4.2.5. Laplaceov operator

- vekt polje $\vec{v}$ potencijalno i solenoidalno

$\vec{v} = \text{grad}\, p$ $\qquad$ $\text{div}\,\vec{v} = 0$

pripadni potencijal

$$\text{div}\,\vec{v} = \text{div}\,(\text{grad}\, p) = \frac{\partial}{\partial x}\left(\frac{\partial p}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial p}{\partial z}\right)$$

$$\text{div}\,\vec{v} = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} = 0 \rightarrow \underbrace{\nabla \cdot \nabla}_{\text{Laplaceov operator } = \Delta} = 0$$

- za nepoznatu skalarnu fiju $\Delta p = 0$

Primjer: $\Delta \vec{V} = ?$ $T(1,0,-1)$, $\vec{v} = (x^2, xy, xz^3)$

$$\Delta \vec{V} = \left(\frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y} + \frac{\partial^2}{\partial z}\right)(x^2, xy, xz^3)$$

$$\Delta \vec{V} = \frac{\partial}{\partial x}(2x, y, z^3) + \frac{\partial}{\partial y}(0, x, 0) + \frac{\partial}{\partial z}(0, 0, 3xz^2)$$

$$\Delta \vec{V} = (2, 0, 0) + (0, 0, 0) + (0, 0, 6xz)$$

$$\Delta \vec{V} = (2, 0, 6xz) \xrightarrow{T} \boxed{\Delta \vec{V} = (2, 0, -6)}$$

$$\Delta F(r) = (\nabla \cdot \nabla)(F(r)) = \nabla \cdot (\nabla F(r)) = \nabla \cdot (F'(r)\,\vec{r}_0) = \nabla \cdot \left(\frac{F'(r)}{r} \cdot \vec{r}\right)$$

$$\Delta F(r) = \vec{r}\left(\nabla \cdot \frac{F'(r)}{r}\right) + \frac{F'(r)}{r}(\nabla \cdot \vec{r})$$

$$= \vec{r} \cdot \frac{rF''(r) - F(r)}{r^2}\,\hat{\vec{r}}_0 + 3\frac{F'(r)}{r} = \left(\frac{F''(r)}{r} - \frac{F'(r)}{r^2}\right) r + \frac{3}{r}F'(r)$$

$$\boxed{\Delta F(r) = F''(r) + \frac{2}{r}F'(r)}$$

Primjer: $\Delta(r^3)$

$$\Delta(r^3) = (\nabla \cdot \nabla)(r^3) = \nabla(\nabla r^3) = \nabla(3r^2 \cdot \hat{\vec{r}}_0) = \nabla \cdot (3r \cdot \vec{r})$$

$$\Delta(r^3) = 3\left[\vec{r}(\nabla r) + r(\nabla \vec{r})\right] = 3\left[\vec{r} \cdot \hat{\vec{r}}_0 + r \cdot 3\right] = 3\left[\frac{\vec{r} \cdot \vec{r}}{|\vec{r}|} + 3r\right]$$

$$\Delta(r^3) = 3r + 9r = 12r$$

4.2.6. Primjena operatora ∇ na složenije izraze

i) $\vec{\nabla}\varphi \neq \varphi\vec{\nabla} \longrightarrow \frac{\partial\varphi}{\partial x}, \frac{\partial\varphi}{\partial y}, \frac{\partial\varphi}{\partial z} \quad \neq \varphi\frac{\partial}{\partial x}i + \varphi\frac{\partial}{\partial y}j + \varphi\frac{\partial}{\partial z}k$

množi sprijeda

ii) $(\varphi\vec{\nabla})\cdot\vec{f} \neq \vec{\nabla}\cdot(\varphi\vec{f})$

iii) $\vec{\nabla}\cdot\vec{f} \neq \vec{f}\cdot\vec{\nabla}$ — // —

iv) $\vec{\nabla}\times\vec{f} \neq -\vec{f}\times\vec{\nabla}$

REZIME: $\nabla\cdot[A\times B] = \nabla\cdot[A\times\underline{B}] + \nabla\cdot[\underline{A}\times B]$

$\frac{\partial}{\partial\vec{s}} = \vec{s}\cdot\nabla \longrightarrow \vec{s}\cdot\nabla = \|\vec{s}\|\cdot\hat{\vec{s}}_0\cdot\nabla = \|\vec{s}\|\frac{\partial}{\partial\vec{s}_0}$

pravila složenih vektorskih produkata:

$\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{c}\cdot(\vec{a}\times\vec{b}) = \vec{b}\cdot(\vec{c}\times\vec{a})$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\cdot\vec{b} - (\vec{a}\cdot\vec{b})\cdot\vec{c}$

$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$

Gradijent, divergencija, rotacija složenih izraza

1) Gradijent djeluje na skalarno polje · skalarno polje

$\mathrm{grad}(\varphi\cdot\psi) = \vec{\nabla}(\varphi\cdot\psi) = \psi(\vec{\nabla}\varphi) + \varphi(\vec{\nabla}\psi) = \psi\,\mathrm{grad}\,\varphi + \varphi\,\mathrm{grad}\,\psi$

2) Gradijent dvaju vekt. polja

$\mathrm{grad}(\vec{f}\cdot\vec{g}) = \underline{\underline{\nabla(\vec{f}\cdot\vec{g})}} = \vec{g}(\vec{\nabla}\vec{f}) + \vec{f}(\vec{\nabla}\vec{g})$

$* \longrightarrow \vec{f}\times(\vec{\nabla}\times\vec{g}) = \vec{\nabla}(\vec{f}\cdot\vec{g}) - \vec{g}(\vec{f}\cdot\vec{\nabla})$

$\nabla(\vec{f}\cdot\vec{g}) = \vec{f}\times(\vec{\nabla}\times\vec{g}) + \vec{g}(\vec{f}\cdot\vec{\nabla})$

$\Rightarrow \mathrm{grad}(\vec{f}\cdot\vec{g}) = \vec{f}\times(\vec{\nabla}\times\vec{g}) + (\vec{f}\cdot\vec{\nabla})g + \vec{g}\times(\vec{\nabla}\times\vec{f}) + \vec{f}(\vec{\nabla}\vec{g})$

$\mathrm{grad}(\vec{f}\cdot\vec{g}) = \vec{f}\times\mathrm{rot}(\vec{g}) + \|\vec{f}\|\cdot\frac{\partial g}{\partial f} + \vec{g}\times\mathrm{rot}(\vec{f}) + \|\vec{g}\|\cdot\frac{\partial\vec{f}}{\partial\vec{g}}$

3) Divergencija vektorskog produkta dvaju vekt polja

$$\mathrm{div}(\vec{f}\times\vec{g}) = \vec{\nabla}\cdot(\vec{f}\times\vec{g}) = \vec{\nabla}\cdot(\underline{\vec{f}}\times\vec{g}) + \vec{\nabla}\cdot(\vec{f}\times\underline{\vec{g}})$$

$$= \vec{g}\cdot(\vec{\nabla}\times\vec{f}) + \vec{f}\cdot(\vec{g}\times\vec{\nabla}) = \vec{g}\cdot(\vec{\nabla}\times\vec{f}) - \vec{f}(\vec{\nabla}\times\vec{g})$$

4) Rotacija vekt produkta polja $\vec{f}$ i $\vec{g}$

$$\mathrm{rot}(\vec{f}\times\vec{g}) = \vec{\nabla}\times(\vec{f}\times\vec{g}) = \vec{\nabla}\times(\underline{\vec{f}}\times\vec{g}) + \vec{\nabla}\times(\vec{f}\times\underline{\vec{g}})$$

$$= \vec{f}(\vec{\nabla}\cdot\vec{g}) - \vec{g}(\vec{\nabla}\cdot\underline{\vec{f}}) + \vec{f}(\vec{\nabla}\cdot\underline{\vec{g}}) - \vec{g}(\vec{\nabla}\cdot\vec{f})$$

$$= \vec{f}(\vec{\nabla}\cdot\vec{g}) - (\vec{f}\cdot\vec{\nabla})\vec{g} + (\vec{g}\cdot\vec{\nabla})\vec{f} - \vec{g}(\vec{\nabla}\cdot\vec{f})$$

$$= \vec{f}\,\mathrm{div}\,\vec{g} - \|\vec{f}\|\frac{\partial\vec{g}}{\partial\vec{f}} + \|\vec{g}\|\frac{\partial\vec{f}}{\partial\vec{g}} - \vec{g}\,\mathrm{div}\,\vec{f}$$

* radijalna polja su posebnog značaja: $\vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{a}$, $\vec{\nabla}\cdot(\vec{a}\times\vec{r}) = 0$

Primjer: $\nabla\left(\dfrac{1}{\vec{a}\cdot\vec{r}}\right) \rightarrow \nabla(a\cdot r)^{-1} \rightarrow$ deriviranje složene f-je

$$\bar{\nabla}\left(\frac{1}{\vec{a}\cdot\vec{r}}\right) = \frac{-1}{(\vec{a}\cdot\vec{r})^2}\underbrace{\nabla(\vec{a}\cdot\vec{r})}_{a} = -\frac{\vec{a}}{(\vec{a}\cdot\vec{r})^2}$$

Primjer: $$\mathrm{div}\left(\frac{\vec{a}\times\vec{r}}{\vec{a}\cdot\vec{r}}\right) = \vec{\nabla}\cdot\left(\underbrace{(\vec{a}\times\vec{r})}_{I}\overbrace{\frac{1}{\vec{a}\cdot\vec{r}}}^{II}\right)$$

$$= \frac{1}{\vec{a}\cdot\vec{r}}\underbrace{\cancel{\nabla(\vec{a}\times\vec{r})}}_{0} + \underbrace{(\vec{a}\times\vec{r})}_{0}\nabla\frac{1}{\vec{a}\cdot\vec{r}} = 0\cdot\left(\frac{-\vec{a}}{(\vec{a}\cdot\vec{r})^2}\right) = \underline{\underline{0}}$$

Primjer: $$\vec{\nabla}[r\vec{\nabla}\cdot(r\vec{r})] = \nabla[r(\vec{r}\cdot\nabla r + r\overbrace{(\nabla\cdot\vec{r})}^{3})]$$

$$= \nabla[r(\vec{r}\cdot\vec{r}_0 + 3r)] = \nabla[r(r+3r)] = \nabla 4r^2 = 8r\hat{\vec{r}}_0 = 8\vec{r}$$

Primjer: $$\nabla\left[(\vec{a}\cdot\vec{r})\nabla\left(\frac{1}{r}\right)\right] = \nabla\left[(\vec{a}\cdot\vec{r})\frac{-1}{r^2}\hat{\vec{r}}_0\right] = \nabla\left[(\vec{a}\cdot\vec{r})\frac{-\vec{r}}{r^3}\right]$$

$$= \frac{-\vec{r}}{r^3}\nabla(\vec{a}\cdot\vec{r}) + (\vec{a}\cdot\vec{r})\nabla\left(\frac{-\vec{r}}{r^3}\right) = -\frac{\vec{r}}{r^3}\nabla(\vec{a}\cdot\vec{r}) - \underbrace{(\vec{a}\cdot\vec{r})}_{a}\nabla\left(\frac{\vec{r}}{r^3}\right)$$

$$= \frac{-\vec{r}}{r^3}\vec{a} - \vec{a}\left[\vec{r}\left(\nabla\frac{1}{r^3}\right) + \frac{1}{r^3}\underbrace{(\nabla\vec{r})}_{3}\right] = \frac{-\vec{r}}{r^3}\vec{a} - \vec{a}\left[\vec{r}\cdot\left(\frac{-3}{r^4}\right)\cdot\hat{\vec{r}}_0 + \frac{3}{r^3}\right]$$

$$= \frac{-\vec{r}}{r^3}\vec{a} - \vec{a}\left[\underbrace{\cancel{\frac{-3}{r^3} + \frac{3}{r^3}}}_{0}\right] = \boxed{-\frac{\vec{r}\cdot\vec{a}}{r^3}}$$

Cilindrične koordinate $F(r, \varphi, z)$ skalarno

$\vec{f} = f_r \vec{e}_r + f_\varphi \vec{e}_\varphi + f_z \cdot \vec{e}_z$ vekt polje

$$\nabla F = \frac{\partial F}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial F}{\partial \varphi}\vec{e}_\varphi + \frac{\partial F}{\partial z}\vec{e}_z$$

$$\nabla \cdot \vec{f} = \frac{1}{r}\frac{\partial}{\partial r}(r f_r) + \frac{1}{r}\frac{\partial f_\varphi}{\partial \varphi} + \frac{\partial f_z}{\partial z}$$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{e}_r & \vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_r & f_\varphi & f_z \end{vmatrix}$$

$$\Delta F = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial F}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 F}{\partial \varphi^2} + \frac{\partial^2 F}{\partial z^2}$$

Sferne koordinate $F(r, \varphi, \vartheta)$ skalarno

$\vec{f} = f_r \vec{e}_r + f_\varphi \vec{e}_\varphi + f_\vartheta \vec{e}_\vartheta$

$$\nabla F = \frac{\partial F}{\partial r}\vec{e}_r + \frac{1}{r\sin\vartheta}\cdot\frac{\partial F}{\partial \varphi}\vec{e}_\varphi + \frac{1}{r}\frac{\partial F}{\partial \vartheta}\vec{e}_\vartheta$$

$\nabla \cdot f$

$\nabla \times f$

ΔF