithm (see Best 1975) used to solve the investment problems does not attempt to evaluate an infeasible policy as it searches for the optimum On the other hand, the solvency constraints may well be binding for the less risk-averse members of the MV and quadratic utility families addressed in the next section While we solved the quadratic approximations to the power problems, see (10) below, with and without the solvency constraints, we chose not to impose them on the MV problem, (7) below, as the imposition of the constraints runs counter to the original intent of the model—to economize on the amount of input data and to ease the computational problem

[5] The realized borrowing rate r^r_{Bt} was calculated as the average of monthly observations

[6] Note that if $n = 32$ under quarterly revision, then the first quarter for which a portfolio can be selected is the 33rd quarter of the data set since the previous 32 quarters are required to develop the estimated return distribution used for any quarter's portfolio choice

since the proper treatment of these items is nontrivial, they are better left to a later study.

3. Mean-variance and Quadratic Approximations to Power Functions

It is well known that the MV model is consistent with von Neumann-Morgenstern utility if investors have quadratic utility functions, or if returns are multivariate normal *provided that the utility function is defined over the whole wealth axis*. Furthermore, the MV model can be expected to produce asymptotically valid approximations to expected utility problems when risks are small, as is implied by Samuelson's (1970) "compact" distributions, or when the length of the trading interval is close to zero (see e.g., Ohlson 1975). These observations provide the impetus for three approaches to approximating expected utility problems in terms of means and variances.

The first approach approximates the utility function with a quadratic utility function. However, quadratic utility generates several implausible results, such as negative marginal utility beyond some finite wealth level, as well as increasing absolute risk aversion. Therefore, a second approach, which assumes returns are multivariate normal, provides a more popular way of reconciling the expected utility and MV models. However, with exact multivariate normal return distributions, the return on any portfolio containing risky assets will be normally distributed, which means that end-of-period wealth will be normally distributed. This, in turn, means that the expected utility does not exist for the logarithmic and power functions generated by dynamic investment theory, as they are not defined over negative wealth levels. The third approach to approximating expected utility in terms of means and variances, which is based on a Taylor-series expansion of the expected utility problem for short holding periods, is discussed below.

Hakansson (1971a) was the first to compare MV portfolios with logarithmic utility (or growth-optimal) portfolios. Employing an example with two risky securities (that exhibited nonsymmetric returns) and a riskfree asset, he showed that the characteristics of growth-optimal portfolios were quite different from MV efficient portfolios. Fama and MacBeth (1974) took issue with the results. They assumed multivariate normality and (ignoring the impossibility of integrating a logarithmic utility function with a normal probability distribution) proceeded to "show" there was no difference in the *ex post* geometric mean of a specially-levered and an unlevered proxy for the market portfolio. Ziemba, Parkan, and Brooks-Hill (1974) solved the portfolio selection problem using MV analysis, numerical integration, normal distributions, a riskfree asset, and power or logarithmic utility functions—with linear segments appended in order to overcome the problem of integrability. Under these approaches the MV approximation appears to work well. Grauer (1981a, 1981b, 1986), on the other hand, focused on "exact" power utility functions and opportunity sets where returns were constructed either to be "approximately" normal or from historical joint frequency distributions. The results showed marked differences in (1) the mix of risky assets, and in (2) the risk of bankruptcy for the levered policies of MV and power utility investors. In contrast, there were small differences in the expected returns and standard deviations of the unlevered portfolios.

Loistl (1976), Levy and Markowitz (1979), Pulley (1981, 1983), and Kroll, Levy, and Markowitz (1984) investigated how closely portfolios chosen on the basis of functions of means and variances can approximate portfolios chosen by maximizing expected utility, while Kallberg and Ziemba (1983), Pulley (1981, 1983), and Grauer (1986) considered whether portfolios with "similar" risk-aversion characteristics would hold "similar" portfolios. With the exception of Loistl, who discussed the difficulties associated with the use of Taylor-series approximations to expected utility problems (particularly logarithmic and power utility problems), and Grauer (1986), the consensus drawn from this literature is that portfolios chosen on the basis of mean and variance can closely approximate portfolios chosen by maximizing expected utility, especially when the investors have similar risk-aversion characteristics.

In this paper we employ two approaches to approximating power function models: the classic MV quadratic programming approach (which gives a continuous-time approximation to expected utility) and the quadratic utility approach. In its classic form, the MV model is formulated as a parametric quadratic programming problem in order to economize on the amount of input data and to ease the computational problem (see, e.g., Markowitz 1959 and Sharpe 1970). Let μ_{it} be the expected rate of return on security i at time t and σ_{ijt} be the covariance between the returns on securities i and j at time t. Then the MV investment problem is

$$\max_{x_t} \left\{ T(1 + \mu_t) - \frac{1}{2}\sigma_t^2 \right\}, \tag{7}$$

subject to (3)–(5), where the mean and variance of the portfolio are

$$\mu_t = \sum_i x_{it}\mu_{it} + x_{Lt} r_{Lt} + x_{Bt} r^d_{Bt},$$

$$\sigma_t^2 = \sum_i \sum_j x_{it} x_{jt} \sigma_{ijt},$$

respectively, and T may be interpreted either as a parametric quadratic programming parameter or an investor's MV "risk tolerance" parameter. In this latter case, the larger T is, the more tolerant the investor is to risk. However, (7) is also open to an interesting interpretation that links it to von Neumann-Morgenstern utility based on the isoelastic functions in (1). Using a Taylor-series approximation to expected utility for short holding periods, it can be shown that T is equal to the reciprocal of the Pratt-Arrow measure of relative risk aversion (RRA), hence the terminology risk tolerance parameter (see, e.g., Ohlson 1970, Pulley 1981 and Grauer 1986). That is, $T = 1/\text{RRA}$, where $\text{RRA} = -wU''(w)/U'(w)$ is evaluated at a zero rate of return.[7] As noted, for the power functions in (1), $\text{RRA} = 1 - \gamma$, and the MV approximations to them are given with

$$T = 1/(1 - \gamma). \tag{8}$$

Under certain conditions this result holds exactly in continuous time (see Merton 1973, 1980).

[7] Alternatively we could rewrite (7) as

$$(1 + \mu_t) - \frac{1}{2T}\sigma_t^2$$

This formulation is consistent with von Neumann-Morgenstern utility if the investor has negative exponential preferences and makes normal probability assessments. In this case T is the investor's (constant) absolute risk tolerance. (See Lintner 1968 and Sharpe 1987.)

The second approach to approximating power utility is in terms of quadratic utility. The family of quadratic utility functions is given by

$$u(1 + r) = a(1 + r) - (1 + r)^2, \tag{9}$$

where the parameter "a" reflects differing degrees of risk aversion. The optimization problem now becomes

$$\max_{x_t} E[a(1 + r_t) - (1 + r_t)^2]$$

$$= \max_{x_t} \sum_s \pi_{ts}[a(1 + r_{ts}) - (1 + r_{ts})^2], \tag{10}$$

subject to (3)–(5) (or (3)–(6)). More generally, a quadratic utility function is of the form $u(w) = a^*w - w^2$, where w is end-of-period wealth. If the utility function is written in return form, as in (9), then $a = a^*/w_0$, where w_0 is initial wealth. Suppose we let $w_0 = 1$ and the return on the portfolio approach zero. Then a quadratic utility function will have the same degree of relative risk aversion as a power utility function if

$$a = 2 + \frac{2}{1 - \gamma}. \tag{11}$$

However, any quadratic utility approximation must be approached with caution. For example, beyond the value $a/2$, the quadratic utility function exhibits decreasing marginal utility. If a is set to a value less than twice unity plus the riskfree return, i.e., $1 + r_{Lt}$, a quadratic utility function will attempt to lose money and will not choose MV-efficient portfolios. Thus, we should anticipate difficulties in approximating very risk averse power functions using (10) and (11) (see Grauer 1986), although where the line is drawn is an empirical question.

4. Data

The data used to estimate the probabilities of next period's returns on risky assets, and to calculate each period's realized returns on risky assets, came from several sources. The (monthly and annual) returns series for the U.S. asset categories were obtained from Ibbotson Associates (1988). The quarterly data base on non-U.S. equity returns for 1960–1987 covering seven countries (Canada, France, Germany, Japan, Netherlands, Switzerland, and the United Kingdom) used in our quarterly portfolio revision runs was provided by First Chicago Investment Advisors. We obtained the annual returns for 14 non-U.S. equity and government bonds from Ibbotson, Carr, and Robinson (1982) for the period 1960–1980 and from Ibbotson Associates for the 1981–1985 period for stocks and for the 1981–1984 period for government bonds. The remainder of the annual non-U.S. data set was updated through 1987 using data provided by First Chicago Investment Advisors. All returns are expressed in U.S. dollars and represent total returns since both dividends (net of foreign taxes withheld) and capital appreciation or depreciation are taken into account.

The riskfree asset used for quarterly revision was assumed to be 90-day U.S. Treasury bills maturing at the end of the quarter; we used the *Survey of Current Business* and *The Wall Street Journal* as sources. In the annual portfolio revision case, the riskfree return was obtained from the yield, as of the beginning of the year, on that U.S. government obligation (note, bond, or bill) that matured on the date closest to the end of the year in question; we obtained the 1968–1976 data privately from Roger Ibbotson and the remainder from *The Wall Street Journal*.

Margin requirements for stocks were obtained from the *Federal Reserve Bulletin*. These requirements were assumed to apply to non-U.S. equities as well. Initial margins were set at 10% for U.S. government bonds, at 20% for non-U.S. government bonds, and at 35% for corporate bonds. These levels are on the conservative side and designed to compensate for the absence of maintenance requirements.[8]

The borrowing rate was assumed to be the call money rate +1% for *decision* purposes (but not for rate of return calculations); the applicable beginning of period rate, r^d_{Bt}, was viewed as persisting throughout the period and thus as riskfree. For 1934–1976, the call money rates were obtained from the *Survey of Current Business*; for later periods *The Wall Street Journal* was the source.

Table 1 **Asset Category and Fixed-Weight Portfolio Symbols**

RL	Riskfree lending (quarterly or 1-year U S Treasury bills or notes)	E10	Equal-weighted portfolio of risky assets
GB	Long-term U S government bonds	E2	20% in E10, 80% in RL
CB	Long-term U S corporate bonds	E4	40% in E10, 60% in RL
CS	U S common stocks (S & P 500)	E6	60% in E10, 40% in RL
SS	U S small stocks	E8	80% in E10, 20% in RL
B	Borrowing	E12	120% in E10, 20% in B
GEB	German government bonds	E14	140% in E10, 40% in B
JAB	Japanese government bonds	E16	160% in E10, 60% in B
CA	Canadian equities	E18	180% in E10, 80% in B
FR	French equities	E20	200% in E10, 100% in B
GE	German equities	IN	U S inflation
JA	Japanese equities		
NE	Dutch equities		
SI	Swiss equities		
UK	British equities		

5. Results

Because of space limitations, only a portion of the results are reported here. However, Tables 2 through 7 and Figures 1 through 5 provide a fairly representative sample of our findings. For comparison with the active strategies, we have, in the global setting, calculated and included the returns on the equal-weighted portfolio of the risky assets (E10) as well as up- and down-levered versions of E10. The compositions of these portfolios,[9] along with an enumeration of the asset categories included in the study, are reported in Table 1.

Quarterly Revision

We first examine quarterly revision strategies choosing among the four basic U.S. asset categories, i.e., riskfree lending, long-term government and corporate bonds, and common stocks (represented by the S&P 500 index). Table 2 shows, and Figure 1 plots, the geometric means and standard deviations of the realized returns of the basic asset categories (see squares); of the 16 (active) power function strategies (see dots) corresponding to values of γ in (1) ranging from -75 (extremely risk averse) to 1 (risk neutral); and of the corresponding MV approximations to the 16 power strategies (see diamonds), for the 22-year period 1966–1987.

[8] There was no practical way to take maintenance margins into account in our programs In any case, it is evident from the results that they would come into play only for the more risk-tolerant strategies, and even for them only occasionally, and that the net effect would be relatively neutral

[9] Portfolio E6, for example rebalances to 60% invested in E10 and 40% in RL (riskfree lending) at the beginning of each period while E14 places 140% of its assets in E10 by borrowing 40% (category B)

The estimating period is 32 quarters and borrowing is permitted up to posted margin requirements.

Over the period, the geometric mean returns of the asset categories were quite similar, ranging from 6.28% (per annum) for government bonds to 8.93% for common stocks. Treasury bills earned 7.43% with a standard deviation of 2.65%. Turning to the power policies, we observe that the geometric mean returns were not spectacular, ranging from 7.56% for the −75 power to 10.00% for the .25 power. On the other hand, the low standard deviations are noteworthy, e.g., the −3 power attained a higher geometric mean rate of return than common stocks with less than half the standard deviation. For our purposes, however, the most important observation to be drawn from Figure 1 and Table 2 is that, for each power γ, the returns of the exact policy

Figure 1 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1966–1987, Four U.S. Asset Categories**

(quarterly portfolio revision, 32-quarter estimating period, with borrowing)

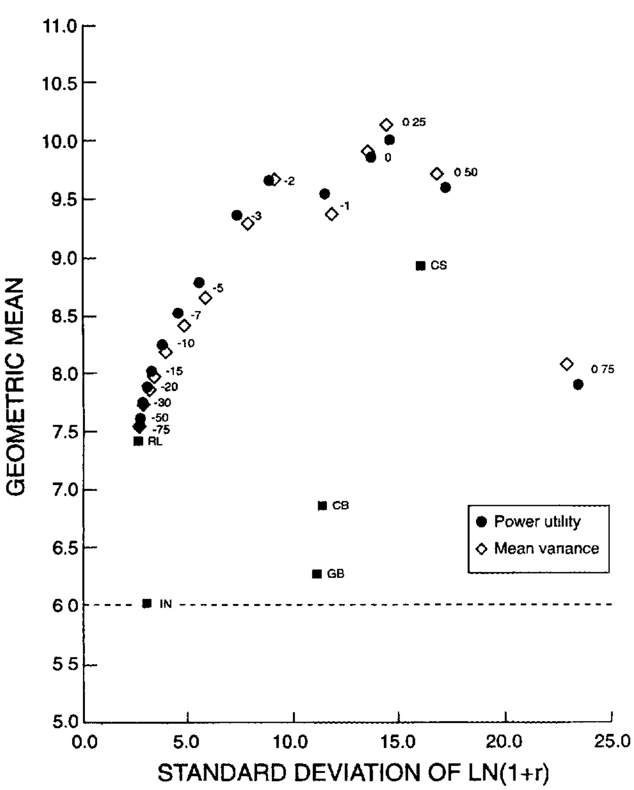


Table 2 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1966–1987, Four U.S. Asset Categories**

(quarterly portfolio revision, 32-quarter estimating period, with borrowing)

	Power Function		MV Approximation	
Portfolio	Geom Mean	Standard Deviation*	Geom Mean	Standard Deviation*
Common Stocks	8.93	16 06	8 93	16 06
Government Bonds	6 28	11 15	6 28	11 15
Corporate Bonds	6 86	11 37	6 86	11 37
Riskfree Lending	7 43	2 65	7 43	2 65
Inflation	6 02	3 13	6 02	3 13
Power −75	7 56	2 69	7 55	2 70
Power −50	7 62	2 73	7 61	2 75
Power −30	7 75	2 85	7.72	2 88
Power −20	7 89	3 04	7 85	3 09
Power −15	8 02	3 26	7 97	3 36
Power −10	8 26	3 79	8 19	3 96
Power −7	8 52	4 52	8 43	4 80
Power −5	8 79	5 51	8 67	5 89
Power −3	9 36	7 38	9 29	7 80
Power −2	9 66	8 86	9 67	9 13
Power −1	9 54	11 50	9 36	11.76
Power 0	9 85	13 70	9 91	13 52
Power 25	10 00	14 58	10 13	14 42
Power 50	9 59	17 24	9 71	16 78
Power 75	7 88	23 47	8 08	23 00
Power 1	2 33	39 63	−9 19	90 22

* Standard deviation is for the variable $\ln(1 + r_t)$

and of the corresponding MV approximation to it were almost identical.

Next we examine quarterly revision strategies when the universe is composed of the four basic U.S. asset categories as well as French, German, Dutch, Swiss, British, Japanese, and Canadian equities. The returns of the power function strategies and the returns of the corresponding MV approximations are reported in Table 3 and in Figure 2 for the 18-year period from 1970–1987. The estimating period is 40 quarters and borrowing is permitted.

During this period, the international arena provided a most generous environment for the U.S. investor. With the exception of Canada, all the international equity

Table 3 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1970–1987, Eleven U.S. and non-U.S. Asset Categories**

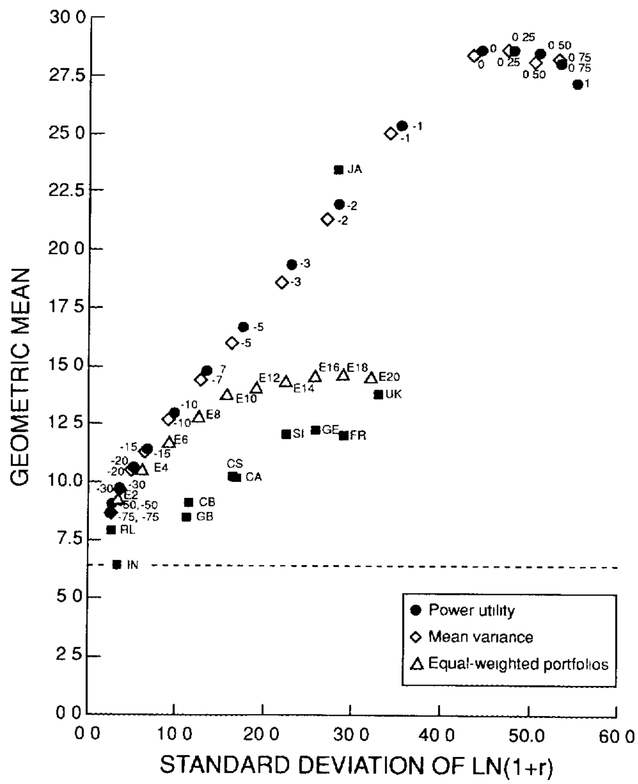
(quarterly portfolio revision 40-quarter estimating period with borrowing)

Portfolio	Power Function Geom Mean	Power Function Standard Deviation*	MV Approximation Geom Mean	MV Approximation Standard Deviation*
U.S. Common Stocks	10.25	16.36	10.25	16.36
U.S. Government Bonds	8.42	11.15	8.42	11.15
U.S. Corporate Bonds	9.10	11.42	9.10	11.42
French Equities	12.02	28.98	12.02	28.98
German Equities	12.19	25.81	12.19	25.81
Dutch Equities	15.25	18.22	15.25	18.22
Swiss Equities	12.06	22.50	12.06	22.50
British Equities	13.72	32.91	13.72	32.91
Japanese Equities	23.45	28.27	23.45	28.27
Canadian Equities	10.22	16.77	10.22	16.77
Riskfree Lending	7.89	2.74	7.89	2.74
Inflation	6.41	3.32	6.41	3.32
Power −75	8.66	2.51	8.61	2.48
Power −50	9.03	2.73	8.96	2.66
Power −30	9.76	3.62	9.64	3.42
Power −20	10.62	5.08	10.45	4.73
Power −15	11.45	6.63	11.22	6.15
Power −10	12.99	9.71	12.67	8.98
Power −7	14.76	13.44	14.33	12.43
Power −5	16.67	17.43	15.99	16.06
Power −3	19.36	23.03	18.63	21.71
Power −2	21.90	28.28	21.28	26.86
Power −1	25.31	35.40	25.01	33.89
Power 0	28.57	44.29	28.35	43.42
Power .25	28.57	47.96	28.57	47.22
Power .50	28.43	50.86	28.07	50.34
Power .75	27.98	53.27	28.21	53.03
Power 1	27.15	55.07	27.15	55.07
E2	9.28	3.28	9.28	3.28
E4	10.57	6.00	10.57	6.00
E6	11.75	9.11	11.75	9.11
E8	12.84	12.32	12.84	12.32
E10	13.82	15.58	13.82	15.58
E12	14.05	18.98	14.05	18.98
E14	14.33	22.30	14.33	22.30
E16	14.55	25.57	14.55	25.57
E18	14.69	28.74	14.69	28.74
E20	14.52	31.84	14.52	31.84

* Standard deviation is for the variable $\ln(1 + r_t)$

Figure 2 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1970–1987, Eleven U.S. and non-U.S. Asset Categories**

(quarterly portfolio revision, 40-quarter estimating period, with borrowing)



markets earned higher geometric mean returns than the U.S. stock and bond markets, albeit with higher standard deviations. Japan was the clear winner with an annual geometric mean return of over 23%. Note that the active strategies did very well compared to the passive (rebalancing) strategies. With virtually the same standard deviation, the −50 power earned 9.03% compared to 7.89% for Treasury bills, while the 0, .25, and .5 powers attained geometric mean returns in excess of 28%. Turning to the power function-MV comparison, we note again that the two return series are strikingly similar. Thus, the MV translation (7)–(8) appears to provide a very good approximation to the power policies, for the full range of risk attitudes γ, even in the more volatile global environment when portfolios are revised quarterly.

Annual Revision

We now turn to annual revision strategies. The first case considered is a universe composed of the four basic U.S. asset categories as well as U.S. small stocks. The returns from the power function strategies and from the corresponding MV approximations are reported in Table 4 and in Figure 3 for the 54-year period 1934–1987. The estimating period is 8 years and margin purchases are permitted.

During the period, equities clearly outdistanced fixed income securities. Common and small stocks earned geometric mean returns of 11.1% and 15.1% versus 4.3% and 4.8% for government and corporate bonds.

Table 4 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1934–1987, Five U.S. Asset Categories**

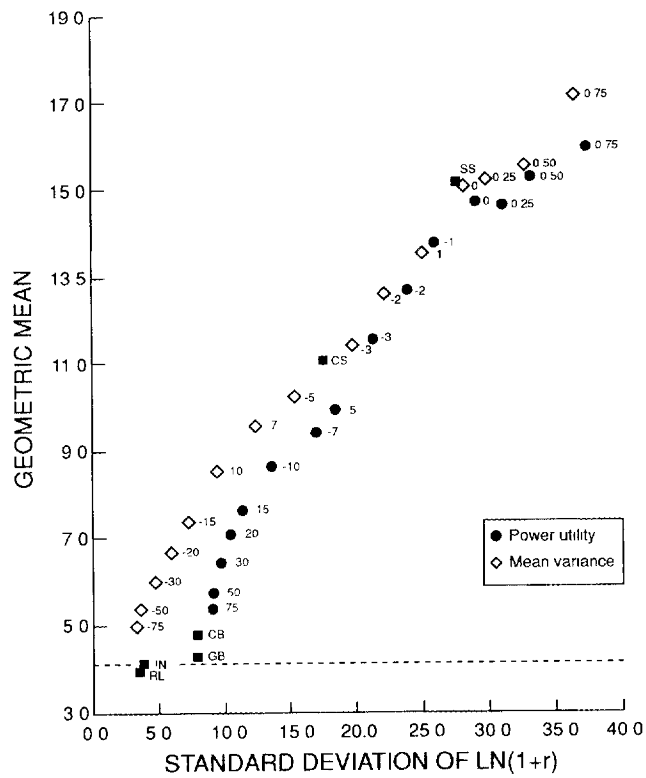
(annual portfolio revision, 8-year estimating period, with borrowing)

	Power Function		MV Approximation	
Portfolio	Geom Mean	Standard Deviation*	Geom Mean	Standard Deviation*
Common Stocks	11 05	17 38	11 05	17 38
Government Bonds	4 26	7 92	4 26	7 92
Corporate Bonds	4 78	7 94	4 78	7 94
Small Stocks	15 14	27 54	15 14	27 54
Riskfree Lending	3 93	3 46	3 93	3 46
Inflation	4 10	3 80	4 10	3 80
Power −75	5 38	8 89	4 99	3 18
Power −50	5 75	9 06	5 36	3 55
Power −30	6 43	9 58	6 00	4.65
Power −20	7 08	10 33	6 68	5 82
Power −15	7 63	11 20	7.37	7 12
Power −10	8 63	13 42	8 52	9 26
Power −7	9 41	16 84	9 58	12 21
Power −5	9 94	18 29	10 23	15 11
Power −3	11 57	21 04	11 43	19 50
Power −2	12 68	23 77	12 61	21 96
Power −1	13 75	25 77	13.53	24 78
Power 0	14 70	28 83	15 05	27 94
Power 25	14 61	30 88	15 21	29 50
Power 50	15 25	32 89	15 52	32 48
Power 75	15 93	37 17	17 13	36 11
Power 1	6 69	81 32	−100 00	–

* Standard Deviation is for the variable $\ln(1 + r_t)$

Figure 3 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1934–1987, Five U.S. Asset Categories**

(annual portfolio revision, 8-year estimating period, with borrowing)



However, the standard deviations of the equities were between two and three and a half times as large as those for bonds. We now observe much larger differences between the power policies and the corresponding MV approximations than in the quarterly revision case, with the MV "frontier" substantially higher than the power policy "frontier" for the highly risk-averse and highly risk-tolerant strategies.

Finally, we investigate the annual revision case when the universe is composed of U.S., German, and Japanese common stocks and government bonds, and Canadian equities. The estimating period is again 8 years with margin purchases permitted. The returns of the exact power function strategies and those of the corresponding MV and quadratic approximations are reported in Table 5 for the 20-year period 1968–1987. The results

Table 5 Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function, Mean-Variance, and Quadratic Decision Makers, 1968–1987, Eight U.S. and non-U.S. Asset Categories

(annual portfolio revision, 8-year estimating period, with borrowing)

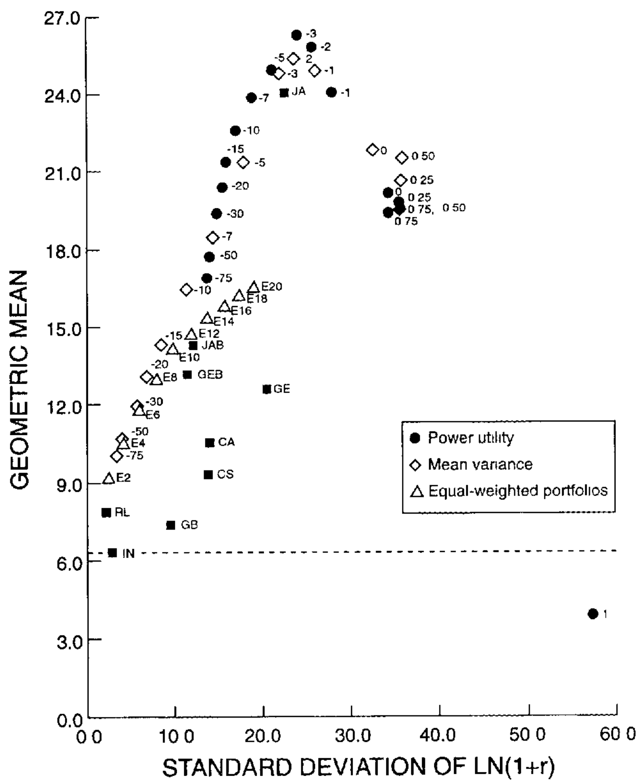
	Power Function		MV Approximation		Quadratic Approx	
Portfolio	Geom Mean	Std Dev *	Geom Mean	Std Dev *	Geom Mean	Std Dev *
U S Common Stocks	9 27	16 03	9 27	16 03	9.27	16.03
U S Gov't Bonds	7 31	10 95	7 31	10 95	7 31	10 95
German Equities	12 57	23.94	12 57	23.94	12 57	23 94
Japanese Equities	24 02	26 28	24 02	26 28	24 02	26 28
Canadian Equities	10 52	16.19	10 52	16.19	10 52	16 19
German Gov't Bonds	13 13	13 27	13 13	13 27	13 13	13 27
Japanese Gov't Bonds	14 27	14.29	14.27	14 29	14.27	14 29
Riskfree Lending	7 84	2 49	7 84	2 49	7 84	2 49
Inflation	6 31	3.16	6 31	3 16	6 31	3 16
Power −75	16 85	15 96	10 07	3 66	1 21	3 23
Power −50	17 71	16 20	10 71	4.39	1 92	2.84
Power −30	19.33	17 14	11.93	6 36	3 32	2 12
Power −20	20 38	17 71	13 05	7 71	5 01	1 32
Power −15	21 35	18 38	14 31	9 58	6 80	1 10
Power −10	22 56	19 57	16 45	13 10	9 59	2 88
Power −7	23 80	21 73	18 44	16 58	11 84	4 14
Power −5	24 91	24 59	21 38	20 52	13 65	6 96
Power −3	26 27	27 75	24 77	25.30	17 01	12 60
Power −2	25 80	29 73	25 36	27 21	19 53	17 19
Power −1	24 03	32 47	24 91	30 27	22.83	24 34
Power 0	20 15	39 76	21 84	37 77	23 30	33 79
Power 25	19 76	41.23	20 65	41 64	21.23	39.16
Power 50	19 53	41 46	21 53	41 68	21 53	41 96
Power 75	19 33	39.71	19.51	41 25	19 70	41 51
Power 1	3 84	66 73	−100 00	—	—	—
E2	9 23	2 70	9.23	2 70	9 23	2 70
E4	10 56	4 56	10 56	4 56	10 56	4 56
E6	11 82	6 75	11.82	6 75	11 82	6 75
E8	13 03	8 98	13 03	8 98	13 03	8 98
E10	14 18	11.19	14 18	11 19	14.18	11 19
E12	14 76	13 56	14.76	13.56	14.76	13 56
E14	15.35	15 87	15 35	15 87	15 35	15 87
E16	15 84	18 09	15 84	18 09	15 84	18 09
E18	16 26	20 15	16 26	20 15	16 26	20 15
E20	16.46	22 04	16 46	22 04	16.46	22 04

* Standard deviation is for the variable $\ln(1 + r_t)$

are also shown in pictorial form in Figure 4, which compares the returns of the power function strategies to those of the corresponding MV approximations, and in Figure 5, which compares the returns of the power function strategies to those of the respective quadratic approximations.

Figure 4 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Mean-Variance Decision Makers, 1968–1987, Eight U.S. and non-U.S. Asset Categories**

(annual portfolio revision, 8-year estimating period, with borrowing)



It is again apparent that the international arena provided a most generous environment for the U.S. investor. For example, both German and Japanese government bonds provided much higher geometric mean rates of return, and smaller standard deviations, than U.S. stocks.[10] Moreover, the power function investors fared very well: with the exception of the risk neutral investor, all earned in excess of 16.8% per annum, with the −3 power attaining a 26.3% geometric mean return. However, two observations overshadow the rest. First, the low power strategies generated very high geometric mean rates of return accompanied by fairly substantial standard deviations. Second, the returns earned by the more risk-averse power function strategies bore almost no resemblance to the returns attained by the corresponding MV and quadratic approximation strategies. In addition, the returns generated by the MV approximations are very different from those earned by the quadratic approximations with the exception of the very risk-tolerant strategies. Perhaps the most dramatic example of these observations is provided by the −75 power strategy, where the power, MV, and quadratic investors achieved geometric mean rates of return of 16.8%, 10.1%, and 1.2% per annum accompanied by standard deviations of 16%, 3.7%, and 3.2%, respectively.

In order to shed light on why the returns were so different, we examine the portfolio compositions and realized returns for the −75 and −3 power strategies

[10] A portion of these differences is due to currency movements

Figure 5 **Comparison of Geometric Means and Standard Deviations of Annual Portfolio Returns for Power Function and Quadratic Decision Makers, 1968–1987, Eight U.S. and non-U.S. Asset Categories**

(annual portfolio revision, 8-year estimating period, with borrowing)

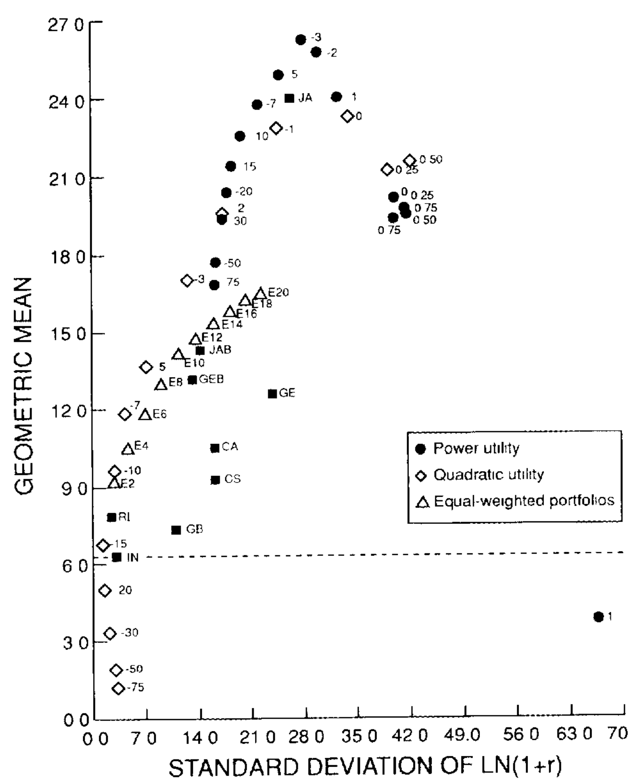


Table 6 Portfolio Compositions and Realized Returns for Power −75 and the Corresponding Mean-Variance and Quadratic Approximations, Eight U.S. and non-U.S. Asset Categories

(annual revision, 8-year estimating period, with borrowing)

DATE	RETURN	LEND	BORROW	CS	GB	GE	JA	CA	GEB	JAB
				Power −75 Compositions and Returns						
1968	8.15	0.67		0.03			0.07		0.18	0.06
1969	6.75	0.89		0.01			0.01	0.02		0.07
1970	7.00	0.92		0.01			0.07			
1971	79.07		−3.55				0.02	0.18	1.42	2.92
1972	36.50		−3.77					0.13	1.45	3.19
1973	20.37		−3.68				0.01	0.14	1.56	2.97
1974	13.77						0.01	0.04	0.71	0.23
1975	7.78						0.02	0.20	0.77	
1976	21.91						0.04	0.22	0.74	
1977	59.48		−2.54				0.12	0.86	2.56	
1978	9.59		−4.26	0.22	2.22		0.05	0.30	2.48	
1979	−12.29		−2.34			1.00	0.10		2.23	
1980	−4.76					0.22		0.15	0.63	
1981	13.48	0.99							0.01	
1982	13.09	0.99					0.01			
1983	12.13	0.71			0.06		0.15	0.08		
1984	9.91	0.87			0.02		0.07	0.04		
1985	12.72	0.84			0.03		0.08	0.04		
1986	26.05	0.63			0.09		0.18	0.10		
1987	26.39				0.19		0.54	0.27		
				MV Approximation Compositions and Returns						
1968	7.92	0.68		0.03			0.05	0.01	0.16	0.07
1969	6.75	0.88		0.01			0.02	0.02	0.01	0.06
1970	7.30	0.93		0.01			0.06			
1971	23.60						0.06	0.06	0.44	0.44
1972	8.57	0.68						0.06		0.26
1973	5.64	0.78						0.02	0.01	0.19
1974	8.96	0.78					0.02	0.02	0.18	
1975	7.85	0.58				0.02	0.01	0.08	0.31	
1976	12.85	0.55				0.02	0.01	0.10	0.33	
1977	12.40	0.59				0.02	0.01	0.08	0.30	
1978	10.35	0.64			0.05		0.01	0.05	0.25	
1979	7.62	0.48				0.13	0.04		0.35	
1980	5.23	0.59				0.07	0.01	0.07	0.26	
1981	13.51	0.99							0.01	
1982	13.11	0.99					0.01			
1983	9.89	0.83		0.03	0.03		0.04	0.04	0.01	0.02
1984	9.78	0.91		0.01			0.04	0.03		
1985	11.07	0.91			0.02		0.04	0.02		
1986	13.90	0.83		0.03	0.04		0.06	0.04		
1987	6.53	0.85		0.08	0.03		0.02	0.02		

Table 7 Portfolio Compositions and Realized Returns for Power −3 and the Corresponding Mean-Variance and Quadratic Approximations, 1968–1987, Eight U.S. and non-U.S. Asset Categories

(annual revision, 8-year estimating period, with borrowing)

DATE	RETURN	LEND	BORROW	CS	GB	GE	JA	CA	GEB	JAB
				Power −3 Compositions and Returns						
1968	20.37		−0.60	0.58			0.56			0.46
1969	7.12			0.31			0.23	0.33		0.13
1970	−6.85					0.03	0.97			
1971	80.06		−3.13				0.26	0.13		3.74
1972	60.78		−3.75				0.14			4.60
1973	−16.35		−2.72				0.57			3.15
1974	11.00		−0.39				0.39		1.00	
1975	5.30		−1.54			0.01	0.35		2.19	
1976	73.54		−2.95				0.57	0.13	3.25	
1977	98.55		−3.29				0.47		3.82	
1978	48.82		−3.69		0.54		0.38		3.77	
1979	−14.35		−2.80			0.29	0.51		3.01	
1980	−2.63						0.19		0.81	
1981	8.18	0.75							0.25	
1982	10.22	0.78					0.22			
1983	24.91			0.35			0.56	0.09		
1984	7.44		−0.34	0.08			0.83	0.42		
1985	53.70		−0.94		0.23		1.18	0.54		
1986	112.13		−1.19	0.58	0.24		1.02	0.35		
1987	39.86		−1.60	0.17	0.76		1.09	0.59		
				MV Approximation Compositions and Returns						
1968	20.05		−0.53	0.66			0.53			0.34
1969	6.27			0.46			0.27	0.23		0.03
1970	−5.55			0.08			0.92			
1971	68.47		−2.24				0.47	0.31		2.46
1972	77.43		−2.41				0.28	0.62		2.51
1973	−4.54		−1.59				0.22	0.09		2.29
1974	13.63		−0.07				0.24		0.84	
1975	7.73		−1.16			0.11	0.25		1.80	
1976	69.46		−2.94				0.34	0.37	3.24	
1977	97.71		−3.27				0.49		3.78	
1978	53.67		−3.14	0.07	0.04		0.39	0.13	3.53	
1979	−13.72		−2.92			0.23	0.49		3.21	
1980	−3.49						0.17		0.83	
1981	8.58	0.76							0.24	
1982	10.57	0.81					0.19			
1983	24.83			0.39			0.51	0.10		
1984	7.70			0.08			0.58	0.35		
1985	33.15			0.10			0.64	0.26		
1986	89.68		−1.04	0.67	0.24		0.76	0.37		
1987	13.67		−1.18	1.48	0.23		0.35	0.12		

Table 7 *Continued*

DATE	RETURN	LEND	BORROW	CS	GB	GE	JA	CA	GEB	JAB
					Quadratic Approximation Compositions and Returns					
1968	15.21			0.40			0.33			0.27
1969	7.31			0.21			0.16	0.20		0.42
1970	−0.89	0.29		0.12			0.59			
1971	47.13		−1.14				0.33	0.30	0.03	1.49
1972	42.23		−0.71				0.14	0.37		1.21
1973	2.93		−0.19				0.10	0.04		1.05
1974	16.73						0.13		0.87	
1975	8.34					0.05	0.11		0.84	
1976	31.83		−0.55				0.11	0.19	1.25	
1977	33.37		−0.37			0.01	0.11	0.11	1.14	
1978	21.82		−0.24	0.02			0.08	0.13	1.00	
1979	3.46		−0.20			0.28	0.10		0.82	
1980	−3.43					0.06	0.07	0.12	0.76	
1981	11.48	0.90							0.10	
1982	12.02	0.91					0.09			
1983	19.41			0.28	0.06		0.31	0.17	0.13	0.05
1984	8.06	0.31		0.11	0.04		0.30	0.21		0.03
1985	26.12	0.24		0.01	0.16		0.39	0.21		
1986	45.55			0.18	0.24		0.34	0.25		
1987	9.59	0.08		0.50	0.18		0.12	0.12		

and for the corresponding MV and quadratic approximations in each of the 20 years 1968 through 1987. Table 6 shows the results for the ultra-conservative −75 power investor. This investor lent heavily in 9 of the 20 years as one might expect. But the same investor also borrowed from 2.34 to as much as 4.26 times initial wealth to invest in bonds six times, realizing a 28.5% geometric mean rate of return in those years. On the other hand, the portfolio compositions of the MV approximation to the −75 power strategy were dominated by investment in Treasury bills in all but one year (as were the other highly risk-averse MV approximation strategies).[11]

While the compositions and returns earned by the −75 power function and the corresponding MV strategy were sharply different, these differences are overshadowed in comparison to the difference between either one and the corresponding quadratic approximation. During the (20-year) period, the quadratic investor borrowed anywhere from 1.46 to 7.96 times initial wealth to invest in Treasury bills! While such a strategy is clearly bizarre, it is easily explained. Recall that a quadratic function attains its maximum at $1 + r = a/2$ (see (9)). Thus, if a is small, the maximizing strategy is simply to borrow to lend! We also note that this strategy is not confined to very risk-averse investors (see (11)). With annual revision, even the quadratic approximation to the −7 power borrowed to lend at times. On the other hand, with quarterly revision, the phenomenon of borrowing to lend was confined to the approximations related to powers −30 or lower.

Table 7 contains the results for the −3 power strategy. In this case, the MV approximations to the power strategies were much closer than they were for the more risk-averse powers. The primary differences in portfolio composition occurred in the 1971–1973 and 1985–1987 periods, when the power investor used more leverage to buy Japanese bonds in the first subperiod and to acquire Japanese equities in the second subperiod. In terms

[11] Similar results were obtained for the 1968–85 period with the expanded 17 risky asset international data set employed in Grauer and Hakansson (1987), and when the estimating period was lengthened to ten years

of realized returns, this resulted in a slight advantage to the MV approximation in the first subperiod and a very distinct advantage to the power function investor in the second. On the other hand, while the quadratic approximations were not as obviously different from the power strategies as in the more risk-averse cases, they were nevertheless distinctly different. For example, the quadratic approximation borrowed less both in terms of amounts and the number of years that leverage was employed. The quadratic investor, for example, borrowed seven times while the power investor did so fourteen times. This departure in investment policy explains most of the large differences in return patterns.

6. Summary and Concluding Comments

Our results indicate that in various asset allocation settings under quarterly portfolio revision, the MV model approximations (7)–(8) replicated the power function model (2) very well. This is true across the full spectrum of risk attitudes γ. However, with annual revision, there were clear differences between the power and MV models. The portfolio compositions and returns generated by the more risk-averse power function strategies bore little resemblance to those obtained via the corresponding MV approximations (7)–(8). In particular, the MV approximations were far more conservative in their use of leverage. Finally, with annual revision, the quadratic approximations (10)–(11) performed even less satisfactorily than the MV approximations. The full extent to which these findings are data dependent remains to be ascertained. However, for well-diversified asset categories and holding periods of one quarter or less, the first two moments do appear to leave little room for the higher moments to assert themselves.[12]

[12] Presented at the European Finance Association meetings in Stockholm, the Northern Finance Association meetings in Ottawa, and the Conference on Financial Optimization at the University of Pennsylvania The authors would like to thank the participants for helpful comments, especially Alan Kraus, Harry Markowitz, Ieuan Morgan, and William Ziemba

Financial support from the Social Sciences and Humanities Research Council of Canada and the Natural Science and Engineering Council of Canada, and the capable research assistance of Ruth Cornale, Jack Merkeley, Simon Ng, and Frederick Shen, are gratefully acknowledged.

References

Bawa, Vijay, Stephen Brown, and Roger Klein, *Estimation Risk and Optimal Portfolio Choice*, North-Holland, Amsterdam, 1979

Best, Michael J , "A Feasible Conjugate Direction Method to Solve Linearly Constrained Optimization Problems," *J Optimization Theory and Applications*, 16 (1975), 25–38.

Fama, Eugene F., and James D. MacBeth, "Long-Term Growth in a Short-Term Market," *J. Finance*, 29 (1974), 857–885

Grauer, Robert R., "A Comparison of Growth Optimal and Mean Variance Investment Policies," *J. Financial and Quantitative Analysis*, 16 (1981a), 1–21

——, "Investment Policy Implications of the Capital Asset Pricing Model," *J Finance*, 36 (1981b), 127–141.

——, "Normality, Solvency and Portfolio Choice," *J. Financial and Quantitative Analysis*, 21 (1986), 265–278

—— and Nils H Hakansson, "Higher Return, Lower Risk: Historical Returns on Long-Run, Actively Managed Portfolios of Stocks, Bonds and Bills, 1936–1978," *Financial Analysts J*, 38 (1982), 39–53

—— and ——, "Returns on Levered, Actively Managed Long-Run Portfolios of Stocks, Bonds, and Bills, 1934–1984," *Financial Analysts J.*, 41 (1985), 24–43.

—— and ——, "A Half-Century of Returns on Levered and Unlevered Portfolios of Stocks, Bonds, and Bills, With and Without Small Stocks," *J Business*, 59 (1986), 287–318.

—— and ——, "Gains from International Diversification. 1968–85 Returns on Portfolios of Stocks and Bonds," *J. Finance*, 42 (1987), 721–739.

—— and ——, "On Timing the Market: The Empirical Probability Approach With an Inflation Adapter," Manuscript (1991a)

—— and ——, "Stein and CAPM Estimators of the Means in Asset Allocation: A Case of Mixed Success," Manuscript (1991b).

—— and Frederick C. Shen, "Do Constraints Improve Portfolio Performance?" Manuscript (1992)

Nils H Hakansson, "Capital Growth and the Mean-Variance Approach to Portfolio Selection," *J. Financial and Quantitative Analysis*, 6 (1971a), 517–557.

——, "On Optimal Myopic Portfolio Policies, With and Without Serial Correlation of Yields," *J. Business*, 44 (1971b), 324–334.

——, "Convergence to Isoelastic Utility and Policy in Multiperiod Portfolio Choice," *J. Financial Economics*, 1 (1974), 201–224

Huberman, Gur and Stephen Ross, "Portfolio Turnpike Theorems, Risk Aversion and Regularly Varying Utility Functions," *Econometrica*, 51 (1983), 1345–1361.

Ibbotson Associates, Inc., *Stocks, Bonds, Bills and Inflation. 1987 Yearbook*, Ibbotson Associates, Inc., Chicago, IL, 1988

Ibbotson, Roger, Richard Carr, and Anthony Robinson, "International Equity and Bond Returns," *Financial Analysts J.*, 38 (1982), 61–84

Kallberg, J G. and W. T. Ziemba, "Comparison of Alternative Utility Functions in Portfolio Selection Problems," *Management Sci.*, 9 (1983), 1257–1276

Kroll, Yorham, Haim Levy and Harry Markowitz, "Mean-Variance versus Direct Utility Maximization," *J. Finance*, 39 (1984), 47–75.

Leland, Hayne, "On Turnpike Portfolios," In Karl Shell, G P. Szego, (Eds.), *Mathematical Methods in Investment and Finance*, North-Holland, Amsterdam, 1972

Levy, Haim and Harry Markowitz, "Approximating Expected Utility by a Function of Mean and Variance," *American Economic Rev.*, 69 (1979), 308–317.

Lintner, John, "The Market Price of Risk, Size of Market and Investor's Risk Aversion," *J. Business* (1968)

Loistl, Otto, "The Erroneous Approximation of Expected Utility by Means of a Taylor's Series Expansion: Analytic and Computational Results," *American Economic Rev.*, 66 (1976), 904–910.

Markowitz, Harry M., *Portfolio Selection: Efficient Diversification of Investments*, John Wiley & Sons, Inc., New York, NY, 1959

Mossin, Jan, "Optimal Multiperiod Portfolio Policies," *J. Business*, 41 (1968), 215–229.

Merton, Robert C., "An Intertemporal Capital Asset Pricing Model," *Econometrica*, 41 (1973), 867–887.

——, "On Estimating the Expected Return on the Market: An Exploratory Investigation," *J. Financial Economics*, 8 (1980), 323–361.

Ohlson, James A , "The Asymptotic Validity of Quadratic Utility as the Trading Interval Approaches Zero," In W. T. Ziemba and R G. Vickson (Eds), *Stochastic Optimization Models in Finance*, Academic Press, New York; NY, 1975

Pulley, Lawrence B., "A General Mean-Variance Approximation to Expected Utility for Short Holding Periods," *J Financial and Quantitative Analysis*, 16 (1981), 361–373.

——, "Mean-Variance Approximation to Expected Logarithmic Utility," *Operations Res* , 31 (1983), 685–696.

Ross, Stephen, "Portfolio Turnpike Theorems for Constant Policies," *J. Financial Economics*, 1 (1974), 171–98.

Sharpe, William F., *Portfolio Theory and Capital Markets*, McGraw-Hill, New York, NY, 1970.

——, "Integrated Asset Allocation", *Financial Analysts J.*, (1987), 25–32.

Samuelson, Paul A , "The Fundamental Approximation Theorem of Portfolio Analysis in Terms of Means, Variances, and Higher Moments", *Rev. Economic Studies*, 36 (1970), 537–542

Ziemba, William T , C. Parkan, and R. Brooks-Hill, "Calculation of Investment Portfolios with Risk-Free Borrowing and Lending," *Management Sci* , 21 (1974), 209–222

Accepted by William T. Ziemba, former Departmental Editor; received December 1990. *This paper has been with the authors* 10 *months for* 2 *revisions*