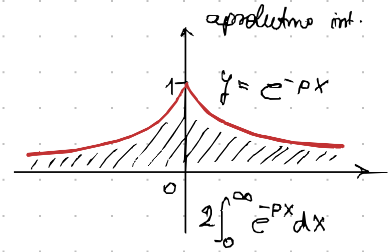
3.1 Fourierov integral

F. red	F. integ.
• periodička funkcija	apsolutno integrabilna fja tj. $\int_{-\infty}^{\infty} \lvert f(x)\rvert dx < \infty$
• a_n, b_n (diskretni spektar), $n \in \mathbb{N}$	• $A(\lambda), B(\lambda)$, $\lambda \in \mathbb{R}$ (kontinuirani spektar)
• $\sum_n$	• $\int \ldots d\lambda$

apsolutno int.

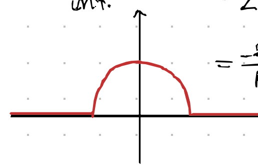


$2\int_0^{\infty} e^{-px} dx$

$= 2 \cdot \frac{e^{-px}}{-p} \Big|_0^{\infty}$

$= \frac{-2}{p}(0 - e^0) = \frac{2}{p}$

apsolutno int.



Formula za formalni prijelaz s konačnog na beskonačni interval

$f: \mathbb{R} \longrightarrow \mathbb{R}$ je po dijelovima glatka. Zadovoljava Dirichletove uvjete na svakom konačnom intervalu. U točkama neprekinutosti $f(x) = S(x)$.

Za svaki $L>$ vrijedi: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$, $-L < x < L$

gdje koef: $a_0 = \frac{1}{L}\int_{-L}^{L} f(\xi)d\xi$ $\quad a_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}d\xi$ $\quad b_n = \frac{1}{L}\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}d\xi$

=> Uvrštavamo:

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(\xi)d\xi + \frac{1}{L}\sum_{n=1}^{\infty}\Bigg[\underbrace{\left(\int_{-L}^{L} f(\xi)\cos\frac{n\pi\xi}{L}d\xi\right)}_{a_n}\cos\frac{n\pi x}{L} + \underbrace{\left(\int_{-L}^{L} f(\xi)\sin\frac{n\pi\xi}{L}d\xi\right)}_{b_n}\sin\frac{n\pi x}{L}\Bigg]$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(\xi)d\xi + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(x)\Big[\underbrace{\cos\frac{n\pi\xi}{L}\cos\frac{n\pi x}{L} + \sin\frac{n\pi\xi}{L}\sin\frac{n\pi x}{L}}\Big]d\xi$$

$$\cos\frac{n\pi}{L}(\xi - x) = \cos\left(\frac{n\pi\xi}{L} - \frac{n\pi x}{L}\right)$$

Nove oznake:

$\lambda_n = \frac{n\pi}{L}$

$\lambda_{n+1} - \lambda_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L}$

$\Downarrow$

$\Delta\lambda_n = \frac{\pi}{L}$

ideja: $L \longrightarrow \infty$

$\lambda_n \longrightarrow \lambda$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(\xi)d\xi + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(\xi)\cos\underbrace{\frac{n\pi}{L}}_{\lambda_n}(\xi - x)d\xi$$

ako je f aps. integrab. $\to 0$ $\quad \frac{\Delta\lambda_n}{\pi}$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(\xi)d\xi + \frac{1}{\pi}\sum_{n=1}^{\infty}\Delta\lambda_n\int_{-L}^{L} f(\xi)\cos\lambda_n(\xi - x)d\xi$$

*apsolutna integ.: $\int_{-\infty}^{\infty}\lvert f(x)\rvert dx < \infty \Longrightarrow \int_{-\infty}^{\infty} f(x)dx < \infty$

Fourierov integral:

$$\boxed{f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(\xi)\cos\lambda(\xi - x)d\xi}$$

$L \to \infty$

TM Fourierov integral

Ako je $f: \mathbb{R} \to \mathbb{R}$ po dijelovima glatka na svakom konačnom intervalu i apsolutno integrabilna, tada postoji njezin Fourierov integral i vrijedi:

$$\frac{1}{\pi}\int_0^{\pi} d\lambda \int_{-\infty}^{\infty} f(\xi)\cos\lambda(x-\xi)\, d\xi = \begin{cases} f(x) & \text{ako je } f \text{ neprekinuta u } x \\ \frac{1}{2}\left[f(x-0)+f(x+0)\right] & \text{ako je } x \text{ točka prekida za } f \end{cases}$$

Sinusni i kosinusni spektar

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^{\infty} dx \int_{-\infty}^{\infty} f(\xi)\cos\lambda(\xi - x)\, d\xi$$

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi)\left[\cos\lambda\xi \cos\lambda x + \sin\lambda\xi \sin\lambda x\right] d\xi$$

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^{\infty}\left(\underbrace{\left(\int_{-\infty}^{\infty} f(\xi)\cos\xi\, d\xi\right)}_{A(\lambda)}\cos\lambda x + \underbrace{\left(\int_{-\infty}^{\infty} f(\xi)\sin\lambda\xi\, d\xi\right)}_{B(\lambda)}\sin\lambda x\right) dx$$

F. red:

$$\tilde{f}(x) = \cancel{\frac{a_0}{2}} + \overset{A(\lambda)}{\underset{B(\lambda)}{\sum_{n=1}^{\infty}}}\left(\underbrace{a_n}_{A(\lambda)}\cos\underbrace{\frac{n\pi x}{L}}_{\lambda x} + \underbrace{b_n}_{B(\lambda)}\sin\underbrace{\frac{n\pi x}{L}}_{\lambda x}\right)$$

Fourierov integral funkcije f.

$$\Rightarrow \tilde{f}(x) = \frac{1}{\pi}\int_0^{\infty}\left(A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x\right) d\lambda$$

$\lambda \mapsto A(\lambda)$

$\lambda \mapsto B(\lambda)$

$$A(\lambda) = \int_{-\infty}^{\infty} f(x)\cos\lambda x\, dx \qquad B(\lambda) = \int_{-\infty}^{\infty} f(x)\sin\lambda x\, dx$$

$A(\lambda)$, $B(\lambda)$ kontinuirani kosinusni i sinusni spektar

$$am = \sqrt{A^2(\lambda) + B^2(\lambda)} \quad \text{amplitudni spektar}$$

* varijablu integracije ξ smo preimenovali u x

$\boxed{f \text{ parna}}$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} A(\lambda)\cos\lambda x\, d\lambda$$

$$A(\lambda) = 2\int_0^{\infty} f(\xi)\cos\lambda\xi\, d\xi$$

$$am(\lambda) = |A(\lambda)|$$

$\boxed{f \text{ neparna}}$

$$f(x) = \frac{1}{\pi}\int_0^{\infty} B(\lambda)\sin\lambda x\, d\lambda$$

$$B(\lambda) = 2\int_0^{\infty} f(\xi)\sin\lambda\xi\, d\xi$$

$$am(\lambda) = |B(\lambda)|$$

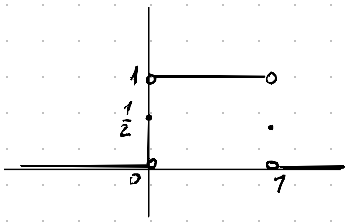
TM | Furierov integral

Neka je $f:\mathbb{R}\to\mathbb{R}$ po dijelovima glatka na svakom beskonačnom intervalu i apsolutno integrabilna. Tada postoji njezin Furierov integral i vrijedi

$$\frac{1}{\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty} f(\xi)\cos\lambda(x-\xi)\,d\xi =$$

$$= \begin{cases} f(x), & \text{ako je } f \text{ neprekinut u } x \\ \frac{1}{2}\big(f(x-0)+f(x+0)\big), & \text{ako } f \text{ ima prekid u } x \end{cases}$$

Primjer: $f(x) = \begin{cases} 1, & 0<x<1 \\ 1/2, & x=0,\ x=1 \\ 0, & \text{inače} \end{cases}$



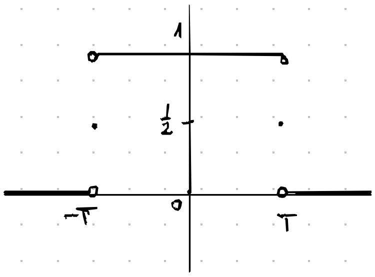
Rj: $\tilde{f}(x) \overset{?}{\doteq} f(x)$

$\tilde{f} = f(x) \quad \forall x\in\mathbb{R}$

$$f(x) = \frac{1}{\pi}\int_0^\infty d\lambda \int_0^1 \cos\lambda(x-\xi)\,d\xi = \frac{1}{\pi}\int_0^\infty d\lambda\, \frac{\sin\lambda(x-\xi)}{-\lambda}\Big|_0^1$$

$$= \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{-\lambda}\big(\sin\lambda(x-1)-\sin(\lambda x)\big) = \frac{1}{\pi}\int_0^\infty \frac{\sin\lambda x - \sin\lambda(x-1)}{\lambda}\,d\lambda = f(x)$$

Primjer: Pomoću F. integrala prikazati i odrediti amplitudni spektar f-je zadane slikom



Rj: $\tilde{f}(x) = f(x), \ \forall x\in\mathbb{R}$

parna $B(\lambda)=0$

$$am(\lambda) = \sqrt{A^2(\lambda)+B^2(\lambda)} = \sqrt{A^2(\lambda)} = |A(\lambda)|$$

$$A(\lambda) = \int_{-\infty}^{\infty} f(\xi)\cos(\lambda\xi)\,d\xi = 2\int_0^\infty f(x)\cos(\lambda x)\,dx$$

$$= 2\int_0^T 1\cdot\cos(\lambda x)\,dx = 2\cdot\frac{\sin\lambda x}{\lambda}\Big|_0^T$$

$$\boxed{= \frac{2\sin(\lambda T)}{\lambda}}$$

$$A(\lambda) = \frac{2\sin\lambda T}{\lambda}, \quad \lambda\neq 0$$

$$\lim_{\lambda\to 0}\frac{2\sin(\lambda T)}{\lambda} = 2\lim_{\lambda\to 0} T\cdot\frac{\sin(\lambda T)}{\lambda T} = 2T$$

$$\boxed{am(\lambda) = \frac{2|\sin(\lambda T)|}{\lambda}, \quad \lambda\neq 0, \quad am(0)=2T}$$

$$f(x) = \int_0^\infty \frac{2\sin\lambda T}{\lambda}\cos\lambda x\,d\lambda$$

$$f(0) = 1 \Rightarrow \frac{1}{\pi}\int_0^\infty \frac{2\sin\lambda T}{\lambda}\,1\cdot d\lambda = f(0) = 1 \ \Big/ \cdot\frac{\pi}{2}$$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin\lambda T}{\lambda}\,d\lambda = \left| \begin{matrix} \lambda T = u \\ T\cdot d\lambda = \end{matrix} \right| = \int_0^\infty \frac{\sin u}{\lambda}\,\frac{du}{T}$$

u

$$\boxed{\frac{\pi}{2} = \int_0^\infty \frac{\sin u}{u}\,du}$$

3.2 Furierova transformacija - definicija

$f: \mathbb{R} \to \mathbb{R}$ po djelovima, aps. integ., Furierov int postoji

$$\lambda \longmapsto \int_{-\infty}^{\infty} f(\xi) \cos\lambda(x-\xi)\, d\xi \quad , \text{ parna po } \lambda$$

$$\lambda \longmapsto \int_{-\infty}^{\infty} f(\xi) \sin\lambda(x-\xi)\, d\xi \quad , \text{ neparna po } \lambda$$

do sada smo promatrali isključivo $\lambda > 0$, ali sada ćemo $\lambda \in \mathbb{R}$

$$f(x) = \frac{1}{\pi} \int_0^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi) \cos\lambda(x-\xi)\, d\xi$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi) \cos(x-\xi)\, d\xi$$ ← parna

+

$$\underbrace{0}_{\text{jer je neparna}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi) \sin(x-\xi)\, d\xi \quad / \cdot i$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi) \underbrace{\left[\cos\lambda(x-\xi) + i \sin\lambda(x-\xi)\right]}_{e^{i\lambda(x-\xi)} = e^{i\lambda x} \cdot e^{-\xi i \lambda}} d\xi$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \boxed{\left(\int_{-\infty}^{\infty} \hat{f}(\xi) e^{-i\lambda\xi}\, d\xi\right)} d\lambda$$

Furierov transformat od f

$$\hat{f}(\lambda) = \mathcal{F}(f(x)) := \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx$$

DEF Furierov transformat

Fija $\hat{f}(x)$ definirana s $\hat{f}(x) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx$

naziva se Furierov transformat funkcije f.

Obrnuta veza dana je formulom $f(x) = \mathcal{F}^{-1}(f(x)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x} \hat{f}(x)\, dx$

Napomena: $\hat{f}(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx = \int_{-\infty}^{\infty} (\cos\lambda x - i \sin\lambda x) f(x)\, dx$

$$= \underbrace{\int_{-\infty}^{\infty} f(x) \cos\lambda x}_{A(\lambda)} - i \underbrace{\int_{-\infty}^{\infty} f(x) \sin\lambda x\, dx}_{B(\lambda)}$$

* f ne mora bit neprekinuta dovoljno da je po dijelovima glatka i apsolutno integrabilna

$\hat{f}(x) = A(\lambda) - i B(\lambda)$

$A(\lambda), B(\lambda)$ kontinuirani spektar F. int.

→ preslikavanje $(f \longmapsto \hat{f})$: F. transformacija $\mathcal{F}\{f(x)\} = \hat{f}(\lambda)$

→ inverzno preslikavanje $(\hat{f} \longmapsto f)$: inverzna F. transform $\mathcal{F}\{\hat{f}(\lambda)\} = f(x)$

PODSJETNIK:

$$\tilde{f}(x) = \frac{1}{\pi}\int_0^\infty (A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x)\,d\lambda$$

F. int: $A(\lambda) = \int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx$

$B(\lambda) = \int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx$

F. transformat $\mathcal{F}(f(x)) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx = \int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx - i\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx$

iz vremenske u frekvencijsku domenu $= A(\lambda) - iB(\lambda)$

amplitudni spektar

$am(\lambda) = |\hat{f}(\lambda)| = \sqrt{A^2(\lambda) + B^2(\lambda)}$

$\hat{f}(\lambda) = |\hat{f}(\lambda)|e^{i\arg\hat{f}(\lambda)}$ $\quad tg\,\varphi(\lambda) = \frac{-B(\lambda)}{A(\lambda)}$ $\quad \varphi(\lambda)$ fazni spektar

TM Furierova transformacija — teorem o postojanju F. transforma.

Ako je $f: \mathbb{R} \to \mathbb{R}$ po dijelovima glatka na svakom konačnom int. i apsolutno integrabilna, tada postoji njezin F. transformat $\mathcal{F}(f(x)) = \hat{f}(\lambda)$ i vrijedi formula inverzije

$$\mathcal{F}^{-1}(\hat{f}(\lambda)) = \begin{cases} f(x), & \text{ako je neprekinuta u } x \\ \frac{f(x-0)+f(x+0)}{2}, & \text{ako je prekinuta u } x \end{cases}$$

→ izvedimo vezu između f-je $\hat{f}$ i f-ja A i B:

$$\hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx = \int_{-\infty}^{\infty} (f(x)\cos\lambda x\,dx - i f(x)\sin\lambda x\,dx)$$

$$= \underbrace{\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx}_{A(\lambda)} - i\underbrace{\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx}_{B(\lambda)}$$

$$\hat{f}(\lambda) = A(\lambda) - i\cdot B(\lambda)$$

→ ako $\hat{f}$ zapišemo u trigonometrijskom obliku:

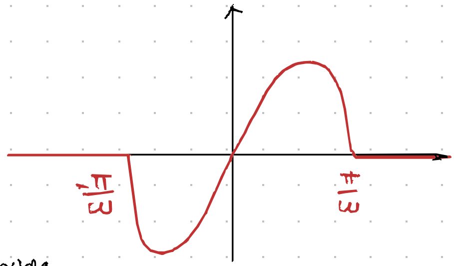
$$\hat{f}(\lambda) = |\hat{f}(\lambda)|e^{i\arg\hat{f}(\lambda)} =: am(\lambda)e^{i\varphi\lambda}$$

$\Rightarrow am(\lambda) = |\hat{f}(\lambda)| = |A(\lambda) - iB(\lambda)| = \sqrt{A^2(\lambda) + B^2(\lambda)}$ amplitudni spekt.

$\varphi(\lambda) = \arg\hat{f}(\lambda) = arc\,tg\frac{-B(\lambda)}{A(\lambda)} = -arc\,tg\frac{B(\lambda)}{A(\lambda)}$ fazni spektar

Primjer: $\quad A, \omega > 0$

$$f(x) = \begin{cases} A \sin \omega x, & |x| \le \frac{\pi}{\omega} \\ 0, & \text{inače} \end{cases}$$



$\mathcal{F}(f(x)) = ?$ traži se F. transformat jednog vala sinusoide

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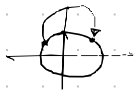
$$\hat{f} = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\, dx$$ $\to$ definicija F transformate

$$\hat{f} = \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} e^{-i\lambda x} \cdot A \sin \omega x \, dx = A \int_{-\pi/\omega}^{\pi/\omega} (\cos \lambda x - i \sin \lambda x) \sin \omega x \, dx$$



$$\hat{f} = A \int_{-\pi/\omega}^{\pi/\omega} \underbrace{\cos \lambda x \cdot \sin \omega x \, dx}_{\substack{\text{neparna f-ja} \\ \| \\ 0}} - i A \int_{-\pi/\omega}^{\pi/\omega} \underbrace{\sin \lambda x \cdot \sin \omega x \, dx}_{\text{parna f-ja}}$$

$$\hat{f}(\lambda) = 2 \cdot (-i) A \int_0^{\frac{\pi}{\omega}} \sin \lambda x \cdot \sin \omega x \, dx$$ $\longrightarrow$ formule: $\frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$

$$\hat{f}(\lambda) = -iA \int_0^{\frac{\pi}{\omega}} \cos x(\lambda - \omega) - \cos x(\lambda + \omega)\, dx$$

$$= -iA \left[\frac{1}{\lambda - \omega} \sin x(\lambda - \omega) - \frac{1}{\lambda + \omega} \sin x(\lambda + \omega) \right] \Bigg|_0^{\frac{\pi}{\omega}}$$

$$= -iA \left[\frac{1}{\lambda - \omega} \sin \frac{\lambda \pi}{\omega} - 0 + \frac{1}{\lambda + \omega} \sin \frac{\lambda \pi}{\omega} + 0 \right]$$

$\sin \frac{\pi}{\omega}(\lambda - \omega)$

$\sin\left(\frac{\lambda\pi}{\omega} - \pi\right)$

$\sin\left(\frac{\lambda\pi}{\omega}\right)$

$\sin\left(\frac{\lambda\pi}{\omega} + \pi\right) = -\sin\left(\frac{\lambda\pi}{\omega}\right)$

$$= -iA \sin \frac{\pi \cdot \lambda}{\omega} \left(\frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} \right) = -iA \sin \frac{\lambda \pi}{\omega} \left(\frac{1}{\omega - \lambda} + \frac{1}{\lambda + \omega} \right)$$

$$\hat{f}(\lambda) = -iA \frac{2\omega}{\lambda^2 - \omega^2} \sin \frac{\pi \lambda}{\omega}$$

$\lambda \neq \pm \omega \to f$ je neparna
↳ ostaje samo imaginarni dio

$\longrightarrow$ Direktno uvrstimo $\lambda = \omega$

$$\hat{f}(\omega) = A \int_{-\pi/\omega}^{\pi/\omega} (\cos \omega x - i \sin \omega x) \sin \omega x \, dx = -2iA \int_0^{\frac{\pi}{\omega}} \sin^2 \omega x \, dx$$

$$\hat{f}(\omega) = -\frac{iA\pi}{\omega}$$

$$\hat{f}(-\omega) = 2iA \int_0^{\frac{\pi}{\omega}} \sin^2 \omega x \, dx \to \hat{f}(-\omega) = \frac{iA\pi}{\omega}$$

$$\to \text{am}(\lambda) = \begin{cases} A \frac{2\omega}{\lambda^2 - \omega^2} \sin \frac{\pi \lambda}{\omega}, & \lambda \neq \pm \omega \\ \frac{A\pi}{\omega}, & \lambda = \pm \omega \end{cases}$$

Budući da je $\hat{f}(\lambda)$ čisto imaginaran broj za svaki λ
↳ $\varphi = \pm \frac{\pi}{2}$ ovisno o $\operatorname{Im} \hat{f}(\lambda)$
(ili $\varphi = 0$ ako $\hat{f}(\lambda)$)

$$\varphi(\lambda) = \arg \hat{f}(\lambda) = \frac{\pi}{2} \operatorname{sgn}(\operatorname{Im} \hat{f}(\lambda))$$

Primjer: (10.)

$$f(x) = \begin{cases} 1, & |x| \leq a \\ 0, & |x| > a \end{cases}$$

$\hat{f}(\lambda) = ?$ + skica $\quad$ * $a > 0$

$$\hat{f}(\lambda) = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx$$

$$\hat{f}(\lambda) = \int_{-a}^{a} e^{-i\lambda x}\,dx = \int_{-a}^{a} \cos\lambda x - i\sin\lambda x\,dx$$

$$\hat{f}(\lambda) = \int_{-a}^{a} \cos\lambda x\,dx - i\int_{-a}^{a} \sin\lambda x\,dx$$

$$\hat{f}(\lambda) = \frac{1}{\lambda}\sin(\lambda x)\Big|_{-a}^{a} + i\,\frac{1}{\lambda}\cos(\lambda x)\Big|_{-a}^{a}$$

$\cos(x) = \cos(-x)$

$$= \frac{1}{\lambda}\Big(\sin a\cdot\lambda - \underbrace{\sin(-a\lambda)}_{-\sin(a\lambda)}\Big) + i\cdot\frac{1}{\lambda}\underbrace{\cancel{\big(\cos a\lambda - \cos(-a\lambda)\big)}}_{=0}$$

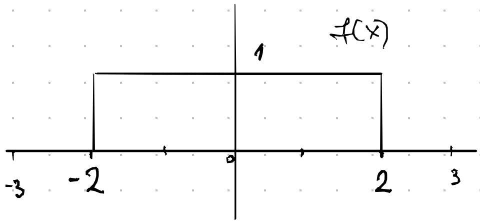
$$= \frac{1}{\lambda}\,2\sin(a\lambda)$$

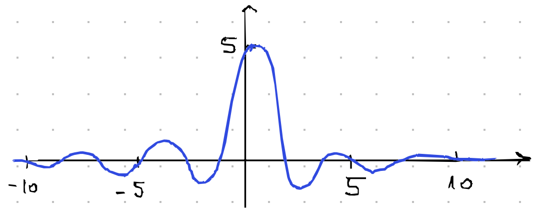
za $\lambda \neq 0$

$\longrightarrow$ napadnemo limesom $\longrightarrow$ $\lim\limits_{\lambda\to 0} \frac{1}{\lambda}\,2\sin a\lambda = 2a = \hat{f}(0)$

$$am(\lambda) = 2\cdot\left|\frac{\sin(a\lambda)}{\lambda}\right| \longrightarrow am(0) = 2$$

$\varphi = 0$ za





Primjer 13.)

$$f(x) = e^{-ax}\,u(x) = \begin{cases} e^{-ax}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$\hat{f} = \mathcal{F}(f(x)) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx$$

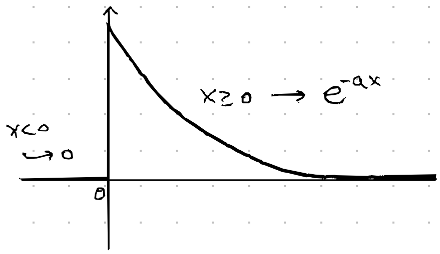
$$\hat{f}(\lambda) = \int_0^{\infty} e^{-i\lambda x}\,e^{-ax}\,dx = \int_0^{\infty} e^{(-i\lambda - a)x}\,dx$$

$$\hat{f}(\lambda) = \frac{-1}{i\lambda + a}\,e^{-(i\lambda + a)x}\Big|_0^{\infty}$$

$$= \frac{-1}{i\lambda + a}\Big(\underbrace{\lim_{x\to\infty}\frac{1}{e^{(i\lambda+a)x}}}_{0} - 1\Big) = \frac{1}{i\lambda + a}$$

$$\lim_{x\to\infty} e^{-cx} = (c = a + ib) = \lim_{x\to\infty} e^{-(ax + ibx)} = \lim_{x\to\infty} |e^{-ax}|\,\big|\underbrace{e^{-ibx}}_{1}\big| = \lim_{x\to\infty}\frac{1}{e^{ax}} = 0$$

0 za $a > 0$



Primjer: F. transformacija gate funkcije

$$g[a,b](x) = \begin{cases} 1, & a \le x \le b \\ 0, & \text{inače} \end{cases} \qquad a<b$$

$$\rightarrow \hat{f}(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x} f(x)\,dx \longrightarrow \int_a^b e^{-i\lambda x}\,dx = \int_a^b \cos\lambda x - i\sin\lambda x\,dx$$

$$\hat{f}(\lambda) = \frac{1}{\lambda}\sin\lambda x\Big|_a^b + i\frac{1}{\lambda}\cos\lambda x\Big|_a^b = \frac{1}{\lambda}\left(\sin bx - \sin ax + i\cos bx + i\cos ax\right)$$

$$\longrightarrow f(\lambda) = \frac{\sin bx - \sin ax}{\lambda} + i\,\frac{\cos bx - \cos ax}{\lambda}$$

$$\Rightarrow \text{mogli smo direktno pisati: } \hat{g}_{[a,b]}(\lambda) = \int_a^b e^{-i\lambda x}\,dx = \frac{e^{-ia\lambda} - e^{-ib\lambda}}{i\lambda}$$