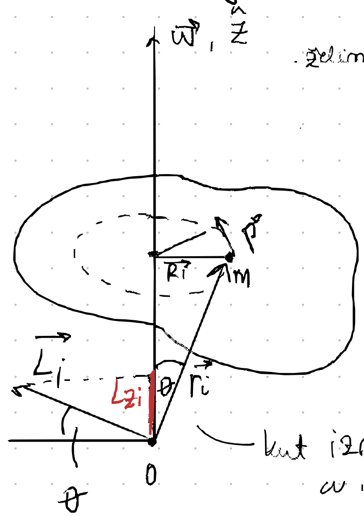
*Do sada smo o gibanju krutog tijela naučili:

translacija: gibanje CM $\quad M\vec{a}_{CM} = \sum_i \vec{F}_{vi} = \vec{F}_v$

rotacija: pretp. nepomična os $\rightarrow$ bez translacije $\quad I\vec{\alpha} = \sum_i \vec{M}_{vi} = \vec{M}_v$

Rotacija krutog tijela oko nepomične osi



želimo izvest formula koja povezuje $\vec{L}$ sa $\vec{\omega}$

$$\vec{L_i} = \vec{r_i} \times \vec{p_i} = |\vec{r_i}||\vec{p_i}| \cdot \sin 90° = r_i \cdot p_i$$

$$\vec{L_i} = r \cdot m\, v_i = r \cdot m \cdot |\vec{\omega} \times \vec{r_i}| \longrightarrow L_i = r_i^2 \cdot m_i \omega \sin\theta$$

$$\Rightarrow v_i = \omega \cdot r_i \cdot \sin\theta$$

$$\vec{L} = \sum_i \vec{L_i} \Rightarrow L_z = \sum L_{zi}$$

$$L_{zi} = L_i \cos(90° - \theta) = L_i \sin(\theta)$$

kut između ω i r_i

$$L_{zi} = r_i^2 \cdot m_i \cdot \omega \cdot \sin^2\theta$$

nije nužno uvijek z os

$$\sin\theta = \frac{R_i}{r_i}$$

$$\sum_i M_i \hat{z} \longrightarrow L_{zi} = \cancel{r_i^2} \cdot \left(\frac{R_i}{\cancel{r_i}}\right)^2 \cdot m_i \cdot \omega \cdot \Longrightarrow \boxed{L_{zi} = R_i^2 \cdot m_i \cdot \omega \cdot \hat{z}}$$

$$L_z = \sum_i L_{zi} = \sum R_i^2 \cdot m_i \cdot \omega = \omega \cdot \underbrace{\sum R_i^2 \cdot m_i}_{\text{moment tromosti } I_z} \Longrightarrow \boxed{L_z = \omega \cdot I_z}$$

*ako tijelo nije savršeno (a rijetko ikad je) rotacija nije oko osi simetrije

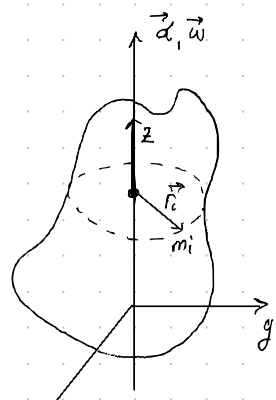
- rotaciju može uzrokovati samo sila okomita na os rotacije
- sila usmjerena duž osi može uzrokovati samo translaciju (ili rotaciju oko neke druge osi)

Moment tromosti (inercije) krutog tijela

I - moment tromosti (inercije) –||–

- prelazak u kontinuirane varijable –> dif. i int. račun

$$I = \lim_{m_i \to 0} \sum_i m_i \vec{r}_i^{\,2} = \int \vec{r}^{\,2} dm = \int \vec{r}^{\,2} \rho dV$$



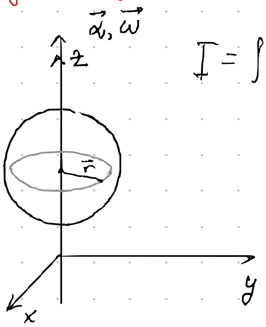
$\Longrightarrow$ moment sile je: $\vec{M} = M_z \hat{k} = I_z \alpha_z \hat{k}$

$\Rightarrow$ kutna kol gibanja: $\vec{L} = L_z \hat{k} = I_z \omega \hat{k}$

mijenjaju se ovisno o promatranoj točki tijela

* moment tromosti $\Rightarrow$ matrica 3×3 („tenzor") $\hookrightarrow$ I ovisi o osi oko koje se odvija rotacija

Kako je tijelo pravilno, a os rotacije se poklapa s nekom od osi simetrije tijela, onda je aksijalna komponenta i jedina komponenta vektora $\vec{L}$ i $\vec{M}$



$I = \int r^2 \rho dV$ – budući da je def za „neku" os rotacije

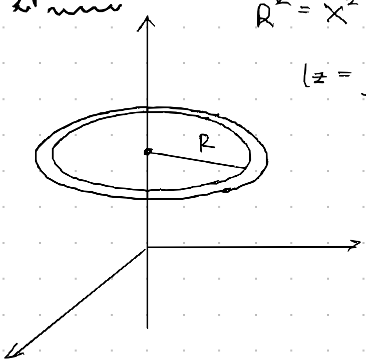
$\Downarrow$

Za z-os: $I_z = \int \vec{r}^{\,2} \rho dV = \int (x^2+y^2) \rho dV$

PRIMJERI RAČUNANJA TROMOSTI

*obično su momenti (I) izraženi kao „broj · masa · R^2"

1) prsten



$R^2 = x^2 + y^2$

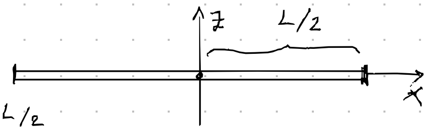
$$I_z = \int r^2 \rho dV = \rho \int (x^2+y^2) dV = \rho \int R^2 dV = \rho \int R^2 dxdydz$$

$$I_z = \rho R^2 V \qquad \rho = \frac{m}{V}$$

$$\boxed{I_z = m \cdot R^2}$$

PRIMJERI RAČUNA TROMOSTI

2) Štap



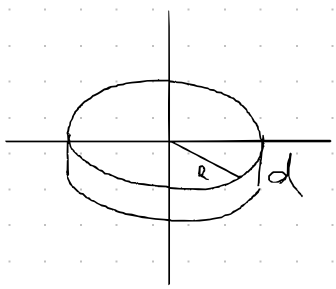
$$I_z = \rho \int x^2 \overbrace{dx\,dy\,dz}^{dV} =$$

$$I_z = \rho \underbrace{\int dy\,dz}_{S} \int_{-L/2}^{L/2} x^2\,dx$$

$$I_z = \rho \cdot S \cdot \frac{x^3}{3}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \rho S \frac{1}{3}\left(\frac{L^3}{8} + \frac{L^3}{8}\right)$$

$$I_z = \frac{1}{12}\rho S L^3 \Rightarrow I_z = \overbrace{\rho \cdot S \cdot L}^{M}\,\frac{L^2}{12} \rightarrow \boxed{I_z = \frac{1}{12}\cdot M \cdot L^2}$$

3) kružna ploča (oko osi kroz središte okomito na ploču)

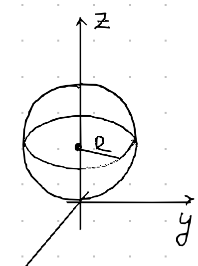


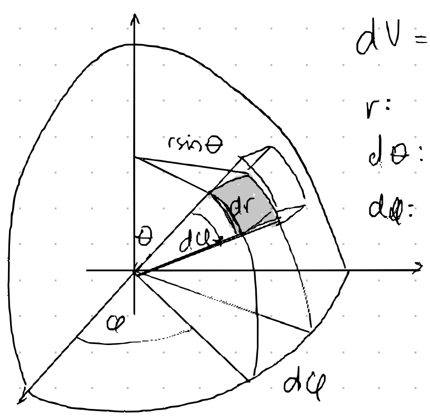
$$I_z = \rho \int r^2\,dV = \rho \int_0^{2\pi} d\theta \int_0^R r^2\,r\,dr \int_0^d dz \quad = \int dx\,dy$$

$$I_z = \rho d\,2\pi \cdot \frac{1}{4}R^4 = \rho d\pi \cdot \frac{1}{2}R^4$$

$$I_z = \underbrace{\rho d\,\pi \cdot R^2}_{M}\cdot \frac{R^2}{2} \Rightarrow \boxed{I_z = \frac{1}{2}MR^2}$$

4) puna kugla





$dV = r\,d\theta\, r\sin\theta\, d\varphi\, dr$

r: $0 - R$

$d\theta$: $0 - \pi$

$d\varphi$: $0 - 2\pi$

$x^2 + y^2 = r^2 \sin\varphi$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$t = \cos^2\theta$

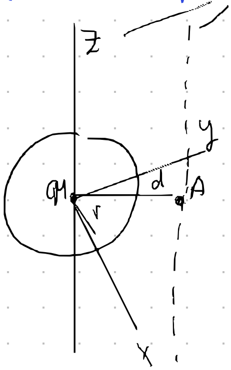
$dt = 2\cos\theta\sin\theta$

$$I_z = \rho \int dV\,(r\sin\theta)^2 = \rho\int (r^2\sin^2\theta)(r\,d\theta\, r\sin\theta\, d\varphi\, dr) =$$

$$I_z = \rho \int_0^{2\pi} d\varphi \int_0^{\pi} \sin^3\theta\,d\theta \int_0^R r^4\,dr = \rho\cdot 2\pi \cdot \int_0^{\pi} (1-\cos^2\theta)\sin\theta \cdot \frac{1}{5}R^5\,d\theta$$

$$I_z = \frac{2}{5}\rho\pi R^5\left(\cos\theta\Big|_{\pi}^{0} - \frac{\cos^3\theta}{3}\Big|_{\pi}^{0}\right) = \underbrace{\underbrace{\frac{4}{3}R^3\pi}_{V}\,\rho}_{M}\left[\frac{3R^2}{5} - \frac{R^2}{5}\right] = \boxed{\frac{2}{5}MR^2}$$

Teorem o paralelnim osima - STEINEROV POUČAK



· vrijedi općenito za moment tromosti oko osi kroz težište i kroz neku paralelnu os udaljenu od nje za d

$$\vec{r}\,' = (x-x_A)\hat{x} + (y-y_A)\hat{y}$$

$$x_{CM} = \frac{\int x\,dm}{M} \rightarrow \int x\,dm = M x_{cm}$$

x^2+y^2 — oko CM

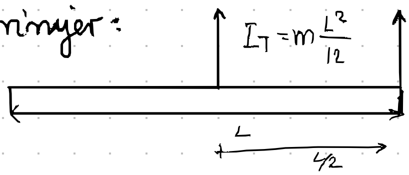
$x'^2+y'^2$ → oko A

$$I_{A,z} = \int\left[(x-x_A)^2+(y-y_A)^2\right]dm = \int\left[x^2-2xx_A+x_A^2+y^2-2yy_A+y_A^2\right]dm$$

$$= \underbrace{\int\left[x^2+y^2\right]dm}_{I_{z_{CM}}} + \int d^2\,dm - 2x_A\underbrace{\int x\,dm}_{0} - 2y_A\underbrace{\int y\,dm}_{0}$$

$$\boxed{I_{Az} = I_{CM,z} + d^2 M}$$

Primjer:



$I = ?$ (na kraju štapa, u sredini smo već računali)

$$I = I_T + md^2 \longrightarrow I = m\frac{L^2}{12} + m\cdot\left(\frac{L}{2}\right)^2 = mL^2\left(\frac{1}{12}+\frac{1}{4}\right) = mL^2\cdot\frac{\cancel{4}}{\cancel{12}}$$

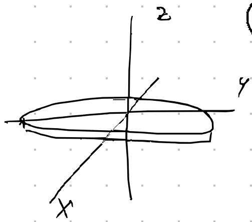
$$I = \frac{1}{3}mL^2$$

Teorem o okomitim osima

· vrijedi za tanka plošna tijela

L zbroj momenata inercije tijela oko 2 osi => mom. inerc. oko osi koja je okomita na ravninu + prolazi sjecištem drugih dviju osi

$$I_x + I_y = I_z$$

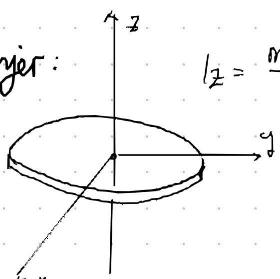


$I_z = \int r^2\,dm \quad \int(x^2+y^2)\,dm$

$I_x = \int y^2\,dm \quad I_y = \int x^2\,dm$

$$\Rightarrow I_x + I_y = \int y^2\,dm + \int x^2\,dm = \int(x^2+y^2)\,dm$$

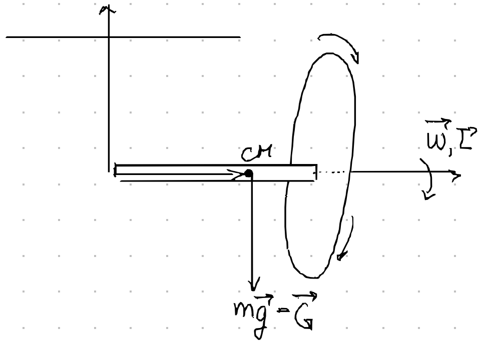
Primjer:

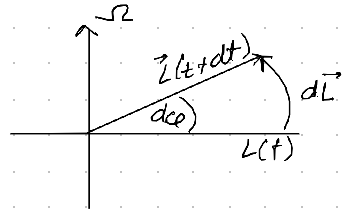


$I_z = \frac{mR^2}{2} \quad I_x = I_y$

$I_x + I_y = I_z = 2I_x \quad I_x = I_y = \frac{mR^2}{4}$

Gibanje zvrka





$$\vec{M} = \frac{d\vec{L}}{dt} = \frac{d|L|}{dt} \cdot \hat{L} + \vec{\Omega} \times \vec{L}$$

$$dL = |\vec{L}| \cdot d\varphi$$

$$\boxed{M = \frac{dL}{dt} = |\vec{L}| \cdot \frac{d\varphi}{dt}}$$

$$\boxed{M = \vec{r}_A \times \vec{G}}$$

$$\Omega = \lim \frac{\Delta\varphi}{\Delta t} = \frac{d\varphi}{dt}$$

$$M = \Omega \cdot \underbrace{I_z \cdot \omega}_{\hookrightarrow z}$$

$$M = r\,mg$$

kutna brzina precesije

$$\boxed{\Omega = \frac{r_{CM} \cdot mg}{I\omega}}$$