The Effect of Errors in Means, Variances, and Covariances on Optimal Portfolio Choice

Good mean forecasts are critical to the mean-variance framework.

Vijay K. Chopra and William T. Ziemba

VIJAY K. CHOPRA is Senior Research Analyst at the Frank Russell Company in Tacoma (WA 98401).

WILLIAM T. ZIEMBA is the Alumni Professor of Management Science at the University of British Columbia in Vancouver (B.C., Canada V6T 1Y8).

There is considerable literature on the strengths and limitations of mean-variance analysis. The basic theory and extensions of MV analysis are discussed in Markowitz [1987] and Ziemba and Vickson [1975]. Bawa, Brown, and Klein [1979] and Michaud [1989] review some of its problems.

MV optimization is very sensitive to errors in the estimates of the inputs. Chopra [1993] shows that small changes in the input parameters can result in large changes in composition of the optimal portfolio. Best and Grauer [1991] present some empirical and theoretical results on the sensitivity of optimal portfolios to changes in means. This article examines the relative impact of estimation errors in means, variances, and covariances.

Kallberg and Ziemba [1984] examine the question of misspecification in normally distributed portfolio selection problems. They discuss three areas of misspecification: the investor's utility function, the mean vector, and the covariance matrix of the return distribution.

They find that utility functions with similar levels of Arrow-Pratt absolute risk aversion result in similar optimal portfolios, irrespective of the functional form of the utility.[1] Thus, misspecification of the utility function is not a major concern because several different utility functions (quadratic, negative exponential, logarithmic, power) result in similar portfolio allocations for similar levels of risk aversion.

Misspecification of the parameters of the return distribution, however, does make a significant difference. Specifically, errors in means are about ten times as important as errors in variances and covariances.

We show that it is important to distinguish between errors in variances and covariances. The relative impact of errors in means, variances, and covariances also depends on the investor's risk tolerance. For a risk tolerance of 50, errors in means are about eleven times as important as errors in variances, a result similar to that of Kallberg and Ziemba.[2] Errors in variances are about twice as important as errors in covariances.

At higher risk tolerances, errors in means are even more important relative to errors in variances and covariances. At lower risk tolerances, the relative impact of errors in means, variances, and covariances is closer. Even though errors in means are more important than those in variances and covariances, the difference in importance diminishes with a decline in risk tolerance.

These results have an implication for allocation of resources according to the MV framework. The primary emphasis should be on obtaining superior estimates of means, followed by good estimates of variances. Estimates of covariances are the least important in terms of their influence on the optimal portfolio.

THEORY

For a utility function U and gross returns r_i (or return relatives) for assets i = 1, 2, ..., N, an investor's optimal portfolio is the solution to:

$$\text{maximize} \quad Z(x) = E[U(W_0 \sum_{i=1}^{N} (r_i)x_i)]$$

$$\text{such that} \quad x_i \geq 0, \sum_{i=1}^{N} x_i = 1,$$

EXHIBIT 1

List of Ten Randomly Chosen DJIA Securities

1. Aluminum Co. of America
2. American Express Co.
3. Boeing Co.
4. Chevron Co.
5. Coca Cola Co.
6. E.I. Du Pont De Nemours & Co.
7. Minnesota Mining and Manufacturing Co.
8. Procter & Gamble Co.
9. Sears, Roebuck & Co.
10. United Technologies Co.

where $Z(x)$ is the investor's expected utility of wealth, W_0 is the investor's initial wealth, the returns r_i have a distribution F(r), and x_i are the portfolio weights that sum to one.

Assuming a negative exponential utility function U(W) = -exp(-aW) and a joint normal distribution of returns, the expected utility maximization problem is equivalent to the MV-optimization problem:

$$\text{maximize} \quad Z(x) = \sum_{i=1}^{N} E[r_i]x_i -$$

$$\frac{1}{t}\sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j E[\sigma_{ij}]$$

$$\text{such that} \quad x_i \geq 0, \sum_{i=1}^{N} x_i = 1,$$

where $E[r_i]$ is the expected return for asset i, t is the risk tolerance of the investor, and $E[\sigma_{ij}]$ is the covariance between the returns on asset i and asset j.[3]

A natural question arises: How much worse off is the investor if the distribution of returns is estimated with an error? This is an important consideration because the future distribution of returns is unknown. Investors rely on limited data to estimate the parameters of the distribution, and estimation errors are unavoidable. Our investigation assumes that the distribution of returns is stationary over the sample period. If it is time-varying or non-stationary, the estimated parameters will be erroneous.

To measure how close one portfolio is to another, we compare the cash equivalent (CE) values of the two portfolios. The cash equivalent of a risky portfolio is the certain amount of cash that provides the same utility as the risky portfolio, that is, U(CE) = Z(x) or CE = U⁻¹[Z(x)] where, as defined before, $Z(x)$ is the expected utility of the risky portfolio.[4] The cash equivalent is an appropriate measure because it takes into account the investor's risk tolerance and the inherent uncertainty in returns, and it is independent of utility units. For a risk-free portfolio, the cash equivalent is equal to the certain return.

Given a set of asset parameters and the investor's risk tolerance, a MV-optimal portfolio has the largest CE value of any portfolio of those assets. The percentage cash equivalent loss (CEL) from holding an arbitrary portfolio x instead of an optimal portfolio o is

$$CEL = \frac{CE_o - CE_x}{CE_o}$$

where CE_o and CE_x are the cash equivalents of portfolio o and portfolio x, respectively.

DATA AND METHODOLOGY

The data consist of monthly observations from January 1980 through December 1989 on ten randomly selected Dow Jones Industrial Average (DJIA) securities. We use the Center for Research in Security Prices (CRSP) data base, having deleted one security (Allied-Signal, Inc.) because of lack of data prior to 1985. Each of the remaining twenty-nine securities had an equal probability of being chosen. The securities are listed in Exhibit 1.

MV optimization requires as inputs forecasts for: mean returns, variances, and covariances. We computed historical means $(\bar{r}_i)$, variances (σ_{ii}), and covariances (σ_{ij}), and assumed that these are the "true" values of these parameters. Thus, we assumed that $E[r_i] = \bar{r}_i, E[\sigma_{ii}] = \sigma_{ii}$, and $E[\sigma_{ij}] = \sigma_{ij}$. A base optimal portfolio allocation was computed on the basis of these parameters for a risk tolerance of 50 (equivalent to the parameter a = 0.04).

Our results are independent of the source of the inputs. Whether we use historical inputs or those based on a complex forecasting scheme, the results continue to hold as long as the inputs have errors.

Exhibit 2 gives the input parameters and the optimal base portfolio resulting from these inputs. To examine the influence of errors in parameter estimates, we change the true parameters slightly and compute the resulting optimal portfolio. This portfolio will be suboptimal for the investor because it is not based on the true input parameters.

Next we compute the cash equivalent values of the base portfolio and the new optimal portfolio. The percentage cash equivalent loss from holding the suboptimal portfolio instead of the true optimal portfolio measures the impact of errors in input parameters on investor utility.

To evaluate the impact of errors in means, we replaced the assumed true mean $\bar{r}_i$ for asset i by the approximation $\bar{r}_i(1 + kz_i)$ where z_i has a standard normal distribution. The parameter k is varied from 0.05 through 0.20 in steps of 0.05 to examine the impact of errors of different sizes. Larger values of k represent larger errors in the estimates. The variances and covariances are left unchanged in this case to isolate the influence of errors in means.

The percentage cash equivalent loss from holding a portfolio that is optimal for approximate means $\bar{r}_i (1 + kz_i)$ but is suboptimal for the true means r_i then

EXHIBIT 2
Inputs to the Optimization and the Resulting Optimal Portfolio for a Risk Tolerance of 50 (January 1980-December 1989)

	Alcoa	Amex	Boeing	Chev.	Coke	Du Pont	MMM	P&G	Sears	U Tech
Means (% per month)	1.5617	1.9477	1.907	1.5801	2.1643	1.6010	1.4892	1.6248	1.4075	1.4537
Std. Dev. (% per month)	8.8308	8.4585	10.040	8.6215	5.9886	6.8767	5.8162	5.6385	8.0047	8.2125
Correlations										
Alcoa	1.0000									
Amex	0.3660	1.0000								
Boeing	0.3457	0.5379	1.0000							
Chev.	0.1606	0.2165	0.2218	1.0000						
Coke	0.2279	0.4986	0.4283	0.0569	1.0000					
Du Pont	0.5133	0.5823	0.4051	0.3609	0.3619	1.0000				
MMM	0.5203	0.5569	0.4492	0.2325	0.4811	0.6167	1.0000			
P&G	0.2176	0.4760	0.3867	0.2289	0.5952	0.4996	0.6037	1.0000		
Sears	0.3267	0.6517	0.4883	0.1726	0.4378	0.5811	0.5671	0.5012	1.0000	
U Tech	0.5101	0.5853	0.6569	0.3814	0.4368	0.5644	0.6032	0.4772	0.6039	1.0000
Optimal Port. Weights	0.0350	0.0082	0.0	0.1626	0.7940	0.0	0.0	0.0	0.00	0.00

is computed. This procedure is repeated with a new set of z values for a total of 100 iterations for each value of k.

To investigate the impact of errors in variances, each variance forecast σ_{ii} was replaced by $\sigma_{ii}(1 + kz_i)$. To isolate the influence of variance errors, the means and covariances are left unchanged.

Finally, the influence of errors in covariances is examined by replacing each covariance σ_{ij} (with i not equal to j) by $\sigma_{ij}(1 + kz_{ij})$, where z_{ij} has a standard normal distribution, while retaining the original means and variances. The procedure is repeated 100 times for each value of k, each time with a new set of z values, and the cash equivalent loss computed. The entire procedure is repeated for risk tolerances of 25 and 75 to examine how the results vary with investors' risk tolerance.

RESULTS

Exhibit 3 shows the mean, minimum, and maximum cash equivalent loss over the 100 iterations for a risk tolerance of 50. Exhibit 4 plots the average CEL as a function of k. The CEL for errors in means is approximately eleven times that for errors in variances and over twenty times that for errors in covariances. Thus, it is important to distinguish between errors in variances and errors in covariances.[5] For example, for k = 0.10, the CEL is 2.45 for errors in means, 0.22 for errors in variances, and 0.11 for errors in covariances.

Our results on the relative importance of errors in means and variances are similar to those of Kallberg and Ziemba [1984]. They find that errors in means are approximately ten times as important as errors in variances and covariances considered together (they do not distinguish between variances and covariances).

Our results show that for a risk tolerance of 50 the importance of errors in covariances is only half as much as previously believed. Furthermore, the relative importance of errors in means, variances, and covariances depends upon the investor's risk tolerance.

Exhibit 5 shows the average ratio (averaged over errors of different sizes, k) of the CELs for errors in means, variances, and covariances. An investor with a high risk tolerance focuses on raising the expected return of the portfolio and discounts the variance more relative to the expected return. To this investor, errors in expected returns are considerably more

EXHIBIT 3
Cash Equivalent Loss (CEL) for Errors of Different Sizes

k (size of error)	Parameter with Error	Mean CEL	Min. CEL	Max. CEL
0.05	Means	0.66	0.01	5.05
0.05	Variances	0.05	0.00	0.34
0.05	Covariances	0.02	0.00	0.25
0.10	Means	2.45	0.01	15.61
0.10	Variances	0.22	0.00	1.39
0.10	Covariances	0.11	0.00	0.66
0.15	Means	5.12	0.15	24.35
0.15	Variances	0.55	0.00	3.35
0.15	Covariances	0.27	0.00	1.11
0.20	Means	10.16	0.17	36.09
0.20	Variances	0.90	0.01	4.16
0.20	Covariances	0.47	0.00	1.94

important than errors in variances and covariances. For an investor with a risk tolerance of 75, the average CEL for errors in means is over twenty-one times that for errors in variances and over fifty-six times that for errors in covariances.

Minimizing the variance of the portfolio is more important to an investor with a low risk tolerance than raising the expected return. To this investor, errors in means are somewhat less important than

EXHIBIT 4
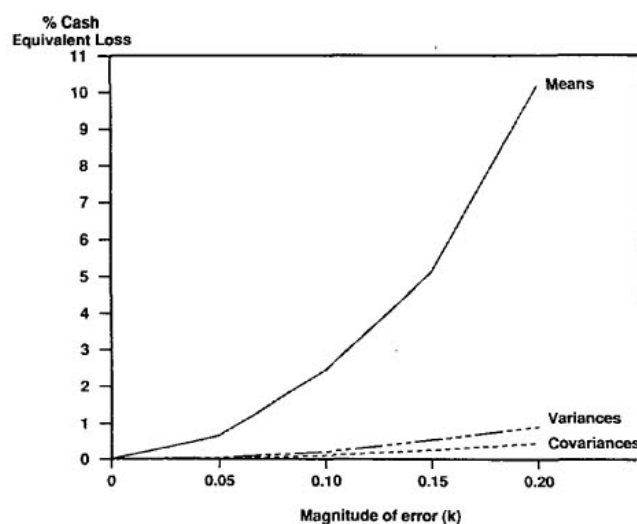**MEAN PERCENTAGE CASH EQUIVALENT LOSS DUE TO ERRORS IN INPUTS**

EXHIBIT 5

Average Ratio of CELs for Errors in
Means, Variances, and Covariances

Risk Tolerance	Errors in Means versus Variances	Errors in Means versus Covariances	Errors in Variances versus Covariances
25	3.22	5.38	1.67
50	10.98	22.50	2.05
75	21.42	56.84	2.68

errors in variances and covariances. For an investor with a risk tolerance of 25, the average CEL for errors in expected returns is about three times that for errors in variances and about five times that for errors in covariances.

Most large institutional investors have a risk tolerance in the 40 to 60 range. Over that range, there is considerable difference in the relative importance of errors in means, variances, and covariances. Irrespective of the level of risk tolerance, errors in means are the most important, followed by errors in variances. Errors in covariances are the least important in terms of their influence on portfolio optimality.

IMPLICATIONS AND CONCLUSIONS

Investors have limited resources available to spend on obtaining estimates of necessarily unknowable future parameters of risk and reward. This analysis indicates that the bulk of these resources should be spent on obtaining the best estimates of expected returns of the asset classes under consideration.

Sometimes, investors using the MV framework to allocate wealth among individual stocks set all the expected returns to zero (or a non-zero constant). This can lead to a better portfolio allocation because it is often very difficult to obtain good forecasts for expected returns. Using forecasts that do not accurately reflect the relative expected returns of different securities can substantially degrade MV performance.

In some cases it may be preferable to set all forecasts equal.[6] The optimization then focuses on minimizing portfolio variance and does not suffer from the error-in-means problem. In such cases it is important to have good estimates of variances and covariances for the securities, as MV optimizes with respect to these characteristics only.

Of course, if investors truly believe that they have superior estimates of the means, they should use them. In this case it may be acceptable to use historical values for variances and covariances.

For investors with moderate to high risk tolerance, the cash equivalent loss for errors in means is an order of magnitude greater than that for errors in variances or covariances. As variances and covariances do not influence the optimal MV allocation much (relative to the means), investors with moderate to high risk tolerance need not expend considerable resources to obtain better estimates of these parameters.

ENDNOTES

[1]For an investor with utility function U and wealth W, the Arrow-Pratt absolute risk aversion is $ARA = -U''(W)/U'(W)$. Friend and Blume [1975] show that investor behavior is consistent with decreasing ARA; that is, as investors' wealth increases, their aversion to a given risk decreases.

[2]The risk tolerance reflects the investor's desired trade-off between extra return and extra risk (variance). It is the inverse slope of the investor's indifference curve in mean-variance space. The greater the risk tolerance, the more risk an investor is willing to take for a little extra return. Under fairly general input assumptions, a risk tolerance of 50 describes the typical portfolio allocations of large U.S. pension funds and other institutional investors. Risk tolerances of 25 and 75 characterize extremely conservative and aggressive investors, respectively.

[3]Although the exponential utility function is convenient for deriving the MV problem with normally distributed returns, the MV framework is consistent with expected utility maximization for any concave utility function, assuming normality.

[4]For negative exponential utility, Freund [1956] shows that the expected utility of portfolio x is $Z(x) = 1 - \exp(-aE[x] + (a^2/2)Var[x])$, where $E[x]$ and $Var[x]$ are the expected return and variance of the portfolio. The cash equivalent is $CE_x = (-1/a)\log(1 - Z(x))$. If returns are assumed to have a multivariate normal distribution, this is also the cash equivalent of a MV-optimal portfolio. See Dexter, Yu, and Ziemba [1980] for more details.

[5]The result for covariances also applies to correlation coefficients, as the correlations differ from the covariances only by a scale factor equal to the product of two standard deviations.

[6]This approach is in the spirit of Stein estimation and is discussed in Chopra, Hensel, and Turner [1993]. As a practical matter, it should be used for assets that belong to the same asset class, e.g., equity indexes of different countries or stocks within a country. It would be inappropriate to apply it to financial instruments with very different characteristics, stocks and T-bills, for example.

REFERENCES

Bawa, Vijay S., Stephen J. Brown, and Roger W. Klein. "Estimation Risk and Optimal Portfolio Choice." *Studies in Bayesian Econometrics*, Bell Laboratories Series. New York: North Holland, 1979.

Best, Michael J., and Robert R. Grauer. "On the Sensitivity of Mean-Variance-Efficient Portfolios to Changes in Asset Means: Some

Analytical and Computational Results." *Review of Financial Studies*, 4, No. 2 (1991), pp. 315-342.

Chopra, Vijay K. "Mean-Variance Revisited: Near-Optimal Portfolios and Sensitivity to Input Variations." *Russell Research Commentary*, 1991.

Chopra, Vijay K., Chris R. Hensel, and Andrew L. Turner. "Massaging Mean-Variance Inputs: Returns from Alternative Global Investment Strategies in the 1980s." Forthcoming in *Management Science*, 1993.

Dexter, Albert S., Johnny N.W. Yu, and William T. Ziemba. "Portfolio Selection in a Lognormal Market When the Investor Has a Power Utility Function: Computational Results." In M.A.H. Dempster, ed., *Proceedings of the International Conference on Stochastic Programming*. New York: Academic Press, 1980, pp. 507-523.

Freund, Robert A. "The Introduction of Risk into a Programming Model." *Econometrica*, 24 (1956), pp. 253-263.

Friend, I., and M. Blume. "The Demand for Risky Assets." *The American Economic Review*, December 1975, pp. 900-922.

Kallberg, Jarl G., and William T. Ziemba. "Mis-specification in Portfolio Selection Problems." In G. Bamberg and K. Spremann, eds., *Risk and Capital: Lecture Notes in Economics and Mathematical Systems*. New York: Springer-Verlag, 1984.

Klein, Roger W., and Vijay S. Bawa. "The Effect of Estimation Risk on Optimal Portfolio Choice." *Journal of Financial Economics*, 3 (June 1976), pp. 215-231.

Markowitz, Harry M. *Mean-Variance Analysis in Portfolio Choice and Capital Markets*. New York: Basil Blackwell, 1987.

Michaud, Richard O. "The Markowitz Optimization Enigma: Is 'Optimized' Optimal?" *Financial Analysts Journal*, 45 (January-February 1989), pp. 31-42.

Ziemba, William T., and Raymond G. Vickson, eds. *Stochastic Optimization Models in Finance*. New York: Academic Press, 1975.