Putting Markowitz theory to work

Some simple and tractable methods to improve the applicability of Markowitz theory.

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Although Markowitz-Sharpe [8] [10] portfolio theory is an extremely appealing normative criterion for forming portfolios of securities, traditional implementation of the theory can fail in three important respects. First, it does not reliably predict the actual mean and standard deviation of return to an investor following its rules. Second, it does not reliably predict the true optimal portfolio's mean and standard deviation. Third, naive portfolio formation rules, such as the equal weight rule, can outperform the Markowitz rule. Both academics and practitioners have made these observations individually, if not collectively.

The failure of the theory is not necessarily due to such common suggestions as changing market conditions or incorrect assumptions about the distributional properties of security returns, although these factors are certainly important. This article identifies the major reasons why traditional application of Markowitz theory is unworkable and fails to predict either the actual or optimal portfolio risk-return parameters. It also proposes some very simple and tractable remedies to overcome the problems of implementing the theory.

TRADITIONAL IMPLEMENTATION

A rational investor's objective is to choose a portfolio of n securities that has the largest expected return for some selected level of risk. Investors who consider investment in a larger number of securities have risk-return opportunities that dominate the risk-return combinations arising from fewer securities. The situation is shown in Figure 1, where the preferable efficient combinations are represented by lines to the northwest.

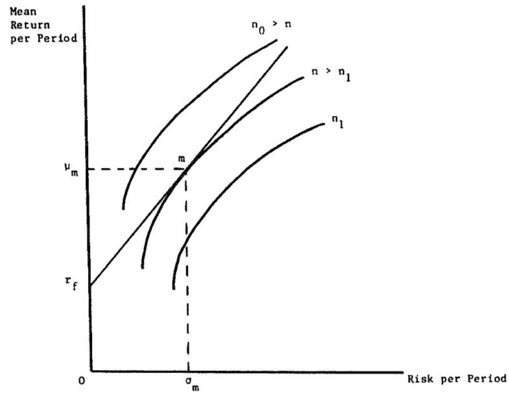

FIGURE 1: Markowitz Efficient Sets for Varying Numbers (n) of Securities.

The Markowitz theory identifies the compositions of the portfolios in an efficient set, given the set of possible securities identified by the investor. All stocks need not be purchased, but these n stocks are subjected to security analysis that identifies the expected holding period returns, variabilities, and correlations among returns. Unfortunately, we do not know the true values of these parameters and must estimate them. One important input into this estimation is historical data.

Typically, if monthly returns are observed on the securities, four to seven years of historical data is the suggested sample size. This rule of thumb is particularly prevalent in the computation of a security's beta.[1] The best sample size represents a trade-off be-

1. Footnotes appear at the end of the article.

tween statistical confidence in the estimates, and using irrelevant data. We then use the data to compute the estimates of the expected return vector and the variance-covariance matrix. (These formulas are shown in Appendix I.) The investor may also continue these historical estimates with an analyst's prior assessments of the parameters.

Next the holding period yield-to-maturity on federal government securities r_f is observed. This risk-free rate is the spot rate of interest prevailing at the beginning of the holding period.

The optimal risky asset portfolio from the n stocks is shown as point m in Figure 1. Combinations of this portfolio and the government security dominate all other combinations. The investment weights for this portfolio are obtained straightforwardly from the expected returns and covariances and are given by the formula $\underline{X}_m = \underline{Y}/b$, which is explained in Appendix I.[2]

Of course, this is only an estimate of the true best portfolio, since the true expected returns and covariances are unknown and are estimated. This estimation risk causes investors to put less credence in the estimated risky security portfolio. Consequently, research and common sense indicate that investors will tend to place a larger fraction of their investable funds in the riskless security versus the risky portfolio. Portfolio managers are naturally interested in ways of reducing this estimation risk.

The presence of estimation risk means that there are three sets of risk-return values germane to the portfolio problem. The first is the true risk-return of the unknown optimal portfolio. The second is the predicted or estimated risk-return of the optimal portfolio. The third is the actual risk-return obtained from the implemented portfolio formation rule. We now discuss the latter two risk-return values.

PREDICTION OF OPTIMAL PORTFOLIO RETURN

Consider an investor who has utilized a historical sample of T monthly returns to estimate the investment weights $\underline{X}_m$ for n stocks, which are then purchased in those proportions. The usual estimators of the portfolio's expected return and risk are given by the formulas $\mu_m = a/b + r_f$ and $\sigma_m^2 = a/b^2$, which are also shown in Appendix I. The a and b are referred to as efficient set constants, which are functions of the securities' expected returns and covariances. Unfortunately, the mean return and risk predicted by the Markowitz equations will, in general, be nowhere near the unknown optimal portfolio parameters aimed for by the investor, because the estimators obtained from the Markowitz equations are extremely biased and subject to large variation, particularly when less than ten years of monthly data are used in the estimation process.[3]

For example, consider twenty stocks with the *true* monthly expected returns and covariances shown in Table 1 and assume for simplicity that the one-month riskless interest rate is zero.[4] The optimal risky portfolio weights $\underline{X}_m$, shown in Table 1, result in a portfolio with an expected return of 1.38% per month and a standard deviation of 4.04% per month. In practice, these numbers are unknown and subject to estimation.

Suppose the investor were to use sixty months of historical return data, drawn from that same stock population, to *estimate* the weights, expected return, and standard deviation of the optimal portfolio. The estimate of the expected return and standard deviation

TABLE 1: Twenty Stock Population Expected Returns, Covariances and Optimal Investment Proportions

Variable in % or $\%^2$

	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	r_{20}
Means	.50	.90	1.10	1.74	1.82	1.11	.91	1.18	1.35	1.07	1.16	1.23	.81	1.18	.88	1.20	.72	1.16	.92	1.25
Covariance	53.64																			
Matrix	6.60	29.84																		
	19.84	16.68	82.88																	
	34.14	20.66	48.01	178.07																
	6.32	6.48	18.66	27.54	118.09															
	5.76	11.83	21.04	19.28	26.29	57.07														
	16.92	8.43	22.16	32.35	23.88	20.24	52.05													
	15.26	9.26	16.21	26.87	12.41	11.69	15.29	48.25												
	9.57	10.80	16.26	18.29	14.16	15.23	12.10	9.69	29.80											
	10.12	11.22	18.94	22.39	23.13	16.27	17.74	9.37	11.21	35.12										
	10.33	9.59	21.59	21.75	31.03	13.72	17.95	8.59	13.05	22.58	47.64									
	18.89	8.76	27.01	41.73	13.08	19.29	21.39	14.42	13.83	12.96	16.56	65.62								
	9.45	13.50	8.80	17.34	5.40	7.78	9.58	9.86	7.27	7.92	5.97	7.92	23.51							
	14.58	14.06	23.31	42.93	20.36	21.53	26.36	11.34	16.74	17.62	19.75	23.10	12.03	51.20						
	14.64	16.47	17.40	26.27	9.88	11.33	16.24	13.33	11.42	10.70	9.26	11.55	14.27	16.42	28.72					
	14.34	8.76	22.27	30.30	14.33	13.21	15.17	16.96	8.21	12.57	13.46	25.84	8.54	14.72	12.20	56.03				
	27.85	14.93	36.71	65.96	17.07	13.45	25.64	32.08	15.70	16.21	20.46	35.76	15.22	26.17	19.87	32.30	109.46			
	25.08	16.74	41.43	47.56	20.20	12.32	24.80	21.66	20.61	21.51	18.76	26.41	14.23	25.60	24.34	24.46	50.78	131.75		
	11.77	22.82	21.44	20.67	13.46	16.84	15.50	14.34	14.80	14.23	13.32	16.96	15.80	20.42	22.36	13.14	18.58	26.97	44.66	
	16.92	10.26	27.74	43.93	18.45	18.08	25.27	15.75	10.72	14.74	17.69	[illegible]	9.72	20.93	13.76	14.48	32.30	29.18	16.09	58.69
Optimal Proportions	.070	.116	.017	.016	.090	.032	.071	.178	.379	.077	.046	.086	.193	.021	.059	.131	.157	.016	.155	[illegible]

would on average be approximately 10% per month and 35% per month. In other words, the optimal risk and return would be overstated by roughly a factor of eight. If the investor were to use 100 months of sample data, the estimates of the optimal risk and return would be on average about 5% per month and 15% per month. In this case, the estimates overstate by a factor of about four. These numbers were obtained by averaging the estimation results of 500 drawings of sixty returns from the stock population of Table 1. The experiment and its results are documented in [4] and [6].[5]

The number of historical observations of monthly returns required to give reasonably unbiased estimates of the optimal risk and return is at least two hundred. This requirement is larger for larger risk-free rates of interest. It should also be noted that any single estimate of the risk and return could be off by more or less than the preivous factors, since those factors are for the average estimate.

The traditional Markowitz procedure for predicting the optimal risk-return is extremely poor with conventional sample sizes of four to seven years. Increasing sample sizes to the requirements suggested here is perhaps untenable because of changing market conditions over such extended periods. Therefore, traditional Markowitz estimation is not very satisfactory for estimating optimal portfolio parameters.

THE EFFECT OF ESTIMATION RISK ON ACTUAL PORTFOLIO RETURNS

The discussion of the prior section dealt with the unreliability of the Markowitz estimators of the optimal portfolio's risk-return. It is also true that the actual risk-return parameters faced by an investor who must estimate the optimal portfolio weights are far from both the predicted optimal and true optimal risk-return.

To demonstrate, consider a portfolio manager who somehow estimates the best allocation among the prospective securities. The input into the decision process is some set of information, available prior to the holding period. As shown in Appendix II, the actual expected return of the portfolio for the holding period is the sum of (i) the true optimal mean return and (ii) an additional amount (positive or negative) due to the bias in estimating the optimal portfolio weights. Similarly, the actual risk (variance) of return in the holding period is the sum of (i) the variability of the optimal portfolio, (ii) the additional variability due to estimated portfolio weights changing from information set to information set, and (iii) the additional variability from the bias (if any) in estimating the portfolio weights. There is only one situation in which the optimal expected return and risk will ever be achieved by a portfolio manager, barring of course the lucky event. The manager must estimate exactly the true optimal weights regardless of the information set.

The pragmatic portfolio manager must practice an investment strategy that keeps the actual portfolio risk-return close to the optimal. He will achieve this goal either by unbiasedly estimating the optimal portfolio weights, or by not changing the portfolio weights too frequently, or by some combination of the two. Placing effort on the first suggestion will keep the mean return close to the optimal mean as well as reducing risk that stems from bias. The second suggestion will reduce portfolio risk that comes from weights variation, such as the reduction achieved by a fixed weight rule like equal or market weights. It seems that the most desirable strategy is one that maximizes the criterion used to evaluate the manager's performance.

One common performance measure is Sharpe's reward-to-variability index, defined as the portfolio's expected return premium (above the risk-free rate) divided by the portfolio's return standard deviation. For the security population of Table 1, the optimal performance index value is $(1.38 - 0)/4.04 = 0.34$.

Consider once again the investor using historical samples of sixty returns on the twenty securities to estimate the optimal portfolio weights by the Markowitz formula. The portfolio is purchased and held for the holding period. The actual value of the performance index (actual expected return premium divided by the actual standard deviation) facing such an investor is not 0.34 but rather approximately 0.08.

This is a rather terrible state of affairs in need of drastic improvement in order to salvage the traditional Markowitz procedure. If the investor uses one hundred months of historical data to form the weight estimates, the actual value of the performance index is larger, but an unsatisfactory 0.16. These numbers were again obtained from a large simulation experiment of the Table 1 stock population. The experiment and its results are documented in [6].

Hence, the reasonably large suggested sample sizes will never (except by chance) bring the portfolio manager anywhere close to the optimal risk-return portfolio. By contrast, if the manager used a passive naive equal weight rule on the same twenty securities, the actual performance index would only be slightly less than the optimal (i.e., 0.27 compared to 0.34). The desirable feature of the equal weight rule is that it is fixed, does not depend upon the sample data, and therefore does not contribute to the risk of the portfolio's return from changing weights. Reducing the portfolio's standard deviation increases the performance measure, *ceteris paribus*. Its undesirable feature is that it is biased but not sufficiently so as to offset its relative advantage over traditional Markowitz estimation.

In conclusion, traditional implementation of the Markowitz diversification strategy is categorically a bad strategy. It is important to remember that this is caused by the unreliability of the estimates obtained from historical data of reasonable size and not from any intrinsic error in Markowitz-Sharpe theory.

REMEDIES FOR REDUCING ESTIMATION RISK

Fortunately, some alternative manipulations of historical data can vastly improve the traditional Markowitz procedure. The estimation technique is known as James-Stein estimation.[6] We need not delve into the statistical theory involved; rather, we examine the application and spirit of the technique, which is remarkably simple for the portfolio problem.

The James-Stein approach to collective return estimation suggests that our best estimate of stocks' returns is the grand mean of the historical returns on all the stocks. This is equivalent to assuming that you cannot confidently decide that one stock has an expected return different from that of any other security.[7] Hence, we average the historical mean returns from each stock to form the grand mean that is allocated to each stock. We leave the estimated covariance matrix unchanged. This adjustment to the expected return estimator results in some new formulas for the Markowitz tangency portfolio. The estimated investment weights now become $\underset{\sim}{X}_0 = \underset{\sim}{Y}_0/c$; the estimated expected return and variance become $\mu_0 = b/c + r_f$ and $\sigma_0^2 = 1/c$.

Recall that the optimal portfolio's risk-return in our example are 1.38 and 4.04. For the investor using sixty or one hundred historical returns from the twenty stocks of Table 1, the average estimates of the optimal return and risk using the new formulas are 1.6 and 4.6. The actual performance measure in the holding period is about 0.32 regardless of whether 60 or 100 historical returns are used in the estimation of the portfolio weights. Table 2 provides a summary of the comparison of the new and traditional approaches to Markowitz estimation.

In summary, the practical implementation of Markowitz-Sharpe portfolio theory is greatly enhanced by assuming that the grand mean of return for all securities represents the best estimator of their individual respective expected returns. Such a simple assumption in the estimation process will greatly reduce the required number of historical return observations for reliable estimation. This conclusion seems unaffected by the riskless rate or the number of securities. The often suggested five years of monthly data seem quite reasonable providing this new procedure is used. In addition, the Sharpe performance measure of the portfolio formed from the new Markowitz weight estimator substantially dominates the portfolio formed from the traditional Markowitz estimator. This new portfolio is also not dominated by the equal weight portfolio.

TABLE 2: COMPARISONS OF TRADITIONAL MARKOWITZ ESTIMATION AND JAMES-STEIN ESTIMATION

	TRADITIONAL		JAMES-STEIN		OPTIMAL
Historical Sample Size	60	100	60	100	∞
Estimate of Optimal Mean	10%	5%	1.6%	1.6%	1.38%
Estimate of Optimal Standard Deviation	35%	15%	4.6%	4.6%	4.04%
Sharpe Performance Index for holding period	.08	.16	.32	.32	.34

APPENDIX I

This appendix provides some detailed explanations of the Markowitz formulas. Let the T historical return premium (above the current risk-free rate r_f) observations on n securities be denoted by r_{jt}, where j = 1, 2, . . . n and t = 1, 2, . . . T. The estimate of the mean return premium on security j is computed as

$$\bar{r}_j = \sum_{t=1}^{t} \frac{r_{jt}}{T},$$

and the covariance s_{ij} between securities i and j returns is computed as:

$$s_{ij} = \sum_{t=1}^{T} \frac{(r_{it} - \bar{r}_i)(r_{jt} - \bar{r}_j)}{(T - 1)}, i = 1, 2, \ldots n, j = 1, 2, \ldots n.$$

The traditional formula for computing the estimated investment weights of the optimal risky portfolio m (shown in Figure 1) is:

$$x_{mj} = \frac{y_{mj}}{b} = \sum_{i=1}^{n} \frac{v_{ij}\bar{r}_i}{b},$$

where v_{ij} is the ij[th] element of the inverse of the covariance matrix:

$$\underset{\sim}{S} = [s_{ij}] \text{ and } b = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{r}_i v_{ij}.$$

The formulas for the estimated mean return and variance of portfolio m are $\mu_m = a/b + r_f$ and $\sigma_m{}^2 = a/b^2$, where:

$$a = \sum_{i=1}^{n}\sum_{j=1}^{n} \bar{r}_i \bar{r}_j v_{ij}.$$

Alternatively, if the portfolio manager assumes that estimated security mean returns are not different, then each security has the estimated mean return given by:

$$\bar{r} = \sum_{j=1}^{n} \frac{\bar{r}_j}{n} + r_f = \sum_{j=1}^{n}\sum_{t=1}^{T} \frac{r_{jt}}{nT} + r_f.$$

The formula for the estimated investment weights becomes:

$$x_{oj} = \frac{y_{oj}}{c} = \sum_{i=1}^{n} \frac{v_{ij}}{c},$$

where:

$$c = \sum_{i=1}^{n} \sum_{j=1}^{n} v_{ij}.$$

The new estimators of the optimal portfolio's mean premium return and variance are given by $\mu = b/c$ and $\sigma_0^2 = 1/c$. This portfolio is often called the global minimum variance portfolio.

APPENDIX II

Let the true Markowitz portfolio weights be given by x_{mj}, $j = 1, 2, \ldots, n$, and let x_{pj}, $j = 1, 2, \ldots, n$ be a set of estimated optimal weights. The elements x_{pj} are random variables since they are functions of the sample data.

Let the observed returns for the next holding period be given by r_j, $j = 1, 2, \ldots, n$. The portfolio return from employing the strategy x_{pj}, $j = 1, 2, \ldots p$ is given by:

$$r_p = \sum_{j=1}^{n} r_j x_{pj}.$$

The expectation of r_p can be written as:

$$\sum_{j=1}^{n} \mu_j x_{mj} + \sum_{j=1}^{n} \mu_j b_j,$$

where μ_j, $j = 1, 2, \ldots, n$ are the means on the n stocks and the b_j, $j = 1, 2, \ldots, n$ are the biases of the estimates x_{pj} as estimators of the x_{mj}, $j = 1, 2, \ldots, p$.

The variance of the portfolio return can be written as:

$$\sigma_p^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} \sigma_{ij} + \sum_{i=1}^{n} \sum_{j=1}^{n} \theta_{ij} \mu_i \mu_j$$

$$+ \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} (b_i + x_{mi})(b_j + x_{mj}),$$

where the elements θ_{ij}, $i, j, = 1, 2, \ldots, n$ are the covariances of the estimated weights x_{pj}, $j = 1, 2, \ldots, n$, and the elements σ_{ij} are the covariances of the returns r_j, $j = 1, 2, \ldots, n$.

The first two terms of the expression for σ^2_p represent the variance contributed by the variation in the x_{pj}, $j = 1, 2, \ldots, n$, while the last term shows the remaining variance is composed of the variance of the true Markowitz portfolio,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} x_{mi} x_{mj},$$

plus an amount due to the bias b_j, $j = 1, 2, \ldots, n$.

For the conventional estimator of the Markowitz portfolio the first two terms and the bias terms are large, resulting in a large σ^2_p. For the equal weight portfolio the first two terms are zero. For the global minimum portfolio all terms are non-zero. In general, however, the last term of the expression for σ^2_p for the global minimum is sufficiently smaller than the last term for the equal weight portfolio, that is, σ^2_p for the global minimum portfolio is less than σ^2_p for the equal weight case. In fact the global minimum portfolio weights are the weights that minimize the last term of σ^2_p.

[1] For some undocumented reasons, the *Financial Analysts Handbook* [2, 1224] suggests eight to ten years of data.

[2] For a detailed explanation of these formulas and their development see Roll [9].

[3] For a discussion and proof see Jobson and Korkie [5]. The subsequent illustrative data are taken from the source.

[4] These mean returns and covariances are the actual values obtained from twenty NYSE stocks, over the 313 months commencing January 1950. The zero risk-free rate is arbitrarily chosen for simplicity.

[5] These estimated values are sample dependent when the number of historical returns is small. One will find erratic predictions when less than ten to fifteen years of monthly data are used. The cause of this behavior is described in [5].

[6] For an intuitive description of this approach see Efron and Morris [1]. For more rigorous explanations see Stein [1], Stein [11], and James and Stein [3].

[7] Studies of the behavior of samples of stock returns have shown that in general the hypothesis of equality of mean returns cannot be rejected. This results from return means that are low in magnitude relative to the return variances, in short date intervals.

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