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THE RELIABILITY OF ESTIMATION PROCEDURES IN PORTFOLIO ANALYSIS

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I. Introduction

The Markowitz model for the efficient diversification of investments [12] has, over the years since its original formulation, provided the basis for many investigations into the question of portfolio selection. Amongst the more notable contributions to the theory are the works of Fama [6] and Mandelbrot [11], Smith [17], Latané [10], Arditti [1], and Blume [2].

Perhaps the most significant work, however, was carried out by Sharpe [16]. He criticized the basic Markowitz theory on the grounds of its extremely complex and unwieldy nature when applied in a practical situation involving a portfolio of many investments, and he proposed a modification known as the diagonal model. Sharpe suggested that the covariance between the returns on any pair of securities is due only to the dependence of each return on the same "market index." He further postulated that the return of each security depended linearly on this market index. By removing this common source of variation, the portfolio analyst is left with residual variations in the two returns which are independent of each other. Extending this idea to a portfolio of n securities results in an expression for portfolio risk which has (n+1) components (i.e., the individual residual variances of the n securities, and a term involving the market index variance).

Undoubtedly, the Sharpe model provides a significant improvement, although the practical results given by either model leave a great deal to be desired, and are, in fact, sufficiently poor for the investment analyst to be forgiven

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for relying on his intuition. The criticisms leveled at the theory are, in general, based upon empirical evidence, and--valuable as this evidence might be--are not based upon criticism of the basic theory itself. In particular, the present writer can find in the literature no examination of the theory from a statistical viewpoint. The purpose of this article is to illustrate why, statistically, we might expect Markowitz's theory to provide results of dubious reliability in general. In Sections II and III, discussion is confined to the portfolio of investments having minimum risk, irrespective of the portfolio rate of return. In Sections IV - VII, the analysis is extended to include the portfolio rate of return in the comparatively simple case of a two-security portfolio.

II. Distribution of Portfolio Risk

Assuming that the historical values of the return on investment are collected over n time periods, and with the notation

E_i = expected value of return,

r_{it} = return on investment i during the time period t,

σ_{ii} = variance of return on investment i,

σ_{ij} = covariance of returns on investments i and j,

we may estimate E_i by $\hat{E}_i$, σ_{ii} by $\hat{\sigma}_{ii}$, σ_{ij} by $\hat{\sigma}_{ij}$

where
$$\hat{E}_i = (1/n) \sum_{t=1}^{n} r_{it}$$

$$\hat{\sigma}_{ii} = (1/n) \sum_{t=1}^{n} (r_{it} - \hat{E}_i)^2$$

$$\hat{\sigma}_{ij} = (1/n) \sum_{t=1}^{n} (r_{it} - \hat{E}_i)(r_{jt} - \hat{E}_j).$$

The well-known result given by Markowitz [12] states that if $w_1, \ldots, w_m$ are the proportions of the total capital invested in each of m securities respectively, then estimates, $\hat{w}_i$, of those weights which define the portfolio of minimum variance (i.e., minimum risk) are given by

$$\hat{w}_i = S^{-1} I_m (I_m' S^{-1} I_m)^{-1} \tag{1}$$

where

$$S = \begin{bmatrix} \hat{\sigma}_{11} & \cdots\cdots & \hat{\sigma}_{1m} \\ \vdots & & \\ \hat{\sigma}_{m1} & \cdots\cdots & \hat{\sigma}_{mm} \end{bmatrix},$$

$I_m = (1, \ldots\ldots, 1)$,

I_m' = transpose of I_m, and

$\hat{w}_i$ = column vector of weight estimates.

Furthermore, it may be shown [3] that the risk, σ_{pp}, of this minimum risk portfolio is estimated by

$$\hat{\sigma}_{pp} = (I_m' S^{-1} I_m)^{-1}. \tag{2}$$

It has been suggested (Fama [5]) that the assumption that the distribution of returns is normal may not be justified, and that the distribution may, in fact, be asymmetrical and of infinite variance. Evidence to support or refute Fama's suggestion is not conclusive, and it is not the purpose of this paper to enter this controversy. Rather, the object is to examine in detail the implications of previous analysis, assuming that returns have normal (or nearly normal) distributions. On this assumption and with the notation

$$\{n^2/(n-1)\}\ \Sigma = \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{bmatrix},$$

it is possible to deduce that $(I_m' \Sigma^{-1} I_m)\hat{\sigma}_{pp}$ has a chi-squared distribution with (n-m+1) degrees of freedom. (The analysis required here is by no means trivial; the interested reader is referred to [4].) Knowledge of this distribution now enables us to construct confidence intervals for the minimum risk.

The following simple example illustrates the procedure when three securities are under consideration for the portfolio.

	Security(i)	$\hat{E}_i$	$\hat{\sigma}_{ii}$	
U.S. Steel	1	0.0167	0.2400	$\hat{\sigma}_{12} = 0.1497$
Olin-Mathieson	2	0.0534	0.2200	$\hat{\sigma}_{23} = 0.0855$
Parke-Davis	3	0.1314	0.2700	$\hat{\sigma}_{31} = 0.1631$

The estimates are based on quarterly returns over a ten-year period (i.e., n = 40) from 1957 to 1966 inclusively (see [8]).

Thus,

$$S = \begin{bmatrix} 0.2400 & 0.1497 & 0.1631 \\ 0.1497 & 0.2200 & 0.0855 \\ 0.1631 & 0.0855 & 0.2700 \end{bmatrix}, \text{ and}$$

$$S^{-1} = \begin{bmatrix} 10.9120 & -5.5459 & -4.8355 \\ -5.5459 & 8.0020 & 0.8162 \\ -4.8355 & 0.8162 & 6.3662 \end{bmatrix}, \text{ and}$$

$I_m' S^{-1} I_m = 6.1499$. Thus $\hat{\sigma}_{pp} = 0.1626$.

Now the true value of the minimum risk is

$$\sigma_{pp} = \{I_m' \begin{bmatrix} \sigma_{11} & \cdots & \sigma_{1m} \\ \sigma_{m1} & \cdots & \sigma_{mm} \end{bmatrix}^{-1} I_m\}^{-1} = \{I_m' (\frac{n-1}{n^2}) \Sigma^{-1} I_m\}^{-1}$$

$$= (\frac{n^2}{n-1})(I_m' \Sigma^{-1} I_m)^{-1}$$

and, for example, a 90 percent confidence interval for σ_{pp} is provided by

$$(n^2\hat{\sigma}_{pp})/\{(n-1)\chi^2_{0.95}(38)\} < \sigma_{pp} < (n^2\hat{\sigma}_{pp})/\{(n-1)\chi^2_{0.05}(38)\},$$

i.e., $0.1255 < \sigma_{pp} < 0.2678$.

It is clear that, even with estimates of expected return and risk based on a large number of historic values, the reliability of the estimate of the minimum risk is very low. In this case, we can only be 90 percent confident that the risk is between 0.1255 and 0.2678.

Theoretically, it is possible to show that the portfolio cannot have greater risk than any of its component securites [3]. Unfortunately, this theoretical conclusion is based upon exact knowledge of $\sigma_{ij}(i,j=1, \ldots, m)$, whereas, in practice, we have at our disposal only estimates $\hat{\sigma}_{ij}$ of these quantities. As a consequence, it is likely that our estimated "minimum risk" portfolio will not, in fact, be the portfolio with least risk. In general, it will have less risk than any of the component securities, although it is possible--albeit very unlikely--that it will have a greater risk than a particular component.

III. Reliability of Weight Estimates

Taking the particularly simple case of a two-security portfolio, the estimated weights to be attached to the securities are provided by equation (2), viz.:

$$\begin{bmatrix} \hat{w}_1 \\ \hat{w}_2 \end{bmatrix} = \begin{bmatrix} \hat{\sigma}_{11} \hat{\sigma}_{12} \\ \hat{\sigma}_{21} \hat{\sigma}_{22} \end{bmatrix}^{-1} I_m \{ I_m' \begin{bmatrix} \hat{\sigma}_{11} & \hat{\sigma}_{12} \\ \hat{\sigma}_{21} & \hat{\sigma}_{22} \end{bmatrix}^{-1} I_m \}^{-1}$$

$$= \begin{bmatrix} \hat{\sigma}_{22} - \hat{\sigma}_{12} \\ -\hat{\sigma}_{21} \ \hat{\sigma}_{11} \end{bmatrix} I_m \{ I_m' \begin{bmatrix} \hat{\sigma}_{22} - \hat{\sigma}_{12} \\ -\hat{\sigma}_{21} \ \hat{\sigma}_{11} \end{bmatrix} I_m \}^{-1}$$

$$\hat{w}_1 = \frac{(\hat{\sigma}_{22} - \hat{\sigma}_{12})}{(\hat{\sigma}_{11} + \hat{\sigma}_{22} - 2\hat{\sigma}_{12})}$$

$$\hat{w}_2 = \frac{(\hat{\sigma}_{11} - \hat{\sigma}_{12})}{(\hat{\sigma}_{11} + \hat{\sigma}_{22} - 2\hat{\sigma}_{12})}$$

In general, the distributions of $\hat{w}_1$, $\hat{w}_2$ are very complex. If, however, the additional assumption is made that the securities are diversified in the sense that the correlation between their rates of return is not significant (i.e., $\hat{\sigma}_{12} = 0$), then

$$\hat{w}_1 = \hat{\sigma}_{22}/(\hat{\sigma}_{11} + \hat{\sigma}_{22}), \text{ and}$$

$$\hat{w}_2 = \hat{\sigma}_{11}/(\hat{\sigma}_{11} + \hat{\sigma}_{22}).$$

i.e.,

(3) $$\hat{w}_1 = 1/(1 + y)$$

where

$$y = \{\Sigma_{t=1}^{n}(r_{2t}-\hat{E}_2)^2\}/\{\Sigma_{t=1}^{n}(r_{1t}-\hat{E}_1)^2\}.$$

Now $\{\Sigma_{t=1}^{n}(r_{it}-\hat{E}_i)^2\}/\sigma_{ii}$ for $i = 1, 2$ are independently distributed chi-squared distributions, each with (n-1) degrees of freedom (see, for example, [15]).

Therefore, the probability density of $\hat{w}_1$ is

(4) $$p(\hat{w}_1) = \frac{\Gamma(2n-2)}{\Gamma(n-1)\Gamma(n-1)}\,\alpha^{\frac{n-1}{2}}\,\frac{(1-\hat{w}_1)^{\frac{n-1}{2}-1}\hat{w}_1^{\frac{n-1}{2}-1}}{\{(\alpha-1)\hat{w}_1 + 1\}^{n-1}}$$

where $\alpha = \sigma_{11}/\sigma_{22}$, and we assume--without any loss of generality--that $\sigma_{11} < \sigma_{22}$. Confidence intervals for $\hat{w}_1$ can now be found using expression (4) for the probability density. These are given in Tables 1 and 2 for some selected values of n and α.

From equation (3), we note that $\hat{w}_1$ lies between 0 and 1, as it should, since it represents the proportion of the total investment to be made in the first security. The Tables 1 and 2 indicate that, for example, if the estimates $\hat{E}_i$, $\hat{\sigma}_{ii}$, $\hat{\sigma}_{ij}$ are based on five historic values of the returns on the securities, and if the ratio of the risks is, say, 0.5, then we can only have a 98 percent degree of confidence that $\hat{w}_1$ lies between 0.1112 and 0.9697, and a 90 degree of confidence that it lies between 0.2384 and 0.9274. Clearly, even for five historic observations, our degree of confidence in estimating the weights accurately is very low.

Comparison of Tables 1 and 2 for a particular value of α, say, 0.3, indicates that the introduction of more observations narrows the 90 percent confi-

TABLE 1

PERCENTAGE POINTS OF DISTRIBUTION OF ESTIMATE $\hat{w}_1$ OF WEIGHT OF FIRST SECURITY, FOR n = 3

α $(=\sigma_{11}/\sigma_{22})$	True Value	1%	5%	95%	99%
0.1	0.9091	0.0917	0.3448	0.9948	0.9990
0.2	0.8333	0.0481	0.2083	0.9896	0.9980
0.3	0.7692	0.0326	0.1493	0.9845	0.9970
0.4	0.7143	0.0246	0.1163	0.9794	0.9960
0.5	0.6667	0.0198	0.0990	0.9744	0.9950
0.6	0.6250	0.0166	0.0806	0.9694	0.9940
0.7	0.5882	0.0142	0.0699	0.9645	0.9930
0.8	0.5556	0.0125	0.0617	0.9596	0.9920
0.9	0.5263	0.0111	0.0552	0.9548	0.9910
1.0	0.5000	0.0100	0.0500	0.9500	0.9900

TABLE 2

PERCENTAGE POINTS OF DISTRIBUTION OF ESTIMATE $\hat{w}_1$ OF WEIGHT OF FIRST SECURITY, FOR n = 5

	True Value	1%	5%	95%	99%
0.1	0.9091	0.3851	0.6102	0.9846	0.9938
0.2	0.8333	0.2384	0.4390	0.9696	0.9876
0.3	0.7692	0.1726	0.3429	0.9551	0.9816
0.4	0.7143	0.1353	0.2813	0.9411	0.9756
0.5	0.6667	0.1112	0.2384	0.9274	0.9697
0.6	0.6250	0.0945	0.2070	0.9141	0.9638
0.7	0.5882	0.0821	0.1828	0.9012	0.9580
0.8	0.5556	0.0726	0.1637	0.8887	0.9520
0.9	0.5263	0.0650	0.1482	0.8766	0.9466
1.0	0.5000	0.0589	0.1354	0.8646	0.9411

dence interval from (0.1493, 0.9845) to (0.3429, 0.9551). Unfortunately, however, it would be unwise to base the estimates on too many past values, since the variation with time of the risks and returns will then affect the estimates.

Admittedly, the estimates used by analysts are based upon a larger number of past values of return--typically quarterly figures over, say, four years might be used, giving a total of sixteen values rather than the five suggested above. Assuming that time dependence considerations are unimportant, the reliability of the estimates is still poor. Although the derivation of exact sampling distributions of $\hat{w}_1$, $\hat{w}_2$ for large values of n is tedious in general, this point may be illustrated quite simply by an example of two securities of equal risk (i.e., $\sigma_{11} = \sigma_{22}$, or $\alpha = 1$).

In this situation, the probability density of $\hat{w}_1$ is (putting $\alpha = 1$ in (4)):

$$p(\hat{w}_1) = \frac{\Gamma(2n-2)}{\Gamma(n-1)\Gamma(n-1)}(1-\hat{w}_1)^{\frac{n-1}{2}-1}\hat{w}_1^{\frac{n-1}{2}-1}$$

Table 3 gives the percentage points of this distribution for several values of n. With equal risk, uncorrelated securities the minimum risk portfolio is made up of equal investments in each security, i.e., $w_1 = w_2 = 0.5$. It is of interest to note that with as many as 121 historic values of return on which to base $\hat{w}_1$, $\hat{w}_2$, we can be only 90 percent confident of obtaining an estimate of $\hat{w}_1$ in the range 0.4252 to 0.5748.

IV. Analysis Incorporating Portfolio Rate of Return

In this, and the succeeding sections, we turn our attention to the two-security portfolio and introduce the portfolio rate of return into our analysis.

We wish, therefore, to minimize σ_{pp} for a preselected portfolio rate of return E.

TABLE 3

PERCENTAGE POINTS OF DISTRIBUTION OF ESTIMATE $\hat{w}_1$ OF WEIGHT OF FIRST SECURITY FOR DIFFERENT VALUES OF n WHEN $\alpha=1$ (i.e., $\sigma_{11}=\sigma_{22}$)

n	1%	5%	95%	99%
2	0.0002	0.0062	0.9938	0.9998
3	0.0100	0.0500	0.9500	0.9900
4	0.0328	0.0973	0.9027	0.9672
5	0.0589	0.1354	0.8646	0.9411
6	0.0836	0.1653	0.8347	0.9164
7	0.1056	0.1893	0.8107	0.8944
8	0.1251	0.2089	0.7911	0.8749
9	0.1423	0.2253	0.7747	0.8577
10	0.1575	0.2393	0.7607	0.8425
11	0.1710	0.2514	0.7486	0.8290
13	0.1940	0.2713	0.7287	0.8060
16	0.2211	0.2938	0.7062	0.7789
21	0.2540	0.3201	0.6799	0.7460
25	0.2733	0.3352	0.6648	0.7267
31	0.2953	0.3520	0.6480	0.7047
41	0.3211	0.3714	0.6286	0.6789
61	0.3526	0.3946	0.6054	0.6474
121	0.3948	0.4252	0.5748	0.6052

(5) Now, $\sigma_{pp} = w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22}$.

The object is then to minimize (5) subject to

(6) $w_1 + w_2 = 1$, and

(7) $w_1E_1 + w_2E_2 = E$.

Introducing the Lagrange undetermined multipliers λ_1, λ_2, this object is achieved by differentiating

(8) $\sigma_{pp} = w_1^2\sigma_{11} + 2w_1w_2\sigma_{12} + w_2^2\sigma_{22} + \lambda_1(w_1+w_2-1) + \lambda_2(w_1E_1+w_2E_2-E)$

with respect to w_1, w_2, λ_1, λ_2, in turn equating the derivatives to zero and solving the resulting four simultaneous equations. Details may be found, for example, in Francis and Archer [8].

The same result, however, may be obtained quite trivially from (6), (7), since here we have two equations in two unknowns. Thus,

(9) $w_1 = (E-E_2)/(E_1-E_2)$, $w_2 = (E_1-E)/(E_1-E_2)$.

The usual methods of approach are now to estimate E_1, E_2 from the past data and insert the estimates $\hat{E}_1$, $\hat{E}_2$ in (9). Assuming that the variance of the returns is finite (although some writers have claimed empirical evidence to the contrary--notably Fama [5])--and that there exists no serial correlation in the returns, we note that, by the Central Limit Theorem, $E - \hat{E}_2$ is approximately normally distributed with mean $(E - E_2)$ and variance σ_{11}/n.

Similarly, $\hat{E}_1 - \hat{E}_2 = \{(r_{11} - r_{21}) + \ldots +(r_{1n} - r_{2n})\}/n$ is approximately normal with mean $(E_1 - E_2)$ and variance $(\sigma_{11} + \sigma_{22} - 2\sigma_{12})/n$.

Thus the estimate $\hat{w}_1$ of w_1 is approximately distributed as the ratio of two normal variables with covariance between the denominator and numerator, which is easily shown to be $(\sigma_{22} - \sigma_{12})/n$.

For simplicity, we write henceforth,

$$E(E - \hat{E}_1) = \theta_1 \qquad E(\hat{E}_2 - \hat{E}_1 = \theta_2$$

$$\mathrm{var}(E - \hat{E}_1) = s_{11} = s_1^2 \qquad \mathrm{var}(\hat{E}_2 - \hat{E}_1) = s_{22} = s_2^2, \text{ and}$$

$$\mathrm{cov}\{(E - \hat{E}_1), (\hat{E}_2 - \hat{E}_1)\} = (\sigma_{22} - \sigma_{12})/n = s_{12} = \rho s_1 s_2.$$

V. Sampling Distribution of Weight Estimates

It has been shown (Hinkley [9]) that the frequency function $F(w) = \text{prob}$ $(\hat{w}_1 < w)$ is given exactly by

$$F(w) = l(h,k;\gamma) + l(-h,-k;\gamma)$$

$$h = (\theta_1 - \theta_2 w)/\{s_1 s_2 (w^2 s_{11}^{-1} - 2\rho w s_1^{-1} s_2^{-1} + s_{22}^{-1})^{1/2}\}$$

$$k = -\theta_2/s_2$$

$$= (s_2 w - \rho s_1)/\{s_1 s_2 (w^2 s_{11}^{-1} - 2\rho w s_1^{-1} s_2^{-1} + s_{22}^{-1})^{1/2}\}, \text{ and}$$

$$l(h,k;\gamma) = \{2\pi\sqrt{(1-\gamma^2)}\}^{-1} \int_h^\infty \int_k^\infty \exp\{-\frac{x^2 - 2\,xy + y^2}{2(1-\gamma^2)}\}.$$

The exact values of $F(w)$ can thus be calculated using, for example, tables of the bivariate normal integral [14].

However, a useful approximation is provided by

$$F(w) = \Phi[(\theta_2 w - \theta_1)/\{s_1 s_2 (w^2 s_{11}^{-1} - 2\rho w s_1^{-1} s_2^{-1} + s_{22}^{-1})^{1/2}\}].$$

The condition under which this approximate form can be used is $0 < s_2 << \theta_2$ which implies that the denominator of $\hat{w}_1$ must be nearly always positive (i.e., prob. $\{\hat{E}_2 - \hat{E}_1) < 0\}$ 0).

In general, this may not be the case, particularly if $|E_1 - E_2|$ is small. However, although this caveat must be borne in mind, it is generally found with the data available in practice that, assuming $E_2 > E_1$ without loss of generality, $(\hat{E}_2 - \hat{E}_1)$ is almost always positive. Alternatively, since both $\hat{w}_1$, $\hat{w}_2$ cannot be negative from the nature of the constraints imposed upon them, either $(E - \hat{E}_1)$ or $(\hat{E}_2 - E)$ will be almost always positive (and of smaller variance than $(\hat{E}_2 - \hat{E}_1)$. The approximation and following argument could then be applied to the reciprocal of the weight estimate.

VI. Confidence Intervals for $\hat{w}_1$

To calculate the mean and variance of $\hat{w}_1$ is by no means trivial, and an alternative approach for examining the sampling distribution, through the use of confidence intervals, is suggested.

Suppose a two-sided α% confidence interval for $\hat{w}_1$ is required. This is given by $(\hat{w}_1^1, \hat{w}_1^{11})$ where

$$F(\hat{w}_1^1) = (1-\alpha)/2, \; f(\hat{w}_1^{11}) = (1 + \alpha)/2$$

Therefore $\hat{w}_1^1$, $\hat{w}_1^{11}$ are given by the roots of

$$(10) \quad \frac{(\theta_2 w - \theta_1)^2}{s_{11}s_{22}\{s_1 s_2 (w^2 s_{11}^{-1} - 2\rho w s_1^{-1} s_2^{-1} + s_{22}^{-1})\}} = p^2$$

where $\Phi(p) = (1 + \alpha)/2$, $\Phi(-p) = (1 - \alpha)/2$.

Arranging (10) as a quadratic equation in w gives

$$(11) \quad (\theta_2^2 - s_{22}p^2)w^2 + 2(s_{12}p^2 - \theta_1\theta_2)w + (\theta_1^2 - s_{11}p^2) = 0.$$

It is, of course, possible to solve this equation in terms of θ_1, θ_2, s_{11}, s_{22}, s_{12}. However, a more useful approach is to consider the length of the confidence interval as a fraction of $\hat{w}_1^1$, $\hat{w}_1^{11}$ through the expression

$$(12) \quad \left\{ \frac{(\hat{w}_1^1 - \hat{w}_1^{11})}{\hat{w}_1^1} \, \frac{(\hat{w}_1^1 - \hat{w}_1^{11})}{\hat{w}_1^{11}} \right\}$$

which gives the geometric mean of the extreme values of the relative error.

The advantage of (12) is that it can be written as

$$\left\{\frac{(\hat{w}_1^{1} + \hat{w}_1^{11})^2 - 4\hat{w}_1^{1}\hat{w}_1^{11}}{\hat{w}_1^{1}\hat{w}_1^{11}}\right\}^{1/2} = \left\{\frac{\hat{w}_1^{1} + \hat{w}_1^{11}}{\hat{w}_1^{1}\hat{w}_1^{11}}^{2} - 4\right\}^{1/2}$$

$$= 2\left\{\frac{(p^2 s_{12} - \theta_1\theta_2)^2}{(p^2 s_{11} - \theta_1^2)(p^2 s_{22} - \theta_2^2)} - 1\right\}^{1/2}$$

(from (11))

(13) $$= 2\left[(1 - \frac{p^2 s_{12}}{\theta_1\theta_2})^2 / (1 - \frac{p^2 s_{11}}{\theta_1^2})(1 - \frac{p^2 s_{22}}{\theta_2^2}) - 1\right]^{1/2} .$$

Typically, s_{11}, s_{12}, $s_{22} << \theta_1$, θ_2 and using the binomial theorem, (13) becomes approximately

$$2\{1 + p^2(s_{11}\theta_1^{-2} - 2s_{12}\theta_1^{-1}\theta_2^{-1} + s_{22}\theta_2^{-2}) - 1\}^{1/2}$$

(14) $$= 2p(s_{11}\theta_1^{-2} - 2s_{12}\theta_1^{-1}\theta_2^{-1} + s_{22}\theta_2^{-2})^{1/2}.$$

VII. Numerical Example

On the basis of 20 past values of two securities, the following estimates were calculated:

$\hat{E}_1 = 1.67\%$ $\qquad$ $\hat{E}_2 = 5.34\%$

$\hat{\sigma}_{11} = 0.24$ $\qquad$ $\hat{\sigma}_{22} = 0.22$.

$\hat{\sigma}_{12} = 0.15$

Setting E = 5%, we may now take the following (estimated) values:

$\theta_1 = 2.33\%$ $\qquad$ $\theta_2 = 3.67\%$

$s_{11} = 0.24/20$ $\qquad$ $s_{22} = 0.22/20$

$= 0.012$ $\qquad$ $= 0.011$

$s_{12} = (0.22\text{-}0.15)/20$ $\qquad$ $= 0.0035$

Then, computation of (14) gives a value for relative error (for a 95 percent confidence interval) of

$$2 \times 1.96[\{0.012/(2.33)^2\} - \{(2 \times 0.0035)/(3.67 \times 2.33)\} + \{(0.011)/(3.67)^2\}]^{1/2} = 18.4\%.$$

The more lengthy analysis to find the roots of (11) proceeds as follows:

$$p^2 s_{12} - \theta_1\theta_2 = -8.5377$$

$$p^2 s_{11} - \theta_1^2 = -5.3828$$

$$p^2 s_{22} - \theta_2^2 = -13.4266.$$

The roots are

$$(17.0754 \pm 1.5743)/(26.8532), \text{ or}$$

$$0.6945,\ 0.5773.$$

VII. Conclusion

The preceding discussion has been restricted to the confidence with which we can present a minimum risk portfolio on the basis of historic values of returns. In general, it has been shown in Section II that to estimate the value of this risk with any reasonable degree of certainty is impossible using other than a very large number of past values. Whereas the Markowitz method will nearly always give a portfolio of lower risk than any of its component securities, it is difficult to give an accurate value of the portfolio risk.

Secondly, it has been shown in Section III that, using an even larger number of historic returns, the estimated weights in a two-security, equal-risk portfolio are unreliable. There seems little reason to doubt that the unreliability will be as great when we are constructing portfolios of a larger number of securities in general.

The analysis of Sections IV-VII rests on two approximations. In the first instance, the numerator and denominator of $\hat{w}_1$ are assumed to be normally distributed through an application of the Central Limit Theorem. In view of the fact that estimates of returns and variance of returns are normally based on a fairly large number of past values (20 in the example), this assumption is reasonable.

The second approximation, to the ratio of two correlated normal variates, has been examined in some detail by Fieller [7], Hinkley [9], and Marsaglia [13]. In particular, Hinkley showed that the error in F(w) is of the order of $\Phi(-\theta_2/s_2)$. This error in many practical situations is very small. In the numerical example, for instance, θ_2/s_2 is of the order of 300. Clearly, the approximation is quite appropriate when $\theta_2/s_2 > 2$. If $\theta_2/s_2 < 2$, then it should be obvious that estimation of weights is likely to be highly suspect.

The statistic (14) is, of course, based on sample values, and is not, therefore, exact. However, it is suggested that it might provide a useful rapid test for the reliability of the weight estimates.

The inputs of security analysis (viz. estimates of risks and returns) are subject to sampling fluctuations. In this paper, the consequences of this sampling error have been traced through the subsequent portfolio analysis, and the effect on the estimation of portfolio weights and risk illustrated explicitly. Since the procedures of portfolio analysis, given the inputs, are shown to lack robustness, the estimates of weights and risk are shown to be unreliable to an extent that casts doubt on the usefulness of the accepted combination of security and portfolio analyses.

Clearly, this article only attempts to point out possible weaknesses. Work is now in hand to extend the results to larger portfolios and possibly to introduce serial correlation between the historic returns on an investment into the analysis.

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