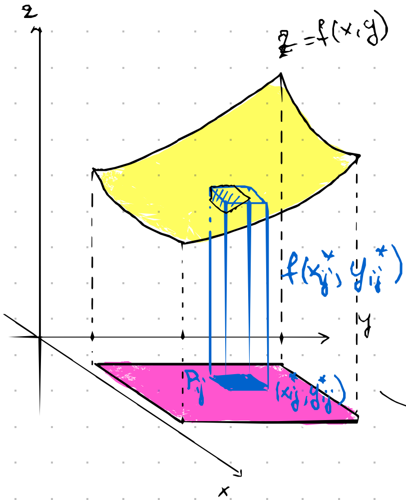
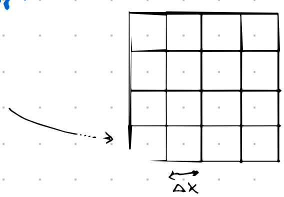
5.1. DVOSTRUKI INTEGRALI



→ Def. dvostrukog integrala na pravokutniku P

→ $V = \iint_P f(x,y)\,dx\,dy = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij})\,\Delta x \Delta y$

- dvostruki integral po pravokutniku P predstavlja volumen tijela iznad P omeđen s gornje strane plohom $z = f(x,y)$



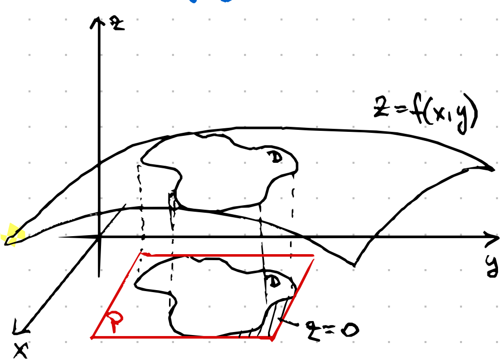
Rj. dvostrukog integrala je uvijek BROJ!

TM Ako je f neprekinuta na pravokutniku P, onda je integrabilna na P.

TM | Fubinijev TM Ako postoji dvostruki integral funkcije f na pravokutniku

$P = [a,b] \times [c,d] \xrightarrow{\text{tada}} \iint_P f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy$

Dvostruki integral na omeđenom skupu D



pomoćna funkcija $F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in P \setminus D \end{cases}$

↳ tamo gdje nije D stavimo 0

⇒ tada dvostruki integral definiramo:

$$\iint_D f(x,y)\,dx\,dy = \iint_P F(x,y)\,dx\,dy$$

→ gledamo projekciju

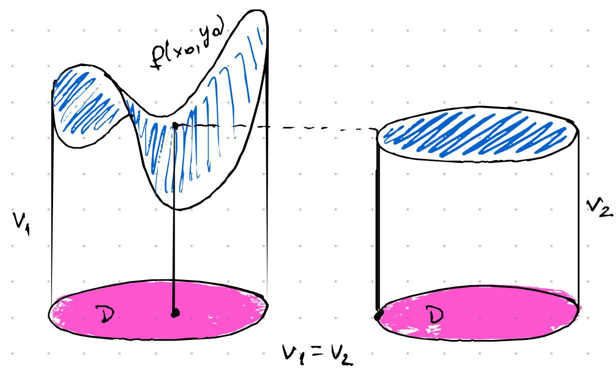
Računanje: $\iint_D f(x,y)\,dx\,dy = \iint_P F(x,y)\,dx\,dy = \int_a^b dx \int_c^d F(x,y)\,dy \Rightarrow \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$

TM | TM srednje vrijednosti za 2 dimenzije

Neka je f neprekinuta na zatvorenom području D. Tada postoji točka $(x_0,y_0)\in D$ takva da je $\iint_D f(x,y)\,dx\,dy = f(x_0,y_0)\cdot\mu(D)$ gdje je $\mu(D)$ površina od D.

↳ Geometrijska interpretacija

- postoji točka $(x_0,y_0)\in D$ za koju je volumen valjka visine $f(x_0,y_0)$ i baze D jednak volumenu plohe $z=f(x,y)$



DOKAZ:

Neka je $m=\min_D f(x,y)$, $M=\max_D f(x,y)$

↳ Tada je $m\underbrace{\iint_D dx\,dy}_{\mu_D} \le \underbrace{\iint_D f(x,y)\,dx\,dy}_{\text{volumen}} \le M\underbrace{\iint_D dx\,dy}_{\mu_D} \quad /:\mu_D$

$$m \le \boxed{\frac{1}{\mu(D)}\iint_D f(x,y)\,dx\,dy} \le M$$

Zbog neprekinutosti funkcije $f(x,y)$ sigurno postoji $(x_0,y_0)\in D$ tako da vrijedi

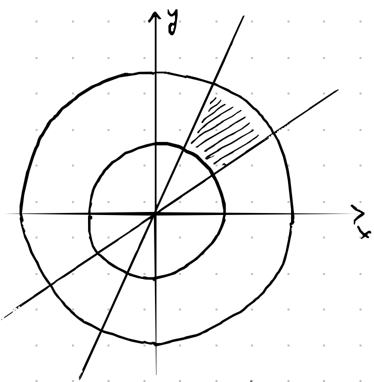
$$f(x_0,y_0)=\frac{1}{\mu(D)}\iint_D f(x,y)\,dx\,dy$$

5.1.4. Zamjena varijabli

$$\iint_D f(x,y)\,dx\,dy = \left| \begin{matrix} (x,y)=\vec{g}(u,v) \\ dx,dy = (??)\,du\,dv \end{matrix} \right| = \iint f(g(u,v))\,\boxed{J}\,du\,dv$$

↳ Jakobijan $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{bmatrix}$

▶ POLARNE KOORDINATE

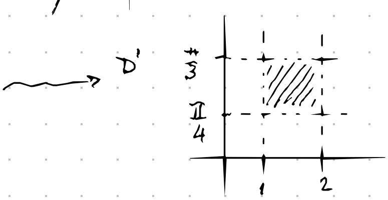


$$\left.\begin{matrix} x = r\cos\varphi \\ y = r\sin\varphi \end{matrix}\right\} \iint_D f(x,y)\,dx\,dy = \iint_D (r\cos\varphi, r\sin\varphi)\,\boxed{r}\,dr\,d\varphi$$

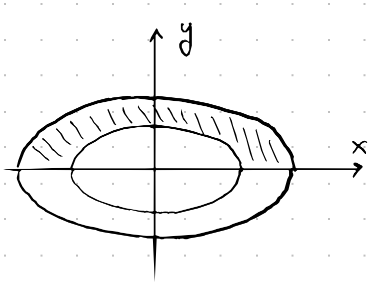
$$J \rightarrow \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \varphi} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

- Jakobijan polarnih je r

prelaskom na polarne koordinate dobivamo □



▶ ELIPTIČKE KOORDINATE



$$\left.\begin{matrix} x = ar\cos\varphi \\ y = br\sin\varphi \end{matrix}\right\} \rightarrow J = abr$$

$$\iint_D f(ar\cos\varphi, br\sin\varphi)\,abr\,dr\,d\varphi$$

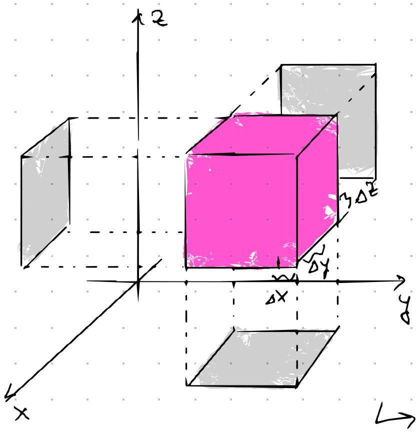
5.2. TROSTRUKI INTEGRALI

$$\underbrace{\iiint}_{\text{tijelo u 3D}} \underbrace{f(x,y,z)\,dx\,dy\,dz}_{\text{graf u 4D}}$$

interpretiramo fizikalno $\Rightarrow$ GUSTOĆA

$\Rightarrow$ masa tijela V s gustoćom $f(x,y,z)$

DEF na kvadru:



$$\iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$= \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{e} f(x_{ijk}, y_{ijk}, z_{ijk})\Delta x \Delta y \Delta z$$

$$\Rightarrow \iiint_V f(x,y,z)\,V$$

$\boxed{TM}$ Fubinijev TM — uzastopno integriranje
— nije bitan redoslijed

↳ integracija po kvadru $V=[a,b]\times[c,d]\times[e,f]$ se svodi na 3 iterirana integrala u bilokojem poretku

Općenito:

$$\underbrace{\int_a^b dx \int_{y_1(x)}^{y_2(x)} dy}_{\text{postavljamo kao za 2D}} \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\,dz$$

konstante

neprekinuta funkcija

neprekinuta ploha

$\boxed{TM}$ TM Srednje vrijednosti

Postoji takva točka da vrijedi $\iiint_V f(x,y,z)\,dV = f(x_0,y_0,z_0)\cdot\mu(V)$

→ ako f predstavlja gustoću tijela V, onda postoji točka tijela u kojoj se gustoća podudara s prosječnom gustoćom

5.22. Zamjena varijabli

$x = x(u,v,w)$

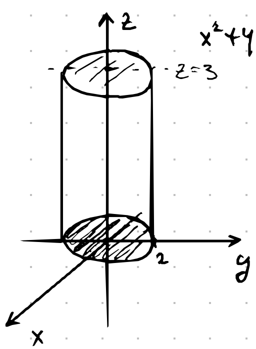
$y = y(u,v,w)$

$z = z(u,v,w)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix} \Rightarrow \iiint_V f(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_V f(u,v,w)\,|J|\,du\,dv\,dw$$

▸ CILINDRIČNE KOORDINATE



$x^2+y^2=4$

$$\int dx \int dy \int_0^3 dz$$

$$J = |\dots| = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$x = r\cos\varphi$

$y = r\sin\varphi$

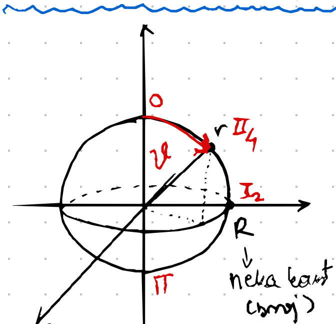
$z = z$

$$J = 1(r\cos^2\varphi + r\sin^2\varphi)$$

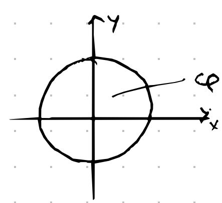
$$J = r$$

$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_V f(r\cos\varphi, r\sin\varphi, z)\,r\,dr\,d\varphi\,dz$$

▸ SFERNE KOORDINATE



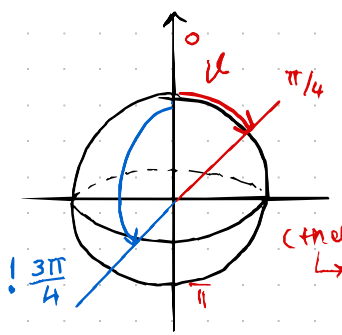
$\varphi \in [0, 2\pi]$



$(x,y,z) \to (\varphi, \vartheta, r)$

$r > 0$, trodimenzionalna udaljenost od ishodišta

$$\boxed{\begin{aligned} x &= r\cos\varphi\sin\vartheta \\ y &= r\sin\varphi\sin\vartheta \\ z &= r\cos\vartheta \end{aligned}}$$



ϑ - kut s pozitivnim dijelom z osi (prema dolje raste)

(theta)
↳ kuda idemo gore i dolje po z

→ ! NE MOŽE BITI VEĆI od π, niti manji od 0