4.2.1. Gradijent skalarnog polja

$\varphi(x,y,z)$ – derivabilno skalarno polje

→ gradijent od skalarnog polja je VEKTORSKO POLJE

$$\vec{\nabla}\varphi = \frac{\partial \varphi}{\partial x}\vec{i} + \frac{\partial \varphi}{\partial y}\vec{j} + \frac{\partial \varphi}{\partial z}\vec{k} \rightarrow \vec{\nabla}\varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right)$$

njegova vrijednost u točki $T(x,y,z)$ je vektor $\nabla\varphi(T)$

Primjer: $\vec{\nabla}f = ?$, $f(x,y,z) = \ln\frac{yz}{x} + 2$

Odredi T tako da je $\vec{\nabla}f(T) = (2,-1,3)$

$$\frac{\partial f}{\partial x} = \frac{\cancel{x}}{\cancel{y}z}\cdot \cancel{y}z \cdot \left(\frac{-1}{x^{\cancel{2}}}\right) = \frac{-1}{x} \qquad \frac{\partial f}{\partial z} = \frac{\cancel{x}}{\cancel{y}z}\cdot\frac{\cancel{y}}{\cancel{x}} = \frac{1}{z}$$

$$\frac{\partial f}{\partial y} = \frac{\cancel{x}}{y\cancel{z}}\cdot\frac{\cancel{z}}{\cancel{x}} = \frac{1}{y} \qquad \vec{\nabla}f = \left(\frac{-1}{x}, \frac{1}{y}, \frac{1}{z}\right)$$

$$T = ? \rightarrow \left.\begin{matrix}\frac{-1}{x} = 2 & \frac{1}{y} = -1 & \frac{1}{z} = 3\end{matrix}\right\} \boxed{T\left(\frac{-1}{2}, -1, \frac{1}{3}\right)}$$

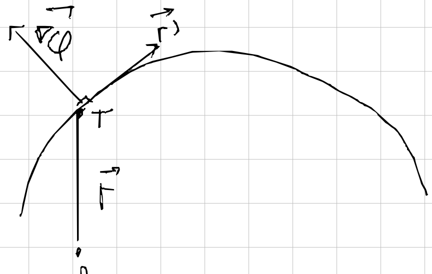
→ gradijent skalarnog polja ima svojstvo da je $\perp$ na nivo krivulje / plohe zadanog skalarnog polja

$$\vec{\nabla}f(T) = \left(\left.\frac{\partial \varphi}{\partial x}\right|_T, \left.\frac{\partial \varphi}{\partial y}\right|_T\right)$$

$\varphi(x,y) = C$ (konstanta) → zadana nivo krivulja funkcije $\varphi(x,y)$

↳ totalni diferencijal: $d\varphi = \frac{\partial \varphi}{\partial x}dx + \frac{\partial \varphi}{\partial y}dy = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}\right)(dx, dy)$

⟹ $d\varphi = \vec{\nabla}f \cdot \vec{r} = 0$ (okomitost) $\quad * r = x\vec{i} + y\vec{j}$

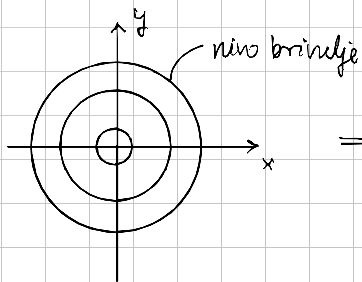


Primjer 2.2 Odredi pripadne nivo krivulje i grad φ

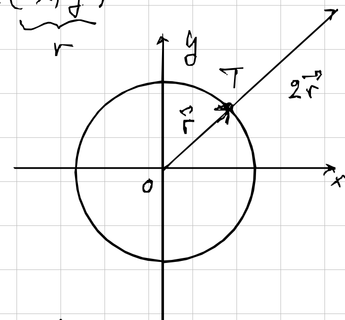
$\varphi(x,y) = x^2 + y^2$ $\qquad \varphi(x,y) = x^2 + y^2 = C \qquad C > 0$

$x^2 + y^2 = (\sqrt{C})^2 \rightarrow \nabla\varphi = (2x, 2y) = 2\underbrace{(x,y)}_{r}$

$\vec{\nabla}\varphi = 2 \cdot \vec{r}$



$\Longrightarrow$



Primjer 2.3

$\varphi(x,y,z) = x^2 + y^2 + z^2$

Pripadne nivo plohe?

$\nabla\varphi = ?$

$C = x^2 + y^2 + z^2$ $\qquad \vec{\nabla\varphi} = (2x, 2y, 2z) = 2 \cdot \vec{r}$

* nivo plohe $\qquad$ * $\nabla\varphi$

Primjer 2.4) Jedinični vektor smjera normale na plohu $z^2 = 4(x^2+y^2)$ u točki $T(1,0,2)$

$4(x^2+y^2) - z^2 = 0$

$\nabla\varphi = (8x, 8y, -2z) \qquad \vec{n} = \nabla\varphi(T) \rightarrow \vec{n} = (8, 0, -4)$

jedinični vektor : $\hat{\vec{n}} = \dfrac{\vec{n}}{|\vec{n}|} = \dfrac{(8, 0, -4)}{\sqrt{64+16}} = \dfrac{(8,0,-4)}{4\sqrt{5}} = \boxed{\left(\dfrac{2}{\sqrt{5}}, 0, \dfrac{-1}{\sqrt{5}}\right)}$

<u>Pravila koja vrijede za gradijent</u>

I) $\nabla(C\varphi) = C\nabla\varphi$

II) $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$

III) $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$

Primjer: $\vec{\nabla} r$ i $\nabla f(r) = ?$

r - radijvektor $\quad r^2 = x^2+y^2+z^2 \longrightarrow r = \sqrt{x^2+y^2+z^2}$

$$\vec{\nabla} r = \left(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}\right) \qquad \frac{\partial r}{\partial x} \; \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}$$

$$\vec{\nabla} r = \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k} = \frac{1}{r}(\vec{r}) \qquad \boxed{\vec{\nabla} r = \hat{\vec{r_0}}}$$

$$\nabla f(r) = \frac{\partial f}{\partial x}(r)\vec{i} + \frac{\partial f}{\partial y}(r)\vec{j} + \frac{\partial f}{\partial z}(r)\vec{k} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} i + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} j + \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial z} k$$

$$\boxed{\nabla f(r) = f'(r)\cdot\vec{\nabla} r = f'(r)\cdot\hat{\vec{r_0}}}$$

Primjer: Nađite vektor u čijem smjeru iz točke $T(1,1)$ funkcija $f(r) = r^2 \sin r$, $r = \sqrt{x^2+y^2}$, najbrže raste

$$\nabla f(r) = f'(r)\cdot\vec{\nabla} r = (2r\sin r + r^2\cdot\cos r)\cdot\hat{\vec{r_0}}$$

$$r(T) = \sqrt{2} \rightarrow \hat{\vec{r_0}} = \frac{\vec{r}}{|\vec{r}|} = \frac{(1,1)}{\sqrt{2}} \Rightarrow r_0 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\rightarrow \nabla f(\sqrt{2}) = (2\sqrt{2}\cdot\sin\sqrt{2} + 2\cdot\cos\sqrt{2})\cdot\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

$$\boxed{\nabla f(\sqrt{2}) = S = (2\sin\sqrt{2} + \sqrt{2}\cos\sqrt{2})(i+j)}$$

Primjer: Električno polje točkastog naboja

el. polje $\quad \vec{E}(x,y,z) = \frac{q}{4\pi\varepsilon_0}\;\frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{(x^2+y^2+z^2)^3}} \longrightarrow \vec{E}(r) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{\hat{\vec{r_0}}}{r^2}$

elektrostatski potencijal: $\quad U(x,y,z) = \frac{q}{4\pi\varepsilon_0}\;\frac{1}{\sqrt{x^2+y^2+z^2}} \Rightarrow \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{r}$

$\rightarrow$ veza između $E(r)$ i $U(r)$ $\qquad \boxed{\vec{E} = -\nabla U}$

DEF Vektorsko polje v je <u>potencijalno ili konzervativno</u> ako postoji <u>skalarno polje p</u> takvo da vrijedi $\vec{v} = \nabla p$. Skalarno polje p naziva se potencijal polja $\vec{v}$.

4.2.2. Usmjerene derivacije

Skalarno polje $f(x,y,z)$, u smjeru zadanog vektora $\vec{s}$ u $T(x_0,y_0,z_0)$

$$\frac{\partial f}{\partial \vec{s}}(x_0,y_0,z_0) = \lim_{h\to 0} \frac{f(x_0+hs_1,\ y_0+hs_2,\ z_0+hs_3)-f(x_0,y_0,z_0)}{h}$$

$* \ \vec{s} = s_1 i + s_2 j + s_3 k$

→ derivacije skalarne funkcije $f(x,y,z)$ u smjeru i, j, k su upravo njene parcijalne derivacije $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$z=0 \longrightarrow$ pravac p, $\vec{s}$

$$t \longmapsto (x_0,y_0) + t(s_1, s_2) = (x_0+ts_1,\ y_0+ts_2)$$

def. fja $F:\mathbb{R}\to\mathbb{R}$; $F(t) = f(x_0+ts_1,\ y_0+ts_2)$

$* \ F(0) = f(x_0, y_0)$ ⟶ izrazimo usmjerenu derivaciju skalarnog polja $f(x,y)$ pomoću funkcije $F(t)$

$$\frac{\partial f}{\partial \vec{s}}(x_0,y_0) = \lim_{h\to 0}\frac{F(h)-F(0)}{h} = \frac{dF}{dt}(0) = \frac{\partial f(x_0,y_0)}{\partial x}s_1 + \frac{\partial f(x_0,y_0)}{\partial y}s_2$$

$$\boxed{\frac{\partial f}{\partial \vec{s}}(x_0,y_0) = \vec{\nabla} f(x_0,y_0)\cdot\vec{s}\ ,\ \text{ili}\ \ \frac{\partial f}{\partial \vec{s}}(x_0,y_0,z_0) = \vec{\nabla} f(x_0,y_0,z_0)\vec{s}}$$

Funkcija se najbrže mijenja u smjeru svog gradijenta

→ ukoliko je $\|s\|=1$

$$\frac{\partial f}{\partial \vec{s}} = (x_0,y_0,z_0) = \nabla f(x_0,y_0,z_0)\cdot\vec{s} \Rightarrow \|\nabla f|_{T_0}\|\cdot\cos\varphi$$

kut između $\vec{\nabla} f$ i $\vec{s}$

→ najveću vrijednost postiže pri $\cos\varphi = 1$ → kada su $\vec{\nabla} f|_{T_0}$ i $\vec{s}$ kolinearni

$*$ Usmjerenu derivaciju još možemo pisati kao $\frac{\partial f}{\partial \vec{s}} = (\vec{s}\cdot\vec{\nabla})f$

$$\vec{s}\cdot\vec{\nabla} = s_1\frac{\partial}{\partial x} + s_2\frac{\partial}{\partial y} + s_3\frac{\partial}{\partial z}$$

Primjer 2.8) Izračunajte usmjerenu derivaciju polja

$f(x,y,z) = x - y^2 z \qquad \vec{s} = \vec{AB} \qquad A(1,-1,3)$ i $B(4,1,3)$

$s = (x_B - x_A,\ y_B - y_A,\ z_B - z_A) \rightarrow \underbrace{s = (3,2,0)}$

$$\vec{s}_0 = \frac{(3,2,0)}{\sqrt{13}} = \frac{3}{\sqrt{13}}\vec{i} + \frac{2}{\sqrt{13}}\vec{j} = \hat{\vec{s}}$$

$$\frac{\partial f}{\partial \vec{s}_0} = (\vec{s}_0 \cdot \nabla) f = \frac{(3,2,0)}{\sqrt{13}} \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \cdot (x - y^2 z)$$

$$\frac{\partial f}{\partial \vec{s}_0} = \frac{3}{\sqrt{13}} + \frac{2}{\sqrt{13}}(-2y \cdot z) \Rightarrow \boxed{\frac{\partial f}{\partial \vec{s}_0} = \frac{3 - 4yz}{\sqrt{13}}}$$

Primjer 2.10 Usmjerena deriv.

$f(r) = \ln r \qquad s = (1,1,-1) \rightarrow \qquad s_0 = \frac{(1,1,-1)}{\sqrt{3}}$

$$\frac{\partial(\ln r)}{\partial s_0} = (s_0 \cdot \nabla) f = \frac{(1,1,-1)}{\sqrt{3}} \cdot \underbrace{(\nabla(\ln r))}_{(\ln r)' \cdot \vec{r}_0} =$$

$$\frac{\partial(\ln(r))}{\partial s_0} = s_0 \cdot \frac{1}{r} \cdot \vec{r}_0 = s_0 \cdot \frac{1}{\underset{\sqrt{x^2+y^2+z^2}}{r}} \cdot \frac{\vec{r}}{r} = \frac{(1,1,-1)}{\sqrt{3}} \cdot \frac{xi + yj + zk}{r^2}$$

$$\boxed{\frac{\partial(\ln(r))}{\partial s_0} = \frac{x + y - z}{\sqrt{3}(x^2+y^2+z^2)}}$$

- Usmjerena derivacija vektorskog polja

$\vec{f}(x,y,z)$ u $T(x_0, y_0, z_0)$, u smjeru $\vec{s} = s_1\vec{i} + s_2\vec{j} + s_3\vec{k}$

$$\frac{\partial \vec{f}}{\partial \vec{s}}(x_0, y_0, z_0) = \lim_{h \to 0} \frac{\vec{f}(x_0 + hs_1,\ y_0 + hs_2,\ z_0 + hs_3)}{h}$$

$$\vec{f}(x,y,z) = f_1(x,y,z)\vec{i} + f_2(x,y,z)\vec{j} + f_3(x,y,z)\vec{k}$$

$$\Rightarrow \frac{\partial \vec{f}}{\partial \vec{s}} = \frac{\partial f_1}{\partial \vec{s}}\vec{i} + \frac{\partial f_2}{\partial \vec{s}}\vec{j} + \frac{\partial f_3}{\partial \vec{s}}\vec{k}$$

$$= (s \cdot \nabla) f_1 i + (s \cdot \nabla) f_2 j + (s \cdot \nabla) f_3 \cdot k$$

$$\boxed{\frac{\partial \vec{f}}{\partial \vec{s}} = (\vec{s} \cdot \nabla)\vec{f}}$$

Primjer Usmjerena derivacija polja $\vec{V}(x,y,z)=(xz, 3, -y\sqrt{z})$ u točki $T(0,-4,1)$ u smjeru $\vec{s}=2i-2j-k$

$$s_0=\frac{(2,-2,-1)}{3}=\left(\frac{2}{3},\frac{-2}{3},\frac{-1}{3}\right)$$

$$\frac{\partial v}{\partial s}=s_0\cdot\nabla v=\left[\left(\frac{2}{3},\frac{-2}{3},\frac{-1}{3}\right)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)\right]\left(xz, 3, -y\sqrt{z}\right)$$

$$\frac{\partial V}{\partial s_0}=\left(\frac{2}{3}\frac{\partial}{\partial x}-\frac{2}{3}\frac{\partial}{\partial y}-\frac{1}{3}\frac{\partial}{\partial z}\right)(xz, 3, -y\sqrt{z})$$

$$=\left(\frac{2}{3}z-\frac{1}{3}x\right)i+\left(\frac{2}{3}\sqrt{z}+\frac{y}{6\sqrt{z}}\right)k$$

u točki $T\rightarrow$ $$\frac{\partial V}{\partial s_0}=\frac{2}{3}i+\cancel{\left(\frac{2}{3}-\frac{4}{6\cdot 1}\right)k}=\boxed{\frac{2}{3}i}$$

Primjer: Usmjerenu derivaciju radijvektora $\vec{r}$ u smjeru zadanog vektora s

$$\frac{\partial r}{\partial s}=(s\cdot\nabla)r=\left(s_1\frac{\partial}{\partial x}+s_2\frac{\partial}{\partial y}+s_3\frac{\partial}{\partial z}\right)(xi+yj+zk)$$

$$=s_1 i+s_2 j+s_3 k=\vec{s} \implies \frac{\partial\vec{r}}{\partial\vec{s}}=\vec{s}$$

▸ kao što kod derivacije produkta: $(fg)'=f'g+fg'$
odnosno $d(fg)=d(fg)+d(fg)$

$$\Rightarrow \frac{\partial F(r)\cdot\vec{r}}{\partial\vec{s}}=(\vec{s}\cdot\nabla)(\vec{F}(r)\cdot\vec{r})=(\vec{s}\cdot\nabla)(F(r)\cdot\vec{r})+(\vec{s}\cdot\nabla)(F(r)\cdot\vec{r})$$

$$=\vec{r}\left((\vec{s}\cdot\nabla)F(r)\right)+F(r)\left((\vec{s}\cdot\nabla)\vec{r}\right)=\vec{r}\,\frac{\partial F(r)}{\partial\vec{s}}+F(r)\frac{\partial\vec{r}}{\partial\vec{s}}$$

$$\boxed{\frac{\partial F(r)\cdot\vec{r}}{\partial\vec{s}}=F'(r)\cdot(\vec{s}\cdot\vec{r}_0)\vec{r}+F(r)\cdot\vec{s}}$$

Primjer: Usmjerenu derivaciju vekt. polja $\vec{r}(\ln r)$ u smjeru vektora $\vec{s}=\frac{1}{\sqrt{3}}(i+j+k)$ u točki $T(1,-2,1)$

$$\frac{\partial}{\partial\vec{s}}\vec{r}\cdot\overbrace{\ln(r)}^{F(r)}=(\vec{s}\cdot\nabla)(\ln(r)\cdot\vec{r})=\vec{r}\cdot[(\vec{s}\cdot\nabla)\cdot\ln(r)]+\ln(r)[(\vec{s}\cdot\nabla)\cdot\vec{r}]$$

$$=\vec{r}\left(\vec{s}\cdot\frac{1}{r}\cdot\vec{r}_0\right)+\ln r\cdot\vec{s}=\vec{r}\left(\frac{(1,1,1)}{\sqrt{3}}\cdot\frac{1}{r}\cdot\vec{r}_0\right)+\ln(r)\cdot\frac{(1,1,1)}{\sqrt{3}}$$

$$\frac{\partial}{\partial\vec{s}}\cdot\vec{r}\ln(r)\Big|_T=(1,-2,1)\left[\frac{(1,1,1)}{\sqrt{3}}\cdot\frac{1}{\sqrt{6}}\cdot\frac{(1,-2,1)}{\sqrt{6}}\right]+\ln(r)\cdot\frac{(1,1,1)}{\sqrt{3}}$$

$$=(1,-2,1)\cdot\cancel{\frac{1-2+1}{6\sqrt{3}}}+\ln(\sqrt{6})\frac{(1,1,1)}{\sqrt{3}}$$

$$\boxed{\frac{\partial}{\partial\vec{s}}\vec{r}\ln(r)\Big|_T=\frac{\ln(\sqrt{6})}{\sqrt{3}}(i+j+k)}$$