11. JEDNADŽBE KRUGOVA U UVJETIMA STACIONARNE SINUSNE POBUDE

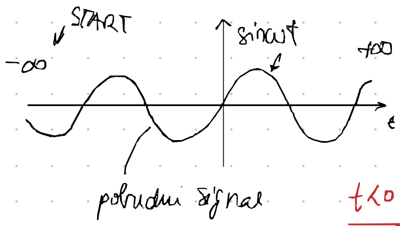
• poseban režim rada el. kruga je u uvjetima stacionarnog sinusnog signala

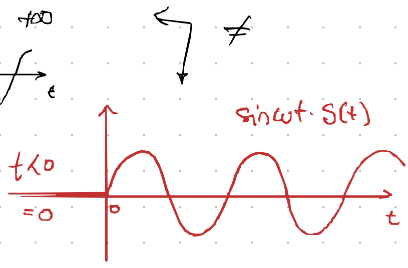
* sve pobude sinusne

u takvim uvjetima:

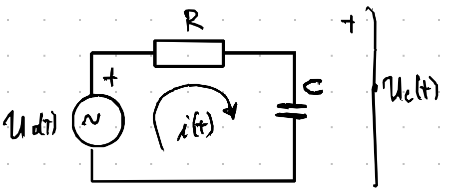
— $\boxed{W}$ svugdje ista

— $j\omega$-račun ⟶ fazor





Primjer: serijski RC krug



$u_0(t) = U_m \cos(\omega t)$

$u_C(t) = ?$

• u vremenskoj domeni ⟶ dif. jed.:

$u_0(t) = i(t) \cdot R + u_C(t)$

$u_0(t) = C \cdot \frac{du_C(t)}{dt} \cdot R + u_C(t)$

pobuda - ima sinusni valni oblik

⟶ Metoda oblika desne strane ⟶ odziv ima isti oblik kao pobuda

* moramo pretpostaviti stupnjeve slobode (U_C, φ)

$u_C(t) = U_C \cos(\omega t + \varphi)$

$\rightarrow u_C(t) = \underbrace{U_C \cdot \cos(\varphi)}_{konst\ (A)} \cdot \cos\omega t - \underbrace{U_C \cdot \sin(\varphi)}_{konst\ (B)} \cdot \sin(\omega t)$

$u_C(t) = A \cdot \cos\omega t + B \sin\omega t$ .... uvrstimo

$\Rightarrow u_0(t) = R \cdot C \cdot \frac{d}{dt} (u_C(t)) + u_C(t)$

$u_0(t) = RC(-A\omega \sin\omega t + B\omega \cos\omega t) + A\cos\omega t + B\sin\omega t = \overbrace{U_m \cos(\omega t)}^{u_0(t)}$

* izjednačavanje lijeve i desne strane

$U_m \cdot \cos(\omega t) = RC \cdot B\omega \cos(\omega t) + A\cos\omega t \longrightarrow U_m = RCB\omega + A$

$0 = -RCA\omega \sin\omega t + B\sin\omega t \qquad 0 = -RCA\omega + B$

izjednačavanje koeficijenata

$\ldots \Rightarrow A = U_C \cdot \cos(\varphi) = \frac{U_m}{(RC\omega)^2 + 1}$

$B = -U_C \cdot \sin(\varphi) = -\frac{RC\omega U_m}{(RC\omega)^2 + 1}$

• napon $u_C(t)$ je:

$u_C(t) = \frac{U_m}{(RC\omega)^2 + 1} \cos(\omega t) - \frac{\omega RC U_m}{(RC\omega)^2 + 1} \sin(\omega t)$

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* $Z_{uk} = Z_C + R = \frac{1}{j\omega C} + R$

$U_C = U_m \cdot \frac{Z_C}{Z_{uk}}$

$U_C = U_m \cdot \frac{1}{\sqrt{1 + (R\omega C)^2}}$

$\varphi = -\mathrm{arctg}(\omega RC)$

* ali želimo sve u modulima:

$|U_C| = U_m \cdot \frac{|Z_C|}{|Z_{uk}|}$

$|Z_C| = \frac{1}{\omega C}$, $Z_{uk} = \sqrt{\left(\frac{1}{\omega C}\right)^2 + R^2}$

φ ⟶ fazna razlika između ulaznog napona i napona na kondenzatoru; fazna komponenta u kompleksnoj analizi: $\frac{Im}{Re}$

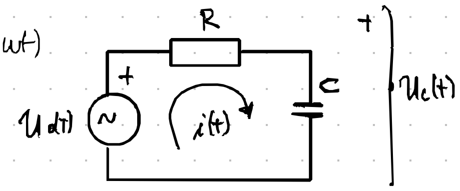
$\Rightarrow u_C(t) = U_C \cdot \cos(\omega t + \varphi) \rightarrow \boxed{u_C(t) = \frac{U_m}{\sqrt{1 + (R\omega C)^2}} \cdot \cos(\omega t - \mathrm{arctg}(\omega RC))}$

• analiza jednostavnog kruga je prilično složena

↳ kompliciraniji ⟶ složeniji

Primjer: serijski RC krug $u_0(t) = U_m \cos(\omega t)$

$u_c(t) = ?$



DRUGI NAČIN: umjesto sinusne pobude uvodimo
↪ eksponencijalnu funkciju

$U[e^{j\omega t}] = U\cdot[\cos(\omega t)+j\cdot\sin(\omega t)]$

nova pobuda $u(t) = U\cdot\cos(\omega t) = \mathrm{Re}[U\cdot e^{j\omega t}]$

novi oblik odziva: $U_c\cdot[e^{j\omega t+\varphi}]$

$u_c(t) = U_c\cdot\cos(\omega t+\varphi) = \mathrm{Re}\cdot U_c\, e^{j\omega t+\varphi}$

·diferencijalna jednadžba:

$u_0(t) = R\cdot C\cdot\frac{du_c(t)}{dt} + u_c(t)$

$U_m\cdot\cos(\omega t) \Rightarrow \mathrm{Re}[U_m e^{j\omega t}] = RC\frac{d}{dt}(U_c\cdot\cos(\omega t+\varphi)) + U_c\cdot\cos(\omega t+\varphi)$

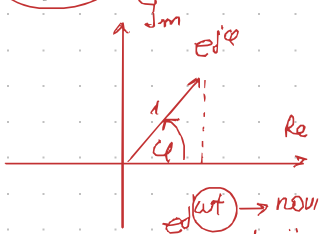
$\mathrm{Re}[U_m\cdot e^{j\omega t}] = \mathrm{Re}[RC\frac{d}{dt}(U_c\cdot e^{j(\omega t+\varphi)}) + U_c\cdot e^{j(\omega t+\varphi)}]$

$U_m\cdot e^{j\omega t} = RC\, j\omega(U_c\cdot e^{j(\omega t+\varphi)}) + U_c\cdot e^{j(\omega t+\varphi)} \;/ : e^{j\omega t}$

$U_m = RC\, j\omega U_c\cdot e^{j\varphi} + U_c\cdot e^{j\varphi}$

$u_c(t) = U_c\cdot\cos(\omega t+\varphi)$

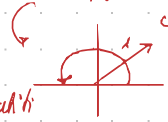
$U_c e^{j\varphi} = \frac{U_m}{1+j\omega RC}$



$|e^{j\varphi}| = 1$

$\measuredangle e^{j\varphi} = \varphi$

$e^{j\omega t}$ → novi φ koji rotira kutnom brzinom ω



možemo pokrenuti

• Množenjem s $e^{j\omega t}$ → odziv na pobudu $U_m e^{j\omega t}$

$U_c e^{j(\omega t+\varphi)} = \frac{U_m\cdot e^{j\omega t}}{1+j\omega RC} = \frac{U_m}{\sqrt{(RC\omega)^2+1}}\cdot e^{j(\omega t-\arctg(RC\omega))}$

$\ast\; U_c = \frac{U_m}{\sqrt{(RC\omega)^2+1}}$ $\varphi = -\arctg(RC\omega)$

• Vremenski odziv $u_c(t) \Rightarrow \mathrm{Re}[U_c e^{j(\omega t+\varphi)}]$

$$\boxed{u_c(t) = \frac{U_m}{\sqrt{(RC\omega)^2+1}}\cdot\cos(\omega t-\arctg(RC\omega))}$$

ZAKLJUČAK:

> Umjesto opće funkcije $f(t) = A\cos(\omega t+\varphi) \longrightarrow f(t) = A\, e^{j(\omega t+\varphi)}$

$f(t) = A\cdot e^{j(\omega t+\varphi)} = A\cdot e^{j\varphi}\, e^{j\omega t} = \underline{A}\cdot e^{j\omega t}$

fazor (mamo označeno)

FAZOR: $U_c = a + jb$

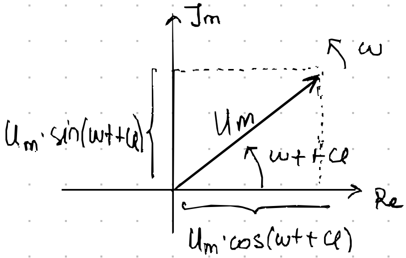
$a \to \mathrm{Re}$, $b \to \mathrm{Im}$ → (fazor) $U_c = \sqrt{a^2+b^2}\,\measuredangle\,\arctg\frac{b}{a}$

$\Rightarrow U_c = |U_c|\,\underbrace{\measuredangle U_c}_{\varphi}$ — sadrži info o amplitudi i fazi

Fazor - complex br, pridružen sinusnoj veličini

• sadrži informacije o njezinoj amplitudi i faznom pomaku

▸ signal moguće prikazati $u(t) = Re[U_m \cdot e^{j(\omega t+\varphi)}]$



→ vektor duljine U_m koji rotira oko ishodišta kutnom brzinom ω

→ φ – kut koji vektor čini s osi u $t=0$

↳ rotirajući fazor → dobivamo sinusoidu

• svi signali su rotirajući fazori → sinusoidalni signali s frekvencijom ω

Rotirajući fazor: $U_m \cdot e^{j(\omega t+\varphi)} = U \cdot e^{j\omega t}$

fazor koji ne rotira (običan)

• u trenutku $t=0$; $e^{j\omega t} = 1$

↳ $U = U_m \cdot e^{j\varphi}$

• **trig. oblik:** $U = U_m e^{j\varphi} = U_m(\cos\varphi + j\sin\varphi)$

• **algebarski oblik:** $U = Re[U] + j\,Im[U]$

$Re[U] = U_m \cdot \cos\varphi$

$Im[U] = U_m \cdot \sin\varphi$

$$U_m = \sqrt{Re^2[U] + Im^2[U]}$$

$$\varphi = \operatorname{arctg} \frac{Im[U]}{Re[U]}$$

} zbrajanje i oduzimanje

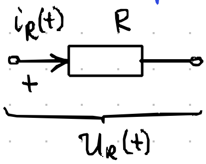
• **polarni oblik:** $U = U_m \cdot e^{j\varphi}$

$U = U_m \cdot \angle\varphi$

} množenje i dijeljenje

Odnos između napona i struja na elementima krugova

OTPOR



$u_R(t) = R \cdot i(t)$

* kasnije ćemo otkriti vezu:
$s = j\omega$
(Laplace i fazori)

• struja sinusnog valnog oblika: $i_R(t) = I\cos(\omega t + \varphi)$

tada je napon $u_R(t) = R \cdot i_R(t) \Rightarrow \underline{u_R(t) = RI\cos(\omega t + \varphi)}$

isti oblik

• u fazorskom prikazu $\quad \dot{I}_R = I\, e^{j\varphi}$

$\dot{U}_R = R \cdot I\, e^{j\omega} \Rightarrow \dot{U}_R = R \cdot \dot{I}_R$

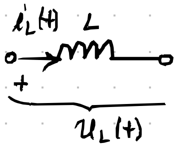
ili drugi oblik:

$u_R(t) = U\cos(\omega t + \varphi)$; $\quad i_R(t) = \frac{1}{R} u_R(t) = \underline{\underline{\frac{1}{R} \cdot U \cdot \cos(\omega t + \varphi)}}$

• u fazorskom prikazu $\quad \dot{U}_R = U \cdot e^{j\varphi}$

$\dot{I}_R = \frac{1}{R} U e^{j\varphi} = \underline{\underline{\frac{1}{R} \cdot \dot{U}_R}}$

INDUKTIVITET



$u_L(t) = L\,\frac{di_L(t)}{dt}$

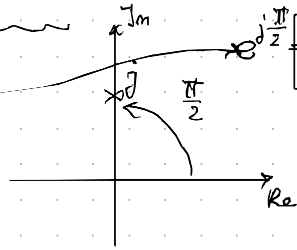
$i_L(t) \cdot I\cos(\omega t + \varphi)$ $\qquad u_L(t) = L \cdot I(-\omega)\sin(\omega t + \varphi)$

$\underline{\dot{I}_L = I \cdot e^{j\varphi}}$ $\qquad u_L(t) = \underline{\underline{\omega L \cdot I}}\cos\left(\omega t + \left(\varphi + \frac{\pi}{2}\right)\right)$

$\underline{\dot{U}_L = \omega L\, I\, e^{j\left(\varphi + \frac{\pi}{2}\right)}}$

$\dot{U}_L = \omega \cdot L \cdot I\, e^{j\varphi}\; e^{j\frac{\pi}{2}}$

$\underline{\dot{U}_L = j\omega L \cdot \dot{I}_L}$



$\boxed{e^{j\frac{\pi}{2}} = j}$

• Impedancija dvopola $\quad Z_L = \frac{\dot{U}_L}{\dot{I}_L} = j\omega L$

→ obrnuti smjer (od napona → integriramo) $\quad u_L(t) = U\cos(\omega t + \varphi)$

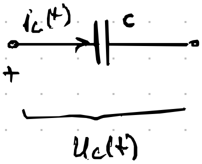
struja: $i_L(t) = \frac{1}{L}\int_{-\infty}^{t} u_L(\tau)\,d\tau = \frac{1}{\omega L} \cdot U\cos\left(\omega t + \varphi - \frac{\pi}{2}\right)$

• fazorski prikaz $\quad \dot{U}_L = U e^{j\varphi}$

$\dot{I}_L = \frac{1}{\omega L} U e^{j\left(\varphi - \frac{\pi}{2}\right)} = \frac{1}{j\omega L} \cdot U e^{j\varphi} = \frac{1}{j\omega L} \cdot \dot{U}_L$

* Laplace: $\quad Z_L(s) = \frac{U_L(s)}{I_L(s)} = sL$

KAPACITET



$$i_c(t) = C\,\frac{du_c(t)}{dt}$$

$\rightarrow$ $u_c(t) = U_c\cos(\omega t + \varphi)$

$\longrightarrow$ $i_c(t) = C\cdot\frac{d}{dt}u_c(t) = C\cdot\frac{d}{dt}\left(U_c\cdot\cos(\omega t+\varphi)\right) = \omega C\cdot U\cos\left(\omega t+\varphi+\frac{\pi}{2}\right)$

fazorski: $\dot U_c = U e^{j\varphi}$

$$\dot I_c = \omega C U e^{j\left(\varphi+\frac{\pi}{2}\right)} = j\omega C U e^{j\varphi} = j\omega C\cdot\dot U$$

Sažetak fazorskih izraza:

$\dot U_R = R\cdot\dot I_R$ $\quad\left(\dot I_R = \frac{\dot U_R}{R}\right)$

$\dot U_C = \frac{1}{j\omega C}\cdot\dot I_C$ $\quad\left(\dot I_C = j\omega C\dot U_C\right)$

$\dot U_L = j\omega L\cdot\dot I_L$ $\quad\left(\dot I_L = \frac{\dot U_L}{j\omega L}\right)$

VREMENSKA domena	FREKV. (FAZORSKA) domena
$\frac{d}{dt}f(t)$	$j\omega\cdot F(j\omega)$
$\int f(t)\,dt$	$\frac{1}{j\omega}\cdot F(j\omega)$

$\longrightarrow$ zbog toga se analiza kruga primjenom fazora često naziva analizom u frekvencijskoj domeni

fazor: $F(j\omega) = |F(j\omega)|\cdot e^{j\varphi(\omega)}$

funkcije od ω

Impedancija: $Z = \frac{\dot U}{\dot I}$

Admitancija $Y = \frac{\dot I}{\dot U}$

$Y = \frac{1}{Z}$ $\rightarrow$ kompleksne veličine

• Za 3 osnovna dvopola:

$Z_R = R$

$Z_L = j\omega L = \omega L\, e^{j90^\circ} = \omega L\angle 90^\circ$

$Z_C = \frac{1}{j\omega C} = \frac{1}{\omega C}\cdot e^{-j90^\circ} = \frac{1}{\omega L}\angle -90^\circ$

$Y_R = \frac{1}{R}$

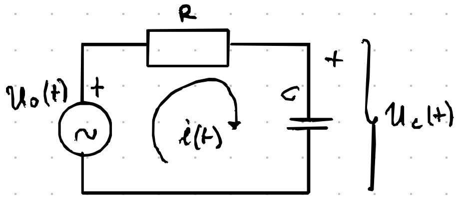
$Y_L = \cdots = \frac{1}{\omega L}\angle -90^\circ$

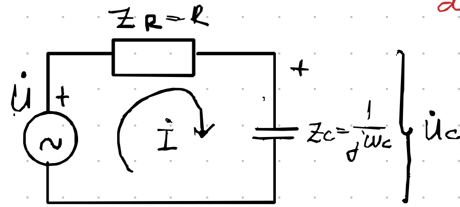
$Y_C = \cdots = \omega L\angle 90^\circ$

Primjer: Izračunati pomoću fazora napona $u_c(t)$ ako je

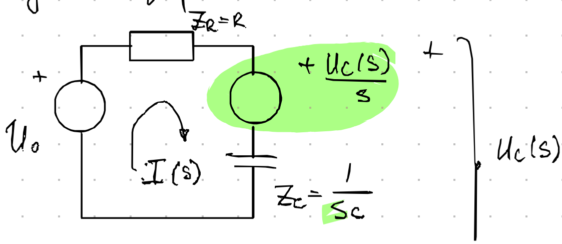
$u_0(t) = U_m \cos \omega t$

*cilj nam je izbjeći dif. jed.





*Sjeti se: Laplace



ZA FAZORE:

$$\boxed{\begin{matrix} s = j\omega \\ u_c(0) = 0 \end{matrix}}$$

sve prijelazne pojave smatramo da ako su krenule u $-\infty$ sada su završile

„ne trpimo početne uvjete"

Primjena KZN: $\dot U = \dot U_c + \dot I \cdot R$

$\rightarrow \dot I = \frac{\dot U}{Z_c} = j\omega c \cdot \dot U_c$

$\Rightarrow \dot U = \dot U_c + j\omega c \cdot \dot U_c$

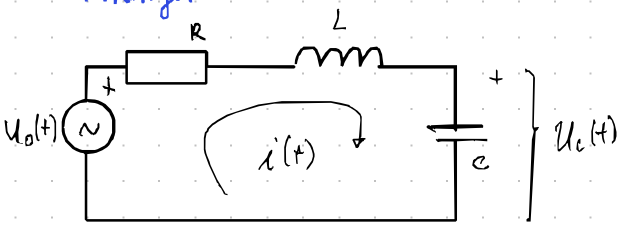
$\dot U = \dot U_c(1 + j\omega c) \Rightarrow \dot U_c = \frac{\dot U}{j\omega CR + 1} \rightarrow$ zgodnije u polarnom obliku

$\dot U_c = \frac{1 - j\omega RC}{1 + (\omega RC)^2} \dot U = \frac{\dot U \angle -\mathrm{arctg}(\omega RC)}{\sqrt{1 + (\omega RC)^2}}$

$\mathrm{tg}\,\varphi = \frac{Im\{\}}{Re\{\}} = \frac{-\omega RC}{1}$ — nazivnik

$$\boxed{u_c(t) = \frac{|\dot U|}{\sqrt{1 + (\omega R C)^2}} \cdot \cos(\omega t + \varphi) \quad * \varphi = -\mathrm{arctg}(\omega RC)}$$

Primjer:



$R = 2\,\Omega \quad L = 1\,H \quad C = 1\,F$

$u_0(t) = 2\cos\left(t + \frac{\pi}{4}\right)$

$\omega = \frac{1\,rad}{s}$

- fazor naponskog izvora je $\dot{U}_0 = |\dot{U}_0|\, e^{j\varphi} = 2\cdot e^{j\left(\frac{\pi}{4}\right)}$
- jednadžba kruga: $\dot{U}_0 = \dot{U}_c + R\cdot \dot{I} + j\omega L\cdot \dot{I}$
- fazor struje $\dot{I} = j\omega C\cdot \dot{U}_c$

$$\Rightarrow \quad \dot{U}_0 = \dot{U}_c + R\cdot j\omega C\cdot \dot{U}_c + j\omega L\cdot j\omega C\cdot \dot{U}_c$$

$$\dot{U}_0 = \dot{U}_c\left(1 + Rj\omega C - \omega^2 LC\right)$$

$$\dot{U}_c = \frac{\dot{U}_0}{1 + Rj\omega C - \omega^2 LC} = \frac{\cancel{2}\, e^{j\left(\frac{\pi}{4}\right)}}{\cancel{1} + \cancel{2}\cdot j\cdot 1 - \cancel{1}} = \frac{e^{j\left(\frac{\pi}{4}\right)}}{j} = \frac{1}{j}\left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2}\right)$$

$$\dot{U}_c = \left(\frac{\sqrt{2}}{2} - j\cdot\frac{\sqrt{2}}{2}\right) \longrightarrow \dot{U}_c = 1\cdot e^{-j\cdot \mathrm{arctg}(1)} = 1\cdot e^{-j\frac{\pi}{4}}$$

$$\boxed{u_c(t) = \cos\left(t - \frac{\pi}{4}\right)}$$

Zaključak:

- jednadžbe krugova dobivene Laplaceovom transf imaju isti oblik kao i dobivene primjenom fazora

  * poč. uvj $\Rightarrow 0$

  $s = j\omega$