2. RISK AVERSION OVER TIME IMPLIES STATIC RISK AVERSION

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Multiperiod Consumption-Investment Decisions

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I. *The Problem*

The simplest version of the multiperiod consumption-investment problem considers a consumer with wealth w_1, defined as the market value of his assets at the beginning of period 1, which must be allocated to consumption c_1 and a portfolio investment w_1-c_1. The portfolio will yield an uncertain wealth level w_2 at the beginning of period 2 which must be divided between consumption c_2 and investment w_2-c_2. Consumption-investment decisions must be made at the beginning of each period, until the consumer dies and his wealth is distributed among his heirs. The consumer's objective is to maximize the expected utility of lifetime consumption.

Uncertainty models of the multiperiod consumption-investment problem have been considered by Edmund Phelps, Nils Hakansson, and Jan Mossin (1968). But their quite similar treatments place severe restrictions on both the form of the consumer's utility function and the process generating investment returns. For example, the most general model is Hakansson's. He assumes that the probability distributions of one-period portfolio wealth relatives[1] that will be available at any future period t are known for certain at period 1 and thus are independent of events that will occur between periods 1 and t. The consumer's utility function is assumed to be of the additive form

$$u(c_1, \ldots, c_t, \ldots) = \sum_{t=1}^{\infty} \alpha^{t-1} U(c_t), \qquad 0 < \alpha < 1,$$

so that the utility provided by consumption in period t cannot be affected by levels of consumption attained in other periods. Moreover, the one-period utility function $U(c_t)$ is assumed to be monotone increasing and strictly concave (i.e., marginal utility is positive and the consumer is a risk averter),[2] and $U(c_t)$ must imply either "constant risk aversion" or "constant proportional risk aversion," where these terms are as defined by John Pratt.

Hakansson shows that in this model the optimal consumption for any period t is a linear increasing function of wealth w_t. In addition, the portfolio opportunities that will be available in periods after t affect the optimal split of w_t between current consumption and investment, but the optimal *proportions* of portfolio funds invested in different assets at t depend only on the consumer's one-period utility function $U(c_t)$ and on the distributions of one-period wealth relatives associated with currently available portfolios. In essence, the choice of an optimal portfolio *mix* is "myopic" in the sense that it depends only on one-period utilities and returns.

But these appealingly simple results are direct consequences of the restrictions

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[1] The one-period wealth relative from t to t+1 is defined as total dollars of market value at t+1 per dollar nvested at t. It is thus one plus the one-period (percentage) return from t to t+1

[2] A general definition of concavity is provided in fn. 9.

imposed on utility functions and the process generating investment returns by the Phelps-Hakansson-Mossin models. The goal in this paper is to present a more general multiperiod consumption-investment model, but one which nevertheless leads to interesting hypotheses about observable aspects of consumer behavior. The main result is the proposition that if the consumer is risk averse (i.e., his utility function for *lifetime* consumption is strictly concave) and markets for consumption goods and portfolio assets are perfect,[3] then the consumer's observable behavior in the market in any period is indistinguishable from that of a risk averse expected utility maximizer who has a one-period horizon.

With this result it is then possible to provide a *multiperiod* setting for hypotheses about consumer behavior derived from *one-period* wealth allocation models, and these have been studied extensively.[4] One-period models assume, of course, that consumers have one-period horizons, but in most cases their behavioral propositions only require that consumers behave *as if* they are risk averse one-period expected utility maximizers, and this will be the case in the multiperiod model to be presented here. Thus perhaps the major contribution of this paper is in providing a means for bridging the gap between one-period and multiperiod models.

As a specific illustration we will later consider in detail the adjustments to the multiperiod model that are necessary to provide a multiperiod setting for the major propositions about consumer behavior associated with the one-period, two-parameter wealth allocation models of Harry Markowitz, James Tobin (1958, 1965), William Sharpe (1963, 1964), John Lintner (1965a, 1965b), and Fama (1965, 1968a). Indeed it will be shown that a multiperiod model in which the optimal portfolio for any period is "efficient" in terms of distributions of *one-period* portfolio wealth relatives requires few assumptions beyond those already made in the one-period models.[5]

[3] That is: (a) consumption goods and portfolio assets are infinitely divisible, (b) reallocations of consumption and investment expenditures are costless, and (c) the consumer's activities in any market have a negligible effect on prices. Such a "perfect markets" assumption is common to almost all wealth allocation models (one-period and multiperiod) and will be maintained throughout the discussion here.

[4] See, for example, Peter Diamond, Jacques Drèze and Franco Modigliani (1966), Fama (1965, 1968a, 1968b), Jack Hirshleifer (1965, 1966), Michael Jensen, John Lintner (1965a, 1965b).

II. *The Wealth Allocation Model*

First the multiperiod model must itself be developed. Let β_t, the "state of the world," signify the set of events (current and past prices, etc.) that constitutes history up to t. Thus β_{t-1} is a subset of β_t. If the state of the world at t is β_t, there will be $n(\beta_t)$ investment assets available to the consumer; the wealth relatives from t to t+1 for these assets are represented by the vector

$$(1)\quad \begin{aligned} R &= R(\beta_{t+1}) \\ &= (r_1(\beta_{t+1}), r_2(\beta_{t+1}), \dots, \\ &\qquad r_{n(\beta_t)}(\beta_{t+1})), \end{aligned}$$

so that a value of β_{t+1} implies a value of the vector of wealth relatives. If

$$H_{\beta_t} = (h_1, h_2, \cdots, h_{n(\beta_t)})$$

is the (nonnegative) vector of dollars invested in each asset at t in state β_t, the consumer's wealth at t+1 will be

$$(2)\qquad w_{t+1} = H_{\beta_t} R(\beta_{t+1})',$$

where w_{t+1} and $R(\beta_{t+1})$ are random vari-

[5] Measuring the dispersion of a distribution of wealth relatives with a single parameter such as the standard deviation or semi-interquartile range, a portfolio is efficient in the Markowitz sense if no other portfolio with the same expected one-period wealth relative has lower dispersion, and no other portfolio with higher expected wealth relative has the same or lower dispersion.

ables at t, and the primed variable denotes the transpose.

The consumer's behavior is assumed to conform to the von Neumann-Morgenstern expected utility model. Thus if for simplicity we initially assume he will die for certain[6] at the beginning of period $\tau+1$, and if the state of the world at $\tau+1$ is $\beta_{\tau+1}$, the consumer's utility for lifetime consumption is given by the "cardinal" function

$$U_{\tau+1}(C_{\tau+1} \mid \beta_{\tau+1}) = U_{\tau+1}(c_{1-k}, \ldots, c_1, \ldots, c_{\tau+1} \mid \beta_{\tau+1}),$$

where in general

$$C_t = (c_{1-k}, \ldots, c_1, \ldots, c_t)$$

is consumption from the beginning of his life, period $1-k$, through period t, and the consumption $c_{\tau+1}$ is in the form of a bequest.[7] The goal of the consumer in his consumption-investment decisions is to maximize the expected utility of *lifetime* consumption.

The consumer must make an optimal consumption-investment decision for period 1, taking into account that decisions must also be made at the beginning of each future period prior to $\tau+1$, and that these future decisions will depend on future events. Dynamic programming, with its "backward optimization," provides a natural approach. That is, to solve the decision problem for period 1, the consumer first determines optimal decisions for all contingencies for the decision problem to be faced at period τ. Then he determines optimal decisions for $\tau-1$, under the assumption that he will always make optimal decisions at τ. And so on, until he works his way back to the decision at period 1, which is then based on the assumption that optimal decisions will be made at each future period for any possible contingency.

Formally, optimal decisions for all w_τ and β_τ can be summarized by the function

$$U_\tau(C_{\tau-1}, w_\tau \mid \beta_\tau) = \max_{c_\tau, H_{\beta_\tau}} \int_{\beta_{\tau+1}} U_{\tau+1}(C_\tau, HR' \mid \beta_{\tau+1}) \cdot dF_{\beta_\tau}(\beta_{\tau+1}), \tag{3}$$

subject to the constraints

$$0 \le c_\tau \le w_\tau, \quad H_{\beta_\tau} i' = w_\tau - c_\tau,$$
$$H_{\beta_\tau} \ge 0_{n(\beta_\tau)},$$

where $F_{\beta_\tau}(\beta_{\tau+1})$ is the distribution function of $\beta_{\tau+1}$ given state β_τ at τ; $0_n(\beta_\tau)$ is the null vector (i.e., a vector of zeros) with dimension $1 \times n(\beta_\tau)$; i is the sum vector (i.e., a vector of ones) which will always be assumed to have whatever dimension is needed for the purpose at hand.[8] The function $U_\tau(C_{\tau-1}, w_\tau \mid \beta_\tau)$ is the maximum of expected utility at τ as a function of realized past consumption $C_{\tau-1}$ and current wealth w_τ, given that the state of the world is β_τ.

More generally, for $t=1, 2, \ldots, \tau$, the process of backward optimization is summarized by the recursive relation

$$U_t(C_{t-1}, w_t \mid \beta_t) = \max_{c_t, H_{\beta_t}} \int_{\beta_{t+1}} U_{t+1}(C_t, HR' \mid \beta_{t+1}) \cdot dF_{\beta_t}(\beta_{t+1}), \tag{4}$$

subject to

$$0 \le c_t \le w_t, \quad H_{\beta_t} i' = w_t - c_t,$$
$$H_{\beta_t} \ge 0_{n(\beta_t)}.$$

[6] Later the model will be extended to allow for an uncertain period of death, (Section IV).

[7] Since the utility of a given $C_{\tau+1}$ can depend on the state of the world $\beta_{\tau+1}$, the model is consistent with the so-called "state preference" model introduced by Kenneth Arrow. As we shall see later, the more traditional framework in which utilities are not state dependent is just a special case.

For an axiomatic development of the expected utility model which implies the existence of subjective probabilities and allows for state dependent utilities (see Drèze). Drèze's analysis is in turn an extension of Savage's work.

[8] Keep in mind that, given (2), integrating over $\beta_{\tau+1}$ implies averaging over $R=R(\beta_{\tau+1})$ and thus over $w_{\tau+1}=H_{\beta_\tau}R'$.

The function $U_t(C_{t-1}, w_t | \beta_t)$ provides the maximum expected utility of *lifetime* consumption if the consumer is in state β_t at period t, his wealth is w_t, his past consumption was C_{t-1}, and optimal consumption-investment decisions are made at the beginning of period t and all future periods.

Expression (4) exemplifies a common feature of dynamic programming models. In general it is possible to *represent* the decision problem of any period t in terms of a derived objective function (in this case U_{t+1}) which is *explicitly* a function only of variables for t+1 and earlier periods, but which in fact summarizes the results of optimal decisions at t+1 and subsequent periods for all possible future events. Thus the recursive relation (4) *represents* the multiperiod problem as a sequence of "one-period" problems, though at any stage in the process the objective function used to solve the one-period problem summarizes optimal decisions for all future periods.

Representing the multiperiod consumption-investment problem as a sequence of one-period problems in itself says nothing about the characteristics of an optimal decision for any period. The main result of this paper is, however, the following.

Proposition 1. If the utility function for lifetime consumption $U_{\tau+1}(C_{\tau+1} | \beta_{\tau+1})$ has properties characteristic of risk aversion (specifically, if for all $\beta_{\tau+1}$, $U_{\tau+1}(C_{\tau+1} | \beta_{\tau+1})$ is monotone increasing and strictly concave in $C_{\tau+1}$), then for all t the derived functions $U_t(C_{t-1}, w_t | \beta_t)$ will also have these properties.[9]

[9] The monotonicity of $U_{\tau+1}$ says that the marginal utility of consumption in any period is positive, while strict concavity implies that for $0<\alpha<1$,

$$U_{\tau+1}(\alpha C_{\tau+1} + (1-\alpha)\hat{C}_{\tau+1} | \beta_{\tau+1}) > \alpha U_{\tau+1}(C_{\tau+1} | \beta_{\tau+1}) + (1-\alpha) U_{\tau+1}(\hat{C}_{\tau+1} | \beta_{\tau+1}),$$

where $C_{\tau+1}$ and $\hat{C}_{\tau+1}$ are any two consumption vectors that differ in at least one element. Geometrically, concavity says that a straight line between any two points on the function $U_{\tau+1}$ lies below the function. As in the case of the more familiar utility of money function, the concavity of $U_{\tau+1}$ implies risk aversion.

The proof of the proposition is presented in the Appendix.

III. *Implications*

Though at this point its importance is far from obvious, it is the concavity of the functions $U_t(C_{t-1}, w_t | \beta_t)$ for all t and β_t, as stated in Proposition 1, that will now allow us to bridge the gap between one-period and multiperiod wealth allocation models.

A. *The Utility of Money Function*

A foretaste of the discussion can be obtained by using the multiperiod model to derive the familiar utility of money function, most often discussed in the literature in connection with the expected utility model. If the state of the world at period 1 is β_1 and the consumer's past consumption has been $\hat{C}_0$, then for t=1 expression (4) yields

$$v_1(w_1 | \beta_1) = U_1(\hat{C}_0, w_1 | \beta_1) = \max_{c_1, H_{\beta_1}} \int_{\beta_2} U_2(C_1, HR' | \beta_2) dF_{\beta_1}(\beta_2),$$

subject to

$$0 \leq c_1 \leq w_1, \quad H_{\beta_1} i' = w_1 - c_1, \quad H_{\beta_1} \geq 0_{n(\beta_1)}.$$

v_1 is the relevant utility function for timeless gambles taking place at period 1: that is, gambles where the outcome is known before the consumption-investment decision of period 1 is made. From Proposition 1, v_1 has the characteristics of a risk averter's utility of money function: that is, it is monotone increasing and strictly concave in w_1. Thus, though he obtains his utility of money function by a complicated process of backward optimization, and though his utility of money function in fact shows the expected utility of lifetime consumption associated with a given level of wealth at period 1, the consumer's behavior in choosing among timeless gambles is indistinguishable from that of a risk averter making a once-and-for-all decision. In other words, our analysis provides a multiperiod setting for the more traditional discussions of utility of money functions for risk averters, most of which abstract from the effects of future decisions.

B. *One-Period and Multiperiod Models: General Treatment*

More generally, when it comes time to make a decision at the beginning of any period t, t = 1, 2, . . . , τ, past consumption (equal, say, to $\hat{C}_{t-1}$) is known, so that the decision at t can be based on the function

$$v_{t+1}(c_t, w_{t+1} \mid \beta_{t+1}) = U_{t+1}(\hat{C}_{t-1}, c_t, w_{t+1} \mid \beta_{t+1})$$

Thus, for given wealth w_t and state of the world β_t, the consumer's problem at t can be expressed as

$$(5) \quad \max_{c_t, H_{\beta_t}} \int_{\beta_{t+1}} v_{t+1}(c_t, HR' \mid \beta_{t+1}) dF_{\beta_t}(\beta_{t+1}),$$

subject to

$$0 \leq c_t \leq w_t, \quad H_{\beta t} i' = w_t - c_t,$$
$$H_{\beta t} \geq 0_{n(\beta_t)}.$$

Since, from Proposition 1, U_{t+1} is monotone increasing and strictly concave in (C_t, w_{t+1}), v_{t+1} is monotone increasing and strictly concave in (c_t, w_{t+1}). Thus, though the consumer faces a τ period decision problem, the function $v_{t+1}(c_t, w_{t+1} | \beta_{t+1})$, which is relevant for the consumption-investment decision of period t, has the properties of a risk averter's *one-period* utility of consumption-terminal wealth function. Though the consumer must solve a multiperiod problem, given v_{t+1} his observed behavior in the market is indistinguishable from that of a risk averse expected utility maximizer who has a one-period horizon.[10]

In itself, this result says little about consumer behavior. Its value derives from the fact that it can be used to provide a multiperiod setting for more detailed behavioral hypotheses usually obtained from specific one-period model. Since by design the multiperiod model is based on less restrictive assumptions than most one-period models,[11] adapting it to any specific one-period model will require additional assumptions. But as we shall now see, these are mostly restrictions already implicit or explicit in the one-period models. Little generality is lost in going from a one-period to a multiperiod framework.

C. *A Multiperiod Setting for One-Period, "Two-Parameter" Portfolio Models*

Given the concavity of v_{t+1}, (5) is formally equivalent to the consumption-investment problem of a risk averse consumer with state dependent utilities and a one-period horizon.[12] As such it can be used to provide a multiperiod setting for a wide

[10] But we must keep in mind that though $v_{t+1}(c_t, w_{t+1} | \beta_{t+1})$ is only explicitly a function of variables for periods t and t+1, it shows the maximum expected utility of *lifetime* consumption, given optimal consumption-investment decisions in periods subsequent to t. Thus v_{t+1} depends both on tastes, as expressed by the function $U_{\tau+1}(C_{\tau+1} | \beta_{\tau+1})$, *and* on the consumption-investment opportunities that will be available in future periods.

The notion of summarizing market opportunities in a utility function should not cause concern. Indeed this is done when utility is written (as we have done throughout) as a function of consumption dollars; then we are implicitly summarizing the consumption opportunities (in terms of goods and services and their anticipated prices) that will be available in each period. We shall return to this point in the Appendix where the utility function $U_{\tau+1}(C_{\tau+1} | \beta_{\tau+1})$ for dollars of consumption will be derived from a more basic utility function for consumption goods.

[11] In particular, we have essentially assumed only that markets for consumption goods and portfolio assets are perfect, and that the consumer is a risk averter in the sense that his utility function for lifetime consumption is strictly concave.

[12] The term "state dependent utilities" refers to the fact that the function $v_{t+1}(c_t, w_{t+1} | \beta_{t+1})$ allows the utility of a given combination (c_t, w_{t+1}) to depend on β_{t+1}. Hirshleifer (1965, 1966) uses instead the term "state preference" to refer to this condition.

variety of one-period models such as, for example, the one-period model analyzed in detail by Hirschleifer.

But the theories of wealth allocation most thoroughly discussed in the literature are the one-period, two-parameter portfolio models of Markowitz, Tobin (1958, 1965), Sharpe (1963), and Fama (1965, 1968a). These have in turn been used by Sharpe (1964), Lintner (1965a, 1965b), Mossin (1966), and Fama (1968a, 1968b) as the basis of one-period theories of capital market equilibrium. The market equilibrium relationships between the one-period expected wealth relatives and risks of individual securities and portfolios derived from these models have in turn been given some empirical support by Marshall Blume and Michael Jensen. The remainder of this section will be concerned with using our model to provide a multiperiod setting for the apparently useful results of these one-period models.

The two-parameter portfolio models start with the assumption that one-period wealth relatives on asssets and portfolios conform to two-parameter distributions of the same general type. That is, the distribution for any asset or portfolio can be fully described once its expected value and a dispersion parameter, such as the standard deviation or the semi-interquartile range, are known.[13] It is then shown that if investors behave as if they try to maximize expected utility with respect to one-period utility functions $v_{t+1}(c_t, w_{t+1})$ that are strictly concave in (c_t, w_{t+1}), optimal portfolios will be efficient in terms of the two parameters of distributions of one-period wealth relatives.[14] The fact that optimal portfolios must be efficient then makes it possible to derive market equilibrium relations between expected wealth relatives and measures of risk for individual assets and portfolios.

But these models assume somewhat more about the utility function v_{t+1} than our multiperiod model. In particular, in the multiperiod model the function v_{t+1} $(c_t, w_{t+1}|\beta_{t+1})$, which is relevant for the consumption-investment decision of period t, is strictly concave in (c_t, w_{t+1}), but utility can be a function of the state β_{t+1} (i.e., utility can be state dependent). Thus to provide a multiperiod setting for the one-period two-parameter models it is sufficient to determine conditions under which v_{t+1} will be independent of β_{t+1}.

State dependent utilities in the derived functions v_{t+1} have three possible sources. First, tastes for given bundles of consumption goods can be state dependent. Second, as will be shown in Proposition 2 of the Appendix, utilities for given dollars of consumption depend on the available consumption goods and services and their prices, and these are elements of the state of the world. Finally, the investment opportunities available in any given future period may depend on events occurring in preceding periods, and such uncertainty about investment prospects induces state dependent utilities. Thus the most direct way to exclude state dependent utilities

[13] Since assets and portfolios must have distributions of the same two-parameter "type," the analysis is limited to the class of symmetric stable distributions, which includes the normal as a special case. Properties of these distributions are discussed, for example, in Fama (1965) and Benoit Mandelbrot, and a discussion of their role in portfolio theory can be found in Fama (1968a).

Alternatively, the results of the mean-standard deviation version of the two-parameter portfolio models can be obtained by assuming that one-period utility functions are quadratic in w_{t+1}. But strictly speaking, since the quadratic implies negative marginal utility at high levels of w_{t+1}, it is not a legitimate utility function. Moreover, the empirical evidence (see Marshall Blume, Fama (1965), Mandelbrot, and Richard Roll) that distributions of security and portfolio wealth relatives conform well to the infinite variance members of the symmetric stable class casts doubt on any model that relies on the existence of variances. Since the approach based on general two-parameter return distributions by-passes these problems, it seems simplest to lay the quadratic to rest, at least for the purpose of portfolio models.

[14] This concept of efficiency was defined in fn. 5.

is to asume that the consumer behaves as if the consumption opportunities (in terms of goods and services and their prices) and the investment opportunities (distributions of one-period portfolio wealth relatives) that will be available in any future period can be taken as known and fixed at the beginning of any previous period, and that the consumer's tastes for given bundles of consumption goods and services are independent of the state of the world.[15]

With these assumptions, the utility of a given (c_t, w_{t+1}) is independent of β_{t+1}, so that β_{t+1} can be dropped from $v_{t+1}(c_t, w_{t+1} | \beta_{t+1})$. Thus for given wealth, w_t, the decision problem facing the consumer at the beginning of any period t can be written as

$$\max_{c_t, H_{\beta_t}} \int_{R_{t+1}} v_{t+1}(c_t, HR')dF(R_{t+1})$$

Subject to

$$0 \leq c_t \leq w_t, \quad H_{\beta t}i' = w_t - c_t,$$
$$H_{\beta t} \geq 0_{n(\beta t)},$$

where $F(R_{t+1})$ is the distribution function for the vector of wealth relatives R_{t+1}.

Since Proposition 1 applies directly to this simplified version of the multiperiod model, at any period t the function $v_{t+1}(c_t, w_{t+1})$ is monotone increasing and strictly concave in (c_t, w_{t+1}) and is thus formally equivalent to the one-period utility function used in the standard treatments of the one-period, two-parameter portfolio models. If distributions of one-period security and portfolio wealth relatives are of the same two-parameter type, we have a multiperiod model in which the consumer's behavior each period is indistinguishable from that of the consumer in the traditional one-period, two-parameter portfolio models. From here it is a short step to develop a multiperiod setting for period-by-period application of the major results of the one-period, two-parameter models of market equilibrium.

[15] It is important to note that some such assumptions are implicit in the one-period, two-parameter models themselves since they do not allow for the effects of state dependent utilities on the consumption-investment decision. Exactly these assumptions are quite explicit in the Phelps-Hakansson-Mossin models.

IV. *Extensions*: *Uncertain Period of Death*

For simplicity, the development of Proposition 1 and its implications made use of the simplest version of the multiperiod consumption-investment model. In particular, it was assumed that (a) the consumer's resources at the beginning of any period t consist entirely of w_t, the value of the marketable assets carried into the period from previous periods; and that (b) the period of the consumer's death is known. But it is not difficult (indeed the major complications are notational) to extend the model to take account of the fact that the consumer has an asset, his "human capital," which will generate income in periods subsequent to t, but which cannot be sold outright in the market. The extended model would allow the ways that the consumer employs his human capital during t—his choice of occupation (s) and the division of his time between labor and leisure—to be at his discretion. It is also easy to extend the model to allow for opportunities the consumer may have to borrow against future labor income or against his portfolio.

But these extensions will not be pursued here.[16] We shall consider instead how the possibility of an uncertain period of death can be introduced into the simple wealth allocation model of Section II.

For simplicity, the analysis so far has assumed that the consumer dies for certain at the beginning of period $\tau+1$. But the model, exactly as stated in (4), is consistent with the probabilistic occurrence of

[16] They are discussed in detail in Fama (1969), an earlier version of this paper, which will be made available to readers on request.

death (and the distribution of the consumer's wealth among his heirs) in earlier periods. A subset of the events that comprise the state of the world, β_t, is the set of all events up to t that could affect the consumer's utility for any vector of lifetime consumption. A subset of these events could in turn be the life-death status of the consumer. Thus the state of the world at t might be defined as

$$\beta_t = (z_t, \hat{\beta}_t),$$

where the variable z_t represents the life-death status of the consumer and can take either the value a_t (indicating that the consumer is alive at t), or the value $d_{\bar{t}}$ (indicating that death occurred in some period $\bar{t} \leq t$), and where $\hat{\beta}_t$ is the set of all other elements of the state of the world. With this interpretation of β_t, it is easy to see that the model presented above (specifically, in (4)) is consistent with the possibility of probabilistic occurrence of death in periods prior to $\tau+1$.

Nevertheless some interesting insights into the role of the horizon period $\tau+1$ can be obtained by examining the effects of "probabilistic death" in a little more detail. When the period of death is uncertain, the consumer must make an optimal consumption-investment decision for period 1, taking into account that decisions must also be made at the beginning of any future period at which he is alive, but that the decision process will terminate as soon as he dies.

If the consumer is alive at τ, optimal decisions for all w_τ and β_τ can be summarized by the function

$$U_\tau(C_{\tau-1}, w_\tau \mid \beta_\tau) = U_\tau(C_{\tau-1}, w_\tau \mid a_\tau, \hat{\beta}_\tau)$$

$$(6) \quad = \max_{c_\tau, H_{\beta_\tau}} \int_{\beta_{\tau+1}} U_{\tau+1}(C_\tau, HR' \mid d_{\tau+1}, \hat{\beta}_{\tau+1}) \cdot dF_{\beta_\tau}(\beta_{\tau+1}),$$

subject to

$$0 \leq c_\tau \leq w_\tau, \quad H_{\beta_\tau} i' = w_\tau - c_\tau,$$
$$H_{\beta_\tau} \geq 0_{n(\beta_\tau)},$$

where in this case $\beta_\tau = (a_\tau, \hat{\beta}_\tau)$ and $\beta_{\tau+1} = (d_{\tau+1}, \hat{\beta}_{\tau+1})$. Expression (6) is just (4) when $t=\tau$ and the consumer is alive at τ.

On the other hand, if the consumer dies at the beginning of any period t ($t=1, 2, \ldots, \tau+1$), his wealth is immediately distributed among his heirs, and expression (4) for the expected utility of his lifetime consumption becomes

$$U_t(C_{t-1}, w_t \mid \beta_t) = U_t(C_{t-1}, w_t \mid d_t, \hat{\beta}_t)$$

$$(7) \quad = \int_{\beta_{\tau+1}} U_{\tau+1}(C_t, 0_{\tau+1-t} \mid \beta_{\tau+1}) dF_{\beta t}(\beta_{\tau+1}),$$

where $w_t = c_t$ is his bequest, and in this case $\beta_t = (d_t, \hat{\beta}_t)$ and $\beta_{\tau+1} = (d_t, \hat{\beta}_{\tau+1})$.[17]

In his consumption-investment decision for any period prior to τ, the consumer must consider that he could be either alive or dead at the beginning of the following period. Assuming for simplicity that the occurrence of death is independent of other elements of the state of the world, let x_t be the conditional probability that the consumer will be alive at t, given that he is alive at $t-1$. Then, with (6) as a starting point, for $t=1, 2, \ldots, \tau-1$, the process of backward optimization summarized by (4) is now expressed by (7) and the recursive relations

$$U_t(C_{t-1}, w_t \mid z_t, \hat{\beta}_t)$$
$$(8) \quad = x_t U_t(C_{t-1}, w_t \mid a_t, \hat{\beta}_t) + (1 - x_t) U_t(C_{t-1}, w_t \mid d_t, \hat{\beta}_t),$$

$$U_t(C_{t-1}, w_t \mid a_t, \hat{\beta}_t)$$
$$(9) \quad = \max_{c_t, H_{\beta_t}} \int_{\beta_{t+1}} U_{t+1}(C_t, HR' \mid z_{t+1}, \hat{\beta}_{t+1}) \cdot dF_{\beta t}(\beta_{t+1}),$$

subject to

$$0 \leq c_t \leq w_t, \quad H_{\beta t} i' = w_t - c_t,$$
$$H_{\beta t} \geq 0_{n(\beta t)}.$$

[17] If the consumer is not concerned with events subsequent to his death, then

$$U_t(C_{t-1}, w_t \mid d_t, \beta_t) = U_{\tau+1}(C_t, O_{\tau+1-t} \mid d_t, \hat{\beta}_{\tau+1})$$

for $w_t = c_t$ and all $\hat{\beta}_{\tau+1}$ such that $\hat{\beta}_t$ is a subset of $\hat{\beta}_{\tau+1}$.

The function $U_t(C_{t-1}, w_t | a_t, \beta_t)$ in (9) provides the maximum expected utility of lifetime consumption and bequests if the consumer is alive in state β_t at period t, his wealth is w_t, his past consumption was C_{t-1}, and optimal consumption-investment decisions are made at the beginning of period t and all future periods at which he is alive.

Since (7)–(9) are just a special case of the model summarized by (4), Proposition 1 applies directly to the probabilistic death model. In this case the proposition implies that for all t and $\hat{\beta}_t$, $U_t(C_{t-1}, w_t | z_t, \hat{\beta}_t)$, $U_t(C_{t-1}, w_t | a_t, \hat{\beta}_t)$, and $U_t(C_{t-1}, w_t | d_t, \hat{\beta}_t)$ are monotone increasing and strictly concave in (C_{t-1}, w_t).

Finally, expressions (7)–(9) suggest an alternative to the "sure death" interpretation of the horizon $\tau+1$. For given wealth w_1, and state of the world β_1, the consumer's problem at period 1 is to choose c_1 and H_{β_1} which

$$\max_{c_1, H_{\beta_1}} \int_{\beta_2} U_2(C_1, HR' \mid z_2, \hat{\beta}_2) dF_{\beta_1}(\beta_2)$$
$$= \max_{c_1, H_{\beta_1}} \int_{\beta_2} [x_2 U_2(C_1, HR' \mid a_2, \hat{\beta}_2)$$
$$+ (1 - x_2) U_2(C_1, HR' \mid d_2, \beta_2)] dF_{\beta_1}(\beta_2).$$

Using (8) and (9) to expand this expression, it can be shown that the decision problem of period t has weight

$$x_t x_{t-1} \cdots x_2 = x(a_t \mid a_1)$$

in the expected utility for the decision of period 1. The probability $x(a_t | a_1)$ of being alive at t will decrease with t. Thus in general for some $t = \tau+1$, the decisions of periods $\tau+1$ and beyond will have negligible weight in the expected utility for the decision at period 1, so that in the decision at period 1 it is unnecessary to look beyond $\tau+1$.

More simply, since the consumer is likely to be dead, the effects of distant future decisions, which are unlikely to be made, can be ignored. And this result does not arise from discounting of future consumption, though the effect is the same. At period 1, consumption in period $\tau+1$ may be regarded as equivalent to consumption in period 1. But in the decision of period 1 the decision of $\tau+1$ is weighted by the probability that the consumer will be alive at that time, which reduces the importance of the future consumption in the current decision.

V. *Conclusion*

In sum, assuming only that markets for consumption goods and portfolio assets are perfect and that the consumer is risk averse in the sense that his utility function for lifetime consumption is strictly concave, it has been shown that though he faces a multiperiod problem, in his consumption-investment decision for any period the consumer's behavior is indistinguishable from that of a risk averter who has a one-period horizon. It was then shown how this result can be used to provide a multiperiod setting for the more detailed hypotheses about risk averse consumer behavior that are traditionally derived in a one-period framework.

Appendix

Proposition 1. If $U_{t+1}(C_t, w_{t+1} | \beta_{t+1})$ is monotone increasing and strictly concave (henceforth m.i.s.c.) in (C_t, w_{t+1}), then $U_t(C_{t-1}, w_t | \beta_t)$ is m.i.s.c. in (C_{t-1}, w_t).

Proof: The proof of the proposition relies primarily on straightforward applications of well-known properties of concave functions (cf. Iglehart). We first establish:

Lemma 1. If $U_{t+1}(C_t, w_{t+1} | \beta_{t+1})$ is m.i.s.c. in (C_t, w_{t+1}), the expected utility function

$$\int_{\beta_{t+1}} U_{t+1}(C_t, w_{t+1} \mid \beta_{t+1}) dF_{\beta t}(\beta_{t+1})$$
$$(10) \qquad = \int_{\beta_{t+1}} U_{t+1}(C_t, H_{\beta t} R(\beta_{t+1})' \mid \beta_{t+1}) \cdot dF_{\beta t}(\beta_{t+1})$$

is strictly concave in $(C_t, H_{\beta t})$.

Proof: For any given value of β_{t+1}, and thus of $R=R(\beta_{t+1})$,

$$w_{t+1} = H_{\beta t}R(\beta_{t+1})'$$

is a linear and thus concave (though not strictly concave) function of H_{β_t}. Since by assumption $U_{t+1}(C_t, w_{t+1}|\beta_{t+1})$ is m.i.s.c. in (C_t, w_{t+1}), $U_{t+1}(C_t, H_{\beta_t}R(\beta_{t+1})'|\beta_{t+1})$ is strictly concave in (C_t, H_{β_t}).[18] Integrating over β_{t+1} in (10) preserves this concavity.

The remainder of the proof of Proposition 1 is then as follows. Let c_t^*, $H_{\beta t}^*$ and $\bar{c}_t^*$, $\bar{H}_{\beta t}^*$ be the optimal values of c_t and H_{β_t} in (4) for any two vectors (C_{t-1}, w_t) and $(\bar{C}_{t-1}, \bar{w}_t)$ that differ in at least one element. Let

$$\begin{aligned} C_{t-1} &= \alpha C_{t-1} + (1-\alpha)C_{t-1}, \\ \hat{w}_t &= \alpha w_t + (1-\alpha)\bar{w}_t, \\ \hat{c}_t &= \alpha c_t^* + (1-\alpha)\bar{c}_t^*, \\ \hat{H}_{\beta t} &= \alpha H_{\beta t}^* + (1-\alpha)\bar{H}_{\beta t}^*, \quad 0<\alpha<1. \end{aligned}$$

To establish the concavity of $U_t(C_{t-1}, w_t|\beta_t)$, we must show that

$$\begin{aligned} &U_t(C_{t-1}, \hat{w}_t \mid \beta_t) \\ (11) \qquad &> \alpha U_t(C_{t-1}, w_t \mid \beta_t) \\ &\quad + (1-\alpha)U_t(C_{t-1}, \bar{w}_t \mid \beta_t). \end{aligned}$$

From Lemma 1, for $0<\alpha<1$,

$$\begin{aligned} &\int_{\beta_{t+1}} U_{t+1}(C_{t-1}, \hat{c}_t, \hat{H}_{\beta t}R(\beta_{t+1})' \mid \beta_{t+1})dF_{\beta t}(\beta_{t+1}) \\ &> \alpha \int_{\beta_{t+1}} U_{t+1}(C_{t-1}, c_t^*, H_{\beta t}^*R(\beta_{t+1})' \mid \beta_{t+1}) \\ (12) \qquad &\qquad \cdot dF_{\beta t}(\beta_{t+1}) \\ &+ (1-\alpha)\int_{\beta_{t+1}} U_{t+1}(C_{t-1}, \bar{c}_t^*, \bar{H}_{\beta t}^*R(\beta_{t+1})' | \beta_{t+1}) \\ &\qquad \cdot dF_{\beta t}(\beta_{t+1}) \\ &= \alpha U_t(C_{t-1}, w_t \mid \beta_t) \\ &\quad + (1-\alpha)U_t(C_{t-1}, \bar{w}_t \mid \beta_t). \end{aligned}$$

Since the consumption-investment decision implied by $\hat{c}_t$, H_{β_t} is not necessarily optimal for the wealth level $\hat{w}_t$,

$$\begin{aligned} &U_t(C_{t-1}, \hat{w}_t \mid \beta_t) \\ &\geq \int_{\beta_{t+1}} U_{t+1}(\hat{C}_{t-1}, \hat{c}_t, \\ &\qquad \hat{H}_{\beta t}R(\beta_{t+1})' \mid \beta_{t+1})dF_{\beta t}(\beta_{t+1}), \end{aligned}$$

which, with (12) implies (11).[19]

The monotonicity of $U_t(C_{t-1}, w_t|\beta_t)$ in (C_{t-1}, w_t) follows straightforwardly from the monotonicity of $U_{t+1}(C_t, w_{t+1}|\beta_{t+1})$ in C_t. Thus the proposition is established.

Finally, as noted earlier (fn. 10), when utility is written (as we have done throughout) as a function of consumption dollars, we are implicitly summarizing the consumption opportunities (in terms of goods and services and their anticipated prices) that will be available in each period. We shall now conclude the paper by showing how a von Neumann-Morgenstern "cardinal" utility function for consumption dollars can be derived from a cardinal utility function for consumption commodities.

Let $q(\beta_t)=(q_1, q_2, \ldots, q_N(\beta_t))$ be the vector of quantities of $N(\beta_t)$ available commodities consumed during t in state β_t and let $p(\beta_t)=(p_1, p_2, \ldots, p_N(\beta_t))$ be the corresponding price vector. In *any* period or state one of the available consumption commodities is always "dollar gifts and bequests" which has price \$1 per unit. At the horizon $\tau+1$, dollar gifts and bequests, denoted $w_{\tau+1}$, is the only available consumption good. Let

$$Q_\tau = (q(\beta_{1-k}), \ldots, q(\beta_1), \ldots, q(\beta_\tau))$$

be the vector representing lifetime consumption of commodities, and let $V(Q_\tau, w_{\tau+1}|\beta_{\tau+1})$ be the consumer's utility of lifetime con-

[18] If $f(x_1, x_2, \ldots, x_N)=f(X)$ is m.i.s.c. in X, and if $x_i=g_i(y_1, y_2, \ldots, y_n)=g_i(Y)$, $i=1, 2, \ldots, N$, is concave (though not necessarily strictly concave) in Y, then $f(g_1(Y), g_2(Y), \ldots, g_N(Y))=f(G(Y))$ is strictly concave in Y. (See, e.g., H. G. Eggleston, p. 52.)

[19] It is assumed that $(c_t, H_{\beta t})$ is a feasible consumption-investment decision for the wealth level $\hat{w}_t$, or equivalently, that the set of feasible values of $(c_t, H_{\beta t})$ is convex. But this is a weak assumption that will be met, for example, when the constraints on c_t and $H_{\beta t}$ are equations such as $H_{\beta t}i'=w_t-c_t$ or linear inequalities such as $0\leq c_t\leq w_t$ or $\underline{H}\leq H\leq \overline{H}$, where $\underline{H}$ and $\overline{H}$ are vectors of lower and upper bounds on quantities invested in each asset.

sumption, given state $\beta_{\tau+1}$ at $\tau+1$, and where $\beta_{1-k}\subset\ldots\subset\beta_1\subset\ldots\subset\beta_{\tau+1}$. The utility function for *dollars* of consumption can then be defined as

$$U_{\tau+1}(C_{\tau+1}\mid\beta_{\tau+1}) = \max_{Q_\tau} V(Q_\tau, w_{\tau+1}\mid\beta_{\tau+1}) \tag{13}$$

Subject to

$$C_{\tau+1} = (p(\beta_{1-k})q(\beta_{1-k})', \ldots, p(\beta_\tau)q(\beta_\tau)', w_{\tau+1}) = (c_{1-k}, \ldots, c_\tau, c_{\tau+1}).$$

The role of $\beta_{\tau+1}$ in $U_{\tau+1}$ is twofold. First, psychological attitudes towards current and past consumption (or "tastes") may depend on the state of the world. Second, even if tastes for consumption *commodities* are not state dependent (so that $\beta_{\tau+1}$ can be dropped from V), the utility of any stream of *dollar* consumption expenditures depends on the history of the set of available consumption commodities and their prices, both of which are subsumed in $\beta_{\tau+1}$.

A utility function $U_{\tau+1}(C_{\tau+1}|\beta_{\tau+1})$ which has the properties required by Proposition 1 can then be obtained from $V(Q_\tau, w_{\tau+1}|\beta_{\tau+1})$ as follows.

Proposition 2. If $V(Q_\tau, w_{\tau+1}|\beta_{\tau+1})$ is m.i.s.c. in $(Q_\tau, w_{\tau+1})$, then $U_{\tau+1}(C_\tau, w_{\tau+1}|\beta_{\tau+1})$ is m.i.s.c. in $(C_\tau, w_{\tau+1})$.

Proof: Let Q_τ^* be the optimal value of Q_τ in (13) for $(C_\tau, w_{\tau+1})$ and let $\tilde{Q}_\tau^*$ be optimal for $(\tilde{C}_\tau, \tilde{w}_{\tau+1})$, where the vectors $(C_\tau, w_{\tau+1})$ and $(\tilde{C}_\tau, \tilde{w}_{\tau+1})$ differ in at least one element. For $0<\alpha<1$, let

$$(\hat{Q}_\tau, \hat{w}_{\tau+1}) = \alpha(Q_\tau^*, w_{\tau+1}) + (1-\alpha)(\tilde{Q}_\tau^*, \tilde{w}_{\tau+1}),$$

$$(\hat{C}_\tau, \hat{w}_{\tau+1}) = \alpha(C_\tau, w_{\tau+1}) + (1-\alpha)(\tilde{C}_\tau, \tilde{w}_{\tau+1}).$$

Then the strict concavity of V implies

$$V(\hat{Q}_\tau, \hat{w}_{\tau+1}\mid\beta_{\tau+1}) > \alpha V(Q_\tau^*, w_{\tau+1}\mid\beta_{\tau+1}) + (1-\alpha)V(\tilde{Q}_\tau^*, \tilde{w}_{\tau+1}\mid\beta_{\tau+1}).$$

Or equivalently,

$$V(\hat{Q}_\tau, \hat{w}_{\tau+1}\mid\beta_{\tau+1}) > \alpha U_{\tau+1}(C_\tau, w_{\tau+1}\mid\beta_{\tau+1}) + (1-\alpha)U_{\tau+1}(\tilde{C}_\tau, \tilde{w}_{\tau+1}\mid\beta_{\tau+1}).$$

Since $\hat{Q}_\tau$ is a feasible but not necessarily an optimal allocation of $\hat{C}_\tau$, an optimal allocation must have utility at least as high as that implied by $\hat{Q}_\tau$, so that

$$U_{\tau+1}(\hat{C}_\tau, \hat{w}_{\tau+1}\mid\beta_{\tau+1}) > \alpha U_{\tau+1}(C_\tau, w_{\tau+1}\mid\beta_{\tau+1}) + (1-\alpha)U_{\tau+1}(\tilde{C}_\tau, \tilde{w}_{\tau+1}\mid\beta_{\tau+1}),$$

and the concavity of $U_{\tau+1}$ is established.

To establish the monotonicity of $U_{\tau+1}$ in C_τ, simply note that if the dollars available for consumption in any period are increased, consumption of at least one commodity can be increased without reducing consumption of any other commodity, so that utility must be increased. An optimal reallocation of consumption expenditures must do at least as well.

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