7.4. NEHOMOGENE LDJ s KK

- sada rješavamo $Ly = f(x)$, već smo dokazali $y = y_h + y_p$

7.4.1. Metoda varijacije konstanti

1. riješimo pripadnu HLDJ $Ly = 0 \rightarrow y_h = c_1 y_1 + \dots + c_n y_n$
2. variramo konstante $c_i \rightarrow C_i(x) \Rightarrow$ opće rješ.: $\boxed{y = C_1(x) y_1 + \dots + C_n(x) y_n}$
3. treba odrediti $C_1(x) \dots C_n(x) = ?$

ovdje je množenje
↳ deriv. umnoška

⟶ gradimo sustav s n nepoznanica

zbog derivacije umnoška to izgleda:

$$y' = \underbrace{\sum_{i=1}^{n} C_i'(x) y_i(x)}_{\text{mora biti } 0} + \boxed{\sum_{i=1}^{n} C_i(x) \cdot y_i'(x)}$$

$$y'' = \underbrace{\sum_{i=1}^{n} C_i'(x) y_i'(x)}_{0} + \boxed{\sum_{i=1}^{n} C_i(x) y_i''(x)}$$

$\vdots$

$$y^{(n)} = \underbrace{\sum_{i=1}^{n} C_i'(x) y_i^{(n-1)}(x)}_{0} + \boxed{\sum_{i=1}^{n} C_i(x) y_i^{(n)}(x)}$$

uvrstimo u početnu LDJ

$$y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_1 y' + a_0 y = f(x)$$

$$\Downarrow$$

$$f(x) + \sum_{i=1}^{n} C_i(x) \cdot \underbrace{[y_i^{(n)} + a_{n-1} y_i^{(n-1)} + \dots + a_1 y_i' + a_0 y_i]}_{=0}$$

$f(x) = f(x)$ jer su y_i rješenja hom.

- dobili smo sustav:

$$C_1'(x) y_1 + C_2'(x) y_2 + \dots + C_n'(x) y_n = 0$$
$$C_1'(x) y_1' + C_2'(x) y_2' + \dots + C_n'(x) y_n' = 0$$
$$\vdots$$
$$C_1'(x) y_1^{(n-1)} + C_2'(x) y_2^{(n-1)} + \dots + C_n'(x) y_n^{(n-1)} = f(x)$$

$\Rightarrow$ Nepoznanice su $C_1'(x), \dots, C_n'(x) = ?$

Determinanta ovog nehomogenog sustava je Wronskijana $(y_1, \dots, y_n)$. Budući da su y_1 do y_n rješenja HLDJ oni su linearno nezavisni pa je $W \neq 0$.

Det $\neq 0$ pa je matrica regularna i ima jedinstveno rješenje
$\Rightarrow C_1'(x) \dots C_n'(x)$ su jednoznačno određene pa se $C_1(x)$ dobije integriranjem

Zadatak 21-22-6)

$$y''' + y' = \frac{1}{\sin x}$$

$e^0 = 1 \rightarrow$ prvi član $c_1(x) \cdot 1$

① $y''' + y' = 0$ $\qquad r_1 = 0 \qquad r_{2,3} = \pm i \qquad \alpha = 0, \ \beta = 1$

$r^3 + r = 0$

$r(r^2+1) = 0 \qquad y_h = C_1 + C_2 \sin x + C_3 \cos x$

② MVK: $y = C_1(x) + C_2(x)\sin x + C_3(x)\cos x$

3x3 sustav

$$C_1'(x) + C_2'(x)\sin x + C_3'(x)\cos x = 0$$

$$\cancel{0 \cdot C_1(x)} + C_2'(x)\cos(x) + C_3'(x)(-\sin x) = 0$$

$$\cancel{C_1(x)}\, 0 + C_2'(x)(-\sin x) + C_3'(x)(-\cos x) = \frac{1}{\sin x}$$

sve jednadžbe su 0 osim zadnje

$\Rightarrow C_1'(x) = \frac{1}{\sin x}$

$\Rightarrow -C_3'(x) = \frac{\cos x}{\sin x} \longrightarrow C_2'(x) = -1$

int.

$C_1(x) = \int \frac{1}{\sin x} dx$

$C_1(x) = \ln\left|\operatorname{tg} \frac{x}{2}\right| + D_1$

$C_2(x) = \int -dx = -x + D_2$

$C_3(x) = \int \frac{-\cos x}{\sin x} dx = -\ln|\sin x| + D_3$

Opće rješenje:

$$y = \ln\left(\operatorname{tg}\frac{x}{2}\right) + D_1 + (-x + D_2)\sin x + (-\ln(\sin x))\cos x$$

$$y = \ln\left(\operatorname{tg}\frac{x}{2}\right) - x\sin x - \ln(\sin x)\cos x + D = y_h + y_p$$

$y_h \rightarrow D_1 + D_2 \sin x + D_3 \cos x \longrightarrow$ homogeno

$+\ln\left(\operatorname{tg}\frac{x}{2}\right) - x\sin x - \ln|\sin x|\cos x$

LJIR - 20 - 8)

1) $y'' - 2y' + y = 0$ $\qquad r_{1,2} = 1 \ (k=2)$

$r^2 - 2r + 1 = 0$ $\qquad y_h = C_1 e^x + C_2 x e^x$

2) $y = C_1(x) e^x + C_2(x) x e^x$

$C_1'(x) = \frac{1}{x}$

$C_1'(x) e^x + C_2'(x) x e^x = 0 \quad / : e^x$

$- \quad C_1'(x) e^x + C_2'(x)(1e^x + x e^x) = \frac{-e}{x^2}$

$C_1(x) = \ln(x) + D_1$

$-C_2'(x) = \frac{1}{x^2} \longrightarrow C_2(x) = \int -\frac{1}{x^2}\,dx = \frac{1}{x} + D_2$

opće rješenje: $y = (\ln(x) + D_1) e^x + (\frac{1}{x} + D_2) x e^x$

$y = D_1 e^x + D_2 x e^x + \ln|x| e^x + e^x$ $\Uparrow$ početni uvjet

$y' = D_1 e^x + D_2 (x e^x + e^x) + \frac{1}{x} e^x + \ln|x| e^x + e^x$

$y_1(1) = 0 \longrightarrow D_1 e + D_2 e + 0 + e = 0$ $\qquad D_2 = -1$

$y(1) = 0 \longrightarrow D_1 e + D_2 2e + e + 0 + e = 0$ $\qquad D_1 = 0$