3. tjedan

# 2.4. DIFERENCIJAL i PRIMJENA

Znamo: ako je $f$ diferenc. $\Longrightarrow$ $\overbrace{f(x+\Delta x, y_0+\Delta y) - f(x_0, y_0)}^{\Delta f \text{ (promjena funkcije)}}$

$$= \frac{\partial f}{\partial x}(T_0)\Delta x + \frac{\partial f}{\partial y}(T_0)\Delta y + \mathcal{O}(\Delta x, \Delta y)$$

kada su $\Delta x$ i $\Delta y$ dovoljno mali $\longrightarrow$ $\Delta f \approx \underbrace{\frac{\partial f}{\partial x}(T_0)\Delta x + \frac{\partial f}{\partial y}(T_0)\Delta y}_{\text{diferencijal f-je } df}$

(kao da je $\mathcal{O}(\Delta x, \Delta y) \approx 0$)

MATAN 1: diferencijal $\to$ $df = f'(x)dx$

$\boxed{\text{DEF}}$ prvi (totalni) diferencijal funkcije $f(x,y)$ je

$$df(x_0,y_0) = \frac{\partial f}{\partial x}(T_0)dx + \frac{\partial f}{\partial y}(T_0)dy \longrightarrow dz = 2dx - 3dy$$

je jednadžba tang. ravnine!

$$z - z_0 = 2(x-x_0) - 3(y-y_0)$$

OSNOVNA PRIMJENA:

$$f(x,y) \approx f(x_0,y_0) + df(x_0,y_0)$$

odabiremo najjednostavniju točku

$\boxed{\text{matan 1: } f(x) \approx f(x_0) + f'(x)\Delta x}$

$$\Downarrow$$

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(T_0)dx + \frac{\partial f}{\partial y}(T_0)dy$$

Linearna aproksimacija

$\boxed{\text{MI-23-2a}}$ $f(x,y) = \sqrt[3]{x^5 y} + x\,\text{arctg}\,y$

$x = 0{,}98$

$y = 1{,}1$

najbliža točka nam je $(1,1)$

$f(0.98, 1.1) \approx ?$

$(0{,}98, 1{,}1)$ $(1,1)$

$\longrightarrow$ $\Delta x = x - x_0 = -0{,}02$

$\Delta y = y - y_0 = 0{,}1$

$$\frac{\partial f}{\partial x} = \left(3\sqrt[3]{x^2}\cdot\sqrt[3]{y}\cdot\frac{5}{3} + \text{arctg}\,y\right)T_0 = \frac{5}{3} + \frac{\pi}{4}$$

$$\frac{\partial f}{\partial y} = \left(\frac{1}{3}\frac{1}{\sqrt[3]{y}}\cdot\sqrt[3]{x^5} + \frac{x}{1+y^2}\right)T_0 = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$$

$$f(0.98, 1{,}1) \approx f(1,1) + \frac{\partial f}{\partial x}(T_0)\Delta x + \frac{\partial f}{\partial y}(T_0)\Delta y$$

$$f(0.98, 1{,}1) \approx \left(1+\frac{\pi}{4}\right) + \left(\frac{5}{3}+\frac{\pi}{4}\right)\cdot(-0{,}02) + \left(\frac{5}{6}\right)\cdot(0.1)$$

$$\approx 1 + \frac{\pi}{4} - \frac{1}{\cancel{50}}\cdot\frac{\cancel{5}}{3} - \frac{1}{50}\cdot\frac{\pi}{4} + \frac{5}{60}$$

Za 3 varijable:

$$f(x,y,z) \approx f(x_0,y_0,z_0) + \underbrace{\frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}dz}_{df(x_0,y_0,z_0)}$$

gradijent: $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\Downarrow$ vektor

$$\longrightarrow df = \nabla f \cdot (\Delta x, \Delta y, \Delta z)$$

diferencijal uvijek možemo zapisati kao gradijent $\cdot$ vektor pomaka, odnosno skalarni produkt $\nabla f$ i $(\Delta x, \Delta y, \Delta z)$

$$\boxed{\text{OPĆENITO: } df(T_0) = \nabla f(T_0) \cdot \Delta \vec{x}}$$

skalarni produkt

Zadaci za vježbu s kraja skripte.

25. $a = 5\,cm$ $\quad f(a,b,c) = O(a,b,c) = 2ab + 2ac + 2bc$

$b = 3\,cm$ $\quad \Delta a = 0{,}1\,cm$

$c = 6\,cm$ $\quad \Delta b = -0{,}2\,cm$

$\Delta c = 0\,cm$

$$\Delta O \approx \frac{\partial O}{\partial a}\Big|_{T_0}\Delta a + \frac{\partial O}{\partial b}\Big|_{T_0}\Delta b + \frac{\partial O}{\partial c}\Big|_{T_0}\Delta c$$

$$\frac{\partial O}{\partial a} = 2b + 2c$$

$$\frac{\partial O}{\partial b} = 2a + 2c$$

$$\frac{\partial O}{\partial c} = 2a + 2b$$

$$\Delta O \approx 18 \cdot \frac{1}{10} - 22 \cdot \frac{2}{10} + 0$$

$$\Delta O \approx 1{,}8 - 4{,}4$$

$$\underline{\Delta O \approx -2{,}6\,cm^2}$$

Oplošje se približno smanji za $2{,}6\,cm^2$

# 2.6. DERIVACIJA SLOŽENE FUNKCIJE (ulančano deriviranje)

MATAN I:

$$[(f \circ g)(x)]' = [f(g(x))]' = f'(g(x)) \cdot g'(x)$$

Zad: $f(x,y) = \sin(3xy^2)$, $\left.\begin{matrix} x = x(t) = t \\ y = y(t) = e^t \end{matrix}\right\} \vec{r}(t)$ vektorska funkcija

$$f(x,y) = \underbrace{f(x(t), y(t))}_{\vec{r}(t)} = (f \circ \vec{r})(t)$$

(Matan I) 1. način $f(\vec{r}(t)) = \sin(3te^{2t})$

$$= \cos(3te^{2t})(3e^{2t} + 3te^{2t} \cdot 2)$$

$$= \cos(3te^{2t})(3e^{2t} + 6te^{2t})$$

(parc. der) 2. način: $\frac{\partial f}{\partial x} = \cos(3xy^2) \cdot (3y^2)$ $\qquad r'(t) = (1, e^t)$

$$\frac{\partial f}{\partial y} = \cos(3xy^2)(6xy)$$

$$f(\vec{r}(t))' = \underbrace{(\cos(3xy^2)3y^2, \cos(3xy^2)6xy)}_{f'(\vec{r}(t))} \cdot \underbrace{(1, e^t)}_{\vec{r}\,'(t)}$$

skalarni produkt

$$= \cos(3te^{2t}) \cdot 3e^{2t} \cdot 1 + \cos(3te^{2t}) \cdot 6te^t \cdot e^t$$

dobili smo isto ✓

$$\boxed{[f(\vec{r}(t))]' = \nabla f(\vec{r}(t)) \cdot \vec{r}\,'(t)}$$

## TM Lančano deriviranje

$f(x_1, \dots, x_n)$ , $\vec{r}(t) = (x_1(t), \dots, x_n(t))$

Tada: $[(f \circ \vec{r})(t)]' = [f(\vec{r}(t))]' = \nabla f(\vec{r}(t)) \cdot \vec{r}'(t)$

tj: $\frac{du}{dt} = \frac{\partial u}{\partial x_1} \cdot \frac{dx_1}{dt} + \dots + \frac{\partial u}{\partial x_n} \cdot \frac{dx_n}{dt} = \underline{\sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot \frac{dx_i}{dt}}$

## JIR - 19 - 2b

b) $w(x,y,z) = 5\cos(xy) - \sin(xz)$

$x = \frac{1}{t} \quad y = t \quad z = t^3$ , $t = \sqrt{\pi}$

$x = \frac{1}{\sqrt{\pi}}$

$y = \sqrt{\pi}$

$z = \pi\sqrt{\pi}$

$$\frac{dw}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dy}{dt}$$

$$= (-5\sin(xy) \cdot y - \cos(xz) \cdot z) \cdot \frac{-1}{t^2} + (-5\sin(xy) \cdot x) \cdot 1$$

$$+ (-\cos(xz) \cdot x) \cdot 3t^2$$

$$\frac{dw}{dt}(\sqrt{\pi}) = (-5\sin(1) \cdot \sqrt{\pi} - \cos(\pi) \cdot \pi\sqrt{\pi}) \cdot \frac{-1}{\pi}$$

$$+ (-5\sin(1) \cdot \frac{1}{\sqrt{\pi}}) \cdot 1 + (-\cos(\pi) \cdot \frac{1}{\sqrt{\pi}}) \cdot 3 \cdot \pi$$

$$\frac{dw}{dt}(\sqrt{\pi}) = 2\sqrt{\pi}$$

## TM općenito lančano deriviranje

Neka je $u = f(x_1, \dots x_n)$, te neka su $x_i = x_i(t_1, \dots, t_m)$, $i = 1, \dots n$

$\vec{r}(t_1, \dots, t_m) : \mathbb{R}^m \longrightarrow \mathbb{R}^n$

Sada: $\frac{\partial u}{\partial t_j} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_j} + \dots \frac{\partial u}{\partial x_n} \cdot \frac{\partial x_n}{\partial t_j}$, $j = 1, \dots, m$

$$\boxed{\frac{\partial u}{\partial t_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial x_i} \cdot \frac{\partial x_i}{\partial t_j}, \quad j = 1, \dots m}$$

* drugačiji način je napisati u obliku umnoška matrica (Jacobijan)

Pr.) $f(x,y) = \ln(x^2 + y^2)$

$x = r\cos\varphi = x(r, \varphi)$

$y = r\sin\varphi = y(r, \varphi)$

$$\frac{\partial f}{\partial r} = \underbrace{\frac{\partial f}{\partial x}}_{\text{fja po } x} \cdot \underbrace{\frac{\partial x}{\partial r}}_{x \text{ po } r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\Longrightarrow \frac{\partial f}{\partial r} = \frac{2x}{x^2+y^2} \cdot \cos\varphi + \frac{2y}{x^2+y^2} \cdot \sin\varphi = \frac{2\cancel{r}\cos^2\varphi}{r^{\cancel{2}}} + \frac{2\cancel{r}\sin^2\varphi}{r^{\cancel{2}}} = \underline{\underline{\frac{2}{r}}}$$

$$\frac{\partial f}{\partial \varphi} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial \varphi} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial \varphi} = \frac{2x}{x^2+y^2} \cdot (-r\sin\varphi) + \frac{2y}{x^2+y^2}(r\cos\varphi)$$

$$= \frac{-2\cancel{r^2}\sin\varphi\cos\varphi}{\cancel{r^2}} + \frac{2\cancel{r^2}\sin\varphi\cos\varphi}{\cancel{r^2}} = \underline{\underline{0}}$$

M1-19-1c $\quad u = x^2 + y^2 \quad v = \frac{x}{y} \quad G(x,y) = g(u,v)$

$$\frac{\partial G}{\partial x} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial x} = \underline{\frac{\partial g}{\partial u} \cdot 2x + \frac{\partial g}{\partial v} \cdot \frac{1}{y}}$$

i to je to!

$$\frac{\partial G}{\partial y} = \frac{\partial g}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \cdot \frac{\partial v}{\partial y} = \frac{\partial g}{\partial u} \cdot 2y + \frac{\partial g}{\partial v} \cdot \left(\frac{-x}{y^2}\right)$$

Zad.) $\quad y \cdot \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0 \qquad u = x, \quad v = x^2 + y^2$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot 2x$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = 0 + \frac{\partial z}{\partial v} \cdot 2y$$

$$y \cdot \left(\cancel{\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \cdot 2x}\right) - x \cdot \left(\cancel{\frac{\partial z}{\partial v} \cdot 2y}\right) = 0$$