This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2018.2829463, IEEE Transactions on Fuzzy Systems

A novel hybrid ICA-FA algorithm for multi-period uncertain portfolio optimization model based on multiple criteria

Wei Chen, Dandan Li, and Yong-Jun Liu

***Abstract*—This paper deals with a multi-period portfolio selection problem in an uncertain investment environment, in which the returns of securities are assumed to be uncertain variables and determined by experts' subjective evaluation. Based on uncertain theory, we present a novel multi-period multi-objective mean-variance-skewness model by considering multiple realistic investment constraints, such as transaction cost, bounds on holdings, and cardinality etc. For solution, we first apply a weighted max-min fuzzy goal programming approach to convert the proposed multi-objective programming model into a single-objective one. After that, we design a novel hybrid of imperialist competitive algorithm (ICA) and firefly algorithm (FA), termed ICA-FA to solve it. Finally, we provide a numerical example to demonstrate the effectiveness of the proposed model and corresponding algorithm.**

***Index Terms*—multi-period portfolio optimization, uncertain variables, multiple criteria decision-making, ICA-FA algorithm**

I. Introduction

In 1952, Markowitz [1] proposed a fundamental basis work (i.e., mean-variance model or M-V model) for modern finance theory, which discussed a single-period portfolio selection problem. However, in the real world, investors often need to dynamically adjust investment strategies according to market environment. Thus, much work has be done to study multi-period portfolio selections; see, for example, Mossin [2], Samuelson [3], Hakansson [4], Elton and Gruber [5], Li et al. [6], Li and Ng [7], Leippold et al. [8], Zhu et al. [9], Wei and Ye [10], Yu et al. [11], Sun et al. [12], Bodnar [13], Yao et al. [14].

The above studies have assumed that the security returns are random variables. As is known, the fundamental premise of applying probability theory is that enough historical data should be provided for the estimation of the probability distribution. Nevertheless, since the complexity of the real financial market and economic environment is changing from time to time, we sometimes can not obtain sufficient historical data. In such situations, we have to rely on the experts' subjective evaluations for predicting the returns of securities. With the introduction of fuzzy set theory, a number of scholars have studied portfolio selection problems under assumption that the security returns are fuzzy numbers, eg., Carlsson et al. [15], Zhang and Nie [16], Vercher et al. [17], Zhou et al. [18], Guo et al. [19] etc. However, recent researches (see, e.g., [20], [21]) showed that several paradoxes will appear if we employ fuzzy variable to describe the security returns. In order to handle this subjective uncertainty in a quantitative method, Liu [21] founded uncertainty theory. After that, Liu [22] refined it in 2010. Especially, uncertain programming proposed by Liu [23] is a efficient method for handling optimization problems with uncertain variables. Since then, many researchers have employed it to handle many optimization problems, such as shortest path, machine scheduling, facility location, project selection, etc. In particular, some researchers have employed uncertain programming for portfolio optimization problem. For example, Huang [24], [25] proposed two uncertain mean-risk portfolio models based on uncertain risk curve and risk ratio, respectively. Li and Qin [26] presented a portfolio model with uncertain interval variables, in which semiabsolute deviation was used as risk measure. Zhang et al. [27] proposed an uncertain expected-variance-chance model, and an uncertain chance-expected-variance model, respectively. Qin et al. [28] developed an uncertain mean semiabsolute deviation model with transaction costs. Chen [29] presented two uncertain portfolio models where semivariance and entropy constraint are taken into account. Recently, Chen et al. [30] proposed an uncertain portfolio model in the mean-variance-skewness-kurtosis framework. Further, Chen et al. [31] proposed an uncertain portfolio selection model under consideration of the skewness, transaction costs, cardinality and minimum transaction lots constraints. Qin et al. [32] presented an uncertain random portfolio selection model based on value-at-risk. More researches on single-period uncertain portfolio selections can be found in Qin [33]. Apart from the above single-period uncertain portfolio selections, Huang and Qiao [34] proposed a multi-period uncertain portfolio model, in which risk index was introduced as risk measure. Li et al. [35] proposed an uncertain multi-period portfolio selection model, in which transaction cost and bankruptcy of investor were considered. It should be noted that, in addition to the above two studies, the uncertainty theory has been rarely applied to multi-period portfolio selection problem.

In the most existing researches, return and risk are two basic factors for constructing portfolio selection model. However, in real financial market, only considering return and risk as decision criteria may lead to loss some relative information for making the right decision. Therefore, in addition to the return and variance, more criteria should be considered for portfolio selections. Recently, many researchers

Manuscript received X.X. 2017; revised X.X. 2018.

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have studied multicriteria portfolio selections. Taking skewness into account, different mean-variance-skewness models were proposed by Li et al. [36], Liu et al. [37], Nguyen and Gordon-Brown [38], Vercher and Bermúdez [39], Li et al. [40]. In addition, several researchers have developed portfolio selection models by considering several realistic investment factors such as transaction cost, liquidity, cardinality and the minimum transaction lots, etc; see for example, Mansini and Speranza [41], Chang et al. [42], Soleimani et al. [43], Cura [44], Krink and Paterlini [45], Liu and Zhang [46], Chen [47], [48], Liu et al. [49]. Note that, with more practical constraints included in the portfolio model, it becomes more difficult for solution. Especially, the optimization problem with cardinality constraint is a NP-Complete problem (see [50]). In this case, evolutionary algorithm is a good alternative to solve complex portfolio optimization problems. In the aforementioned studies, genetic algorithm (GA), particle swarm optimization (PSO), differential evolution (DE) algorithm, firefly algorithm (FA), etc were applied for solution.

Imperialist competitive algorithm (ICA) is a new evolutionary algorithm designed by Atashpaz-Gargari and Lucas [51] in 2007. This algorithm mimics the competition between imperialist countries to control more colonies in order to strengthen their empires through a process of imperialistic. The ICA has shown potential superiority to some widely used evolutionary algorithms. Atashpaz-Gargari and Lucas [51] compared ICA with GA and PSO by using 4 benchmark functions, and the results showed that ICA has great performance in both convergence rate and better global optima. Dossary and Nasrabadi [52] concluded that ICA can achieve a better solution than the PSO and GA in a fixed number of simulation runs. Nowadays, ICA and its variants have aroused much interest and have been widely applied to many fields, such as job-shop scheduling [53], transportation [54], and parameters estimation [55], etc. A comprehensive survey of ICA can be found in [56], [57].

Though there are some researches on portfolio selections based on the uncertainty theory, none has constructed a multi-period uncertain portfolio model under simultaneous consideration of some real-world constraints, such as transaction cost, bounds on holdings, and cardinality etc. The main purpose of this paper is two folds. We propose a new uncertain multi-period portfolio optimization model, in which several criteria, viz., return, risk, transaction costs, skewness of portfolio return, bounds on holdings, and cardinality of the portfolio. The resultant portfolio selection problem in presence of the above mentioned constraints becomes a mixed integer nonlinear programming problem, which can not be efficiently solved by exact solution approaches. Therefore, the second contribution of this paper is to develop a hybrid ICA-FA algorithm for solving the proposed model.

The remainder of this paper is organized as follows. Section 2 introduces some basic knowledge related to uncertainty theory. Section 3 formulates a multi-period multi-objective uncertain portfolio model, and then transfers it into a single-objective model. In Section 4, the proposed algorithm is designed. Section 5 provides experimental results. Section 6 concludes this paper.

II. PRELIMINARIES

In this section, we review some basic concepts and properties about uncertainty theory, which will be used throughout the paper.

Definition 1 [21]. For an uncertain variable ξ, if at least one of the two integrals $\int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr$ and $\int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$ is finite, then its expected value is defined by

$$E(\xi) = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr. \quad (1)$$

Since the operations of uncertain variables are mainly in the form of inverse uncertainty distributions, Liu [22] and Yao [58] gave the below formulas to calculate the expected value and variance of an uncertain variable via inverse uncertainty distribution, respectively,

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)d\alpha, \quad (2)$$

and

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 d\alpha. \quad (3)$$

Definition 2 [21]. Assume that ξ is an uncertain variable with finite expected value e. Then, its variance is defined by

$$V[\xi] = E\{(\xi - e)^2\}. \quad (4)$$

Theorem 1 [22]. Let ξ and η be two independent uncertain variables. Then, $\forall a, b \in R$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \quad (5)$$

According to these results mentioned above, we can deduce the following theorems.

Theorem 2. Assume that ξ and η are independent uncertain variables with finite expected values e_1 and e_2. Then, $\forall a, b \in R$, we have

$$V[a\xi + b\eta] = a^2V[\xi] + b^2V[\eta]. \quad (6)$$

Proof. It follows from Definition 2 and Theorem 1 that

$$\begin{aligned} V[a\xi + b\eta] &= E\{a\xi + b\eta - (ae_1 + be_2)\}^2 \\ &= E\{a(\xi - e_1) + b(\eta - e_2)\}^2 \\ &= E\{a(\xi - e_1)^2 + b(\eta - e_2)^2 \\ &\quad + 2ab(\xi - e_1)(\eta - e_2)\} \\ &= E\{a(\xi - e_1)^2\} + E\{b(\eta - e_2)^2\} \\ &\quad + E\{2ab(\xi - e_1)(\eta - e_2)\} \\ &= a^2V[\xi] + b^2V[\eta]. \end{aligned}$$

The theorem is proved. □

Definition 3 [31]. If ξ is an uncertain variable with finite expected value e, then its skewness is defined by

$$S[\xi] = E[(\xi - e)^3]. \quad (7)$$

Theorem 3. If ξ is an uncertain variable with uncertainty distribution Φ and finite expected value e, then its skewness

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is expressed by

$$S[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^3 d\alpha. \tag{8}$$

Proof. Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, we have

$$\begin{aligned}
S[\xi] &= E[(\xi - e)^3] \\
&= \int_0^{+\infty} M\{(\xi - e)^3 \geq x\}dx - \int_{-\infty}^0 M\{(\xi - e)^3 \leq x\}dx \\
&= \int_0^{+\infty} [1 - \Phi(\sqrt[3]{x} + e)]dx - \int_{-\infty}^0 \Phi(\sqrt[3]{x} + e)dx \\
&= \int_{-\infty}^{+\infty} \big\{[1 - \Phi(\sqrt[3]{x} + e)] + \Phi(\sqrt[3]{x} + e)\big\}\, dx \\
&= \int_{-\infty}^{+\infty} (t - e)^3 d\Phi(t) \\
&= \int_0^1 (\Phi^{-1}(\alpha) - e)^3 d\alpha.
\end{aligned}$$

This completes the proof. □

Theorem 4 [31]. Assume that ξ is an uncertain variable with finite expected value. Then, $\forall a, b \in R$, we have

$$S[a\xi + b] = a^3 S[\xi]. \tag{9}$$

III. The Proposed Model

In this section, we first formulate a multi-period multi-objective uncertain portfolio selection model, in which several criteria, viz., return, risk, transaction costs, skewness of portfolio return, bounds on holdings, and cardinality of the portfolio are considered. Then, we transfer our proposed multi-objective model into a single-objective one.

A. Assumption and notations

We suppose that a investor will invest initial wealth W_0 in n securities, and he can reallocate his wealth at the beginning of each investment period. Additionally, due to insufficiency of historical data, the returns of securities are regarded as uncertain variables. For convenience, the notations used hereafter are presented below:

$r_{t,i}$	the uncertain return of security i at period t;
$x_{t,i}$	the investment proportion of security i at period t;
$d_{t,i}$	the transaction cost of security i at period t;
W_t	the available wealth at the end of period t;
$R_{t,N}$	the net return of the portfolio at period t;
$l_{t,i}$	the lower bound of $x_{t,i}$;
$u_{t,i}$	the upper bound of $x_{t,i}$;
m_t	the number of securities that investors wish to hold at period t, $1 \leq m_t \leq n$;
$z_{t,i}$	the binary variable, $z_{t,i} \in \{0, 1\}$. If security i is included at period t, $z_{t,i} = 1$, and $z_{t,i} = 0$ otherwise.

$$i = 1, 2, \ldots, n; \quad t = 1, 2, \ldots, T.$$

B. Multi-period multi-objective uncertain portfolio model

1) Decision objectives: Three investment objectives are introduced as follows.

• **Terminal wealth**

In real-world investment, transaction cost is one of the main considerations of investment decision. Similar to [46], [47], V-shaped function is used for the transaction costs. Therefore, the transaction costs of the portfolio $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,n})$ at period t is described as

$$D_t = \sum_{i=1}^n d_{t,i}|x_{t,i} - x_{t-1,i}|, \quad t = 1, 2, \ldots, T. \tag{10}$$

Then, the net return of the portfolio x_t at period t is

$$R_{t,N} = \sum_{i=1}^n \Big(r_{t,i}x_{t,i} - d_{t,i}|x_{t,i} - x_{t-1,i}|\Big). \tag{11}$$

As discussed earlier, we know that in real finical market there is sometimes lack of historical data for estimating the security returns. In such cases, security returns can be obtained according to experts' subjective evaluations. As in [28], [30], the security returns $r_{t,i}$ $(i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T)$ are assumed as zigzag uncertain variables, denoted by $\mathcal{Z}(a_{t,i}, b_{t,i}, c_{t,i})$, where $a_{t,i}, b_{t,i}$ and $c_{t,i}$ are real numbers with $a_{t,i} < b_{t,i} < c_{t,i}$. The zigzag uncertainty distribution of $r_{t,i}$ is

$$\Phi(r_{t,i}) = \begin{cases} 0, & r_{t,i} \leq a_{t,i}, \\ \frac{r_{t,i} - a_{t,i}}{2(b_{t,i} - a_{t,i})}, & a_{t,i} \leq r_{t,i} \leq b_{t,i}, \\ \frac{r_{t,i} + c_{t,i} - 2b_{t,i}}{2(c_{t,i} - b_{t,i})}, & b_{t,i} \leq r_{t,i} \leq c_{t,i}, \\ 1, & r_i \geq c_{t,i}. \end{cases}$$

And, the inverse uncertainty distributions of $r_{t,i}$ $(i = 1, 2, \ldots, n;\ t = 1, 2, \ldots, T)$ are

$$\Phi^{-1}(\alpha_{t,i}) = \begin{cases} (1 - 2\alpha_{t,i})a_{t,i} + 2\alpha_{t,i}b_{t,i}, & 0 < \alpha_{t,i} < 0.5, \\ (2 - 2\alpha_{t,i})b_{t,i} + (2\alpha_{t,i} - 1)c_{t,i}, & 0.5 \leq \alpha_{t,i} < 1. \end{cases} \tag{12}$$

Then, from Eqs. (2) and (12) , we have

$$E[r_{t,i}] = \frac{a_{t,i} + 2b_{t,i} + c_{t,i}}{4}. \tag{13}$$

Since $x_{t,i}$ are nonnegative real numbers, by the operational rules of uncertain variables, the total uncertain return is still a zigzag uncertain variable with the following form

$$\begin{aligned}
\sum_{i=1}^n r_{t,i}x_{t,i} &= \sum_{i=1}^n (a_{t,i}x_{t,i}, b_{t,i}x_{t,i}, c_{t,i}x_{t,}) \\
&= \mathcal{Z}\Big(\sum_{i=1}^n a_{t,i}x_{t,i}, \sum_{i=1}^n b_{t,i}x_{t,i}, \sum_{i=1}^n c_{t,i}x_{t,i}\Big).
\end{aligned} \tag{14}$$

By Theorem 1 and Eq. (14), the mean value of portfolio at period t is given by

$$\begin{aligned}
E(R_{t,N}) &= E\Big(\sum_{i=1}^n r_{t,i}x_{t,i} - D_t\Big) \\
&= \frac{1}{4}\Big(\sum_{i=1}^n a_{t,i}x_{t,i} + \sum_{i=1}^n 2b_{t,i}x_{t,i} + \sum_{i=1}^n c_{t,i}x_{t,i}\Big) - D_t.
\end{aligned} \tag{15}$$

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Furthermore, the wealth at the end of period $t+1$ is given by

$$\begin{aligned} W_{t+1} &= W_t(1+R_{t,N}) \\ &= W_t\bigg(1+\sum_{i=1}^{n} r_{t,i}x_{t,i} - D_t\bigg). \end{aligned} \tag{16}$$

Solving Eq. (16) recursively, we have

$$W_T = W_0 \prod_{t=1}^{T}\bigg(1+\sum_{i=1}^{n} r_{t,i}x_{t,i} - D_t\bigg). \tag{17}$$

Thus, the expected value of the terminal wealth at the end of period T can be expressed as

$$\begin{aligned} E(W_T) &= W_0 \prod_{t=1}^{T}\bigg[1+\sum_{i=1}^{n}\bigg(x_{t,i}E(r_{t,i}) \\ &\qquad - d_{t,i}|x_{t,i}-x_{(t-1),i}|\bigg)\bigg] \\ &= W_0 \prod_{t=1}^{T}\bigg[1+\frac{1}{4}\bigg(\sum_{i=1}^{n} a_{t,i}x_{t,i} + 2\sum_{i=1}^{n} b_{t,i}x_{t,i} \\ &\quad + \sum_{i=1}^{n} c_{t,i}x_{t,i}\bigg) - d_{t,i}|x_{t,i}-x_{(t-1),i}|\bigg)\bigg]. \end{aligned} \tag{18}$$

- **Cumulative investment risk**

This paper use variance as risk measurement. To obtain the cumulative risk over T, we first introduce the following theorem.

Theorem 5. Let ξ be a zigzag uncertain variable denoted by $\mathcal{Z}(a,b,c)$, with the finite expected value e. Then, we have

$$\begin{aligned} V[\xi] &= \frac{1}{2}(a-e)^2 + \frac{1}{2}(a-e)(b-a) + \frac{1}{6}(b-a)^2 \\ &\quad + \frac{1}{2}(b-e)^2 + \frac{1}{2}(c-b)(b-e) + \frac{1}{6}(c-b)^2. \end{aligned} \tag{19}$$

Proof. According to Eqs. (4) and (13), we have

$$\begin{aligned} V[\xi] &= \int_0^1 (\Phi^{-1}(\alpha)-e)^2 d\alpha \\ &= \int_0^{0.5} \big[(1-2\alpha)a + 2\alpha b - e\big]^2 d\alpha \\ &\quad + \int_{0.5}^{1} \big[(2-2\alpha)b + (2\alpha-1)c - e\big]^2 d\alpha \\ &= J_1 + J_2, \end{aligned}$$

$$\begin{aligned} J_1 &= \int_0^{0.5} \big[(2b-2a)^2\alpha^2 + 2(2b-2a)(a-e)\alpha \\ &\qquad + (a-e)^2\big]\, d\alpha \\ &= \frac{8}{3}(2b-2a)^2 + \frac{1}{4}(2b-2c)(a-e) + \frac{1}{2}(a-e)^2, \end{aligned}$$

$$\begin{aligned} J_2 &= \int_{0.5}^{1} \big\{(2c-2b)^2\alpha^2 + 2(2c-2b)(2b-c-e)\alpha \\ &\qquad + (2b-c-e)^2\big\}\, d\alpha \\ &= \frac{7}{6}(c-b)^2 + \frac{3}{2}(c-b)(2b-c-e) + \frac{1}{2}(2b-c-e). \end{aligned}$$

At last, we can obtain that

$$\begin{aligned} V[\xi] &= J_1 + J_2 \\ &= \frac{1}{2}(a-e)^2 + \frac{1}{2}(a-e)(b-a) + \frac{1}{6}(b-a)^2 \\ &\quad + \frac{1}{2}(b-e)^2 + \frac{1}{2}(c-b)(b-e) + \frac{1}{6}(c-b)^2. \end{aligned}$$

The theorem is proved. □

Furthermore, from Theorems 2 and 5, we obtain the cumulative risk over T period as follows

$$\begin{aligned} V(x) &= \sum_{t=1}^{T} V\bigg[\sum_{i=1}^{n}\big(r_{t,i}x_{t,i} - d_{t,i}|x_{t,i}-x_{t-1,i}|\big)\bigg] \\ &= \sum_{t=1}^{T}\bigg\{\frac{1}{2}\bigg(\sum_{i=1}^{n} a_{t,i}x_{t,i} - E[R_{t,N}]\bigg)^2 \\ &\quad + \frac{1}{2}\bigg(\sum_{i=1}^{n} a_{t,i}x_{t,i} - E[R_{t,N}]\bigg)\bigg[\sum_{i=1}^{n}(b_{t,i}-a_{t,i})x_{t,i}\bigg] \\ &\quad + \frac{1}{6}\bigg[[\sum_{i=1}^{n}(b_{t,i}-a_{t,i})x_{t,i}]^2\bigg] + \frac{1}{2}\bigg(\sum_{i=1}^{n} b_{t,i}x_{t,i} - E[R_{t,N}]\bigg)^2 \\ &\quad + \frac{1}{2}\bigg(\sum_{i=1}^{n} b_{t,i}x_{t,i} - E[R_{t,N}]\bigg)\bigg[\sum_{i=1}^{n}(c_{t,i}-b_{t,i})x_{t,i}\bigg] \\ &\quad + \frac{1}{6}\bigg[[\sum_{i=1}^{n}(c_{t,i}-b_{t,i})x_{t,i}]^2\bigg]\bigg\}. \end{aligned} \tag{20}$$

- **Cumulative investment skewness**

As it is well known, the standard Markowitz's model relies strictly on the assumptions that the returns of assets are normally distributed. However, empirical studies show that asset returns typically have heavy-tailed feature than implied by the normal assumption, and are often not symmetrically distributed [59]. In particular, Scott and Horvath [60] showed that the third moment of return must be considered when the distributions of the returns are not symmetric. Therefore, in this paper, we will consider the skewness of return in portfolio selection. Next, we will provide a crisp form of the uncertain cumulative skewness.

Theorem 6. Suppose that ξ is a zigzag uncertain variable. Denote by $\xi = \mathcal{Z}(a,b,c)$ and with the finite expected value e. Then, the skewness of ξ can be formulated as

$$S[\xi] = \frac{(b-e)^4}{8(b-a)} - \frac{(a-e)^4}{8(b-a)} + \frac{(c-e)^4}{8(c-b)} - \frac{(b-e)^4}{8(c-b)}. \tag{21}$$

Proof. By Definition 3 and Theorem 3, we can obtain

$$\begin{aligned} S[\xi] &= E[(\xi - E(\xi))^3] \\ &= \int_0^1 (\Phi^{-1}(\alpha) - E(\xi))^3 d\alpha \\ &= \int_0^{0.5} \big[(1-2\alpha)a + 2\alpha b - e\big]^3 d\alpha \\ &\quad + \int_{0.5}^{1} \big[(2-2\alpha)b + (2\alpha-1)c - e\big]^3 d\alpha \\ &= J_3 + J_4. \end{aligned}$$

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For the first term, we have

$$\begin{aligned}
J_3 &= \int_0^{0.5} \left[(2b-2a)\alpha + (a-e)\right]^3 d\alpha \\
&= \int_0^{0.5} (2b-2a)^3(\alpha + \frac{a-e}{2b-2a})^3 d\alpha \\
&= (2b-2a)^3 \int_0^{0.5} (\alpha + \frac{a-e}{2b-2a})^3 d\alpha \\
&= (2b-2a)^3 \int_0^{0.5} (\alpha + \frac{a-e}{2b-2a})^3 d(\alpha + \frac{a-e}{2b-2a}) \\
&= (2b-2a)^3 \int_k^s t^3 dt \\
&= (2b-2a)^3 \frac{t^4}{4}|_k^s,
\end{aligned}$$

where $k = 0 + \frac{a-e}{2b-2a}$, $s = 0.5 + \frac{a-e}{2b-2a}$.
Due to

$$(a+b)^n = C_n^i a^i b^{n-i},$$

we have

$$\begin{aligned}
J_3 &= \frac{(2b-2a)^3}{4}\Bigg[\alpha^4 + 4\alpha^3 \frac{a-e}{2b-2a} + \\
&\quad 6\alpha^2(\frac{a-e}{2b-2a})^2 + 4\alpha(\frac{a-e}{2b-2a})^3 + (\frac{a-e}{2b-2a})^4\Bigg]\Bigg|_0^{0.5} \\
&= 2(b-a)^3\Bigg[\frac{1}{16} + \frac{1}{2}\frac{a-e}{2b-2a} + \frac{3}{2}(\frac{a-e}{2b-2a})^2 \\
&\quad + 2(\frac{a-e}{2b-2a})^3\Bigg] \\
&= \frac{(b-a)^3}{8} + 2(b-a)^2(a-e) \\
&\quad + \frac{3(b-a)(a-e)^2}{4} + \frac{(a-e)^3}{2} \\
&= \frac{(b-e)^4}{8(b-a)} - \frac{(a-e)^4}{8(b-a)}.
\end{aligned}$$

Now for the second term, we have

$$\begin{aligned}
J_4 &= \int_{0.5}^1 \Bigg[(2c-2b)\alpha + (2b-c-e)\Bigg]^3 d\alpha \\
&= \int_{0.5}^1 (2c-2b)^3(\alpha + \frac{2b-c-e}{2c-2b})^3 d\alpha \\
&= (2c-2b)^3 \frac{(\alpha + \frac{2b-c-e}{2c-2b})^4}{4}\Bigg|_{0.5}^1 \\
&= 2(c-b)^3\Bigg[\alpha^4 + 4\alpha^3\frac{2b-c-e}{2c-2b} \\
&\quad + 6\alpha^2(\frac{2b-c-e}{2c-2b})^2 + 4\alpha(\frac{2b-c-e}{2c-2b})^3 \\
&\quad + (\frac{2b-c-e}{2c-2b})^4\Bigg]\Bigg|_{0.5}^1 \\
&= \frac{15(c-b)^3}{8} + \frac{7(c-b)^2(2b-c-e)}{2} \\
&\quad + \frac{9(c-b)(2b-c-e)^2}{4} + \frac{(2b-c-e)^3}{2} \\
&= \frac{(c-e)^4}{8(c-b)} - \frac{(b-e)^4}{8(c-b)}.
\end{aligned}$$

Thus, we can obtain that

$$\begin{aligned}
S[\xi] &= J_3 + J_4 \\
&= \frac{(b-e)^4}{8(b-a)} - \frac{(a-e)^4}{8(b-a)} + \frac{(c-e)^4}{8(c-b)} - \frac{(b-e)^4}{8(c-b)}.
\end{aligned}$$

The theorem was proved. □

Furthermore, according to Theorems 4 and 6, we obtain the cumulative skewness over the period T as follows

$$\begin{aligned}
S[x] &= \sum_{t=1}^T S\left[R_{t,N}\right] \\
&= \sum_{t=1}^T S\Bigg[\sum_{i=1}^n \left(r_{t,i}x_{t,i} - d_{t,i}|x_{t,i} - x_{t-1,i}|\right)\Bigg] \\
&= \frac{1}{8}\sum_{t=1}^T \Bigg\{ \frac{(\sum_{i=1}^n b_{t,i}x_{t,i} - E[R_{t,N}])^4}{\sum_{i=1}^n (b_{t,i} - a_{t,i})x_{t,i}} \\
&\quad - \frac{(\sum_{i=1}^n a_{t,i}x_{t,i} - E[R_{t,N}])^4}{\sum_{i=1}^n (b_{t,i} - a_{t,i})x_{t,i}} \\
&\quad + \frac{(\sum_{i=1}^n c_{t,i}x_i - E[R_{t,N}])^4}{\sum_{i=1}^n (c_{t,i} - b_{t,i})x_{t,i}} \\
&\quad - \frac{(\sum_{i=1}^n b_{t,i}x_{t,i} - E[R_{t,N}])^4}{\sum_{i=1}^n (c_{t,i} - b_{t,i})x_{t,i}} \Bigg\}.
\end{aligned} \tag{22}$$

2) Investment constraints:

- **Cardinality constraint**. A given number of securities are included in the portfolio to reduce the transaction costs. The total number of securities in the portfolio at period t is expressed as

$$\sum_{i=1}^n z_{t,i} = m_t, \quad t = 1, 2, \ldots, T, \tag{23}$$

where $z_{t,i} \in \{0, 1\}$, and if $z_{t,i} = 1$, ith security is chosen to be invested, otherwise it is not invested.
- **Bound constraints**. If security i is not held at period t, $z_{t,i} = 0$, then the resulting proportion $x_{t,i}$ is also zero. If security i is held at period t, $z_{t,i} = 1$, then the investment proportion lies between the lower bound $l_{t,i}$ and the upper bound $u_{t,i}$, i.e., $l_{t,i} \leq x_{t,i} \leq u_{t,i}$. Thus, in the presence of cardinality constraint, bound constraints can be represented by

$$l_{t,i}z_{t,i} \leq x_{t,i} \leq u_{t,i}z_{t,i}, \tag{24}$$

- **Budget constraint**. It requires that all the available capital at period t is invested, i.e.,

$$\sum_{i=1}^n x_{t,i} = 1, \quad t = 1, 2, \ldots, T. \tag{25}$$

- **No short selling**. At period t $(t = 1, 2, \ldots, T)$, this constraint is presented by

$$x_{t,i} \geq 0. \tag{26}$$

For notational simplicity, we denoted the real-world constraints $(23) - (26)$ as $x \in D$.

3) Model formulation: Assume that the investment goals are maximizing the investment's terminal wealth, minimiz-

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ing the risk, and maximizing the portfolio skewness over the whole investment horizon, we formulate the multi-period tri-objective portfolio model as follows:

$$
\begin{aligned}
\max \quad & W_T(x) = W_0 \prod_{t=1}^{T} \Bigg[1 + \frac{1}{4} \Bigg(\sum_{i=1}^{n} a_{t,i} x_{t,i} + 2 \sum_{i=1}^{n} b_{t,i} x_{t,i} \\
& \quad + \sum_{i=1}^{n} c_{t,i} x_{t,i} \Bigg) - d_{t,i} |x_{t,i} - x_{(t-1),i}| \Bigg) \Bigg] \\
\min \quad & V_T(x) = \sum_{t=1}^{T} \Bigg\{ \frac{1}{2} \Bigg(\sum_{i=1}^{n} a_{t,i} x_{t,i} - E[R_{t,N}] \Bigg)^2 \\
& \quad + \frac{1}{2} \Bigg(\sum_{i=1}^{n} a_{t,i} x_{t,i} - E[R_{t,N}] \Bigg) \Bigg[\sum_{i=1}^{n} (b_{t,i} - a_{t,i}) x_{t,i} \Bigg] \\
& \quad + \frac{1}{6} \Bigg[\sum_{i=1}^{n} (b_{t,i} - a_{t,i}) x_{t,i} \Bigg]^2 \\
& \quad + \frac{1}{2} \Bigg(\sum_{i=1}^{n} b_{t,i} x_{t,i} - E[R_{t,N}] \Bigg)^2 \\
& \quad + \frac{1}{2} \Bigg(\sum_{i=1}^{n} b_{t,i} x_{t,i} - E[R_{t,N}] \Bigg) \Bigg[\sum_{i=1}^{n} (c_{t,i} - b_{t,i}) x_{t,i} \Bigg] \\
& \quad + \frac{1}{6} \Bigg[[\sum_{i=1}^{n} (c_{t,i} - b_{t,i}) x_{t,i}]^2 \Bigg] \Bigg\} \\
\max \quad & S_T(x) = \frac{1}{8} \sum_{t=1}^{T} \Bigg\{ \frac{(\sum_{i=1}^{n} b_{t,i} x_{t,i} - E[R_{t,N}])^4}{\sum_{i=1}^{n} (b_{t,i} - a_{t,i}) x_{t,i}} \\
& \quad - \frac{(\sum_{i=1}^{n} a_{t,i} x_{t,i} - E[R_{t,N}])^4}{\sum_{i=1}^{n} (b_{t,i} - a_{t,i}) x_{t,i}} \\
& \quad + \frac{(\sum_{i=1}^{n} c_{t,i} x_i - E[R_{t,N}])^4}{\sum_{i=1}^{n} (c_{t,i} - b_{t,i}) x_{t,i}} \\
& \quad - \frac{(\sum_{i=1}^{n} b_{t,i} x_{t,i} - E[R_{t,N}])^4}{\sum_{i=1}^{n} (c_{t,i} - b_{t,i}) x_{t,i}} \Bigg\} \\
\text{s.t.} \quad & x \in D.
\end{aligned} \tag{27}
$$

C. Turning the tri-objective model into a single-objective one

The proposed model (27) is a multi-objective optimization problem. In recent years, several researchers have applied multiobjective evolutionary algorithms (MOEAs) for multi-objective optimization problems, such as Zitzler and Thiele [61], Deb [62], Coello [63], Zhou et al. [64]. In addition, Cruz-Reyes et al. [65] pointed out that, in spite of obtaining the Pareto frontier, the decision-maker still needs to make a choice which compromise solution are the best. However, as the number of the decision objectives increase, it is very difficult to make the right judgment among conflicting objectives (see [66]). Thus, to overcome the above obstacle, many approaches have been proposed by incorporating decision-maker's preferences. In this paper, the proposed model (27) is converted into a single-objective model based on weighted max-min fuzzy goal programming approach [67], in which investor's aspiration levels for each objective are considered.

According to the method in [67], the membership functions for wealth, risk and skewness are given by

$$
\mu(W_T(x)) = \begin{cases} 1, & \text{if } W_T(x) \geq W_{max}, \\ \frac{W_T(x) - W_{min}}{W_{max} - W_{min}}, & \text{if } W_{min} \leq W_T(x) \leq W_{max}, \\ 0, & \text{if } W_T(x) \leq W_{min}, \end{cases}
$$

$$
\mu(V_T(x)) = \begin{cases} 1, & \text{if } V_T(x) \leq V_{max}, \\ \frac{V_{min} - V(x)}{V_{min} - V_{max}}, & \text{if } V_{max} \leq V(x) \leq V_{min}, \\ 0, & \text{if } V(x) \geq V_{min}, \end{cases}
$$

$$
\mu(S_T(x)) = \begin{cases} 1, & \text{if } S_T(x) \geq S_{max}, \\ \frac{S_T(x) - S_{min}}{S_{max} - S_{min}}, & \text{if } S_{min} \leq S_T(x) \leq S_{max}, \\ 0, & \text{if } S_T(x) \leq S_{min}, \end{cases}
$$

It should be noted that, W_{max}, W_{min}; V_{max}, V_{min}; S_{max}, S_{min} are the maximum and minimum values for the wealth, risk, and skewness, respectively.

Furthermore, the proposed multi-period multi-objective model (27) can be rewritten as

$$
\begin{aligned}
& \max \lambda \\
& \text{s.t.} \quad \omega_1 \lambda \leq \frac{W_T(x) - W_{min}}{W_{max} - W_{min}}, \\
& \qquad \omega_2 \lambda \leq \frac{V_{min} - V(x)}{V_{min} - V_{max}}, \\
& \qquad \omega_3 \lambda \leq \frac{S_T(x) - S_{min}}{S_{max} - S_{min}}, \\
& \qquad x \in D,
\end{aligned} \tag{28}
$$

where ω_1, ω_2 and ω_3 represent the investor's preferences for the objectives of terminal wealth, the cumulative risk and the cumulative skewness, respectively. Here, $\omega_i \in [0, 1]$ $(i = 1, 2, 3)$ satisfy $\sum_{i=1}^{3} \omega_i = 1$. If $\omega_1 = 1$, it means that the investor only considers the objective of terminal wealth. If $\omega_2 = 1$, it means that the investor only considers the objective of the cumulative risk. If $\omega_3 = 1$, it means that the investor only considers the cumulative skewness. Especially, if $\omega_i = \frac{1}{3}$ $(i = 1, 2, 3)$, it represents that the investor shows indifferent attitude to the three objectives. By varying the values of ω_i $(i = 1, 2, 3)$ in the model (28), the investor's preferences for the aforementioned three objectives can be freely expressed.

IV. HYBRID ICA-FA ALGORITHM

This section presents a novel hybrid ICA-FA algorithm for solving the model (28).

A. Imperialist competitive algorithm

Imperialist competitive algorithm (ICA) is one of the evolutionary algorithms proposed by tashpaz and Lucas in 2007 [51]; the algorithm is inspired by the concept of imperialistic competition process. ICA begins with an initial population called *countries*. The countries is then divided into some sub-populations called *empires*. Every empire is composed of one *imperialist* and some *colonies*. By a assimilation operation, empires begin to evolve where

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all colonies move toward the imperialists. Namely, the positions of imperialist and the colony can be changed under the condition that a colony has more powerful than its imperialist. Next, the empires starts to compete to acquire more colonies belonging to the weakest empires. At the end of the algorithm, the weakest empires lose their colonies and collapse. An overview of the main steps of the ICA is presented next.

In the initialization step, N_{pop} countries are generated randomly. Then, N_{imp} of the most powerful countries be the imperialist, and N_{col} $(N_{pop} - N_{imp})$ be the colonies. So, an empire consists of an imperialist and set of colonies.

In the process of assimilation, the imperialist countries absorb their colonies toward themselves by

$$x^{t+1} = x^t + \beta \cdot d \cdot \gamma \cdot U(-\varphi, \varphi), \tag{29}$$

where β is a control parameter, and it is usually set to be 2. d is the distance between colony and imperialist, γ is the assimilation coefficient, φ is deviation parameter.

Similar to mutation operation in GA, the revolution operator is introduced in ICA to increases exploration and avoid trapping in local optimum. In this process, the number of colonies to be mutated is equal to the product of the "revolution rate" and the number of colonies of that empire. As the revolution rate get larger, the exploration ability will increases. On the contrary, the exploitation ability will enhances gradually with the decrease of revolution rate.

After assimilation and revolution operations, the position of the imperialist and the colony will be exchanged on condition that a colony is better than its imperialist.

The total power of an empire is determined by the cost of its imperialist plus its average colonies' cost as

$$TC_n = C(imp_n) + \xi * mean\{C(col_{emp_n})\}, \tag{30}$$

where $C(imp_n)$ is the cost of the nth imperialist, ξ is a constant between 0 and 1, and col_{emp_n} is the colonies of the nth empire.

In the imperialistic competition process, the powerful empires try to own colonies from the weakest empires. If all of colonies possed by the weakest imperialist are lost, then this imperialist is collapsed and controlled by other imperialists as a colony.

The ICA stops when a set number of iterations is reached, or all empires have collapsed and only one remains, i.e., $N_{imp} = 1$.

B. Hybrid ICA-FA algorithm

1) New search strategies: In the basic ICA, colonies move toward the imperialist by the assimilation operator. After movement, the structures of some colonies maybe similar to those of the empire. Thus, some solutions may be unexplored during assimilation process. That's to say, the basic assimilation operator not behave well for the exploration. Moreover, firefly algorithm (FA) was proposed by Yang in 2008 [68]. At present, FA has been successfully applied to many optimization problems including NP-hard problems [68]. Besides, it has been demonstrated in [69], [70] that, compared to GA, PSO and DE, FA has better performance. Therefore, inspired by the FA search mechanism, we will hybridize ICA with FA, which could be demonstrated as i) a new assimilation operator is proposed, where each colony steals the search space by employing the FA movement strategy, and ii) the imperialists can be absorbed and move toward the other stronger imperialists by employing the FA movement strategy. The hybrid ICA-FA is shown in details as follows.

i) **Movement of colonies**

In FA, a new position of firefly i is updated as follows

$$x_i^{t+1} = x_i^t + \beta_0 e^{-\gamma r_{ij}^2}(x_j^t - x_i^t) + \alpha(rand - \frac{1}{2}), \tag{31}$$

where α is a randomization parameter, and $rand$ is a random number generator uniformly distributed in [0, 1]. It is worth noting that the first term is the current firefly position, the second term is used to update the firefly position based on the brightness of the fireflies, and the third term used to randomize the movement of firefly. In addition, the attractiveness β is defined as

$$\beta(r) = \beta_0 e^{-\gamma r^2}, \tag{32}$$

where β_0 is the attractiveness at $r = 0$, and γ is called light absorption coefficient. Moreover, the distance r_{ij} between any two fireflies x_i and x_j can be computed according to Cartesian distance:

$$r_{ij} = \|x_i - x_j\| = \sqrt{\sum_{k=1}^{N}(x_{ik} - x_{jk})^2}, \tag{33}$$

where x_{ik} and x_{jk} are the kth element of the ith and jth firefly positions, and N is the dimension of the objective function to be optimized.

In the basic ICA, the colonies move toward its' imperialist according to Eq. (29). But, this assimilation operator may cause slow convergence or even makes the search easily trapped in undesired local minima. Thus, inspired by the movement of fireflies, a new strategy is proposed to improve the movement of the colonies toward imperialist, which is defined as follows:

$$\begin{aligned} col_i^{t+1} = {} & col_i^t + \beta_0 e^{-\gamma r_{ij}^2}\left(col_i^t - imp_{col_i}^t\right) \\ & + \alpha(rand - \frac{1}{2}), \end{aligned} \tag{34}$$

where col_i^t refers to the position of ith colony at t iteration, and $imp_{col_i}^t$ refers to the position of relevant imperialist of the ith colony at t iteration.

ii) **Movement of imperialists**

As we all know, in the basic ICA, there do not exist the movements among the imperialists. Thus, to improve the globe search ability of ICA, we assume that, for any two imperialists, the weaker one will be attracted by the stronger one; and every imperialist moves based on the current position and other better imperialists. Furthermore, the movement of imperialists is defined

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as

$$imp_i^{t+1} = imp_i^t + \beta_0 e^{-\gamma r_{ij}^2}\left(imp_i^t - imp_j^t\right) + \alpha(rand - \frac{1}{2}), \quad (35)$$

where imp_i^t refers to the position of ith imperialist at t iteration. It should be noted that, after movement, if the new country is not better than old one, let $imp_i^{t+1} = imp_i^t$.

2) Constraint satisfaction: A hybrid representation is used to define a portfolio where two vectors are defined as

$$\Delta = \{z_{1,1}, z_{1,2}, \ldots, z_{1,n}; \ldots; z_{T,1}, z_{T,2}, \ldots, z_{T,n}\},$$
$$X = \{x_{1,1}, x_{1,2}, \ldots, x_{1,n}; \ldots; x_{T,1}, x_{T,2}, \ldots, x_{T,n}\},$$

where $z_{t,i} \in \{0,1\}$, $x_{t,i} \in [0,1]$, $i = 1,2,\ldots,n$, $t = 1,2,\ldots,T$.

Similar to Chang et al. [42], and Liu et al. [49], to satisfy the cardinality constraint, the following methods are employed. If the number of securities in the portfolio at each period exceeds the maximum allowed number m_t, the investment proportions for those securities with the $(n-m)$ smallest weights are set to zero.

Moreover, to meet the remain constraints, the normalization operation is performed as:

$$x'_{t,i} = \frac{x_{t,i}z_{t,i}}{\sum_{i=1}^{n} x_{t,i}z_{t,i}}. \quad (36)$$

Considering that the normalize investment proportion $x'_{t,i}$ may not satisfy the upper and lower bounds constraints, the following three cases should be discussed.

Case 1: If the lower and upper constraints are both given, then the investment proportions are determined by

$$x'_{t,i} = l_{t,i}z_{t,i} + \frac{x_{t,i}z_{t,i}}{\sum_{i=1}^{n} x_{t,i}z_{t,i}}\left(u_{t,i}z_{t,i} - \sum_{i=1}^{n} l_{t,i}z_{t,i}\right). \quad (37)$$

Case 2: If the lower bound is not satisfied, and there is no restriction on the upper bound, then the investment proportions are determined by

$$x'_{t,i} = l_{t,i}z_{t,i} + \frac{x_{t,i}z_{t,i}}{\sum_{i=1}^{n} x_{t,i}z_{t,i}}\left(1 - \sum_{i=1}^{n} l_{t,i}z_{t,i}\right). \quad (38)$$

Case 3: If the upper bound is not satisfied, and there is no restriction on the lower bound, then the investment proportions are determined by

$$x'_{t,i} = u_{t,i}z_{t,i} - \frac{x_{t,i}z_{t,i}}{\sum_{i=1}^{n} x_{t,i}z_{t,i}}\left(\sum_{i=1}^{n} u_{t,i}z_{t,i} - 1\right). \quad (39)$$

3) Stop criteria: Two stopping criteria are considered. Namely, the program will terminate when the number of imperialist $N_{imp} = 1$ or the maximum number of iterations is reached.

The pseudo-code of the hybrid ICA-FA is presented as Algorithm 1.

Algorithm 1 The hybrid algorithm

```
Define the optimization parameter
Generate N_pop initial countries randomly
Initialize the imperialists, colonies and evaluate empires
While stop condition is not satisfied do
  for i = 1 to N_imp
    for j = 1 to N_col  /* movement of colonies */
      Move colony j towards their imperialist according to Eq. (34)
    end  /* movement of imperialists */
    for k = 1 to i
      if fit(imp_i) < fit(imp_k)
        Move imperialist i towards imperialist k according to Eq. (35)
      end if
    end
    Evaluate new solutions and update countries.
    Imperialist competition
    Perform revolution on new colony
    if fit(col') > fit(imp)
      Exchange the position of colony and imperialist
    end if
    Select the weakest colony from the weakest empire and let it to be the empire
    if there is imperialist with no colonies
      Eliminate the imperialist
    end if
  end
end While
```

V. Numerical Example

In this section, we present a numerical example to illustrate the proposed methods.

A. Parameters setting

Assume that an investor will invest in the 10 securities with initial wealth 10,000 RMB at three consecutive periods. In addition, we assume that the return rates $r_{i,t}$ are all zigzag uncertain variables, which are shown in Table I.

TABLE I
Zigzag uncertain return rates of 10 securities.

Security i	$t=1$			$t=2$			$t=3$		
	$a_{1,i}$	$b_{1,i}$	$c_{1,i}$	$a_{2,i}$	$b_{2,i}$	$c_{2,i}$	$a_{3,i}$	$b_{3,i}$	$c_{3,i}$
1	-0.4	0.2	0.5	-0.5	0.1	0.7	-0.2	0.001	0.5
2	-0.6	-0.2	0.2	-0.8	0.2	0.3	-0.9	-0.1	0.2
3	-0.8	0.1	0.3	-0.7	0.1	0.4	-0.6	0.3	0.5
4	-0.7	-0.1	0.8	-0.8	0.1	0.6	-0.9	-0.2	0.6
5	-0.9	0.4	0.8	-0.6	0.2	0.9	-0.5	-0.3	0.8
6	-0.3	-0.2	0.5	-0.5	-0.1	0.6	-0.4	-0.3	0.7
7	-0.6	0.1	0.9	-0.9	0.1	0.7	-0.8	0.1	0.6
8	-0.5	0.3	0.7	-0.6	0.2	0.7	-0.4	0.1	0.8
9	-0.8	0.002	0.8	-0.6	0.003	0.9	-0.8	0.02	0.7
10	-0.6	0.2	0.6	-0.7	0.2	0.7	-0.5	0.2	0.6

Moreover, we assume that the initial portfolio at the beginning of period is $x_{0,i} = 0.1$ for $i = 1,2,\ldots,10$. The desired number of securities in the portfolio at each period is $m_t = 5,\ \ t = 1,2,3$. The transaction costs of each security is $d_{t,i} = 0.003$, and the lower and upper bounds are $l_{t,i} = 0.005$ and $u_{t,i} = 0.5,\ i = 1,2,\ldots,10,\ t = 1,2,3$.

The parameters of the proposed hybrid ICA-FA are listed in Table II. Additionally, to show the performance of the

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ICA-FA, we compare it with basic ICA, FA, GA and PSO. Corresponding parameters are also shown in Table II.

TABLE II
THE PARAMETER SETTINGS FOR COMPARISON OF THE ALGORITHMS.

Algorithms	Parameters	Value
ICA-FA	Population size	30
	Maximum number of generations	200
	Imperialists number	15
	Colonies number	15
	Revolution rate	0.2
	Absorption coefficient	0.00001
	Attractiveness	3
ICA	Population size	30
	Maximum number of generations	200
	Imperialists number	15
	Colonies number	15
	Assimilation coefficient	0.6
	Revolution rate	0.2
FA	Population size	30
	Maximum number of generations	200
	Absorption coefficient	0.00001
	Attractiveness	3
GA	Population size	30
	Maximum number of generations	200
	Crossover rate	0.5
	Mutation rate	0.08
PSO	Population size	30
	Maximum number of generations	200
	Cognitive constant	2
	Social constant	2
	Inertia constant	1

B. Experimental results

In this example, we consider four typical kinds of investor's preferences as follows. Case 1: $\omega_1 = 1/2$, $\omega_2 = 1/3$ and $\omega_3 = 1/6$; Case 2: $\omega_1 = 1/3$, $\omega_2 = 1/3$ and $\omega_3 = 1/3$; Case 3: $\omega_1 = 1/3$, $\omega_2 = 1/2$ and $\omega_3 = 1/6$; Case 4: $\omega_1 = 1/2$, $\omega_2 = 1/2$ and $\omega_3 = 0$. Using the ICA-FA, we solve the corresponding models respectively. The results are shown in Table III. It is clear that the investor adopts different investment strategies with the change of preferences. More investment strategies can be obtained by adjusting the preference weights. In addition, for case 2, varying the transaction costs from $0 \sim 0.005$ with step of 0.001, the values of terminal wealth are depicted in Fig. 1. We can see that as the transaction costs decrease, the terminal wealth has a downward trend.

Given $m_t = 3, 5, 7$, and 9 $(t = 1, 2, 3)$, respectively, Fig. 2 shows the cardinality has a impact on the the investment strategy. From Fig. 2, we can see that, with the changes of the m_t, the investment strategies are different. For example, when $m_t = 3$, the investor holds securities 1, 2 and 4 at period 1, holds securities 1, 6 and 10 at period 2 and holds securities 1, 4 and 5 at period 3. Moreover, when $m_t = 7$, except that securities 1 and 2 are both invested in three periods, in most cases, other securities selected to be invested are different.

Let $d_{t,i} = 0.003$ $(t = 1, 2, 3;\ i = 1, 2, \ldots, 10)$, $m_1 = m_2 = m_3 = 5$, and $\omega_1 = \omega_2 = \omega_3 = 1/3$. Then the optimal portfolio strategies under different bounds constraints are presented in Tables IV and V. In Table IV, the lower bounds are set as $l_{t,i} = 0.002$, 0.005 and 0.01, respectively, while in Table V, $u_{t,i} = 0.4$, 0.5 and 0.6, respectively. From Tables IV and V, it is obvious that with the changes of the upper and lower bounds, portfolio strategies are different. For example, when $l_{t,i} = 0.002$, investor just allocate his initial wealth on securities 1, 2, 5, 6 and 8, and the optimal investment strategy is $x_{11} = 0.336,\ x_{12} = 0.307,\ x_{15} = 0.032,\ x_{16} = 0.239,\ x_{18} = 0.086$ and others are zero, while when $l_{t,i} = 0.01$, investor allocate his initial wealth on securities 2, 3, 4, 5, and 7, and the optimal investment strategy is $x_{12} = 0.122,\ x_{13} = 0.273,\ x_{14} = 0.276,\ x_{15} = 0.278,\ x_{17} = 0.05$ and others are zero. In addition, from Table V, we can see that for three different upper bounds, although the security 1 is all selected at the period 3, the proportions of the investment are different, namely, $x_{31} = 0.164,\ 0.159,\ 0.22$, respectively.

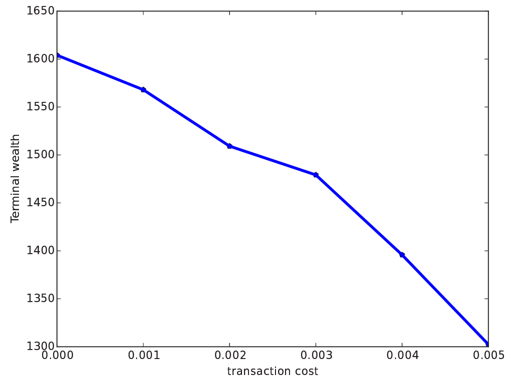

Fig. 1. The terminal wealth under different transaction costs.

C. Performances of the hybrid ICA-FA algorithm

To show the performance of ICA-FA, we compare it with the basic ICA, FA, GA and PSO. It should be noted that, for the proposed model, we assume that $d_{t,i} = 0.003$, $l_{t,i} = 0.005$, $u_{t,i} = 0.5$, $i = 1, 2, \ldots, 10$, $t = 1, 2, 3$, and $\omega_1 = \omega_2 = \omega_3 = 1/3$.

Given $m_t = 3, 5, 7$, and 9 $(t = 1, 2, 3)$, using different algorithms over 20 runs, the values of return, risk, skewness and objective values of the model (28) are presented in Table VI. The best results are bolded. From Table VI, we can see that, 1) in all cases, the objective values and terminal wealths obtained by the hybrid ICA-FA algorithm are the best; 2) in most cases, the values of cumulative skewness by the ICA-FA are better than those by other algorithms.

By varying the number of maximum generations (Gen_{max}) from 100 to 500, the experiments are performed 20 times for each algorithm. Performance indicators such as maximum, minimum, mean, standard deviation (SD) and range values of the optimal objective are tabulated in Table VII. The best results are bolded. From Table VII, we can see that, 1) in all cases, ICA-FA is better than other other algorithms in term of the minimum, SD and Range; 2) in most cases, ICA-FA is better than other other algorithms according to the maximum and mean.

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TABLE III
THE OPTIMAL PORTFOLIOS UNDER DIFFERENT INVESTMENT PREFERENCES

Case i	t	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	$E(W)$	$V(x)$	$S(x)$	ObV
Case1	$t=1$	0.293	0	0.277	0.146	0.07	0	0	0.214	0	0				
	$t=2$	0.029	0.274	0.254	0	0	0	0.207	0	0	0.236	1894.676	2.897	0.621	0.267
	$t=3$	0.196	0	0.191	0	0.346	0	0.07	0	0	0.197				
Case2	$t=1$	0.229	0	0	0.119	0	0.202	0.251	0	0.199	0				
	$t=2$	0.37	0.291	0	0	0.126	0	0.047	0	0	0.166	1325.383	0.425	0.137	0.351
	$t=3$	0.159	0.27	0	0	0.217	0.126	0.228	0	0	0				
Case3	$t=1$	0.421	0	0	0	0.255	0.028	0	0.135	0	0.261				
	$t=2$	0.175	0	0	0.115	0.11	0	0	0.307	0	0.293	1157.699	0.302	0.246	0.277
	$t=3$	0	0.073	0	0.366	0	0.05	0	0.396	0	0.115				
Case4	$t=1$	0.203	0.232	0	0	0	0.251	0.293	0	0.021	0				
	$t=2$	0	0	0.204	0.094	0	0	0.408	0.095	0	0.199	1128.156	0.323	0.117	0.193
	$t=3$	0.151	0	0	0.049	0.194	0.567	0	0	0	0.038				

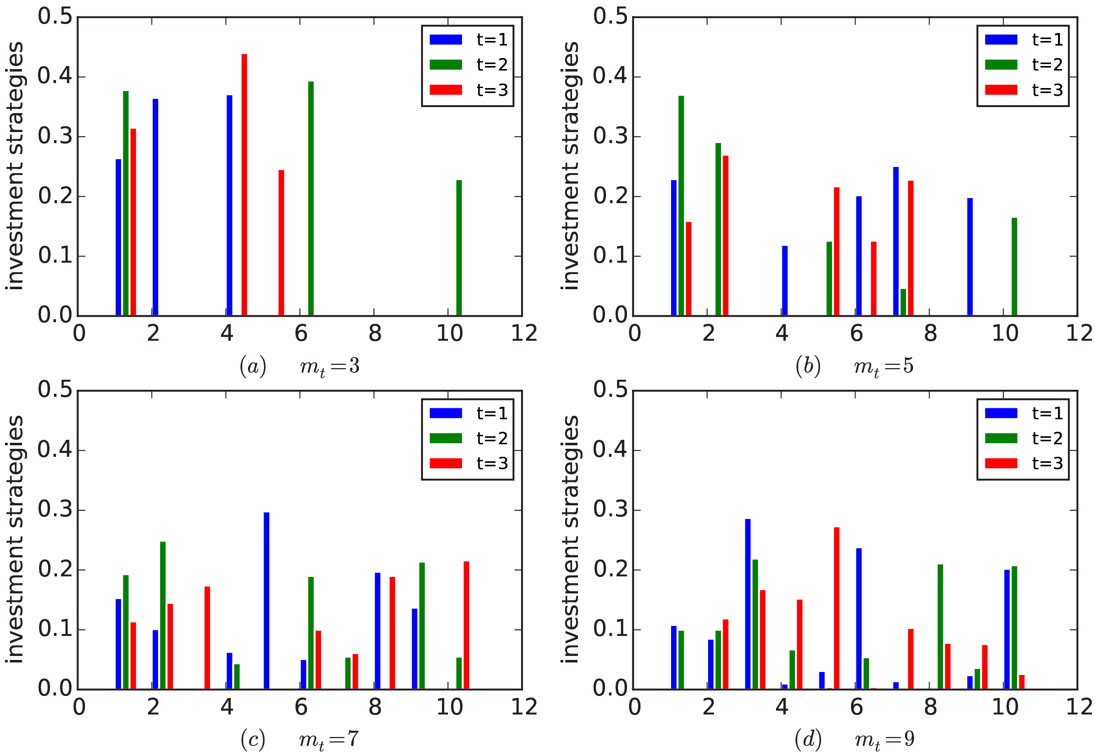

Fig. 2. The optimal investment strategies under different cardinality constrants.

TABLE IV
THE OPTIMAL INVESTMENT STRATEGIES UNDER DIFFERENT LOWER BOUNDS ($u_{t,i} = 0.5$)

Security i	$l_{t,i}=0.002$			$l_{t,i}=0.005$			$l_{t,i}=0.01$		
	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$	$t=1$	$t=2$	$t=3$
1	0.336	0	0.228	0.229	0.37	0.159	0	0	0.303
2	0.307	0.335	0	0	0.291	0.27	0.122	0	0.076
3	0	0.198	0.104	0	0	0	0.273	0.444	0
4	0	0	0	0.119	0	0	0.276	0	0.196
5	0.032	0	0.301	0	0.126	0.217	0.278	0	0
6	0.239	0	0	0.202	0	0.126	0	0.218	0.302
7	0	0.003	0.266	0.251	0.047	0.228	0.05	0.155	0
8	0.086	0.405	0	0	0	0	0	0.161	0.123
9	0	0.059	0.101	0.199	0	0	0	0.022	0
10	0	0	0	0	0.166	0	0	0	0

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TABLE V
THE OPTIMAL INVESTMENT STRATEGIES UNDER DIFFERENT UPPER BOUNDS ($l_{t,i} = 0.005$)

Security i	$u_{t,i} = 0.4$			$u_{t,i} = 0.5$			$u_{t,i} = 0.6$		
	$t = 1$	$t = 2$	$t = 3$	$t = 1$	$t = 2$	$t = 3$	$t = 1$	$t = 2$	$t = 3$
1	0	0	0.164	0.229	0.37	0.159	0.302	0.244	0.22
2	0	0	0.271	0	0.291	0.27	0	0.22	0
3	0	0	0.09	0	0	0	0.21	0	0
4	0.161	0	0	0.119	0	0	0.232	0.13	0
5	0	0.063	0	0	0.126	0.217	0	0	0.199
6	0.27	0.255	0	0.202	0	0.126	0.143	0	0
7	0.308	0.291	0.31	0.251	0.047	0.228	0	0.229	0.228
8	0.144	0	0	0	0	0	0.113	0	0
9	0	0.012	0.165	0.199	0	0	0	0	0.126
10	0.116	0.377	0	0	0.166	0	0	0.177	0.227

TABLE VI
PERFORMANCE COMPARISONS UNDER DIFFERENT CARDINALITY CONSTRANITS

m_t	Algorithm	Terminal wealth	Cumulative Risk	Cumulative Skewness	Objective value
$m_t = 3$	ICA-FA	**1203.391641**	0.403393	**0.308253**	**0.368216**
	ICA	1193.568558	0.437426	0.261283	0.355920
	FA	1135.571752	**0.070638**	0.104361	0.362856
	GA	1189.319995	0.377038	0.211973	0.353627
	PSO	1095.771977	0.085614	-0.467916	0.359642
$m_t = 5$	ICA-FA	**1325.383194**	0.424862	**0.137213**	**0.380579**
	ICA	1277.530007	0.382059	0.293507	0.361587
	FA	1221.973382	**0.021991**	-5.190801	0.378325
	GA	1195.270189	0.384522	0.742000	0.359682
	PSO	1217.198105	3.137401	0.227890	0.365672
$m_t = 7$	ICA-FA	**1507.019201**	**0.364658**	**0.570494**	**0.386190**
	ICA	1390.487571	0.365656	0.247402	0.373067
	FA	1448.713000	0.463503	0.527095	0.382851
	GA	1336.166366	3.250868	0.179774	0.367926
	PSO	1439.650719	2.244482	0.141092	0.365267
$m_t = 9$	ICA-FA	**1691.597006**	0.407112	0.472342	**0.398683**
	ICA	1559.327388	**0.389335**	**0.571671**	0.3859260
	FA	1382.073334	3.687964	0.155166	0.3950867
	GA	1658.506568	1.1126890	0.158072	0.387529
	PSO	1645.965324	3.759292	0.209374	0.389858

Fig. 3 displays the convergence characteristic of the different algorithms. We can see that the hybrid ICA-FA algorithm converges fast and can achieve a higher accuracy than other algorithms.

VI. CONCLUSION

In this paper, we discuss a multi-period portfolio selection problem, in which security returns are given by experts' evaluations. Based on the uncertainty theory, we propose a multi-period multi-objective uncertain portfolio model, in which some criteria such as return, transaction cost, risk, skewness, bounds on holdings, and cardinality are considered. After that, we convert the proposed multi-objective model into a single one. Then, we develop a hybrid ICA-FA algorithm to solve the proposed model. Finally, we present a numerical example to illustrate the effectiveness of the proposed approaches.

For future research, we will consider more real constraints on portfolio decision making such as minimum transaction lots, and sector capitalization constraints. Besides, we will apply the proposed ICA-FA algorithm to solve more real-world optimization problems, such as vehicle routing problem [71], assignment problem [72], etc.

ACKOWLEDGEMENTS

This research was supported by the National Natural Science Foundation of China (No. 71720107002), the Beijing Municipal Education Commission Foundation of China (No. KM201810038001), and the Special Research Funds from the Quantitative Finance Research Center of School of Information, Capital University of Economics and Business.

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TABLE VII
PERFORMANCE COMPARISONS UNDER DIFFERENT ITERATIONS.

Gen_{max}	Algorithm	Minimum	Maximum	Mean	SD	Range
	ICA-FA	**0.328765**	0.331587	0.330176	**0.000753**	**0.002822**
	ICA	0.324524	0.328210	0.326367	0.001495	0.003686
100	FA	0.32832	**0.332567**	**0.3304435**	0.001020	0.004247
	GA	0.292901	0.325920	0.3094105	0.012992	0.033019
	PSO	0.159380	0.338480	0.248930	0.054936	0.179100
	ICA-FA	**0.356629**	**0.365629**	**0.36079**	0.003129	0.009
	ICA	0.340586	0.348308	0.344447	0.002126	0.007722
200	FA	0.348417	0.349144	0.348781	**0.000251**	**0.000727**
	GA	0.33876	0.346561	0.342661	0.002558	0.007801
	PSO	0.328777	0.358190	0.343484	0.009132	0.023113
	ICA-FA	**0.379727**	**0.382398**	**0.381063**	**0.000729**	**0.002671**
	ICA	0.352157	0.355549	0.353853	0.000931	0.003392
300	FA	0.356209	0.360086	0.358148	0.001013	0.003877
	GA	0.357395	0.361918	0.3596565	0.001033	0.004523
	PSO	0.348776	0.361882	0.355329	0.003773	0.013106
	ICA-FA	**0.386747**	**0.386747**	**0.386747**	**0.00**	**0.00**
	ICA	0.373362	0.374067	0.373715	0.000236	0.000705
500	FA	0.365188	0.369217	0.3672025	0.00114	0.004029
	GA	0.361758	0.371760	0.366759	0.002766	0.010002
	PSO	0.361918	0.361918	0.361918	0.00	0.00

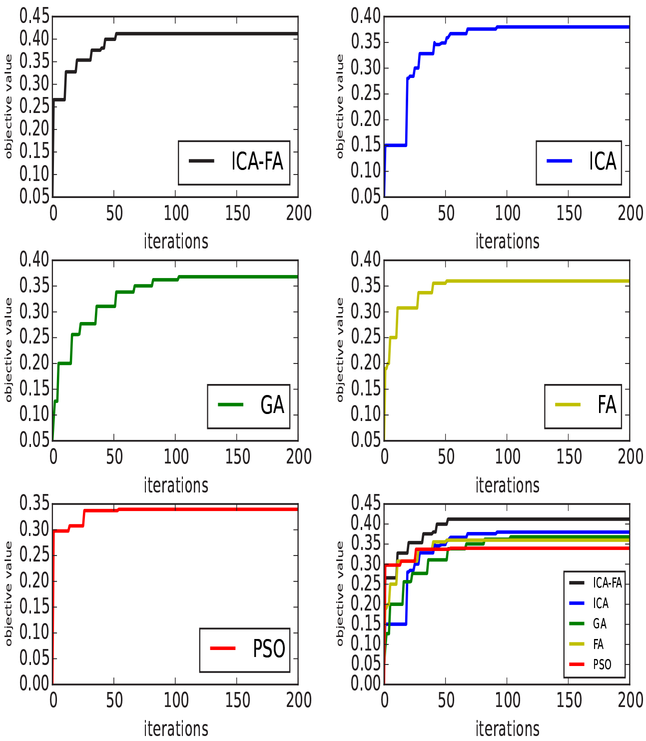

Fig. 3. The convergence curves of different algorithms.

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This article has been accepted for publication in a future issue of this journal, but has not been fully edited. Content may change prior to final publication. Citation information: DOI 10.1109/TFUZZ.2018.2829463, IEEE Transactions on Fuzzy Systems

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