2.3. DIFERENCIJABILNOST

matan1: ako postoj der. u nekoj točki, ona je diferencijabilna

$(f'(a) \Rightarrow f$ dif. u $a)$ + tada postoji i tangenta

ne vrijedi za f-je više varijable

$$\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} = f'(x) \quad / \cdot \Delta x$$

kada $\Delta x \to 0$ onda imamo jednakost, ali ako je samo izrazito mali onda je samo približno

$$\Rightarrow f(x_0+\Delta x) - f(x_0) \approx f'(x) \cdot \Delta x$$

$$\Rightarrow f(x_0+\Delta x) - f(x_0) = f'(x_0)\Delta x + \theta(\Delta x) \quad \text{– greška aproksimacije}$$

pri čemu $\lim\limits_{\Delta x \to 0} \frac{\theta(\Delta x)}{\Delta x} = 0$ → $\theta(\Delta x)$ mora brže težit u 0 (da bi lim = 0) nego Δx

MATAN2: ako postoje parc. der.

⇓

ne mora f-ja biti dif. niti postojati tang. ravnina

$\boxed{\text{DEF}}$ $f(x,y)$ je diferencijabilna u (x_0,y_0) ako postoje parc. deriv. u (x_0,y_0) te ako vrijedi:

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \theta(\Delta x, \Delta y)$$

pri čemu $\lim\limits_{(\Delta x,\Delta y)\to(0,0)} \frac{\theta(\Delta x,\Delta y)}{\underbrace{\sqrt{(\Delta x)^2 + (\Delta y)^2}}_{\text{udaljenost točke od } 0,0}} = 0$ → opet vrijedi isto (mora težit brže)

$\boxed{\text{M1-2022-1)}}$ $f(x,y) = x \cdot y$ dif $\forall_{x,y}$

$$\frac{\partial f}{\partial x} = y \qquad \frac{\partial f}{\partial y} = x$$

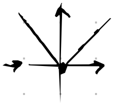
$$(x+\Delta x)(y+\Delta y) - xy = y \cdot \Delta x + x \cdot \Delta y + \theta(\Delta x, \Delta y) \rightarrow \text{trebamo izračunat } \theta$$

$$\cancel{\Delta y \cdot x} + \cancel{\Delta x \, y} + \Delta x \Delta y = \cancel{\Delta x \, y} + \cancel{\Delta y \, x} + \theta(\Delta x, \Delta y)$$

$$\theta(\Delta x, \Delta y) = \Delta x \cdot \Delta y$$

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x \cdot \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} \overset{\text{pol.}}{=} \lim_{r\to 0} \frac{r\cos\varphi \cdot r\sin\varphi}{\sqrt{r^2}} = \lim_{r \to 0} \frac{r^{\cancel{2}}(\cos\varphi \cdot \sin\varphi)}{\cancel{r}} = 0$$

$\boxed{\text{TM}}$ f. je diferencijabilna ako postoji tang. ravnina u T_0.

*MATI: f-ja je dif. $\longrightarrow$ f. je neprekinuta (OBRAT NE VRIJEDI → )

$\boxed{\text{TM}}$ Ako je $f(x,y)$ dif. u $T(x_0,y_0) \Rightarrow$ tada je neprekinuta u T_0

DOKAZ: Ako je f. dif., tada vrijedi sve ono iz def ↑.

• Ključni limes iz definicije:

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{o(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = 0, \quad \text{tj.} \quad \underbrace{\lim_{(\Delta x,\Delta y)\to 0,0} o(\Delta x,\Delta y) = 0}$$

*(sigurno)

• Napadnimo def. sa lim kada $(\Delta x, \Delta y) \longrightarrow (0,0)$.

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \left[f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)\right] = \frac{\partial f}{\partial x}\cdot 0 + \frac{\partial f}{\partial y}\cdot 0 + 0$$

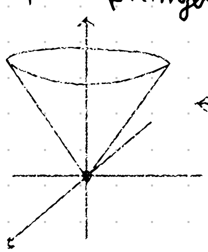
NUŽAN UVJET

$$\lim_{\Delta x,\Delta y\to 0,0} f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0)$$

tj. f-ja je neprekinuta po definiciji

Napomene:

1. Ako f ima prekid u T_0, f nije diferencijabilna u T_0 (kao obrat po kontrapoziciji)
2. Obrat naravno ne vrijedi, tj. ako je f neprekinuta u T_0, ne mora biti dif.
   Protuprimjer: STOŽAC! $z=\sqrt{x^2+y^2}$

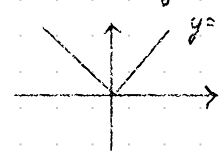


f-ja je neprekinuta
ali nije dif!!

jer ne postoje parc. der. (zbog vrha 0,0)

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{\Delta x^2+\Delta y^2}} = \cancel{\frac{0}{0}} \;!!$$

* u 2D je to $y=|x|$



3. Ako postoje parc. der. u $T_0 \Longrightarrow$ tada je f. neprekinuta u T_0
   TVRDNJA NIJE ISTINITA!!

Postoje f koje imaju parc. der. u T_0, ali imaju prekid u T_0

UIR 2021.)

1. Dodefinirajte:

a) $f(x,y)=\dfrac{2xy}{\sqrt{x^2+y^2}}$ $(x,y)\neq(0,0)$ mora biti neprekinuta.

$$\lim_{x,y\to 0,0}\frac{2xy}{\sqrt{x^2+y^2}} \overset{\text{pol.}}{=} \lim_{r\to 0}\frac{2r^2\cos\varphi\sin\varphi}{\cancel{\sqrt{r^2}}} = \lim_{r\to 0} r\cdot\overbrace{\boxed{2\sin 2\varphi}}^{\text{ograničeno}} = 0$$

postoji limes

$$\Rightarrow f(x,y)=\begin{cases}\dfrac{2xy}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Dedefinirati ovako!

b) izračunajte po def po $\dfrac{\partial f}{\partial x}(0,0)$ i $\dfrac{\partial f}{\partial y}(0,0)$.

$$\frac{\partial f}{\partial x}(0,0)=\lim_{\Delta x\to 0}\frac{f(\Delta x+x,y)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\dfrac{2(\Delta x+\overset{=0}{x})\overset{=0}{y}}{\sqrt{(\Delta x+x)^2+y^2}}\overset{=0}{-}\ \underset{0}{0}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\dfrac{2\Delta x\cdot 0}{\Delta x}=0}{\Delta x} \Rightarrow \lim_{\Delta x\to 0} 0 = \underline{\underline{0}}$$

$\dfrac{\partial f}{\partial y}(0,0)=DZ-0$.

Iz def. dif.

c) $f(0+\Delta x, 0+\Delta y)-f(0,0)=\dfrac{\partial f}{\partial x}\Delta x+\dfrac{\partial f}{\partial y}\Delta y+\sigma(\Delta x,\Delta y)$

$$\frac{2\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}-\cancel{0}=\cancel{0\cdot\Delta x}+\cancel{0\cdot\Delta y}+\sigma(\Delta x,\Delta y)$$

$$\Rightarrow \sigma(\Delta x,\Delta y)=\frac{2\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}$$

$$\lim_{\Delta x\Delta y\to 0,0}\frac{\sigma(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\Delta x\Delta y\to 0,0}\frac{\dfrac{2\Delta x\Delta y}{\sqrt{\Delta x^2+\Delta y^2}}}{\sqrt{\Delta x^2+\Delta y^2}} \overset{\text{pol.}}{=} \frac{\dfrac{2\cancel{r^2}\cos\varphi\sin\varphi}{\cancel{r}}}{\cancel{r}}=\sin 2\varphi$$

limes ne postoji jer nije zadan uvjet $\Rightarrow$ <u>f nije dif!</u>

$\boxed{\text{MI - 2019 - 1)}}$

a) $f(x,y) = \begin{cases} \frac{4xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

i)

$$\lim_{x,y\to 0,0} \frac{4xy}{x^2+y^2} \overset{pol}{=} \lim_{x,y\to 0,0} \frac{4r^2\cos\varphi\sin\varphi}{r^2} = \underline{\underline{2\sin 2\varphi}}$$

limes ne postoji $\longrightarrow$ f ima prekid u (0,0)

ii)

$$\frac{\partial f}{\partial x}(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x+0,0) - f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\frac{4\Delta x\cdot 0}{\Delta x^2+0}}{\Delta x} = \lim_{\Delta x\to 0} 0 = \underline{0}$$

fija ima prekid, ali parc. der. postoji!

b) Tvrdnja nije točna: protuprimjer je f iz a)
jer u a) postoje parc. der., ali f ima prekid u (0,0) pa nije dif u (0,0) (po teoremu)

TM | Dovoljan uvjet za diferencijab. fije dviju varijabla

Ako postoje parc. der. i ako su parc. der. neprekinute (ne funkcija nego baš parc. der.) $\rightarrow$ f je dif.

-> nema dokaza niti TT

=> OBRAT NE VRIJEDI

npr. valovi na vodi
· fija je dif i neprekinuta, ali parc. der. imaju prekid

DEF Neka $f(x,y)$ ima parc. der. u $T(x,y)$. Tada je **GRADIJENT** od f vektor oblika:

$$\boxed{\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}}$$

(nabla)

Za $f(x,y,z) \Rightarrow \nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$$

∇ (Nabla) je LINEARAN OPERATOR

$\hookrightarrow \nabla(\alpha f + \beta g) = \alpha\nabla f + \beta\nabla g, \quad \alpha,\beta \in \mathbb{R}$