PORTFOLIO ANALYSIS UNDER UNCERTAIN MEANS, VARIANCES, AND COVARIANCES

CHRISTOPHER B. BARRY*

THE USE OF mean-variance analysis in portfolio selection involves the estimation of means, variances, and covariances for the returns of all securities under consideration. Typically the estimates of those parameters are used as if they were the "true" (i.e., known) parameter values. Kalymon [3] has recently indicated the effects of uncertainty about means upon the investor's probability distribution of future security returns. The estimation of the covariance matrix is a more difficult task than the estimation of the mean vector, however, so uncertainty about the covariance matrix may be greater and may therefore have a stronger impact upon the investor's probability distribution of future security returns than has uncertainty about the mean vector.

In Section One of the paper the effects of three contrasting assumptions about prior knowledge of the parameters of the distribution of future security returns upon the investor's subjective probability distribution of security returns are considered. The effects of these assumptions upon estimates of portfolio risk are described, and a comparison is made of portfolio risk in the three cases. In Section Two it is shown that the set of efficient portfolios does not change as the assumptions of knowledge about $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ change, i.e., the efficient set consists of the same portfolios in all three cases. In Section Three it is shown that although the efficient set does not change as additional uncertainty is considered, the portfolio which would be selected from among those in the efficient set does change, and the type of change that occurs is an intuitively reasonable one. Finally, a summary and discussion is presented in Section Four.

I. UNKNOWN PARAMETERS AND PORTFOLIO RISK

Assume that future returns on a set of $K \geqslant 2$ securities are normally distributed with mean μ[1] and covariance matrix $\underset{\sim}{\Sigma}$, where $\underset{\sim}{\mu}$ is a $K \times 1$ vector, Σ is a $k \times k$ positive definite symmetric matrix, and $\underset{\sim}{\Sigma} = (\sigma_{ij})$. Let the $K \times 1$ vector x denote the vector of returns for the K risky securities being considered by the investor for his portfolio. Define the $K \times 1$ vector $\underset{\sim}{a}$ such that a_i is the proportion of the portfolio invested in security i, $a_i \geqslant 0$ for $i = 1, 2, \ldots, K$, and $\sum_i a_i = 1$.

In the discussion to follow, assume that the investor's information about $\underset{\sim}{\tilde{x}}$ is equivalent to a sample[2] of $n > K + 1$ observations $\underset{\sim}{x}_i$, $i = 1, 2, \ldots, n$ with sample mean vector

* The author is Assistant Professor of Management at the University of Florida. He is grateful to Robert L. Winkler, Indiana University, for helpful comments on an earlier draft of this paper.

1. Tildes below a variable indicate a vector or matrix, and tildes above variables denote random variables.

2. From the Bayesian point of view this may mean that the investor actually has a sample of n

$$\underset{\sim}{m} = \frac{1}{n} \sum_{i=1}^{n} \underset{\sim}{x}_i \tag{1}$$

and sample covariance matrix

$$\underset{\sim}{S} = \frac{1}{n-1} \sum_{i=1}^{n} (\underset{\sim}{x}_i - \underset{\sim}{m})(\underset{\sim}{x}_i - \underset{\sim}{m})'. \tag{2}$$

The sample mean and sample covariance matrix defined above are commonly used to estimate $\tilde{\underset{\sim}{\mu}}$ and $\tilde{\underset{\sim}{\Sigma}}$ respectively.

Three states of prior knowledge may be distinguished: $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ both known, μ unknown and $\underset{\sim}{\Sigma}$ known, and $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ both unknown. The first case is the case typically assumed in mean-variance analysis, although $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ are both estimated in some fashion. The second case is the case studied by Kalymon, and the third case represents the most general situation of the three.

Case One: $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ known

Suppose the investor estimates μ with m and Σ with S, and that he treats these estimates of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ as if they were the "known" values of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$. Then the investor's distribution of $\underset{\sim}{x}$ is normal with mean m and covariance matrix $\underset{\sim}{S}$. For any given portfolio, denoted by $\underset{\sim}{a}$, the expected portfolio return is $\underset{\sim}{a}'\underset{\sim}{m}$ and the portfolio variance is computed as $\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a}$.

Case Two: $\underset{\sim}{\Sigma}$ known and $\underset{\sim}{\mu}$ unknown

Suppose the investor treats $\underset{\sim}{S}$ as the "known" value of $\underset{\sim}{\Sigma}$, but that he treats μ as unknown and uses $\underset{\sim}{m}$ and n in forming a normal distribution of $\tilde{\underset{\sim}{\mu}}$. This is the case treated in detail by Kalymon. If the investor's distribution of $\tilde{\underset{\sim}{\mu}}$ is normal with mean $\underset{\sim}{m}$ and covariance matrix $(1/n)\underset{\sim}{S}$, then his distribution[3] of $\underset{\sim}{x}$ is normal with mean vector $\underset{\sim}{m}$ and covariance matrix $[(n+1)/n]\underset{\sim}{S} = \underset{\sim}{S} + (1/n)\underset{\sim}{S}$. The expected return on any given portfolio $\underset{\sim}{a}$ is $\underset{\sim}{a}'\underset{\sim}{m}$, and the variance of the portfolio's return is

$$\underset{\sim}{a}'[\underset{\sim}{S} + (1/n)\underset{\sim}{S}]\underset{\sim}{a} = \left(1 + \frac{1}{n}\right) \underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a}. \tag{3}$$

Assuming that $\underset{\sim}{S}$ is positive definite[4] (which is required in order to assure that the distribution of $\tilde{\underset{\sim}{\mu}}$ and $\tilde{\underset{\sim}{x}}$ are proper), the quantity $\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a}$ is positive. Thus, as Kalymon points out, the expression for portfolio risk under the assumption of a known covariance matrix and unknown mean vector exceeds the expression for portfolio risk under the assumption of a known mean vector and covariance matrix by the positive amount $(1/n)\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a}$. Notice that this discussion concerns the effects of the different assumptions upon portfolio

---

observations of $\underset{\sim}{x}$ or that he has information of some other form which he feels is equivalent to such a sample.

3. This distribution is called the predictive distribution of $\underset{\sim}{x}$ in Bayesian terminology. See Winkler [6] for a discussion of the predictive distribution.

4. See Press [5] for a definition and discussion of positive definiteness.

risk for a ***given*** portfolio; it says nothing about how the assumptions might affect the *composition* of an *optimal* portfolio.

Case Three: $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ unknown

Now suppose the investor treats both $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ as unknown and that his joint distribution of $\underset{\sim}{\tilde{\mu}}$ and $\underset{\sim}{\tilde{\Sigma}}$ is a member of the normal-inverted Wishart family.[5] This implies that his distribution of $\underset{\sim}{\tilde{\mu}}$, conditional upon $\underset{\sim}{\Sigma}$, is normal with mean vector $\underset{\sim}{m}$ and covariance matrix $(1/n)\underset{\sim}{\Sigma}$ and that his marginal distribution of $\underset{\sim}{\tilde{\Sigma}}$ is inverted Wishart with parameters $\underset{\sim}{S}$ and $d = n - K + 1$, i.e.,

$$f_{NiW}(\underset{\sim}{\mu}, \underset{\sim}{\Sigma} \, n, \underset{\sim}{m}, \underset{\sim}{S}, d) = f_N\left(\underset{\sim}{\mu}\, \underset{\sim}{m}, \frac{1}{n}\underset{\sim}{\Sigma}\right) f_{iW}(\underset{\sim}{\Sigma}\, \underset{\sim}{S}, d). \tag{4}$$

If the investor's distribution of $(\underset{\sim}{\mu}, \underset{\sim}{\Sigma})$ is normal-inverted Wishart with parameters $(n, \underset{\sim}{m}, \underset{\sim}{S}, d)$, then his predictive distribution of $\underset{\sim}{\tilde{x}}$ is a multivariate t distribution[6] with

$$E(x) = m \tag{5}$$

and

$$V(\underset{\sim}{\tilde{x}}) = \left(\frac{d}{d-2}\right)\left(\frac{n+1}{n}\right)\left(\frac{n-1}{n}\right)\underset{\sim}{S} = \frac{(n^2-1)(n-K+1)}{n^2(n-K-1)}\underset{\sim}{S}, \tag{6}$$

where $V(\underset{\sim}{\tilde{x}})$ denotes the predictive covariance matrix of security returns. Once again the predictive covariance matrix is $\underset{\sim}{S}$ multiplied by a constant. For a given portfolio a, expected portfolio return is $\underset{\sim}{a}'\underset{\sim}{m}$ and the variance of the portfolio's return is

$$\underset{\sim}{a}'\left[\frac{(n^2-1)(n-K+1)}{n^2(n-K-1)}\right]\underset{\sim}{S}\underset{\sim}{a} = \left[\frac{(n^2-1)(n-K+1)}{n^2(n-K-1)}\right]\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a}. \tag{7}$$

The expected return on a given portfolio is the same in all three cases considered here. However, portfolio risk as measured by variance is greater in Case Three than in either of the first two cases, and is greater in Case Two than in Case One. This is easily shown since $\underset{\sim}{S}$ is positive definite (and therefore $\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a} > 0$ for $\underset{\sim}{a} \neq \underset{\sim}{0}$) and

$$\frac{(n^2-1)(n-K+1)}{n^2(n-K-1)} > \frac{(n^2-1)(n+1)}{n^2(n-1)} = \frac{(n+1)^2}{n^2} > \frac{n+1}{n} = 1 + \frac{1}{n} > 1. \tag{8}$$

The fact that the portfolio variance increases from Case One to Case Two and again from Case Two to Case Three is consistent with the intuitive notion that as increased uncertainty is introduced (in the form of additional parameters that are assumed unknown) the perceived risk should increase.

5. See LaValle [4] for a discussion of the normal-inverted Wishart family. This family is conjugate for sampling from a multivariate normal process with unknown μ and $\underset{\sim}{\Sigma}$. Winkler [6] discuss the normal-inverted Wishart in relation to forecasts of security prices.

6. See LaValle [4] or Winkler [6].

II. Effects of the Assumptions on the Efficient Set

Case One represents the usual assumption made in mean-variance analysis, i.e., that $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ are known. As stated above, if $\underset{\sim}{m}$ and $\underset{\sim}{S}$ are treated as the "true" values of $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$, the estimates of portfolio expected return and variance of return for the portfolio denoted by $\underset{\sim}{a}$ are $\underset{\sim}{a}'\underset{\sim}{m}$ and $\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a}$, respectively. A given portfolio a_o is said to be efficient under the assumptions of Case One if

$$\underset{\sim}{a}_o'\underset{\sim}{m} = \max_{\underset{\sim}{a} \epsilon A_1} (\underset{\sim}{a}'\underset{\sim}{m}), \tag{9}$$

where

$$A_1 = \{\underset{\sim}{a} | \underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a} = \underset{\sim}{a}_o'\underset{\sim}{S}\,\underset{\sim}{a}_o\}. \tag{10}$$

Let

$$K_1 = 1 + 1/n \tag{11}$$

and

$$K_2 = [(n^2 - 1)(n - K + 1)]/[n^2(n - K - 1)]. \tag{12}$$

Then a_o is efficient under the assumptions of Case Two if

$$\underset{\sim}{a}_o'\underset{\sim}{m} = \max_{\underset{\sim}{a} \epsilon A_2} (\underset{\sim}{a}'\underset{\sim}{m}) \tag{13}$$

where

$$A_2 = \{\underset{\sim}{a} | K_1\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a} = K_1\underset{\sim}{a}_o'\underset{\sim}{S}\,\underset{\sim}{a}_o\}. \tag{14}$$

Similarly a_o is efficient under the assumptions of Case Three if

$$\underset{\sim}{a}_o'\underset{\sim}{m} = \max_{\underset{\sim}{a} \epsilon A_3} (\underset{\sim}{a}'\underset{\sim}{m}) \tag{15}$$

where

$$A_3 = \{\underset{\sim}{a} | K_2\underset{\sim}{a}'\underset{\sim}{S}\,\underset{\sim}{a} = K_2\,\underset{\sim}{a}_o'\underset{\sim}{S}\,\underset{\sim}{a}_o\}. \tag{16}$$

Note that the sets A_1, A_2, and A_3 are identical since K_1 and K_2 are constants. Thus, criteria (9), (11), and (13) are equivalent. Therefore, a portfolio which is efficient under the assumptions of Case One is also efficient under the assumptions of Cases Two and Three, and *vice versa*.

Since the efficient sets identified using mean-variance analysis are the same in all three cases, an investor willing to make decisions on the basis of mean-variance analysis would be selecting an optimal portfolio from the same set regardless of which case represented his assumptions about the distribution of security returns. However, there is no reason to expect him to choose the same portfolio from among those in the efficient set under the assumptions of each of the three cases since the risk characteristics of the portfolios are perceived differently in each case. That is, the "optimal" portfolio may change even though the efficient set does not. In the next section this point is demonstrated and is discussed in some detail.

III. Uncertainty and the Optimal Portfolio

In the previous section it was pointed out that the efficient sets consist of the same portfolios in each of the three cases considered. Suppose the efficient set in Case One can be described by the continuous function f relating R and σ^2 (portfolio expected return and variance), with $\frac{df}{d\sigma^2} > 0$ and $\frac{d^2f}{d(\sigma^2)^2} < 0$. An example of f is shown in Figure 1.

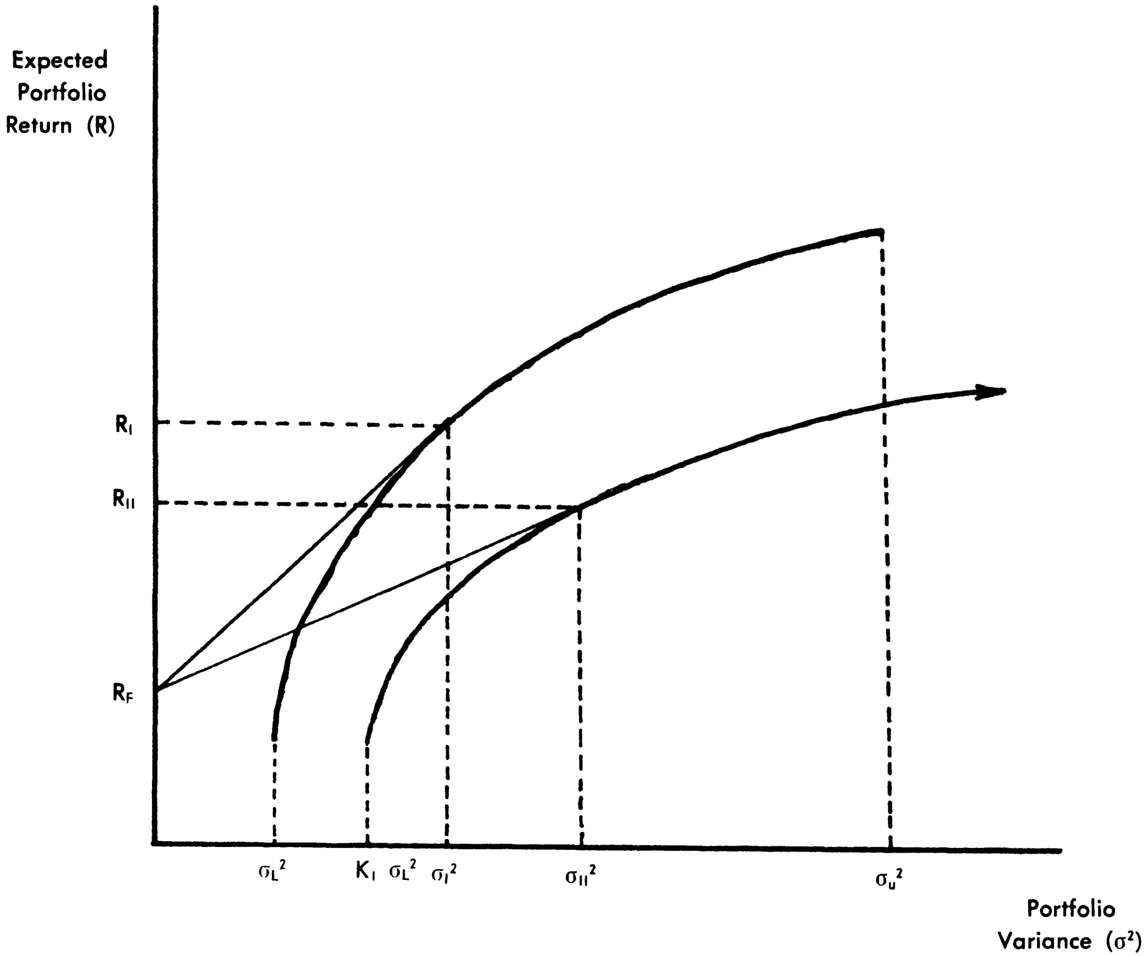

FIGURE 1

It is assumed that f is defined for values of σ^2 on the interval $[\sigma_L^2, \sigma^2_U]$. A portfolio with expected return R and variance of σ^2 in Case One has expected return R and variance $K_1\sigma^2$, $K_1 > 1$, in Case Two. The efficient set in Case Two can be described by the function $R = g(\sigma^2)$, where

$$f(\sigma^2) = g\,(K_1\,\sigma^2). \tag{17}$$

An example of one such function (g) is shown in Figure 1.

Assume the existence of a risk-free rate R_F at which unlimited borrowing or lending can occur. Then the optimal portfolio lies on the ray beginning at $(0, R_F)$ and tangent to f. Suppose this tangency occurs at (σ_I^2, R_I) as shown in Figure 1. Then the portfolio with expected return R_I is the optimal portfolio; the investor invests in this package in combination with borrowing or lending at the rate R_F.

Now consider Case Two, with efficient set $R = g(\sigma^2)$. The optimal portfolio in this case has expected return R_{II}, and it will now be shown that $R_{II} < R_I$, implying that the optimal portfolio decision differs as uncertainty about μ is introduced (recall that a *given* portfolio has the same expected return in all three cases).

Consider any point $R_o \geqslant R_I$, and let σ_o^2 be the point that satisfies $R_o = f(\sigma_o^2)$, which implies that $R_o = g(K_1\sigma_o^2)$. The tangent line to g at R_o has intercept R_1 satisfying[7]

$$\frac{R_o - R_1}{K_1\,\sigma_o^2} = \frac{dg(K_1\sigma_o^2)}{d\sigma^2} = \frac{1}{K_1}\,\frac{df(K_1\sigma_o^2)}{d\sigma^2} < \frac{1}{K_1}\,\frac{df(\sigma_o^2)}{d\sigma^2}, \tag{18}$$

which implies

$$R_o - R_1 < \sigma_o^2\,\frac{df(\sigma_o^2)}{d\sigma^2}, \tag{19}$$

or

$$R_1 > R_o \;-\sigma_o^2\,\frac{df(\sigma_o^2)}{d_o^2} \geqslant R_F. \tag{20}$$

The last inequality falls out of the assumptions that $R_o \geqslant R_I$, $\dfrac{df}{d\sigma^2} > 0$, and $\dfrac{d^2f}{d(\sigma^2)^2} < 0$. Equation (20) states that $R_1 > R_F$, which implies that the point of tangency of the tangent line from R_F to g must occur at some $R_{II} < R_I$, which shows that the optimal portfolio in Case One is not the optimal in Case Two.

By a similar argument it can be shown that the optimal portfolio in Case Three has an expected return $R_{III} < R_{II} < R_I$. For Case Three the additional assumption that the investor's utility function is quadratic is required. This assumption is necessary because in Case Three the predictive distribution of security returns is multivariate t and, strictly speaking, mean-variance analysis is no longer appropriate unless quadratic utility is assumed.

The result that the optimal portfolio changes as uncertainty about $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ is introduced can be demonstrated directly without the assumption of a risk-free rate R_F if some specific assumption about the form of the investor's utility function is made. The proof is quite long even for the seemingly simple case

7. The notation $\dfrac{dg(K_1\,\sigma_o\,2)}{d_\sigma 2}$ is used here to mean $\dfrac{dg}{d_\sigma 2}$ evaluated at $K_1\sigma_o^2$.

of quadratic utility and only two risky securities, however, so it will not be shown here.[8]

The extent to which the optimal portfolios in the three cases will differ depends largely upon the amount of information available concerning returns, i.e., it depends upon n. As n approaches infinity, K_1 and K_2 both approach 1 [see equations (11) and (12)], which implies that the covariance matrices for the predictive distributions in Cases Two and Three converge to the predictive covariance matrix in Case One. In other words, for large n the three efficient frontiers are nearly identical, so they should lead to roughly the same portfolio decisions. Another aspect of the problem as n becomes large is that the multivariate t predictive distribution becomes approximately normal, implying that even in the absence of quadratic utility mean-variance analysis may not be inappropriate in Case Three for sufficiently large n.

Thus if n is allowed to increase without bound, the three cases described herein become indistinguishable. What might prevent n from growing without bound? One answer which has received rather limited attention in the literature is that due to structural changes in firms, markets, and the economy, security price distributions may be nonstationary,[9] i.e., the distributions may change over time. If this is true, then recent information is more relevant than older data for making inferences about security prices, and so the amount of information available in a given historical series would be limited. If n is in fact limited, then the differences in the optimal portfolios across the three cases may be important.

IV. Summary

In this paper three contrasting assumptions about prior knowledge concerning $\underset{\sim}{\mu}$ and $\underset{\sim}{\Sigma}$ have been considered. It was shown that as increased uncertainty is introduced, the risk associated with a given portfolio increases, but that the portfolios identified as efficient in one case remain efficient in the other cases. Although the efficient set contains the same portfolios in all three cases, the map of the set in risk-return space shifts, and this shift causes a change in the portfolio identified as optimal. In cases with only limited prior information differences in the optimal portfolios may be significant and, therefore, should not be ignored.

REFERENCES

1. Christopher B. Barry and Robert L. Winkler. "Portfolio Analysis under Nonstationarity," paper to be presented at the 1974 annual meetings of the Western Economic Association.
2. A. J. Boness, A. H. Chen, and S. Jatusipitak. "A Hypothesis of Non-Stationary Common

8. Barry and Winkler [1] present proofs of the proposition with quadratic and exponential utility functions in a related situation involving unknown and non-stationary means rather than unknown means. The mechanics would be nearly identical to those for the situation discussed in this paper.

9. Boness, Chen, and Jatusipitak [2] recently presented empirical evidence of nonstationarity in the distribution of log price relatives and cited capital structure changes as the principal cause of the nonstationarity they observed. Barry and Winkler [1] have recently considered the problem of forecasting security prices under nonstationarity and investigated the effects of nonstationarity on portfolio choice.

Share Price Changes," unpublished manuscript, State University of New York at Buffalo, 1972.
3. Basil A. Kalymon. "Estimation Risk and the Portfolio Selection Model," *Journal of Financial and Quantitative Analysis,* VI (January, 1971), pp. 559-582.
4. Irving H. LaValle. *An Introduction to Probability, Inference, and Decision* (New York: Holt, Rinehart and Winston, Inc., 1970).
5. S. James Press. *Applied Multivariate Analysis* (New York: Holt, Rinehart and Winston, 1972).
6. Robert L. Winkler. "Bayesian Models for Forecasting Future Security Prices," *Journal of Financial and Quantitative Analysis.* VIII (June, 1973), pp. 387-406.