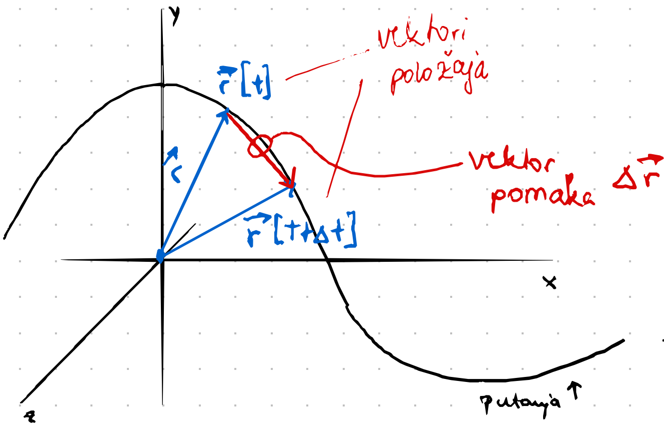
KINEMATIKA ČESTICA U 3D PROSTORU

PUTANJA Č. I POLOŽAJ Č.



$$\vec{r}[t+\Delta t] = \vec{r}[t] + \Delta\vec{r}$$

$$\Downarrow$$

Pomak:

$$\Delta\vec{r} = \vec{r}[t+\Delta t] - \vec{r}[t]$$

- koliko god smanjimo Δt, smanjit ćemo i duljinu $\Delta\vec{r}$ (modul)

avg. velocity

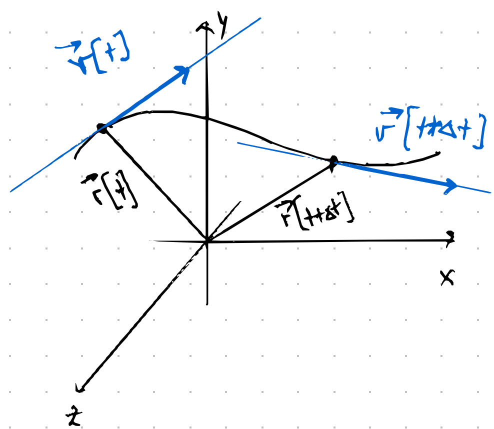
brzina: (vektor) $\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta\vec{r}}{\Delta t}$ → vektor smijemo pomnožiti skalarom $\left(\frac{1}{\Delta t}\right)$ ali nije potrebno pa možemo napisati ovako

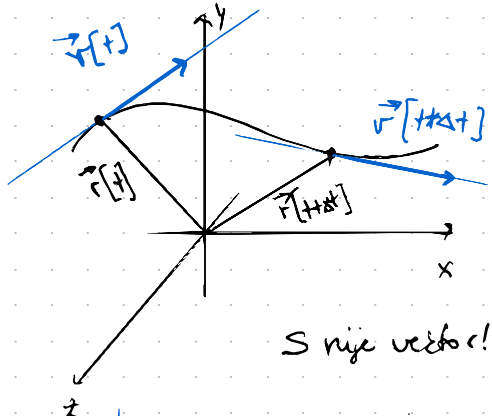
→ shvatimo $r[t]$ kao funkciju vremena

$$\Rightarrow \vec{v} = \lim_{\Delta t \to 0} \frac{\vec{r}[t+\Delta t] - \vec{r}[t]}{\Delta t}$$ derivacija!

$$\vec{v} = \frac{d}{dt}\vec{r}[t]$$

iznos brzine: $v = |\vec{v}|$ avg. speed!





element duljine prevaljenog puta:

→ matematički → diferencijal

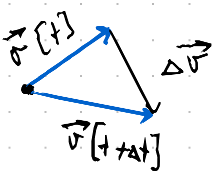
$$ds = |\vec{v}|\,dt \Rightarrow ds = v\,dt$$

S nije vektor!

iznos brzine: $v = |\vec{v}| = \frac{ds}{dt}$

vektore brzine možemo translatirati u istu točku da ih usporedimo

put funkcija po vremenu → $v = \frac{d}{dt} s[t]$



$\Delta\vec{v}$ je razlika brzina koja je nastala u intervalu vremena Δt

v je f-ja vremena

→ kada taj Δt teži u 0:

akceleracija: $\lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d}{dt} v[t]$

budući da znamo da je $\vec{v}[t] = \frac{d}{dt}\vec{r}[t]$

onda znamo, tj. možemo napisati $\vec{a} = \frac{d^2}{dt^2}\vec{r}[t]$

Položaj brzina i akceleracija u pravok. koord. sust.

položaj $\vec{r}[t] = x[t] \cdot \hat{x} + y[t] \cdot \hat{y} + z[t] \cdot \hat{z}$

→ x koord. položaja č. u t.

konstanta (ne deriviramo)

brzina: $\vec{v}[t] = \frac{d}{dt}\vec{r}[t] = \left(\frac{d}{dt}x[t]\right)\hat{x} + \left(\frac{d}{dt}y[t]\right)\hat{y} + \left(\frac{d}{dt}z[t]\right)\cdot\hat{z}$

$$= v_x[t]\,\hat{x} + v_y[t]\,\hat{y} + v_z[t]\,\hat{z}$$

akceleracija: $\vec{a}[t] = \frac{d}{dt}\vec{v}[t] = \left(\frac{d}{dt}v_x(t)\right)\hat{x} + \ldots = a_x[t]\,\hat{x} \ldots$

$$a_x[t] = \frac{d}{dt}v_x[t] = \frac{d v_x}{dt} = \dot{v}_x \Rightarrow \frac{d^2 x}{dt^2} = \ddot{x}$$

Inverzne relacije za $\vec{v}$ i $\vec{r}$

BRZINA

* akceleracija $= \frac{d\vec{v}}{dt} \Rightarrow \boxed{d\vec{v} = \vec{a} \cdot dt}$

$$d\vec{v} = \vec{a}[t']\,dt' \quad / \int_{t'=t_0}^{t'=t}$$

$$\int_{t'=t_0}^{t'=t} d\vec{v} = \int_{t'=t_0}^{t'=t} \vec{a}[t']\,dt'$$

$$\vec{v}[t] - \vec{v}[t_0] = \int_{t'=t_0}^{t'=t} \vec{a}[t']\,dt'$$

$$\vec{v}[t] = \vec{v}[t_0] + \int_{t'=t_0}^{t'=t} \vec{a}[t']\,dt'$$

* brzina $\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow d\vec{r} = \vec{v} \cdot dt \quad / \int_{t_0}^{t'=t}$

$$\vec{r}[t] - r[t_0] = \int_{t'=t_0}^{t'=t} \vec{v}[t']\,dt'$$

$$\boxed{\Rightarrow \vec{r}[t] = \vec{r}[t_0] + \int_{t'=t_0}^{t'=t} \vec{v}[t']\,dt'}$$

GIBANJE STALNOM BRZINOM

$\vec{a} = 0$ $\qquad$ $\vec{v} = \vec{v_0} = \overrightarrow{konst.}$ $\qquad$ ako je brzina konstantna, jedino se mijenja položaj

$$\vec{r}[t] = \vec{r}[t_0] + \int_{t_0}^{t} \underbrace{\vec{v}[t']}_{\vec{v}_0} dt' = \vec{r}[t_0] + \int_{t_0}^{t} \vec{v}_0 dt'$$

$$= \vec{r}[t_0] + v_0 \int_{t_0}^{t} dt' = \vec{r}[t_0] + v_0 (t - t_0)$$

GIBANJE STALNOM AKCELERACIJOM

$$\vec{a} = \vec{a_0} = \overrightarrow{konst.}$$

$$\vec{v}(t) = \vec{v}[t_0] + \int_{t_0}^{t} \underbrace{\vec{a}[t']}_{\vec{a}_0} dt = \vec{v}[t_0] + a_0 \int_{t_0}^{t} dt = \vec{v}[t_0] + \vec{a_0}(t - t_0)$$

$$\vec{r}[t] = \vec{r}[t_0] + \int_{t_0}^{t} \vec{r}[t'] dt'$$

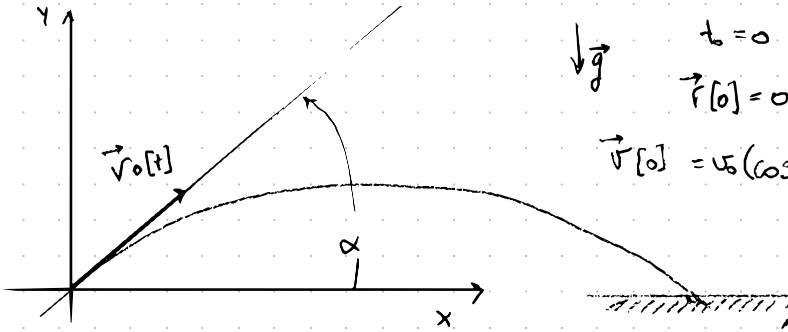
$$\vec{r}[t] - \vec{r}(t_0] = \int_{t_0}^{t} (\vec{v}[t_0] + \vec{a_0}(t' - t_0)) dt'$$

$$= \int_{t_0}^{t} \vec{v}[t_0] dt' + \int_{t_0}^{t} \vec{a_0}(t' - t_0) dt' = \vec{v_0}[t_0] \int_{t_0}^{t} dt + \vec{a_0} \int_{t_0}^{t} (\underbrace{t' - t_0}_{= \tau}) dt'$$

$$= \vec{v_0}[t_0](t - t_0) + \vec{a_0} \int_{0}^{t - t_0} \tau \, d\tau = \vec{v_0}[t_0](t - t_0) + \vec{a_0} \frac{\tau^2}{2} \Big|_{0}^{t - t_0}$$

$$\Rightarrow \vec{r}[t] - \vec{r}[t_0] = \vec{v_0}[t_0](t - t_0) + \frac{\vec{a_0}}{2}(t - t_0)^2$$

Primjer gibanja stalnom akc. Kosi hitac



$\downarrow \vec{g}$

$t_0 = 0$

$\vec{r}[0] = 0$

$\vec{v}[0] = v_0(\cos\alpha\, \hat{\vec{x}} + \sin\alpha\, \hat{\vec{y}})$

$\vec{a}_0 = -g\hat{\vec{y}}$

položaj: $\vec{r}[t_0] = 0$ (iz ishodišta)

$$\vec{r}[t] = v_0(\cos\alpha\, \hat{\vec{x}} + \sin\alpha\, \hat{\vec{y}})t + \frac{(-g\hat{\vec{y}})}{2}t^2$$

$$= \underbrace{v_0 \cos\alpha t}_{x\text{-koord}} \cdot \hat{\vec{x}} + \underbrace{v_0 \sin\alpha t}_{y\text{-koord}} \cdot \hat{\vec{y}} - \frac{g}{2} \cdot \hat{\vec{y}}\, t^2$$

x: $v_0 \cos\alpha \cdot t \cdot \hat{\vec{x}}$

y: $\left(v_0 \sin\alpha t - \frac{g}{2} t^2\right)\hat{\vec{y}}$

putanja:

$$x[t] = v_0 \cos\alpha \cdot t \longrightarrow t = \frac{x[t]}{v_0 \cos\alpha}$$

$$y[t] = v_0 \sin\alpha\, t - \frac{g}{2} t^2$$

$$* \ \frac{1}{\cos^2\alpha} = \frac{\sin^2\alpha + \cos^2\alpha}{\cos^2\alpha}$$

$$\Rightarrow y[t] = \cancel{v_0} \sin\alpha \cdot \frac{x[t]}{\cancel{v_0}\cos\alpha} - \frac{g}{2} \cdot \left(\frac{x^2[t]}{v_0^2 \cos^2\alpha}\right)$$

$$\underbrace{-\,tg^2\alpha + 1}$$

$$y[t] = tg\alpha \cdot x[t] - \frac{1}{2} g \cdot \frac{x^2[t]}{v_0^2 \cos^2\alpha} \Rightarrow \text{parabola}$$

$u = tg\alpha$

$$y[t] = tg\alpha \cdot x[t] - \frac{g \cdot x^2[t]}{2 v_0^2}(tg^2\alpha + 1) \Rightarrow \boxed{x \cdot u - \frac{g \cdot x^2}{2 v_0}(u^2+1)}$$

brzina: $\vec{v} = \frac{d}{dt}\vec{r}$ $\underbrace{(\overbrace{v_0 \cos\alpha}^{\text{konstanta}}\, t) \cdot \hat{x}}_{x}$ $\underbrace{\left(v_0 \sin\alpha\, t - \frac{g}{2} t^2\right)\hat{y}}_{y}$

$v_x[t] = \frac{d}{dt} x[t] = \underline{v_0 \cos\alpha}$

$v_y[t] = \frac{d}{dt} y[t] = \underline{v_0 \sin\alpha - g t}$

akceleracija: $\vec{a} = \frac{d}{dt}\vec{v}$

$a_x[t] = \frac{d}{dt} v_x = \underline{0}$

$a_y[t] = 0 - g = \underline{-g}$

Najveća visina: kada je y-komponenta brzine $= 0$ (leti vodoravno)

uvjet: $v_y[t'] = 0$

$\hookrightarrow v_y[t'] = v_0 \sin\alpha - g t' \longrightarrow t' = \frac{v_0 \sin\alpha}{g}$

Najveća visina $= y[t'] = v_0 \sin\alpha t - \frac{g}{2} t^2$

$\longrightarrow y[t'] = v_0 \sin\alpha \cdot \frac{v_0 \sin\alpha}{g} - \frac{\cancel{g}}{2} \cdot \frac{v_0^2 \sin^2\alpha}{g^{\cancel{2}}} = v_0^2 \sin^2\alpha \cdot \frac{1}{g} - \frac{v_0^2 \sin^2\alpha}{2g}$

$y[t'] = \left(\frac{v_0^2}{g} - \frac{v_0^2}{2g}\right) \sin^2\alpha = \boxed{\frac{v_0^2}{2g} \sin^2\alpha}$

x-koordinata:

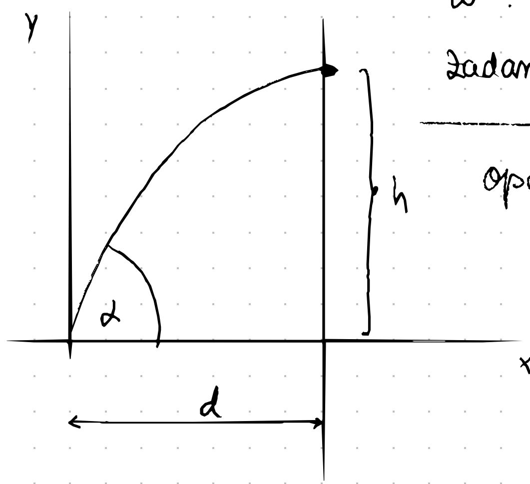
$x[t'] = v_0 \cos\alpha \frac{v_0 \sin\alpha}{g} = \frac{v_0^2}{g} \sin\alpha \cos\alpha = \boxed{\frac{v_0^2}{g} \cdot \frac{\sin 2\alpha}{2}}$

Domet na vodoravnom tlu

uvjet $y[t'] = 0 \implies t' = \frac{2}{g} v_0 \sin\alpha$

Domet: $x[t'] = v_0 \underline{\cos\alpha} \frac{2}{g} v_0 \underline{\sin\alpha} = \boxed{\frac{v_0^2}{g} \cdot \sin 2\alpha}$

Zadatak



$\alpha = ?$ za maks. h

Zadane su d i v_0 (i g)

Općenito: $y(x) = x \cdot u - \dfrac{g x^2}{2 v_0^2}(u^2+1)$

$u^* = tg\,\alpha$

$$h = d \cdot u - \frac{g d^2}{2 v_0^2}(u^2+1)$$

Max $h[u]$ nalazimo uvjetom $\dfrac{d}{du} h[u] = 0$

$$h' \Rightarrow d - \frac{g d^2}{\cancel{2} v_0^2}(\cancel{2}u + 0) = 0$$

$$d - u \cdot \frac{g d^2}{v_0^2} = 0 \longrightarrow u = \frac{\cancel{d} \cdot v_0^2}{g\, d^{\cancel{2}}} = \frac{v_0^2}{g d} \Rightarrow tg\,\alpha = \frac{v_0^2}{g d}$$

$$\Rightarrow \alpha = \text{arctg}\left(\frac{v_0^2}{g d}\right)$$

Primjer: Gibanje u 1D $(y=z=0)$

Stalna akcelaracija

$$x[t]\,\cancel{\hat{\vec{x}}} = x[t_0]\,\cancel{\hat{\vec{x}}} + v_x(t_0)(t-t_0)\,\cancel{\hat{\vec{x}}} + \frac{a_x}{2}(t-t_0)^2\,\cancel{\hat{\vec{x}}}$$

$$x[t] = x[t_0] + v_x[t_0](t-t_0) + \frac{a_x}{2}(t-t_0)^2$$

Rastav akceleracije čestice na tangencijalnu i centripetalnu komponentu (akceleracije)

brzina: $\vec{v} = v\,\hat{\vec{v}}$

v — iznos brzine (speed)

$\hat{\vec{v}}$ — smjer brzine (direction of motion)

$v = |\vec{v}|$ $\qquad \hat{\vec{v}} = \frac{1}{v}\cdot\vec{v}$

akceleracija: $\vec{a} \equiv \frac{d}{dt}\vec{v}$

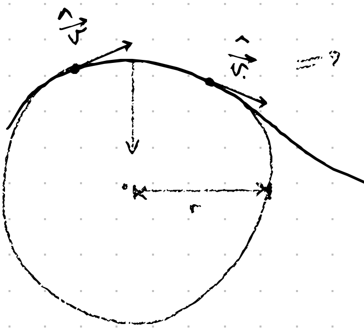
$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}\left(v\cdot\hat{\vec{v}}\right) = \underbrace{\frac{dv}{dt}\hat{\vec{v}}}_{\text{tangencijalna akceleracija}} + \underbrace{\boxed{\frac{d\hat{\vec{v}}}{dt}\cdot v}}_{\text{centripetalna akceleracija}}$$

na tangenti na putanji

isti smjer => prvi član zadržava smjer brzine

$\parallel$ sa $\vec{v}$

promjena jed. vekt. je okomita na njega => $\perp$ na $\vec{v}$



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