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A fuzzy portfolio selection model with background risk

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ABSTRACT

In financial markets, the presence of background risk may affect investors' investments. This article develops a fuzzy portfolio selection model with background risk, based on the definitions of the possibilistic return and possibilistic risk. For the returns of assets obey LR-type possibility distribution, we propose a specific portfolio selection model with background risk. Then, a numerical study is carried out by using the data concerning some stocks. Based on the data, we obtain the efficient frontier of the possibilistic portfolio with background risk, and compare it with the efficient frontier of the portfolio without background risk. Finally, we conclude that the background risk can better reflect the investment risk of the real economy environment which make the investors choose a more suitable portfolio to them.

1. Introduction

Markowitz [1] proposed a mean-variance model for portfolio selection in the earliest stage. Over the last 50 years, his theory has played an important role in the development of modern portfolio selection theory. In Markowitz's portfolio theory, it is assumed that all the investors are averse to risk, asset returns are random variables, the expected return of a portfolio is defined by the mean, and the risk is characterized by the variance. The objective of an investor is to minimize the variance of a portfolio for a fixed expected return. Therefore, the portfolio selection problem can be stated as follows:

$$\begin{cases} \min \quad V(X) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_i x_j, \\ s.t. \quad \sum_{i=1}^{n}\mu_i x_i \geqslant \bar{r}, \\ \qquad \sum_{i=1}^{n} x_i = 1, \\ \qquad i = 1, 2, \ldots, n, \end{cases}$$

where $\bar{r}$ denotes a required return of the portfolio, σ_{ij} is the covariance between assets i and j, μ_i is the expected return rate of an asset i, and x_i is the weight of an asset i.

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Researches on the mean-variance portfolio selection problem typically include Sharpe [2], Merton [3, Perold [4], Pang [5] etc. However, because of the complexity of financial systems, in several cases the input data is not precise but fuzzy. Therefore the decision maker should not consider parameters (goals and constraints) using crisp numbers or unique distribution functions, but use fuzzy numbers or fuzzy probability distribution functions [6]. Possibility theory was proposed by Zadeh [7] and was advanced by Dubois and Prade [8]. In Zadeh's theory, fuzzy variables are associated with possibility distributions, which is in the similar way that random variables are associated with probability distribution. Carlsson and Fullér [9,10] introduced the notions of possibilistic mean, possibilistic variance and covariance of fuzzy numbers. Georgescu [11] presented an approach to some topics of risk theory in the context of Zadeh's possibility theory. Inuiguchi and Tanino [12] introduced a possibilistic programming approach to the portfolio selection problem based on the minimax regret criterion. Zhang [13] and Zhang et al. [14] discussed the portfolio selection problems based on the lower, upper and crisp possibilistic means and possibilistic variances when short sales were not allowed on all risky assets. Huang [15] discussed fuzzy chance-constrained portfolio selection.

However, most of the above-mentioned papers do not address the issue of background risk because, under the assumption of complete markets, background risk can be priced and capitalized into wealth [16]. The statistical properties of background risk are then irrelevant to the allocation of wealth between risky stocks and risk-free bonds [16]. In practice, investors often face background risk such as those arising from labor income and real estate which are not insurable in financial markets. Heaton and Lucas [16] focused on how the presence of background risk influenced portfolio allocation. Eeckhoudt et al. [17] analyzed the effect of background risk. Gollier [18] analyzed risk vulnerability and the tempering effect of background risk. Tsanakas [19] constructed a distortion-type risk measure which evaluated the risk of any uncertain position in the context of a portfolio with a fixed background risk. Baptista [20] proposed the optimal delegated portfolio management with background risk. Eichner and Wagener [21] analyzed the effects on portfolio composition of changes in the mean, the variance and the covariance between assets and background risk, and studied the tempering effect of increasing background risk. Georgescu [22,23] proposed the investment models with background risk combining probabilistic and possibilistic aspects. Cardak and Wilkins [24] studied the portfolio allocation decisions of Australian households, and analyzed the effect of labor income uncertainty and health risk. Jiang et al. [25] investigated the impact of background risk on an investor's portfolio choice, and analyzed the investor's hedging behavior in the presence of background risk. Alghalith [26] introduced an incomplete-market dynamic investment model with a correlated background risk. Menoncin [27] analyzed the portfolio problem with the stochastic investment opportunities and background risk.

Although a considerable number of research papers have been published for portfolio selection problem in fuzzy environment, there is few research on portfolio selection problem with background risk based on possibility theory. The purpose of this paper is to discuss a fuzzy portfolio with background risk based on possibilistic theory. Different from Zhang [13,14], we discuss a possibilistic portfolio selection model with background risk by assuming the returns of assets obey LR-type possibility distributions. We consider constraints on holdings of assets in the portfolio model. Finally, we carry out a study by using the data concerning some stocks. We conclude that the investors are able to choose a more suitable portfolio for them with background risk.

The rest of the paper is organized as follows. In Section 2, we introduce indicators of fuzzy numbers and their properties. In Section 3, we propose a fuzzy portfolio selection model with background risk. Assuming that the expected rate of returns is a LR-type distribution fuzzy variable, we derive a crisp equivalent form of the possibilistic portfolio with background risk. In Section 4, a numerical example is given to illustrate our proposed effective approaches. Finally, Section 5 presents our conclusions.

2. Indicators of fuzzy numbers

Let us introduce some definitions, which will be used in the following section. A fuzzy number $\widetilde{A}$ is a fuzzy set of the real line $\mathcal{R}$ with a normal, fuzzy convex and continuous membership function of bounded support. The family of fuzzy numbers is denoted by $\mathcal{F}$. Let A be a fuzzy number with γ-level set $[\widetilde{A}]^{\gamma} = [a_1(\gamma), a_2(\gamma)](\gamma > 0)$. In 2001, Carlsson and Fullér [28] defined the upper and lower possibilistic mean values of a fuzzy number $\widetilde{A}$ as

$$M_U(\widetilde{A}) = 2\int_0^1 \gamma a_2(\gamma)d\gamma, \tag{1}$$

and

$$M_L(\widetilde{A}) = 2\int_0^1 \gamma a_1(\gamma)d\gamma. \tag{2}$$

Carlsson and Fullér [28] also introduced the crisp possibilistic mean value of a fuzzy number as

$$\overline{M}(\widetilde{A}) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma = \frac{M_U(\widetilde{A}) + M_L(\widetilde{A})}{2}. \tag{3}$$

The crisp possibilistic mean value of $\widetilde{A}$ is the arithmetic mean of its lower possibilistic and upper possibilistic mean values.

Corresponding to the upper and lower possibilistic means, Carlsson and Fullér [28] and Zhang [29] introduced the crisp possibilistic variance and covariance of fuzzy numbers. The crisp possibilistic variance of a fuzzy number $\widetilde{A}$ with γ-level set $[\widetilde{A}]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$ $(\gamma > 0)$ is defined as

$$\overline{V}(\widetilde{A}) = \int_0^1 \gamma\left[\overline{M}(\widetilde{A}) - a_2(\gamma)\right]^2 d\gamma + \int_0^1 \gamma\left[\overline{M}(\widetilde{A}) - a_1(\gamma)\right]^2 d\gamma. \tag{4}$$

The crisp possibilistic covariance between fuzzy numbers $\widetilde{A}$ with γ-level set $[\widetilde{A}]^{\gamma} = [a_1(\gamma), a_2(\gamma)]$ and $\widetilde{B}$ with γ-level set $[\widetilde{B}]^{\gamma} = [b_1(\gamma), b_2(\gamma)]$ $(\gamma \in [0,1])$ is defined as

$$\overline{Cov}(\widetilde{A},\widetilde{B}) = \int_0^1 \gamma\left[\left(\overline{M}(\widetilde{A}) - a_2(\gamma)\right)\left(\overline{M}(\widetilde{B}) - b_2(\gamma)\right) + \left(\overline{M}(\widetilde{A}) - a_1(\gamma)\right)\left(\overline{M}(\widetilde{B}) - b_1(\gamma)\right)\right] d\gamma. \tag{5}$$

Lemma 1. *Let $\lambda_1, \lambda_2 \in R$ and let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy numbers, then*

$$\overline{M}\left(\lambda_1\widetilde{A} + \lambda_2\widetilde{B}\right) = \lambda_1\overline{M}(\widetilde{A}) + \lambda_2\overline{M}(\widetilde{B}).$$

Lemma 2. *Let $\lambda_1, \lambda_2 \in R$ and let $\widetilde{A}$ and $\widetilde{B}$ be fuzzy numbers, then*

$$\overline{V}\left(\lambda_1\widetilde{A} + \lambda_2\widetilde{B}\right) = \lambda_1^2\overline{V}(\widetilde{A}) + \lambda_2^2\overline{V}(\widetilde{B}) + 2|\lambda_1\lambda_2|\overline{Cov}\left(\phi(\lambda_1)\widetilde{A}, \phi(\lambda_2)\widetilde{B}\right),$$

where $\phi(x)$ is a sign function of $x \in R$.

3. Possibilistic portfolio selection model with background risk

3.1. Mathematics modeling

In this section, we suppose that investors in the economy face not only portfolio risk but also non-hedgeable background risk. These background risks are associated with labor income or holdings of non-tradeable assets. Let $\widetilde{\xi}_i$ be the return rates of risky assets $i, i = 1, 2, \ldots, n$, which are fuzzy numbers. Let x_i denote the investment proportions in assets $i, i = 1, 2, \ldots, n$. If the return on background asset is denoted by $\widetilde{r}_b$, which is a fuzzy number, then the sum of portfolio return and non-tradeable income can be expressed as

$$\widetilde{r}_p = \sum_{i=1}^{n} x_i\widetilde{\xi}_i + \widetilde{r}_b.$$

From Lemma 1, the crisp possibilistic mean value of $\widetilde{r}_p$ is given by

$$\overline{M}(\widetilde{r}_p) = \overline{M}\left(\sum_{i=1}^{n} x_i\widetilde{\xi}_i\right) + \overline{M}(\widetilde{r}_b) = \sum_{i=1}^{n} x_i\overline{M}(\widetilde{\xi}_i) + \overline{M}(\widetilde{r}_b).$$

From Lemma 2, the crisp possibilistic variance of $\widetilde{r}_p$ is given by

$$\overline{V}(\widetilde{r}_p) = \overline{V}\left(\sum_{i=1}^{n} x_i\widetilde{\xi}_i + \widetilde{r}_b\right) = \overline{V}\left(\sum_{i=1}^{n} x_i\widetilde{\xi}_i\right) + \overline{V}(\widetilde{r}_b) + 2\sum_{i=1}^{n}|x_i|\overline{Cov}\left(\widetilde{\xi}_i, \widetilde{r}_b\right).$$

Similar to the mean-variance principle for selecting the optimal portfolio proposed by Markowitz [1], the crisp possibilistic mean corresponds to the return, while the crisp possibilistic variance corresponds to the risk. The objective of the investor is to choose a portfolio that minimizes the risk on the investment subject to some constraints on the return of the investment and assets holdings. Therefore, the possibilistic portfolio selection model with background risk can be formulated as

$$\begin{cases} \min & \overline{V}(\widetilde{r}_p) = \overline{V}\left(\sum_{i=1}^{n} x_i\widetilde{\xi}_i\right) + \overline{V}(\widetilde{r}_b) + 2\sum_{i=1}^{n}|x_i|\overline{Cov}\left(\widetilde{\xi}_i, \widetilde{r}_b\right) \\ s.t. & \sum_{i=1}^{n} x_i\overline{M}(\widetilde{\xi}_i) + \overline{M}(\widetilde{r}_b) \geqslant \bar{r}, \\ & \sum_{i=1}^{n} x_i \leq 1, \\ & 0 \leqslant l_i \leqslant x_i \leqslant u_i, \quad i = 1, 2, \ldots, n, \end{cases} \tag{6}$$

where $\bar{r}$ is the required return rate. l_i and u_i represent the lower bound and the upper bound on investment in asset $i, i = 1, 2, \ldots, n$, respectively.

In our modeling approach, we assume that $l_i \geqslant 0$ holds, which implies that short-selling of securities is not allowed. Moreover, in order to ensure feasibility of the portfolio selection problem, $\sum_{i=1}^{n} l_i < 1$ holds. This model is an expansion of the fuzzy possibilistic mean-variance model by Zhang [13,14].

3.2. LR-type possibility distribution

In this section, we assume that $\widetilde{\xi}_i, i = 1, 2, \ldots, n$, are general LR-type fuzzy numbers with the following membership function:

$$f_{\widetilde{\xi}_i}(t) = \begin{cases} L(\frac{a_i - t}{\alpha_i}), & \text{if} \quad a_i - \alpha_i \leqslant t \leqslant a_i, \\ 1, & \text{if} \quad a_i \leqslant t \leqslant b_i, \\ R\left(\frac{t - b_i}{\beta_i}\right), & \text{if} \quad b_i \leqslant t \leqslant b_i + \beta_i, \\ 0, & \quad \text{otherwise}, \end{cases}$$

where $L, R : [0, 1] \to [0, 1]$ with $L(0) = R(0) = 1$ and $L(1) = R(1) = 0$ are non-increasing, continuous functions. If L and R are strictly decreasing functions, then the γ-level set of $\widetilde{\xi}_i$ can be computed as

$$[\widetilde{\xi}_i]^\gamma = \left[a_i - \alpha_i L^{-1}(\gamma), b_i + \beta_i R^{-1}(\gamma)\right], \quad \forall \gamma \in [0, 1], \ i = 1, 2, \ldots, n.$$

Then let us denote the return on background asset by $\widetilde{r}_b$, from the following membership function:

$$f_{\widetilde{r}_b}(t) = \begin{cases} L^b\left(\frac{a_b - t}{\alpha_b}\right), & \text{if} \quad a_b - \alpha_b \leqslant t \leqslant a_b, \\ 1, & \text{if} \quad a_b \leqslant t \leqslant b_b, \\ R^b\left(\frac{t - b_b}{\beta_b}\right), & \text{if} \quad b_b \leqslant t \leqslant b_b + \beta_b, \\ 0, & \quad \text{otherwise}. \end{cases}$$

The γ-level set of $\widetilde{r}_b$ can easily be computed as

$$[\widetilde{r}_b]^\gamma = \left[a_b - \alpha_b (L^b)^{-1}(\gamma), b_b + \beta_b (R^b)^{-1}(\gamma)\right], \quad \forall \gamma \in [0, 1].$$

Using the definitions of the crisp possibilistic mean and variance of fuzzy numbers, we easily obtain

$$\overline{M}(\widetilde{\xi}_i) = \int_0^1 \gamma(a_1(\gamma) + a_2(\gamma))d\gamma = \frac{a_i + b_i}{2} + A_i,$$

$$\overline{V}(\widetilde{\xi}_i) = \int_0^1 \gamma\left[\overline{M}(\widetilde{A}) - a_2(\gamma)\right]^2 d\gamma + \int_0^1 \gamma\left[\overline{M}(\widetilde{A}) - a_1(\gamma)\right]^2\Big) d\gamma = \frac{(b_i - a_i)^2}{4} + C_{ii} + (b_i - a_i)B_i - A_i^2,$$

$$\begin{aligned} \overline{Cov}\left(\widetilde{\xi}_i, \widetilde{\xi}_j\right) &= \int_0^1 \gamma\left[\left(\overline{M}(\widetilde{A}) - a_2(\gamma)\right)\left(\overline{M}(\widetilde{B}) - b_2(\gamma)\right) + \left(\overline{M}(\widetilde{A}) - a_1(\gamma)\right)\left(\overline{M}(\widetilde{B}) - b_1(\gamma)\right)\right] d\gamma \\ &= \frac{(b_i - a_i)(b_j - a_j)}{4} + C_{ij} + (b_i - a_i)B_j + (b_j - a_j)B_i - A_i A_j, \end{aligned}$$

where $A_i = \beta_i E_R - \alpha_i E_L$, $B_i = \beta_i E_R + \alpha_i E_L$, $C_{ii} = \alpha_i^2 F_{LL} + \beta_i^2 F_{RR}$, $C_{ij} = \alpha_i \alpha_j F_{LL} + \beta_i \beta_j F_{RR}$, $E_L = \int_0^1 \gamma L^{-1}(\gamma) d\gamma$, $E_R = \int_0^1 \gamma R^{-1}(\gamma) d\gamma$, $F_{LL} = \int_0^1 \gamma (L^{-1}(\gamma))^2 d\gamma$ and $F_{RR} = \int_0^1 \gamma (R^{-1}(\gamma))^2 d\gamma$.

By Lemma 1, the crisp possibilistic mean of $\sum_{i=1}^{n} \widetilde{\xi}_i x_i$ is given by

$$\overline{M}\left(\sum_{i=1}^{n} \widetilde{\xi}_i x_i\right) = \sum_{i=1}^{n} \left(\frac{a_i + b_i}{2} + A_i\right) x_i.$$

By Lemma 2, the crisp possibilistic variance of $\sum_{i=1}^{n} \widetilde{\xi}_i x_i$ is given by

$$\begin{aligned} \overline{V}\left(\sum_{i=1}^{n} \widetilde{\xi}_i x_i\right) &= \sum_{i=1}^{n} x_i^2 \overline{V}\left(\widetilde{\xi}_i\right) + 2\sum_{i>j=1}^{n} x_i x_j \overline{Cov}\left(\widetilde{\xi}_i, \widetilde{\xi}_j\right) \\ &= \sum_{i=1}^{n} x_i^2 \left[\frac{(b_i - a_i)^2}{4} + C_{ii} + (b_i - a_i)B_i - A_i^2\right] \\ &\quad + 2\sum_{i>j=1}^{n} x_i x_j \left[\frac{(b_i - a_i)(b_j - a_j)}{4} + C_{ij} + (b_i - a_i)B_j + (b_j - a_j)B_i - A_i A_j\right], \end{aligned}$$

for all $x_i \geqslant 0, i = 1, 2, \ldots, n$.

On the other hand, we can obtain

$$\overline{M}(\widetilde{r}_b) = \frac{a_b + b_b}{2} + A_b,$$

$$\overline{V}(\widetilde{r}_b) = \frac{(b_b - a_b)^2}{4} + C_{bb} + (b_b - a_b)B_b - A_b^2,$$

$$\overline{Cov}(\widetilde{\xi}_i, \widetilde{r}_b) = \frac{(b_i - a_i)(b_b - a_b)}{4} + C_{ib} + (b_i - a_i)B_b + (b_b - a_b)B_i - A_iA_b,$$

where $A_b = \beta_b E_R^b - \alpha_b E_L^b$, $B_b = \alpha_b E_L^b + \beta_b E_R^b$, $C_{bb} = \alpha_b^2 F_{LL}^b + \beta_b^2 F_{RR}^b$, $C_{ib} = \alpha_i \alpha_b F_{LL}^b + \beta_i \beta_b F_{RR}^b$, $E_L^b = \int_0^1 \gamma (L^b)^{-1}(\gamma) d\gamma$, $E_R^b = \int_0^1 \gamma (R^b)^{-1}(\gamma) d\gamma$, $F_{LL}^b = \int_0^1 \gamma L^{-1}(\gamma)(L^b)^{-1}(\gamma) d\gamma$, $F_{RR}^b = \int_0^1 \gamma R^{-1}(\gamma)(R^b)^{-1}(\gamma) d\gamma$.

Thus we can obtain the crisp form of the model (6), for a LR-type distribution fuzzy variable, and rewrite the possibilistic portfolio selection model with background risk as

$$\begin{cases} \min \quad \overline{V}(\widetilde{r}_p) = \sum_{i=1}^{n} x_i^2 \left[\frac{(b_i - a_i)^2}{4} + C_{ii} + (b_i - a_i)B_i - A_i^2\right] + \\ \qquad 2\sum_{i>j=1}^{n} x_i x_j \left[\frac{(b_i - a_i)(b_j - a_j)}{4} + C_{ij} + (b_i - a_i)B_j + (b_j - a_j)B_i - A_iA_j\right] + \\ \qquad \frac{(b_b - a_b)^2}{4} + C_{bb} + (b_b - a_b)B_b - A_b^2 + \\ \qquad 2\sum_{i=1}^{n} x_i \left[\frac{(b_i - a_i)(b_b - a_b)}{4} + C_{ib} + (b_i - a_i)B_b + (b_b - a_b)B_i - A_iA_b\right] \\ s.t. \quad \sum_{i=1}^{n} \left(\frac{a_i + b_i}{2} + A_i\right) x_i + \frac{a_b + b_b}{2} + A_b \geqslant \bar{r}, \\ \qquad \sum_{i=1}^{n} x_i \leq 1, \\ \qquad 0 \leqslant l_i \leqslant x_i \leqslant u_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{7}$$

Furthermore, the problem (7) can be simplified to some special forms of possibility distributions.

For $i = 1, 2, \ldots, n$, if $\widetilde{\xi}_i = (a_i, b_i, \alpha_i, \beta_i)$ and $\widetilde{r}_b = (a_b, b_b, \alpha_b, \beta_b)$ are trapezoidal fuzzy numbers, that is

$$E_L = E_R = \frac{1}{6}, \quad F_{LL} = F_{RR} = \frac{1}{12}, \quad E_L^b = E_R^b = \frac{1}{6}, \quad F_{LL}^b = F_{RR}^b = \frac{1}{12},$$

then the model (7) is equal to the following programming problem:

$$\begin{cases} \min \quad \overline{V}(\widetilde{r}_p) = \sum_{i=1}^{n} x_i^2 \left[\frac{(b_i - a_i)^2}{4} + \frac{(b_i - a_i)(\alpha_i + \beta_i)}{6} + \frac{\alpha_i^2 + \beta_i^2 + \alpha_i \beta_i}{18}\right] + \\ \qquad \sum_{i>j=1}^{n} x_i x_j \left[\frac{\alpha_i \alpha_j + \beta_i \beta_j}{18} + \frac{1}{2}\left(b_i - a_i + \frac{\alpha_i + \beta_i}{3}\right)\left(b_j - a_j + \frac{\alpha_j + \beta_j}{3}\right)\right] + \\ \qquad \frac{(b_b - a_b)^2}{4} + \frac{(b_b - a_b)(\alpha_b + \beta_b)}{6} + \frac{\alpha_b^2 + \beta_b^2 + \alpha_b \beta_b}{18} + \\ \qquad \sum_{i=1}^{n} x_i \left[\frac{\alpha_i \alpha_b + \beta_i \beta_b}{18} + \frac{1}{2}\left(b_i - a_i + \frac{\alpha_i + \beta_i}{3}\right)\left(b_b - a_b + \frac{\alpha_b + \beta_b}{3}\right)\right] \\ s.t. \quad \sum_{i=1}^{n} x_i \left(\frac{a_i + b_i}{2} + \frac{\beta_i - \alpha_i}{6}\right) + \frac{a_b + b_b}{2} + \frac{\beta_b - \alpha_b}{6} \geqslant \bar{r}, \\ \qquad \sum_{i=1}^{n} x_i \leq 1, \\ \qquad 0 \leqslant l_i \leqslant x_i \leqslant u_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{8}$$

Especially, if $a_i = b_i$, we can obtain, $\widetilde{\xi}_i = (a_i, \alpha_i, \beta_i)$ and $\widetilde{r}_b = (a_b, \alpha_b, \beta_b)$ are triangular fuzzy numbers, then the program (7) can be simplified as

$$\begin{cases} \min & \overline{V}(\widetilde{r}_p) = \sum_{i=1}^{n} \frac{\alpha_i^2+\beta_i^2+\alpha_i\beta_i}{18} x_i^2 + \sum_{i>j=1}^{n} \frac{\alpha_i\alpha_j+\beta_i\beta_j+(\alpha_i+\beta_i)(\alpha_j+\beta_j)}{18} x_i x_j + \\ & \frac{\alpha_b^2+\beta_b^2+\alpha_b\beta_b}{18} + \sum_{i=1}^{n} \frac{\alpha_i\alpha_b+\beta_i\beta_b+(\alpha_i+\beta_i)(\alpha_b+\beta_b)}{18} x_i \\ s.t. & \sum_{i=1}^{n} (a_i + \frac{\beta_i-\alpha_i}{6}) x_i + a_b + \frac{\beta_b-\alpha_b}{6} \geqslant \bar{r}, \\ & \sum_{i=1}^{n} x_i \leq 1, \\ & 0 \leqslant l_i \leqslant x_i \leqslant u_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{9}$$

If $\widetilde{\xi}_i$ and $\widetilde{r}_b$ are bell shaped fuzzy numbers, and $f_{\widetilde{\xi}_i}(t) = exp\left\{\frac{-(t-\mu_i)^2}{\sigma_i^2}\right\}, f_{\widetilde{r}_b}(t) = exp\left\{\frac{-(t-\mu_b)^2}{\sigma_b^2}\right\}$, that is

$$E_L = E_R = \frac{\sqrt{\pi}}{4\sqrt{2}}, \quad F_{LL} = F_{RR} = \frac{1}{4}, \quad E_L^b = E_R^b = \frac{\sqrt{\pi}}{4\sqrt{2}}, \quad F_{LL}^b = F_{RR}^b = \frac{1}{4},$$

then the model (7) is equal to the following programming problem:

$$\begin{cases} \min & \overline{V}(\widetilde{r}_p) = \left(\frac{1}{2} - \frac{\pi}{8}\right) \left(\sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2 \sum_{i>j=1}^{n} x_i x_j \sigma_i \sigma_j \right) + \left(\frac{1}{2} - \frac{\pi}{8}\right) \left(\sigma_b^2 + 2 \sum_{i=1}^{n} x_i \sigma_i \sigma_b \right) \\ s.t. & \sum_{i=1}^{n} x_i \mu_i + \mu_b \geqslant \bar{r}, \\ & \sum_{i=1}^{n} x_i \leq 1, \\ & 0 \leqslant l_i \leqslant x_i \leqslant u_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{10}$$

4. Numerical example

In order to illustrate our proposed effective approaches of the portfolio selection problem in this paper, we give a numerical example. In this example, the return of asset i is regard as a bell shaped possibility distributions with a center μ_i and a spread σ_i. We select six stocks from Shanghai Stock Exchange. Based on the historical data, the corporations' financial reports and the future information, we can estimate their returns with the following possibility distributions shown in Table 1.

In this example, in order to simplify the calculation, non-tradeable income can be assumed to have zero expected value. The results in the case when its expected value differs from zero are identical to those when the expected value is zero [20].

Therefore, the γ-level sets of $\widetilde{\xi}_i, i = 1, \ldots, 6$ are given by

$$[\widetilde{\xi}_1]^\gamma = \left[0.0605 - 0.0810\sqrt{\ln \gamma^{-1}}, 0.0605 + 0.0810\sqrt{\ln \gamma^{-1}}\right],$$

$$[\widetilde{\xi}_2]^\gamma = \left[0.0778 - 0.0905\sqrt{\ln \gamma^{-1}}, 0.0778 + 0.0905\sqrt{\ln \gamma^{-1}}\right],$$

$$[\widetilde{\xi}_3]^\gamma = \left[0.0810 - 0.1130\sqrt{\ln \gamma^{-1}}, 0.0810 + 0.1130\sqrt{\ln \gamma^{-1}}\right],$$

$$[\widetilde{\xi}_4]^\gamma = \left[0.1067 - 0.1920\sqrt{\ln \gamma^{-1}}, 0.1067 + 0.1920\sqrt{\ln \gamma^{-1}}\right],$$

$$[\widetilde{\xi}_5]^\gamma = \left[0.1245 - 0.2300\sqrt{\ln \gamma^{-1}}, 0.1245 + 0.2300\sqrt{\ln \gamma^{-1}}\right],$$

$$[\widetilde{\xi}_6]^\gamma = \left[0.1302 - 0.2570\sqrt{\ln \gamma^{-1}}, 0.1302 + 0.2570\sqrt{\ln \gamma^{-1}}\right].$$

Table 1
The bell shaped possibility distributions of returns of six stocks.

Stock	Stock1	Stock2	Stock3	Stock4	Stock5	Stock6	Non-tradable
μ_i	0.0605	0.0778	0.0810	0.1067	0.1245	0.1302	0.0000
σ_i	0.0810	0.0905	0.1130	0.1920	0.2300	0.2570	0.0530

Table 2
Some possibilistic efficient portfolios with background risk.

$\bar{r}$	0.0400	0.0620	0.0680	0.0740	0.0800	0.0860	0.0920	0.0970	0.1000	0.1100
$\overline{V}$	0.0024	0.0025	0.0028	0.0031	0.0033	0.0036	0.0041	0.0047	0.0051	0.0068
x_1	0.1000	0.1076	0.1000	0.1000	0.1439	0.1361	0.1000	0.1000	0.1000	0.1000
x_2	0.1000	0.1441	0.2276	0.2810	0.3223	0.4000	0.3852	0.3296	0.2685	0.1009
x_3	0.1000	0.1005	0.1000	0.1228	0.1244	0.1297	0.1571	0.1020	0.1000	0.1000
x_4	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1085	0.1060	0.1000	0.1000
x_5	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1340	0.2543	0.3315	0.1179
x_6	0.1000	0.1000	0.0000	0.1000	0.1000	0.1000	0.1152	0.1081	0.1000	0.4812
$\sum_{i=1}^{6} x_i$	0.6000	0.6522	0.7276	0.8038	0.8906	0.9658	1	1	1	1

A Genetic Algorithm is to solve the proposed model (10), and the parameter setting of the GA is given below. The crossover and mutation probabilities are set to 0.95 and 0.05, respectively. The population size is 100, iteration times are 100. The calculation is programmed using MATLAB 7.10, the experiments are conducted on a PC with Intel(R) Core(TM), i3 CPU M350 @ 2.27 GHZ, and the average execution time is about 1–2 minutes.

We assume that the lower bound of investment ratio x is $[0.1, 0.1, 0.1, 0.1, 0.1, 0.1]$, the upper bound is $[0.40, 0.40, 0.40, 0.50, 0.60, 0.50]$. By solving the model (10), when the required return $\bar{r}$ has different value, the corresponding possibilistic efficient portfolios are obtained as shown in Table 2. From Table 2 we can see that when the required return $\bar{r}$ is small (e.g. $\bar{r} = 0.04$), the investment proportions of assets are exactly the lower bound value given in advance, and the sum of the proportions is 0.6. As the required return $\bar{r}$ increases, the proportions of assets increase consequently. But the investor does not need to invest total capital to six risky assets for $\bar{r} \leq 0.086$, since there exist limits on minimum holdings. If the investor is not satisfied with any of the obtained portfolios, more portfolios can be calculated by varying the value of $\bar{r}$. Based on Table 2, the efficient frontier of the possibilistic portfolios with background risk is shown in Fig. 1.

Next, in order to illustrate the effect of background risk on the optimal portfolio selection, we use a case of the portfolio selection without background risk. Based on the model (10), the possibilistic portfolio selection model without background risk can be easily represented as follows:

$$\begin{cases} \min \quad \overline{V}(\tilde{r}_p) = \left(\frac{1}{2} - \frac{\pi}{8}\right)\left(\sum_{i=1}^{n} x_i^2 \sigma_i^2 + 2\sum_{i>j=1}^{n} x_i x_j \sigma_i \sigma_j\right), \\ s.t. \quad \sum_{i=1}^{n} x_i \mu_i \geqslant \bar{r}, \\ \qquad \sum_{i=1}^{n} x_i \leq 1, \\ \qquad 0 \leqslant l_i \leqslant x_i \leqslant u_i, \quad i = 1, 2, \ldots, n. \end{cases} \tag{11}$$

The possibilistic efficient portfolios of the model (11) are shown in Table 3. From Tables 2 and 3, it shows that when the required return is the same, if background risk is considered, the variance is larger. For example, when the required return $\bar{r}$ is 0.0860, the variance with background risk is 0.0036, while the variance without background risk is 0.0019. Fig. 2 illustrates the efficient frontiers of two possibilistic portfolios. The asterisk parabola shows the efficient frontier of the model (11), and the dotted parabola shows the efficient frontier of the model (10). The locus of the portfolio with background risk are on the right of the locus of the portfolio without background risk. When the required return is same, the variance with background risk is higher than that without background risk. That is to say, background risk has a great effect on making the optimal strategies.

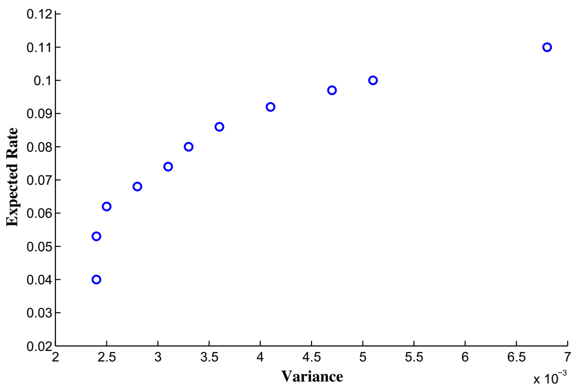

Fig. 1. Some possibilistic efficient portfolios with background risk.

Table 3
Some possibilistic efficient portfolios without background risk.

$\bar{r}$	0.0400	0.0620	0.0680	0.0740	0.0800	0.0860	0.0920	0.0970	0.1000	0.1100
$\overline{V}$	0.0010	0.0011	0.0013	0.0014	0.0016	0.0019	0.0022	0.0026	0.0029	0.0042
x_1	0.1000	0.1168	0.1356	0.1441	0.1340	0.2044	0.1000	0.1000	0.1000	0.1000
x_2	0.1000	0.1211	0.1815	0.2462	0.3224	0.2831	0.3475	0.3106	0.2692	0.1012
x_3	0.1000	0.1155	0.1178	0.1233	0.1317	0.1910	0.1992	0.1239	0.1027	0.1000
x_4	0.1000	0.1001	0.1000	0.1000	0.1000	0.1000	0.1001	0.1012	0.1000	0.1000
x_5	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1530	0.2594	0.3017	0.1150
x_6	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000	0.1003	0.1049	0.1264	0.4838
$\sum_{i=1}^{6} x_i$	0.6000	0.6536	0.7349	0.8136	0.8881	0.9785	1	1	1	1

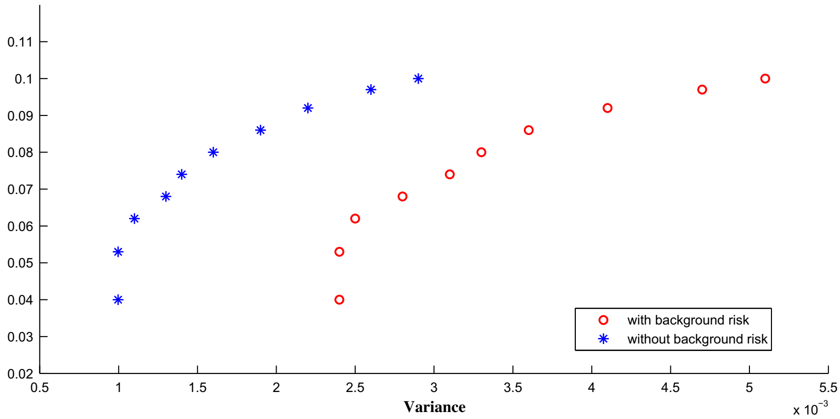

Fig. 2. Some possibilistic efficient portfolios with background risk and without background risk.

5. Conclusion

In this paper, a fuzzy portfolio selection model with background risk is proposed based on LR-type fuzzy returns, which can be deduced into any specific form by investors' estimation and practical situation. We carry out an empirical study based on a portfolio of six assets. A Genetic Algorithm which is coded by MATLAB 7.10 software is produced to solve our model. We compare the possibilistic portfolio model with background risk with the possibilistic portfolio model without background risk. The experiments indicate that when the required return is the same, the variance with background risk is higher than that without background risk. But in order to fit changes in financial markets, the investors often need to revise an existing portfolio. Our proposed models were limited to a portfolio added problem. In future work, we will not only study the models which other constraints of real markets are considered, but also expand the model to the multi period portfolio selection model and verify its performance in a real market.

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