On the Optimality of Some Multiperiod Portfolio Selection Criteria

Author(s): Edwin J. Elton and Martin J. Gruber

Source: *The Journal of Business*, Apr., 1974, Vol. 47, No. 2 (Apr., 1974), pp. 231-243

Published by: The University of Chicago Press

Stable URL: https://www.jstor.org/stable/2353382

*Edwin J. Elton and Martin J. Gruber**

On the Optimality of Some Multiperiod Portfolio Selection Criteria

Criteria for portfolio selection have received a great deal of attention in the recent economic literature. Modern portfolio analysis has its origin in the work of Markowitz, who specified the portfolio problem in terms of the one-period means and variances of returns.[1] However, most portfolio problems are multiperiod. The appropriateness of one-period analysis for this class of problems has been seriously questioned in recent years.[2] As a result, several alternative decision rules and modification of the one-period analysis have been proposed.

The purpose of this paper is to evaluate two proposals that have received wide attention in the economic literature. The first involves selecting portfolios on the basis of the geometric mean of future multiperiod returns. The second involves selecting portfolios on the basis of the expected utility of multiperiod returns.[3] We shall show that, when the ability of the investor to revise his portfolio is considered, each of these rules is only appropriate under a very restrictive set of conditions.

In order to evaluate any portfolio selection rule, it is necessary to establish a goal for portfolio management. In this paper, we will assume that the goal of portfolio managers is to maximize the expected utility of the investor's wealth at some terminal date. We have selected this goal because (1) the decision rules we will investigate were devised to achieve this goal, (2) the goal is appropriate for many real portfolio problems (such as retirement portfolios), and (3) this goal has been assumed by many other authors.[4]

Once this goal has been established, we can define the portfolio problem as the development of decision rules for selecting a sequence of

* The authors are senior research fellows at the International Institute of Management, Berlin, and professors of finance, New York University. The initial research for this paper was funded by the Institute for Quantitative Research in Finance.

1. Harry Markowitz, *Portfolio Selection* (New York: John Wiley & Sons, 1959).

2. See Nils H. Hakansson, "Capital Growth and the Mean-Variance Approach to Portfolio Selection," *Journal of Financial and Quantitative Analysis* 6 (January 1971): 517–55; "Optimal Investment and Consumption Strategies under Risk for a Class of Utility Functions," *Econometrica* 38 (September 1970): 587–607; and Jan Mossin, "Optimal Multi-Period Portfolio Policies," *Journal of Business* 41 (April 1968): 215–29. In addition, Markowitz laid the groundwork for some of this latter attack on his one-period model by exploring the dynamic nature of portfolio management policy.

3. This can be thought of as consistent with a Markowitz analysis where the risk and return for a portfolio is defined as the multiperiod risk and return.

4. Hakansson, "Optimal Investment," and Mossin.

portfolios over time to maximize the expected utility of terminal wealth. The optimum sequence of portfolios can depend on the expected results of portfolios held prior to the horizon as well as the distribution of returns over the entire horizon.[5]

The rest of this paper will examine the two decision rules mentioned above and will delineate conditions under which they will lead to the maximization of the expected utility of terminal wealth. In Section I we examine the geometric mean return criteria both when return distributions are unchanging over time and when they change in a regular pattern. In Section II we perform the same analysis, for the criteria maximize the utility of T-period returns. The analysis in this paper differs from much of the previous work that has been done on portfolio management in that we explicitly allow for portfolio revision over time.

I. CONDITIONS FOR OPTIMALITY OF THE GEOMETRIC MEAN RETURN

A criterion for ranking portfolios that has received a great deal of attention in the economic literature and has attracted an increasing number of supporters is the geometric mean return.[6] The purpose of this section is to delineate those types of investor utility functions for which the geometric mean return is an appropriate portfolio selection criterion. Throughout this paper the appropriateness of a criterion will be judged by its ability to select that sequence of portfolios that maximizes the expected utility of terminal wealth. This section will be divided into two parts. In the first part, we will examine the suitability of the geometric mean under the assumption that the distribution of returns in all future periods (until the horizon) is expected to be the same and the return in one period is independent of the return in all others. These are the assumptions that proponents of the geometric mean have usually made in defending its use. Furthermore, these assumptions have received theoretical and empirical support from proponents of the strict form of the random walk theory.[7] In the second part of this section, we will relax the assumption of equal return distributions and reexamine the suitability of the geometric mean.

5. A decision rule that naturally suggests itself is to solve a sequence of one-period portfolio problems. The fact that this decision rule generally leads to incorrect decisions, as well as the specific conditions under which it will lead to optimum decisions, has already been documented (see Hakansson, "Capital Growth" and "Optimal Investment," and Mossin).

6. Among its supporters are Henry Latane, "Criteria for Choice among Risky Ventures," *Journal of Political Economy* 67 (April 1959): 144–55; Henry Latané and Donald Tuttle, *Security Analysis and Portfolio Analysis* (New York: Ronald Press, 1970); Hakansson, "Optimal Investment"; and J. B. Williams, "Speculation and the Carry-over," *Quarterly Journal of Economics* 50 (May 1936): 436–55.

7. See Eugene F. Fama, "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance* 20 (January 1965): 383–417.

To define the geometric mean, let R_{it} be a random variable equal to 1 plus the return, where t is a period index and i denotes a particular return; P_{it} be the probability of R_{it} occurring; and T be the number of periods in the investor's time horizon. Then for the one-period horizon, the geometric mean return is given by

$$\prod_{i1} R_{i1}{}^{P_{i1}}. \tag{1}$$

The geometric mean return over a two-period horizon is given by

$$\left(\prod_{i1} \prod_{i2} R_{i1}{}^{P_{i1}} R_{i2}{}^{P_{i2}}\right)^{1/2}.$$

For a T-period horizon this generalizes as

$$\left(\prod_{i1} \prod_{i2} \prod_{i3} \ldots \prod_{iT} R_{i1}{}^{P_{i1}} R_{i2}{}^{P_{i2}} R_{i3}{}^{P_{i3}} \ldots R_{iT}{}^{P_{iT}}\right)^{1/T}. \tag{2}$$

A. Return Distributions Stable over Time

If the expected distribution of returns is the same in each period, equation (2) becomes

$$\left(\prod_{i1} \prod_{i1} \ldots \prod_{i1} R_{i1}{}^{P_{i1}} R_{i1}{}^{P_{i1}} \ldots R_{i1}{}^{P_{i1}}\right)^{1/T} = \prod_{i1} R_{i1}{}^{P_{i1}}. \tag{3}$$

Note that equation (3) is identical to equation (1). Furthermore, equation (3) represents the ranking of all portfolios, regardless of the number of years left to the horizon. Thus, the ranking associated with any portfolio will be the same in each future period. The investor who uses the geometric mean would choose that portfolio which maximizes (3) and revise his portfolio at the beginning of each period (2 through T) in order to restore the percentage of each asset in his portfolio to the initial level. From this point on, a policy which calls for portfolios to be revised so as to hold the percentage invested in each asset constant over time will be called a constant percentage portfolio policy. Thus, given constant return distributions over time, a necessary but not sufficient condition for the geometric mean to be an optimum decision criterion is the presence of a utility function that calls for a constant percentage portfolio policy to be optimum. Mossin has shown that there are only two classes of utility functions (log and power) that have this property.[8] Furthermore, he has shown that these functions produce the same ranking of portfolios whether utility is expressed in terms of return or wealth. We have chosen to perform our analysis in terms of return distributions because this is consistent with the bulk of the literature on portfolio analysis.

Most economists would further restrict utility functions in order to ensure increasing utility of wealth and risk avoidance [i.e., $u'(w) > 0$

8. Mossin.

and $u''(w) < 0$]. If we assume these properties, then the functions of interest can be written as[9]

$$\begin{array}{lll} U(w) = R, & \gamma = 1 & \text{log function,} \\ U(w) = -R^{1-\gamma}, & \gamma > 1 & \multirow{2}{*}{\Big\} power function.} \\ U(w) = R^{1-\gamma}, & 0 < \gamma < 1 & \end{array}$$

While the existence of any of these utility functions is a necessary condition for the geometric mean to be an optimum criterion, the sufficient condition for optimality is that the maximization of the geometric mean lead to the maximization of the utility of terminal wealth. In investigating the sufficient condition, we shall make use of another special property of the log and power utility functions. As Mossin has shown for either of these functions, the portfolio which maximizes the expected utility of first-period returns is identical to the optimal portfolio resulting from a T-period dynamic programming solution which maximizes the expected utility of terminal wealth.[10] This is true, though the dynamic programming solution considers all possible portfolios over all possible holding periods. Therefore, if the geometric mean leads to the same ranking of portfolios as the expected utility of first-period returns using the log or power utility function, then the geometric mean also leads to the same ranking as that produced by a T-period dynamic programming solution that allows the percentage of the portfolio invested in each asset to be changed over time.

1. *Log utility function.*—The expected utility of first-period returns for log utility functions is

$$\sum_{i1} P_{i1} \ln R_{i1}. \tag{4}$$

The geometric mean will produce identical rankings of portfolios, providing there is some monotonic transformation of equation (3) that yields equation (4). If the expected utility produced by equation (4) is a monotonic transformation of a decision criterion (such as the geometric mean), then the ranking of portfolios by the utility function will be identical to the ranking by the decision criterion and the same optimum will exist for both functions. One such monotonic transformation is the log function.[11] Since the log of a larger number will always be larger than the log of a smaller number, if the expected utility produced by a utility function is the log of the rating produced by a decision criterion, order will be preserved and the same portfolio will be selected as opti-

9. These functions involve constant relative risk aversion in the Pratt-Arrow sense. That is, $[U''(R)\,R]/[U'(R)] = -\gamma$, where γ is the coefficient of relative risk aversion and is a constant.

10. See Mossin.

11. If R could take on negative values, the log function would not be monotonic. However, since R is 1 plus the rate of return and since the rate of return can never be smaller than -100 percent, the smallest value R can take on is zero. Since R can never take on negative values, the log function is monotonic.

mum by both criteria. If we take logs of the geometric mean (eq. [3]), we get

$$\ln \prod_{i1} R_{i1}^{P_{i1}} = \sum_{i1} P_{i1} \ln R_{i1}. \tag{5}$$

Since equation (5) equals equation (4) and since the log function is monotonic, the ranking given by the geometric mean must always be identical to the ranking given by the log utility function. Thus, when portfolio revision is considered, the geometric mean is an appropriate decision criteria when the investor's utility function is of the log form and the distribution of returns in all future periods are the same.

2. *Power function.*—In the following discussion we will show that, when portfolio revision is allowed, the geometric mean can lead to a ranking of portfolios different from that of a utility function which has the form of a power function. It is more difficult to prove that a decision criterion and a utility function lead to different rankings than it is to prove that they lead to the same ranking. Since it would be tedious (and an almost infinite task) to prove inequality under every possible monotonic transform, we will present an example of two portfolios and show that the geometric mean and the power function can lead to different rankings and are therefore not equivalent.

Consider the two portfolios described in table 1. Furthermore, as-

Table 1

Return	Probability
Portfolio A	
1.0	.25
1.5	.50
2.0	.25
Portfolio B	
1.2	.25
1.4	.50
1.6	.25

sume that the appropriate utility function is $-R^{-2}$, that is, $\gamma = 3$. The geometric mean return for portfolio A is 1.459, while for portfolio B it is 1.400. The value of the power utility function for portfolio A is —0.535, while for B it is —0.526. Thus, the geometric mean selected A as a superior portfolio while the power utility function selected B. In this example, if the investor's utility function were of the form $R^{1-\gamma}$ with γ equal to 3, the geometric mean would lead to an incorrect ranking.

B. *Unequal Return Distributions over Time*

Up to this point, we have shown that if returns in all future periods are identically distributed, maximizing the geometric mean leads to the selection of the same portfolio as a T-period dynamic programming problem with an objective of maximizing terminal wealth if, and only if, the utility function is of the log form. Now we will extend the analysis to show under what alternative return patterns the geometric mean is appropriate when the investor's utility is of the log form.[12]

Mossin has shown that when utility functions are of the log form, myopic portfolio decision can be made. If utility is of the log form, then the ranking of portfolios for the tth period is given by

$$\Sigma P_{it} \ln R_{it}. \tag{6}$$

If we use the geometric mean criteria, then the ranking for the tth period is

$$\prod_{it} \prod_{it+1} \prod_{it+1} \dots \prod_{iT} (R_{it}{}^{P_{it}} R_{it+1}{}^{P_{it+1}} R_{it+2}{}^{P_{it+2}} \dots R_{it}{}^{P_{iT}})^{1/(T-t+1)}. \tag{7}$$

If all $R_{it} = C_t R_{i1}{}^{C'_t}$ and $P_{it} = P_{i1}$, where C_t and C'_t are constants, then log utility functions and the geometric mean will lead to the same ranking of portfolios.

To show this, note that if the two functions are to lead to the same ranking of portfolios, they must do so in every period. We can take logs of equation (7) without changing its ranking, since the log is a monotonic function. So (7) becomes

$$1/(T-t+1)(\sum_{it} P_{it} \ln R_{it} + \sum_{it+1} P_{it+1} \ln R_{it+1} + \sum_{it+2} P_{it+2} \ln R_{it+2} + \dots + \sum_{iT} P_{iT} \ln R_{iT}).$$

If all $R_{it} = C_t R_{i1}{}^{C'_t}$, then this equation can be written as

$$1/(T-t+1)(\sum_{i1} P_{i1} \ln C_t R_{i1}{}^{C'_t} + \sum_{i1} P_{i1} \ln C_{t+1} R_{i1}{}^{C'_{t+1}} + \dots \Sigma P_{i1} \ln C_T R_{i1}{}^{C'_T}).$$

Simplifying yields

$$1/(T-t+1)(\ln C_t + \ln C_{t+1} + \dots + \ln C_T) + 1/(T-t+1)(C'_t + C'_{t+1} + \dots + C'_T) \sum_{i1} P_{i1} \ln R_{i1}. \tag{8}$$

Substituting $R_{it} = C_t R_{i1}{}^{C'_t}$ into equation (6) yields

$$\ln C_t + C'_t \sum_{i1} P_{i1} \ln R_{i1}. \tag{9}$$

Since (8) is just a linear transform of (9), the geometric mean is a valid criterion for portfolio choice when the investor's utility function is logarithmic and the distribution of returns in each future period are

12. For almost any utility function, there is a specific pattern of returns which will cause the geometric mean to give the same ranking of portfolios as does the utility function. However, in the case of the log function, there is a class of return patterns that lead to the same ranking.

first-period returns raised to a power, or multiplied times a constant, or both.

In this section we have shown that, if return distributions are constant over time, the geometric mean will only lead to maximization of investor wealth provided that the investor's utility function is logarithmic.[13] Furthermore, if the investor's utility function is logarithmic, the geometric mean will be an appropriate decision criterion provided that the distribution of returns in any period is expected to be the first-period returns multiplied by a constant, or raised to a power, or both multiplied by a constant and raised to a power. All of these proofs have assumed that the investor has the ability to revise his portfolio at the end of each period.

II. CONDITIONS FOR THE OPTIMALITY OF T-PERIOD RETURNS

A second criterion which has been suggested to maximize the expected utility of terminal wealth is to base the investment decisions on the maximization of the expected utility of T-period returns. In this section we will show that, when portfolio revision is allowed, only under very special conditions will the use of T-period returns lead to optimal investment decisions. To do so, we will examine three forms of utility functions: logarithmic, power, and quadratic.[14] We will initially examine the case where the probability distributions of returns are equal in each period and where returns are independent from period to period.[15] This will be followed by a section discussing more general results. Note that the assumption of equal return distributions encompasses many situations discussed in standard portfolio theory. For example, assume that an individual wishes to maximize wealth over 2 years, short-run prospects equal long-run prospects (same distribution of returns in each period), and no new information is forthcoming over the 2 years. There is general agreement that in this situation the same portfolio should be held over the full period and, if utility is quadratic, the optimum portfolio can be determined using standard mean-variance analysis of 2-year returns. It is shown in this section that this result is incorrect and that generating a mean-variance efficient frontier using 2-year returns, while appropriate if utility is logarithmic or power and returns normally distributed, is inappropriate if utility is quadratic.

13. We are assuming the security outcomes are independent of the investor's personal wealth (i.e., the investor is not large enough to affect the market). If this is not true, serious complications occur.

14. The log and power functions were chosen because they produce optimum decisions that are independent of wealth. The quadratic was chosen because it is the most commonly used utility function in portfolio theory and because it is representative of that class of utility functions under which relative risk aversion is not constant.

15. The assumption of independence is consistent with the random walk theory.

The expected utility of T-period returns can be expressed as

$$E[U(z)] = \sum_{i1}\sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} U(R_{i1} R_{i2} \ldots R_{iT}), \quad (10)$$

where z is T-period returns and all other terms are as previously defined.

Mossin has shown that for logarithmic and power utility functions, maximization of the expected utility of the first-period returns leads to the same portfolio decisions as maximization of the expected utility of terminal wealth at the end of T-periods.[16] If for these functions we can extend Mossin's analysis to show that the expected utility of T-period returns can be expressed as a monotonic transform of the expected utility of one-period returns, then maximization of the expected utility of T-period returns also leads to the same portfolio decisions as the maximization of the expected utility of terminal wealth.

If utilities are logarithmic, equation (10) becomes

$$E[U(z)] = \sum_{i1}\sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} \ln (R_{i1} R_{i2} \ldots R_{iT}).$$

Using the assumption of independent and identical return distributions over time, this equation becomes[17]

$$E[U(z)] = T \sum_{i1} P_{i1} \ln R_{i1}. \quad (11)$$

The right-hand side of equation (11) is simply T times the expected utility of first-period returns.

16. See Mossin. The possible combinations of terminal wealth that are considered are those that are produced by all combinations of securities for all possible holding periods.

17. With the log utility function, the expected utility of T-period returns is

$$E[U(z)] = \sum_{i1}\sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} \ln (R_{i1} R_{i2} \ldots R_{iT})$$

or

$$E[U(z)] = \sum_{i1}\sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} (\ln R_{i1} + \ln R_{i2} + \ldots + \ln R_{iT}).$$

Consider the first term:

$$\sum_{i1}\sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} \ln R_{i1}.$$

Since $\sum_{it} P_{it} = 1$ and since we are assuming independence, this expression simplifies to

$$\sum_{i1} P_{i1} \ln (R_{i1}).$$

Applying the same simplification to each term in the initial equation yields

$$E[U(z)] = \sum_{i1} P_{i1} \ln R_{i1} + \sum_{i2} P_{i2} \ln R_{i2} + \ldots + \sum_{iT} P_{iT} \ln R_{iT}.$$

If $R_{ij} = R_{i1}$ and $P_{ij} = P_{i1}$ for all i (constant probability distribution of returns over time), this equation reduces to

$$E[U(z)] = T \sum_{i1} P_{i1} \ln R_{i1}.$$

If utilities are power functions, equation (10) becomes

$$E[U(z)] = \sum_{i1} \sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} \; (R_{i1} R_{i2} \ldots R_{iT})^{1-\gamma}.$$

Using the assumption of independent and identical return distributions over time, this equation becomes[18]

$$E[U(z)] = (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma})^{T}. \qquad (12)$$

The right-hand side of equation (12) is simply the expected utility of first-period returns raised to the Tth power.

Since multiplying any series of positive numbers by a constant or raising the series to a power greater than 1 preserves order, the ranking of expected utility by T-period returns is identical to the ranking by one-period returns when utility functions are either of the log or the power form. Since ranking on the basis of the expected utility of one-period returns is consistent with maximizing the expected utility of terminal wealth, ranking on the basis of the expected utility of T-period returns is also consistent with maximizing the expected utility of terminal wealth.

A. Quadratic Utility Functions with Constant Returns

To show that maximizing the utility of T-period returns with a quadratic utility function leads, in general, to different portfolio decisions from that set of portfolios which maximizes the expected utility of terminal wealth, we will consider a two-period two-asset case. This requires a change in notation, since we will be working directly with the return on individual assets. Let R_{it} = the return in period t on asset i, where $i = 1$, 2, and $t = 1, 2$; W_0 = the initial wealth of the investor; W_2 = the final wealth of the investor; R = the return earned by investing over the two periods; E_i = the expected one-period return of asset i (no period subscript is necessary since the distribution of returns are assumed equal in each period); and V_i = the variance of the one-period return of asset i (no period subscript is necessary since the distributions of returns are assumed equal in each period).

18. With the power utility function, the expected utility of T-period returns is

$$E[U(z)] = \sum_{i1} \sum_{i2} \ldots \sum_{iT} P_{i1} P_{i2} \ldots P_{iT} \; (R_{i1} R_{i2} \ldots R_{iT})^{1-\gamma}.$$

This simplifies to

$$E[U(z)] = (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma}) \; (\sum_{i2} P_{i2} R_{i2}{}^{1-\gamma}) \ldots (\sum_{iT} P_{iT} R_{iT}{}^{1-\gamma}).$$

Setting $R_{ij} = R_{i1}$ and $P_{ij} = P_{i1}$ for all j, we have

$$E[U(z)] = (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma}) \; (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma}) \ldots (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma}),$$

$$E[U(z)] = (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma})^{T}.$$

We can represent the investor's quadratic utility function as

$$U(W_2) \doteq W_2 - \alpha W_2^2.$$

By defining B as αW_0, we can state this in the equivalent form

$$U(R) = R - BR^2. \tag{13}$$

Mossin has shown that the dynamic programming solution that maximizes this utility function calls for the fraction (k) of the initial assets invested in security 1 to be given by[19]

$$k = \frac{(E_1 - E_2)[V_1(1 + E_2) + V_2(1 + E_1)]}{2\beta[V_1 + E_2)^2 + V_2(1 + E_1)^2 + V_1V_2][V_1 + V_2 + (E_1 - E_2)^2]} - \frac{(1 + E_2)(E_1 - E_2) - V_2}{V_1 + V_2 + (E_1 - E_2)^2}. \tag{14}$$

If maximizing the expected utility of two-period returns leads to the same initial portfolio, it is a satisfactory decision rule. If it leads to any other initial portfolio, it will produce lower expected utility of terminal wealth and must be rejected as a decision rule.

We can define the two-period return for any constant percentage portfolio policy as

$$R = [(1 + X_{21}) + k(X_{11} - X_{21})][(1 + X_{22}) + k(X_{12} - X_{22})]. \tag{15}$$

Substituting (15) into (13), taking expected values and differentiating with respect to k, yields

$$\begin{aligned}
&-k^3[V_1 + V_2 + (E_1 - E_2)^2]^2\, 2\beta \\
&-k^2[V_1 + V_2 + (E_1 - E_2)^2][-V_2 + (1 + E_2)(E_1 - E_2)]\, 6\beta, \\
&-k\{[V_1 + V_2 + (E_1 - E_2)^2][V_2 + (1 + E_2)^2]\, 2\beta \\
&\qquad + [-V_2 + (1 + E_2)(E_1 - E_2)]^2\, 4\beta - (E_1 - E_2)^2\}, \\
&-[-V_2 + (1 + E_2)(E_1 - E_2)][V_2 + (1 + E_2)^2]\,(-2\beta) \\
&\qquad + (1 + E_2)(E_1 - E_2).
\end{aligned} \tag{16}$$

Substituting equation (14) into equation (16) reveals that the optimum k as defined by (14) is not a root of equation (16). Thus, maximizing the utility of T-period returns will lead to nonoptimum portfolios if the investor's utility function is quadratic.

The above analysis has important implications for portfolio selection. The selection of portfolios that maximize the expected value of a utility function in terms of return and risk (standard deviation or variance) has been deemed appropriate whenever either utility functions are quadratic or returns are normally distributed.[20] The appropriate distribu-

19. While Mossin does not explicitly present this expression, he describes the methodology needed to derive it and presents a numerical example utilizing this expression.

20. Originally it was believed that any two-parameter distribution was appropriate. However, it has been pointed out that the normal distribution is the only

tion of returns to examine has often been defined in terms of the horizon with which the investor is concerned.[21]

As shown in Part A of this section, if an investor can revise his portfolio before the horizon and has a quadratic utility function, then maximizing the expected utility of T-period returns is inconsistent with maximizing terminal wealth at the end of T-periods). In fact, if return distributions are uniform through time, it is only appropriate to maximize utility in terms of horizon returns if the utility function is either of the log or the power form. Furthermore, restricting the analysis to means and variances for log and power functions is only consistent with maximizing utility if returns are normally distributed. Thus, for the standard Markowitz efficient frontier to be meaningful in terms of an investor maximizing the utility of terminal wealth over any time span within which he can trade, (1) his utility function must be of the power or log form (it cannot be quadratic) and (2) returns must be normally distributed.

B. *The Optimality of Maximizing N-Period Returns When the Distribution of Returns Varies from Period to Period*

In this section we will relax the assumption that return distributions are constant and show that there are still situations where the maximization of T-period returns leads to optimum decisions.[22] This result conflicts with Mossin's assertion that return distribution must be the same in each period for a stationary policy.[23]

Earlier we showed that if utility is logarithmic the utility of T-period returns is[24]

$$E[U(z)] = \sum_{i1} P_{i1} \ln R_{i1} + \sum_{i2} P_{i2} \ln R_{i2} + \ldots + \sum_{iT} P_{iT} \ln R_{iT}. \quad (17)$$

If $R_{it} = C_t\, R_{i1}{}^{C't}$ and $P_{it} = P_{i1}$, then maximizing the expected utility of T-period returns leads to the same portfolio decisions as maximizing terminal wealth. To show this, substitute $R_{it} = C_t\, R_{i1}{}^{C't}$ into equation (17):

$$E[U(z)] = \sum_{i1} P_{i1} \ln R_{i1} + \sum_{i1} P_{i1} \ln C_2\, R_{i1}{}^{C_2} + \ldots + \sum_{i1} P_{i1} \ln C_T\, R_{i1}{}^{C'T} = C' \sum_{i1} \ln R_{i1} + C,$$

distribution that has the property that the form of the distribution is maintained under linear transform (see K. Borch, "A Note on Uncertainty and Indifference Curves," *Review of Economic Studies* 36 [1969]: 473–507; M. S. Feldstein, "Mean-Variance Analysis in the Theory of Liquidity Preference and Portfolio Analysis," *Review of Economic Studies* 36 [January 1969]: 5–12; and James Tobin, "Comment on Borch and Feldstein," *Review of Economic Studies* 36 [January 1969]: 13–15).

21. The alternative formulation is in terms of one-period returns. This also leads to incorrect decisions (see Hakansson, "Capital Growth," and Mossin).

22. We will continue to assume that returns are independent over time.

23. See Mossin, p. 228.

24. See n. 17 above.

where

$$C = 1 + \sum_{i1} P_{i1} \ln C_2 + \sum_{i1} P_{i1} \ln C_3 \ldots \sum_{i1} P_{i1} \ln C_T = \ln (C_2 C_3 \ldots C_T),$$

$$C' = 1 + C'_2 + C'_3 + \ldots C'_T.$$

Since $E[U(z)]$ is a linear function of the expected utility of one-period returns, portfolio decisions based on T-period returns are identical to portfolio decisions based on one-period returns. As discussed earlier, for logarithmic utility functions this implies that decisions based on T-period returns are identical to those based on terminal wealth.

Note also that the optimal portfolio decision is not a function of T. Therefore, subsequent decisions would be identical to the initial decision if expectations regarding return distributions were maintained. A similar result holds if utility functions are a power series. The utility of T-period returns is[25]

$$E[u(z)] = (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma}) (\sum_{i2} P_{i2} R_{i2}{}^{1-\gamma}) \ldots (\sum_{iT} \sum_{iT} R_{iT}{}^{1-\gamma}). \quad (18)$$

Assume that $R_{it} = C''_t R_{i1}$ and $P_{it} = P_{i1}$. Substituting into equation (18) yields

$$E[u(z)] = (\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma}) (\sum_{i1} P_{i1} C''_2 R_{i1}{}^{1-\gamma}) \ldots (\sum_{i1} P_{i1} C''_T R_{i1}{}^{1-\gamma}),$$
$$E[U(z)] = C''(\sum_{i1} P_{i1} R_{i1}{}^{1-\gamma})^T = C''\{E[u(R_i)]\}^T, \quad (19)$$

where $C'' = (C''_2 C''_3 \ldots C''_T)$.

By now-familiar reasoning, equation (19) shows that the ranking of portfolios by T-period returns is identical to the ranking by one-period returns and hence identical to the ranking by terminal wealth.[26]

Once again the optimal portfolio is not a function of T, and if expectations concerning return distributions are maintained, the policy is stationary over time.[27]

III. CONCLUSION

The analysis performed in this paper demonstrates that, when portfolio revision is considered, portfolio decisions based on either the expected utility of multiperiod returns or the geometric mean of multiperiod returns are often different from and inferior to decisions based on consideration of returns sequentially over time. This is true even when the distribution of returns is expected to be identical in each future period. The conditions under which the geometric mean and the utility of multi-

25. See n. 18 above.

26. There are, of course, a number of patterns that can cause eq. (22) to be a monotonic transform of first-period returns. If T were an odd number and if $R_{ij} = 1/R_{i(j+1)}$, then eq. (22) would be identical to first-period returns. Other such patterns are possible.

27. Again, this conflicts with Mossin's result (see Mossin, p. 228).

period returns are appropriate portfolio selection criteria are delineated in the paper.

Closely associated with the analysis in this paper is the problem of deciding the optimum frequency for portfolio revision. Many authors have contended that the frequency of portfolio revision should be exclusively a function of the frequency with which new information arises or expectations concerning return distributions change.[28] However, as shown in this paper, even if no new information is expected to arise over time, and even if the distribution of returns is expected to be identical in each future period, changing the portfolio will increase the expected utility of terminal wealth if the investor's utility is neither of the log nor of the power function form.[29] This results from changes in investor wealth over time which cause the investor to change his risk preferences and so the composition of his portfolio. In fact, in the absence of revision costs, the maximization of the expected utility of terminal wealth calls for a continuous revision of the investor's portfolio.

28. See Keith Smith, *Portfolio Management* (New York: Holt, Rinehart & Winston, 1971).

29. Even if it is of the log or power function form, expected changes in the distribution of returns lead to revision in the absence of new information.