1. CJELINA

Vektorska funkcija i funkcija više varijabli

2. CJELINA

Diferencijalni račun funkcija više varijabli

2.2. PARCIJALNE DERIVACIJE

$$\rightarrow \frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$ konstanta

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \rightarrow$$ Schwarzov TM

2.3 DIFERENCIJABILNOST

TM Ako je $f(x,y)$ diferencijabilna u To (x_0, y_0) tada je neprekinuta u To.

$* \; f(x_0+\Delta x, y_0+\Delta y) - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \vartheta(\Delta x, \Delta y)$ → linearna aproks

$* \; f(x,y)$ je dif. ako postoji parc. deriv u (x_0, y_0) te ako vrijedi

pri čemu: $$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\vartheta(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$

DOKAZ: primjenimo

$$\lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\vartheta(\Delta x, \Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$$ ⟷ mora brže težiti u 0!

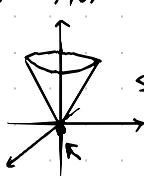
znači sigurno je $\vartheta(\Delta x, \Delta y) = 0$

„napadnemo" definiciju sa lim $(\Delta x, \Delta y) \to (0,0)$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} [f(x_0+\Delta x, y_0+\Delta x) - f(x_0, y_0)] = \frac{\partial f}{\partial x} \cdot 0 + \frac{\partial f}{\partial y} \cdot 0 + 0 = 0$$

$$\lim_{(\Delta x, \Delta y) \to (0,0)} f(x_0+\Delta x, y_0+\Delta x) = f(x_0, y_0)$$ ⟹ ZNAČI, neprekinuta je po definiciji.

OBRAT NE VRIJEDI:



stožac

ne postoji parc. deriv zbog vrha (0,0)

$\frac{\partial f}{\partial x} = \cancel{\frac{0}{0}}$!!

2.4. DIFERENCIJAL I PRIMJENA

prvi dif $\Rightarrow df(T_0) = \frac{\partial f}{\partial x}\Big|_{T_0} dx + \frac{\partial f}{\partial y}\Big|_{T_0} dy \Rightarrow^{*} \nabla f$

↳ za linearnu aproksimaciju!

2.6. DERIV. SLOŽENE FJE

TM Lančano deriviranje

$f(x_1, \dots x_n)$, $\vec{r}(t) = (x_1(t), \dots, x_n(t))$

Tada: $[(f \circ \vec{r})(t)]' = [f(\vec{r}(t))]' = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)$

↳ $\frac{du}{dt} = \frac{\partial u}{\partial x_i} \cdot \frac{\partial x_i}{\partial t}, \dots \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t}$ AKA Jacobijan

2.7 IMPLICITNO DERIVIRANJE

TM: imamo zadanu $f(x,y) \rightarrow z = e^{xz} + xy$

Ako kažemo da je $F(x,y,z) = e^{xz} + xy - z$; $f(x,y) = e^{xz} + xy$,

onda ako je $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0 \Rightarrow f(x,y)$ je jedinstvena implicitno zadana fja

I.) $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0 \longrightarrow$ postoji jed. impl. zad. $y = y(x)$

· zadovoljava $f(x,y) = 0 \rightarrow y' = -\frac{\frac{\partial f}{\partial x}(T_0)}{\frac{\partial f}{\partial y}(T_0)}$

↳ ako je $y(x)$ zadana implicitno s $f(x,y) = 0 \longrightarrow$ tang je $\frac{\partial f}{\partial x}\Big|_{T_0}(x - x_0) + \frac{\partial f}{\partial y}\Big|_{T_0}(y - y_0) = 0$

II.) $f(x,y,z) = 0$, ako je $\frac{\partial f}{\partial z}(T_0) \neq 0 \longrightarrow$ postoji jed. impl. zad. $z = z(x,y)$ u T_0

↳ $\frac{\partial z}{\partial x}(T_0) = -\frac{\frac{\partial f}{\partial x}(T_0)}{\frac{\partial f}{\partial z}(T_0)}$

2.8 USMJERENA DERIVACIJA

TM $\quad \frac{\partial f}{\partial \vec{v}}(\vec{x}_0) = \lim_{t \to 0} \frac{f(\vec{x}_0 + t\cdot\vec{v}_0) - f(\vec{x}_0)}{t}$ $\quad$ gdje je $\vec{v}_0 = \frac{\vec{v}}{\|\vec{v}\|}$

deriv. ne smije ovisiti o duljini vekt. u čijem smjeru ide fja

što također možemo iskazati kao $\frac{\partial f}{\partial \vec{v}}(T_0) = \nabla f(T_0)\cdot\vec{v}_0 = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\cdot v_{0i}$

$* \vec{v}_0 = v_{01}\vec{i} + v_{02}\vec{j}$

izvod za $f(x,y)$ - (paramet. pravac na kojem je v_0)

$p \ldots \left.\begin{aligned} x &= x_0 + t\cdot v_{01} \\ y &= y_0 + t\cdot v_{02} \end{aligned}\right\}$ $\quad z = f(x,y) = f(x_0 + t\cdot v_{01},\ y_0 + t\cdot v_{02})$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{\partial z}{\partial x(t)}\cdot v_{01} + \frac{\partial z}{\partial y(t)}\cdot v_{02}$$

(lančano deriv.)

uvrstimo $t=0 \longrightarrow \frac{\partial z}{\partial t}(T_0) = \frac{\partial f}{\partial \vec{v}}(T_0) = \underline{\underline{\nabla f(T_0)\cdot v}}$

TM a) $\nabla f(T_0) = \vec{0}$

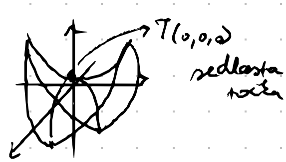
- sve usmjerene derivacije su NULA $\rightarrow$ STOJIMO = STACIONARNA TOČ.

b) $\nabla f(T_0) \neq \vec{0}$

- najbrže raste u smjeru ∇f, $\quad$ + f najbrže pada u smjeru $-\nabla f$

izonos max rasta je $\|\nabla f(T_0)\|$

$\rightarrow$ nije svaka stac. točka ekstrem!



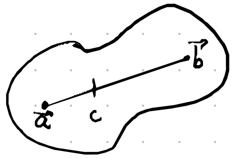
c) $\nabla f \perp$ nivo krivulja

2.9. TEOREM SREDNJE VRIJEDNOSTI

Lagrangeov TM

. $f: U \to \mathbb{R}$ je dif. na $U \subseteq \mathbb{R}^n$

. $\vec{a}, \vec{b} \in U$, tdj. je spojnica $\vec{a}$ i $\vec{b}$ u U.



* skal. prod.

→ Tada na spojnici postoji takav $\vec{c}$ da vrijedi $f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c}) \cdot (\vec{b} - \vec{a})$

!ne dijeliti sa $(\vec{b} - \vec{a})$!

DOKAZ:

vekt. smjera $\quad \vec{a} \quad \vec{b}$

1) Parametriziramo spojnicu $\vec{a}$ i $\vec{b}$: $g(t) = \vec{a} + t(\vec{b} - \vec{a})$ $\quad t \in [0, 1]$

2) parametrizaciju uvrstimo u f

$g(t) = f(\vec{a} + t(\vec{b} - \vec{a}))$ → diferencijab. je jer je f dif, a g linearna

* → dobili smo fiju 1 varijable možemo MATAN 1

$f(\vec{b}) - f(\vec{a}) = g(1) - g(0)$

$g(1) - g(0) = g'(t_c)(1 - 0)$

- g(t) lančano deriviramo → $g'(t) = \nabla f(\vec{a} + t(\vec{b} - \vec{a}))(\vec{b} - \vec{a})$

$f(\vec{b}) - f(\vec{a}) = g(1) - g(0) = g'(t_c)(1-0) = g'(t_c)$

$f(\vec{b}) - f(\vec{a}) = \nabla f(\underbrace{\vec{a} + t_c(\vec{b} - \vec{a})}_{\vec{c}})(\vec{b} - \vec{a})$

$$\Rightarrow \boxed{f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{c})(\vec{b} - \vec{a})}$$

KOROLAR

I.) $\nabla f \equiv \vec{0}$ → fija je konstantna

- za bilokoje dvije točke spojnica je u U → $f(\vec{b}) - f(\vec{a}) = \vec{0}(\vec{b} - \vec{a})$

$f(\vec{b}) - f(\vec{a}) = 0$

$f(\vec{b}) = f(\vec{a})$ ✓

II.) $\nabla f \equiv \nabla g$, f i g se razlikuju za C

$\nabla f \equiv \nabla g \to \nabla f - \nabla g \equiv \vec{0}$

$f - g = C$ ✓

3. CJELINA

Primjena diferencijalnog
računa

3.1. INTEGRALI OVISNI O PARAMETRU

Leibnizovo p. TM Derivacija integrala ovisnog o parametru

imamo $I(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx$ dif i neprekinute!

$$\Longrightarrow I'(\alpha) = \frac{d}{d\alpha}\int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx$$

$$I'(\alpha) = f[\psi(\alpha),\alpha](\psi(\alpha))' + f[\varphi(\alpha),\alpha]\cdot(\varphi(\alpha))' + \int_{\varphi(\alpha)}^{\psi(\alpha)} \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx$$

▶ **Ako granice ne ovise o parametru:**

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial f(x,\alpha)}{\partial\alpha}\,dx$$

*primjenjujemo i ako je jedna od granica u ∞

3.2. TAYLOR

- za dvije varijable

$$\underline{T_n(x,y)} = f(x_0+y_0) + \frac{\partial f}{\partial x}(T_0)(x-x_0) + \frac{\partial f}{\partial y}(T_0)(y-y_0) +$$

$$+\frac{1}{2!}\left(f''_{xx}(x-x_0)^2 + 2f''_{xy}(x-x_0)(y-y_0) + f''_{yy}(y-y_0)^2\right) +$$

$$+ \ldots + \frac{1}{n!}\left(\frac{\partial}{\partial x}(x-x_0) + \frac{\partial}{\partial y}(y-y_0)\right)^n \cdot f(x_0,y_0)$$

$$\underline{R_n(x,y)} = \frac{1}{(n+1)!}\left(\frac{\partial}{\partial x}(x-x_0) + \frac{\partial}{\partial y}(y-y_0)\right)^{(n+1)} \cdot f(T_c)$$

↳ T_c je točka na spojnici $T(x,y)$ i $T(x_0,y_0)$

$$f(x,y) = T_n(x,y) + R_n(x,y)$$

3.3. KVADRATNE FORME

$Q(h,k) = ah^2 + 2bhk + ck^2$, svakoj formi je pridružena sim mat $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$\hookrightarrow Q = A \cdot \begin{bmatrix} h \\ k \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

(d^2f) je kvad forma $\rightarrow \overbrace{f''_{xx}}^{a}(dx)^2 + \overbrace{2f''_{xy}}^{b}(dx)(dy) + \overbrace{f''_{yy}}^{c}(dy)^2 \Rightarrow H_f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix}$

Sylvesterov TM

a) $\det A > 0$, $a > 0 \longrightarrow Q$ poz. def.

b) $\det A > 0$, $a < 0 \longrightarrow Q$ neg. def.

c) $\det A < 0 \longrightarrow Q$ indefinitna

} regularna mat

d) $\det A = 0 \longrightarrow$ nema odluke

3.4. LOKALNI EKSTREMI

Fermateov TM – NUŽAN UVJET

$\nabla f = \vec{0}$, tj $\dfrac{\partial f}{\partial x} = 0$, $\dfrac{\partial f}{\partial y} = 0$

OBRAT ne vrijedi : sedlasta točka

Dovoljan uvjet za lok ext TM

a) $d^2f(T_0) > 0 \longrightarrow$ poz. def & MIN

b) $d^2f(T_0) < 0 \longrightarrow$ neg. def & MAX

c) $d^2f(T_0) \lessgtr \longrightarrow$ indefinitna & SEDLO

Sylvesterov TM dovoljan uvjet za kvad forme

$H_f = \begin{bmatrix} f''_{xx}(T_0) & f''_{xy}(T_0) \\ f''_{xy}(T_0) & f''_{yy}(T_0) \end{bmatrix}$

a) $(f''_{xx})_0 > 0$, $\det H_f(T_0) > 0$ STROGI MIN

b) $(f''_{xx})_0 < 0$, $\det H_f(T_0) > 0$ STROGI MAX

c) $\det H_f(T_0) < 0$ SEDLASTA TOČKA

* ako je $\det H_f = 0$ ovaj teorem je useless

Sylvesterov TM za 3 var

$\begin{bmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{xy} & f''_{yy} & f''_{yz} \\ f''_{xz} & f''_{yz} & f''_{zz} \end{bmatrix} \longrightarrow$

a	mala	velika	
+	+	+	MIN
−	+	−	MAX
	ostalo		SEDLO

3.6. UVJETNI EKSTREMI

NUŽAN UVJET TM $\nabla f(\vec{a}) + \lambda \nabla f(\vec{a}) = \vec{0}$

također poznat kao: $\nabla f(x_0, y_0, z_0) + \lambda \varphi(x_0, y_0, z_0) = \vec{0}$

λ – Lagrangeov multiplikator $L(\vec{x}, \lambda) = f(\vec{x}) + \lambda f(x)$

I.) uvedemo $\lambda \rightarrow L(x, y, \lambda) = f(x,y) + \lambda \varphi(x,y)$

II.) $\frac{\partial L}{\partial x} = 0 \quad \frac{\partial L}{\partial y} = 0 \quad \frac{\partial L}{\partial \lambda} = 0$ * stac točke moraju zadovoljiti $\varphi(x,y) = 0$

III.) Provjeriti definitnost $d^2 L$

$$d^2 f = L''_{xx}(dx)^2 + L''_{yy}(dy) + L''_{\lambda\lambda}(d\lambda)^2$$

$$+ 2L''_{xy}(dx)(dy) + \underbrace{2L''_{x\lambda}(dx)(d\lambda) + 2L''_{y\lambda}(dy)(d\lambda)}_{2(\varphi'_x dx + \varphi'_y dy)d\lambda} \Rightarrow 0$$

*prvi dif

uvjet!

$$\Rightarrow d^2 f = L''_{xx}(dx)^2 + L''_{yy}(dy)^2 + L''_{\lambda\lambda}(d\lambda)^2 + 2L''_{xy}(dx)(dy)$$

& $\varphi'_x(dx) + \varphi'_y(dy) = 0$

4. CJELINA

Diferencijalne jednadžbe prvog reda

Separacija

ako rj. sadrži C = OPĆE RJ.

ako ima konkretno = PARTIKULARNO rješenje

$y' = f(x,y)$, $y(x) = ?$

separacija $\Rightarrow y' = f(x,y) = f_1(x) \cdot f_2(y)$

$\longrightarrow$ supstitucija & direktno integriranje

Linearna $y' + f(x)y = g(x)$

• METODA VARIJACIJE KONSTANTI

I.) separacija homogene dj $(y' + f(x,y)y = 0)$

II.) direktna integracija – dobijemo konstantu C

III.) varijacija konstante $\Rightarrow C \longrightarrow C(x)$ (postaje funkcija od x)

↳ uvrštavanje u početnu $y' + f(x,y)y = g(x)$

IZVOD: $y' + f(x,y)y = g(x)$

I.) $y' + f(x,y)y = 0$

$$dy = -f(x,y)\, y \cdot dx$$

$$\frac{dy}{y} = -f(x,y)\,dx$$

II.) $\displaystyle\int \frac{dy}{y} = -\int f(x,y)\,dx$

$$\ln|y| = -\int f(x,y)\,dx + \overset{\ln|c|}{C} \quad / e$$

$$\underline{y = C e^{-\int f(x,y)dx}}$$

III.) $C \longrightarrow C(x)$

$$y = C(x) \cdot e^{-\int f(x,y)dx}$$

$$\Rightarrow \left[C(x) e^{-\int f(x,y)dx}\right]' + f(x,y) \cdot C \cdot e^{-\int f(x,y)dx} = g(x)$$

$$C'(x)\, e^{-\int f(x,y)dx} + \cancel{C(x) \cdot e^{-\int f(x,y)dx}(-f(x,y)) + f(x,y) \cdot C \cdot e^{-\int f(x,y)dx}} = g(x)$$

$$C'(x) \cdot \underline{e^{-\int f(x,y)dx}} = g(x)$$

$$C'(x) = g(x) \cdot e^{\int f(x,y)dx} \quad / dx$$

$$C(x) = \int g(x) \cdot e^{\int f(x,y)dx} + C$$

$\Rightarrow$ OPĆE RJEŠENJE LDJ

$$\boxed{y = \left[\int \left(g(x) e^{\int f(x,y)dx}\right) + C\right] \cdot e^{-\int f(x,y)dx}}$$

Bernoullijeva $y' + f(x,y)\,y = g(x)\cdot y^{\alpha}$ $\alpha \in \mathbb{R} \setminus \{0,1\}$

- supstitucija : $z = y^{1-\alpha}$, $z' = (1-\alpha)\,y^{-\alpha}$ } uvrstimo i dobijemo LDJ za z

Homogena $f(tx,ty) = t^{\alpha} f(x,y)$

- lijeva i desna strana moraju biti istog stupnja
- prikažemo u obliku $P(x,y)\,dx + Q(x,y)\,dy = 0$
- supstitucija $\Rightarrow$ $z = \frac{y}{x}$, $y' = z'\cdot x + z$

↳ separacija

— transformacija homogene: $y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right)$

$\Rightarrow$ uvedemo supstituciju : $u = x - x_0$, $v = y - y_0$

Egzaktne $P(x,y)\,dx + Q(x,y)\,dy = 0$

↳ ako je $du(x,y) = P(x,y)\,dx + Q(x,y)\,dy$

$\longrightarrow$ $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = Q$

- **NUŽAN UVJET TM** $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ $\Rightarrow$ Schwarzov TM

- **dovoljan uvjet TM** $\quad u(x,y)=\int_{x_0}^{x}P(x,y)\,dx+\int_{y_0}^{y}Q(x_0,y)\,dy+C$

DOKAZ:

$$\frac{\partial u}{\partial x}=P \quad \Big/ \int_{x_0}^{x} dx$$

$$u(x,y)=\int_{y_0}^{x}P(x,y)\,dx+c(y) \quad \Big/ \frac{\partial}{\partial y}$$

prema nužnom uvjetu $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

$$\frac{\partial u}{\partial y}=Q(x,y)\rightarrow Q(x,y)=\int_{x_0}^{x}\frac{\partial P(x,y)}{\partial y}+c'(y)$$

$$Q(x,y)=\int_{x_0}^{x}\frac{\partial Q(x,y)}{\partial x}+c'(y)$$

$$\cancel{Q(x,y)}=\cancel{Q(x,y)}-Q(x_0,y)+c'(y)\implies c'(y)=Q(x_0,y)\Big/\int_{y_0}^{y}dy$$

$$c(y)=\int_{y_0}^{y}Q(x_0,y)\,dy$$

$$\boxed{\implies u(x,y)=\int_{x_0}^{x}P(x,y)\,dx+\int_{y_0}^{y}Q(x_0,y)\,dy}$$

Egzaktna DJ - Eulerov multiplikator npr. $\mu(x)$

$$P(x,y)\,dx+Q(x,y)\,dy=0 \quad \Big/\cdot\mu(x)$$

$$\underbrace{\mu(x)\cdot P(x,y)}_{P}\,dx+\underbrace{\mu(x)\cdot Q(x,y)}_{Q}\,dy=0$$

Znamo da $\frac{\partial \mu P}{\partial y}=\frac{\partial \mu Q}{\partial x}$

$$\underbrace{\cancel{\mu_y'(x)\cdot P}}_{0}+\mu(x)\cdot P_y'=\mu_x'(x)\cdot Q+\mu(x)\cdot Q_x'$$

$$\mu_x'(x)\,Q=\mu(x)\left(P_y'-Q_x'\right)$$

$$\frac{d\mu}{dx}=\mu(x)\,\frac{P_y'-Q_x'}{Q}\quad\Big/\cdot\frac{dx}{\mu(x)}$$

$$\frac{d\mu}{\mu(x)}=\left(\frac{P_y'-Q_x'}{Q}\right)\cdot dx\quad\Big/\int$$

$$\ln|\mu(x)|=\int\frac{P_y'-Q_x'}{Q}\,dx$$

$\rightsquigarrow$ često se traži izvod po $\mu(y)$!

4.7. EGZISTENCIJA I JEDINSTVENOST RJ.

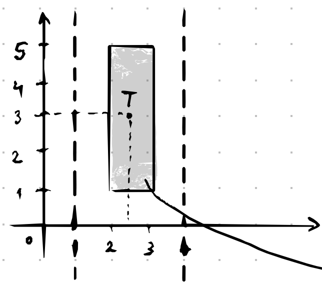
Peanov TM o lok. egzistenciji

Neka $f: \mathbb{R} \to \mathbb{R}^2$ je neprekinuta na pravokutniku oko točke (x_0, y_0) (središte pravokut.)

$$D = \{(x,y) \in \mathbb{R}^2 : |x - x_0| < a,\ |y - y_0| < b\}$$

Tada postoji $\langle x_0 - h, x_0 + h \rangle$ na kojem CP ima rješenje.

Primjer:



$$y' = \frac{xy^2 + 4}{(x-1)(x-4)}$$

$$y\left(\frac{5}{2}\right) = 3 \quad \longrightarrow T(x_0, y_0) = \left(\frac{5}{2}, 3\right)$$

bilokoji pravokutnik unutar uzeta sa središtem u (x_0, y_0) je rješenje

Picardov TM o lok. jedinstvenosti

Neka je $f: \mathbb{R} \to \mathbb{R}^2$ definirana na pravokutniku $D = \{(x,y) \in \mathbb{R}^2 : |x - x_0| < a,\ |y - y_0| < b\}$ *Pean

te neka ima svojstva:

- f je neprekinuta na D
- $\frac{\partial f}{\partial y}$ je omeđena na D

$\Rightarrow$ Tada postoji interval $\langle x_0 - h, x_0 + h \rangle$ na kojem CP ima jedinstveno rj.