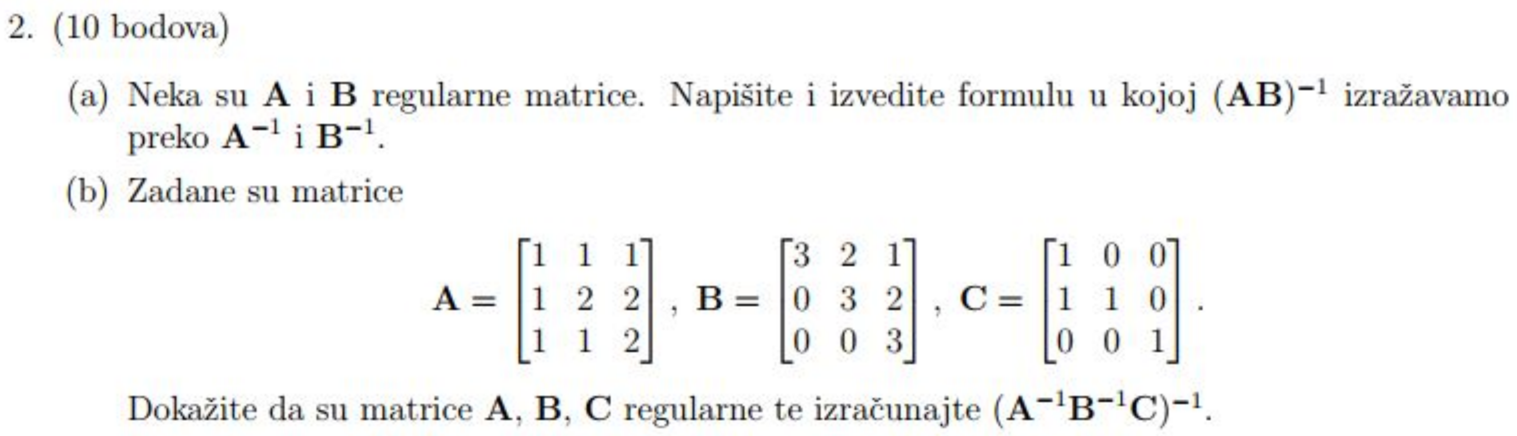
3. Rang i inverz

zadaci sa ispita

MI 2019 2



2. (10 bodova)

(a) Neka su **A** i **B** regularne matrice. Napišite i izvedite formulu u kojoj $(\mathbf{AB})^{-1}$ izražavamo preko $\mathbf{A}^{-1}$ i $\mathbf{B}^{-1}$.

(b) Zadane su matrice

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Dokažite da su matrice **A**, **B**, **C** regularne te izračunajte $(\mathbf{A}^{-1}\mathbf{B}^{-1}\mathbf{C})^{-1}$.

2. (a) $(AB)^{-1} = B^{-1}A^{-1}$

Izvod:

$$(AB)(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{I}A^{-1} = AA^{-1} = I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I$$

(b) $$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{vmatrix} \begin{matrix} |\cdot(-1) \\ + \\ +\,|\cdot(-1) \end{matrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot 1\cdot 1 = 1 \neq 0$$

$\det B = 3\cdot 3\cdot 3 = 27 \neq 0$, $\det C = 1\cdot 1\cdot 1 = 1 \neq 0$ $\Rightarrow$ A, B, C regularne

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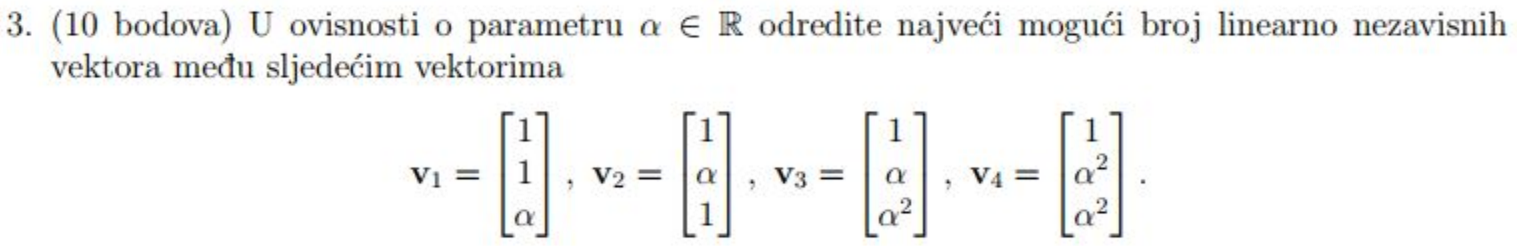
$$(A^{-1}B^{-1}C)^{-1} = C^{-1}BA = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 2 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 9 \\ -1 & 0 & 1 \\ 3 & 3 & 6 \end{bmatrix}$$

MI 2019 3



3. (10 bodova) U ovisnosti o parametru $\alpha \in \mathbb{R}$ odredite najveći mogući broj linearno nezavisnih vektora među sljedećim vektorima

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ \alpha \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 1 \\ \alpha \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ \alpha \\ \alpha^2 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 1 \\ \alpha^2 \\ \alpha^2 \end{bmatrix}.$$

③ U ovisnosti o parametru $\alpha \in \mathbb{R}$ određujemo rang matrice čiji su stupci zadani vektori:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \alpha & \alpha & \alpha^2 \\ \alpha & 1 & \alpha^2 & \alpha^2 \end{bmatrix} \begin{matrix} 1\cdot(-1) \\ + \quad 1\cdot(-\alpha) \\ + \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha-1 & \alpha^2-1 \\ 0 & 1-\alpha & \alpha^2-\alpha & \alpha^2-\alpha \end{bmatrix} \begin{matrix} 1\cdot 1 \\ + \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha-1 & \alpha^2-1 \\ 0 & 0 & \alpha^2-1 & 2\alpha^2-\alpha-1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha-1 & \alpha-1 & (\alpha-1)(\alpha+1) \\ 0 & 0 & (\alpha-1)(\alpha+1) & (2\alpha+1)(\alpha-1) \end{bmatrix}$$

1° $\alpha = 1$

Dobivamo matricu

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow r = 1 \Rightarrow$$ jedan linearno nezavisan vektor među zadanima

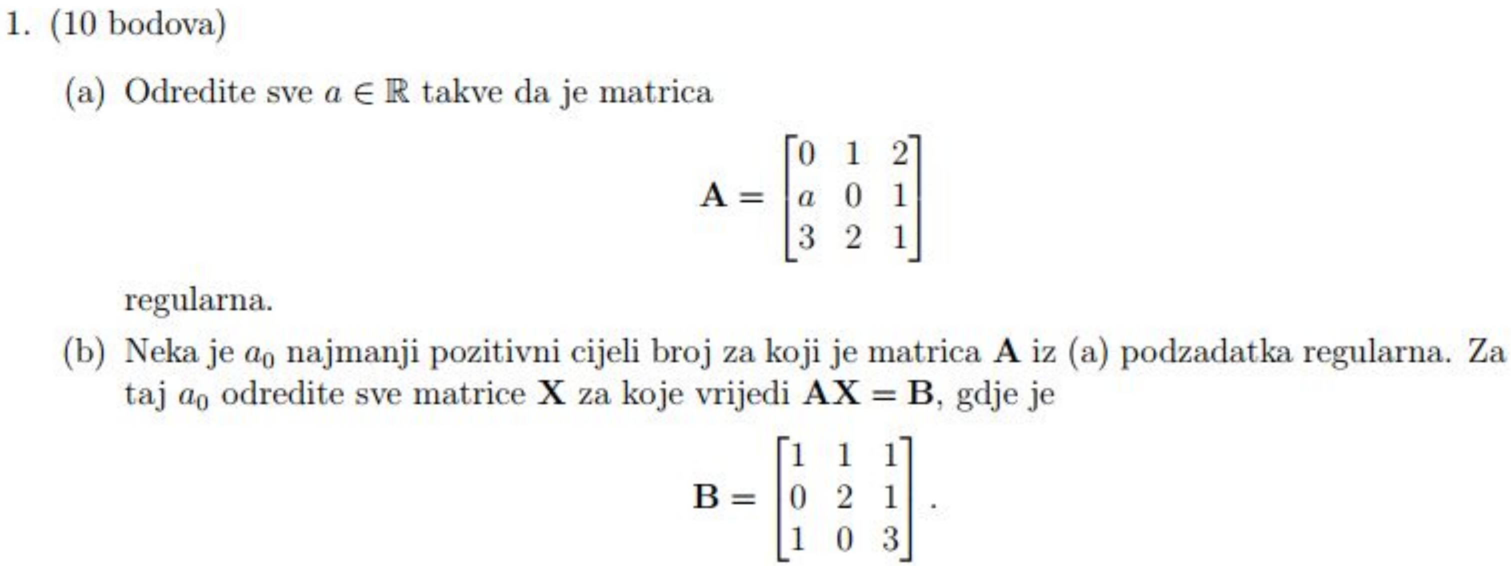
2° $\alpha \neq 1$

Drugi i treći redak možemo podijeliti s $\alpha - 1$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & \alpha+1 \\ 0 & 0 & \alpha+1 & 2\alpha+1 \end{bmatrix} \underset{1\cdot(-1)\,+}{} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & \alpha \\ 0 & 0 & \alpha+1 & \alpha \end{bmatrix} \underset{1\cdot(-1)\,+}{} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & -\alpha+1 & \alpha \\ 0 & 0 & 1 & \alpha \end{bmatrix}$$

$\Rightarrow r = 3 \Rightarrow$ tri linearno nezavisna vektora među zadanima

JIR1 2019 1



1. (10 bodova)

(a) Odredite sve $a \in \mathbb{R}$ takve da je matrica

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ a & 0 & 1 \\ 3 & 2 & 1 \end{bmatrix}$$

regularna.

(b) Neka je a_0 najmanji pozitivni cijeli broj za koji je matrica **A** iz (a) podzadatka regularna. Za taj a_0 odredite sve matrice **X** za koje vrijedi $\mathbf{AX} = \mathbf{B}$, gdje je

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

①. (a)

$$\det A = \begin{vmatrix} 0 & 1 & 2 \\ a & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} \begin{matrix} \\ |\cdot(-2) \\ + \end{matrix} = \begin{vmatrix} 0 & \overset{\downarrow}{1} & 2 \\ a & 0 & 1 \\ 3 & 0 & -3 \end{vmatrix}$$

$$= - \begin{vmatrix} a & 1 \\ 3 & -3 \end{vmatrix} = 3a+3$$

Matrica A je regularna za $\det A = 3a+3 \neq 0$, tj. $a \neq -1$.

(b) Prema (a) podzadatku slijedi $a_0 = 1$. Za taj a_0 tražimo inverz matrice A:

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ |\cdot(-2) \\ + \end{matrix} \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 0 & -3 & -2 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ |:3 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & -\frac{2}{3} & 0 & \frac{1}{3} \end{array}\right] \begin{matrix} \\ |\cdot 1 \\ + \end{matrix} \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & -\frac{2}{3} & 1 & \frac{1}{3} \end{array}\right] \begin{matrix} \\ \\ |:2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{array}\right] \begin{matrix} \\ + \\ |\cdot(-1) \end{matrix} \sim \left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{array}\right] \begin{matrix} + \\ |\cdot(-2) \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & \frac{1}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{3} & -1 & \frac{1}{3} \end{array}\right]$$

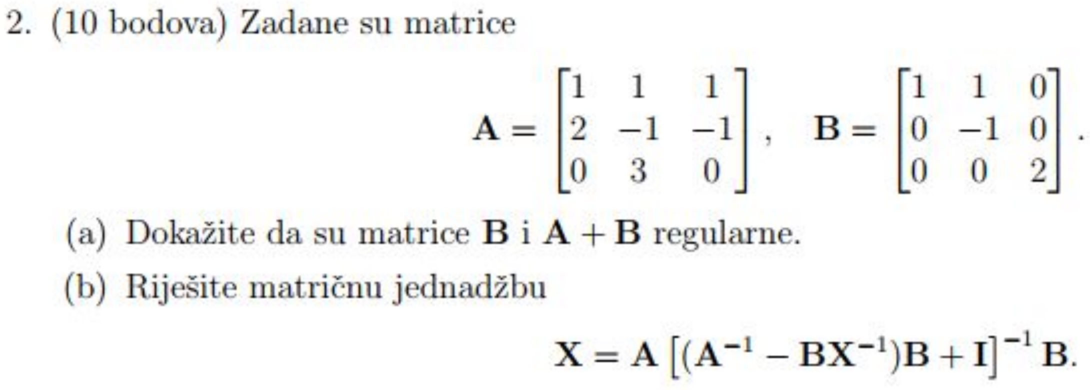
$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{3} & \frac{1}{2} & \frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{3} & -1 & \frac{1}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{2} & -\frac{1}{6} \end{array}\right] \quad \Rightarrow \quad A^{-1} = \frac{1}{6}\begin{bmatrix} -2 & 3 & 1 \\ 2 & -6 & 2 \\ 2 & 3 & -1 \end{bmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{A^{-1}}$$

Zato:

$$AX = B \Rightarrow X = A^{-1}B = \frac{1}{6}\begin{bmatrix} -2 & 3 & 1 \\ 2 & -6 & 2 \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -1 & 4 & 4 \\ 4 & -10 & 2 \\ 1 & 8 & 2 \end{bmatrix}.$$

JIR1 2019 2



2. (10 bodova) Zadane su matrice

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Dokažite da su matrice $\mathbf{B}$ i $\mathbf{A}+\mathbf{B}$ regularne.

(b) Riješite matričnu jednadžbu

$$\mathbf{X} = \mathbf{A}\left[(\mathbf{A}^{-1} - \mathbf{B}\mathbf{X}^{-1})\mathbf{B} + \mathbf{I}\right]^{-1}\mathbf{B}.$$

② (a)

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 0 \end{vmatrix} \leftarrow = -3 \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 9 \neq 0$$

$$\det(A+B) = \begin{vmatrix} 2 & 2 & 1 \\ 2 & -2 & -1 \\ 0 & 3 & 2 \end{vmatrix} \Big)^{\cdot 1}_{+} = \begin{vmatrix} 2 & 2 & 1 \\ 4 & 0 & 0 \\ 0 & 3 & 2 \end{vmatrix} \leftarrow = -4 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = -4 \neq 0$$

$\Rightarrow$ A i A+B su regularne matrice

(b) $X = A\left[(A^{-1} - BX^{-1})B + I\right]^{-1} B$

<u>1. način</u>

$$A^{-1}\cdot \Big| \quad X = A\left[(A^{-1} - BX^{-1})B + I\right]^{-1} B \quad \Big| \cdot B^{-1}$$

$$A^{-1} X B^{-1} = \left[(A^{-1} - BX^{-1})B + I\right]^{-1} \quad \Big/^{-1}$$

$$BX^{-1}A = (A^{-1} - BX^{-1})B + I$$

$$BX^{-1}A = A^{-1}B - BX^{-1}B + I$$

$$BX^{-1}A + BX^{-1}B = A^{-1}B + I$$

$$BX^{-1}(A+B) = A^{-1}B + A^{-1}A$$

$$B^{-1}\cdot \Big| \quad BX^{-1}(A+B) = A^{-1}(B+A) \quad \Big| \cdot (A+B)^{-1}$$

$$X^{-1} = B^{-1}A^{-1} \quad \Big/^{-1}$$

$$\Rightarrow X = AB = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 0 & 3 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{bmatrix}.$$

2. način

$$X = A\left[(A^{-1} - BX^{-1})B + I\right]^{-1} B \qquad /^{-1}$$

$$X^{-1} = B^{-1}\left[(A^{-1} - BX^{-1})B + I\right]A^{-1}$$

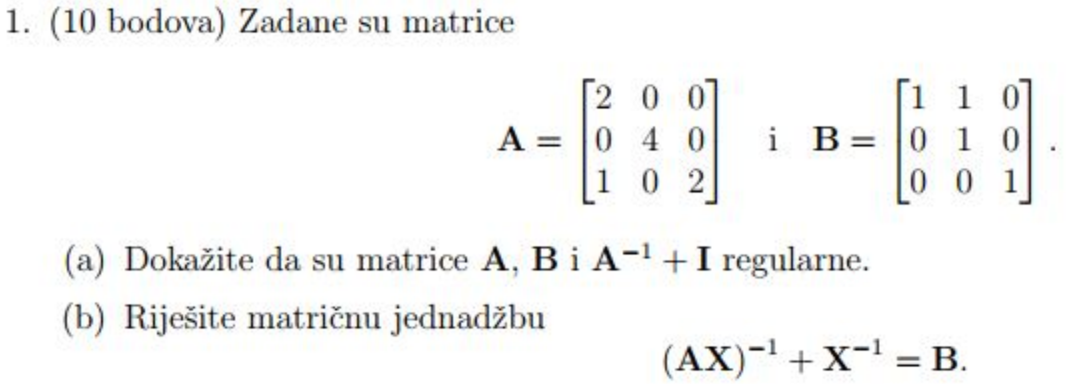
$$X^{-1} = B^{-1}A^{-1}BA^{-1} - X^{-1}BA^{-1} + B^{-1}A^{-1}$$

$$X^{-1}(I + BA^{-1}) = B^{-1}A^{-1}(BA^{-1} + I) \qquad \Big| \cdot (I + BA^{-1})^{-1}$$

$$X^{-1} = B^{-1}A^{-1} \qquad /^{-1}$$

$$X = AB = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 3 & -2 \\ 0 & -3 & 0 \end{bmatrix}.$$

ZIR 2019 1



1. (10 bodova) Zadane su matrice

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 1 & 0 & 2 \end{bmatrix} \quad \text{i} \quad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Dokažite da su matrice $\mathbf{A}$, $\mathbf{B}$ i $\mathbf{A}^{-1} + \mathbf{I}$ regularne.

(b) Riješite matričnu jednadžbu

$$(\mathbf{AX})^{-1} + \mathbf{X}^{-1} = \mathbf{B}.$$

① (a) $\det A = 2 \cdot 4 \cdot 2 = 16 \neq 0 \Rightarrow A$ regularna

$\det B = 1 \cdot 1 \cdot 1 = 1 \neq 0 \Rightarrow B$ regularna

$$\left[\begin{array}{ccc:ccc} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 4 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} |:2 \\ |:4 \\ \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} |\cdot(-1) \\ + \\ \end{array}$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 2 & -\frac{1}{2} & 0 & 1 \end{array}\right] \begin{array}{l} \\ \\ |:2 \end{array} \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 & -\frac{1}{4} & 0 & \frac{1}{2} \end{array}\right]$$

$$\underbrace{\begin{matrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{1}{2} \end{matrix}}_{A^{-1}}$$

$$\Rightarrow \det(A^{-1}+I) = \left(\tfrac{1}{2}+1\right)\left(\tfrac{1}{4}+1\right)\left(\tfrac{1}{2}+1\right) = \frac{45}{16} \neq 0$$

$\Rightarrow A^{-1}+I$ regularna

(b) $(AX)^{-1} + X^{-1} = B$

$X^{-1}A^{-1} + X^{-1} = B$

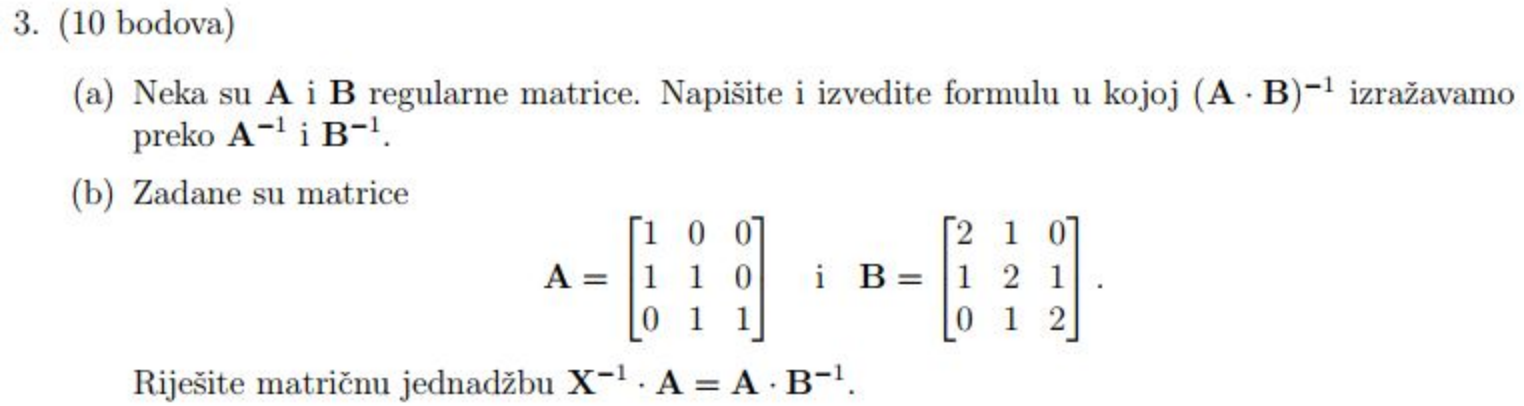
$X^{-1}(A^{-1}+I) = B \qquad |^{-1}$

$(A^{-1}+I)\cdot| \quad (A^{-1}+I)^{-1}X = B^{-1}$

$X = (A^{-1}+I)B^{-1}$

$$= \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & \frac{5}{4} & 0 \\ -\frac{1}{4} & 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & -\frac{3}{2} & 0 \\ 0 & \frac{5}{4} & 0 \\ -\frac{1}{4} & \frac{1}{4} & \frac{3}{2} \end{bmatrix}$$

MI 2020 3



3. (10 bodova)

(a) Neka su $\mathbf{A}$ i $\mathbf{B}$ regularne matrice. Napišite i izvedite formulu u kojoj $(\mathbf{A} \cdot \mathbf{B})^{-1}$ izražavamo preko $\mathbf{A}^{-1}$ i $\mathbf{B}^{-1}$.

(b) Zadane su matrice

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{i} \quad \mathbf{B} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Riješite matričnu jednadžbu $\mathbf{X}^{-1} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{B}^{-1}$.

3. (a) Formula: $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$.

Izvod:

$$(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) = \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}$$

$$(\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) = \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}$$

Gornji račun pokazuje željenu tvrdnju.

(b)

$$\mathbf{X}^{-1}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{B}^{-1}$$

$$\mathbf{X}^{-1} = \mathbf{A}\cdot\mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$

$$\mathbf{X} = (\mathbf{A}\cdot\mathbf{B}^{-1}\cdot\mathbf{A}^{-1})^{-1} = \mathbf{A}\cdot\mathbf{B}\cdot\mathbf{A}^{-1}$$

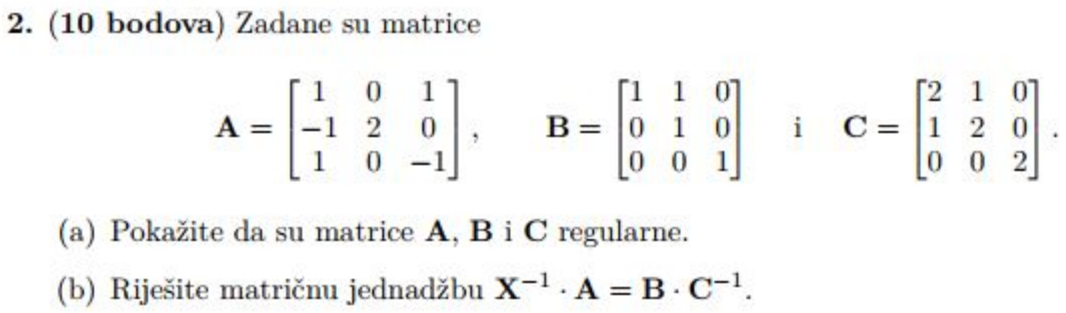
Primijetimo da su **A** i **B** regularne matrice što nam opravdava sav gornji račun.
Izračunajmo sada $\mathbf{A}^{-1}$.

$$\begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 1 & 1 & 0 & : & 0 & 1 & 0 \\ 0 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & -1 & 1 & 0 \\ 0 & 0 & 1 & : & 1 & -1 & 1 \end{bmatrix}$$

$$\Longrightarrow \mathbf{A}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

JIR 2020 2



2. (**10 bodova**) Zadane su matrice

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{i} \quad \mathbf{C} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(a) Pokažite da su matrice $\mathbf{A}$, $\mathbf{B}$ i $\mathbf{C}$ regularne.

(b) Riješite matričnu jednadžbu $\mathbf{X}^{-1} \cdot \mathbf{A} = \mathbf{B} \cdot \mathbf{C}^{-1}$.

② (a) $\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix} \begin{matrix} \cdot 1 \\ + \\ \cdot(-1)\ + \end{matrix} = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{vmatrix} = -4 \neq 0,$

$\det B = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0,$

$\det C = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} \begin{matrix} \cdot(-\frac{1}{2}) \\ + \\ \end{matrix} = \begin{vmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6 \neq 0$

$\Rightarrow$ A, B, C su regularne matrice

(b) $X^{-1}A = B \cdot C^{-1} \quad / \cdot A^{-1}$

$X^{-1} = BC^{-1}A^{-1} \quad /^{-1}$

$X = ACB^{-1}$

Računamo

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{B}\!\!\left[\begin{array}{ccc} \!\!\!\!\!\!\!\!\!\! & & \end{array}\right.\!\!\!\!\!\! \begin{bmatrix} 1 & 1 & 0 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \begin{matrix} + \\ \cdot(-1) \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & | & 1 & -1 & 0 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

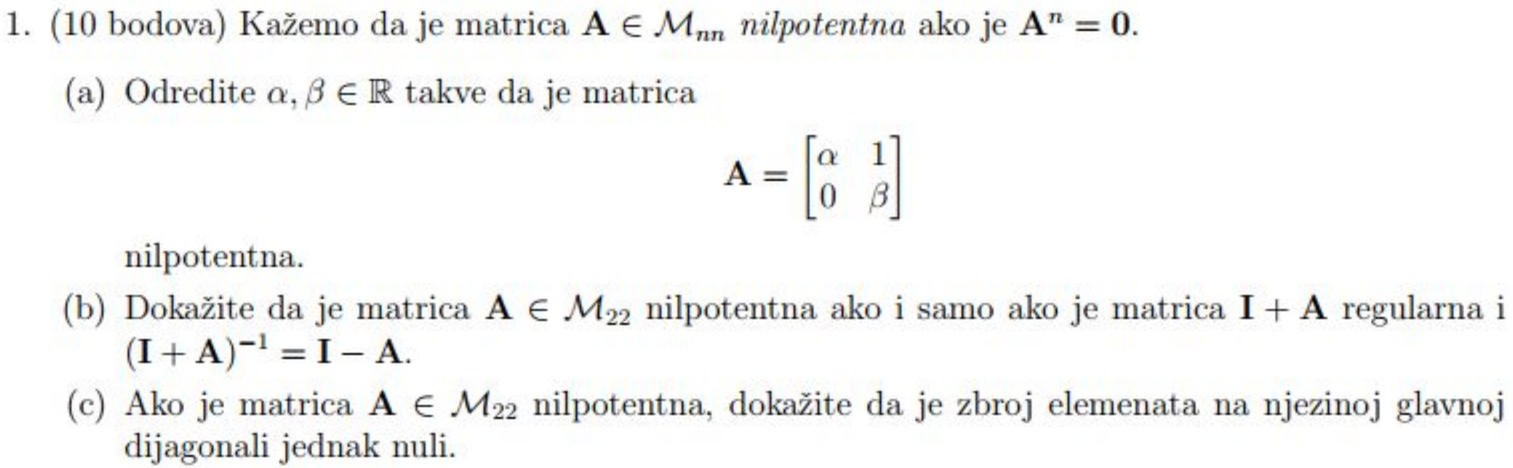
$$\left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{B} & & & 1 & 0 & 0 \\ & & & 0 & 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right] \begin{matrix} + \\ \cdot(-1) \\ \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{= B^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$\Rightarrow X = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 0 \\ 2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -1 & 2 \\ 0 & 3 & 0 \\ 2 & -1 & -2 \end{bmatrix}$$

LJIR 2020 1



1. (10 bodova) Kažemo da je matrica $\mathbf{A} \in \mathcal{M}_{nn}$ *nilpotentna* ako je $\mathbf{A}^n = \mathbf{0}$.
   (a) Odredite $\alpha, \beta \in \mathbb{R}$ takve da je matrica

   $$\mathbf{A} = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}$$

   nilpotentna.

   (b) Dokažite da je matrica $\mathbf{A} \in \mathcal{M}_{22}$ nilpotentna ako i samo ako je matrica $\mathbf{I} + \mathbf{A}$ regularna i $(\mathbf{I} + \mathbf{A})^{-1} = \mathbf{I} - \mathbf{A}$.

   (c) Ako je matrica $\mathbf{A} \in \mathcal{M}_{22}$ nilpotentna, dokažite da je zbroj elemenata na njezinoj glavnoj dijagonali jednak nuli.

1. (a) $$A^2 = \begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix}\begin{bmatrix} \alpha & 1 \\ 0 & \beta \end{bmatrix} = \begin{bmatrix} \alpha^2 & \alpha+\beta \\ 0 & \beta^2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} \alpha^2 = 0 \\ \alpha + \beta = 0 \\ \beta^2 = 0 \end{cases} \Rightarrow \alpha = \beta = 0$$

(b) Uočimo da je

$$(I+A)(I-A) = I - A + A - A^2 = I - A^2. \qquad (*)$$

Dakle,

$$\begin{aligned} A \in \mathcal{M}_{22} \text{ nilpotentna} &\Leftrightarrow A^2 = 0 \\ &\Leftrightarrow I - A^2 = I \\ &\overset{(*)}{\Leftrightarrow} (I+A)(I-A) = I \\ &\Leftrightarrow I+A \text{ regularna matrica i } (I+A)^{-1} = I - A \end{aligned}$$

(po definiciji regularnosti matrice)

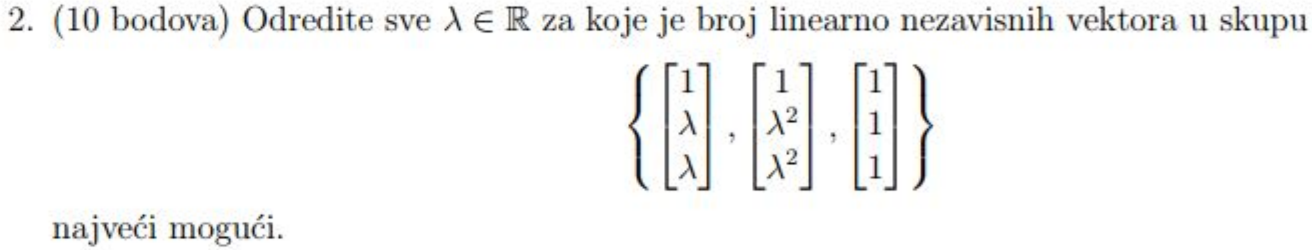
(c) Neka je $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{M}_{22}$ nilpotentna matrica. Imamo

$$A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & b(a+d) \\ c(a+d) & bc+d^2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{cases} a^2+bc = 0 \\ b(a+d) = 0 \\ c(a+d) = 0 \\ bc+d^2 = 0 \end{cases}$$

Pretpostavimo da je $a+d \neq 0$. Tada iz druge i treće jednakosti slijedi $b = c = 0$. Uvrštavanjem u prvu i četvrtu jednakost dobivamo $a^2 = d^2 = 0$, tj. $a = d = 0$ pa $a+d = 0$. Kontradikcija, dakle, $a+d = 0$.

LJIR 2020 2



2. (10 bodova) Odredite sve $\lambda \in \mathbb{R}$ za koje je broj linearno nezavisnih vektora u skupu

$$\left\{ \begin{bmatrix}1\\ \lambda\\ \lambda\end{bmatrix}, \begin{bmatrix}1\\ \lambda^2\\ \lambda^2\end{bmatrix}, \begin{bmatrix}1\\ 1\\ 1\end{bmatrix} \right\}$$

najveći mogući.

② Najveći broj linearno nezavisnih vektora među zadanima jednak je rangu matrice čiji su stupci (ili retci) zadani vektori:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ \lambda & \lambda^2 & 1 \\ \lambda & \lambda^2 & 1 \end{bmatrix} \begin{matrix} \\ \big) | \cdot (-1) \\ \swarrow + \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ \lambda & \lambda^2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \big) | \cdot (-\lambda) \\ \swarrow + \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda(\lambda-1) & 1-\lambda \\ 0 & 0 & 0 \end{bmatrix}$$

Razlikujemo slučajeve u ovisnosti o λ:

1° $\lambda = 1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow r(A) = 1$$

2° $\lambda = 0$

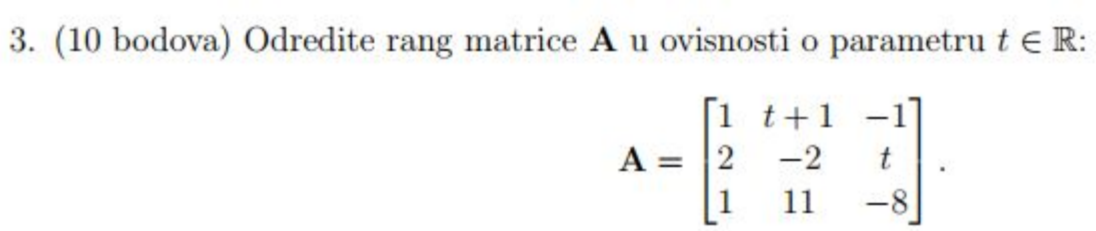
$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow r(A) = 2$$

3° $\lambda \neq 0, 1$

$$A \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda(\lambda-1) & 1-\lambda \\ 0 & 0 & 0 \end{bmatrix} | : \lambda(\lambda-1) \neq 0 \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{\lambda} \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow r(A) = 2$$

Dakle, u zadanom je skupu najveći mogući broj linearno nezavisnih vektora jednak 2 i taj broj se postiže za sve $\lambda \in \mathbb{R} \setminus \{1\}$.

MI 2021 3



3. (10 bodova) Odredite rang matrice $\mathbf{A}$ u ovisnosti o parametru $t \in \mathbb{R}$:

$$\mathbf{A} = \begin{bmatrix} 1 & t+1 & -1 \\ 2 & -2 & t \\ 1 & 11 & -8 \end{bmatrix}.$$

3°)

$$A = \begin{bmatrix} 1 & t+1 & -1 \\ 2 & -2 & t \\ 1 & 11 & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t+1 & -1 \\ 2 & -2 & t \\ 1 & 11 & -8 \end{bmatrix} \cdot -2 \quad \cdot -1 \sim \begin{bmatrix} 0 & t-10 & 7 \\ 0 & -24 & t+16 \\ 1 & 11 & -8 \end{bmatrix} :-24$$

$$\sim \begin{bmatrix} 0 & t-10 & 7 \\ 0 & 1 & \frac{t+16}{-24} \\ 1 & 11 & -8 \end{bmatrix} \cdot -(t-10) \sim \begin{bmatrix} 0 & 0 & 7+\frac{(t-10)(t+16)}{24} \\ 0 & 1 & (t+16)/-24 \\ 1 & 11 & -8 \end{bmatrix}$$

$\wr$ (*)

$$7+\frac{(t-10)(t+16)}{24} = \frac{168+t^2-10t+16t-160}{24}$$

$$= \frac{8+t^2+6t}{24}$$

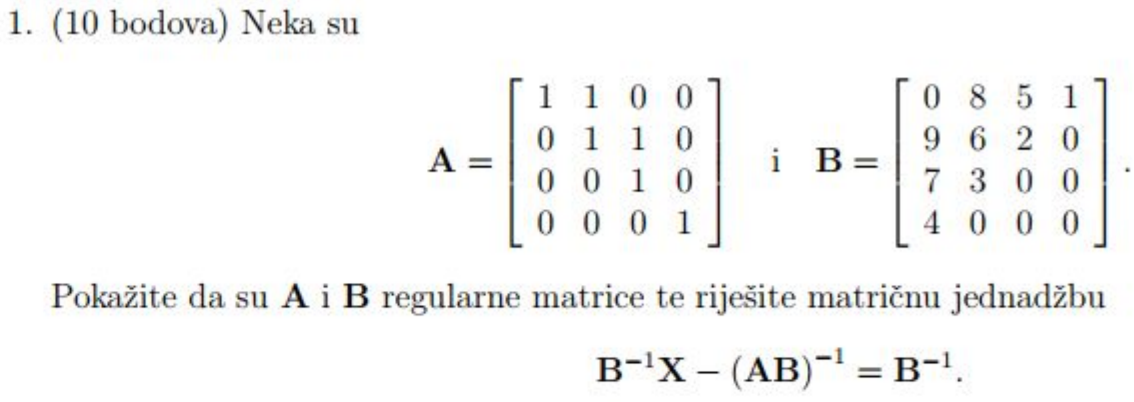
$$(*) \sim \begin{bmatrix} 0 & 0 & t^2+6t+8 \\ 0 & 1 & -(t+16)/24 \\ 1 & 11 & -8 \end{bmatrix}$$

$$t^2+6t+8 = (t+2)(t+4)$$

Za $t=-2$ ili $t=-4$ je $r(A)=2$

inače je $r(A)=3$

LJIR 2021 1



1. (10 bodova) Neka su

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{i} \quad \mathbf{B} = \begin{bmatrix} 0 & 8 & 5 & 1 \\ 9 & 6 & 2 & 0 \\ 7 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

Pokažite da su **A** i **B** regularne matrice te riješite matričnu jednadžbu

$$\mathbf{B}^{-1}\mathbf{X} - (\mathbf{AB})^{-1} = \mathbf{B}^{-1}.$$

1. (10 bodova) Neka su

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{i} \quad \mathbf{B} = \begin{bmatrix} 0 & 8 & 5 & 1 \\ 9 & 6 & 2 & 0 \\ 7 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{bmatrix}.$$

Pokažite da su **A** i **B** regularne matrice te riješite matričnu jednadžbu

$$\mathbf{B}^{-1}\mathbf{X} - (\mathbf{AB})^{-1} = \mathbf{B}^{-1}.$$

Računamo

$$\det A = \begin{vmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \left[\begin{matrix}\text{gornje trokutasta} \\ \text{determinanta}\end{matrix}\right] = 1^4 = 1 \neq 0,$$

$$\det B = \begin{vmatrix} 0 & 8 & 5 & 1 \\ 9 & 6 & 2 & 0 \\ 7 & 3 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 7 & 3 & 0 & 0 \\ 9 & 6 & 2 & 0 \\ 0 & 8 & 5 & 1 \end{vmatrix} = \left[\begin{matrix}\text{donje trokutasta} \\ \text{determinanta}\end{matrix}\right] = 4 \cdot 3 \cdot 2 \cdot 1$$

$$= 24 \neq 0,$$

odakle slijedi da su A i B regularne.

Za zadanu matričnu jednadžbu imamo

$$B^{-1}X - (AB)^{-1} = B^{-1}$$

$$B^{-1}X - B^{-1}A^{-1} = B^{-1}$$

$$B\cdot \Big| \quad B^{-1}(X - A^{-1}) = B^{-1}$$

$$X - A^{-1} = I$$

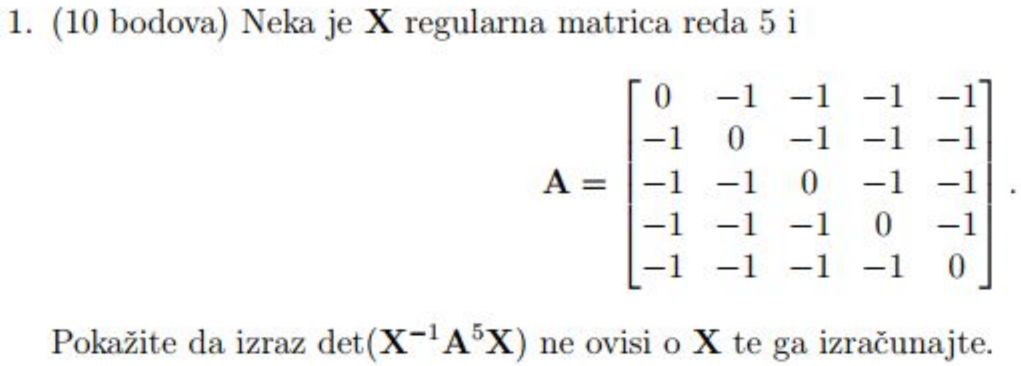
$$X = I + A^{-1}$$

$$\left[\underbrace{\begin{matrix}1&1&0&0\\0&1&1&0\\0&0&1&0\\0&0&0&1\end{matrix}}_{=A}\;\middle|\;\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}\right]\begin{matrix}\\ \uparrow^{+}\\ |\cdot(-1)\\ \\ \end{matrix}\sim\left[\begin{matrix}1&1&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}\;\middle|\;\begin{matrix}1&0&0&0\\0&1&-1&0\\0&0&1&0\\0&0&0&1\end{matrix}\right]\begin{matrix}\uparrow^{+}\\ |\cdot(-1)\\ \\ \\ \end{matrix}$$

$$\sim\left[\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}\;\middle|\;\underbrace{\begin{matrix}1&-1&1&0\\0&1&-1&0\\0&0&1&0\\0&0&0&1\end{matrix}}_{=A^{-1}}\right]$$

$$\Rightarrow X=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+\begin{bmatrix}1&-1&1&0\\0&1&-1&0\\0&0&1&0\\0&0&0&1\end{bmatrix}=\begin{bmatrix}2&-1&1&0\\0&2&-1&0\\0&0&2&0\\0&0&0&2\end{bmatrix}$$

JIR 2021 1



1. (10 bodova) Neka je $\mathbf{X}$ regularna matrica reda 5 i

$$\mathbf{A} = \begin{bmatrix} 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{bmatrix}.$$

Pokažite da izraz $\det(\mathbf{X}^{-1}\mathbf{A}^5\mathbf{X})$ ne ovisi o $\mathbf{X}$ te ga izračunajte.

1. (10 bodova) Neka je $\mathbf{X}$ regularna matrica reda 5 i

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Pokažite da izraz $\det(\mathbf{X}^{-1}\mathbf{A}^5\mathbf{X})$ ne ovisi o $\mathbf{X}$ te ga izračunajte.

Prema Binet-Cauchyjevom teoremu

$$\det(X^{-1}A^5X) = \det(X^{-1}) \det(A^5) \det X$$

$$= \frac{1}{\cancel{\det X}} \cdot (\det A)^5 \cdot \cancel{\det X}$$

$$= (\det A)^5,$$

pa vidimo da zadani izraz ovisi samo o determinanti matrice A.

Računamo

$$\det A = \begin{vmatrix} 0 & -1 & -1 & -1 & -1 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{vmatrix} \begin{matrix} \\ + \\ + \\ + \\ + \end{matrix} = \begin{vmatrix} -4 & -4 & -4 & -4 & -4 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{vmatrix}$$

$$= -4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & 0 \end{vmatrix} \begin{matrix} \\ + \\ + \\ + \\ + \end{matrix} = -4 \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{bmatrix} \text{dobivena determinanta} \\ \text{je gornje} \\ \text{trokutasta} \end{bmatrix} = -4 \quad \Rightarrow \det(X^{-1}A^5X) = (-4)^5 = -1024$$

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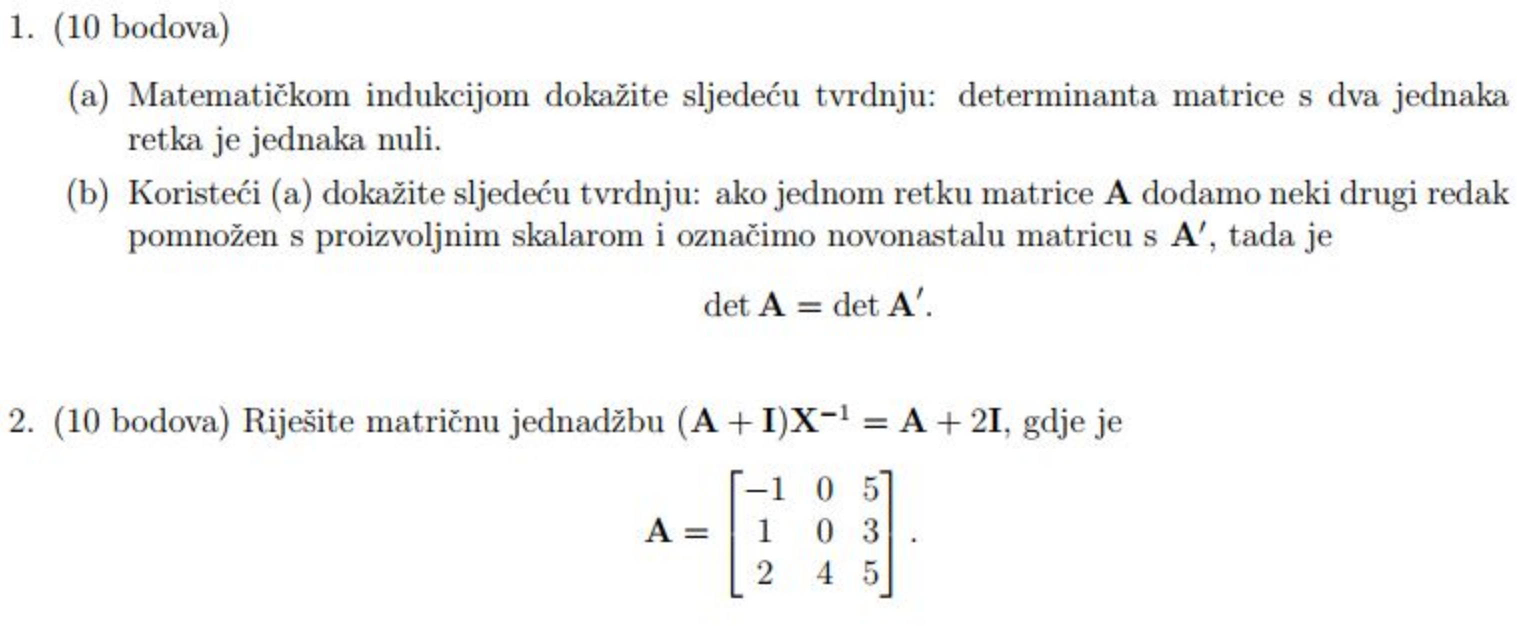

1. (10 bodova)

   (a) Matematičkom indukcijom dokažite sljedeću tvrdnju: determinanta matrice s dva jednaka retka je jednaka nuli.

   (b) Koristeći (a) dokažite sljedeću tvrdnju: ako jednom retku matrice $\mathbf{A}$ dodamo neki drugi redak pomnožen s proizvoljnim skalarom i označimo novonastalu matricu s $\mathbf{A}'$, tada je

$$\det \mathbf{A} = \det \mathbf{A}'.$$

2. (10 bodova) Riješite matričnu jednadžbu $(\mathbf{A}+\mathbf{I})\mathbf{X}^{-1} = \mathbf{A}+2\mathbf{I}$, gdje je

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 5 \\ 1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}.$$

1. (10 bodova)

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$$\det \mathbf{A} = \det \mathbf{A}'.$$

(a) Tvrdnju dokazujemo matematičkom indukcijom po redu n kvadratne matrice $A \in M_n$:

1° Baza $n=2$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{11} & a_{12} \end{vmatrix} = a_{11}a_{12} - a_{11}a_{12} = 0 \quad \text{za sve } a_{11}, a_{12} \in \mathbb{R} \quad \checkmark$$

2° Pretpostavimo da tvrdnja vrijedi za sve kvadratne matrice A reda n i neki $n \in \mathbb{N}$. Neka je $A \in M_{n+1}$ proizvoljna kvadratna matrica reda $n+1$ u kojoj su dva retka, i-ti i j-ti, međusobno jednaki. Laplaceovim razvojem po k-tom retku, $k \neq i$, $k \neq j$, dobivamo

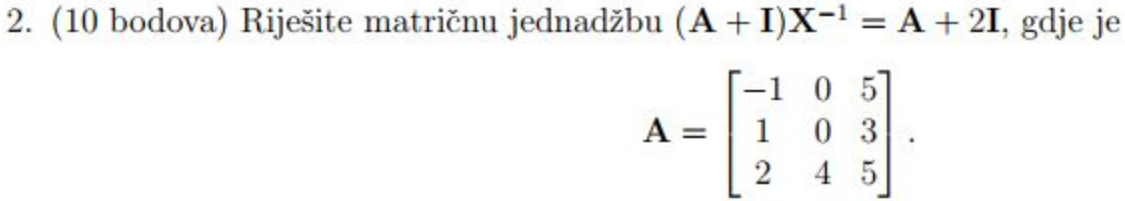
$$\det A = \sum_{l=1}^{n+1} a_{kl} \cdot (-1)^{k+l} M_{kl}. \qquad (*)$$

Budući da je svaka od pripadnih minora M_{kl} determinanta n-tog reda koja ponovno ima dva jednaka retka, one su sve po induktivnoj pretpostavci jednake nuli pa $(*)$ povlači i $\det A = 0$, čime je dokazan korak indukcije.

(b) Neka su vektori $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n \in V^n$ redom retci matrice $A \in M_n$ i neka je matrica $A' \in M_n$ dobivena dodavanjem i-tog retka pomnoženog skalarom $\lambda \in \mathbb{R}$ j-tom retku matrice A. Imamo

$$\det A' = \begin{vmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_j + \lambda a_i \\ \vdots \\ \vec{a}_n \end{vmatrix} = \begin{vmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_n \end{vmatrix} + \lambda \begin{vmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_n \end{vmatrix} = \begin{bmatrix} \text{druga je determinanta} \\ \text{prema (a) podzadatku} \\ \text{jednaka nuli} \end{bmatrix} = \begin{vmatrix} \vec{a}_1 \\ \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_n \end{vmatrix} = \det A'.$$

ZIR 2021 2



2. (10 bodova) Riješite matričnu jednadžbu $(\mathbf{A}+\mathbf{I})\mathbf{X}^{-1} = \mathbf{A}+2\mathbf{I}$, gdje je

$$\mathbf{A} = \begin{bmatrix} -1 & 0 & 5 \\ 1 & 0 & 3 \\ 2 & 4 & 5 \end{bmatrix}.$$

2. (10 bodova) Riješite matričnu jednadžbu $(\mathbf{A}+\mathbf{I})\mathbf{X}^{-1} = \mathbf{A}+2\mathbf{I}$, gdje je

$$\mathbf{A} = \begin{bmatrix}-1 & 0 & 5\\ 1 & 0 & 3\\ 2 & 4 & 5\end{bmatrix}.$$

$$(A+I)^{-1}\cdot \Big|\ (A+I)X^{-1} = A+2I$$

$$X^{-1} = (A+I)^{-1}(A+2I) \quad /^{-1}$$

$$X = \left[(A+I)^{-1}(A+2I)\right]^{-1} = (A+2I)^{-1}(A+I)$$

$$\left[\begin{array}{c|c}\underbrace{\begin{matrix}1 & 0 & 5\\ 1 & 2 & 3\\ 2 & 4 & 7\end{matrix}}_{A+2I} & \begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix}\end{array}\right] \begin{matrix}|\cdot(-1)\\ +\\ + \ |\cdot(-2)\end{matrix} \sim \left[\begin{array}{ccc|ccc}1 & 0 & 5 & 1 & 0 & 0\\ 0 & 2 & -2 & -1 & 1 & 0\\ 0 & 4 & -3 & -2 & 0 & 1\end{array}\right] \begin{matrix}\\ |\cdot(-2)\\ +\end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc}1 & 0 & 5 & 1 & 0 & 0\\ 0 & 2 & -2 & -1 & 1 & 0\\ 0 & 0 & 1 & 0 & -2 & 1\end{array}\right] \begin{matrix}+\\ + \ |\cdot(-5)\\ |\cdot 2\end{matrix} \sim \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 10 & -5\\ 0 & 2 & 0 & -1 & -3 & 2\\ 0 & 0 & 1 & 0 & -2 & 1\end{array}\right] \begin{matrix}\\ |:2\\ \end{matrix}$$

$$\sim \left[\begin{array}{c|c}\begin{matrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{matrix} & \underbrace{\begin{matrix}1 & 10 & -5\\ -\frac{1}{2} & -\frac{3}{2} & 1\\ 0 & -2 & 1\end{matrix}}_{(A+2I)^{-1}}\end{array}\right]$$

$$\Rightarrow X = \begin{bmatrix}1 & 10 & -5\\ -\frac{1}{2} & -\frac{3}{2} & 1\\ 0 & -2 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 5\\ 1 & 1 & 3\\ 2 & 4 & 6\end{bmatrix} = \begin{bmatrix}0 & -10 & 5\\ \frac{1}{2} & \frac{5}{2} & -1\\ 0 & 2 & 0\end{bmatrix}$$

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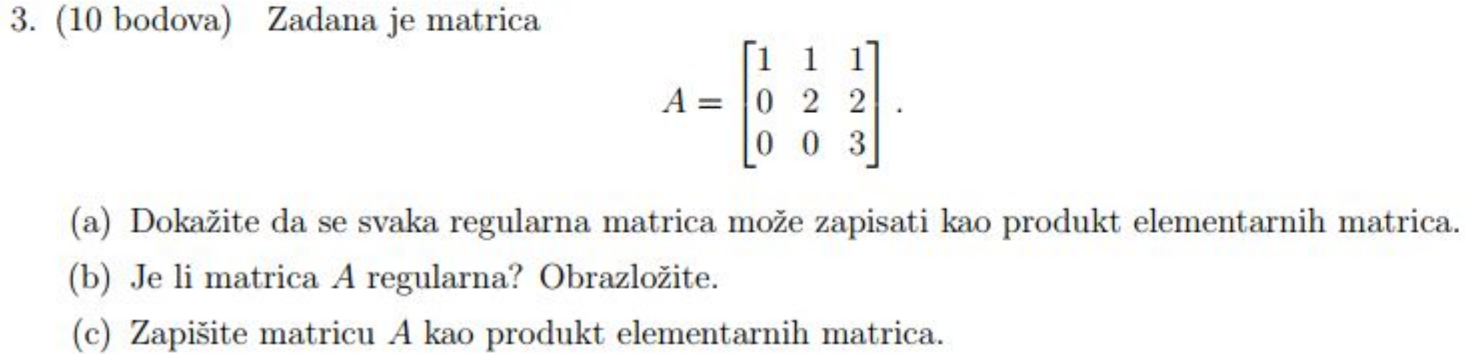

3\. (10 bodova) Zadana je matrica

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

(a) Dokažite da se svaka regularna matrica može zapisati kao produkt elementarnih matrica.

(b) Je li matrica A regularna? Obrazložite.

(c) Zapišite matricu A kao produkt elementarnih matrica.

RJEŠENJE **(a)** Skripta.

(b) Kako je matrica A gornje trokutasta, lako se vidi da je $\det(A) = 6 \neq 0$, pa je A regularna. Ista tvrdnja slijedi iz (c) podzadatka: matrica A je umnožak elementarnih matrica, koje su regularne, pa je i A regularna.

(c) Pomnožimo li matricu A *slijeva* s elementarnom matricom E, rezultat je matrica EA dobivena primjenom elementarne transformacije, koja odgovara toj elementarnoj matrici, na retcima matrice A . Napravimo li k elementarnih transformacija na retcima, koje odgovaraju matricama $E_1, \dots, E_k$, rezultat je matrica $E_k \dots E_1 A$. Ako je $E_k \dots E_1 A = I$, slijedi da je $E_k \dots E_1$ inverz od A te vrijedi

$$A = (E_k \dots E_1)^{-1} = E_1^{-1} \dots E_k^{-1}.$$

Kako je inverz elementarne matrice ponovno elementarna matrica, bit ćemo gotovi jednom kad pokažemo da je A ekvivalentna s jediničnom matricom.

$$\begin{bmatrix}1&1&1\\0&2&2\\0&0&3\end{bmatrix} \sim \begin{bmatrix}1&1&1\\0&1&1\\0&0&3\end{bmatrix} \sim \begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&1\\0&0&1\end{bmatrix} \sim \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Prvo smo pomnožili drugi red s $1/2$, što odgovara elementarnoj matrici $E_2(1/2)$. Potom smo pomnožili treći red s $1/3$, što odgovara matrici $E_3(1/3)$. Zatim smo drugi redak pomnožen s -1 dodali prvom, što odgovara elementarnoj matrici $E_{21}(-1)$. Konačno, dodali smo treći redak pomnožen s -1 drugom retku, što odgovara matrici $E_{32}(-1)$. Stoga se gornji niz ekvivalencija može zapisati kao

$$E_{32}(-1)E_{21}(-1)E_3(1/3)E_2(1/2)A = I \implies$$

$$A = E_2(1/2)^{-1}E_3(1/3)^{-1}E_{21}(-1)^{-1}E_{32}(-1)^{-1} = E_2(2)E_3(3)E_{21}(1)E_{32}(1).$$

□

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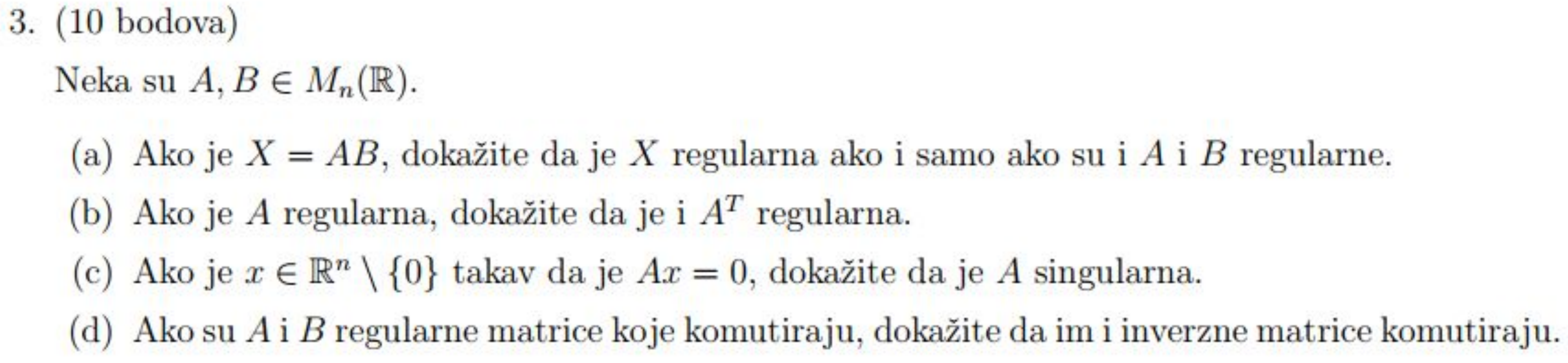

3. (10 bodova)

Neka su $A, B \in M_n(\mathbb{R})$.

(a) Ako je $X = AB$, dokažite da je X regularna ako i samo ako su i A i B regularne.

(b) Ako je A regularna, dokažite da je i A^T regularna.

(c) Ako je $x \in \mathbb{R}^n \setminus \{0\}$ takav da je $Ax = 0$, dokažite da je A singularna.

(d) Ako su A i B regularne matrice koje komutiraju, dokažite da im i inverzne matrice komutiraju.

Zadatak 3.

<u>RJEŠENJE</u> **a)** Uz pomoć Binet-Cauchyjevog teorema imamo

$$\det(X) = \det(A)\det(B).$$

Dakle, X je regularna akko je $\det(X) \neq 0$ akko je i $\det(A) \neq 0$ i $\det(B) \neq 0$ akko su i A i B regularne.

b) Pokazat ćemo da je $(A^T)^{-1} = (A^{-1})^T$:

$$\begin{aligned} A^T(A^{-1})^T &= (A^{-1}A)^T = I^T = I \\ (A^{-1})^T A^T &= (AA^{-1})^T = I^T = I. \end{aligned}$$

c) Pretpostavimo da je A regularna. Tada je

$$x = Ix = A^{-1}Ax = A^{-1}0 = 0,$$

što je kontradikcija.

d) Treba pokazati da je $A^{-1}B^{-1} = B^{-1}A^{-1}$. Množeći prvo dvaput slijeva, pa dvaput zdesna, imamo

$$AB = BA \implies B = A^{-1}BA \implies I = B^{-1}A^{-1}BA \implies A^{-1} = B^{-1}A^{-1}B \implies A^{-1}B^{-1} = B^{-1}A^{-1}.$$

□

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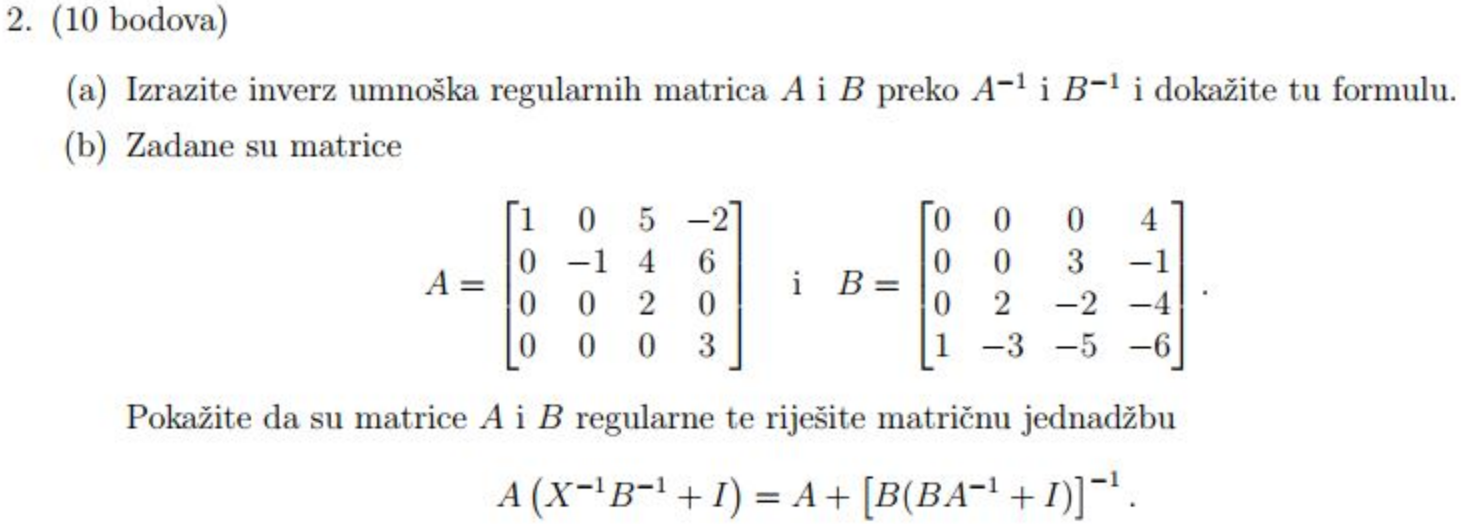

2. (10 bodova)

(a) Izrazite inverz umnoška regularnih matrica A i B preko A^{-1} i B^{-1} i dokažite tu formulu.

(b) Zadane su matrice

$$A = \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \quad \text{i} \quad B = \begin{bmatrix} 0 & 0 & 0 & 4 \\ 0 & 0 & 3 & -1 \\ 0 & 2 & -2 & -4 \\ 1 & -3 & -5 & -6 \end{bmatrix}.$$

Pokažite da su matrice A i B regularne te riješite matričnu jednadžbu

$$A\left(X^{-1}B^{-1} + I\right) = A + \left[B(BA^{-1} + I)\right]^{-1}.$$

Zadatak 2.

RJEŠENJE **a)** Vrijedi $(AB)^{-1} = B^{-1}A^{-1}$. Kako je matrično množenje asocijativno, vrijedi

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I \quad \text{te}$$
$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I.$$

b) Matrica je regularna ako i samo ako joj je determinanta različita od nule. Determinanta od A je umnožak elemenata na dijagonali, što daje $\det A = -6 \neq 0$, dok je

$$\det B = (-1)^{4+1} \begin{vmatrix} 0 & 0 & 4 \\ 0 & 3 & -1 \\ 2 & -2 & -4 \end{vmatrix} = (-1) \cdot 2 \cdot (-1)^{3+1} \begin{vmatrix} 0 & 4 \\ 3 & -1 \end{vmatrix} = 24 \neq 0.$$

Dakle, obje matrice su regularne. Sada riješavamo matričnu jednadžbu.

$$\begin{aligned}
A\left(X^{-1}B^{-1} + I\right) &= A + \left[B(BA^{-1} + I)\right]^{-1} \\
AX^{-1}B^{-1} + A &= A + \left(BA^{-1} + I\right)^{-1} B^{-1} \\
AX^{-1}B^{-1} &= \left(BA^{-1} + I\right)^{-1} B^{-1} && \text{(množimo zdesna s } B) \\
AX^{-1} &= \left(BA^{-1} + I\right)^{-1} && \text{(invertiramo)} \\
XA^{-1} &= BA^{-1} + I && \text{(množimo zdesna s } A) \\
X &= A + B \implies
\end{aligned}$$

$$X = \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & -1 & 7 & 5 \\ 0 & 2 & 0 & -4 \\ 1 & -3 & -5 & -3 \end{bmatrix}.$$

□