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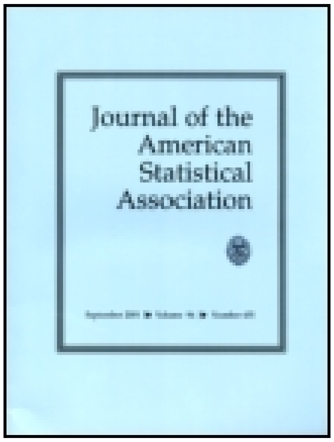

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Estimation for Markowitz Efficient Portfolios

J.D. JOBSON and B. KORKIE*

Given a set of N assets a portfolio is determined by a set of weights x_i, $i = 1, 2, \ldots, N$; $\sum_{i=1}^{N} x_i = 1$ indicating the proportion of the value of the portfolio devoted to each asset. A Markowitz efficient portfolio is the vector of weights $\mathbf{X}_m$ that minimizes the variance σ_m^2 of the total return from the portfolio, subject to the condition that the portfolio mean premium return μ_m has a certain value. The estimators for the $N \times 1$ vector $\mathbf{X}_m$, the return premium μ_m, and the variable σ_m^2 require estimators for the mean premium return vector $\boldsymbol{\mu}$ and for the covariance matrix $\boldsymbol{\Sigma}$. The expectations, variances, and asymptotic distributions of the estimators of $\mathbf{X}_m$, μ_m, and σ_m^2 are derived under the assumption that returns are normally distributed. The use of these sampling properties for statistical inference is also discussed. The derived results are also compared with results obtained from a Monte Carlo simulation for a population of 20 stocks and several sample sizes.

KEY WORDS: Asymptotic distributions; Sampling distributions; Taylor series approximations; Monte Carlo simulation; Distributions of ratios; Fieller-Creasy problem.

1. INTRODUCTION

The theory of portfolio analysis involves the determination of sets of assets that are efficient in a risk-return space. Efficient portfolios are those combinations of assets that have maximum return for a given level of risk or, alternatively, minimum risk for a given level of return. In the Markowitz (1959) formulation of portfolio analysis, the measures of return and risk are the mean and variance of the portfolios' returns. The objects of choice are, therefore, the mean and variance because investors possess quadratic preference functions for return, or, alternatively, the distributions of asset returns are completely specified by their first two moments.

The number of assets available to investors is nearly limitless. The available assets range from riskless zero-variance securities (federal government bills and zero coupon bonds held to maturity) to various other financial securities and real assets. Most financial research has been restricted to financial assets and, more often, common stocks.

An efficient portfolio (allowing unrestricted short sales of assets) is determined by minimizing portfolio variance, subject to a mean portfolio premium return and the additional constraint that investment proportions in risky assets sum to one. The Lagrangian is

$$\min_{\mathbf{X}} L = \mathbf{X}'\boldsymbol{\Sigma}\mathbf{X} - \lambda_1\{\mathbf{X}'\boldsymbol{\mu} - \mu_m\} - \lambda_2\{\mathbf{X}'\mathbf{e} - 1\} \ , \quad (1.1)$$

where $\mathbf{X}$ is an $N \times 1$ vector representing the unknown proportions invested in risky positive-variance assets;

$\boldsymbol{\Sigma}$ is the $N \times N$ covariance matrix of risky assets with rank $r = N$, the number of assets;

$\boldsymbol{\mu} = (\boldsymbol{\mu}^* - \mathbf{e}E_z)$ is the $N \times 1$ vector of asset mean-return premiums, where $\boldsymbol{\mu}^*$ is the mean-return vector, $\mathbf{e}$ is the unit vector, and E_z is the fixed return on the riskless asset z; and

μ_m is the desired mean premium return = (mean portfolio return $-E_z$).

Solution of (1.1) gives the investment proportions or weight vector $\mathbf{X}_m$, mean premium return μ_m, and variance σ_m^2 of the optimal portfolio m. (See Merton 1972 and Roll 1977 for a summary.)

$$\begin{aligned} \mathbf{X}_m &= \boldsymbol{\Lambda}\boldsymbol{\mu}/b = \mathbf{F}_m/b \ , \\ \mu_m &= \boldsymbol{\mu}'\mathbf{X}_m = a/b \ , \\ \sigma_m^2 &= \mathbf{X}_m'\boldsymbol{\Sigma}\mathbf{X}_m = a/b^2 \ , \end{aligned} \quad (1.2)$$

where $\boldsymbol{\Lambda}$ is the inverse of $\boldsymbol{\Sigma}$, $\mathbf{F}_m = \boldsymbol{\Lambda}\boldsymbol{\mu}$ is the nonstandardized weight vector, $a = \boldsymbol{\mu}'\boldsymbol{\Lambda}\boldsymbol{\mu}$, and $b = \mathbf{e}'\boldsymbol{\Lambda}\boldsymbol{\mu}$.

The portfolio m, for given E_z, is a unique Markowitz portfolio, which could be combined with investment in the riskless asset z to produce portfolios that are termed Sharpe (1964) efficient. The solutions (1.2) to this optimization problem do not depend on the existence of a zero-variance asset, with return E_z. In the absence of a zero-variance asset, E_z can be thought of as the mean return from a positive-variance portfolio, whose return is orthogonal to the return on the unique portfolio m. In the parlance of financial economics, E_z is referred to as the return on the zero beta portfolio (see Black 1972 for a discussion).

A major problem, which belies the implementation of this normative theory of portfolio analysis, is the formation of rational expectations regarding the mean-return premium vector $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$ that is appropriate for the investors' holding period. In this article, we assume that the returns from the N stocks are stationary random time series, which are distributed as a multivariate normal with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$. Although the assumption of multivariate normality is suspect for daily and weekly returns, the distribution does not seem to be significantly different from a normal for monthly returns. (See Fama 1976 for a review of the evidence on normality and station-

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arity.) Given that the time series of monthly returns is stationary, sample estimates of the mean-return vector and covariance matrix may be obtained from past returns of the N assets under consideration.

Using a simulation approach for three stocks, Frankfurter, Phillips, and Seagle (1971) conclude that, since sampling error is so large, portfolios selected according to the Markowitz criterion are likely not more efficient than an equally weighted portfolio. For the two-asset case, Dickenson (1974a, b) indicates poor reliability of the estimators of the proportions vector and variance of the global minimum-risk portfolio. In addition, for the two-asset case he provides preliminary indication of the bias in estimating the weight vector of Markowitz portfolios. Other research by Barry (1974, 1975, 1978) and Klein and Bawa (1976, 1977) incorporated estimation risk in the problem with Bayesian procedures and investigated the effect of this risk on final portfolio choice.

In this article we are concerned with the sampling distributions and asymptotic properties for estimators of the weight vector, mean, and variance of Markowitz efficient portfolios for arbitrary numbers of assets. The following section of the article derives the means, variances, covariances, and asymptotic distributions of the estimators. The third section gives the results of a Monte Carlo experiment. Section 4 discusses the effects of changes in $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, N, and E_z, and Section 5 discusses inference for the unknown parameters.

2. EXPECTATIONS, VARIANCES, AND ASYMPTOTIC DISTRIBUTIONS OF THE ESTIMATORS

Assume a random sample of T return premium observations on each of N stocks that is denoted by the $N \times 1$ vector $\mathbf{r}_t$, $t = 1, 2, \ldots, T$. Now assume that $\mathbf{r}_t$ is multivariate normal with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, and denote the sample mean vector and covariance matrix by

$$\bar{\mathbf{r}} = (1/T) \sum_{t=1}^{T} \mathbf{r}_t$$

and

$$\mathbf{V} = [1/(T-1)] \sum_{t=1}^{T} (\mathbf{r}_t - \bar{\mathbf{r}})(\mathbf{r}_t - \bar{\mathbf{r}})' .$$

To estimate $\mathbf{X}_m$, $\mathbf{F}_m$, μ_m, σ_m^2, a, and b, replace $\boldsymbol{\mu}$ and $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$ in (1.2) by estimators based on $\bar{\mathbf{r}}$ and $\mathbf{V}$. It is shown in Anderson (1958, p. 53) that $\bar{\mathbf{r}}$ is an unbiased estimator of $\boldsymbol{\mu}$. Marx and Hocking (1977) have shown that $\mathbf{W} = [(T-N-2)/(T-1)]\mathbf{V}^{-1}$ is an unbiased estimator of $\boldsymbol{\Lambda}$. The sample estimators, of the population parameters identified in (1.2), are therefore

$$\begin{aligned} \hat{a} &= \bar{\mathbf{r}}'\mathbf{W}\bar{\mathbf{r}} , \\ \hat{b} &= \bar{\mathbf{r}}'\mathbf{W}\mathbf{e} , \\ \hat{\mu}_m &= \hat{a}/\hat{b} , \\ \hat{\sigma}_m^2 &= \hat{a}/\hat{b}^2 , \\ \bar{\mathbf{F}}_m &= \mathbf{W}\bar{\mathbf{r}} , \\ \bar{\mathbf{X}}_m &= \mathbf{W}r/\hat{b} . \end{aligned} \qquad (2.1)$$

The columns of $\boldsymbol{\Lambda}$ will be denoted by $\boldsymbol{\Lambda}_j$, $j = 1, \ldots, N$. The elements of $\bar{\mathbf{r}}$, $\mathbf{F}_m$, $\mathbf{X}_m$, $\boldsymbol{\mu}$, $\mathbf{W}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\Lambda}$ will be denoted by $\bar{r}_i$, F_{m_i}, X_{m_i}, μ_i, W_{ij}, σ_{ij}, and λ_{ij}, respectively, $i, j = 1, 2, \ldots, N$. In subsequent sections exact expressions are derived for the expectations and variances of the estimators of the ratio components a, b, and $\mathbf{F}_m$. In addition, approximate expressions are obtained for the expectations and variances of the estimators $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and the elements of $\bar{\mathbf{X}}_m$. Finally, the asymptotic distributions of the estimators are derived.

2.1 Expectations and Variances for $\hat{a}$, $\hat{b}$, and $\bar{F}_m$

The estimators $\hat{b}$ and $\bar{\mathbf{F}}_m$ are unbiased, since $\bar{\mathbf{r}}$ and $\mathbf{W}$ are unbiased and statistically independent. The expectation of $\hat{a}$ is given by

$$E[\hat{a}] = E[\sum_i^N \sum_j^N \bar{r}_i\bar{r}_j W_{ij}]$$

$$= \sum_i^N \sum_j^N \left[\frac{\sigma_{ij}}{T} + \mu_i\mu_j\right][\lambda_{ij}] = a + \frac{N}{T} .$$

Therefore, $\hat{a}$ is biased and the bias is given by (N/T), independent of $\boldsymbol{\mu}$ and $\boldsymbol{\Lambda}$.

It is shown in Marx and Hocking (1977) that the covariance between the (i, j)th and (k, ℓ)th element of $\mathbf{V}^{-1}$ can be written as

$$\left(\frac{T-1}{(T-N-2)}\right)^2 [k_1\lambda_{ij} + k_2(\lambda_{ik}\lambda_{j\ell} + \lambda_{i\ell}\lambda_{jk})]$$

where

$$k_1 = \frac{2}{(T-N-1)(T-N-4)}$$

and

$$k_2 = \frac{(T-N-2)}{(T-N-1)(T-N-4)} .$$

Expressions for the variances of $\hat{a}$ and $\hat{b}$ and the covariance matrix for $\bar{\mathbf{F}}_m$ may therefore be derived. After some simplification, this leads to the following formulas:

$$V(\hat{b}) = \frac{c(T-2)(T-N-2)}{T(T-N-1)(T-N-4)} + \frac{ac(T-N-2)}{(T-N-1)(T-N-4)} + \frac{b^2(T-N)}{(T-N-1)(T-N-4)} , \qquad (2.2)$$

$$\operatorname{cov}(\bar{F}_{m_j}\bar{F}_{m_k}) = \frac{\lambda_{jk}(T-2)(T-N-2)}{T(T-N-1)(T-N-4)} + \frac{F_{m_j}F_{m_k}(T-N)}{(T-N-1)(T-N-4)} + \frac{a\lambda_{jk}(T-N-2)}{(T-N-1)(T-N-4)} . \qquad (2.3)$$

$$V(\hat{a}) = \frac{4a(T-2)}{T(T-N-4)} + \frac{2a^2(T)}{T(T-N-4)} + \frac{2N(T-N)}{T^2(T-N-4)} \cdot \quad (2.4)$$

$$\operatorname{cov}(\hat{a}, \hat{b}) = \frac{2b(T-2)}{T(T-N-4)} + \frac{2ab}{(T-N-4)}, \quad (2.5)$$

$$\operatorname{cov}(\bar{F}_{m_j}, \hat{b}) = \frac{\mathbf{\Lambda}'_j\mathbf{e}(T-2)(T-N-2)}{T(T-N-1)(T-N-4)} + \frac{(T-N)bF_{m_j}}{(T-N-1)(T-N-4)} + \frac{a(\mathbf{\Lambda}'_j\mathbf{e})(T-N-2)}{(T-N-1)(T-N-4)},$$

$$\operatorname{cov}(\bar{F}_{m_j}, \hat{a}) = \frac{2F_{m_j}(T-2)}{T(T-N-4)} + \frac{2aF_{m_j}}{(T-N-4)}, \quad (2.6)$$

where $c = \mathbf{e}'\mathbf{\Lambda}\mathbf{e}$.

For the remainder of the article these moments shall be referred to as exact moments.

The higher moments for $\hat{a}$, $\hat{b}$, and the elements of $\bar{\mathbf{F}}_m$ can be approximated by assuming that the vector $[\hat{a}, \hat{b}, \bar{\mathbf{F}}'_m]$ is multivariate normal. (We shall see in Section 3 that this assumption is reasonable.) With this assumption, all other moments can be expressed as a function of the moments given earlier. In Section 2.2 these moments are required to obtain approximations to mean and variances of $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and the mean vector and covariance matrix for $\bar{\mathbf{X}}_m$.

2.2 Expectations and Variances for $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$

Because $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$ are ratios involving $\hat{a}$, $\hat{b}$, and $\bar{\mathbf{F}}_m$, Taylor series expansions may be employed to approximate the moments of $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$ in terms of the moments of $\hat{a}$, $\hat{b}$, and $\bar{\mathbf{F}}_m$. To obtain higher-order moments of $\hat{a}$, $\hat{b}$, and $\bar{\mathbf{F}}_m$, normality is assumed. The resulting approximate expectations and variances are given by

$$E[\hat{\mu}_m] = \frac{[a+(N/T)]}{b}\left[1+\frac{V(\hat{b})}{b^2}+\frac{3[V(\hat{b})]^2}{b^4}\right] + \frac{1}{b}\left[\frac{-\operatorname{cov}(\hat{a},\hat{b})}{b} - \frac{3V(\hat{b})\operatorname{cov}(\hat{a},\hat{b})}{b^3}\right] + O(T^{-3})$$

$$V[\hat{\mu}_m] = \frac{[a+(N/T)]^2}{b^2}\left[\frac{V(\hat{b})}{b^2}+\frac{8[V(\hat{b})]^2}{b^4}\right] + \frac{2[a+(N/T)]}{b^2}\left[\frac{-\operatorname{cov}(\hat{a},\hat{b})}{b} - \frac{8V(\hat{b})\operatorname{cov}(\hat{a},\hat{b})}{b^3}\right] + \frac{1}{b^2}\left[\frac{3V(\hat{b})V(\hat{a})}{b^2} + V(\hat{a}) + \frac{5[\operatorname{cov}(\hat{a},\hat{b})]^2}{b^2}\right] + O(T^{-3}) .$$

$$E[\hat{\sigma}_m^2] = \frac{[a+(N/T)]}{b^2}\left[1+\frac{3V(\hat{b})}{b^2}+\frac{15[V(\hat{b})]^2}{b^4}\right] + \frac{1}{b^2}\left[-\frac{2\operatorname{cov}(\hat{a},b)}{b} - \frac{12V(\hat{b})\operatorname{cov}(\hat{a},\hat{b})}{b^3}\right] + O(T^{-3})$$

$$V[\hat{\sigma}_m^2] = \frac{[a+(N/T)]^2}{b^4}\left[\frac{4V(\hat{b})}{b^2}+\frac{66[V(\hat{b})]^2}{b^4}\right] + \frac{2[a+(N/T)]}{b^4}\left[-\frac{2\operatorname{cov}(\hat{a},\hat{b})}{b} - \frac{42V(\hat{b})\operatorname{cov}(\hat{a},\hat{b})}{b^3}\right] + \frac{1}{b^4}\left[V(\hat{a}) + \frac{10V(\hat{a})V(\hat{b})}{b^2} + \frac{16[\operatorname{cov}(\hat{a},\hat{b})]^2}{b^2}\right] + O(T^{-3}) .$$

Finally, the approximate expectations and variances of $\bar{X}_{m_j}$ are

$$E[\bar{X}_{m_j}] = \frac{F_{m_j}}{b}\left[1+\frac{V(\hat{b})}{b^2}+\frac{3[V(b)]^2}{b^4}\right] + \frac{1}{b}\left[-\frac{\operatorname{cov}(\hat{b},\bar{\mathbf{F}}_{m_j})}{b} - \frac{3\operatorname{cov}(\hat{b},\bar{F}_{m_j})V(\hat{b})}{b^3}\right] + O(T^{-3})$$

$$V[\bar{X}_{m_j}] = \frac{F_{m_j}^2}{b^2}\left[\frac{V(\hat{b})}{b^2}+\frac{8[V(\hat{b})]^2}{b^4}\right] + \frac{2F_{m_j}}{b^2}\left[-\frac{\operatorname{cov}(\hat{b},\bar{F}_{m_j})}{b} - \frac{8V(\hat{b})\operatorname{cov}(\hat{b},\bar{F}_{m_j})}{b^3}\right] + \frac{1}{b^2}\left[V(\bar{F}_{m_j}) + \frac{3V(\bar{F}_{m_j})V(\hat{b})}{b^2} + \frac{5[\operatorname{cov}(\hat{b},\bar{F}_{m_j})]^2}{b^2}\right] + O(T^{-3}) .$$

For the remainder of the article these moments of $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$ obtained earlier shall be referred to as approximate moments.

Correction for the bias in $\hat{a}$ produces new estimators for μ_m and σ_m^2 given by

$$[\hat{a} - (N/T)]/\hat{b} \quad \text{and} \quad [\hat{a} - (N/T)]/\hat{b}^2 ,$$

respectively. The expectation and variance for these estimators can be found, from the expressions given for $\hat{\mu}_m$ and $\hat{\sigma}_m^2$, by dropping all terms in (N/T). The expression for $\hat{\mu}_m$ is then similar in structure to $\bar{X}_{m_j}$, with $[\hat{a} - (N/T)]$ in place of $\bar{F}_{m_j}$.

2.3 Derivation of Asymptotic Distributions

Applying a theorem given in Rao (1973, p. 387), it can be shown that the asymptotic distributions of the estimators given by (2.1) are normal and asymptotically unbiased. The variances and covariances for the asymptotic distribution of $[\hat{a}, \hat{b}, \bar{\mathbf{F}}'_m]$, as $T \to \infty$, are

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given by

$$A\,V(\hat{a}) = \frac{4a}{T} + \frac{2a^2}{(T-N-4)} = V_{a1} + V_{a2}$$

$$A\,V(\hat{b}) = \frac{c}{T} + \left[\frac{b^2(T-N)}{(T-N-1)(T-N-4)} + \frac{ac(T-N-2)}{(T-N-1)(T-N-4)}\right] = V_{b1} + V_{b2}$$

$$A\,\mathrm{cov}(\bar{F}_{m_j}, \bar{F}_{m_k}) = \frac{\lambda_{jk}}{T} + \left[\frac{(T-N)(F_{m_j})(F_{m_k})}{(T-N-1)(T-N-4)} + \frac{a\lambda_{jk}(T-N-2)}{(T-N-1)(T-N-4)}\right] = C_{1jk} + C_{2jk} \quad (2.7)$$

$$A\,\mathrm{cov}(\hat{a}, \hat{b}) = \frac{2b}{T} + \frac{2ab}{(T-N-4)} = C_{ab1} + C_{ab2}$$

$$A\,\mathrm{cov}(\hat{b}, \bar{F}_{m_j}) = \frac{\mathbf{\Lambda}_j\mathbf{e}}{T} + \left[\frac{(T-N)F_{m_j}b}{(T-N-1)(T-N-4)} + \frac{a(\mathbf{\Lambda}_j\mathbf{e})(T-N-2)}{(T-N-1)(T-N-4)}\right] = C_{bj1} + C_{bj2}\,.$$

The expressions for the asymptotic variances, for $\hat{\mu}_m$ and $\hat{\sigma}_m^2$, and the asymptotic covariances of the elements of $\bar{\mathbf{X}}_m$ also can be written as

$$A\,V(\hat{\mu}_m) = \frac{1}{b^2}\left\{\left[V_{a1} - \frac{2a}{b}C_{ab1} + \frac{a^2}{b^2}V_{b1}\right] + \left[V_{a2} - \frac{2a}{b}C_{ab2} + \frac{a^2}{b^2}V_{b2}\right]\right\}$$

$$A\,V(\hat{\sigma}_m^2) = \frac{1}{b^4}\left\{\left[V_{a1} - \frac{4a}{b}C_{ab1} + \frac{4a^2}{b^2}V_{b1}\right] + \left[4V_{a2} - \frac{8a}{b}C_{ab2} + \frac{4a^2}{b^2}V_{b2}\right]\right\} \quad (2.8)$$

$$A\,\mathrm{cov}(\bar{X}_{m_j}, \bar{X}_{m_k}) = \frac{1}{b^2}\left\{\left[C_{1jk} - \frac{F_{m_j}}{b}C_{bj1} - \frac{F_{m_k}}{b}C_{bk1} + \frac{F_{m_j}F_{m_k}}{b^2}V_{b1}\right] + \left[C_{2jk} - \frac{F_{m_j}}{b}C_{bj2} - \frac{F_{m_k}}{b}C_{bk2} + \frac{F_{m_j}F_{m_k}}{b^2}V_{b2}\right]\right\}.$$

An examination of the expressions for the asymptotic variances and covariances given shows that, unlike the expressions in Section 2.2, terms of $O(T^{-2})$ have been excluded. We shall see in Section 3 that inclusion of the terms of $O(T^{-2})$ provides much greater accuracy for small T.

In later sections all the moments derived in this section—exact, approximate, and asymptotic—will be referred to as theoretical moments. Detailed derivations of the expressions obtained throughout Section 2 are available from the authors on request.

3. RESULTS OF A SAMPLING EXPERIMENT

A Monte Carlo simulation was conducted to compare the sample properties of the estimators in (2.1) with the corresponding theoretical properties obtained in Section 2.

3.1 The Design

An asset population of size 20 was chosen by random selection from the population of common stocks, continuously listed on the New York Stock Exchange during the period December 1949 to December 1975. A total of 313 monthly return observations were obtained for each stock from the Centre for Research in Security Prices (CRSP) financial data tape, where

$$r_{jt} = 100\left[\frac{p_{jt} + d_{jt}}{p_{jt-1}} - 1\right] - E_z\,, \quad \begin{array}{l} j = 1, 2, \ldots, 20 \\ t = 1, 2, \ldots, 313 \end{array}$$

and

r_{jt} = the monthly effective return premium from stock j in month t,
p_{jt} = the closing price of j at month end t, and
d_{jt} = the divided per share paid in month t.

The return was used to compute a population mean-return vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$ of returns from the 20 stocks. These population parameters are shown in Table 1, where the mean returns range from .50 to 1.82 and variances range from 23 to 178. The efficient set parameters a and b and the nonstandardized weight vector $\mathbf{F}_m$ were computed, which then permitted the calculation of the mean μ_m and variance σ_m^2, as well as the proportions vector $\mathbf{X}_m$ for the Markowitz portfolio. Initially the value of E_z was arbitrarily set to zero. At a later stage some results for the case $E_z = .6$ percent are also presented.

The population parameters were then used in the computation of the theoretical moments defined in Section 2. Finally, the simulation experiment employed the population mean vector and covariance matrix to generate sample observations of the parameters. For a fixed number of stocks $N = 20$ and sample size T, ranging from 60 to 1,000 return observations, 100 multivariate normal random deviates with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$ were generated. All computations in the simulation, including the inverse operation, were performed in double-precision FORTRAN. The multivariate normal deviates were generated from subroutine GGNRM of the IMSL Subroutine Library.

The sample estimates of the efficient set parameters a, b, the Markowitz portfolio mean return μ_m and variance σ_m^2, the portfolio weight vectors $\bar{\mathbf{X}}_m$ and $\bar{\mathbf{F}}_m$ were computed by equations (2.1). The mean values and variances of these statistics over the 100 trials were calculated and compared with their theoretical counterparts.

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1. Population Parameters for Simulation Experiment

	Variable in %																			
	r_1	r_2	r_3	r_4	r_5	r_6	r_7	r_8	r_9	r_{10}	r_{11}	r_{12}	r_{13}	r_{14}	r_{15}	r_{16}	r_{17}	r_{18}	r_{19}	r_{20}
Mean	.50	.90	1.10	1.74	1.82	1.11	.91	1.18	1.35	1.07	1.16	1.23	.81	1.18	.88	1.20	.72	1.16	.92	1.25
Covariance matrix	53.64																			
	6.60	29.84																		
	19.84	16.68	82.88																	
	34.14	20.66	48.01	178.07																
	6.32	6.48	18.66	27.54	118.09															
	5.76	11.83	21.04	19.28	26.29	57.07														
	16.92	8.43	22.16	32.35	23.88	20.24	52.05													
	15.26	9.26	16.21	26.87	12.41	11.69	15.29	48.25												
	9.57	10.80	16.26	18.29	14.16	15.23	12.10	9.69	29.80											
	10.12	11.22	18.94	22.39	23.13	16.27	17.74	9.37	11.21	35.12										
	10.33	9.59	21.59	21.75	31.03	13.72	17.95	8.59	13.05	22.58	47.64									
	18.89	8.76	27.01	41.73	13.08	19.29	21.39	14.42	13.83	12.96	16.56	65.62								
	8.45	13.50	8.80	17.34	5.40	7.78	9.58	9.86	7.27	7.92	5.97	7.92	23.51							
	14.58	14.06	23.31	42.93	20.36	21.53	26.36	11.34	16.74	17.62	19.75	23.10	12.03	51.20						
	14.64	16.47	17.40	26.27	9.88	11.33	16.24	13.33	11.42	10.70	9.26	11.55	14.27	16.42	28.72					
	14.34	8.76	22.27	30.30	14.33	13.21	15.17	16.96	8.21	12.57	13.46	25.84	8.54	14.72	12.20	56.03				
	27.85	14.93	36.71	65.96	17.07	13.45	25.64	32.08	15.70	16.21	20.46	35.76	15.22	26.17	19.87	32.30	109.46			
	25.08	16.74	41.43	47.56	20.20	12.32	24.80	21.66	20.61	21.51	18.76	26.41	14.23	25.60	24.34	24.46	50.78	131.75		
	11.77	22.82	21.44	20.67	13.46	16.84	15.50	14.34	14.80	14.23	13.32	16.96	15.80	20.42	22.36	13.14	18.58	26.97	44.66	
	16.92	10.26	27.74	43.93	18.45	18.08	25.27	15.75	10.72	14.74	17.69	23.65	9.72	20.93	13.76	14.48	32.30	29.18	16.09	58.69

3.2 Comparison of Monte Carlo and Theoretical Moments

The means for $\hat{a} - N/T$, $\hat{b}$, and the elements of $\bar{\mathbf{F}}_m$ are in general comparable to the theoretical means for all values of T. In addition, their exact variances are comparable to the simulation variances for all sample sizes. Comparison of the simulation variances of $\hat{a}$, $\hat{b}$, and the elements of $\bar{\mathbf{F}}_m$ to their asymptotic counterparts, however, demonstrated that sample sizes of 300 or more are required before the simulation and asymptotic variances are comparable.

Tables 2a and b compare the theoretical means and variances of $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{X}_{m_j}$ with the simulation means and variances. For sample sizes of at least 300, the approximate means and variances of the estimators $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{X}_{m_j}$, $j = 1, 2, \ldots, 20$, are comparable to their simulation means and variances and, in general, are much closer to the simulation values than the asymptotic values. In some cases the asymptotic values are not comparable to the simulation values at $T = 1{,}000$. For small sample sizes the differences between the approximate means and variances and their simulation counterparts indicate a lack of convergence of the Taylor series expressions for the moments. It can be shown that, when $T > 1/b^2$, the Taylor series expressions for the moments converge. In our simulation experiment $b = .084$ and hence $T > 140$ is required for convergence. This result is consistent with the findings in Tables 2a and b. In conclusion, comparisons between approximate and simulation moments can only be made at sample sizes in excess of 140 for our example.

3.3 Normal Goodness of Fit

Normal goodness-of-fit tests were also performed for each statistic by using the Kolmogorov-Smirnov test. The mean and variance used in the tests were computed from the sample data. The probability of a larger deviation under the assumption of normality was obtained for each statistic. The estimators of a, b and the elements of $\bar{\mathbf{F}}_m$ showed probabilities much larger than .10 for all sample sizes, while the estimators of μ_m and the elements of $\mathbf{X}_m$ were well behaved for sample sizes of 300 or more. In the case of the estimator of σ_m^2, the null hypothesis of normality is only marginally acceptable at sample size 1,000 ($p = .13$). For this estimator both the skewness and kurtosis were relatively large.

For the ratio of two normally distributed random variables $W = X/Y$, Hayya, Armstrong, and Gressis (1975) conclude that W is approximately normal if $|\rho_{XY}| \leq .5$, $\sigma_Y/\mu_Y < .19$, and $\sigma_X/\mu_X > .09$, where μ_X, μ_Y, σ_X^2, σ_Y^2, and ρ_{XY} are the means, variances, and correlation, respectively, of the joint distribution of X and Y. For our study the values of the coefficients of variation and correlations were computed for the terms $\hat{a}$, $\hat{b}$, $\hat{b}^2$, and $\bar{F}_{m_j}$, $j = 1, 2, \ldots, N$, of the ratio estimators $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$. For the numerators $\hat{a}$ and $\bar{F}_{m_j}$, $j = 1, 2, \ldots, N$, the coefficients of variation obtained were well above the critical value of .09. For the denominator $\hat{b}$, the coefficients of variation for sample sizes 60 through 1,000 were .61, .43, .23, .17, and .12, respectively, while for $\hat{b}^2$ the corresponding coefficients were .98, .76, .44, .34, and .24. The correlations between $\hat{a}$ and $\hat{b}$ are .56, .63, .75, .78, and .81 for the respective sample sizes 60, 100, 300, 500, and 1,000, while the corresponding correlations between $\hat{a}$ and $\hat{b}^2$ are .09, .11, .13, .13, and .14. The correlations between $\hat{b}$ and $\bar{F}_{m_j}$, $j = 1, 2, \ldots, N$, ranged from $-.18$ to .32 and did not vary with sample size.

Our earlier finding that the ratios $\bar{X}_{m_j}$, $j = 1, 2, \ldots, N$, and $\hat{\mu}_m$ are normally distributed for sample sizes of 300 or more is consistent with the values of the coefficients of variation obtained. By sample size 300, the coefficient of variation $\hat{b}$ has declined to .23 and seems to be sufficiently close to the required .19 for normality. The correlation coefficient requirement given by Hayya, Armstrong, and Gressis (1975) does not seem to be im-

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2a. Comparison of Simulation and Approximate Means for $\bar{\mathbf{X}}_m$, $\hat{\mu}_m$, and $\hat{\sigma}_m^2$

Parameter	Parameter Value		Means at Various Sample Sizes: 60	100	300	500	1,000
X_1	−.070	SIM	−1.865	−.408	−.070	−.070	−.072
		APP	−.194	−.115	−.080	−.075	−.073
X_2	.116	SIM	−1.489	.042	.110	.111	.121
		APP	.074	.101	.113	.114	.115
X_3	−.017	SIM	.095	−.042	−.008	−.012	−.006
		APP	−.023	−.019	−.017	−.017	−.017
X_4	.016	SIM	.457	.079	.009	.015	.017
		APP	.071	.036	.020	.018	.017
X_5	.090	SIM	.356	.135	.071	.076	.079
		APP	.145	.110	.094	.092	.091
X_6	−.032	SIM	−.406	−.193	−.056	−.046	−.035
		APP	−.069	−.045	−.035	−.034	−.033
X_7	−.071	SIM	.858	−.083	−.084	−.091	−.070
		APP	−.130	−.092	−.075	−.073	−.072
X_8	.178	SIM	−.182	.262	.178	.180	.171
		APP	.243	.201	.183	.181	.179
X_9	.379	SIM	.821	.531	.399	.384	.375
		APP	.518	.428	.389	.384	.381
X_{10}	.077	SIM	.374	.171	.121	.096	.083
		APP	.064	.072	.076	.077	.077
X_{11}	.046	SIM	−.074	.074	.028	.051	.060
		APP	.037	.043	.045	.045	.045
X_{12}	.086	SIM	.314	.125	.088	.091	.083
		APP	.115	.096	.088	.087	.087
X_{13}	.193	SIM	1.536	.347	.210	.209	.194
		APP	.122	.167	.187	.190	.191
X_{14}	.021	SIM	−.027	−.042	.028	.011	.012
		APP	.044	.029	.023	.022	.021
X_{15}	.059	SIM	−.200	−.107	.047	.066	.057
		APP	.040	.052	.057	.058	.058
X_{16}	.131	SIM	−.340	.158	.136	.130	.127
		APP	.173	.146	.134	.133	.132
X_{17}	−.157	SIM	.023	−.193	−.149	−.142	−.147
		APP	−.230	−.183	−.162	−.160	−.158
X_{18}	−.016	SIM	−.426	.003	−.019	−.023	−.022
		APP	−.003	−.011	−.015	−.015	−.016
X_{19}	−.155	SIM	−.221	−.152	−.171	−.173	−.157
		APP	−.174	−.162	−.157	−.156	−.155
X_{20}	.127	SIM	1.394	.295	.133	.136	.129
		APP	.177	.145	.131	.129	.128
μ_M	1.38	SIM	12.98	5.61	2.15	1.83	1.61
		APP	8.07	4.28	2.19	1.85	1.61
σ_M^2	16.33	SIM	1.8×10^4	413.87	26.87	21.54	18.75
		APP	212.66	74.18	27.75	22.68	19.35

portant, in that the correlation between $\hat{a}$ and $\hat{b}$ exceeded the .5 limit for all sample sizes, even though normality of $\hat{\mu}_m$ was not rejected for sample sizes exceeding 300. In addition, the correlations between $\hat{b}$ and the $\bar{F}_{m_j}$, $j = 1, 2, \ldots, 20$, were below .5 in absolute value, and the ratios $\bar{X}_{m_j}$ did not appear to approach normality at sample sizes lower than those for $\hat{\mu}_m$.

In the case of $\hat{\sigma}_m^2$, the coefficient of variation for $\hat{b}^2$ was still above .19 at sample size 1,000, which seems to be consistent with our findings that $\hat{\sigma}_m^2$ could not be assumed to be normally distributed below samples of size 1,000. At a sample size of 1,000, the coefficient of variation of $\hat{b}^2$ was .24, which appears to be sufficiently close to the critical .19, so that the null hypothesis of normality discussed before could not be rejected at significance levels below .13.

4. THE EFFECTS OF CHANGES IN $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, N, AND E_z

From Sections 2 and 3, the parameters a, b, c, and $\mathbf{F}_m$, together with T and N, determine the sampling properties of the estimators $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$. Because a, b, c, and $\mathbf{F}_m$ are functions of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, it is important to determine how the parameters are affected by changes in $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$, N, and E_z.

For simplicity, assume that all the diagonal elements of $\boldsymbol{\Sigma}$ are equal to α and all the off-diagonal elements are equal to β. The matrix $\boldsymbol{\Lambda} = \boldsymbol{\Sigma}^{-1}$ can be written as

$$\boldsymbol{\Lambda} = \frac{1}{(\alpha - \beta)}\left[\mathbf{I} - \frac{\beta}{(\alpha + (N-1)\beta)}\mathbf{J}\right],$$

where $\mathbf{J}$ is a matrix of unities (See Graybill 1969, p. 172). Hence,

$$\lambda_{ii} = \frac{1}{(\alpha - \beta)}\left[1 - \frac{\beta}{(\alpha + (N-1)\beta)}\right], \quad i = 1, 2, \ldots, N \tag{4.1}$$

$$\lambda_{ij} = \frac{-\beta}{(\alpha + (N-1)\beta)}, \quad i \neq j \quad i, j = 1, 2, \ldots, N .$$

The parameters a, b, c, and F_{m_j} may therefore be written as

$$a = \sum_{i=1}^{N}\sum_{j=1}^{N} \mu_i\mu_j\lambda_{ij} = \frac{NV(\mu)}{(\alpha - \beta)} + \frac{N\bar{\mu}^2}{(\alpha + (N-1)\beta)}$$

$$b = \sum_{i=1}^{N}\sum_{j=1}^{N} \mu_i\lambda_{ij} = \frac{N\bar{\mu}}{(\alpha + (N-1)\beta)} \tag{4.2}$$

$$c = \frac{N}{(\alpha + (N-1)\beta)}$$

$$F_{m_j} = \frac{\mu_j}{(\alpha - \beta)} - \frac{N\bar{\mu}\beta}{(\alpha - \beta)(\alpha + (N-1)\beta)}, \quad j = 1, 2, \ldots, N ,$$

where

$$\bar{\mu} = \sum_{i=1}^{N} \mu_i/N \quad \text{and} \quad V(\mu) = \left(\sum_{i=1}^{N} \mu_i^2/N\right) - \bar{\mu}^2$$

are the mean and variance, respectively, of the elements of $\boldsymbol{\mu}$.

Examination of these expressions reveals that, as N increases, b, c, and F_{m_j} approach $\bar{\mu}/\beta$, $1/\beta$, and $(\mu_j - \bar{\mu})/$

$(\alpha - \beta)$, respectively. Thus, if the number of stocks N is large, b grows as $\bar{\mu}$ grows relative to β, while c increases as β decreases. In the case of F_{m_j}, as N increases, F_{m_j} grows as $(\mu_j - \bar{\mu})$ increases relative to $(\alpha - \beta)$.

In the case of a, as N increases a increases without limit. For fixed N, a varies directly with $V(\mu)$, $\bar{\mu}$, and β and inversely with α. Therefore, larger values of a, b, c, and F_{m_j} result if β decreases. Increases in $\bar{\mu}$ bring about increases in a and b, while increases in μ_j relative to $\bar{\mu}$ will cause F_{m_j} to increase.

In Section 3, it was shown that the magnitude of b influenced the sample size required for the convergence of the Taylor series approximations for the moments of $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$. Therefore, increases in $\bar{\mu}$ and/or decreases in β are beneficial in reducing the sample size required to obtain good approximations to the moments.

In Section 1, the mean vector of return premiums $\boldsymbol{\mu}$ was defined as $\boldsymbol{\mu}^* - E_z\mathbf{e}$, where E_z is a constant and $\boldsymbol{\mu}^*$ is a vector of mean returns. Increases in E_z, therefore, bring about decreases in $\bar{\mu}$ and hence corresponding decreases in the parameters a and b. In the simulation experiment of Section 3, E_z was set at .0. For the same $\boldsymbol{\mu}^*$ and $\boldsymbol{\Sigma}$ and sample size $T = 300$, an additional simulation was performed for $E_z = .60$. The sampling properties of the estimators of a, b, and the elements of $\mathbf{F}_m$ were comparable to the asymptotic properties. For the ratios μ_m, σ_m^2 and the elements of $\mathbf{X}_m$, however, the simulation moments far exceeded the asymptotic moments.

2b. Comparison of Simulation, Asymptotic, and Approximate Variances for $\bar{\mathbf{X}}_m$, $\hat{\mu}_m$, and $\hat{\sigma}_m^2$

Table 2b. (Continued)

Parameter		Variances at Various Sample Sizes: 60	100	300	500	1,000
X_1	SIM	260.052	2.579	.022	.010	.004
	ASY	.077	.045	.015	.009	.004
	APP	.252	.089	.018	.010	.005
X_2	SIM	208.183	1.123	.040	.024	.012
	ASY	.184	.107	.035	.021	.010
	APP	.561	.202	.043	.024	.011
X_3	SIM	2.541	.751	.011	.005	.003
	ASY	.053	.031	.010	.006	.003
	APP	.165	.059	.012	.007	.003
X_4	SIM	12.222	.077	.007	.004	.002
	ASY	.027	.016	.005	.003	.002
	APP	.087	.031	.006	.003	.002
X_5	SIM	6.867	.239	.007	.005	.002
	ASY	.033	.019	.006	.004	.002
	APP	.097	.035	.008	.004	.002
X_6	SIM	20.051	.868	.018	.009	.005
	ASY	.074	.043	.014	.008	.004
	APP	.232	.083	.017	.010	.004
X_7	SIM	98.681	.460	.022	.011	.007
	ASY	.096	.056	.018	.011	.005
	APP	.297	.106	.022	.012	.006
X_8	SIM	27.066	1.040	.021	.010	.005
	ASY	.081	.047	.015	.009	.005
	APP	.225	.083	.018	.010	.005
X_9	SIM	21.501	.848	.037	.017	.007
	ASY	.141	.082	.027	.016	.008
	APP	.301	.124	.030	.017	.008
X_{10}	SIM	2.728	.248	.034	.013	.008
	ASY	.138	.080	.026	.016	.008
	APP	.424	.152	.032	.018	.008
X_{11}	SIM	3.804	.889	.021	.011	.005
	ASY	.101	.059	.019	.011	.006
	APP	.311	.111	.024	.013	.006
X_{12}	SIM	6.579	.271	.018	.007	.005
	ASY	.069	.040	.013	.008	.004
	APP	.206	.075	.016	.009	.004
X_{13}	SIM	158.47	1.466	.042	.021	.009
	ASY	.178	.104	.034	.020	.010
	APP	.519	.190	.041	.023	.011
X_{14}	SIM	1.050	1.000	.021	.012	.005
	ASY	.103	.060	.020	.012	.006
	APP	.322	.115	.024	.013	.006
X_{15}	SIM	15.410	.946	.052	.025	.011
	ASY	.225	.131	.043	.026	.013
	APP	.699	.250	.053	.029	.013
X_{16}	SIM	16.492	.213	.016	.008	.004
	ASY	.070	.041	.013	.008	.004
	APP	.202	.074	.016	.009	.004
X_{17}	SIM	5.934	.109	.014	.007	.003
	ASY	.048	.028	.009	.005	.003
	APP	.127	.048	.011	.006	.003
X_{18}	SIM	11.922	.066	.006	.003	.002
	ASY	.031	.018	.006	.004	.002
	APP	.097	.035	.007	.004	.002
X_{19}	SIM	1.353	.547	.032	.019	.008
	ASY	.137	.080	.026	.017	.008
	APP	.402	.147	.032	.018	.008
X_{20}	SIM	182.204	1.240	.019	.008	.005
	ASY	.076	.044	.014	.009	.004
	APP	.222	.082	.018	.010	.005
μ_M	SIM	4244.400	76.000	.140	.065	.026
	ASY	.397	.236	.078	.047	.023
	APP	24.014	3.098	.168	.072	.028
σ_M^2	SIM	3.2×10^{10}	5.7×10^6	99.73	23.70	7.40
	ASY	70.16	40.86	13.34	7.98	3.98
	APP	28,162.1	3,226.00	97.54	29.295	8.49

The decrease in $\boldsymbol{\mu}$ by .60 caused b to decrease from .084 to .029 and a to decrease from .116 to .043. The sample size required for convergence of the Taylor series approximations increased from 140 to $1/b^2 = 1{,}190$. Thus, the lack of agreement between the asymptotic moments and simulation moments is easily accounted for by the lack of convergence of the Taylor series expansions.

Expressions 4.1 and 4.2 may be used to determine the effects of changes in $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ on the variances and coefficients of variation. The coefficient of variation for $\hat{b}$ can be written approximately as

$$\mathrm{CV}(\hat{b}) \approx \left[\frac{1}{(T-N)}\left\{\frac{c}{b^2} + \frac{ac}{b^2} + 1\right\}\right]^{\frac{1}{2}}$$
$$= \left[\frac{1}{(T-N)}\left\{\frac{(\alpha + (N-1)\beta)}{N\bar{\mu}^2} + \frac{V(\mu)}{\bar{\mu}^2}\frac{(\alpha + (N-1)\beta)}{(\alpha - \beta)} + 2\right\}\right]^{\frac{1}{2}}.$$

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Increases in β or decreases in $\bar{\mu}$ cause increases in $CV(\hat{b})$. As mentioned in Section 3, an increase in $CV(\hat{b})$ increases the sample size T required for the distributions of the ratios $\hat{\mu}_m$ and $\bar{X}_{m_j}$, $j = 1, 2, \ldots, N$, to approach normality. We conclude that an increase in E_z of .60 (hence a decrease in $\bar{\mu}$) results in a marked departure from normality for the ratios $\bar{X}_{m_j}$, $j = 1, 2, \ldots, 20$, $\hat{\mu}_m$ and σ_m^2, at $T = 300$.

In addition to this simulation experiment the authors have repeated some of these analyses for other populations with as many as 500 trials. The results of these simulations are quite comparable to our results in this article. When $(1/b^2) > T$, the simulation moments for the estimators of μ_m and σ_m^2 can be extremely large relative to the approximate moments, and the magnitudes of the simulation moments are unstable from experiment to experiment.

5. STATISTICAL INFERENCE FOR μ_m, σ_m^2, AND $\mathbf{X}_m$

It was shown in Sections 2 and 3 that the sampling distributions of the estimators $\hat{a}$, $\hat{b}$, and $\bar{F}_{m_j}$, $j = 1, 2, \ldots, N$ are approximately normal and that closed-form expressions for the means and variances exist. The means and variances for the ratio estimators $\hat{\mu}_m$, $\hat{\sigma}_m^2$, and $\bar{\mathbf{X}}_m$, however, are not obtainable in closed form, and for small samples the Taylor series approximations to the moments do not converge. In addition, the sampling distributions of these ratio estimators differed from normality for small sample sizes.

In the case of bivariate normal random variables X and Y, the distribution of the ratio $W = Y/X$ was first studied by Geary (1930) and Fieller (1932). Later articles by Fieller (1954), Creasy (1954), Marsaglia (1965), and Hinkley (1969) have also discussed various aspects of the distribution of W, as well as inference procedures for W.

It was shown by Hinkley (1969) that the transformed variable

$$Z = (X - WY)/(\sigma_X^2 + W^2\sigma_Y^2 - 2W\sigma_{XY})^{\frac{1}{2}} \quad (5.1)$$

approaches a standard normal random variable, as the coefficient of variation of Y, σ_Y/μ_Y, approaches zero. Monte Carlo simulation studies by Hayya, Armstrong, and Gressis (1975) have shown that, if $\sigma_Y/\mu_Y < .39$ and $\sigma_X/\mu_X > .05$, the distribution of Z is approximately normal. In their simulation, Hayya, Armstrong, and Gressis used the true parameters σ_Y, σ_X, and σ_{XY} to compute Z.

From Section 3, the coefficients of variation for the $\bar{F}_{m_j}$, $j = 1, 2, \ldots, 20$ and $\hat{a}$ are well above the required .05 for all sample sizes. The coefficient of variation of $\hat{b}$ is below the required .39 for samples of size 300 or more, and for $\hat{b}^2$ the coefficient of variation is below .39 for samples of size 500 or more. In addition, at sample sizes 100 for $\hat{b}$ and 300 for $\hat{b}^2$, the coefficient of variation is only slightly above .39, at .43 and .44, respectively.

For each of the 100 samples of Section 3, Z values were obtained by computing the quantities

$$\begin{aligned} Z_\mu &= [(\hat{a} - N/T) - \mu_m\hat{b}]/ \\ &\qquad [V(\hat{a}) + \mu_m^2V(\hat{b}) - 2\mu_m \operatorname{cov}(\hat{a}, \hat{b})]^{\frac{1}{2}} , \\ Z_{\sigma^2} &= [(\hat{a} - N/T) - \sigma_m^2\hat{b}^2]/ \\ &\qquad [V(\hat{a}) + \sigma_m^4V(\hat{b}^2) - 2\sigma_m^2 \operatorname{cov}(\hat{a}, \hat{b}^2)]^{\frac{1}{2}} , \quad (5.2) \\ Z_{X_j} &= [\bar{F}_{m_j} - X_{m_j}\hat{b}]/ \\ &\qquad [V(\bar{F}_{m_j}) + X_{m_j}^2V(\hat{b}) - 2X_{m_j} \operatorname{cov}(\bar{F}_{m_j}, \hat{b})]^{\frac{1}{2}} , \end{aligned}$$

using the true values of μ_m, σ_m^2, and X_{m_j}, $j = 1, 2, \ldots, 20$. Estimates of the variances and covariances were made by replacing a, b, c, F_{m_j}, and λ_{jj} by $\hat{a}$, $\hat{b}$, $\hat{c}$, $\bar{F}_{m_j}$, and W_{ij}, $j = 1, 2, \ldots, 20$, respectively, in the expressions (2.2) through (2.6). As in Section 2, $V(\hat{b}^2)$ and $\operatorname{cov}(\hat{a}, \hat{b})^2$ were obtained by assuming that $\hat{a}$ and $\hat{b}$ are normally distributed. The means and variances of the Z values were computed over the 100 trials and $K - S$ normal goodness-of-fit tests were performed.

Table 3 shows the means and variances of the various Z values. The means and variances of Z_μ and Z_{σ^2} were omitted for sample size 60 and 100 because of several extremely large sample values. Note that, for Z_{X_j}, $j = 1, 2, \ldots, 20$, the means and variances are approximately 0 and 1, respectively. For sample sizes 60 and 100, the variances of the Z_{X_j}, $j = 1, 2, \ldots, 20$ tended to be less than 1. In the case of Z_μ and Z_{σ^2}, the means were negative for all sample sizes. The variances of Z_μ were larger than 1.0 at sample size 300 and 500. The normal goodness-of-fit tests (excluding Z_μ and Z_{σ^2} for $T = 60$, 100) showed that the null hypothesis of normality could not be rejected except at very large significance levels.

It was suggested by Fieller (1932) that confidence intervals for W in (5.1) can be obtained from a confidence interval for Z by solving a quadratic equation in W. This approach has been used in practice by Fuller and Martin (1961) and Beveridge (1975), for example. From our results, we conclude that the transformed variables Z_{X_j}, $j = 1, 2, \ldots, 20$, Z_μ, and Z_{σ^2} given in (5.2) may be used to make inferences about X_{m_j}, $j = 1, 2, \ldots, 20$, μ_m, and σ_m^2 for samples of size 300 or more. In the case of the X_{m_j}, $j = 1, 2, \ldots, 20$, the Z_{X_j} may also be used for samples as small as 60.

It may be of interest to test whether a given portfolio with the proportions vector $\mathbf{X}_p$ is significantly different from any Markowitz efficient portfolio, as opposed to the efficient portfolio obtained from a specified E_Z. If $\mathbf{X}_p$ is a Markowitz efficient portfolio other than the global minimum portfolio, then there exists an E_Z such that $\mathbf{X}_p = \mathbf{\Lambda}(\boldsymbol{\mu}^* - \mathbf{e}E_Z)/\mathbf{e}'\mathbf{\Lambda}(\boldsymbol{\mu}^* - \mathbf{e}E_Z)$. Defining $b^* = \mathbf{e}'\mathbf{\Lambda}\boldsymbol{\mu}^*$, $\mathbf{F}_m^* = \mathbf{\Lambda}\boldsymbol{\mu}^*$ and recalling that $c = \mathbf{e}'\mathbf{\Lambda}\mathbf{e}$ and $\boldsymbol{\mu}^*$ is the mean-return vector defined in Section 1, we may write $(b^*\mathbf{X}_p - \mathbf{F}_m^*) = E_Z(c\mathbf{X}_p - \mathbf{\Lambda}\mathbf{e})$, and, hence, for each j, $j = 1, 2, \ldots, N$, we have $(b^*X_{p_j} - F_{m_j}^*) = E_Z(cX_{p_j} - \mathbf{\Lambda}'_j\mathbf{e})$. For a given $\mathbf{X}_p$, confidence intervals for E_Z may be obtained in a manner similar to (5.2).

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3. Sample Moments of Z Distributions

Parameter	Moment	Sample Size 60	100	300	500	1,000
Z_{x_1}	Mean	−.057	−.193	.074	.069	.013
	var	.951	1.164	1.183	1.149	1.059
Z_{x_2}	Mean	−.134	.009	.012	−.000	.067
	var	.840	.927	.927	1.092	1.114
Z_{x_3}	Mean	−.040	.076	.116	.061	.193
	var	1.054	1.072	.935	.815	1.124
Z_{x_4}	Mean	.144	.090	−.147	−.058	−.026
	var	.685	.916	1.131	1.118	1.131
Z_{x_5}	Mean	−.092	−.131	−.256	−.267	−.273
	var	.771	1.000	.859	1.196	1.144
Z_{x_6}	Mean	.073	−.095	−.154	−.122	−.021
	var	.848	.948	1.024	1.051	1.095
Z_{x_7}	Mean	.078	.013	−.065	−.162	.036
	var	.953	.916	.876	.922	1.225
Z_{x_8}	Mean	−.075	−.132	−.042	−.016	−.110
	var	.961	.861	1.121	1.056	.939
Z_{x_9}	Mean	.128	.170	.044	.013	−.064
	var	.965	.932	1.127	.986	.894
$Z_{x_{10}}$	Mean	.114	.124	.223	.104	.039
	var	.666	.718	.843	.805	1.065
$Z_{x_{11}}$	Mean	−.083	−.107	−.103	.071	.203
	var	.791	.947	.811	.929	.933
$Z_{x_{12}}$	Mean	−.027	.009	−.007	.051	−.040
	var	1.001	.914	1.031	.919	1.185
$Z_{x_{13}}$	Mean	.148	.184	.102	.126	.013
	var	.703	.992	1.001	.938	.873
$Z_{x_{14}}$	Mean	−.000	−.040	.019	−.114	−.128
	var	.801	.754	.948	.993	.803
$Z_{x_{15}}$	Mean	−.065	−.089	−.047	.034	−.026
	var	1.017	1.127	.960	.919	.842
$Z_{x_{16}}$	Mean	−.049	.025	.016	−.017	−.078
	var	.762	.697	1.014	.988	.995
$Z_{x_{17}}$	Mean	−.056	.092	.148	.248	.237
	var	.974	1.129	1.283	1.166	1.141
$Z_{x_{18}}$	Mean	−.072	−.036	−.091	−.133	−.168
	var	.974	1.172	.866	.883	.990
$Z_{x_{19}}$	Mean	−.015	−.092	−.058	−.128	−.014
	var	.836	.966	.964	1.128	1.037
$Z_{x_{20}}$	Mean	−.061	.059	.016	.066	.025
	var	.779	.918	1.076	.916	1.037
Z_{μ}	Mean	—	—	−.321	−.248	−.100
	var	—	—	1.459	1.338	1.031
Z_{σ^2}	Mean	—	—	−.363	−.376	−.311
	var	—	—	1.066	1.076	.970

Define the Z variable $Z_{E_j} = D_j/[V(D_j)]^{\frac{1}{2}}$, where $D_j = (\hat{b}^* X_{p_j} - \bar{F}_{m_j}{}^*) - E_Z(\hat{c} X_{p_j} - \mathbf{W}'_j\mathbf{e})$, $j = 1, 2, \ldots, N$, and $\hat{b}^*$, $\bar{\mathbf{F}}_m{}^*$, and $\hat{c}$ are obtained by replacing $\boldsymbol{\mu}^*$ and $\boldsymbol{\Lambda}$ by the sample quantities $\bar{\mathbf{r}}^*$ and $\mathbf{W}$, where $\bar{\mathbf{r}}^*$ is the sample mean-return vector. It is easily shown that $E[D_j] = 0$ and that

$$\begin{aligned} V[D_j] = {} & X_{p_j}{}^2 V(\hat{b}^*) - 2X_{p_j} \operatorname{cov}(\hat{b}^*, \bar{F}_{m_j}) \\ & + V(\bar{F}_{m_j}{}^*) - 2E_{Z_j}[X_{p_j}{}^2 \operatorname{cov}(\hat{b}^*, \hat{c}) \\ & + \operatorname{cov}(\bar{F}_{m_j}{}^*, \mathbf{W}'_j\mathbf{e}) - X_{p_j} \operatorname{cov}(\hat{c}, \bar{F}_{m_j}) \\ & - X_{p_j} \operatorname{cov}(\hat{b}^*, \mathbf{W}'_j\mathbf{e})] \\ & + E_Z{}^2[X_{p_j}{}^2 V(\hat{c}) - 2X_{p_j} \operatorname{cov}(\hat{c}, \mathbf{W}'_j\mathbf{e}) + V(\mathbf{W}'_j\mathbf{e})] \ . \end{aligned}$$

Expressions for $V(\hat{b}^*)$, $\operatorname{cov}(\hat{b}^*, \bar{F}_{m_j})$, and $V(\bar{F}_{m_j}{}^*)$, are obtained from the expressions for $V(\hat{b})$, $\operatorname{cov}(\hat{b}, \bar{F}_{m_j})$, and $V(\bar{F}_{m_j})$ in Section 2.1 by replacing b, F_{m_j} and a by b^*, $F_{m_j}{}^*$ and $a^* = \boldsymbol{\mu}^{*\prime}\boldsymbol{\Lambda}\boldsymbol{\mu}^*$. The remaining expressions are defined by

$$V[\hat{c}] = \frac{2c^2}{(T - N - 4)}$$

$$\operatorname{cov}(\hat{c}, \mathbf{W}'_j\mathbf{e}) = \frac{2(\boldsymbol{\Lambda}'_j\mathbf{e})}{(T - N - 4)}$$

$$V[\mathbf{W}'_j\mathbf{e}] = \frac{(T - N)(\boldsymbol{\Lambda}'_j\mathbf{e})^2}{(T - N - 1)(T - N - 4)} + \frac{(T - N - 2)c\lambda_{jj}}{(T - N - 1)(T - N - 4)}$$

$$\operatorname{cov}[\hat{b}^*, \hat{c}] = \frac{2b^* c}{(T - N - 4)}$$

$$\operatorname{cov}[\bar{F}_{m_j}{}^*, \mathbf{W}'_j\mathbf{e}] = \frac{(T - N)F_{m_j}{}^*(\boldsymbol{\Lambda}'_j\mathbf{e})}{(T - N - 1)(T - N - 4)} + \frac{(T - N - 2)b^*\lambda_{jj}}{(T - N - 1)(T - N - 4)}$$

$$\operatorname{cov}[\hat{b}^*, \mathbf{W}'_j\mathbf{e}] = \frac{(T - N)b^*(\boldsymbol{\Lambda}'_j\mathbf{e})}{(T - N - 1)(T - N - 4)} + \frac{(T - N - 2)cF_{m_j}{}^*}{(T - N - 1)(T - N - 4)}$$

$$\operatorname{cov}[\hat{c}, \bar{F}_{m_j}{}^*] = \frac{2cF_{m_j}{}^*}{(T - N - 1)(T - N - 4)} + \frac{2(T - N - 2)b^*(\boldsymbol{\Lambda}'_j\mathbf{e})}{(T - N - 1)(T - N - 4)} .$$

Estimators for the components of $V[D_j]$ can be obtained by replacing $\boldsymbol{\mu}^*$ and $\boldsymbol{\Lambda}$ by $\bar{\mathbf{r}}^*$ and $\mathbf{W}$ in the definitions of a^*, b^*, c, and $F_m{}^*$.

The magnitudes of the interval estimators of E_Z may now be judged for reasonableness. In theory $0 < E_Z < b^*/c$, where b^*/c is the mean return for the global minimum-variance portfolio. A confidence interval for $\mu_0 = b^*/c$ can be obtained from the sample information by employing the estimator $\hat{\mu}_0 = \hat{b}^*/\hat{c}$. It can be shown that $E[\hat{\mu}_0] = \mu_0$ and

$$V[\hat{\mu}_0] = \mu_0{}^2 V(\hat{c})/c^2 - 2\mu_0 \operatorname{cov}(\hat{b}^*, \hat{c})/c^2 + V(\hat{b}^*)/c^2 + O(T^{-2}) \ ,$$

and, therefore, we define $Z_0 = (\hat{\mu}_0 - \mu_0)/[V(\hat{\mu}_0)]^{\frac{1}{2}}$. Estimating $V(\hat{\mu}_0)$ by replacing $\boldsymbol{\mu}^*$ and $\boldsymbol{\Lambda}$ by $\bar{\mathbf{r}}^*$ and $\mathbf{W}$ in the expressions for $V(\hat{c})$, $\operatorname{cov}(\hat{b}^*, \hat{c})$, and $V(\hat{b}^*)$, we can obtain a confidence region for μ_0. If μ_{0_u} is the upper limit obtained from this interval than the acceptable range for E_Z is $(0, \mu_{0_u})$.

For each component of $\mathbf{X}_p$, X_{p_j}, $j = 1, 2, \ldots, N$ the procedure shown before gives an interval estimate for E_Z. If a single estimator of E_Z is desired, two conventional estimators that may be used are the ratio estimator

$$\hat{E}_{Z_1} = \sum_{j=1}^{N} (\hat{b}^* X_{m_j} - \bar{\mathbf{F}}_{m_j}{}^*)/(\hat{c} X_{m_j} - \mathbf{W}'_j \mathbf{e})$$

and the regression estimator

$$\hat{E}_{Z_2} = [\sum_{j=1}^{N} (\hat{b}^* X_{m_j} - \bar{F}_{m_j}{}^*)(\hat{c} X_{m_j} - \mathbf{W}'_j \mathbf{e})] / [\sum_{j=1}^{N} (\hat{c} X_{m_j} - \mathbf{W}'_j \mathbf{e})^2] \cdot$$

Both these estimators, however, are biased and for inference purposes suffer from the same shortcomings as the estimators of μ_m, $\sigma_m{}^2$, and $\mathbf{X}_m$ discussed in Section 3.

6. SUMMARY AND CONCLUDING REMARKS

Under the assumption of the normality of asset returns, this article has examined the sampling properties of the conventional estimators for the parameters of an efficient portfolio. The parameters estimated are the efficient set constants a and b, the standardized and nonstandardized weight vectors $\mathbf{X}_m$ and $\mathbf{F}_m$, and the mean and variance μ_m and $\sigma_m{}^2$ of an efficient portfolio. Because many of the derived sampling properties are approximate, a sampling experiment was used to determine the applicability of the derived properties at various sample sizes.

The exact moments of the estimators of a, b, and the elements of F_{m_j} were derived. The estimators of a, b, and the elements of $\mathbf{F}_m$ were found to be comparable to a normal distribution for all sample sizes. The estimators of b and $\mathbf{F}_m$ were shown to be unbiased, while the estimator of a was shown to have bias (N/T). The asymptotic variances for $\hat{a}$, $\hat{b}$, and $\bar{\mathbf{F}}_m$ differed substantially from their exact counterparts at sample size less than 300.

Assuming normality for the distributions of $\hat{a}$, $\hat{b}$, and the elements of $\bar{\mathbf{F}}_m$, Taylor series approximations were obtained from the means and variances of the ratios $\hat{\mu}_m$, $\hat{\sigma}_m{}^2$, and $\bar{X}_{m_j}$, $j = 1, 2, \ldots, N$. The theoretical means and variances and simulation means and variances were compared. For sample sizes of at least 300 the approximate means and variances were found to be comparable to the simulation values. The convergence of the expressions for the means and variances of $\hat{\mu}_m$, $\hat{\sigma}_m{}^2$, and $\bar{X}_{m_j}$, $j = 1, 2, \ldots, N$, were shown to depend on the magnitude of T relative to $1/b^2$. For the $\bar{X}_{m_j}$, $j = 1, 2, \ldots, N$, the asymptotic means and variances were comparable to the simulation means and variances for sample sizes 500 and 1,000. For $\hat{\mu}_m$ and $\hat{\sigma}_m{}^2$, the asymptotic means were comparable to the simulation counterparts at sample size 500 and 1,000. The asymptotic variances of $\hat{\mu}_m$ approximated the simulation variance at sample size 1,000, while for $\hat{\sigma}_m{}^2$ the asymptotic variance was still not comparable at $T = 1{,}000$.

The distributions of $\hat{\mu}_m$ and $\bar{X}_{m_j}$, $j = 1, 2, \ldots, N$, in the simulation were found to be close to normal at sample sizes of 300 or more. For $\hat{\sigma}_m{}^2$ the normality assumption was barely acceptable at sample size 1,000. The effect on these distributions of the coefficient of variation of the denominators of these ratios was shown to be consistent with the findings of Hayya, Armstrong, and Gressis (1975). The correlations between the numerator and denominator of these ratios were found to be unimportant.

The applicability of transformed variables of the form $Z = (X - WY)/(\sigma_X{}^2 - W^2\sigma_Y{}^2 - 2W\sigma_{XY})^{\frac{1}{2}}$ for the ratio $W = X/Y$ was examined. Sample estimates for the variances and covariances were used, and the distributions of Z were studied for each of μ_m, $\sigma_m{}^2$, and the elements of $\mathbf{X}_m$. The Z values corresponding to $\mathbf{X}_m$ were reasonably well behaved for all sample sizes, while for μ_m and $\sigma_m{}^2$ the Z values were applicable for samples of size 300 or more.

The impact on the sampling properties of changes in the mean premium-return vector $\boldsymbol{\mu}$, the covariance matrix $\boldsymbol{\Sigma}$, and the portfolio size N were examined. For simplicity it was assumed that the diagonal and off-diagonal elements of $\boldsymbol{\Sigma}$ have magnitude α and β, respectively, and that the mean of the elements of the vector $\boldsymbol{\mu}$ is given by $\bar{\mu}$. For large N, the magnitude of b and the coefficient of variation of b were found to be dependent on the ratio $\bar{\mu}/\beta$. Thus, the sample size T required for both the convergence of the expressions for the approximate moments and for the normality of the ratio estimators decreases as the ratio $\bar{\mu}/\beta$ increases. Therefore, if the covariances among the assets are relatively small but the mean returns are relatively large the sampling properties of the estimators are improved. The effect of an increase in the risk-free rate E_z is equivalent to a decrease in $\bar{\mu}$. We may therefore conclude that the estimators are not applicable for populations of stocks in which $\bar{\mu}$ is small relative to the off-diagonal elements of $\boldsymbol{\Sigma}$.

The applicability of the conclusions of the Monte Carlo study to the financial market depends in part on the realism of the mean-return and covariance parameters. Because these parameters were computed over 313 months or 26 years, temporal changes in market conditions would cause elements of the covariance matrix, including β, to be unrealistically large. In addition, sample outcomes such that $E_z > b/c$ are included in our results. Merton (1972) has shown that such results are not economically realistic. These sample outcomes tend to reduce the values of the elements of the mean-return vector and therefore $\bar{\mu}$. Sample size requirements mentioned throughout the study therefore would in general be less stringent.

From this study we conclude that the estimators $\hat{\mu}_m$, $\hat{\sigma}_m{}^2$, and $\bar{\mathbf{X}}_m$ do not lend themselves to making inferences in small samples. The key to improving the small-sample properties of these estimators lies in improving the estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Lambda}$. Other studies by the authors (1979, 1980) have shown that the use of James-Stein-

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type estimators of $\mathbf{\mu}$ and $\mathbf{\Lambda}$ can bring about substantial improvements to the estimators of $\hat{\mu}_m$, $\hat{\sigma}_m{}^2$, and $\bar{\mathbf{X}}_m$.

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