For better performance: Constrain portfolio weights

Constraining weights significantly reduces estimation error in optimizing mean–variance selection strategies.

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Portfolio managers commonly impose constraints on the maximum proportion of a portfolio that can be invested in a single security. At first glance, these constraints appear counterproductive. Why restrict the portfolio manager's ability to optimize the portfolio? Imposing constraints certainly does not make sense in a world where portfolio managers know the actual risk and expected return of securities. And portfolio performance will deteriorate if the constraint is binding. In practice, however, the actual risk and expected return of securities are unknown, so imposing constraints can be justified.

Risk and return must be estimated either subjectively or from historical observation of security returns. The consequence is overinvestment in securities with favorable estimation error and underinvestment in securities with unfavorable estimation error. As a result, estimated portfolio expected returns are biased upward, and estimated portfolio variances are biased downward.[1]

Imposing an upper bound on portfolio weights can improve portfolio performance by reducing estimation bias. Constraining the individual security investment weights allows the investor to limit the amount of overinvestment or underinvestment in securities attributable to estimation error. Constraints must not be too severe, or portfolio performance may deteriorate if the portfolio manager is unable to take advantage of valuable information about true differences in security expected returns and risks.

Strangely enough, there is scant research on the potential of upper bound constraints to reduce estimation bias and improve portfolio performance. In one study of the relative performance of alternative portfolio selection models, Cohen and Pogue (1967) impose an upper bound on the maximum percentage that can be invested in a single security. They set upper bounds of 5% and 2.5% on security populations consisting, respectively, of 75 and 150 securities. After noting that institutional reasons frequently justify upper bound constraints, they add that "it may still be desirable to employ upper-bound restrictions as a method of hedging against the risk of biases in the input data" (p. 173).

Other papers by Frost and Savarino (1986a) and Jorion (1986) use empirical Bayes methods to reduce estimation bias and improve portfolio performance. Empirical Bayes procedures limit extreme portfolio weights by reducing large estimation error. These papers provide indirect evidence that constraining security investment weights may improve portfolio performance.

This article uses simulation techniques to explore the potential for reducing estimation bias and improving performance by limiting the proportion of the portfolio that can be invested in a single security. We assume that a portfolio manager in determining the optimal portfolio uses historical sample estimates

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of security risks and expected returns as input to Markowitz's (1959) mean–variance model. While this security evaluation model is admittedly naive, our basic results should apply to any portfolio selection technique that attempts to identify optimal portfolios using estimates containing error. The simulation study assumes as well that security performance can be adequately forecast from historical sample returns. In practice, we do not advocate using historical estimates as the sole input to any portfolio selection model.

We start by describing the data and the methodology of the simulation model. Next we report how imposing upper bound constraints on individual security investment weights reduces estimation bias and improves portfolio performance. Then we discuss the related issue that disallowing short-selling reduces estimation risk and leads to the selection of substantially superior portfolios. We conclude that imposing upper bounds both reduces estimation bias and improves performance.

DATA AND METHODOLOGY

The sample includes 200 securities of the 574 securities that were traded continuously on the New York Stock Exchange (NYSE) during the twenty-year period from January 1966 through December 1985. First we ranked the 574 securities by their four-digit SIC code, then we selected 200 securities using systematic sampling with the probability of selection proportional to each security's market value.[2] A systematic sampling procedure ensures a representative cross section of securities from different industries.

Security expected returns are assumed to be linearly related to their relative market risk. This assumption ensures that the market portfolio, a value-weighted portfolio consisting of all 200 securities, is ex ante efficient. In particular, security j's monthly expected return, μ_j, is given by Equation (1):

$$\mu_j = \text{Risk-Free Interest Rate} + \text{Security Risk Premium}$$
$$= i + [\text{Market Risk Premium}] \cdot [\text{Relative Market Risk}]$$
$$= 0.002612 + [0.006561]\beta_j \qquad (1)$$

The risk-free interest rate is the average monthly return of 0.2612% on U.S. Treasury bills during the 1926 to 1982 historical period. Over the same period, the average monthly return for the Standard & Poor's 500 Composite Index was 0.9173%.[3] The difference of 0.6561% between the historically observed average return on the S&P 500 and U.S. Treasury bills is the market risk premium, the premium required by an average investor to accept market risk. Relative market risk is captured by a security's beta, $\hat{\beta}_j$.

Individual security expected returns are determined by Equation (1), where the security's true beta $\hat{\beta}_j$ is replaced by its historical estimate $\hat{\beta}_j$. We obtained beta estimates $\hat{\beta}_j$ for each security by regressing security j's return on the value-weighted return of the 200-security population during the January 1966 through December 1985 historical period. Estimates of each security's variance and correlation with other security returns during the same historical period are equated with their true values. This multivariate normal distribution is used both to simulate monthly observations on individual security returns and to evaluate the performance of portfolios. International Mathematical and Statistical Subroutine Library (IMSL) subroutine GGNSM is used to simulate 240 monthly observations on each security's return.

We evaluated portfolio performance for three investors of varying risk tolerance. Table 1 provides the expected return, the variance, and the certainty equivalent return for the optimal portfolios held by the three investors. A portfolio's certainty equivalent return depends on the investor's risk tolerance and on the portfolio's true expected return and risk.[4] The optimal portfolio of an investor with a risk tolerance of 0.5571 is the market portfolio, the value-weighted portfolio of the 200 securities with a monthly expected return of 0.9173%. This investor is the market investor. For the remaining two investors we chose risk tolerances of 0.2786 and 1.142, so that these two investors exhibit one-half and twice the risk tolerance of the market investor. These investors' optimal portfolios contain, respectively, 41 and 38 securities.

We evaluated the performance of six investment strategies over twenty-five independent trials. Buying on margin and short-selling were not permitted. The first strategy leaves the portfolio weights unrestricted, allowing investment of the entire portfolio in a single security. In the other strategies, the maximum percentage that can be invested in a single security is 5%, 2%, 1%, ⅔ of 1%, and ½ of 1%. This is equivalent to forcing diversification into at least 20, 50, 100, 150, or 200 securities, respectively.

The minimum upper bound of ½ of 1% dictates a passive management strategy, as it requires equal

TABLE 1

Optimal Portfolios for Three Investors
(Returns in Percent)

Investor	Risk Tolerance	Expected Return	Variance	Certainty Equivalent Return	Number of Securities
Most Risk-Averse	0.2786	0.737	0.111	0.359	41
Market	0.5571	0.917	0.183	0.589	200
Least Risk-Averse	1.1142	1.188	0.394	0.835	38

investment in all 200 securities. Requiring equal investment in the 200 securities must provide an expected portfolio return equaling the average expected return of the security population. Because the portfolio weights are unaffected by estimation error, an equally-weighted investment strategy provides unbiased estimates of both portfolio expected return and variance.

ESTIMATION BIAS AND PORTFOLIO PERFORMANCE

Table 2 reports summary statistics for the portfolios chosen by the three investors. With the exception of the minimum and maximum number of securities in the portfolio and the t-statistics, all figures reported in the table are average values for the twenty-five trials. Estimated expected returns and variances are based on the sample estimates used to choose the portfolios. Actual expected returns and variances are calculated from knowledge of the true distribution of security returns. Estimation bias is represented by the difference between the average estimated and true portfolio expected returns and variances.

The last column in Table 2 reports the minimum and maximum number of securities held in the portfolio during the twenty-five trials. In general, without imposing an upper bound on the portfolio weights, the number of securities held is usually less than the minimum number of twenty required by the weakest constraint. The most risk-averse investor's portfolios included as few as ten to twenty-one securities, the market investor's portfolios included seven to thirteen securities, and the least risk-averse investor's portfolios included three to nine securities.

Each investor plunges heavily into a small subset of securities that have favorable measurement errors, especially those securities with a large positive error in the estimate of their expected return. As a result, estimates of both portfolio expected return and risk are substantially biased. Imposing upper bound constraints helps reduce this bias. When the maximum security investment weight is 100%, the average estimated expected return on the portfolio is almost twice the average actual expected return.

Table 2 reports the t-statistics for paired t-tests of the null hypothesis that two adjacent upper bound constraints provide equally biased estimates of portfolio expected return. Significant negative values indicate that the bias for the indicated upper bound constraint is less than the bias for the constraint immediately above. The bias in the estimates of portfolio expected return decreases as the upper bound constraint is tightened from 100% to ½ of 1%. Estimation bias in the estimates of portfolio variance appears less severe. When the upper bound constraint is 100%,

TABLE 2

Bias in Estimated Portfolio Expected Return and Variance

Upper Bound	Expected Return: Estimated	Actual	Bias	t-Stat	Variance of Return: Estimated	Actual	Bias	t-Stat	Securities in Portfolio: Min-Max
				Most Risk-Averse Investor					
100	1.624	0.893	0.731		0.198	0.226	−0.028		10-21
5	1.461	0.899	0.562	−13.10	0.180	0.202	−0.022	2.83	24-31
2	1.296	0.896	0.400	−22.35	0.174	0.190	−0.016	7.13	52-57
1	1.125	0.900	0.225	−48.61	0.175	0.186	−0.011	8.32	101-106
⅔	1.010	0.918	0.092	−45.50	0.187	0.193	−0.006	8.75	150-154
½	0.891	0.955	−0.064	−36.57	0.214	0.215	−0.001	8.22	200-200
				Market Investor					
100	1.906	0.995	0.911		0.310	0.337	−0.027		7-13
5	1.631	0.985	0.646	−13.40	0.246	0.265	−0.019	2.14	22-27
2	1.404	0.957	0.447	−27.70	0.216	0.229	−0.013	2.64	50-55
1	1.206	0.949	0.257	−42.94	0.207	0.215	−0.008	4.82	101-104
⅔	1.054	0.944	0.110	−48.56	0.204	0.209	−0.005	5.73	150-152
½	0.891	0.955	−0.064	−50.11	0.214	0.215	−0.001	8.14	200-200
				Least Risk-Averse Investor					
100	2.149	1.096	1.053		0.502	0.522	−0.020		3-9
5	1.700	1.041	0.659	−11.86	0.298	0.315	−0.017	1.20	22-27
2	1.458	1.010	0.448	−25.67	0.257	0.269	−0.012	1.67	50-54
1	1.235	0.979	0.256	−42.90	0.228	0.236	−0.008	2.92	100-102
⅔	1.069	0.960	0.109	−49.50	0.215	0.219	−0.004	3.72	150-153
½	0.891	0.955	−0.064	−49.86	0.214	0.215	−0.001	6.32	200-200

the estimated portfolio variance is only 5 to 10% less than the true variance.

Table 2 also reports the t-statistics for paired t-tests of the null hypothesis that two adjacent upper bound constraints provide equally biased estimates of portfolio variance. Significant positive t-values indicate that the absolute value of the bias for the indicated upper bound constraint is less than the absolute value of the bias for the constraint immediately above.[5] As with the estimates of portfolio expected return, the bias in the estimated portfolio variances decreases in absolute value as the upper bound constraints are tightened. The bias in estimating portfolio expected return is highest for the least risk-averse investor and lowest for the most risk-averse investor. The reverse is true for the absolute value of the bias when estimating portfolio variances. This result is expected. Security investment weights are more heavily influenced by security expected return and less by security risk, the less risk-averse the investor.

Does the reduction in estimation risk translate into superior performance? This question can be answered by comparing the average certainty equivalent returns provided by the different strategies. Table 3

TABLE 3

Certainty Equivalent Returns
(Returns in Percent)

Investor	Upper Bound Constraint 100%	5%	2%	1%	⅔%	½%
Most Risk-Averse	0.083	0.173	0.216	0.234	0.225	0.182
Market	0.390	0.510	0.546	0.563	0.569	0.569
Least Risk-Averse	0.627	0.761	0.770	0.768	0.763	0.762

provides the average monthly certainty equivalent return in the twenty-five trials for each investor and strategy. For all investors, imposing an upper bound constraint of 5% provides a large incremental gain in certainty equivalent return. This gain is primarily the result of a reduction in portfolio risk. Tightening the upper bound to 2% is beneficial for all investors.

Whether further tightening is beneficial, however, depends on the risk aversion of the investor. Upper bounds of 1% for the most risk-averse investor, ⅔ of 1% for the market investor, and 2% for the least risk-averse investor provide the highest average certainty equivalent return.[6]

The "optimal" upper bound for each of the three investors reflects the opportunity costs of restricting the portfolio manager's ability to take advantage of information about true differences in security expected returns and risks. Upper bound constraints impose a relatively small cost on the market investor because the market investor's optimal portfolio contains the 200 securities. The gain in certainty equivalent return from constraining the individual security investment weights is 14.3 to 17.9 basis points per month, depending on the investor. This translates into gains in annual returns of 1.73 to 2.17%.

RESTRICTING SHORT SALES

Our analysis permitted neither short-selling nor buying on margin. Previous studies by Brown (1976), Frost and Savarino (1986a, 1986b), Jobson and Korkie (1980), and Jorion (1986) on the harmful effects of estimation error on portfolio performance have allowed unrestricted short-selling. Prohibiting short sales and buying on margin imposes a lower bound of zero on the security investment weights and limits the maximum amount that can be invested in any security to investor wealth.

Table 4 reports the impact of allowing short-

TABLE 4

Estimation Risk and Short-Selling

	Unrestricted		No Short Sales	
	Estimated	Actual	Estimated	Actual
Expected Return	212.90	5.68	1.906	0.995
Variance of Return	49.30	1881.20	0.310	0.337

selling by the market investor. Unrestricted short-selling causes skyrocketing estimation risk.[7] The average estimated expected portfolio return and variance are, respectively, 212.9 and 49.3%. In actuality, the average true portfolio expected return and variance are 5.68 and 1881.2%. The market investor will heavily overinvest in securities with favorable estimation error and short-sell securities with unfavorable estimation error, thinking to receive, on average, 37.5 times the true expected portfolio return at approximately 2.6% of that portfolio's true risk.

Disallowing short sales significantly reduces estimation bias. When short-selling is prohibited, the average estimated expected portfolio return and variance are 1.906 and 0.310%, respectively. In actuality, the corresponding average true expected return and variance are 0.995, and 0.337%. In the absence of short-selling, the market investor thinks that he or she can earn, on average, only 1.9 times the true expected portfolio return and at 92% of the portfolio's actual risk. The results summarized in Table 4 clearly suggest that research that has allowed unrestricted short-selling overstates the adverse effect of estimation error on portfolio selection.

CONCLUDING REMARKS

Constraints on the proportion of the portfolio that may be invested in any security can be justified when the actual risk and expected return of securities are unknown and must be estimated. In the context of a mean–variance model, imposing upper bounds on individual security investment weights both reduces estimation bias and improves performance. The benefit of upper bound constraints depends upon the level of measurement error in return and risk estimates.

In the extreme case, when available information effectively prohibits delineation of true differences between security returns, an investor should hold a passively managed index fund. At the level of estimation error in this study, Markowitz's (1959) mean–variance model with upper bound constraints selects portfolios that outperform an equally-weighted index fund. Even though the gain in certainty equivalent is small, the results are encouraging given that the security analysis uses only historical data.

[1] Assuming the existence of a risk-free investment, Jobson and Korkie (1980) provide strong empirical evidence that the estimates of portfolio expected return and variance are biased estimates of their true values. Frost and Savarino (1986b) show that biased estimation of a portfolio's expected return and variance causes portfolio performance to deteriorate, primarily through an increase in portfolio risk.

[2] See Cochran (1963) for a description of systematic sampling. The 574 ranked securities are divided into n strata, which consist of the first K dollars of market value (based on the stock price and the number of shares outstanding on December 31, 1985), the second K dollars of market value, and so on. K equals the total market value of the 574 securities divided by n. Many of the firms' security market values are sufficiently large to span several strata. The first security is selected by randomly picking a dollar value Y between 0 and K and then selecting the corresponding security in the first group. Thereafter, the procedure selects a security in each stratum with a cumulative market value within that stratum of Y dollars. If the security has been selected previously, no security is selected from that stratum. As a consequence, n must be set equal to 476 to yield 200 different securities.

[3] Average monthly returns on the S&P Composite Index and U.S. T-bills are geometric averages of annual returns obtained from Ibbotson and Sinquefield (1982).

[4] It is assumed that each investor's utility function is negative exponential. Under this assumption, each investor chooses the portfolio that maximizes expected utility

$$E[U] = -e^{-\frac{1}{\lambda}\left[1 + \mu_p - \frac{2}{\lambda}\sigma^2_p\right]},$$

where μ_p and σ^2_p represent the expected portfolio return and the variance of portfolio return. Investor risk aversion is characterized by the risk tolerance parameter λ. For an individual investor, the certainty equivalent return,

$$CER = \mu_p - \frac{1}{\lambda}[\sigma^2_p],$$

measures portfolio performance.

[5] The bias in the estimated variance is distributed as a modified chi-square with 239 degrees of freedom. With 239 degrees of freedom, the modified chi-square is approximately normal, so that a t-test on the average difference between the bias in the estimated portfolio variances is valid.

[6] We used a Wilcoxon Matched Pairs Rank Sign test to determine whether the median certainty equivalent return at the optimal upper bound constraint is different, at a 5% level of significance, from the median certainty equivalent returns given the other upper bound constraints. The differences are significant in all but two cases. For the market investor, we cannot reject the null hypothesis that the median certainty equivalent return with a ⅔ of 1% upper bound is equal to the median certainty equivalent return with a ½ of 1% upper bound. For the least risk-averse investor, we cannot reject the null hypothesis that the median certainty equivalent return with a 2% upper bound is equal to the median certainty equivalent return with a 1% upper bound.

[7] In one randomly selected trial, the minimum and maximum amounts invested in a security by the market investor were respectively, −10,389% and 8,161%.

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