DEFINICIJA DETERMINANTA

Što je determinanta?

→ determinanta : skalar pridružen nekoj kv. matrici

Oznaka : $\det(A)$

▶ Motivacija : veza sa linearnim sustavima
↳ pojam koji se javio pri rj. lin.

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned} \xrightarrow[\text{koeficijenata}]{\text{metoda suprotnih}}$$

* prvu jedn. pomn. sa d, a drugu s $-b$ i zbrojimo ih.

$$(ad-bc)x = ed - bf \implies \frac{ed-bf}{ad-bc} = x$$

$$(ad-bc)y = af - ce \implies \frac{af-ce}{ad-bc} = y$$

primijetimo da su brojnik i nazivnik determinante

$$\Downarrow$$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Determinanta mat. trećg reda:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

* caka: gledamo po redu/stupcu i onda eliminiramo po sjecištu

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \underbrace{a\begin{vmatrix} e & f \\ h & i \end{vmatrix}} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

gledamo koji su elementi preostali i prepišemo ih

→ račun se nastavlja kao da se računaju mat. nižeg reda (2. reda)

$$= a_{11}(a_{22}\cdot a_{33} - a_{23}\cdot a_{32}) - a_{12}(a_{21}a_{33} - a_{23}\cdot a_{31}) + a_{13}(a_{21}\cdot a_{32} - a_{22}\cdot a_{31})$$

Induktivno s obzirom na dimenziju matrice

- $n=1$: $A=[a_{11}]$, $\det A = a_{11}$
- $n=2$: $A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} := a_{11}a_{22}-a_{21}a_{12}$
- $n=3$:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} := a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$+ a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) +$$
$$+ a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

Primjer 1.)

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = -2$$

$$\begin{vmatrix} 1 & 1 \\ 3 & 3 \end{vmatrix} = 1\cdot 3 - 1\cdot 3 = 0$$

Primjer 1.)

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 5 \\ -1 & 2 & -3 \end{vmatrix} = 2\cdot\begin{vmatrix} 1 & 5 \\ 2 & -3 \end{vmatrix} - (-1)\begin{vmatrix} 3 & 5 \\ -1 & -3 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= 2\cdot(-3-10) + 1\,(\overset{-9}{3\cdot(-3)} + 5) + 3\,(6+1)$$

$$= -26 - 4 + 21 = \boxed{-9}$$

Laplaceov razvoj determinante

Minore: M_{ij} minora elementa a_{ij}

↳ determinanta matrice koja se dobije brisanjem i-tog retka i j-tog stupca

$$M_{11} := \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{12} := \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \qquad M_{13} := \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Algebarski komplement: A_{ij} elementa a_{ij}

$$A_{ij} := (-1)^{i+j} M_{ij}$$

$$A_{11} = + M_{11} \qquad A_{12} = - M_{12} \qquad A_{13} = + M_{13}$$

Formula za razvoj determinante mat. tipa 3x3 po 1. retku

$$\det A = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Primjer 2.) Razvoj po 2. retku

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 5 \\ -1 & 2 & -3 \end{vmatrix} = 3 \cdot A_{21} + 1 \cdot A_{22} + 5 \cdot A_{23}$$

$$= 3 \cdot (-1)^{2+1} \cdot M_{21} + 1 \cdot (-1)^{2+2} \cdot M_{22} + 5 \cdot (-1)^{2+3} \cdot M_{23}$$

$$= -3 \begin{vmatrix} -1 & 3 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ -1 & -3 \end{vmatrix} - 5 \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix}$$

$$= -3(-3-6) + (-6+3) - 5(4-1)$$

$$= 9 - 3 - 15 = \underline{\underline{-9}}$$

! nije bitno po kojem retku (ili stupcu)

Laplaceov razvoj det matrice

- razvoj po i-tom retku

$$\det A = \sum^{n}$$

Primjer 3)

i)
$$\begin{vmatrix} a & 3 & 0 & 5 \\ 0 & b & 0 & a \\ 1 & 2 & c & 3 \\ 0 & 0 & 0 & d \end{vmatrix} = d\cdot(-1)^{4\cdot 4}\begin{vmatrix} a & 3 & 0 \\ 0 & b & 0 \\ 1 & 2 & c \end{vmatrix}$$

$$= d\cdot 1\cdot\ c\cdot(-1)^{\overset{+1}{3+3}}\begin{vmatrix} a & 3 \\ 0 & b \end{vmatrix} = -dc\begin{vmatrix} a & 3 \\ 0 & b \end{vmatrix}$$

$$= dc\,(ab - 0\,3) = \boxed{dcab}$$

ii)
$$\begin{vmatrix} 1 & 0 & -1 & 2 \\ -1 & 0 & 0 & 3 \\ 1 & 1 & 0 & -2 \\ 1 & 2 & 0 & -1 \end{vmatrix} = \underbrace{-1\cdot\underbrace{(-1)^{1+3}}_{1}}_{-1}\begin{vmatrix} -1 & 0 & 3 \\ 1 & 1 & -2 \\ 1 & 2 & -1 \end{vmatrix}$$

$$= -1\left(-1\underbrace{(-1)^{1+1}}_{1}\begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + 3\underbrace{(-1)^{1+3}}_{1}\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}\right)$$

SVOJSTVA DETERMINANTA

(1.) Ako matrica A ima redak (stupac) sastavljen od samih nula, onda je $\det A = 0$.

Dokaz:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} \underset{\substack{\| \\ 0}}{a_{ij}} M_{ij} = 0$$

(2.) Determinanta trokutaste matrice jednaka je umnošku elemenata na dijagonali.

BSO (bez smanjenja općenitosti)

Dokaz:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & \\ & & \cdots & a_{44} \end{vmatrix} = a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ 0 & a_{33} & a_{34} \\ 0 & 0 & a_{44} \end{vmatrix}$$

$$= a_{11} \left(a_{22} \cdot (-1)^{2+2} \begin{vmatrix} a_{33} & a_{34} \\ 0 & a_{44} \end{vmatrix} \right) = a_{11}\, a_{22} \cdot (a_{33} \cdot a_{44} + 0)$$

$$\boxed{= a_{11}\, a_{22}\, a_{33}\, a_{44}}$$

3. Ako matrica A ima 2 jednaka retka (stupca) onda je $\det A = 0$.

MAT. IND. BSO dokazujemo samo retke

i.) baza $n=2$

$$A = \begin{bmatrix} a & b \\ a & b \end{bmatrix}, \det = a \cdot b - a \cdot b = 0 \quad \checkmark$$

ii. pretpostavka Tvrdnja vrijedi za sve det. reda n.

iii. korak Neka je $A \in M_{mn}$ s dva jednaka retka $(i \neq 1, j \neq 1)$ razvijena $\det A$ po bilo kojem od preostalih redak $(k \neq 1, k \neq i, j)$.

$$\det A = \sum_{l=1}^{n-1} (-1)^{k+l} a_{kl} \underset{\overset{\shortparallel}{0}}{M_{kl}} = 0 \quad \checkmark$$

Determinanta mat. reda n čija su dva retka ista.

4. Transponiranjem matrice vrijednost det. se ne mijenja.

$$\det A = \det A^T$$

5. Det. se množi skalarom tako da se jedan (bilo koji) njezin redak (stupac) množi skalarom.

Dokaz:

$$\lambda \det A = \lambda \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (\lambda a_{ij}) A_{ij} = \det \tilde{A}$$

$\tilde{A}$ je mat. dobivena iz mat. A t.d.j. i-ti redak mat. A pomnožen s λ.

6. Rastave li se svi elementi nekog retka (stupca) mat. na zbroj dvaju elemenata, onda je det. jednaka zbroju dvaju odg. determinanti.

$\vec{a}_i$ oznaka za i-ti redak matrice

$$\underbrace{\begin{vmatrix} \vec{a}_1' + \vec{a}_1'' \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix}}_{A} \overset{?}{=} \underbrace{\begin{vmatrix} \vec{a}_1' \\ a_2 \\ \vdots \\ \vec{a}_n \end{vmatrix}}_{A'} + \underbrace{\begin{vmatrix} \vec{a}_1'' \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix}}_{A''}$$

$$\det A = \sum_{j=1}^{n} (a_{ij}' + a_{ij}'') A_{ij}$$

$$= \sum_{j=1}^{n} a_{ij}' \cdot A_{ij} + \sum_{j=1}^{n} a_{ij}'' \cdot A_{ij}$$

$$\boxed{= \det A' + \det A''}$$

7. → za njega nam trebaju svojstva 3 i 6.

7. Ako zamijenimo dva retka ~~retka~~ matrice, tada det. mijenja predznak

Dokaz:

$$A = \begin{bmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{bmatrix} \qquad 0 \overset{\text{sv.3}}{=} \begin{array}{c} \\ \\ i \\ j \\ \\ \\ \end{array}\begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vec{a}_i + \vec{a}_j \\ \vec{a}_i + \vec{a}_j \\ \vdots \\ \vec{a}_n \end{vmatrix} \overset{\text{sv.6}}{=} \begin{vmatrix} \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_i + \vec{a}_j \end{vmatrix} + \begin{vmatrix} \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_i + \vec{a}_j \end{vmatrix}$$

ako det ima dva jednaka retka, ona je jednaka nuli

$$\overset{\text{sv.6}}{=} \underbrace{\begin{vmatrix} \vec{a}_i \\ \vdots \\ \vec{a}_i \end{vmatrix}}_{\overset{\text{sv.3}}{=} 0} + \begin{vmatrix} \vec{a}_i \\ \vdots \\ \vec{a}_j \end{vmatrix} + \underbrace{\begin{vmatrix} \vec{a}_i \\ \vdots \\ \vec{a}_i \end{vmatrix}}_{= 0} + \begin{vmatrix} \vec{a}_i \\ \vdots \\ \vec{a}_j \end{vmatrix} \Rightarrow \begin{vmatrix} \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_i \end{vmatrix} = - \begin{vmatrix} \vec{a}_i \\ \vdots \\ \vec{a}_j \end{vmatrix} = \det A$$

8. Ako nekom retku (stupcu) matrice dodamo neki drugi redak (stupac) pomnožen skalarom, vrijednost det se neće promijeniti.

Nap: računanje det. pomoću Laplaceovog razvoja za mat. reda $n \geq 4$ je mukotrpan posao. Zato za rač. mat. višeg reda koristimo navedena svojstva.

Dokaz: BSO, pomnožimo 1 redak s λ i dodajmo 2. retku

$$\begin{vmatrix} \vec{a}_1 \\ \lambda\vec{a}_1+\vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} \underset{\text{sv. 6.}}{=} \begin{vmatrix} \vec{a}_1 \\ \lambda\vec{a}_1 \\ \vdots \\ \vec{a}_n \end{vmatrix} + \begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} \underset{\text{sv. 5}}{=} \lambda \begin{vmatrix} \vec{a}_1 \\ \vec{a}_1 \\ \vdots \\ \vec{a}_n \end{vmatrix} + \begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} = \det(A)$$

Zad. 1.)

svojstvo 8

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 2 & 2 \\ 1 & 2 & -2 & 2 & 2 \\ 1 & 2 & 2 & -2 & 2 \\ 1 & 2 & 2 & 2 & -2 \end{vmatrix} \cdot(-1) = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 1 & -3 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix}$$

svojstvo 8

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix}$$

Zad. 2.) Odredite sve $x \in \mathbb{R}$ t.d. $\det(A) = 0$, ako je

$$A = \begin{vmatrix} 1-x & 2 & 3 & 4 & 5 \\ 1 & 2-x & 3 & 4 & 5 \\ 1 & 2 & 3-x & 4 & 5 \\ 1 & 2 & 3 & 4-x & 5 \\ 1 & 2 & 3 & 4 & 5-x \end{vmatrix} \cdot(-1) = \begin{vmatrix} -x & 0 & 0 & 0 & x \\ 0 & -x & 0 & 0 & x \\ 0 & 0 & -x & 0 & x \\ 0 & 0 & 0 & -x & x \\ 1 & 2 & 3 & 4 & 5-x \end{vmatrix}$$

* ideja dobit što više nula

$$= \begin{vmatrix} -x & 0 & 0 & 0 & 0 \\ 0 & -x & 0 & 0 & 0 \\ 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & -x & 0 \\ 1 & 2 & 3 & 4 & 15-x \end{vmatrix} = (15-x)(-1)^{5+5} \begin{vmatrix} -x & 0 & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & -x \end{vmatrix} = (15-x)(-x)^4 = x^4(15-x)$$

$$= 0 \Rightarrow \boxed{\begin{matrix} x_1 = 0 \\ x_2 = 15 \end{matrix}}$$

Zad. 2)

DZ

$$\begin{vmatrix} x & 1 & 0 & 0 \\ 3 & x & 2 & 0 \\ 0 & 2 & x & 3 \\ 0 & 0 & 1 & x \end{vmatrix} =$$

KORIŠTENJE REKURZIJSKIH FORMULA

Pr.)

A)

$$\Delta_n = \begin{vmatrix} 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} = 1\cdot(-1)^2 \begin{vmatrix} 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix} + 1\cdot(-1)^3 \underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & \cdots & 1 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \end{vmatrix}}_{\Delta_{n-1}}$$

Δ_{n-1} — determinanta nižeg reda od Δ_n

$$\Rightarrow \Delta_n = 1\cdot 1 + (-1)\cdot \Delta_{n-1} = 1 - \Delta_{n-1}$$

* najmanja det. koja ima smisla je 3 (pomak para 11)

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$$

Δ_{2n+1} – neparni indeksi

Δ_{2n} – parni indeksi

→ Šta i zašto?

nuli dokaz mctri $\det(\tilde{A}) = \lambda \det(A)$

BAZA: λ puta cijeli redak

Pretpostavka: da radi na ostalim redovima