Neharmonijski oscilator i titranje malenom amplitudom

Neharmonijski oscilator $\ddot{x} + \omega^2[x]\,x = 0$

j. d. gibanja nekih sustava BEZ GUBITAKA EN. ali složeniji oblik

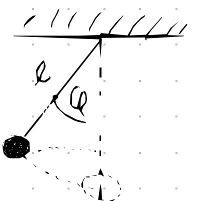
- $x = x[t]$ otklon
- $\omega^2[x]$ – konačna veličina koja ovisi o x

Titranje malenom amplitudom $\omega_0^2 = \lim\limits_{x\to 0} \omega^2[x]$

tako dobijemo $\omega_0^2 = 0$ → gibanje neharmonijskog oscilatora opisanog ne možemo aproksimirati harm. titranjem

+ isto vrijedi u slučaju kad lijevi i desni limes nisu jednaki

Primjer Frekvencija mat. njihala pri njihanju malenom amplitudom



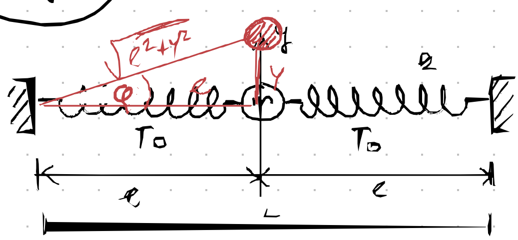
$$\ddot{\varphi} + \omega^2[\varphi]\,\varphi = 0$$

$$\omega^2[\varphi] = \frac{g \sin\varphi}{l\,\varphi}$$ – zadovoljno malo $\varphi \longrightarrow \omega_0 > 0$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}$$

$$\omega_0^2 = \lim_{\varphi\to 0} \omega^2 = \frac{g}{l} \lim_{\varphi \to 0} \underbrace{\frac{\sin\varphi}{\varphi}}_{1} = \frac{g}{l}$$

Primjer Transverzalno titranje mase na dvjema napetim oprugama



$$\sin\varphi = \frac{y}{\sqrt{l^2+y^2}}$$

$l = L$

ravnotežni st.

napetost pri otklonu: $T = T_0 + k\left(\sqrt{y^2 + l^2} - l\right)$

$$m\ddot{y} = -2T\sin\varphi = -2\left(T_0 + k\left(\sqrt{y^2+l^2} - l\right)\right)\frac{l}{\sqrt{l^2+y^2}}$$

$$\Rightarrow \ddot{y} + \omega^2[y]\,y = 0$$

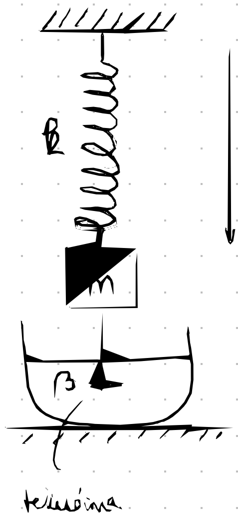
$$\omega^2[y] = \frac{2}{m}\left(T_0 + k\left(\sqrt{a^2+y^2} - a\right)\right)\frac{1}{\sqrt{a^2+y^2}}$$

$$l = \frac{L}{2}$$

$$\omega_0^2 = \lim_{y\to 0} \omega^2[y] = \frac{2}{m}\left(T_0 + k\underbrace{\left(\sqrt{a^2} - a\right)}_{0}\right)\frac{1}{\sqrt{l^2}} = \frac{2T_0}{ml} = \boxed{\frac{4T_0}{mL}}$$

- povećanjem napetosti opruga njihovim produljenjem se zanemar.
- $\sin(\varphi) \simeq \mathrm{tg}(\varphi)$

Prigušeno titranje



- sile otpora nisu prejake: prigušeno titranje
- sile otpora dovoljno jake: aperiodičko titranje

β - konstanta proporcionalnosti između jakosti sile otpora i iznosa brzine tijela

$$m\ddot{x} = -kx - \beta\dot{x}$$ — prigušeno titranje modeliramo silom $\vec{F} = -b\vec{v}$

$$m\ddot{x} + kx + \beta\dot{x} = 0 \quad /:m \qquad \omega_0^2 = \frac{k}{m} \qquad \delta = \frac{\beta}{2m}$$

koeficijent gušenja

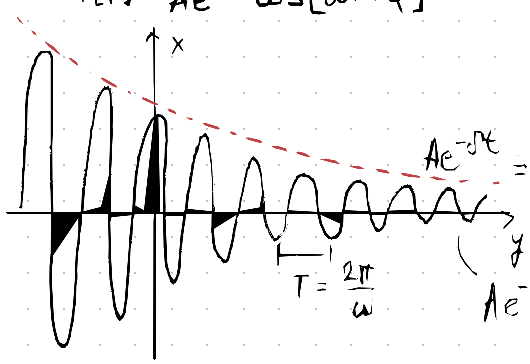
$$\ddot{x} + 2\delta\dot{x} + \omega_0^2 x = 0$$

$$x = e^{\lambda t} \rightarrow \lambda_{1,2} = -\delta \pm \sqrt{\delta^2 - \omega_0^2}$$

Podkritično prigušenje $\delta < \omega_0$

$$\Rightarrow x[t] = Ae^{-\delta t}\cos[\omega t + \varphi] \quad \text{uz} \quad \omega^2 = \sqrt{\omega_0^2 - \delta^2}$$

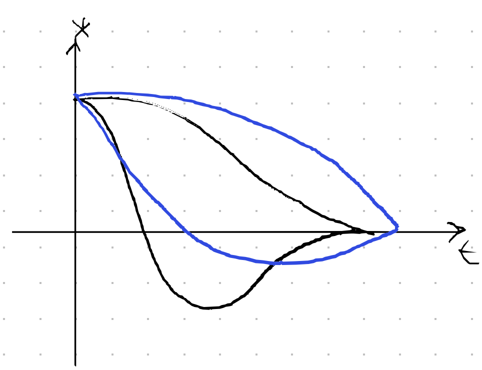
frekv. prigušenog titranja
- manja od neprigušene



$Ae^{-\delta t}$ = ovojnica (envelopa)

Kritično prigušenje $\delta = \omega_0$

$$x[t] = e^{-\delta t}(At + B)$$



Nadkritično prigušenje $\delta > \omega_0$

$$*\ \vartheta^2 = \delta^2 - \omega_0^2$$

$$\hookrightarrow x[t] = e^{-\delta t}(Ae^{\vartheta t} + Be^{-\vartheta t})$$

$$x[t] = A_0 e^{\delta t}\sinh(\vartheta t + \varphi)$$

- Energija prigušenog titranja ! meh en. nije očuvana! (smanjuje zbog otpor)

- kada se tijelo zaustavlja => $K=0$, $E=U>0$; otklon je $x[t] \simeq A e^{-\delta t}$

$$\Rightarrow E=U \simeq \frac{1}{2} k A^2 \underbrace{e^{-2\delta t}}$$

en. prig titranja je razmjerna tom f-u

- Logaritamski dekrement - jakost <u>podkritičnog</u> gušenja

$$\lambda = \ln \frac{x[t]}{x[t+T]} = \ln \frac{e^{-\delta t}}{e^{-\delta (t+T)}} = \delta T$$

ili $\lambda = \frac{2\pi\delta}{\omega}$ + pri vrlo slabom prigušenju $\omega \simeq \omega_0 \rightarrow \lambda = \frac{2\pi\delta}{\omega_0}$

- Q-faktor ili faktor kakvoće (podkritično)
  - recipročna vrijednost prosječnog relativnog gubitka en. oscilatora

$$Q^{-1} = \frac{1}{2\pi}\left|\frac{\Delta E}{E}\right| = \frac{1}{2\pi} \frac{E(t)-E(t+T)}{E(t)} = \frac{1}{2\pi}\left(1-e^{-2\delta T}\right) \simeq \frac{\delta T}{\pi}$$

$$\Rightarrow Q = \frac{\pi}{\delta T} = \frac{\omega}{2\delta} \simeq \frac{\omega_0}{2\delta}$$

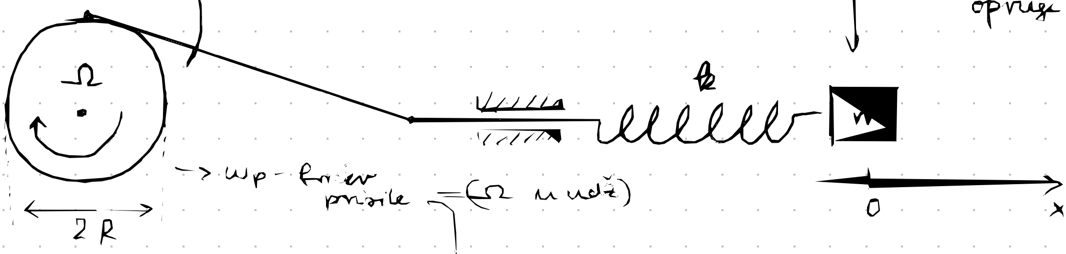
Titranje pod djelovanjem vanjske harmonijske sile

$$F_p \cos(\Omega t + \overset{\approx 0}{\psi}) \;\to\; \text{djel. vanjska sila}$$

$F_0 = k \cdot R$

Amplituda F_0 — kružna f $= \omega_p$

vrtnjom kotača, dolazi do sabijanja i rastezanja opruge



$\to \omega_p$ - frekv. pobude $= (\Omega$ u nož$)$

$$m\ddot{x} = -kx - \beta\dot{x} - F_p \cos \omega_p t \quad / m$$

$$\ddot{x} + \underbrace{\omega_0^2}_{\frac{k}{m}} x + \underbrace{2\delta}_{\frac{b}{m}} \dot{x} = \underbrace{f_p}_{\frac{F_p}{m} = \frac{kR}{m}} \cos(\omega_p t)$$

$\Rightarrow$ najjednostavnije rješenje:

$$x(t) = A\cos(\omega_p t - \varphi)$$

$$x = Ae^{i(\omega_p t - \varphi)}$$

$$\dot{x} = -A\sin(\omega_p t - \varphi) \cdot \omega_p$$

$$\ddot{x} = -A\cos(\omega_p t - \varphi) \cdot \omega_p^2$$

$$-A\cos(\omega_p t - \varphi)\cdot \omega_p^2 + 2\delta \cdot (-A\sin(\omega_p t - \varphi)\omega_p) + \omega_0^2 \cdot A\cos(\omega_p t - \varphi) = f_p \cos(\omega_p t)$$

$$\Rightarrow (-\omega_p^2 + 2\delta\omega_p i + \omega_0^2) A e^{-i\varphi} = f_p$$

$$|A| = \frac{f_p}{\sqrt{\underbrace{(\omega_0^2 - \omega_p^2)}_{\to 0} + \underbrace{(2\delta\omega_p)^2}_{\to 0}}} \qquad f_p = \frac{kR}{m} = \omega_0 R$$

$Re \Rightarrow \omega_0^2 - \omega_p^2$

$Im \to 2\delta\omega_p$

* tražimo modul za vrijednost

$$A = \frac{\omega_0 R}{\sqrt{\omega_0^2}} = R \Rightarrow \boxed{A = R}$$

* maks. pri rezonantnoj frekvenciji dobijemo deriviranjem po frekv vanjske sile

$$\omega_{REZ} = \sqrt{\omega_0^2 - (2\delta)^2} \qquad A_{REZ} = \frac{R\omega_0^2}{2\delta\sqrt{\omega_0^2 - \delta^2}}$$