

Large-scale Time-Varying Portfolio Optimisation using Graph Attention Networks *

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Abstract

Apart from assessing individual asset performance, investors in financial markets also need to consider how a set of firms performs collectively as a portfolio. Whereas traditional Markowitz-based mean-variance portfolios are widespread, network-based optimisation techniques have built upon these developments. However, most studies do not contain firms at risk of default and remove any firms that drop off indices over a certain time. This is the first study to incorporate risky firms and use all the firms in portfolio optimisation. We propose and empirically test a novel method that leverages Graph Attention networks (GATs), a subclass of Graph Neural Networks (GNNs). GNNs, as deep learning-based models, can exploit network data to uncover nonlinear relationships. Their ability to handle high-dimensional features and accommodate customised layers for specific purposes makes them particularly appealing for large-scale problems such as mid- and small-cap portfolio optimization. This study utilises 30 years of data on mid-cap firms, creating graphs of firms using distance correlation and the Triangulated Maximally Filtered Graph approach. These graphs are the inputs

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to a GAT model that we train using custom layers which impose weight and allocation constraints and a loss function derived from the Sharpe ratio, thus directly maximising portfolio risk-adjusted returns. This new model is benchmarked against a network characteristic-based portfolio, a mean variance-based portfolio, and an equal-weighted portfolio. The results show that the portfolio produced by the GAT-based model outperforms all benchmarks and is consistently superior to other strategies over a long period while also being informative of market dynamics.

Keywords—

Portfolio optimisation; mid-caps; correlation networks; distance correlation; filtered graphs; deep learning; graph attention networks

1 Introduction

Portfolio optimisation is crucial in modern risk management, as performance correlations between firms in a portfolio bring unforeseen risks. Given that every investor’s risk profile differs, the portfolio construction model must also account for different objectives. One such model, based on the classic work by Markowitz (1952, 1959), is mean-variance optimisation, where returns are optimised while volatility is minimised. In this paper, too, we look to optimise with a mean-variance objective, but we do so over a large set of firms that could also default or go bankrupt. This problem must be solved for an investable universe that is expanding, as financial markets continue to develop, and emerging and private markets are becoming increasingly accessible. All these assets have different risk and liquidity profiles. Against this backdrop, which firms to select and what proportion of capital to allocate to each is an increasingly high-dimensional problem.

The classical mean-variance measure is not without its problems, and portfolios optimised using it have been shown to have poor out-of-sample performance (Siegel and Woodgate, 2007). Assumptions about the normality of returns, absence of transaction costs and presence of regimes in markets make the classical model difficult to implement (Guidolin and Ria, 2011). Even if these assumptions are fulfilled, the expected mean of the portfolio and the covariance matrix cannot be easily estimated as they are not observed in practice, which means that, instead, the sample mean and covariance matrix are commonly used (Ao et al., 2019). Furthermore, it is challenging for such models to cope with high dimensionality, which is a common characteristic of modern portfolios (DeMiguel et al., 2009b). To better address these challenges, different methods

to solve the portfolio optimisation problem continue to be developed, borrowing from different techniques in other domains, such as fuzzy programming (Arenas Parra et al., 2001), cluster analysis (Puerto et al., 2020), quantum annealing (Venturelli and Kondratyev, 2019) and deep reinforcement learning (Shi et al., 2022).

Of particular interest to this work are topological or network studies for portfolio optimisation (Pozzi et al., 2013; Li et al., 2019). Network models exploit graph data structures to identify relationships that are impossible to detect by Euclidean data-based models. In our case, the network nodes are the firms, and each edge measures the relationship between the two firms. More formally, the network at a given time t is represented as an undirected graph $\mathcal{G} = (\mathcal{V}_t, \mathcal{E}_t)$ where \mathcal{V}_t are the nodes or firms and \mathcal{E}_t is the set of edges, often represented by an adjacency matrix A of dimension $|\mathcal{V}_t| \times |\mathcal{V}_t|$.

The aforementioned network studies consistently find that allocating capital to firms in the peripheries of the networks produces better returns, due to low correlations with other parts of the network. Similarly, to produce a diversified portfolio, mean-variance models also tend to prefer firms in the peripheries of the network (Onnela et al., 2003). However, studies using the former methods have only been deployed to small portfolios or were limited to a specific sector. Here, we look to extend these topological analyses to the whole market of US mid-cap companies, a much more challenging and realistic problem setting.

Mid-cap firms (in short ‘mid-caps’) are defined as firms with a 1 to 10 billion USD market capitalisation and are likely constituents of the Dow Jones Wilshire Mid-cap or S&P 400 Mid-cap indexes. Modelling the performance of these firms is complicated by the illiquid and sporadic nature of trading, which makes their return distributions non-normal (Castellano and Cerqueti, 2014). They are also far more numerous compared to large-cap firms, which makes mid-caps unsuitable for analysts to cover. However, as they behave as a separate autonomous asset class, they can further improve the diversification aspect of a portfolio if included (Petrella, 2005). Over the long term, mid-caps also provide a premium in return for the same risk, which is desirable for any portfolio seeking financial returns (Ge, 2018). The indices of such firms contain several constituents, making simple index creation strategies, such as the popular market-weighted methodology of the Russell 2000 Index (or S&P 500), suboptimal for investors. As investors have different horizons and risk tolerance combined with churn in index constituents leads to under performing the market (Cai and Houge, 2008; Cremers et al., 2020). A comprehensive and intensive approach is needed to generate

portfolio weights for such firms that yield a better risk-adjusted return. Another area of practical interest that our study is applicable to, would be in devising new automated ETF strategies for such companies. Large-scale portfolio optimisation with the strategy constraints of the particular ETF needs to be modelled.

Studying the correlation of firms is an integral part of any portfolio optimisation procedure. Pearson correlation approaches, generally used by mean-variance models, can only capture linear dependencies and pairwise correlations. In this work, we deploy the distance correlation measure (Székely et al., 2007). This can capture non-linear relationships between pairs of firms. Sun et al. (2019) compared distance correlation with Pearson correlation for portfolio optimisation and found that the distance correlation strategy performs well. The portfolios in most studies, however, do not have a natural churn of firms which, we believe, makes distance correlation even more useful and deploy this for our data. Also, we optimise the metric further for a large-scale portfolio. As mid-cap companies are illiquid when compared to large-caps, the available trading history may be shorter, less uniform or missing for some period of time. The distance correlation measure can handle such time series features and still produce a quantitative measure of relationship between firms. This allows us not to drop any firms, which mitigates any selection bias. We use the covariance matrix generated by distance correlations to generate a fully connected network of firms. Furthermore, we deploy the Triangulated Maximally Filtered Graph (TMFG) method introduced by Massara et al. (2017) to filter the dense correlation matrices generated by the distance correlation measure. This process generates a filtered network with fewer edges without loss of information and represents the state of relationships between firms. Deep learning techniques further build on these representations and using all the historical data can create more complex higher-order representations.

Once data from stock prices and networks is included, traditional methods fall short on handling the complex resulting data structure. Deep learning techniques however excel in this area. They have produced state-of-the-art results in several domains, such as speech recognition, natural language processing, object detection, drug discovery, and genomics (LeCun et al., 2015). As they do not impose restrictions on data distribution and handle exotic data types by design, they are applicable in a wide range of situations. This suggests that deep learning methods may be well-suited to mid-cap with high default rates and reclassifications into large-caps, their joint distribution of returns, or other temporal features that may not be suitable for classical

models. Deep learning models also scale well to big datasets with numerous features. They can also find optima with constraints for different problems, which makes them again suitable for large-scale portfolio optimisation which involves similar objectives with constraints, as a recent study on midcap default prediction has shown (Korangi et al., 2022). In this paper, we propose a particular class of deep learning networks, Graph Neural Networks(GNN), which can find higher-order representations taking the relationships between various firms into account. This is an important and valuable feature for portfolio optimisation as the relationships between firms change over time and also behave differently in different macroeconomic environments. The optimising procedures of deep learning techniques are particularly suitable in a higher-dimensional space, which also supports its use in this problem real. They can also be trained to any complex objective function and are flexible in their output, which we exploit to design a model that delivers optimal weights for a large set of mid-cap stocks. The distance correlation and TMFG filtering provide an important step for the GNN models where a suitable initial relationship between firms is provided, which the deep learning models build on to find higher-order relationships.

Portfolio optimisation often involves estimating the expected returns and volatility of the set of stocks, or its corresponding covariance matrix, and then using a constrained optimisation method to find the asset allocation weights that maximise the portfolio objective. Topological studies have shown that network structure plays a vital role in this process. In this study, instead of relying on just a few measures, as done in previous studies, we allow the graph neural network to distil information from the relationships to produce the portfolio weights. We use the widely applied Sharpe ratio, i.e. the ratio of returns over volatility, to measure the portfolio’s performance (Sharpe, 1966). The models are tuned to predict the Sharpe ratio directly without predicting returns separately as (Zhang et al., 2020) did. Training the networks is also based on the same measure, using a custom loss function derived from the Sharpe ratio. We are the first ones to deploy Graph Attention networks (GATs) for portfolio optimisation in a computationally efficient manner, allowing different aspects of the network structure to be learnt at scale (Veličković et al., 2018). One of the important aspects of this study is the dynamic nature of the problem, the set of active mid-cap firms varies over time and so the resultant graphs are dynamic. Hence, the output, i.e. the weights allocated to each firm, varies with each graph. This further complicates the problem compared to most portfolio studies, which often assume a constant number of firms. This requires additional data preprocessing and a suitable definition of the problem for training.

We use a different graph for each forecast period and different returns data for training, validation, and testing. Using a rolling window approach, we move the forecast period; this changes both the graph inputs and returns data.

Therefore, the three key research questions addressed in the paper are the following:

1. Can an effective network topology be constructed from sparse historical data on a large collection of firms?
2. Are graph attention networks able to generate higher-order representations of this network to construct an optimal portfolio for mid-caps?
3. How does the model perform under different market conditions, and can we infer useful strategies from the model results?

In so doing, the paper makes three main contributions. First, we develop topological portfolio optimisation models, extending the literature that hitherto focused chiefly on stochastic models for the segment, applying our result to the challenging set of mid-cap firms. Second, we optimise distance correlation measures, use TMFG to generate large-scale networks, and describe the complex interactions among these firms. Finally, we use these networks as inputs to graph-based deep learning models that are capable of producing portfolio weight for the large number of firms with which we are faced.

The remainder of the paper is organised as follows. Section 2 reviews the relevant literature, focusing on large-scale portfolio optimisation, graphs and graph-based deep learning models. Section 3 describes the data and the process by which this data is converted into graphs and relevant measures used in the study. The proposed models, and the baseline models against which they are compared, are described in Section 4. Section 5 discusses how we set up our experiments, including the data flow from input to output, the loss function used, and other key experimental design choices. Section 6 then presents the results of the various models applied to different data types. Finally, Section 7 summarises the contributions and suggests future steps.

2 Related literature

In this section, we present the relevant literature for the study, focusing on large-scale portfolio optimisation studies and GNNs.

2.1 Correlation Networks and Portfolio models

Portfolio optimisation has been the subject of a large body of research, and novel methods, with various constraints and objectives, continue to be developed (DeMiguel et al., 2009a; Branch et al., 2019). Of particular interest to our work are studies that focus on optimising large-scale portfolios effectively. Perold (1984) was the first to do so, by considering the specific nature of dense covariance matrices, and recommending strategies to make them sparse so they become computationally feasible. The first branch of large-scale portfolio optimisation studies follow a similar approach in devising algorithms to efficiently use computational resources, such as computing time and memory.

Later studies on large-scale portfolio optimisation focus on generating weights for the firms and measuring the model portfolio performance against metrics such as the Sharpe ratio. Bonami and Lejeune (9 06) proposed an exact solution approach for a portfolio of up to 200 firms by fixing the expected returns with a probabilistic confidence level and imposed several trading constraints to extend the mean-variance framework. On a portfolio of 100 firms within the S&P 500, Ao et al. (2019) converted mean-variance portfolio optimisation into a sparse linear regression, and combined them with Fama-French factors to study portfolio performance. Liu (2017) used a combination of efficient computing resources with evolutionary algorithms for portfolio optimisation. Bian et al. (2020); Dong et al. (2020) applied regularisation methods for portfolio optimisation as, without such methods, a large portfolio would lead to overly small allocations; they also estimated these covariance matrices using factor models depending on size, book-to-market values and industry sector, and found regularization helped to arrive at better optima. Performance-based regularization where the sample variance is restricted was also found to perform better on portfolios from Fama-French data sets (Ban et al., 2018). Recently, Bertsimas and Cory-Wright (2022) studied the size of portfolios of previous large-scale portfolio optimisation studies and proposed a ridge regression-based regularisation algorithm which could theoretically scale to a portfolio of over 3000 firms by converging to a solution. However, we argue that the portfolio scale considered thus far is still not large enough when using real data or otherwise studies use simulated data to measure the efficacy of portfolio optimisation algorithms. Additionally, in all these studies, the universe of firms or assets to select from is always constant, thus eliminating firms that default, get acquired or liquidate, eliminating a large driver of idiosyncratic portfolio risk. Instead, we take a dynamic approach and allow the portfolio to vary over time, allowing us to use real data and use

them on smaller firms instead of focusing on large firms with readily available data. Our approach mitigates some of the selection bias and is closer to a real-world scenario where investors want to invest in a certain sector or asset class of the market, and would like to make an optimal decision considering all the firms available within that asset class.

In order to create a sparse covariance matrix, another alternative explored in the literature is information filtering using graphs or network data. Information filtering in large data sets by building sparse networks is an active area of study in various domains such as internet search (Xie et al., 2018), social networks (Berkhout and Heidergott, 2019) and, same as our paper, finance (Fan et al., 2013). Specifically, to optimise the portfolio, Onnela et al. (2003); Cho and Song (2023) showed that investing in the peripheries identified by the graph provided benefits for portfolio diversification. They used a static slice of S&P 500 companies, starting at a larger base than previous studies, but the firms are unchanged over the long time frame of 20 years they studied. In the area of information filtering of large-scale networks there continue to be major improvements, such as social network analysis for link prediction (Zhang and Chen, 2018), recommender systems (Fan et al., 2019), and the study of object interactions in complex systems (Battaglia et al., 2016). The goal of these studies is to maintain the most relevant information by constraining the topology of the graphs. The Minimum-spanning tree is a filtering mechanism for dense graphs that keeps the edges with the highest weights and allows no cycles or loops in the graph (Mantegna, 1999). Planar Maximally Filtered Graph (PMFG) uses a different constraint on the graph's topology, requiring it to be planar; i.e., there should be no edge crossing on a plane (Tumminello et al., 2005). They are found to be robust for financial market networks as market conditions change without losing much information content (Yan et al., 2015). Triangulated Maximally Filtered Graph (TMFG) is a more computationally efficient algorithm since it can be parallelized compared to PMFG (Massara et al., 2017). In this work, we adapt this methodology for correlation networks of stocks, making them suitable for the large datasets we work with.

As we mentioned earlier, we use the distance correlation measure to account for the strength of the relationship between firms. Diebold and Yilmaz (2012, 2014) developed the connectedness metric, using VAR (Vector Auto Regression) decomposition methods, to quantify the systemic relationship between firms. They can identify clusters and central firms crucial to networks and quantify the direction of risk spillovers. However, for large networks such as our mid-cap firms, these methods have limitations due to the amount of historical data they need. Given these

limitations, we instead used distance correlation measures for pairs of firms and TMFG for filtering, to provide the sparse network which serves as the input to our deep learning-based models.

2.2 Graph Neural Networks

GNNs were first proposed by Scarselli et al. (2009) for node classification tasks. Similar to recurrent neural networks (RNNs), they employ recursion, which is used to learn higher-order representations for a node from its neighbours. As the deep learning field for Euclidean data evolved with RNNs for sequential data, Convolutional Neural Networks (CNNs) for primarily image processing, and Attention-based models Vaswani et al. (2017) for spatial analysis of unstructured data, non-euclidean data models based on GNNs also developed in parallel. Graph Convolutional Networks (GCNs) borrow concepts from CNNs, such as kernel filter size and stride, to generate representations for graphs (Kipf and Welling, 2017). They produced state-of-the-art results on popular graph datasets such as citation networks and knowledge graphs over semi-supervised or skip-gram-based graph embeddings, label propagation and regularization approaches. Graph Attention Networks (GATs) further improved on these results by introducing variable aggregation of neighbours, and they also proved successful in transductive learning tasks where the data is not fully labelled (Veličković et al., 2018). We refer the reader to a comprehensive survey by Wu et al. (2021) for a general introduction to GNNs and their various flavours. Thus far, GNNs have been applied in some application areas related to finance or financial markets. For example, in consumer finance, they were used for fraud detection (Xu et al., 2021) and credit risk prediction (Óskarsdóttir and Bravo, 2021) in consumer finance. Avramov et al. (2023) applied other deep learning techniques for listed NYSE firms and found it can produce complex signals that are profitable but become costly with high turnover. Feng et al. (2022) used a variant of GNNs and applied self-attention separately for the stock recommendation out of 738 stocks which is to predict top-3 stocks for the next period. We could have given the fully connected graph as an input to the GAT model. However with such high-dimensionality with an average 6000 companies to look at every quarter, the training would have been lot complicated but by taking advantage of econometric-based preprocessing of data to extract initial relationships we initialize the model with better input. Overall, we hypothesise the deep learning methods are better deployed at large scale with illiquid, risky firms than on a small set of established firms, as they are better at extracting complex relationships and with a lower churn the portfolio which reduces transaction costs, but need large data to arrive at a stable

solution.

3 Methods

In this section, we describe the data used for this study, the distance correlation measure and the graph filtering algorithm, which all together create the inputs for the GNN and other benchmark models.

3.1 Data

We collected the daily closing prices of all mid-cap companies listed in the US over 30 years, from 1990 to 2021. There are approximately 20,000 firms active for at least part of this period. For portfolio selection, we use a three-year rolling window, so the portfolio allocation problem is limited to around 5,000 firms at any given time point. This number changes substantially over time, however, due to firms entering and leaving the market. We convert the prices to daily returns for each firm, defined as r_{ut} for a firm u at time t . These return series are sequentially divided into a training (50%), validation(25%), and test set(25%), and serve as the main input to our models.

To create the relationships between firms, we also calculate the return-volatility series using the standard deviation of these returns and a 30-day look back period. For a firm u at time t it is defined as $c_{ut} = \sigma(r_{ut}, r_{ut-1}, \dots r_{ut-30})$.

The return-volatility series is input to calculate the covariance matrix between firms using distance correlation. We used a three-year look back period between a pair of firms to derive the covariance matrix. This look back period allows for a relatively new firm to also be part of the portfolio, as mid-cap firms generally have some previous trading history as small-cap firms. The return-volatility series also has interesting features compared to return series, such as clustering where volatilities are asymmetric with short periods of co-movements particularly during risk-off situations in the market, while during benign times markets show more independence in relationships. This series also has strong serial correlation, and also it tracks market investors sentiment (e.g.: the volatility index, VIX), which are essential for identifying crises (Huang et al., 2019). These are the situations where the performance of different portfolio allocation strategies can diverge substantially.

We define the universe of firms as \mathcal{V} . We sample them over N periods where $\mathcal{V}_t \subseteq \mathcal{V}$ denotes

the set of firms active at time t , $t = 1, 2, \dots, N$. We use a lookback period of T (3 years) for network creation over the set of firms \mathcal{V}_t . Each firm $u \in \mathcal{V}_t$ has a daily return series,

$$x_u = [r_{u1}, r_{u2}, \dots, r_{uT}] \in \mathbb{R}^T, \quad (1)$$

and a daily return-volatility series,

$$l_u = [c_{u1}, c_{u2}, \dots, c_{uT}] \in \mathbb{R}^T. \quad (2)$$

To prepare the input to the models, we stack the individual return series from (1) for all firms in \mathcal{V}_t to get the features $X_t \in \mathbb{R}^{|\mathcal{V}_t| \times T}$

$$X_t = [\mathbf{x}_{1t}, \mathbf{x}_{2t}, \dots, \mathbf{x}_{ut}, \dots, \mathbf{x}_{|\mathcal{V}_t|t}]^\top \quad (3)$$

where \top denotes the transpose of a matrix and \mathbf{x}_{ut} denotes the vector of returns of firm u at time t .

3.2 Distance correlation

We quantify the strength of relationships between two firms using the distance correlation measure. In this section, we describe this measure. Distance correlation is a generalised measure of dependence, which captures non-linear dependencies and performs well in domains such as signal processing (Brankovic et al., 9 12) and computational biology (Mendes and Beims, 2018). The distance correlation using the estimation procedure is defined below, $dcor(u, v)$ between two firms using the volatility series (l_u, l_v) defined earlier in (2). For each firm, we get the absolute change in volatility over all periods and scale these changes by removing the averages to get firm-level scaled change matrices, which are compared with other firm-level matrices to derive the distance correlation measure. First, we define two matrices A and B for a pair of firms u and v .

$$a_{i,j} = \|l_{ui} - l_{uj}\|$$

$$b_{i,j} = \|l_{vi} - l_{vj}\|$$

where $i, j = 1, 2, \dots, T$, and $\|\cdot\|$ is the Euclidean distance. This captures times when the firm has

higher or lower volatility compared to each time period. By capturing across all times, we get a quantified value of the firm's volatility changes over the time period we are measuring. This sets up a comparison with another firm over the same period, for that, we further define two matrices A' , B' that normalise the matrices A , B . This normalisation would make the values comparable between two firms with different risk characteristics.

$$A'_{i,j} = a_{i,j} - a_{\cdot j} - a_{i\cdot} + a_{\cdot\cdot}$$

$$B'_{i,j} = b_{i,j} - b_{\cdot j} - b_{i\cdot} + b_{\cdot\cdot}$$

where $a_{\cdot j}$ is the mean across rows and $a_{i\cdot}$ is the mean across columns and $a_{\cdot\cdot}$ is the mean across all values in matrix A . The distance covariance $dcov(l_u, l_v)$ is an average of the element-wise multiplication of A' and B' .

$$dcov(l_u, l_v) = 1/T^2 \sum_{i=1}^T \sum_{j=1}^T (A'_{i,j} \times B'_{i,j})$$

$$dcor(l_u, l_v) = dcov(l_u, l_v) / \sqrt{dcov(l_u, l_u) \times dcov(l_v, l_v)} \quad (4)$$

This measure has some valuable properties that are relevant to our problem Székely et al. (2007).

1. $dcor(l_u, l_v) = 0$, if and only if l_u and l_v are independent.
2. $0 \leq dcor(l_u, l_v) \leq 1$, unlike Pearson correlation, which captures the linear dependence as a number between -1 and 1. This is useful because we are interested in the strength of the dependence rather than the direction of the dependence.
3. The measure can produce a value for two series of unequal lengths. Given that firms can drop in and out of the portfolio, resulting in different histories, this feature allows one to nonetheless consider the relationship between them.

A naive implementation of distance correlation calculation has $O(n^2)$ time complexity for a pair of firms, where n is the amount of look back period. There are optimisation methods to implement this in $O(n \log n)$ between two series, where the main problem is to calculate over many firms over different periods (Huo and Székely, 2016). Here, we implement it in a distributed architecture

using mix of GPUs and CPUs. The implementation is parallelized among many firms, which allows the calculation of such large correlation matrices(using SLURM). The resulting code can handle the problem we study in a reasonable time, as discussed in the results section.

The distance correlation between the firms in \mathcal{V}_t produces a dense dependence matrix $D_t \in \mathbb{R}^{|\mathcal{V}_t| \times |\mathcal{V}_t|}$ where the u^{th} row, v^{th} column element of D_t is $dcor(l_u, l_v)$. This matrix leads to a complete graph where every set of firms has an edge regardless of the strength of the dependence between the firms. Some studies have used threshold conditions to remove weaker edges, but the choice of threshold value is arbitrary, and the composition of our portfolio changes over time, further complicating using a global threshold mechanism. Instead, we use a more advanced technique from graph theory to filter D_t and remove weaker connections.

3.3 Graph Filtering

The filtering technique we chose is the Triangulated Maximum Filtered Graph (TMFG) method proposed by Massara et al. (2017) which, like PMFG, imposes a planarity constraint on the graph but is more scalable to larger datasets such as ours. Using the topological features of the graph as a constraint, a planar graph retains most of the information with fewer edges. A graph is planar if no edges cross between the nodes on a sphere. Such graphs have attractive features, making, for example, cluster or community detection easier, which reduces the information need and thus making large graphs tractable for analysis.

We define the dense graph before filtering as $\mathcal{K} = (\mathcal{V}_t, F_t)$, where F_t is the adjacency matrix. We let an element $f_{uv} \in F_t$ take the value

$$f_{uv} = \begin{cases} 1, & \text{if } d_{uv} > 0 \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

TMFG along with the adjacency matrix F_t uses the distance correlation values from Equation (4) to filter the graph \mathcal{K} into a sparse planar subgraph $\mathcal{G} = (\mathcal{V}_t, \mathcal{E}_t)$ that is maximal, i.e. the sum of edge weights is maximum. Planarity constraints reduce the edges from $|\mathcal{V}_t|(|\mathcal{V}_t| - 1)/2$ in \mathcal{K} to $(3(|\mathcal{V}_t| - 2))$ in graph \mathcal{G} .

The TMFG algorithm first identifies a clique of four firms with the highest distance correlation measure. A clique is when all the distinct vertices in the sub-graph have an edge between them, i.e, all members of the clique are connected. In our volatility networks, this happens in most

cases, as many firms are correlated. The score function is a function that represents the strength of connections between firms; in this paper, we have chosen the sum of the weights of a pair of firms, obtained from D_t . Of all the remaining firms outside the sub-graph in \sqsubseteq_t , a firm is added to the sub-graph if the new firm produces a higher score than any other firm. Efficiency comes from reduced computational complexity by maintaining a cache of possible combinations of these sub-graphs and tracking them. Finally, the sub-graph $\mathcal{G} \subseteq \mathcal{K}$ is created.

The graph \mathcal{G} serves as input to the deep learning models and graph-based portfolio optimisation models. A sample graph is presented in Figure 1. The planar filtering technique also helps us to visually present these graphs. It shows the complexity of the network with few central node networks and a large periphery for one period. As we roll over, we construct such graphs for every time period.

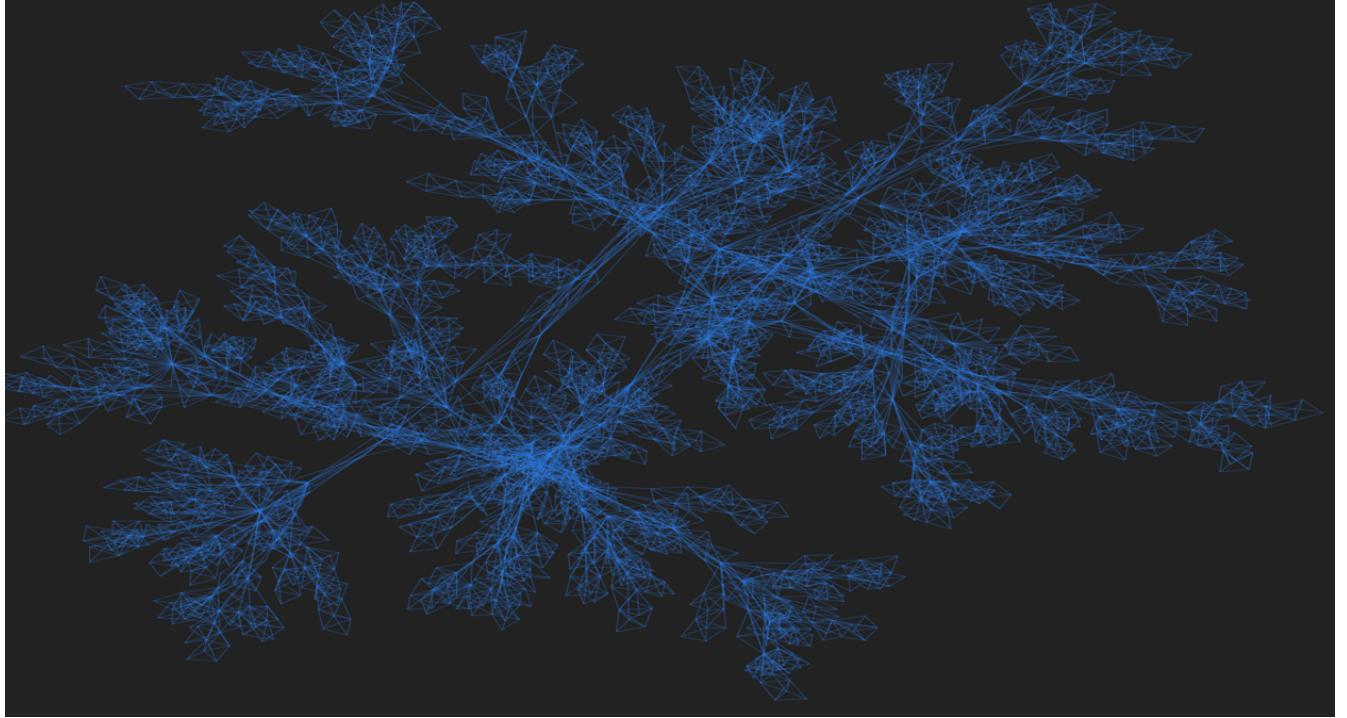


Figure 1: Snapshot of network of mid-cap firms where nodes are all still connected after the TMFG filtering.

3.4 Network measures

The graphs created after TMFG filtering could be used directly for portfolio optimisation, as some studies (Li et al., 2019; Pozzi et al., 2013) have done. We calculate a score for each node in the graph $\mathcal{G} = (\mathcal{V}_t, \mathcal{E}_t)$. This score is transformed as the weight assigned to the firms in the portfolio

through the method that follows.

We define the score for a firm $u \in \mathcal{V}_t$ as p_u , an average of the three measures of the centrality of the networks, the degree centrality, betweenness centrality, and closeness centrality. These measures have been studied since the 1970s (Freeman, 1977) and are still popular in the network literature (Óskarsdóttir and Bravo, 2021). The first measure, degree centrality, is the fraction of nodes to which the node u is directly connected to. We count all the nodes adjacent to node u and normalise it by the total number of nodes $|\mathcal{V}_t| - 1$. These set of nodes are also the neighbours, \mathcal{N}_u , for node u . Formally, we define neighbours as

$$\mathcal{N}_u = \{v | (u, v) \in \mathcal{E}_t\} \quad (6)$$

That is, nodes v that have an edge connecting them to u . This neighbourhood definition is also useful for the description of GNN models, as we will see in Section 4.1 where we define the GAT model.

The degree centrality dc_i for firm i is defined as

$$dc_i = |\mathcal{N}_u| / (|\mathcal{V}_t| - 1) \quad (7)$$

or the total number of neighbours divided by the total number of nodes minus one.

The betweenness centrality measure specifies the activity that passes through the graph node when any changes occur in the network. To measure this activity level, we first compute the shortest paths between all pairs of nodes. Betweenness centrality is the fraction of the shortest paths that pass through a node that does not include the given node. For a firm i , this is defined as

$$bc_i = \sum_{u,v \in \mathcal{V}_t} \frac{s(u, v|i)}{s(u, v)} \quad (8)$$

where $s(u, v)$ is the number of shortest paths between (u, v) and $s(u, v|i)$ is the number of shortest paths that pass through i . $s(u, v|i) = 0$ if $i = u$ or $i = v$ and $s(u, v) = 1$ if $u = v$.

The final centrality measure we use is closeness centrality cc_i . It is the reciprocal of the average shortest paths to all the nodes that are reachable from a given node. Higher scores meaning higher centrality.

$$cc_i = \frac{(n - 1)}{(|\mathcal{V}_t| - 1)} \frac{(n - 1)}{\sum_{u=1}^{n-1} d(u, i)} \quad (9)$$

where n is the number of nodes reachable from i and $d(u, i)$ is the shortest distance between (u, i) . In our setting, the TMFG graph still connects all nodes, so all nodes are reachable from one to another and $d(u, v) = 1$ when there is a direct edge. The distance, however, varies, providing the desired closeness measure.

We use the networkx library implemented by Hagberg et al. (2008) to calculate these measures. From them, we calculate the peripherality score for each firm as defined earlier.

$$p_i = (dc_i + bc_i + cc_i)/3 \quad (10)$$

This measure translates into weights for portfolio optimisation in the network model discussed in section 4.2.

3.5 Model performance and loss metric

We adopt the Sharpe ratio as the final performance measure for the models (Sharpe, 1966). This is a well-studied metric to measure portfolio performance. There are several studies in the portfolio performance literature on further refinements or limitations of the Sharpe ratio (Lo, 2002; Farinelli et al., 2008).

Here, we use the widely accepted form of the metric. From equation (1) for an individual firm u , the Sharpe ratio is estimated from the sample mean and variance of the returns. This same method applies to portfolio returns as well.

$$\hat{\mu}_u = \frac{1}{T} \sum_{i=1}^T r_{ui}$$

$$\hat{\sigma}_u^2 = \frac{1}{T} \sum_{i=1}^T (r_{ui} - \hat{\mu})^2$$

From these quantities, the Sharpe ratio \widehat{SR}_u can be easily estimated

$$\widehat{SR}_u = \frac{\hat{\mu}_u - R_f}{\hat{\sigma}_u}$$

where R_f is the risk-free interest rate. For ease of calculation, we normalize it to zero as it is a constant baseline for all. So, the Sharpe ratio produced cannot be compared to external studies on the Sharpe ratio, but the results are comparable between the models in this study. We use this

form of Sharpe ratio after training to test the performance of models and report this measure over time for all models. For training, we take a modified approach outlined below.

Most supervised deep learning models have a prediction target, but in our problem, we do not aim to predict but to provide a score for each firm, the normalised portfolio allocation. This has implications for training and measuring portfolio performance. We use the Sharpe ratio as a loss metric during training, which the deep learning models look to minimise. A similar approach was used by Zhang et al. (2020) where they used gradient ascent and a differentiable Sharpe ratio. The Sharpe ratio could be expressed as a constrained optimisation problem where the expected returns are to be maximised while minimising the volatility. We use the negative logarithmic Sharpe ratio ($-\ln SR$) as the loss metric. The loss function is thus expressed as

$$LF = -\ln \hat{\mu} + 2 * \ln(\hat{\sigma}_u) \quad (11)$$

4 Models

4.1 Graph Attention Networks

GNNs are applied directly to graph data and successfully exploit the structure within the data for various tasks, such as node classification and edge prediction. In domains with natural graph data, such as protein structures in chemistry or biology, citation networks, or recommendation systems, they produce state-of-the-art results even when compared to other large-scale deep learning models (Ying et al., 2018; Gilmer et al., 2017). Graph convolutional networks were developed along with convolutional neural networks used for image processing. Unlike images with strict structure, graph data are dynamic and have much more complexity and meaning between connections. Graph convolutional networks generate a higher-order representation of input features and neighbours by using the neighbourhood degree centrality to weigh the neighbour's features. As degree centrality is a fixed measure, more complex weighing mechanisms are impossible. Graph Attention networks solve this problem by using the self-attention mechanism, creating learnable parameters to generate the weights for neighbours, making them more dynamic in learning from the neighbourhood features (Veličković et al., 2018). Large graphs such as the mid-cap graph we create in this paper are dynamic and show significant evolution as time passes. Also, the market conditions vary for each time period, meaning fixed weighing of neighbours may not perform well

with these changes. Added to this, the Attention operations are more efficient than alternative approaches, since they can be parallelizable across node neighbour pairs.

Here, we formally define the GAT specific to our problem. Given a graph $\mathcal{G} = (\mathcal{V}_t, \mathcal{E}_t)$ as defined in section 3.3, and the input features defined in (3), GAT transforms the input features X_t into a higher-order representation H_t given by

$$H_t = [h_1, h_2, \dots, h_u, h_v, \dots, h_{|\mathcal{V}_t|}]^\top \quad (12)$$

where $h_u \in \mathbb{R}^{T'}$

Specifically, for a given firm u , the transformation function from x_u to h_u is where each variant of GNN differs.

$$h_u = \parallel_{k=1}^K F(x_u, \Sigma_{v \in N_u} a(u, v)(Wx_v)) \quad (13)$$

N_u defined in equation (6) are the neighbours of firm u . $W \in \mathbb{R}^{(T' \times T)}$ is a weight parameter matrix of the model that is learnt during training. $a(u, v)$ is the weighted importance score of firm v on firm u . A softmax layer does the weighing after a feed-forward network over each neighbour firm v with the present firm u . The function $a(u, v)$ is the one that causes the differences, where it is a convolutional function for GCNs and attention for GATs.

$$a(u, v) = \text{softmax}_v(\sigma(\mathbf{a}[Wx_u || Wx_v])) \quad (14)$$

where σ is a non-linear function, LeakyRelu in this case. $\mathbf{a} \in \mathbb{R}^{2T'}$ and \parallel is the concatenation operator.

$F(\cdot)$ also applies the non-linearity function (usually a Regularized Linear Unit, ReLU) after aggregating all weighted outputs. GAT also allows for multi-head attention, where each head learns a different aspect of the input. For a K -multi-head attention, we concatenate each head's outputs to construct the final representation. We used $T' = 24$ and $K = 8$ in our experiment.

The H_t function is a higher-order representation. We scaled these representations to lower dimensions to make them suitable for predicting weights for the portfolio. We apply batch normalisation, which normalises the feature inputs for the next layer, and use two feedforward networks with dropout to scale the representation as we want to generate a single number for each firm finally. The dropout process helps to improve the stability of the training while reducing complexity. The final feedforward network also uses L1, or LASSO, regularisation to further shrink

and eliminate unnecessary weights. These are applied to each node separately. The output of this layer is s , or a score for each node, used for the portfolio allocation later.

$$s_{iu} = \sigma'(W_{iu} * h_u + b_i)$$

where $i = 1, 2$ and W_i and b_i are the trainable weight and bias terms, with σ' being a nonlinear function (normally ReLU). The final matrix $W_2 \in \mathbb{R}$ is the score that represents the embeddings for all firms in the graph

We developed a new allocation layer to normalise these scores, generate the final weights, and meet the weight constraints. To generate weights, softmax has been the standard solution for such a situation. However, given these are the weights for a large-scale portfolio, softmax generates tiny weights for large numbers of firms, which is impractical and can bring high transaction costs. We choose not to have tiny allocation to firms and concentrate the portfolio in fewer positions, which makes the portfolio manageable at lower costs. To alleviate this problem, we also considered sparsemax, which generates a sparse output that is more suitable to our application scope (Martins and Astudillo, 2016). We develop a reduce-weight mechanism that generates the final weights

$$w_u = \frac{s_{2u}}{\sum_{v=1}^{|V_t|} s_{2v}}$$

We found our reduce-weights mechanism more stable and also simple to implement, followed by softmax and sparsemax. The training loss function reduction is smoother and consistent, whereas the other training mechanisms generated a more noisy convergence, where the loss over the epochs is more volatile. This allocation layer could be easily extended to meet other portfolio constraints (e.g., choosing top K firms).

We use the standard GATs with its original parameters where possible and implemented them using the [NAME] package developed by Grattarola and Alippi (2021). The full model can be seen shown in Figure 2. The data embeddings are shown in colour and the deep learning layers have no fill in the Figure. The GAT layer takes the graph as input. Each node in the graph also has the return series as node features. For each node in the graph, the GAT model generates an embedding which is further processed by Dense layers with non-linear (Relu) activation, dropout and L1 regularisation. This regularisation applies a penalty for too many small positions in the

portfolio and makes the portfolio more manageable and comparatively inexpensive to run, as previously discussed. These scores are then converted to portfolio weights in the importance layer, which collects all the scores from earlier steps and then applies the softmax, sparsemax or the reduce-weight mechanism previously described.

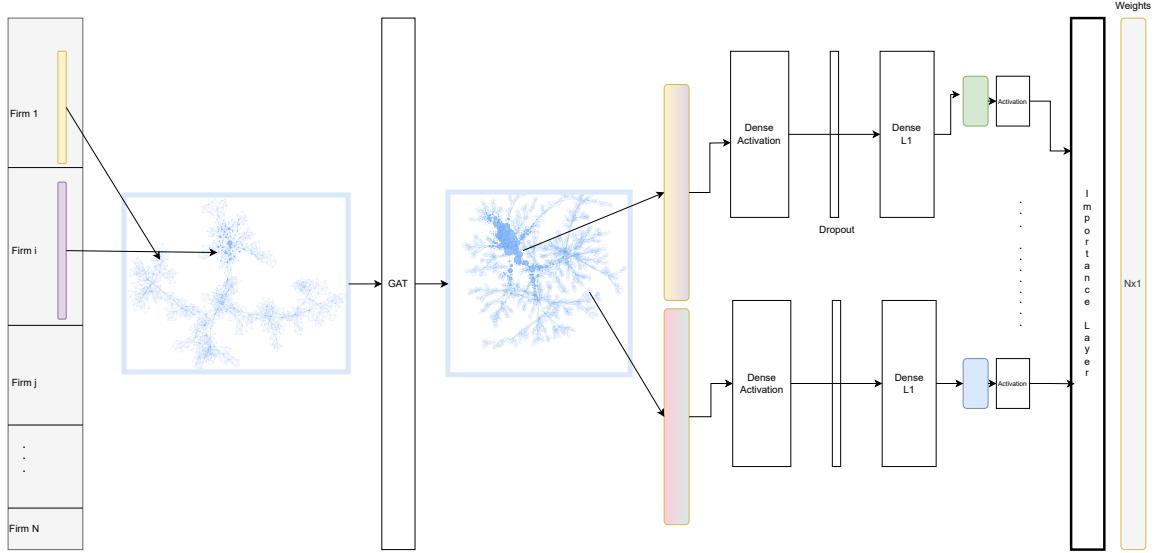


Figure 2: GAT Model: First embeddings are obtained from GAT and for each node new representations are created before they are combined to form portfolio weights

4.2 Network index model

As studies have shown, portfolios invested in peripheral assets outperform portfolios containing more central firms (Pozzi et al., 2013; Giudici et al., 2020). This is possible due to the diversification benefits of peripheral firms, as they are less correlated with market moves. Our model uses the peripherality measure defined in Section 3.4 and allocates capital to the portfolio as the inverse of the network index score. We rescale these weights so that they sum up to one. For each node or asset i in the network,

$$w_i = 1/p_i$$

$$w_i = \frac{w_i}{\sum w_j}$$

The model takes a graph as input at each iteration and calculates the weights as shown above. These weights are used to calculate the next 3-month daily portfolio returns based on the datasets of individual returns of firms. These returns are then used to calculate the Sharpe ratio for each graph and then averaged for each training, validation, and test datasets.

4.3 Benchmark Portfolios

4.3.1 Mean variance model:

Markowitz (1952)'s mean-variance model is widely known as part of Modern Portfolio Theory. Here, we include the model as one of the benchmark models. For $|\mathcal{V}_t|$ firms active at time t , we find the weights w_i for each asset in the portfolio such that $\sum_{i=1}^{|\mathcal{V}_t|} w_i = 1$. Starting with random weights, we aim to optimise the weights by maximising the expected return and minimising the portfolio's volatility.

The expected return and the variance of the portfolio are derived from the sample using the sample mean and covariance. The covariance matrix is for a pair of firms, as the system-wide covariance matrix from methods such as VAR decomposition is infeasible for firm numbers with illiquid trading patterns (Diebold and Yilmaz, 2012). We use a similar pairs-based method for graph input as well, using distance correlation, which makes this covariance calculation strategy comparable.

$$E(R_i) = \sum_{i=1}^{|\mathcal{V}_t|} w_i \mu_i \quad (15)$$

$$V(R_i) = \sum_i \sum_j w_i w_j cov(i, j) \quad (16)$$

where μ_i is the return of i^{th} asset and $cov(i, j)$ the covariance of the return series between firms i and j .

We solve the mean-variance problem for w_i using an optimisation library that uses quadratic programming (Martin, 2021). Intuitively, the model still picks up lowly correlated firms to achieve diversification benefits. However, a high-dimensional portfolio poses challenges to finding an efficient portfolio, and our model running times increase considerably.

4.3.2 The Equal weight portfolio model:

The market portfolio is an important benchmark for measuring models. More importantly, it is well known in the literature that the performance of such portfolios is challenging to beat (DeMiguel et al., 2009b). The strategy consists of simply assigning equal weights to all the firms in the portfolio. For each firm u in a set of firms \mathcal{V}_t the weight w_u corresponds to

$$w_u = |\mathcal{V}_t|^{-1} \quad (17)$$

This allocation may not be feasible when developing a portfolio strategy for many firms, as the transaction costs increase considerably. A modern proxy is to invest a similar amount in the market index of the firms via an ETF or other similar instrument if they exist. We assume no transaction costs, as is common in these instruments¹, and this equal-weight portfolio serves as the market benchmark in our study. The equal-weighted strategy, though not practical, is difficult to outperform, especially as the portfolio size increases because the risk of model misspecification error increases for models that use complicated strategies.

5 Experimental settings

The setup of the model from the raw input of the prices of mid-cap firms to the final result is shown in Figure 3. All models receive the same data for training in different formats. The Figure 3 shows the three stages in the experiment. The first stage is the preparation of Inputs. The raw data is converted to returns and volatilities. For each period the volatility data is used to create the dense covariance matrix using distance correlation and which is filtered by the TMFG algorithm to create a sparse graph. The second stage in the experiment is when the models receive the appropriate inputs. The Mean-Variance model and Equal-weighted model get the expected returns data. The GAT model, in addition to the return series, also has a filtered graph as an additional input. The network model just receives the graph input, which calculates the overall score for each node. All models generate investment weights at the final Output stage, after which they are all measured in a similar method with test, unseen, data, which is two quarters ahead of training data and one quarter ahead of the validation data, i.e., an out-of-time sample. The Sharpe ratio is calculated

¹Although, as discussed earlier, our method is robust when in the presence of transaction costs. This no-cost assumption benefits our benchmarks.

for each quarter, and we slide to the next windows of returns and volatility series to generate new graphs and covariance matrices. We collect the performance over time and report these results in Section 6.

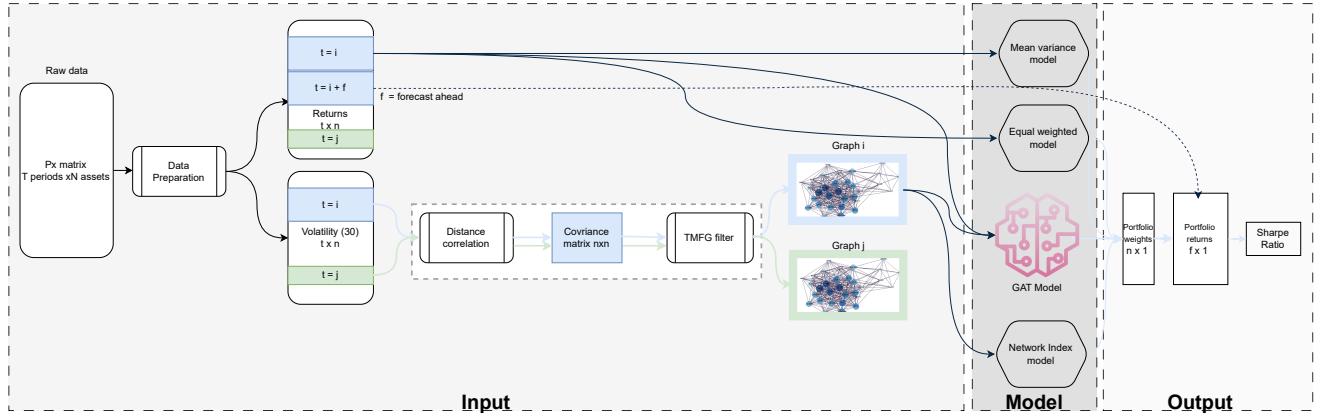


Figure 3: Experiment setup

5.1 Loss function

Unlike training in classification models, the use of the familiar cross-entropy loss function is impossible, as our model generates weights, not predictions. Our work proposes the negative of the Sharpe ratio expressed as the Lagrangian as the loss function. This allows us to leverage the flexibility of neural networks, so we train the model directly on the desired objective.

The inputs to the deep learning models are either the graph network or a time series of historical returns. For a graph constructed with historical data of window size T periods containing N_t firms over T periods, the outputs of the models are the weights of the portfolio w_{it} of shape $N_t \times 1$ where $i = i, 2, \dots, N_t$. To train the model, we use the true returns of the next h periods, y_h of shape $N \times h$.

The loss function is calculated as follows. For each batch, we calculate the predicted portfolio returns y_{port} .

$$y_{port} = w_i^T * y_h$$

We calculate the expected returns $E(y_{port})$ and volatility $V(y_{port})$ and annualise the ratio. To note, we assumed risk-free interest as zero as this assumption is applied for all the models, thus the relative performance we are interested in is unaffected.

$$SR_{port} = E(y_{port})/V(y_{port})$$

This procedure ensures that the output w_i of the final layer of the network learns from changes in asset i . We averaged the Sharpe ratio of all batches to arrive at a final performance measure.

5.2 Training settings

We first split the returns data into training and test data sets, using 60% for training and the remaining as test dataset. The training data set is divided into training and validation data, with 25% of the remaining data for validation. The models are assessed based on the validation dataset to tune the hyperparameters in deep learning models. We also adopt early stopping to prevent overfitting of the data, and a patience of 15 epochs is applied to this early stopping to avoid local minima.

5.3 Evaluation metrics

The Sharpe ratio is a good measure of portfolio performance. However, this large-scale portfolio further challenges understanding why the performance is achieved. We use the two centrality measures discussed earlier, specifically betweenness and degree centrality, to calculate a portfolio-level score. We take a weighted average of the centrality score of each node and their weight in the portfolio. This shows how peripheral the nodes are selected in each model.

We also look at industry-level composition across different portfolios selected by the models. As there are approximately 6,000 firms, the industry-level grouping will give us a better visual of any differences. We calculate a weighted industry score by summing up the portfolio weights of each company in the industry.

We also look at some turnover statistics due to the dynamic nature of the universe of companies we selected, as opposed to selected index constituents, where changes are not common. These mid-

cap companies have a large turnover, with around 15% of companies changing every six months. Due to superior performance, they might move away from mid-cap, or due to higher default rates, they could liquidate or go bankrupt much more often than their blue-chip counterparts. We benchmark the model turnover above this natural rate of change, which can be derived from the equal portfolio. An equal-weighted portfolio will create new positions or close positions only if the companies are new or exit the mid-cap universe, respectively. For each model, we calculate new positions and subtract it from the equal portfolio new positions, and report the total percentage of new positions and similarly closed positions. We define turnover as the sum of new and closed positions.

6 Results and discussion

In this section, we first report the performance of each model based on our Sharpe ratio metric for training, validation, and test splits. We compare and discuss the model performance based on the Sharpe ratio of the test dataset. We also present how each model has performed over time in the last 30 years with a rolling window of a one-quarter period. This measure is important as we recalculate the graph every quarter, and the training, validation, and test data all roll ahead.

We then report an analysis of the possible reasons why the models performed the way we observed them to. We report the overall peripherality score of each model and compare the differences between the models. We also look at changes in industry concentration, as different strategies might overweight certain industries. Finally, we look at the turnover of each portfolio optimisation model. More turnover typically implies more trades, which increases the costs to run a portfolio. We define turnover as the number of new positions in the portfolio either created or closed compared to the natural churn in the portfolio over every period.

6.1 Model performance

Table 1 shows the portfolio performance on all data splits. The GAT model has the highest ratio of 1.082 over the equal portfolio, where higher results mean higher risk-adjusted performance. Deep learning models can learn intricate relationships based on features and the adjacency matrix information, which can beat an equal portfolio. As earlier studies confirm, it is hard for well-designed portfolio models to outperform equal portfolios, especially when the portfolio is large

(DeMiguel et al., 2009b). The Mean-Variance model has the worst performance again, confirming studies on large-scale portfolio optimisation (Ao et al., 2019) . The expected returns and covariance matrix from linear correlation measures do not have the same information contained in the input compared to the whole return series and filtered adjacency matrix, which are the inputs to GAT models.

Sharpe Ratio (annualized)	train	val	test
Equal	0.825	0.925	0.830
Network	0.817	0.917	0.820
Efficient Portfolio	0.779	0.785	0.700
GAT	1.819	1.480	1.082

Table 1: Portfolio optimisation results

The Sharpe ratio reported in Table 1 is the average ratio across different test data sets. We should also examine how the models perform over time with these different test datasets. We report the performance over time in Figure 4. Most of the time, we see GAT models performing consistently better. All the models perform similarly for the first five years, and the divergence becomes consistent later. In the recent past, we have observed that the Mean-variance models underperform, likely due to a higher number of firms, which makes the limitations of these models more apparent. In contrast, the advantage GAT models had in the middle of the range slightly decreased compared to the naïve and network-based models. As the market went through several cycles during this long period, we can confirm the results to be robust to general market conditions and the relative strength of graph-based portfolio optimization methods.

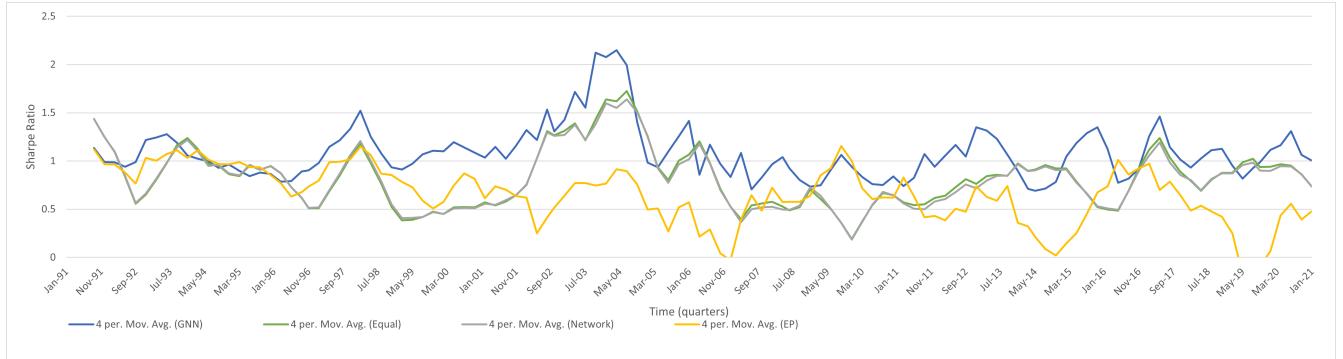


Figure 4: Model performance over time

6.2 Strategy differences

In this section, we look at why such differences exist between the model’s performance by looking at portfolio-level network scores and industry selection. We chart two scores to understand where each model is focusing on. Figure 5 shows each model’s weighted average centrality score defined earlier. The weights are the portfolio weights of the model. The betweenness centrality (BC) is the first set of bars for each model depicted in yellow. The black line is the standard deviation of these scores for each model over time. As we see for the betweenness centrality, the GAT model and Mean-variance model select similar scored nodes. However, there is a lack of stability with high variability in the Mean-variance model, while GAT models are relatively stable. The network models choose the most peripheral nodes as measured by the betweenness centrality score, but they underperform against the GAT models, as they lack diversity compared to them. The mid-cap firms which are too peripheral and carry 10% default risk will impact the portfolio and eliminate the performance gains we have from peripheral nodes. This is a different conclusion from previous studies, especially those based on topological information for portfolio optimisation (Li et al., 2019). However, those studies do not contain firms at-risk of default and are typically indexes or constituent firms of well-tracked indices. They also remove any firms that do not have the full data available. This is the first study to incorporate such features, as deep learning models can handle these inputs effectively. From degree centrality scores, we see that GAT models show the smallest variance while selecting relatively central nodes as they are marginally higher than the equal-weighted portfolio.

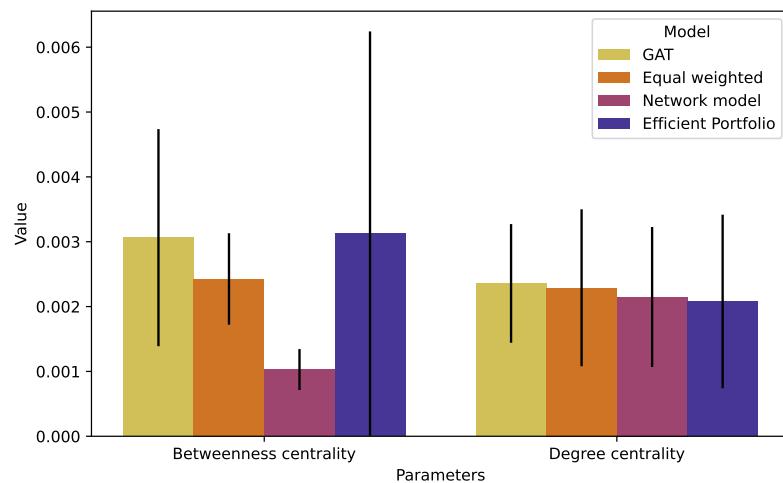


Figure 5: Model performance over time

As the portfolio is extensive, we present the overall weight given to a select industry sector instead of looking at individual firms, as shown in Figure 7. The equal portfolio (orange) is the benchmark that can be used to understand how the firm population is distributed. The largest sector is manufacturing, with around 45% of firms. The network-based model allocates similarly to the equal-weighted portfolio, but as seen in Figure 5 it chooses more peripheral nodes. The mean-variance model allocates weights with more variability than other models. This is expected with a large-scale portfolio as the universe of firms changes every quarter the optimisation of mean-variance portfolio is unstable. Here it underweights the larger manufacturing sector and overweights the transportation and public utilities sectors, but with very high variance, as can be seen by the longer range lines. The GAT model does more rebalancing of existing positions, as we can see a higher range than the equal and network-based portfolios, but this range is not as high as the MV model. The GAT models monitor their positioning better while tweaking the portfolio for better returns. For mid-cap firms, such a strategy seems intuitive as firms face higher business risk than large-cap firms, so an optimal strategy will have more adjustments across time. From a practitioner view, in financial markets, small and midcap firms are more numerous compared to large cap firms, but also need to have more monitoring processes as the individual firm default risk is higher. These are challenges that practitioners undergo in the investment process. The models presented in this study attempt to solve such challenges.

We present a snapshot of the resulting filtered complex network in Figure 1 and show the portfolio weights allocated by GAT model for the same network in Figure 6. It shows the allocations focusing on select branches instead of selecting peripheral nodes. The darker pink tones show where the model did not allocate any weight at all, and as we move higher up in the scale, we see larger weights towards a few peripheral firms but small weights distributed across numerous central firms with the white and light green tones.

% change	new	closed	turnover
Network	2.48	2.36	4.84
Efficient Portfolio	2.83	2.68	5.50
GAT	2.07	2.03	4.10

Table 2: Average turnover by model

Finally, we look at the turnover of each strategy. The equal weighted portfolio model buys

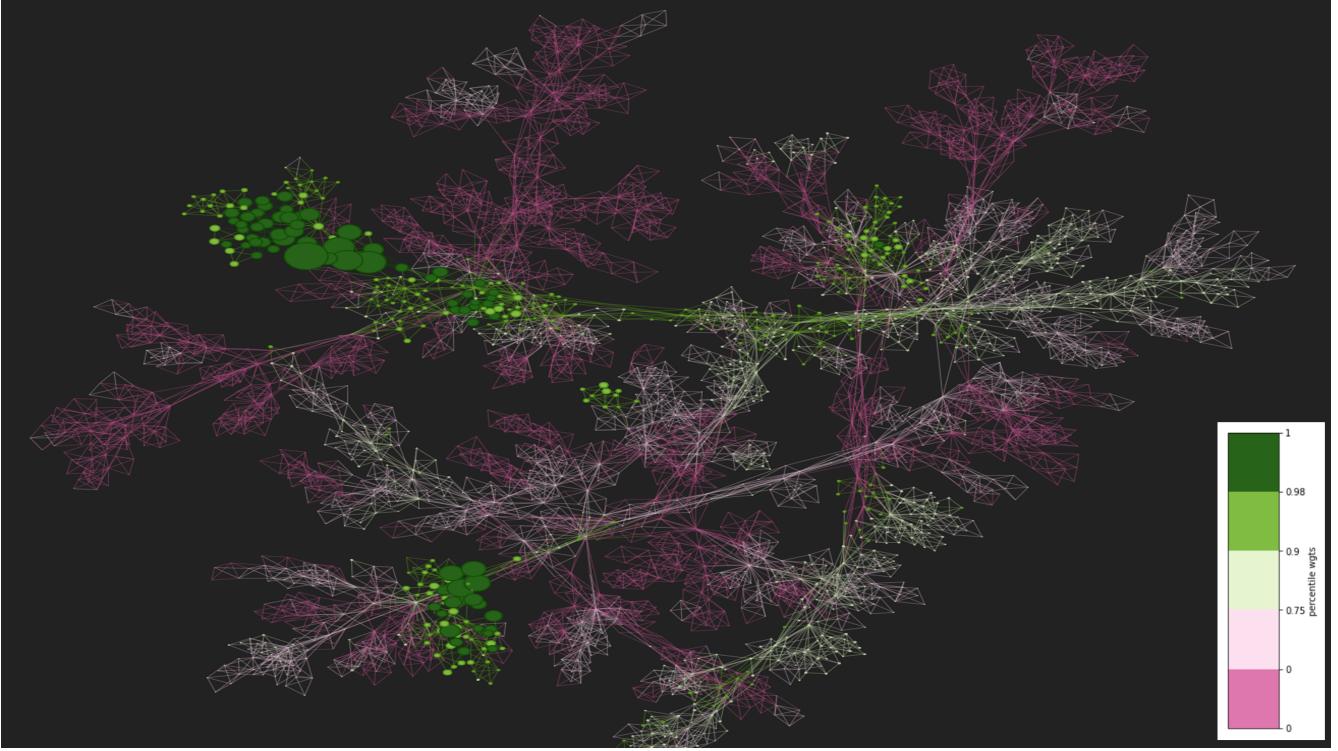


Figure 6: Snapshot of allocation of portfolio weights of mid-cap firms by GATs

into new positions when a firm which was absent in the previous period has entered the mid-cap universe, and sells when the company has exited as a mid-cap. This is the natural churn which is usual for mid-caps. We are interested in the churn specific to the model over the natural rate. We report the average percentage change compared to the previous period for the new and closed positions, and define the turnover as the sum of those two changes. From Table 2 we see that the GNN-based graph model achieves better performance with little turnover compared to other strategies. The average portfolio change is 2.07% new positions every quarter for GNN and 2.83% new positions for the MV portfolio. GNN achieves a 25% reduction compared to MV and a 15% reduction compared to the network model in overall turnover. This strongly suggests that the GAT model will have lower transaction costs compared to traditional models.

7 Conclusion

In this paper, we have shown a solution for large-scale portfolio optimisation using deep graph models. We have seen how GAT-based models, a model within the family of deep learning models, can extract intricate relationships that other traditional models cannot find. These models are

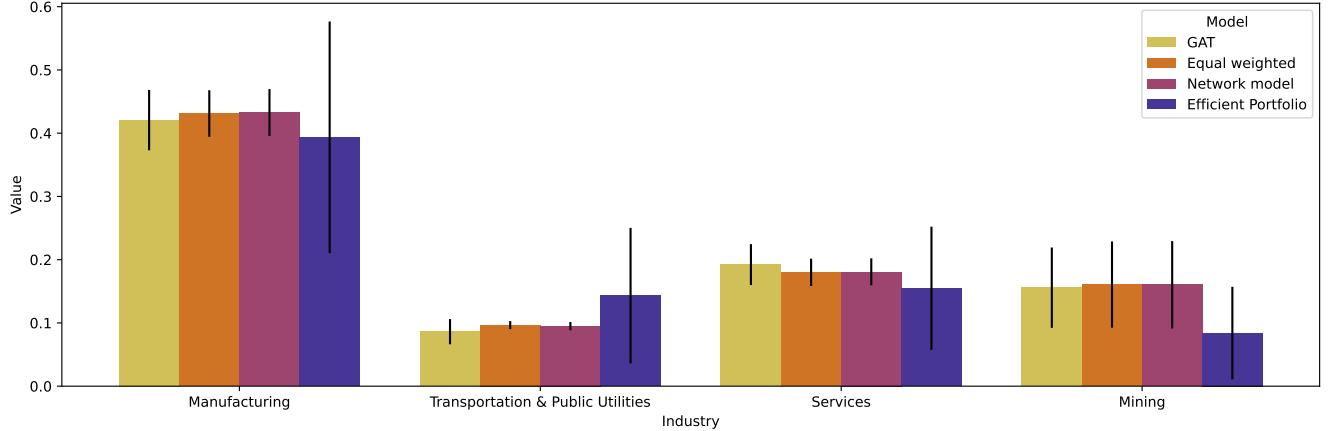


Figure 7: Industry allocations per model

able to make better use of complex input data. While most studies focus on portfolio optimisation for assets that have continuous prices, we focused on problems where the data is difficult to model by analysing the volatile mid-cap market. The study of such firms in itself is critical. As they are far more numerous than their large-part counterparts, their smaller size makes them more vulnerable to market movements. There are also potentially higher returns for commensurate risk for investors, while better risk management of these firms may lead to better access to financial markets for those firms.

We have linked a few areas of studies by applying optimised distance correlation measures to derive the covariance matrix, thus explicitly revealing how firms connect as shown by their volatility pairings, capturing intrinsically more complex dynamics than alternative models. We generate these links using filtering techniques applicable to large-scale graphs, using the Triangulated Maximally Filtered Graph algorithm. We used these filtered graphs for a deep learning-based GAT model, which can identify higher-order relationships. The final allocation layers optimised the embeddings generated by the GAT models and the regularisation parameters used in the deep learning models imposed constraints on possible weights and number of firms capital was allocated to. The loss function set the objective for portfolio performance maximisation through the shape ratio. Other portfolio objectives could be similarly be used and constraints imposed on the portfolio allocations. As deep learning models are adept at optimising large-dimensional problems, we reach better performance from GAT models than other alternatives. We also studied how the results are robust to different market conditions and looked at the distribution of allocation of firms across different strategies, to identify some of the reasons why GAT models reached superior

performance choosing companies that are not too much in the periphery and allocating capital to fewer firms . Finally, we observe that the turnover of the GAT models is lower than alternatives, while maintaining a slightly higher variance. These characteristics result in overall better performance.

As for future work, graph neural network models could be further developed to predict aspects like market regimes, or early warning indicators for financial networks. By changing the objectives and fine-tuning the loss functions, we could extend the graph neural network-based portfolio optimisation models for different objectives, such as the construction of Environmental, Social and Governance (ESG) portfolios or other diversification-orientated portfolios on a much larger scale.

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