# COULOMBOV ZAKON I EL. POLJE

**Coulombov zakon** – sila kojom djeluju dva naboja

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{r^2}\,\hat{r}$$

Sila kojom međudjeluju DVA naboja

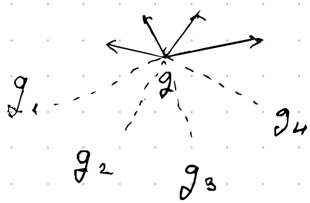
Puno veći od onog $G$ iz gravitacijske sile

$\varepsilon_0$ - permitivnost vakuuma

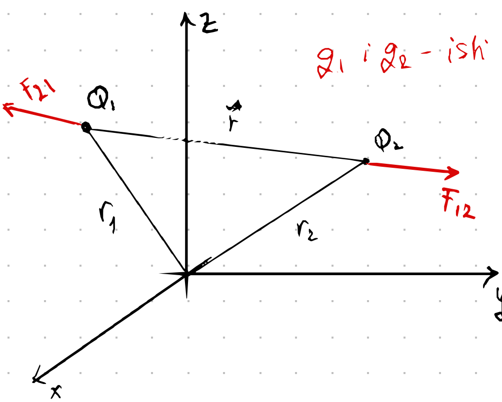
$q_{min} = e = 1{,}602 \times 10^{-19}\ C \longrightarrow [C] = [\text{Sek} \cdot \text{Amp}]$

**naboj je očuvan UVIJEK!** $Q_{uk}$ (prije) = $Q_{uk}$ (poslije)

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i$$




- Coulombova sila je odbojna kada naboji imaju isti predznak
- –||– privlačna –||– suprotni (+ i –)



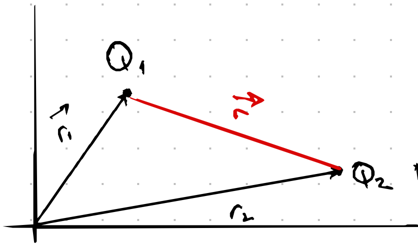

$q_1$ i $q_2$ - isti predznak

- neovisno jesu li naboji čestica istog ili suprotnog predznaka, uvijek vrijedi:

$$\underbrace{F_{12} = -F_{21}}\; = \frac{1}{4\pi\varepsilon_0}\,\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12}$$

# ELEKTRIČNO POLJE

- prostor u kojem postoje zamišljene el. silnice iz izvora $Q_1$




$$E_1 = \frac{F}{Q_2}$$

* za el. polja vrijedi načelo superpozicije

$$E_i = \frac{1}{4\pi\varepsilon_0}\frac{Q_i}{r^2}\hat{r} \Rightarrow \vec{E_i} = \sum_{i\neq j} \vec{E_j} = \vec{E}_{i1} + \vec{E}_{i2} + \dots$$

$$\vec{F_i} = \sum_{i\neq j} \vec{F_j} = \vec{F}_{i1} + \vec{F}_{i2}$$

Coulombova sila koja ovdje djeluje:

$$F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 \cdot q_2}{|r_2 - r_1|^{3/2}} \cdot \underbrace{(\vec{r_2} - \vec{r_1})}_{\text{ostane smjer}} \Rightarrow F_{12} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{|r_2 - r_1|^2} \cdot \hat{r}$$

Ako silu $F_{12}$ napišemo u obliku el. dijela Lorentzove sile: $F_{12} = q_2 E_1 [r_2]$

$$\rightarrow \frac{1}{4\pi\varepsilon_0}\frac{q_1 \cancel{q_2}}{|r_2 - r_1|^2}\vec{r} = \cancel{q_2} E_1$$

$$\boxed{E_1 = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{|r_2 - r_1|^2}\vec{r}}$$

• Ako je raspodjela el. naboja u prostoru opisana volumnom, površinskom ili linijskom gustoćom naboja

$\Rightarrow$ naboj $q \rightarrow dq'$ (nalazi se u okolini točke $r'$)

$\Rightarrow$ sumu zamijenjujemo integralom

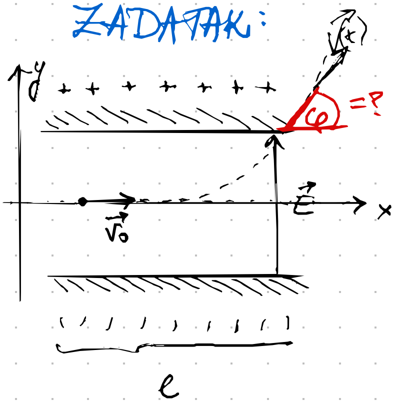
$$E[r] = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{|r - r'|^3}(r - r')$$

linearna gustoća $\rightarrow \frac{dq'}{dl} = \lambda \rightarrow dq' = \lambda[r'] \cdot dl$ — linijska gustoća

površinska gustoća $\Rightarrow \frac{dq'}{dS} = \sigma \rightarrow dq' = \sigma[r'] dS$ — površinska

volumna gustoća $\Rightarrow \frac{dq'}{dV} = \rho \rightarrow dq' = \rho[r'] dV$ — volumna

## ZADATAK:




$$\vec{v}_0 = v_0 \hat{\vec{x}}$$

$$\vec{E} = E \hat{\vec{y}}$$

$$F = m \cdot a = qE$$

**x**

$$ma_x = 0$$

$$\frac{dv_x}{dt} = 0$$

$$\int_{v_{x_0}}^{v_x(t)} dv_x = 0$$

$$v_x - v_0 = 0$$

$$\underline{\underline{v_x = v_0}} \text{ konst.}$$

**y**

$$ma_y = qE$$

$$\frac{dv_y}{dt} = \frac{qE}{m} \quad / \int dt$$

$$\int_0^{v_y(t)} dv_y = \frac{qE}{m}\int_0^t dt$$

$$\underline{\underline{v_y(t) = \frac{qE}{m} \cdot t}}$$

→ neodređeni trenutak

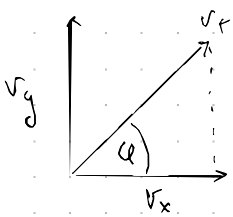
određuje $l$ (put)

$$v_0 = \frac{l}{t} \rightarrow t = \frac{l}{v_0}$$

$$\frac{dx}{dt} = v_0$$

$$x(t) = v_0 t$$

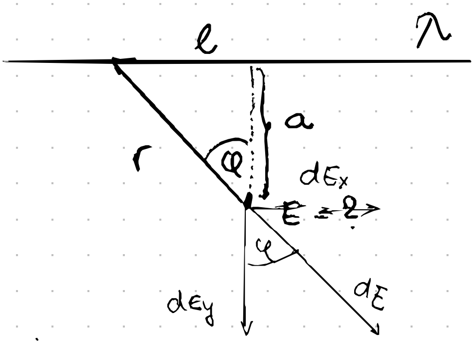
$$v_y(t) = \frac{qE}{m} \cdot \frac{l}{v_0}$$




$$\tg\varphi = \frac{v_y}{v_x}$$

$$\varphi = \arctg\left(\frac{qE}{mv_0} \cdot \frac{1}{v_x}\right) = \underline{\underline{\arctg\left(\frac{qE}{mv_0^2}\right)}}$$

## ZADATAK 3:




$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\,\hat{\vec{r}}$$

$$\frac{dq}{dl} = \lambda \rightarrow dq = \lambda \cdot dl$$

$$E = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dl}{r^2}\,\hat{\vec{r}}$$

$$dE = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda\, dl}{r^2}\,\hat{\vec{r}}$$

$$dE_x = 0$$

$$dE_y = dE\cos\varphi$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\, dl}{r^2}\cdot\cos\varphi$$

r želimo izraziti preko cosφ (a)
jer nam je to konstanta,
a sinus bi bio sa l koji je nedefiniran

$$\cos\varphi = \frac{a}{r} \rightarrow r = \frac{a}{\cos\varphi}$$

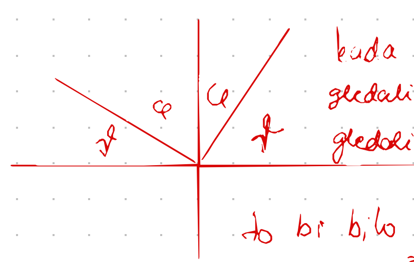
$$\operatorname{tg}\varphi = \frac{l}{a} \rightarrow l = \operatorname{tg}\varphi\cdot a \qquad dl = d\varphi\, a \qquad (\operatorname{tg}\varphi)' = \frac{1}{\cos^2\varphi}$$

$$r^2 = \frac{a^2}{\cos^2\varphi}$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\int \frac{\cos^{\cancel{2}}\varphi}{a^{\cancel{2}}}\cdot\lambda\cdot\frac{\cancel{a}}{\cancel{\cos^2\varphi}}\,d\varphi$$

$$E_y = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda\cos\varphi}{a}\,d\varphi$$

koje kutove za granice umetnuti?




kada bismo gledali thetu ϑ gledali bi 0 − π

to bi bilo $E_x$

$$E_x = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda}{a}\int_0^{\pi}\cos\varphi\, d\varphi$$

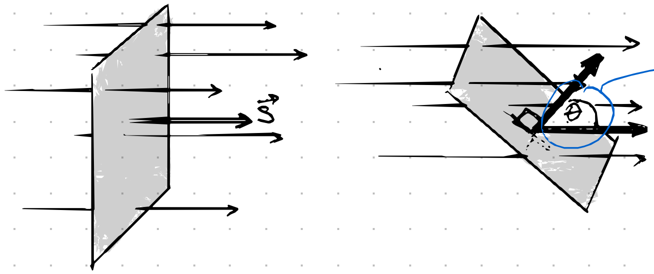
$$E_y = \frac{1}{4\pi\varepsilon_0}\,\frac{\lambda}{a}\int_{-\pi/2}^{\pi/2}\cos\varphi\, d\varphi$$

# TOK električnog polja = $\Phi$

$$\boxed{\Phi_E = \oint_S \vec{E}\, d\vec{S}}$$

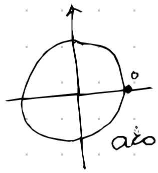
zatvorena plohe

- promatramo površinu S kroz el. polje → kroz nju prolazi više ili manje silnica




kut između silnice el. polja i smjera okomice

skalarni umnožak $|E|\,|dS| \cdot \cos\theta$



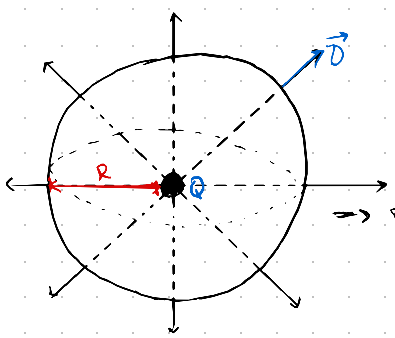
★ vektor $\vec{S}$ ima: • iznos vektora površine
• smjer vektora normale

ako je kut neki $\theta$:

$$\Phi_E = \oint_S E\, dS \cos\theta$$

$$\boxed{\text{TOK EL POLJA TOČKASTOG NABOJA Q (zatvorena ploha S)}}$$

→ zamišljena ploha S je kugla (sfera)




• silnice okomito probadaju kuglinu plohu

• iznos vektora D jednak je u svim točkama kugline plohe

→ tok el. polja jednak umnošku iznosa vektora $\vec{D}$ i ukupne površine kugle: $\Phi = 4R^2\pi D = 4R^2\pi\varepsilon E$

u središnji naboj Q, polje na površini: $E = \dfrac{Q}{4\pi\varepsilon R^2}$

TOK električnog polja kroz neku kuglinu plohu proporcionalan je naboju unutar te kugle (i neovisan je o R)

↓

Gauss

→ $D = \varepsilon E = \dfrac{Q}{4\pi R^2}$

→ $\Phi = D \cdot S = \dfrac{Q}{\cancel{4\pi R^2}} \cdot \cancel{4R^2\pi} = \underline{\underline{Q}}$

$$\boxed{\text{JAKOST EL. POLJA: } \oint \vec{E}\, d\vec{S} = 0}$$

# Gaussov zakon

⇒ el. tok po zatvorenoj plohi ∼ ukupnom naboju koji smo obuhvatili tom plohom

$$\boxed{\oint_S \vec{E}\, d\vec{s} = \frac{Q_{unutra}}{\varepsilon_0}}$$

→ fizikalna posljedica toga što elektrostatska sila opada s kvadratom udaljenosti $F \sim \frac{1}{r^2}$

## Gaussov teorem - teorem o divergenciji

$$\oint \vec{E}\, d\vec{s} = \oint \vec{\nabla} \vec{E}\, dV$$

- zatvorena ploha S obuhvaća neki volumen V

volumna gustoća: $dq = \rho \cdot [r]\, dV$

$q = \int \rho\, dV$

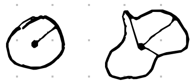
$\oint \vec{E}\, d\vec{s} = \frac{q}{\varepsilon} \Rightarrow \frac{1}{\varepsilon} \int \rho\, dV = \int \vec{\nabla} \vec{E}\, dV \Rightarrow$

PRVA MAXWELLOVA

$$\boxed{\frac{\rho}{\varepsilon_0} = \vec{\nabla} \vec{E}}$$

JEDNADŽBA!

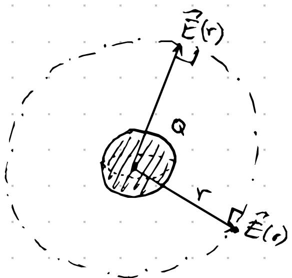
→ Gaussov zakon vrijedi bez obzira na oblik



polje točkastog naboja

## El. polje nabijene čestice

$$\oint_S \vec{E}\, d\vec{s} = \frac{Q}{\varepsilon_0} = \Phi_E$$




$dV = dx\, dy\, dz$

$dS = dx\, dy$

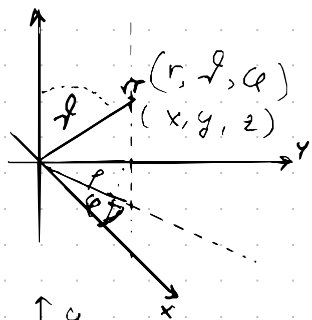
$V = \frac{4}{3} r^3 \pi$

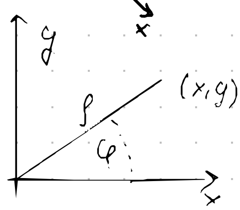
$dS = r^2 \sin\vartheta\, d\vartheta\, d\varphi$

$S = r^2 \int_0^{\pi} \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi$

ploha koja obuhvaća naboj Q

SFERNI KOORD.:

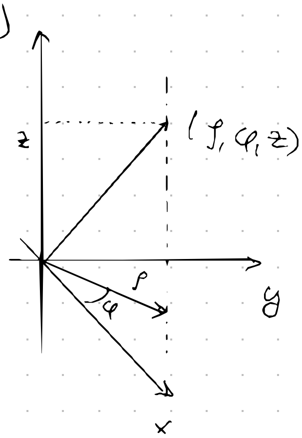



• E(r) je ⊥ na sferu zbog prostorne simetrije

- polje ima isključivo komponentu koja je okomita na q jer tangencijalna ne može biti izražena (fizički sustav ne sadrži niti jedno svojstvo pomoću kojeg bismo odredili smjer)

CILINDRIČNI:

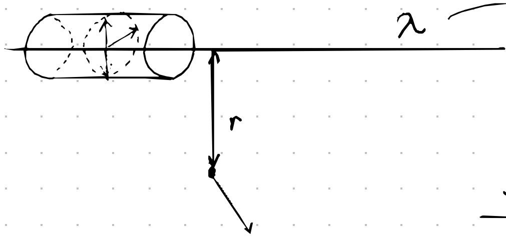



$$\oint_S \vec{E}\, d\vec{s} = \frac{Q}{\varepsilon_0} = \Phi_E$$

$$E S = \frac{Q}{\varepsilon_0} \Rightarrow E r^2 \overbrace{\int_0^{\pi} \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi}^{\text{ubacimo u kalkulator broj.}}$$

$$\boxed{\Phi_E = 4 r^2 E = \frac{Q}{\varepsilon_0}}$$

# POLJE jednoliko nabijene žice



$\lambda$ — linearna gustoća  $\lambda = \frac{Q}{l}$

↳ $Q = \lambda l$

$$E = \frac{\lambda}{2\pi \varepsilon_0 r}$$ (dolje postupak)

- rješavamo pomoću cilindričnih ploha

$d\vec{S} = dx\,dy\,\hat{z}$

$dS = r^2 \sin\theta \, d\theta \, d\varphi$ ?

$\boxed{dS = r\,d\varphi\,dz}$

$dV = dx\,dy\,dz$

$dV = r^2 dr \sin\theta \, d\varphi$

$dV = r\,dr\,d\varphi\,dz$

$\vec{E} \downarrow\downarrow d\vec{S}$

vekt. umnožak

kut je 0, $\cos = 1$

$$\oint \vec{E}\, d\vec{S} = \int E\, dS \cos(E, dS)$$

$$\oint \vec{E}\, d\vec{S} = \oint E\, dS = \oint \underbrace{E(r)\, r}_{\text{konst}}\, d\varphi\, dz$$

$$E(r)\, r \int_0^{2\pi} d\varphi \int_{-z}^{z} = E(r)\, r\, 2\pi\, 2z$$

Gaussov zakon: $\oint \vec{E}\, d\vec{s} = \frac{Q}{\varepsilon_0}$

$E(r)\, r\, 2\pi 2z = \frac{Q}{\varepsilon_0}$    $Q = \lambda l$, $l = 2z \rightarrow Q = 2z \cdot \lambda$

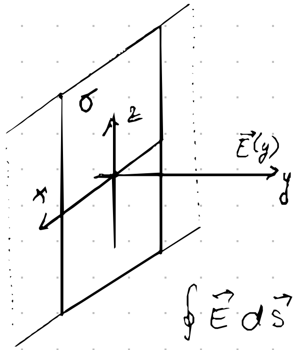
$E(r)\, r\, 2\pi \cancel{2z} = \frac{\lambda \cancel{2z}}{\varepsilon_0}$

$E$   $2\pi r$

$$\boxed{E = \frac{1}{2\pi r \varepsilon_0}}$$

*polje unutar i izvan nabijene kugle => AUDITORNE

# POLJE jednako nabijene plohe




– polje se ne mijenja po x i z, samo po y jer se pomicanjem po y približavamo ili udaljavamo od plohe

$\sigma = \frac{Q}{S}$ – plošna gustoća naboja

↳ okomita je na ravninu

$S = \int_{-x}^{x} dx \int_{-z}^{z} dz$

$S = 2x2z$

$\oint \vec{E} d\vec{S}$

$\sigma = \frac{dq}{dS}$ $\quad dS = dx \cdot dz$ (površina)

$$\oint E(y)\, dx dz \hat{y} = \oint E(y)\, dx dz = \oint 2E(y)\, dx dz = 2E(y) \int_{-x}^{x} dx \int_{-z}^{z} dz$$

$$\Rightarrow 2E(y) \cdot 2x \cdot 2z = \frac{Q}{\varepsilon_0} \rightarrow \sigma \cdot S = \frac{\sigma S}{\varepsilon_0} \quad (S \rightarrow 2x2z)$$

otkud taj 2? zbog zatvorenog $\oint$?

$$2E(y) = \frac{\sigma}{\varepsilon_0} \rightarrow \boxed{E_y = \frac{\sigma}{2\varepsilon_0}}$$

## I. MAXWELLOVA JEDNADŽBA

### ▶ POLJE unutar i izvan jednoliko nabijene kugle

$\rho = \frac{dQ}{dV}$ → gustoća naboja

↳ $Q = \int \rho\, dV$

– svaka zatvorena ploha obuhvaća neki volumen

⇓

otud slijedi Gaussov TM (TM o divergenciji)

$$\Phi_E = \frac{Q}{\varepsilon_0} \qquad \oint_S \vec{E} d\vec{S} = \frac{Q}{\varepsilon_0} = \frac{\int \rho dV}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int \rho\, dV$$

ta ploha obuhvaća neki volumen (zatvoreni $\int$)

Gaussov TM.

$$\rightarrow \oint_S \vec{F} \cdot d\vec{S} = \int \vec{\nabla} \vec{F}\, dV$$

Ako imamo zatvorenu plohu i ako tražimo $\int$ možemo ga naći pomoću divergencije

$$\Rightarrow \oint \vec{E} d\vec{S} = \frac{q}{\varepsilon_0} \qquad \oint \vec{E} d\vec{S} = \int \vec{\nabla} \vec{E}\, dV$$

$$\Rightarrow \frac{1}{\varepsilon_0} \int \rho\, dV = \int \vec{\nabla} \vec{E}\, dV$$

$$\rightsquigarrow \int \underbrace{\frac{\rho}{\varepsilon_0}}\, dV = \int \underbrace{\vec{\nabla} \vec{E}}\, dV$$

$$\Rightarrow \boxed{\vec{\nabla} \vec{E} = \frac{\rho}{\varepsilon_0}}$$

analogno će vrijediti za mag. polje, ali o tome kasnije

$\nabla \quad (\vec{\nabla} \vec{B} = 0)$

# 1. MAXWELLOVA jednadžba

$\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0}$

$\vec{\nabla} = \frac{d}{dx}\hat{x} + \frac{d}{dy}\hat{y} + \frac{d}{dz}\hat{z}$

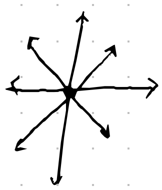
$\vec{\nabla}\cdot U$ - gradijent

$\vec{\nabla}\cdot\vec{E}$ - divergencija !

$\vec{\nabla}\times\vec{E}$ - rotacija

divergencija $\neq 0$

$(\vec{\nabla}\vec{E} \neq 0)$



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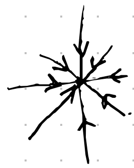
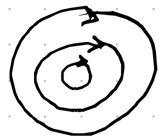


divergencija $= 0$

$(\vec{\nabla}\vec{E} = 0)$



Neko Markovo računanje gradijenata i ostalih magija:

$U(r) = \int \frac{\partial U}{\partial \vec{r}} d\vec{r}'$ → integracija gradijenta koju znamo otprije

↳ isto možemo zapisati: $\int U dr' = \int \frac{\partial U}{\partial \vec{r}} dr'$

vekt

$\int E dr' = \int (\vec{\nabla}\times\vec{E}) d^2 r$

→ $\int \vec{E} d^2 r = \int (\vec{\nabla}\cdot\vec{E}) d^3 r$

skal

HUH?

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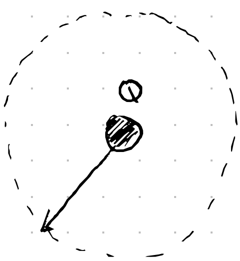
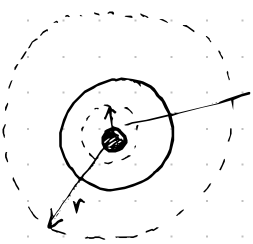
Unutar kugle




$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

* sa ljuskom

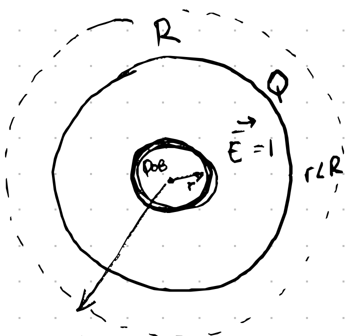



$E = 0$ !

$E(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}$

▸ HOMOGENA KUGLA:

$\rho = \frac{Q}{V} \rightarrow \rho = \frac{dQ}{dV} \rightarrow \rho(\cancel{r, \theta, \varphi}) = \rho$




$\vec{E} = \frac{Q}{4\pi r^2 \varepsilon_0}\hat{r}$

# KONZERVATIVNOST COULOMBOVE SILE & EL. POTENCIJAL

Prva Maxwellova jednadžba: $\underline{\vec{\nabla}\vec{E} = \frac{\rho}{\varepsilon_0}}$

Coulombova sila je konzervativna jer vrijedi: $\vec{\nabla} \times \vec{F} = 0$
└ rotacija je ∅

↳ to vrijedi samo u elektrostatici, ne i u magnetizmu *

kasnije smo nazvali potencijalnom energijom

⟹ slijedi: postoji skalarno polje $\varphi$ za koje vrijedi $F = -\vec{\nabla}\varphi$

analogno vrijedi: $\vec{\nabla} \times \vec{E} = 0$ (el. polje je konzervativno)

⟹ -||- : postoji negativni gradijent $E = -\vec{\nabla} V$ — *potencijal (skal. polje $V(x,y,z)$)

$F = -\vec{\nabla}\varphi = -\frac{\partial \varphi}{\partial l}$ — po duljini vodiča

$E = -\vec{\nabla} V = -\frac{\partial V}{\partial l}$

$E = \frac{F}{q}$

⟹ $V = -\int \vec{E}\, d\vec{l}$

$\varphi = -\int \vec{F}\, d\vec{l}$ — električni potencijal

⇝ $V = -\int \frac{\vec{F}}{q} d\vec{l} = \frac{1}{q}\varphi$ ⟹ $\varphi = qV \rightarrow \boxed{U = qV}$

- el. potencijal po jednakoj analogiji s potencijalnom energijom možemo prikazati kao:

$V = -\int \vec{E}\, d\vec{r}$ → $dr \cdot \hat{r}$

$* \; E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot \hat{r}$ — jed. vektor smjera

$V = -\int \frac{q}{4\pi\varepsilon_0 r^2} \cancel{\hat{r}} \cdot dr \cancel{\hat{r}}$

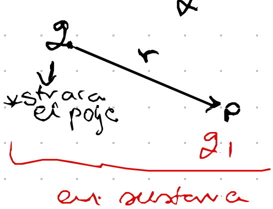
LINALG: $\hat{r} \cdot \hat{r} = 1$

$V = -\int \frac{q}{4\pi\varepsilon_0 r^2} \cdot dr \longrightarrow \boxed{V = \frac{q}{4\pi\varepsilon_0 r} + \cancel{C}^{\,0}}$

*isto kao potencijalna energija

- potencijal koji stvara neki $q$ u udaljenosti $r$

↳ $U = q_1 V$




- dovodimo $q_1$ iz beskonačnosti u točku potencijala P

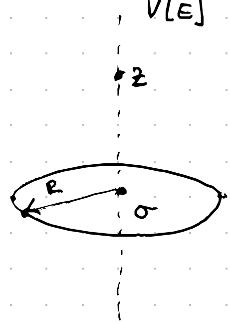
⟹ možemo izračunati energiju sustava

$U = q_1 V = \frac{q_1 q}{4\pi\varepsilon_0 r}$

· veza el. polja i el. potencijala: $\vec{E} = -\vec{\nabla} V$ (gradijent potencijala)

Zad. El. potencijal na bilo kojoj točki sred. osi homogeno nabijenog diska je dan izrazom: $V(z) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)$.

Izvedite izraz za el. polje:




$$V(z) = \frac{\sigma}{2\varepsilon_0}\left(\sqrt{z^2+R^2} - z\right)$$

$$\sigma = \frac{Q}{R^2\pi} \quad \text{(plošna gustoća)}$$

$$\vec{E} = \left(\cancel{\frac{\partial V}{\partial x}\hat{x}} + \cancel{\frac{\partial V}{\partial y}\hat{y}} + \frac{\partial V}{\partial z}\hat{z}\right)$$

$$\vec{E} = -\frac{\partial V}{\partial z}\hat{z}$$

$$E = \frac{-\sigma}{2\varepsilon_0}\left(\frac{1}{\cancel{2}} \cdot \frac{1}{\sqrt{z^2+R^2}} \cdot \cancel{2}z - 1\right) = -\frac{\sigma}{2\varepsilon_0}\left(\frac{z}{\sqrt{z^2+R^2}} - 1\right)$$

Ako je $\Phi_E = \int \vec{E}\,d\vec{S}$ (gaussov zakon)

$$\Phi_E = \oint \vec{E}\,d\vec{S} = \frac{Q_{OBUHVAĆENI}}{\varepsilon_0} \qquad \Phi \sim \int \rho\, dV$$

$\rightarrow \oint \vec{E}\,d\vec{S} = \int \vec{\nabla}\vec{E}\,dV$

nešto???