3.3 Svojstva Fourierove transformacije

1) Linearnost

→ $\mathcal{F}(\alpha f(x) + \beta g(x)) = \alpha \mathcal{F}\{f(x)\} + \beta \mathcal{F}\{g(x)\}$

2. Parnost i neparnost

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx - i\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx$$

1) ako je f parna → $\mathcal{F}(f)$ = parna $\quad \hat{f}(\lambda) = \hat{f}(-\lambda)$

2) ako je f neparna → $\mathcal{F}(f)$ = neparna $\quad \hat{f}(-\lambda) = -\hat{f}(-\lambda)$

3 Pomak u frekvencijskoj i vremenskoj domeni

1) $\hat{\mathcal{F}}(f(ax)) = \int_{-\infty}^{\infty} f(ax)\, e^{-i\lambda x}\,dx = \begin{pmatrix} t = ax \\ dt = a\,dx \end{pmatrix} = \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda \frac{t}{a}} \frac{dt}{a}$

$$\mathcal{F}(f(ax)) = \frac{1}{|a|}\hat{f}\left(\frac{\lambda}{a}\right) \qquad a \neq 0$$

2) $\mathcal{F}(f(x-a)) = \int_{-\infty}^{\infty} f(x-a)\, e^{-i\lambda x}\,dx = \begin{pmatrix} t = x-a \\ dt = dx \end{pmatrix} = \int_{-\infty}^{\infty} f(t)\, e^{-i\lambda t} \cdot e^{-i\lambda a}\,dt =$

$$\mathcal{F}\{f(x-a)\} = e^{-i\lambda a}\hat{f}(\lambda) \qquad a \in \mathbb{R}$$

3) $\mathcal{F}(e^{i\omega x} f(x)) = \int_{-\infty}^{\infty} e^{i\omega x} \cdot f(x) \cdot e^{-i\lambda x}\,dx = \int_{-\infty}^{\infty} e^{ix(\omega - \lambda)} f(x)\,dx$

$$\mathcal{F}\{e^{i\omega x} f(x)\} = \hat{f}(\lambda - \omega) \qquad \omega \in \mathbb{R}$$

4. Transformat derivacije

→ neka je f diferencijabilna i neka $\lim_{x\to\pm\infty} f(x) = 0$; f' je apsolutno integrab.

$$\mathcal{F}(f'(x)) = \int_{-\infty}^{\infty} f'(x)\, e^{-i\lambda x}\,dx = \cancel{f(x)\, e^{-i\lambda x}\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} f(x)\, e^{-i\lambda x}(-i\lambda)\,dx$$

$$= \lim_{x\to\infty} \left| f(x)\, e^{-i\lambda x} \right| = \lim_{x\to\infty} |f(x)| = 0 \qquad \mathcal{F}(f'(x)) = i\lambda \hat{f}(\lambda)$$

5.) Derivacija Fourierovog transformata

$$\frac{d}{d\lambda}\hat{f}(\lambda)=\frac{d}{d\lambda}\int_{\infty}^{\infty} f(x)e^{-i\lambda x}dx=\int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\cdot(-ix)dx=-i\int_{-\infty}^{\infty} xf(x)e^{-i\lambda x}dx$$

$$\mathcal{F}\{xf(x)\}=i\frac{d}{d\lambda}\hat{f}(\lambda) \rightarrow -i\,\mathcal{F}\{xf(x)\}=\hat{f}'(\lambda)$$

$$\mathcal{F}\{x^n f(x)\}=i^n\frac{d^n}{d\lambda^n}\hat{f}(\lambda)$$

Primjer: Odredi $\mathcal{F}\{e^{-\frac{1}{2}x^2}\}$

$\hat{f}(\lambda)=\mathcal{F}\{e^{-\frac{1}{2}x^2}\}$ * zadovoljava uvjete za $\mathcal{F}(f'(x)) \rightarrow \lim_{x\to\infty} f(x)=0$

$\Rightarrow \mathcal{F}(f'(x))=i\lambda\hat{f}(\lambda)$ definicija

$$\mathcal{F}(-xe^{-\frac{1}{2}x^2})=\mathcal{F}\{-x\cdot f(x)\}=i\lambda\hat{f}(\lambda)\overset{*}{=}-i\frac{d}{d\lambda}\hat{f}(\lambda)$$

$$\Rightarrow -\lambda\hat{f}(\lambda)=\frac{d}{d\lambda}\hat{f}(\lambda)\rightarrow \frac{d\hat{f}(\lambda)}{\hat{f}(\lambda)}=-\lambda\, d\lambda \;/\int$$

$$\ln|\hat{f}(\lambda)|=-\frac{1}{2}\lambda^2+c \;/e \rightarrow \hat{f}(\lambda)=ce^{-\frac{1}{2}\lambda^2}$$

→ želimo odrediti konstantu c

$$\hat{f}(0)=c=\int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2}dx=\sqrt{2\pi}=c \longrightarrow \boxed{\hat{f}(\lambda)=\mathcal{F}(e^{-\frac{1}{2}x^2})=\sqrt{2\pi}\cdot e^{-\frac{1}{2}\lambda^2}}$$

(Matan 2)

6.) Konvolucija

$$\mathcal{F}\{(f*g)(x)\}=(\widehat{f*g})(\lambda)=\int_{-\infty}^{\infty}e^{-i\lambda t}(f*g)(x)dx=\int_{-\infty}^{\infty}e^{-i\lambda x}\int_{-\infty}^{\infty}f(t)g(x-t)dt\,dx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}e^{-i\lambda x}f(t)g(x-t)\,dt\,dx=\left[\text{zamjena redoslijeda integracije}\right]$$

$$=\int_{-\infty}^{\infty}e^{-i\lambda t}f(t)\int_{-\infty}^{\infty}e^{-i\lambda(x-t)}g(x-t)\,dx\,dt=\begin{bmatrix}u=x-t\\ du=dx\end{bmatrix}$$

$$=\int_{-\infty}^{\infty}e^{-i\lambda t}f(t)dt\int_{-\infty}^{\infty}e^{-i\lambda u}g(u)du$$

$$\Longrightarrow \mathcal{F}\{(f*g)(x)\}=\hat{f}(\lambda)\cdot\hat{g}(\lambda)$$

$*\ (f*g)*h=f*(g*h)$

Konvolucija u obradi signala

* Fourierova transf. prebacuje signal $f(x)$ iz vremenske u $\hat{f}(\lambda)$ frekv. domenu

$\hat{g}(\lambda)=g_{[-A,A]} \longrightarrow \hat{f}(\lambda)\cdot g_{[-A,A]}$ → dobijemo f-ju koja nema amp veću od $|A|$

→ kako pronaći gate? $\longrightarrow \mathcal{F}^{-1}(\hat{f}(\lambda)\cdot\hat{g}(\lambda))=(f*g)(x)$

$\rightarrow \mathcal{F}^{-1}(\hat{g}(\lambda))=g(x)$

* $\hat{g}(\lambda)$ je gate funkcija

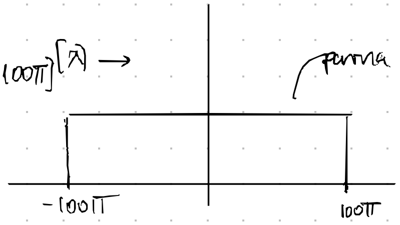
Primjer: Odredimo funkciju $g(x)$ za koju vrijedi $\mathcal{F}\{g(x)\} = \hat{g}(\lambda)$.

$\to \hat{g}$ je gate funkcija $\longrightarrow$ $\hat{g}(\lambda) = \begin{cases} 1, & \lambda \in [-100\pi, 100\pi] \\ 0, & \text{inače} \end{cases}$

$$\mathcal{F}^{-1}\left(\hat{g}_{[-100\pi,100\pi]}(\lambda)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\, g_{[-100\pi,100\pi]}(\lambda)\, d\lambda$$

$$= \frac{1}{2\pi}\int_{-100\pi}^{100\pi} 1 \cdot e^{i\lambda x} d\lambda = \frac{1}{2\pi}\int_{-100\pi}^{100\pi} (\cos\lambda x + i\sin\lambda x)\, d\lambda$$

$g_{[-100\pi,100\pi]}(\lambda) \to$



parna je $\to = \frac{1}{\pi}\int_0^{100\pi} \cos\lambda x \, d\lambda$

$$= \frac{1}{\pi}\cdot\frac{1}{x}\sin\lambda x\Big|_0^{100\pi} = \boxed{\frac{\sin 100\pi x}{\pi x} = g(x)}$$

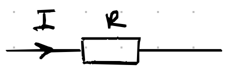
$\to$ Za brisanje frekvencija $|\lambda| > 100\pi$ iz nekog signala, potrebno je taj signal konvulirati s funkcijom $\frac{\sin 100\pi x}{\pi x}$

7. Parsevalova jednakost

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(\lambda)|^2 d\lambda$$

* analogna kao i za Fourierov red $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$

Primjer:



$I(t) = \underbrace{u(t)}_{\text{step. f-ja}} e^{-t}$ vrijeme

$P = R \cdot I^2(t) \quad t \in [a,b]$

$$E = \int_a^b P\, dt = \int_a^b R \cdot I(t) dt = R\int_a^b u^2(t) e^{-2t} dt$$

W (rad) = ? * ukupna potrošnja en.

$$\boxed{E = \frac{R}{2}}$$

Parsevalova jednakost nam govori da tu ukupnu energiju možemo računati preko Fourierovog transformata.

$$\to \hat{I}(\lambda) = \mathcal{F}\{u(t)\cdot e^{-t}\} = \int_0^{\infty} e^{-i\lambda x} u(x) e^{-x} dx = \int_0^{\infty} e^{(-i\lambda-1)x} dx$$

$$\hat{I}(\lambda) = \frac{-1}{\lambda i + 1} e^{-(i\lambda+1)x}\Big|_0^{\infty} = \frac{1}{i\lambda+1}$$

$\longrightarrow$ Parseval: $E = R\int_{-\infty}^{\infty} I^2(t)\, dt \overset{P}{=} \frac{R}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{i\lambda+1}\right|^2 d\lambda = \frac{R}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\lambda^2+1} d\lambda$ *modul

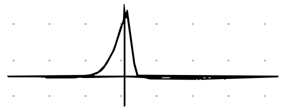
$$E = \frac{R}{2\pi}\operatorname{arctg}(\lambda)\Big|_{-\infty}^{\infty} = \frac{R}{2\pi}\cdot\pi = \boxed{\frac{R}{2}}$$

Diracova delta funkcija

- npr. trenutno nabijanje kondenzatora nekim nabojem

– striktno rečeno, ovo nije funkcija, već derivacija step funkcije $u(x)$

$$\longrightarrow \delta(t) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$$



→ integrabilan objekt: $\int_a^b \delta(t) = \begin{cases} 1, & \text{ako } 0 \in [a,b] \\ 0, & \text{inače} \end{cases}$

=> Diracova delta fja = jedinični impuls

- može biti pomaknutog vrha → $\delta_{x_0}(x) = \delta(x - x_0)$

- $\int_{-\infty}^{\infty} f(x)\delta(x)\,dx = f(0) \xrightarrow{\mathcal{F}(\delta(x))} \int_{-\infty}^{\infty} e^{-i\lambda x}\delta(x)\,dx = e^{-i\lambda \cdot 0} = 1$

* općenito: $\mathcal{F}\{\delta_{x_0}(x)\} = \int_{-\infty}^{\infty} e^{-i\lambda x}\delta(x - x_0)\,dx = e^{-i\lambda x_0}$

- n-ta derivacija delta fje:

$$\mathcal{F}\left(\delta_{x_0}^{(n)}(x)\right) = \int_{-\infty}^{\infty} e^{-i\lambda x}\delta_{x_0}^{(n)}(x)\,dx = (i\lambda)^n e^{-i\lambda x_0}$$

- delta fja je neutralni element za operaciju konvolucije $f * \delta = f$