2.3 SVOJSTVA FURIEROVOG INTEGRALA

Diskretni spektar periodične funkcije

* Furierov red: $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n w_0 x + b_n \sin n \cdot w_0 x)$; $w_0 = \frac{2\pi}{T}$

* $f(x) = C \cdot \sin(wx + \varphi)$

FAZNI POMAK

$\Rightarrow$ F. red: $\pm \frac{1}{2} C_0 + \sum_{n=1}^{\infty} C_n \cdot \sin(n w_0 x + \varphi_n)$

AMPLITUDA

$$\begin{cases} C_0 = |a_0| \\ C_n = \sqrt{a_n^2 + b_n^2} \\ \operatorname{tg} \varphi_n = \frac{a_n}{b_n} \end{cases}$$

a_n kosinusni spektar (parna)

C_n amplitudni spektar $\rightarrow$ s kojim intenzitetom n-ti harmonik ulazi u rastav f-je f

b_n sinusni spektar (neparna)

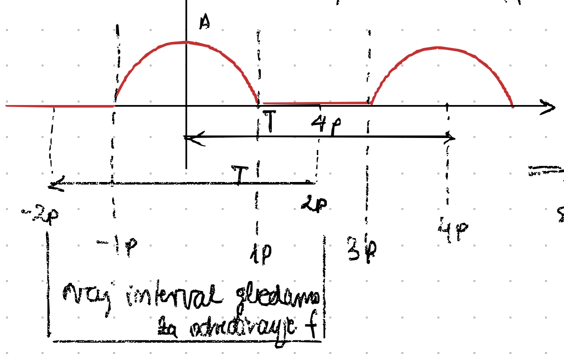
φ_n fazni spektar

Jedinstvenost spektralnog prikaza

TM Ako periodičke f-je f i g zadovoljavaju Dirichletove uvjete i imaju isti diskretni spektar, onda se one podudaraju u svim točkama osim možda u točkama prekida.

Primjer

// funkcija je periodična na periodu T=4p



njezina jednadžba na intervalu $[-2p, 2p]$ glasi:

$$f(x) = \begin{cases} 0, & p \leq |x| \leq 2p \\ A\cos \frac{\pi}{2p} x & |x| \leq p \end{cases}$$

$\Rightarrow$ budući da je funkcija parna, svi b_n koeficijenti (oni uz sinus) su 0!

parnost: $a_0 = \frac{2}{L}\int_0^L f(x)dx$

$T = 2L = 4p \rightarrow L = 2p$

$\Rightarrow a_0 = \frac{1}{L}\int_0^p A\cos\frac{\pi}{2p} x \, dx = \frac{A}{p}\int_0^p \cos\frac{\pi}{2p} x dx = \frac{A}{\cancel{p}} \cdot \frac{2\cancel{p}}{\pi} \cdot \sin\frac{\pi}{2p} x \Big|_0^p$

$= \frac{A \cdot 2}{\pi} \cdot \Big(\underbrace{\sin\frac{\pi}{2\cancel{p}} \cdot \cancel{p}}_{1} - \underbrace{\cancel{\sin 0}}_{0}\Big) \Rightarrow \boxed{a_0 = \frac{2A}{\pi}}$

$\Rightarrow a_n = \overset{\text{parnost}}{\frac{2}{L}}\int_0^p A\cos\frac{\pi x}{2p} \cdot \cos\frac{n\pi x}{L} = \frac{1}{p}\int_0^p A\cos\frac{\pi x}{2p} \cdot \cos\frac{n\pi x}{2p} dx$

$a_n = \frac{A}{p}\int_0^p \frac{1}{2}\Big(\cos\big(\frac{\pi x + n\pi x}{2p}\big) + \cos\big(\frac{\pi x}{2p} - \frac{n\pi x}{2p}\big)\Big)dx$

$a_n = \frac{A}{2p}\int_0^p \Big(\cos\frac{\pi}{2p}x(1+n) + \cos\frac{\pi}{2p}x(1-n)\Big)dx$

$a_n = \frac{A}{\cancel{2p}} \cdot \frac{\cancel{2p}}{\pi}\Big(\frac{1}{1+n} \cdot \sin\frac{\pi}{2p}x(1+n) + \frac{1}{1-n}\sin\frac{\pi}{2p}x(1-n)\Big)\Big|_0^p \rightarrow$ za 0 $\Rightarrow 0$

$a_n = \frac{A}{\pi}\Big(\frac{1}{1+n}\sin\frac{\pi}{2}(n+1) + \frac{1}{n-1}\sin\frac{\pi}{2}(n-1)\Big) = \frac{A}{\pi}\Big(\frac{(n-1)\sin\left[\frac{\pi}{2}(n+1)\right] + (n+1)\sin\left(\frac{\pi}{2}\ldots\right)}{n^2-1}\Big)$

$a_n = \frac{A}{\pi} \cdot \frac{1}{n^2-1} \cdot 2\sin\frac{\frac{\pi}{2}n + \cancel{\frac{\pi}{2}} + \frac{\pi}{2}n - \cancel{\frac{\pi}{2}}}{2} \cdot \cos\frac{\cancel{\frac{\pi}{2}n} + \frac{\pi}{2} - \cancel{\frac{\pi}{2}n} + \frac{\pi}{2}}{2}$

$\Rightarrow \boxed{a_n = -\frac{2A}{\pi} \cdot \frac{\cos\frac{n\pi}{2}}{n^2-1}}$ ako je $n \neq 1$ *

Kosinusni spektar: $a_0 = \frac{2A}{\pi}$ $a_1 = \frac{A}{2}$

$a_1 = \lim_{n\to 1} a_n = \lim_{n\to 1}\Big(-\frac{2A}{\pi}\Big) \cdot \frac{\cos\frac{n\pi}{2}}{n^2-1} = -\frac{2A}{\pi}\lim_{n\to 1}\frac{-\sin\frac{n\pi}{2} \cdot \left(\frac{\pi}{2}\right)}{2n} = \Big(\frac{-2A}{\pi} \cdot \frac{-1}{2} \cdot \frac{\pi}{2}\Big)$

amplitudni spektar: $C_0 = \frac{2A}{\pi}$ $C_1 = \frac{A}{2}$

parni $C_{2n} = \frac{2A}{\pi} \cdot \left|\frac{(-1)^{2n}}{4n^2-1}\right| = \frac{2A}{\pi(4n^2-1)}$

$C_{2n+1} = 0$ neparni spektar

Integriranje i deriviranje Furierovog reda

Integriranje $\int f(x)\,dx = \frac{a_0}{2}\int dx + \sum_{n=1}^{\infty}\int (a_n \cos n\omega_0 x + b_n \sin n\omega_0 x)\,dx$

→ za svaki od koeficijenata integriramo izraz prema x unutar zadanog intervala (npr $[-T/2, T/2]$ ili $[0,T]$)

$$\int f(x)dx = \frac{a_0}{2}\int dx + a_n\int \cos nx\,dx + \dots = \frac{a_0}{2}x + a_n\cdot\frac{1}{n}\sin nx + \dots$$

Primjer: $f(x)=x^2$, $g(x)=x$, $[-\pi,\pi]$ parna f-ja (f)

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx \cancel{+ b_n \sin nx})$ parna je!

$\longrightarrow a_0 = \frac{2}{L}\int_0^L f(x)dx \quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2 dx = \frac{2}{\pi}\cdot\frac{1}{3}x^3\Big|_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x^2\cdot\cos nx = \overset{\text{parc. int.}}{\cdots} = \frac{4}{n^2}(-1)^n$$

$$f(x) = \frac{\pi^2}{3} + \sum \frac{4}{n^2}(-1)^n\cdot\cos nx$$

$$g(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

(g) neparna f-ja: $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$\Rightarrow$ integracija $g(x)$

$$\int g(x) = 2\int\Big(\underbrace{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}}_{\text{konst.}}\sin nx\Big)dx = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int \sin nx\,dx = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\frac{\cos nx}{n} + C$$

Deriviranje $f'(x) = \frac{d}{dx}f(x) = \frac{d}{dx}\left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 x + b_n \sin n\omega_0 x)\right)$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\cdot\cos nx \Rightarrow f'(x) = 2x = 0 + 4\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cdot(-n)\sin nx$$

$$f'(x) = \cancel{2}x = \cancel{4}\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin nx$$ ↙ dobije se g(x)

$$g(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx \Longrightarrow g'(x) = 1 = 2\cdot\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{\cancel{n}}\cdot\cancel{n}\cdot\cos nx$$

$$g'(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\cos nx$$ → ali taj nije konvergentan $\forall n \geq 1$!

Xnovodobiveni red ne mora konvergirati!

TM Deriviranje F. reda

Pretpostavimo da je periodična f-ja f perioda 2π neprekinuta na R i ima sljedeći Furierov prikaz:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos nx + \sum_{n=1}^{\infty}b_n \sin nx$$

Ako f' zadovoljava sve Dirichletove uvjete, onda se ona može prikazati u obliku

$$f'(x) = \sum_{n=1}^{\infty}b_n\cdot n\cdot\cos nx + \sum_{n=1}^{\infty}(-a_n)\,n\sin nx$$

Parsevallova jednakost

Fje su također ortogonalne na intervalu $[a,b]$ duljine T

$$\omega_0=\frac{2\pi}{T} \qquad \left\{\frac{1}{2}, \sin(n\omega_0 x+\varphi_n)\right\} \text{ ortogonalni sustav na } [a,b]$$
ako je $T=b-a$

$\longrightarrow$ Lako se provjeri:

$$\int_a^b \sin^2(n\omega_0 x+\varphi_n)\,dx=\frac{T}{2}$$

$$\int_a^b \sin(n\omega_0 x+\varphi_n)\cdot\sin(m\omega_0 x+\varphi_m)\,dx=0 \quad (\text{lema: za } n\neq m)$$

$$\int_a^b \frac{1}{2}\sin(n\omega_0 x+\varphi_n)\,dx=0,\quad n\geq 1$$

$$\int_a^b \sin^2(n\omega_0 x+\varphi_n)\,dx=\frac{T}{2}$$

$$\int_a^b \left(\frac{1}{2}\right)dx=\frac{1}{2}x\Big|_a^b=\left(\frac{1}{2}\right)(b-a)=\frac{T}{4}$$

$\rightarrow$ Sada imamo:

$$\int_a^b |f(x)|^2\,dx=\int_a^b\left(\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n\sin(n\omega_0 x+\varphi_n)\right)^2dx \quad \left[\text{koristi se } f(x)=\frac{c_0}{2}+\sum_{n=1}^{\infty}c_n\sin(n\omega_0 x+\varphi_n)\right]$$

$$=\int_a^b\frac{c_0^2}{4}\,dx+\int_a^b\sum_{n=1}^{\infty}c_n^2\sin^2(n\omega_0 x+\varphi_n)\,dx$$

$$=c_0^2\cdot\frac{T}{4}+\sum_{n=1}^{\infty}c_n^2\cdot\frac{T}{2}$$

$$\int_a^b |f(x)|^2\,dx=c_0^2\cdot\frac{T}{4}+\sum_{n=1}^{\infty}c_n^2\cdot\frac{T}{2}$$

$$(c_n^2=a_n^2+b_n^2)$$

$$\int_a^b |f(x)|^2\,dx=c_0^2\cdot\frac{T}{4}+\frac{T}{2}\sum_{n=1}^{\infty}c_n^2 \longrightarrow$$

$$\int_a^b |f(x)|^2\,dx=\frac{T}{4}a_0^2+\frac{T}{2}\sum_{n=1}^{\infty}(a_n^2+b_n^2)$$

TH **Parsevalova jednakost:**

Za Furierove koef. $a_0, a_1, \dots a_n$ i $b_0, b_1, \dots b_n$ vrijedi Pars. jed.

$$\frac{1}{2}a_0^2+\sum_{n=1}^{\infty}a_n^2+\sum_{n=1}^{\infty}b_n^2=\frac{2}{T}\int_a^b|f(x)|^2\,dx$$

Primjer: Koristeći razvoj funkcije $f(x)=x^2$, $-\pi < x < \pi$ u Fourierov red i Parsevalovu jednakost, izračunaj sumu reda $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$.

* Znamo otprije: $f(x) = x^2 = \frac{\pi^2}{3} + 4\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cdot \cos nx$ $\quad [-\pi, \pi] \Rightarrow T = 2\pi$

$\rightarrow$ primjenimo Parsevalovu jednakost

$$\frac{2}{T}\int_a^b |f(x)|^2 dx = \frac{2}{2\pi}\int_a^b (x^2)^2 dx = \frac{1}{\pi}\int_a^b x^4 dx = \frac{2\pi^4}{5}$$

Prema teoremu: $^{*}\left(\frac{1}{2}a_0^2 + \sum\limits_{n=1}^{\infty} a_n^2 + \sum\limits_{n=1}^{\infty} b_n^2 = \frac{2}{T}\int_a^b |f(x)|^2 dx\right)$ izjednačimo

$a_0 = \frac{2\pi^2}{3}$ $\quad a_n = \frac{4}{n^2}(-1)^n \cos nx$ $\quad$ a traži se $\sum \frac{1}{n^4}$

$a_0^2 = \frac{4\pi^4}{9}$ $\quad a_n^2 = \frac{16}{n^4}$ $\longrightarrow$ $\frac{1}{2}\frac{4\pi^4}{9} + \sum\limits_{n=1}^{\infty}\frac{16}{n^4} = \frac{2\pi^4}{5}$

$$\Rightarrow 16\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{2\pi^4}{5} - \frac{2\pi^4}{9} \quad /:16$$

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = 2\pi^4\left(\frac{9-5}{45}\right)\cdot\frac{1}{16} = \frac{8\pi^4}{45\cdot 16} = \boxed{\frac{\pi^4}{90}}$$

Jednolika konvergencija Fourierovog reda

*mislim da se ovo ne obrađuje

$\boxed{TM}$ Neka je f neprekinuta f-ja za koju f' zadovoljava Dirichletove uvjete na intervalu $[-\pi, \pi]$ i neka je $f(-\pi) = f(\pi)$ (parna je).

Trig. F. red: $S(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$

konvergira jednoliko i vrijedi $S(x) = f(x)$, $\forall x \in [-\pi, \pi]$

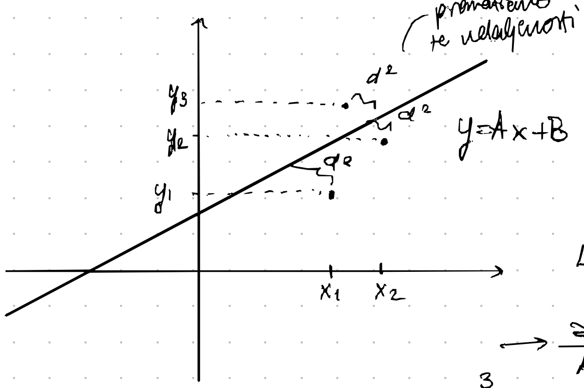
Najbolja aproksimacija

Funkciju f aproksimiramo konačnim Fourierovim redom

$$f(x) \approx S_N(x) := \frac{a_0}{2} + \sum_{k=1}^{N} \left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right)$$ $\rightarrow$ Je li to najbolje moguće?

*Na koji način treba odabrati A_n i B_n $\rightarrow$ $R_N := \frac{A_0}{2} + \sum_{k=1}^{N} \left(A_k \cos\frac{k\pi x}{L} + B_k \sin\frac{k\pi x}{L}\right)$ da aprox. bude najbolja moguća?

$\Longrightarrow$ udaljenost najmanjih kvadrata



IDEJA: Pronaći te A, B t.d. pravac y najbolje aprox. skup točaka $\{(x_1, y_1), (x_2, y_2), (x_3, y_3)\}$

$$(y_1 - Ax_1 - B)^2 + (y_2 - Ax_2 - B)^2 + (y_2 - Ax_3 - B)^2 = 0$$

$\Downarrow$

$E(A, B) \rightarrow$ minimum nađemo preko derivacija

$\rightarrow \frac{\partial E}{A_n} = 0 \qquad \frac{\partial E}{B_n} = 0$

$$E(A, B) = \sum_{n=1}^{3} (y_n - A_n - B_n) \rightarrow MIN \Longrightarrow$$ preko Fouriera

$$E := \int_{-L}^{L} |f(x) - R_N(x)|^2 dx$$

$$= \int_{-L}^{L} \left| f(x) - \frac{A_0}{2} - \sum_{k=1}^{N} \left(A_k \cos\frac{k\pi x}{L} + B_k \sin\frac{k\pi x}{L}\right) \right|^2 dx$$

$$\frac{\partial E}{\partial A_n} = 2 \int_{-L}^{L} \left[f(x) - \frac{A_0}{2} - \sum_{k=1}^{N} \left(A_k \cos\frac{k\pi x}{L} + B_k \sin\frac{k\pi x}{L}\right) \right] \cos\frac{n\pi x}{L} dx$$

$\rightarrow$ zbog ortogonalnosti trigonometrijskih f-ja, pod integralom preostaju samo 2 člana

$$\int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx - A_n \int_{-L}^{L} \cos^2\frac{n\pi x}{L} dx = 0$$

$\rightarrow$ i odavde: $A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\frac{n\pi x}{L} dx = a_n$

$\rightsquigarrow$ slično i za b_n

$\Longrightarrow$ Fourierov red daje najbolji aprox. za f-ju f među svim trig. polinomima reda N

Kompleksni oblik kompleksnog reda

$f(t)$ fja perioda T, $T = b - a$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$
$$e^{-i\alpha} = \cos\alpha - i\sin\alpha \quad /- \quad /+$$

$$e^{i\alpha} - e^{-i\alpha} = 2i\sin\alpha \; /:2i$$
$$\sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

$$e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha \; /:2$$
$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2}$$

+ ideja, da članove F reda ne pišemo kao $\sin\alpha$ i $\cos\alpha$ nego u obliku $e^{i\alpha}$

$$a_n\cos n\alpha + b_n\sin n\alpha = c_n e^{in\alpha} + c_{-n} e^{-in\alpha}$$

$$= a_n \cdot \frac{e^{in\alpha} + e^{-in\alpha}}{2} + b_n \cdot \frac{e^{in\alpha} - e^{-in\alpha}}{2i}$$

$$= (\;\cdot\;)e^{in\alpha} + (\quad)e^{-in\alpha} = \left(\frac{a_n}{2} + \frac{b_n}{2i}\cdot\frac{i}{i}\right)e^{in\alpha} + \left(\frac{a_n}{2} - \frac{b_n}{2i}\cdot\frac{i}{i}\right)e^{-in\alpha}$$

$$= \underbrace{\left(\frac{a_n}{2} - i\frac{b_n}{2}\right)}_{c_n} e^{in\alpha} + \underbrace{\left(\frac{a_n}{2} + i\frac{b_n}{2}\right)}_{c_{-n}} e^{-in\alpha}$$

$$\alpha = \frac{2\pi x}{T} \longrightarrow \text{F red (T formule)}: \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{2n\pi x}{T} + b_n\sin\frac{2n\pi x}{T}\right)$$

$$\Rightarrow c_0 + \sum_{n=1}^{\infty} c_n\left(e^{in\frac{2\pi x}{T}}\right) + c_{-n}\left(e^{-in\frac{2\pi x}{T}}\right) \Rightarrow \boxed{\sum_{n=1}^{\infty} c_n e^{\frac{2in\pi x}{T}}}$$

kompleksni oblik F reda

! ne uzimati pomoćnu varijablu x jer ju imamo u argumentu!

$$a_n = \frac{2}{T}\int_a^b f(\xi)\cos\frac{2n\pi\xi}{T}\,d\xi$$
$$b_n = \frac{2}{T}\int_a^b f(\xi)\sin\frac{2n\pi\xi}{T}\,d\xi$$
$$c_n = \frac{1}{2}(a_n - ib_n)$$

$$\to c_n = \frac{1}{T}\left(\int_a^b f(\xi)\cdot\cos\frac{2\pi\xi}{T}\,d\xi - i\int_a^b f(\xi)\sin\frac{2n\pi\xi}{T}\,d\xi\right)$$

$$= \frac{1}{T}\int_a^b f(\xi)\underbrace{\left[\cos\frac{2\pi n\xi}{T} - i\sin\frac{2\pi n\xi}{T}\right]}_{e^{-i\frac{2n\pi\xi}{T}}}d\xi \Rightarrow \boxed{\frac{1}{T}\int_a^b f(\xi)e^{-i\frac{2n\pi\xi}{T}}d\xi}$$