Quantum Integer Programming: Investigating Quantum Approaches to the Vehicle Routing Problem

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Abstract

Quantum computing, with its inherent speedups, holds the promise of revolutionizing optimization problems by providing faster and more efficient solutions. In this study, we explore the application of quantum optimization techniques to tackle the challenging Vehicle Routing Problem (VRP) [1] and its popular variant, the Capacitated VRP (CVRP).

The VRP is a combinatorial optimization problem that involves finding the most efficient routes for a fleet of vehicles to deliver goods to a set of customers, minimizing the total distance traveled or the overall cost. As the VRP is a well-known NP-hard problem, finding optimal solutions for large instances can be computationally demanding. By leveraging the power of quantum computing, we aim to explore novel approaches and algorithms that can potentially offer significant improvements in solving VRP instances.

To formulate the VRP, we employ the classical two-index approach, where binary variables x_{ij} indicate whether an arc (i, j) is traversed by a vehicle. To ensure meaningful solutions, we incorporate subtour elimination constraints. Although these constraints improve the final solutions, they quickly become problematic when transitioning to QUBO formulation [2], as they introduce many integer slack variables, resulting in an excessive number of variables for gate-based quantum models. To address this issue, we also explore a heuristic two-phase approach.

According to this heuristic, nodes are first clustered into disjoint groups, with each cluster containing the depot. Routing is then performed individually within each cluster.

Clustering and Routing phases are modeled as a Multi Knapsack Problem (MKP) and Travelling Salesman Problem (TSP), respectively.

In the Routing phase, we sequentially add subtour elimination constraints as needed (*DFJ* strategy) rather than adding all the constraints in advance (*MTZ* strategy). This adaptive approach significantly reduces the number of qubits, making the models suitable for both gate-based and annealing-based quantum devices.

For quantum optimization, we explore various quantum algorithms, including Quantum Adiabatic Computation [3], Quantum Approximate Optimization Algorithm (QAOA) [4] and Variational Quantum Eigensolver (VQE) [5] with hardware-efficient ansatz. To benchmark these quantum algorithms, we utilize Gurobi as a classical comparison method. Due to current hardware constraints, we focus on instances of small size, involving around 15 nodes and 3 vehicles for annealers, and around 5 nodes and 2 vehicles for gate-based models.

By analyzing these quantum approaches and comparing them with classical methods, we aim to gain insights into the potential quantum speedups for solving complex VRP instances. With ongoing advancements in quantum hardware development, hopefully, quantum optimization will become a robust strategy for real-world transportation and logistics applications.

***Index Terms*—Combinatorial Optimization, Vehicle Routing Problem, Quantum Computing, Quantum Annealing, QAOA, VQE.**

References

[1] P. Toth and D. Vigo, *Vehicle Routing. Problems, Methods, and Applications*. 2th ed., 2014.

[2] F. Glover, G. Kochenberger, and Y. Du, “A tutorial on formulating and using qubo models,” 2019.

[3] E. Farhi, J. Goldstone, S. Gutmann, and M. Sipser, “Quantum computation by adiabatic evolution,” 2000.

[4] E. Farhi, J. Goldstone, and S. Gutmann, “A quantum approximate optimization algorithm,” 2014.

[5] A. Peruzzo, J. McClean, P. Shadbolt, M.-H. Yung, X.-Q. Zhou, P. J. Love, A. Aspuru-Guzik, and J. L. O’Brien, “A variational eigenvalue solver on a photonic quantum processor,” *Nature Communications*, vol. 5, jul 2014.

[6] A. Lucas, “Ising formulations of many NP problems,” *Frontiers in Physics*, vol. 2, 2014.

[7] N. K. Suresh and P. Ramasamy, “A survey on the vehicle routing problem and its variants,” *Intelligent Information Management*, 2012.

[8] M. Gendreau and J.-Y. Potvin, *Handbook of Metaheuristics*. International Series in Operations Research & Management Science, Springer, 3th ed., 2019.

[9] R. de Wolf, “Quantum computing: Lecture notes,” 2023.

[10] D. E. Bernal, S. Tayur, and D. Venturelli, “Quantum integer programming (quip) 47-779: Lecture notes,” 2021.

[11] H. Alghassi, R. Dridi, and S. Tayur, “Graver bases via quantum annealing with application to non-linear integer programs,” 2019.

[12] H. Alghassi, R. Dridi, and S. Tayur, “Gama: A novel algorithm for non-convex integer programs,” 2019.

[13] G. E. Crooks, “Performance of the quantum approximate optimization algorithm on the maximum cut problem,” 2018.

[14] L. Zhou, S.-T. Wang, S. Choi, H. Pichler, and M. D. Lukin, “Quantum approximate optimization algorithm: Performance, mechanism, and implementation on near-term devices,” *Physical Review X*, vol. 10, jun 2020.

[15] E. Osaba, E. Villar-Rodriguez, and I. Oregi, “A systematic literature review of quantum computing for routing problems,” *IEEE Access*, vol. 10, pp. 55805–55817, 2022.

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