SAFETY FIRST AND THE HOLDING OF ASSETS[1]

By A. D. Roy

This paper considers the implications of minimising the upper bound of the chance of a dread event, when the information available about the joint probability distribution of future occurrences is confined to the first- and second-order moments. The analysis is then applied to the particular problem of holding n assets, either for speculative gain or for the income yielded. Some comments are made about the peculiar role of money, and finally the simple case of two assets is considered in more detail.

INTRODUCTION

Early attempts to deal with the problem of behaviour under uncertainty were based upon the assumption that it is reasonable for an individual to maximise expected gain or profit in either real or money terms. There are two major objections to such an approach. First, the ordinary man has to consider the possible outcomes of a given course of action on one occasion only and the average (or expected) outcome, if this conduct were repeated a large number of times under similar conditions, is irrelevant. Second, the principle of maximising expected return does not explain the well-known phenomenon of the diversification of resources among a wide range of assets.

Recent writers have succeeded in overcoming each of the two difficulties singly, but so far as the author is aware both problems have not been solved simultaneously. Thus, Shackle in his recent contribution [**5**] to the theory of expectation in economics has tackled the first objection at the cost of a break away from orthodox probability theory, but even so he has not succeeded in explaining satisfactorily why people hold more than two assets.

Another school, of which the writings of von Neumann and Morgenstern [**6**], Friedman and Savage [**3**], and Marschak [**4**] are representative, has extended the theory of choice under certainty to problems involving expectation, by the assumption that individuals maximise expected utility. If the utility function is suitably chosen, this approach succeeds in explaining the diversification of resources among different assets. But such a theory is still open to the criticism that the behaviour advocated is only *rational* if individuals are free to expose themselves to independent risks on a large number of occasions.

A third criticism can be leveled against both sets of theory. This objection is not, however, logical but practical. Is it reasonable that

[1] I am indebted to Professor C. F. Carter, Dr. H. E. Daniels, Mr. H. G. Johnson, and Dr. A. R. Prest for their helpful comments and to Mr. R. A. Arnould for the diagrams.

real people have, or consider themselves to have, a precise knowledge of all possible outcomes of a given line of action, together with their respective probabilities or *potential surprise*? Both introspection and observation suggest that expectations are generally framed in a much more vague manner. Nor is it likely that the ordinary individual has much opportunity for extending his knowledge about what the future holds in store for him. What, therefore, appears to be required is some *simple* theory which meets in some measure all three of these major difficulties. An attempt to provide such an analysis will be given in the following discussion.

THE PRINCIPLE OF SAFETY FIRST AND LIMITED KNOWLEDGE

A valid objection to much economic theory is that it is set against a background of ease and safety. To dispel this artificial sense of security, theory should take account of the often close resemblance between economic life and navigation in poorly charted waters or manoeuvres in a hostile jungle. Decisions taken in practice are less concerned with whether a little more of this or of that will yield the largest net increase in satisfaction than with avoiding known rocks of uncertain position or with deploying forces so that, if there is an ambush round the next corner, total disaster is avoided. If economic survival is always taken for granted, the rules of behaviour applicable in an uncertain and ruthless world cannot be discovered.

In the economic world, disasters may occur if an individual makes a net loss as the result of some activity, if his resources are eroded by the process of inflation to, say, 70 per cent of their former worth, or if his income is less than what he would almost certainly obtain in some other occupation. For large numbers of people some such idea of a *disaster* exists, and the principle of Safety First asserts that it is reasonable, and probable in practice, that an individual will seek to reduce as far as is possible the chance of such a catastrophe occurring.[2] At every moment an individual's whole property is necessarily vulnerable to chance events, and the result of the current exposure to risk will determine his stake at fate's next throw.

From a formal standpoint, the minimisation of the chance of disaster can be interpreted as maximising expected utility if the utility function assumes only two values, e.g., one if disaster does not occur, and zero if it does. It would appear, however, that this formal analogy is scarcely helpful, since in the one case an individual is trying to make the expected *proportion* of occurrences of disaster as small as possible, while in maximising expected utility he is operating on a level of *satisfaction*.

[2] Such a principle has been applied to the theory of risk in insurance companies. See Cramér [**1**].

Readers, however, are open to interpret the principle in this way if they so desire, but the purpose of this discussion is not to suggest that individuals may possess a utility function of peculiar form but rather to find out the implication of a certain mode of behaviour, which appears both plausible and simple. In calling in a utility function to our aid, an appearance of generality is achieved at the cost of a loss of practical significance and applicability in our results. A man who seeks advice about his actions will not be grateful for the suggestion that he maximise expected utility.

There would appear to be no valid objection to the discontinuity in the preference scale that the existence of a single disaster value implies. In practice, death, bankruptcy, and a prison sentence are likely to be associated with sharp breaks in both our pattern of behaviour and in our scale of preferences.

It may be possible that the outcome of economic activity which is regarded as disaster is not independent of the expected value of the outcome. Thus, a person may be prepared to revise the level of disaster downwards if the expected return is at the same time raised. For example, he may at one and the same time regard a speculative loss of 10 per cent as a disaster if the expected gain is only 5 per cent, while, if the expected gain is 15 per cent, he will only get excited if his loss exceeds 25 per cent. Once again such individual psychology can no doubt be interpreted in terms of a utility function, but such a development will not be pursued here. In what follows, the disaster level of the outcome is taken to be a constant.

Let us suppose, then, that the principle of Safety First is adopted and that, when confronted with range of possible actions, we are concerned that our gross return should not be less than some quantity d. With every possible action is associated m, which is the expected value of the gross return. Since this outcome is not certain, there is coupled with m a quantity σ (the standard error of m) which is, very roughly, the average amount by which the prediction m is expected to be wrong. In what follows, we assume that we know m and σ precisely, whereas in fact they must be estimated from information about the past. This raises all kinds of problems, which are beyond the scope of this discussion, since estimates of m and σ, say $\hat{m}$ and $\hat{\sigma}$, will themselves have sampling distributions. Thus, a full analysis of the problem should discuss simultaneously not only behaviour under uncertainty but also action under uncertain uncertainty.[3]

In the particular application of the principle of Safety First that is examined here, it is postulated that m and σ are the only quantities that

[3] This brings us to the problem of statistical decision functions. See Wald [7].

can be distilled out of our knowledge of the past. The slightest acquaintance with problems of analysing economic time series will suggest that this assumption is optimistic rather than unnecessarily restrictive.

Given the values of m and σ for all feasible choices of action, there will exist a functional relationship between these quantities, which will be denoted by $f(\sigma, m) = 0$.[4] Since it is not possible to determine with this information, the precise probability of the final return being d or less for a given pair of values of m and σ, the only alternative open is a calculation of the upper bound of this probability. This can be done by an appeal to the Bienaymé-Tchebycheff inequality (see, for instance, H. Cramér [2]). Thus, if the final return is a random variable ξ, then[5]

$$P(|\xi - m| \geqslant m - d) \leqslant \frac{\sigma^2}{(m-d)^2}.$$

Then *a fortiori*

$$P(m - \xi \geqslant m - d) = P(\xi \leqslant d) \leqslant \frac{\sigma^2}{(m-d)^2}.$$

If then in default of minimising $P(\xi \leqslant d)$, we operate on $\sigma^2/(m-d)^2$, this is equivalent to maximising $(m-d)/\sigma$. In the subsequent analysis, we shall maximise this quantity and thus approach as near as is possible, under the circumstances, the true principle of Safety First. If ξ was distributed normally with mean m and standard deviation σ, then this line of conduct would minimise the probability of disaster itself; this fact is both interesting and reassuring.

If we find that σ is constant for all values of m, then the maximising of $(m-d)/\sigma$ is equivalent to maximising expected gain or profit, though not, of course, to maximising expected utility. Thus, the procedure suggested here may be regarded as a generalisation of profit maximisation in an uncertain world.

The implication of the above argument can be represented graphically with great simplicity. The function $f(\sigma, m) = 0$ can be represented by a curve as in Figure 1. If we desire to avoid an outcome of d or worse, we plot the point $D(0, d)$ on the m-axis. A tangent of positive slope is then drawn from the point D to touch the $f(\sigma, m) = 0$ curve at P. Then if we adopt that course of action which produces an estimated outcome m_0, we shall have made the upper bound of the probability of d or worse happening as small as possible. This upper bound is equal to the reciprocal of the square of the gradient of DP and so the steeper the slope

[4] There may be a whole family of such relationships: in this case $f(\sigma, m) = 0$ is their envelope.

[5] If ξ has a unimodal and symmetrical distribution, the right-hand side of the inequality will be multiplied by the factor 4/9. See, once again, Cramér [2].

of DP, the smaller is the probability of disaster[6]. In Figure 1 the upper bound of the probability of d' or worse is clearly less than that of d If, however, the slope of DP was less than 45 degrees (or 1 in 1), we should only be able to say that the chance of disaster was less than or equal to unity,[7] which is not very helpful.

This then is a particular application of the principle of Safety First; its relevance to the problems of expectation is dependent on giving an everyday interpretation to the concept of an expected value and of a standard error. We shall show later that the principle can be used with advantage even if the standard errors are unknown.

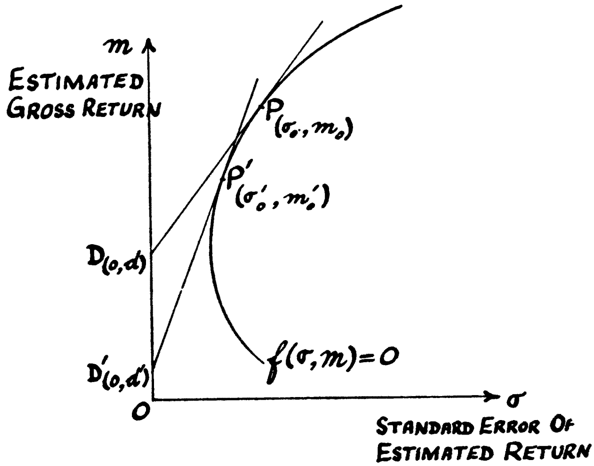

FIGURE 1—The graphical determination of the best σ, m combination.

THE OPTIMUM DISTRIBUTION OF RESOURCES AMONG n ASSETS

Now let us attack the problem of how we should distribute our resources between different assets, given certain expectations about the future. Let us suppose that we possess an amount k of resources measured in real terms. For the moment we shall not discuss what we mean by *real* in this context. It is sufficient that our resources should not be reckoned in terms of any one asset. We wish to ensure that at the end of a given period our resources do not fall to or below an amount d also in real terms.

On the basis of past experience we have estimated what the prices of the various assets are likely to be at the end of a certain period, and we have also investigated how the movements of the price of any one asset have been associated with those of the other assets.

[6] One of the negative gradient would maximise the probability.

[7] Unless we could say that the outcomes had a unimodal probability distribution.

Taking the present prices of all assets to be 1.00, the best estimates we can make of their level at the end of the period are $p_1, p_2, \cdots, p_n$[8] with which are coupled the standard errors $\alpha_1, \alpha_2, \cdots, \alpha_n$. Further, we have estimated that the correlation between the prices of the first two assets is r_{12} and similarly for all other pairs of assets. If $x_1, x_2, \cdots, x_n$ are the amounts of our resources which we hold in the form of each asset, we can now calculate m and σ.

$$m = \sum_{i=1}^{n} x_i p_i, \tag{1}$$

$$\sigma^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j r_{ij} \alpha_i \alpha_j, \tag{2}$$

for all values of x_i such that

$$k = \sum_{i=1}^{n} x_i. \tag{3}$$

In general these three equations determine not one curve relating m and σ but a whole family of such curves. We can, however, determine the curve which forms the outer boundary or envelope of this family and this is the only curve that we need to know. In order to give it in a reasonably simple form we shall employ matrix notation.

Let us write a for the column vector $(p_1/\alpha_1, p_2/\alpha_2, \cdots, p_n/\alpha_n)$ and b for the column vector $(1/\alpha_1, 1/\alpha_2, \cdots, 1/\alpha_n)$. The correlation matrix, of which the elements are r_{ij}, will be denoted by W and its inverse by W^{-1}. The boundary curve which we require (derived in the Appendix) is then the hyperbola with the equation,

$$\left[\frac{(a'W^{-1}a)(b'W^{-1}b) - (a'W^{-1}b)^2}{b'W^{-1}b}\right]\left(\sigma^2 - \frac{k^2}{b'W^{-1}b}\right) = \left(m - k\frac{a'W^{-1}b}{b'W^{-1}b}\right)^2. \tag{4}$$

This rather formidable relationship can be shown graphically as in Figure 2. The shape of the curve shown in the figure holds quite generally and does not depend on any special assumptions.

We wish to know what values of $x_1, x_2, \cdots, x_n$ should be chosen if the outcome associated with P (the point at which the tangent from D touches the hyperbola) is the most safe from our point of view. We should also like to know how secure this most safe position actually is.

It is convenient to give the answer to the second query first. It can be shown (see the Appendix) that the reciprocal of the square of the

[8] These are not money prices but prices measured in terms of the composite numeraire which we use for reckoning our *real* resources.

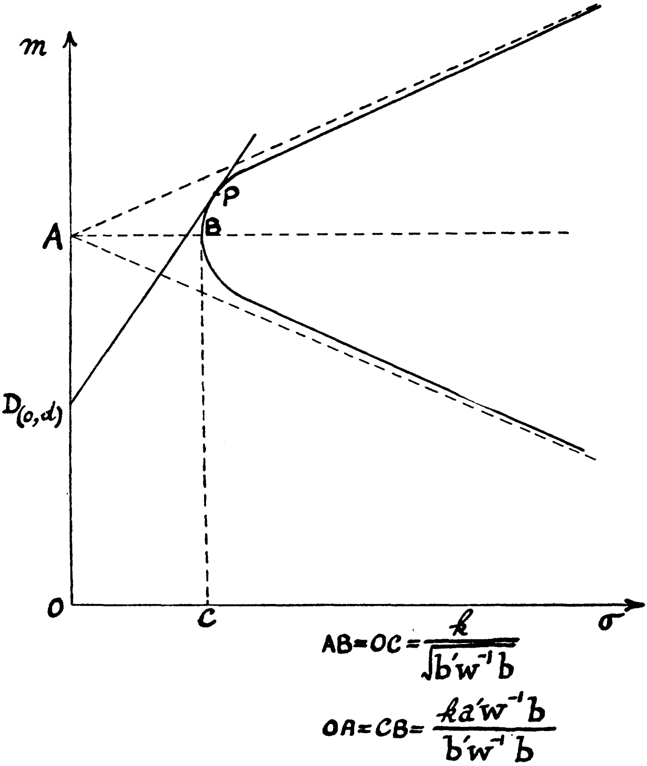

FIGURE 2—The graphical determination of the best σ, m combination in the case of n assets.

gradient of DP, i.e., the upper bound of the probability of disaster, is equal to

$$\frac{1}{\left[a - \left(\frac{d}{k}\right)b\right]' W^{-1} \left[a - \left(\frac{d}{k}\right)b\right]}$$

or

$$(5) \qquad \frac{|W|}{\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\left(p_i - \frac{d}{k}\right)}{\alpha_i} W_{ij} \frac{\left(p_j - \frac{d}{k}\right)}{\alpha_j}},$$

where W_{ij} is the cofactor of r_{ij} in the matrix W and $|W|$ is the determinant of W. This result shows, as might be expected, that if $|W| = 0$ we can find a position of complete and blissful security.

The required values of the x's are given by the equations (derived in the Appendix),

$$x_i = \frac{\lambda}{\alpha_i} \sum_{j=1}^{n} \frac{\left(p_j - \frac{d}{k}\right)}{\alpha_j} \frac{W_{ij}}{|W|} \qquad (i = 1, 2, \cdots, n) \tag{6}$$

where λ is chosen so that $\sum_{i=1}^{n} x_i = k$. Despite the faintly discouraging appearance of the n equations given by (6), they can be given a comparatively simple interpretation.

A cursory inspection of the results given at (5) and (6) above shows that a particular significance attaches to the ratio d/k and this we shall call henceforward the *critical price*. For if we were to commit all our resources to one kind of asset and its price at the end of the period fell to or below (or rose no further than) d/k, then the disaster that we dread would have come about.

To decide which assets to hold and how much of them, we consider each asset in turn. We ask ourselves this question: "If the prices of all the other assets fell to the critical price, what is the best (linear) estimate of price of the asset under examination?" If the estimated price under these conditions exceeds the critical price, then we should hold some of our resources in this form. If the estimated price is less than the critical price, then either we should reject this asset altogether or, better still, we should contract liabilities in this form, if this is possible. Thus in the critical price we have a criterion which enables us to decide what things are eligible as assets and what as liabilities.

We must now find out how much of a particular asset we should hold or how big a liability we should contract. To do this we allocate points to each asset or liability. First, we determine the difference between the estimated price of the asset when all other prices fall to the critical value, and the critical price. Then we calculate the standard error of this estimated price, which will be smaller than the standard error of our original estimate of the price of this asset. The points assigned to a particular asset are then equal to the difference already mentioned over the square of this standard error. The proportion of resources held in this form is obtained by calculating the points allotted to this asset as a proportion of the total points allotted to all assets. In other words we can rewrite (6) as,

$$x_i = \lambda \frac{\left[\begin{pmatrix}\text{best estimate of price of } i\text{th asset} \\ \text{when all other prices are equal to } \frac{d}{k}\end{pmatrix} - \frac{d}{k}\right]}{\begin{pmatrix}\text{Standard error of best estimate of } i\text{th asset's} \\ \text{price when all other prices are equal to } \frac{d}{k}\end{pmatrix}^2} \tag{7}$$

where λ is determined as before and $i = 1, 2, \cdots, n$.

We are thus enabled to discover the best structure of our assets and of our liabilities so that the chance[9] of a particular eventuality, at the end of a given period of time, shall be kept as small as possible. In general it will be desirable to hold our resources in a large number of different forms,[10] because although we may diminish the chance of a large gain somewhat by so doing we also reduce the probability of a really catastrophic outcome. We can also calculate the upper bound of the least chance of disaster and so we can tell how complacent or otherwise we should be in the existing situation.

In the simple case when all the correlations between the prices of assets are thought to be zero, the equations given at (6) and (7) become

$$x_i = \lambda \frac{\left(p_i - \frac{d}{k}\right)}{\alpha_i^2} \qquad (i = 1, 2, \cdots, n). \tag{8}$$

In this instance, we hold those assets the price of which we expect to exceed the critical price d/k. We contract liabilities in the form of those items of which the price is expected to be below d/k. If, further, our expectations about all assets are the same, we are not indifferent as to which assets we hold. For although we cannot affect the expected outcome by varying our holdings, we can affect our uncertainty about the outcome. This will be least when we spread our resources equally among all the assets.

If we hold only one of the exactly similar assets the upper bound of the chance of disaster would be $\alpha^2/(p - d/k)^2$, while if we held all n of them in equal amounts this upper bound would become $\alpha^2/n(p - d/k)^2$. This is a demonstration of the folly of putting all one's eggs in one basket.

Even if we had no means of estimating the necessary standard errors of our estimates of future prices, we should still have a reasonable guide for action. If we decide to assume that all our estimates of prices were equally reliable, (8) would become

$$x_i = \lambda \left(p_i - \frac{d}{k}\right) \qquad (i = 1, 2, \cdots, n). \tag{9}$$

If we thought that the estimates had equal proportionate reliability, which is perhaps a more plausible assumption in the economic world, we could determine our x's from

$$x_i = \lambda \frac{\left(p_i - \frac{d}{k}\right)}{p_i^2} \qquad (i = 1, 2, \cdots, n). \tag{10}$$

[9] Strictly the upper bound of the chance.

[10] This is an important difference between this theory and the theories which depend on the maximisation of expected gain.

When we did not know the absolute values of the standard errors but we had some estimate of their relative magnitude, we could use (8) without modification. We should not know whether we were succeeding in reducing the upper bound of the probability below unity, but it would still be worth seeing that the upper bound was as small as possible.

DISCUSSION OF ASSETS WITH YIELDS AND OF THE SPECIAL POSITION OF MONEY

Our discussion so far has considered only the best procedure to adopt when holding assets to ensure a given speculative gain (i.e., $d > k$) or to prevent an excessive loss ($d < k$). We have implicitly assumed that none of the assets have any appreciable yield in the period of time under consideration.

If for the moment we consider the opposite extreme, we shall have a situation in which we are interested in yields alone and where the prices of the assets are of no immediate concern. In this case the p's become our best estimates of the yields of the various assets during the period which we are considering, and the α's are their standard errors. The quantity $100d/k$ is the percentage yield to or below which we have no desire to sink. We endeavour to minimise the chance that our real income over the period will be less than or equal to d.

When the period is sufficiently long for both yields over the period and prices at the end of the period to be of importance, then the p's become our best estimates of the yields *and* the prices of the various assets and the α's are again the appropriate standard errors. We shall presumably be anxious to maintain our resources intact and to obtain a yield of more than $100\ (d - k)/k$ per cent over the period. If the period were t years this would be equivalent to a return of $100\ (d - k)/kt$ per cent per annum at simple interest. So we see that the results we have obtained can be adapted to deal with problems other than those of pure speculative gain or loss.

Next we must discuss in more detail the implications of measuring our original resources in real terms. Money has certain limitations as a store of value, and it is therefore better to consider what goods our resources at any given moment will buy rather than what sum of money we can obtain by realising our assets. We should be well advised to measure our resources in terms of their purchasing power over consumer goods[11] if we are private individuals, or over capital goods and labour if we are entrepreneurs. Whatever standard we decide to use, money will be just another asset like the rest, about the future value of which we are uncertain. In times of rising prices we shall be encouraged to contract debts in terms of money and to use the proceeds to hold varying

[11] Perhaps durable consumer goods.

quantities of all other assets that appear eligible on the basis of the criterion that we have established. Likewise in times of falling prices we shall be happy to hold a substantial proportion of our resources in the form of money and we shall be anxious to incur liabilities in terms of things, the prices of which are expected to fall in real terms. In either situation there will be some stability in the sense that the amount of money we desire to borrow or hold will be limited by our uncertainty about what the value of money will actually be.

If, however, we reckon our resources in money terms, the whole position is radically altered and becomes most unstable. We might use a money standard of measurement either because we believed it to be an effective store of value or because we were bound by convention to attempt to maintain the money value of our resources intact. In either event the price of money in terms of itself is clearly unity and the standard error of this estimate is zero. Hence in times of rising prices our rules set no limits to the amount of money which should be borrowed in order to increase our holdings of assets with prices that are expected to rise in money terms. In practice the limit of our borrowing powers would be set by our credit worthiness alone and would be unaffected by our desire to minimise the chance that we made no more than a given speculative gain. When prices were expected to fall, there would be an indefinite encouragement to hold as much money as possible and to contract debts in the things of which the prices were falling.

Some sort of stability is restored when we are concerned not with prices alone but with yields as well. Uncertainty about the negative yield, i.e., the rate of interest, associated with borrowing money will set definite limits to the amount of money that we shall wish to borrow in a period of rising prices. This, however, is only true provided that the intervals, at which it is necessary to repay the loans and reborrow, are shorter than the period over which we are planning our distribution of resources. To give an example, if we are planning a scheme which will yield its fruit in ten years time, there are definite limits to the amounts we should like to borrow for periods of less than ten years; limits which are set, *inter alia*, by our uncertainty as to the rates of interest ruling when we have to reborrow. We should, however, be prepared to borrow an indefinitely large amount of money for more than ten years, provided always that the rate of interest charged is not greater than the least return we hoped to get.

We thus see that if we used money as our standard of value, we should be much more enterprising in periods of rising prices but that expectations of a general fall in prices would have a much more severe effect than when we used some kind of composite standard.

When deciding on the best distribution of assets, we have not as yet taken into account the fact that our current asset-holdings have been inherited from the past. Whatever the best distribution determined by our present expectation may be, there are costs, e.g., stamp duties, commissions and the like, involved in shifting our resources from one form of asset to another in order to achieve this distribution. Taking account of these costs causes no serious difficulty and our equations at 6 above can be suitably modified to allow for the pre-existing distribution of our assets. When making actual calculations a good procedure is first to find the desired distribution neglecting all such costs, in order to find the direction in which we should move from our present position. Having done this, the costs can be taken into account and the best distribution of assets calculated. The existence of these costs will merely damp down our desire to change the distribution whenever a change in our expectation takes place.

A DETAILED EXAMINATION OF THE PARTICULAR CASE OF TWO ASSETS

When we examine the artificial but simple case of a world in which there are but two kinds of assets, the results that we have obtained for the case of m assets assume less forbidding forms. Using the same symbols as before, the relation between σ and m given at (4) above now becomes

$$(p_1 - p_2)^2 \left[\sigma^2 - \frac{k^2(1 - r^2)\alpha_1^2 \alpha_2^2}{\alpha_1^2 - 2r\alpha_1\alpha_2 + \alpha_2^2} \right] = (\alpha_1^2 - 2r\alpha_1\alpha_2 + \alpha_2^2) \left[m - \frac{k\{\alpha_1^2 p_2 - r\alpha_1\alpha_2(p_1 + p_2) + \alpha_2^2 p_1\}}{(\alpha_1^2 - 2r\alpha_1\alpha_2 + \alpha_2^2)} \right]^2. \tag{11}$$

Figure 2 can be redrawn as in Figure 3 below. We also have useful relationships between x_1, x_2, and m:

$$x_1 = \frac{kp_2 - m}{p_2 - p_1}, \qquad x_2 = \frac{kp_1 - m}{p_1 - p_2}. \tag{12}$$

The points X, Y in Figure 3 are of some interest. If the point P lies between X and Y on the curve $f(\sigma, m) = 0$, then we shall find it advisable to hold some of both assets. If however P lies above X on the curve and p_1 is greater than p_2, then in order to be as secure as possible we must borrow in terms of the second asset in order to increase our holdings of the first.

It can also be seen that the higher the correlation, either positive or negative, between the prices of the assets, the nearer will the curve move to the m-axis. Thus it might appear that, as the tangent from D to the curve would necessarily become steeper the higher the correlation we assumed, so we should always be happier in a situation involving correlated prices. This, however, is not necessarily so because the existence of a correlation also has the effect of shifting the curve nearer or further from the σ axis. *A priori* we cannot say what will happen since it depends on the relative sizes of p_1 and p_2, and of α_1 and α_2.

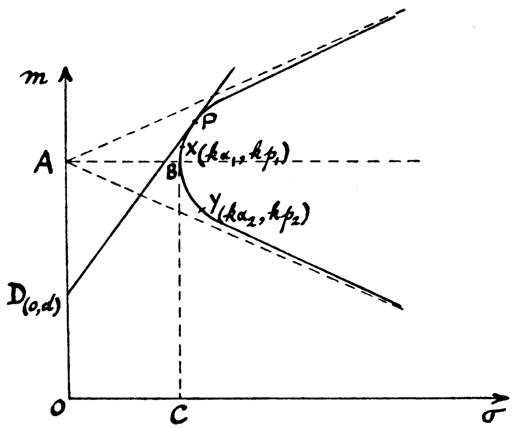

FIG. 3 $OC = \dfrac{\sqrt{1-r^2}\,.\,k\alpha_1\alpha_2}{\sqrt{\alpha_1^2 - 2r\alpha_1\alpha_2 + \alpha_2^2}}$

Figure 3—The graphical determination of the best σ, m combination in the case of two assets only.

Equations (6) can now be written down again in a more simple form:

$$x_1 = \frac{\lambda}{\alpha_1(1-r^2)}\left[\frac{\left(p_1 - \frac{d}{k}\right)}{\alpha_1} - \frac{r\left(p_2 - \frac{d}{k}\right)}{\alpha_2}\right] \tag{13}$$

$$x_2 = \frac{\lambda}{\alpha_2(1-r^2)}\left[\frac{\left(p_2 - \frac{d}{k}\right)}{\alpha_2} - \frac{r\left(p_1 - \frac{d}{k}\right)}{\alpha_1}\right]$$

where λ is such that $x_1 + x_2 = k$.

An increase in the estimated future price of an asset will, other things being equal, always increase our holdings of this asset. Similarly an increase in the reliability of the estimated price will have the same effect, if the correlation is zero or negative. If this is not so, then it may happen that the correlation exceeds a certain limiting positive value and in this case the holdings of the asset will be diminished.

If our total amount of resources, k, is increased, then the proportion of resources held in the form of the asset with the lower estimated future price will increase. In other words an increase in k means that we can more than compensate the fall in m by a fall in σ as well. The wealthier

TABLE—PROPORTIONATE DISTRIBUTION OF ASSETS ACCORDING TO VARIATION IN DESIRES AND RELATIVE FIRMNESS OF EXPECTATIONS[a]

$[P_1 = 1.00; P_2 = 1.10; \alpha_2 = 0.05]$

We desire to avoid:	A speculative loss of 10% or more $d/k = 0.90$			A speculative loss of 5% or more $d/k = 0.95$			No speculative gain or worse $d/k = 1.00$			A speculative gain of 5% or less $d/k = 1.05$		
Ratio of standard errors α_1/α_2	*Percentage holding* Asset 1	Asset 2	(P)	*Percentage holding* Asset 1	Asset 2	(P)	*Percentage holding* Asset 1	Asset 2	(P)	*Percentage holding* Asset 1	Asset 2	(P)
						Zero correlation						
0.6	*58*	*42*	(0.036)	*48*	*52*	(0.085)	*0*	*100*	(0.250)	*0*	*100*	(1.00)
0.8	*44*	*56*	(0.045)	*34*	*66*	(0.095)	*0*	*100*	(0.250)	*0*	*100*	(1.00)
1.0	*33*	*67*	(0.050)	*25*	*75*	(0.100)	*0*	*100*	(0.250)	*0*	*100*	(1.00)
1.2	*26*	*74*	(0.053)	*19*	*81*	(0.103)	*0*	*100*	(0.250)	*−227*	*327*	(0.59)
1.4	*20*	*80*	(0.055)	*15*	*85*	(0.105)	*0*	*100*	(0.250)	*−104*	*204*	(0.66)
						Correlation $=+\frac{1}{2}$						
0.6	*49*	*51*	(0.054)	*11*	*89*	(0.111)	*−500*	*600*	(0.188)	*0*	*100*	(1.00)
0.8	*19*	*81*	(0.061)	*−15*	*115*	(0.110)	*−167*	*267*	(0.188)	*0*	*100*	(1.00)
1.0	*0*	*100*	(0.063)	*−25*	*125*	(0.107)	*−100*	*200*	(0.188)	*0*	*100*	(1.00)
1.2	*−10*	*110*	(0.062)	*−27*	*127*	(0.104)	*−71*	*171*	(0.188)	*−363*	*463*	(0.30)
1.4	*−14*	*114*	(0.061)	*−27*	*127*	(0.102)	*−56*	*156*	(0.188)	*−177*	*277*	(0.34)
						Correlation $=-\frac{1}{2}$						
0.6	*61*	*39*	(0.019)	*58*	*42*	(0.045)	*46*	*54*	(0.188)	*0*	*100*	(1.00)
0.8	*52*	*48*	(0.023)	*49*	*51*	(0.052)	*38*	*62*	(0.188)	*0*	*100*	(1.00)
1.0	*45*	*55*	(0.027)	*42*	*58*	(0.058)	*33*	*67*	(0.188)	*0*	*100*	(1.00)
1.2	*39*	*61*	(0.029)	*36*	*64*	(0.062)	*29*	*71*	(0.188)	*−91*	*191*	(0.87)
1.4	*37*	*63*	(0.032)	*32*	*68*	(0.064)	*26*	*74*	(0.188)	*−31*	*131*	(0.94)

[a] P is the upper bound of the probability that disaster occurs. If we assume distributions are unimodal and symmetrical all the P's can be multiplied by the factor 4/9.

man can rely more for his safety on the diversification of his holdings than the poorer man. Because of the symmetry existing between d and k in the formulae, the downward revision of the outcome, which we want most to avoid, has a precisely similar effect to an increase in total resources, other things being equal.

Some of these results can be illustrated by numerical examples. Suppose that we expected the prices of the assets at the end of a given period to be 1.00 and 1.10, the current prices being both 1.00. Suppose further

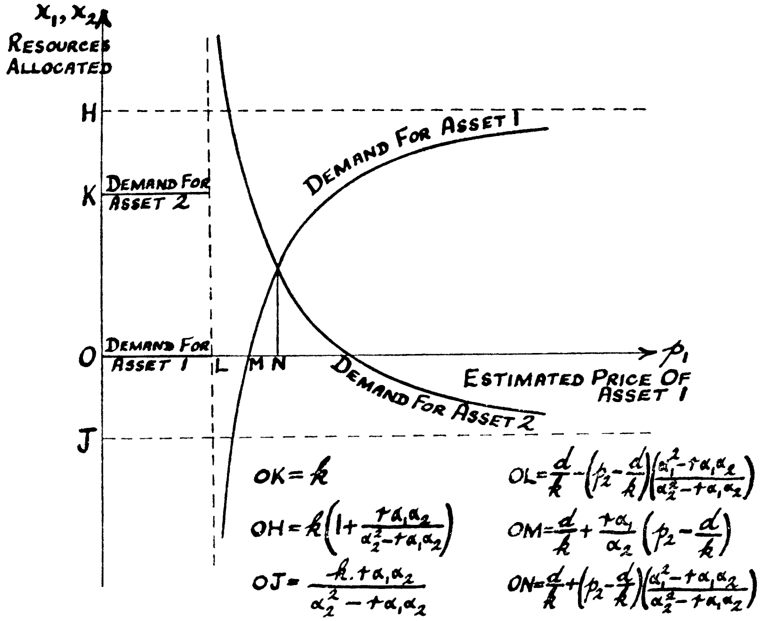

FIGURE 4—Variations in the demand for two assets only in response to changes in the estimated future price of one of them.

that the standard error of the estimated price of the second asset is 0.05. Then we can tabulate the proportionate distribution of assets according to the variation in our desires and expectation as in the Table.

The Table calls for little comment. Perhaps the most interesting fact that it reveals is that small changes in the reliability of an estimated price may cause very heavy borrowing in an attempt to stave off disaster.

The demand for an asset in terms of variations in its own estimated price and the estimated price of the other asset can be shown diagrammatically as in Figure 4. For reasons of convenience the price is plotted along the horizontal axis and not in the usual way. The variations in the upper bound of the probability of disaster can also be shown for variation in p_1 as in Figure 5. The main point of interest here is that although the estimated price of one asset is falling as we proceed from M towards L and no other variables are changing, yet nevertheless we are gradually getting into a safer position. The possibility of doing this depends on the relationship between the standard errors of the two estimated prices.

These diagrammatic and numerical illustrations should serve to clear up some at least of the obscurities that persist in the earlier exposition of the theory.

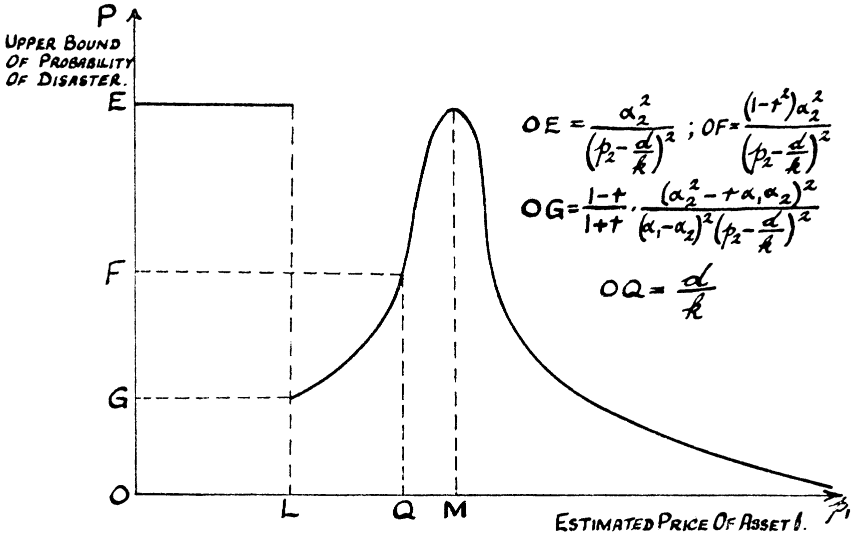

FIGURE 5—Variations in the upper bound of the chance of disaster, in the case of two assets only, in response to changes in the estimated future price of one of them.

CONCLUSION

In conclusion we must consider how far the scheme of rational behaviour that has been sketched out here can be considered to be equivalent to the rules of thumb that guide the ordinary man in everyday life. We cannot expect our laws to be obeyed in even a rudimentary way by the reckless gamblers who never consider the possibility of disaster. Our principles are, however, applicable to all those people in whose make-up caution plays some role however slight. They can be used by those whose main interest is speculative gain and by those who are content with modest yields over a long period. Different kinds of persons will use different ways of estimating future prices and the reliability of such estimates. Sometimes people may contemplate without undue dread the possibility of their wealth falling by ten per cent in real terms because in the existing circumstances such an outcome appears to be a minor evil. At other times the same people may try to make sure that they make a speculative gain of not less than five per cent. In all these various situations our rules remain relevant and useful.

Can we, however, expect people to arrive intuitively at rules of behaviour based on the theory of probability? Such a question cannot be answered directly by an economist. It is pertinent, however, to remark that the assumption that entrepreneurs maximise their profits and consumers their utility is not usually taken to imply that the rational man must be well-grounded in the differential calculus. It is not, therefore, an extravagance to suggest that our theory may well be a rationalisation of an already well-worn procedure. We have shown why it is a good thing for the property owner to disperse widely both his assets and liabilities, a principle that is always accepted in practice but rarely explained satisfactorily on relatively simple theoretical assumptions. We have also demonstrated that small changes in expectations about prices, whether they concern their future level or the reliability of existing predictions, may under certain circumstances produce very big changes in an individual's demand for some assets and liabilities. We have indicated too, how changes in expectations may affect the confidence of an individual or of a group of individuals with similar hopes and fears.

However plausible the explanations of individual behaviour given here may seem, in the last resort their validity must turn on their ability to stand up to statistical tests. Such tests cannot be used until the principles have been applied to large groups of people. To do this, the theory of the individual must be subjected to an extensive simplification. Although it is not at present obvious what kind of modification ought to be made, it seems that the basic theory developed here provides a secure base from which to launch attacks on the problems of aggregate market behaviour.

Sidney Sussex College, Cambridge

APPENDIX

I. The envelope curve of $f(m, \sigma) = 0$ with m, σ as in (1)–(3) of the text is obtained by minimizing σ^2 for given m and k. Using W, a, b as in the text and y for the column vector $(x_1\alpha_1, \cdots, x_n\alpha_n)$ we may write (1)–(3) as

$$m = y'a, \qquad \sigma^2 = y'Wy, \qquad k = y'b. \tag{14}$$

We minimise then

$$y'Wy + 2\mu_1 (m - y'a) + 2\mu_2(k - y'b),$$

where μ_1 and μ_2 are undetermined multipliers. Substituting in (14) the n solutions,

$y = W^{-1}(\mu_1 a + \mu_2 b)$, of the simultaneous equations $Wy = \mu_1 a + \mu_2 b$, and eliminating μ_1 and μ_2 from the resulting three equations, we find the equation of the envelope to be

$$\begin{vmatrix} \sigma^2 & m & k \\ m & a'W^{-1}a & a'W^{-1}b \\ k & a'W^{-1}b & b'W^{-1}b \end{vmatrix} = 0,$$

which is (4) in the text.

II. We next find the least upper bound of the probability that d or worse will occur.

Consider the line of slope h through the point $\sigma = 0$, $m = d$, i.e., D. This line, $h\sigma = m - d$, is tangent to the envelope if

$$\begin{vmatrix} \sigma^2 & h\sigma + d & k \\ h\sigma + d & a'W^{-1}a & a'W^{-1}b \\ k & a'W^{-1}b & b'W^{-1}b \end{vmatrix}$$

is a perfect square. It is necessary then that

$$h^2 = \left[a - \frac{d}{k}\, b\right]' W^{-1} \left[a - \left(\frac{d}{k}\right) b\right].$$

Since on the line we have $\sigma^2/(m - d)^2 = 1/h^2$, the upper bound of the probability that d or worse will occur is $1/h^2$. Thus as the upper bound of the probability of disaster we obtain (5) of the text.

III. The point at which the tangent line touches the envelope is given by

$$\sigma = k \frac{\sqrt{\left[a - \left(\frac{d}{k}\right) b\right]' W^{-1} \left[a - \left(\frac{d}{k}\right) b\right]}}{\left[a - \left(\frac{d}{k}\right) b\right]' W^{-1} b}$$

$$m = k \frac{\left[a - \left(\frac{d}{k}\right) b\right]' W^{-1} a}{\left[a - \left(\frac{d}{k}\right) b\right]' W^{-1} b}.$$

Using these values of σ and m, we may evaluate μ_1 and μ_2 from the simultaneous equations obtained from the expressions for m and k of (14) by the substitution of the solutions of the equations $Wy = \mu_1 a + \mu_2 b$.
We have then

$$\mu_1 = \frac{k}{\left[a - \left(\frac{d}{k}\right) b\right]' W^{-1} b}, \qquad \mu_2 = \frac{-d}{\left[a - \left(\frac{d}{k}\right) b\right]' W^{-1} b}.$$

Substituting in $y = W^{-1}(\mu_1 a + \mu_2 b)$,

$$y = \frac{kW^{-1}\left[a - \left(\frac{d}{k}\right)b\right]}{\left[a - \left(\frac{d}{k}\right)b\right]' W^{-1}b}. \tag{15}$$

which is equivalent to (6) of the text with

$$\lambda = \frac{k}{\left[a - \left(\frac{d}{k}\right)b\right]' W^{-1}b} = \frac{k}{\sum_{j=1}^{n}\sum_{l=1}^{n} \frac{\left(p_j - \frac{d}{k}\right)}{\alpha_j\, \alpha_l} \frac{W_{jl}}{|W|}}.$$

Equation (15) gives us then the distribution of resources which minimises the upper bound of the probability of d or worse occurring.

REFERENCES

[1] Cramér, H., "On the Mathematical Theory of Risk," *Försäkringsaktiebolaget Skandias Festkrift*, Stockholm: Centraltryckeriet, 1930, pp. 7–84.

[2] Cramér, H., *Mathematical Methods of Statistics*, Princeton: Princeton University Press, 1946, pp. 182–183.

[3] Friedman, M., and L. J. Savage, "The Utility Analysis of Choices Involving Risk," *Journal of Political Economy*, Vol. 56, August, 1948, pp. 279–304.

[4] Marschak, J., "Rational Behavior, Uncertain Prospects, and Measurable Utility," Econometrica, Vol. 18, April, 1950, pp. 111–141.

[5] Shackle, G. L. S., *Expectation in Economics*, Cambridge, England: Cambridge University Press, 1949, 146 pp.

[6] von Neumann, J., and O. Morgenstern, *Theory of Games and Economic Behavior*, Princeton: Princeton University Press, 1947, 641 pp.

[7] Wald, A., *Statistical Decision Functions*, New York: John Wiley and Sons, 1950, 179 pp.