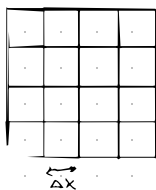
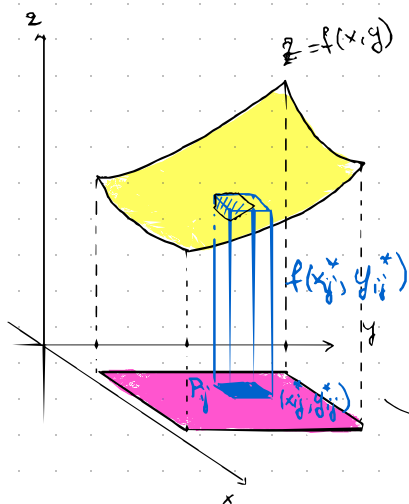


5.1. DVOSTRUKI INTEGRALI

→ Def. dvostrukog integrala na pravokutniku P

$$V = \iint_P f(x,y) dx dy = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n f(x_{ij}^*, y_{ij}^*) \Delta x \Delta y$$

- dvostruki integral po pravokutniku P predstavlja volumen tijela iznad P omeđen s gornje strane plohom $z = f(x,y)$

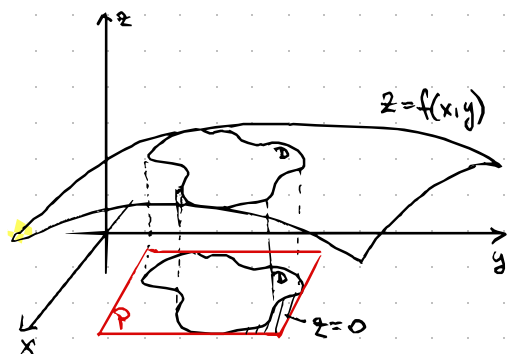


Rj. dvostrukog integrala je uvijek BROJ!

TM Ako je f neprekidna na pravokutniku P, onda je integrabilna na P.

TM Fubinijev TM Ako postoji dvostruki integral funkcije f na pravokutniku $P = [a,b] \times [c,d]$ tada $\iint_P f(x,y) dx dy = \int_a^b \left(\int_c^d f(x,y) dy \right) dx = \int_c^d \left(\int_a^b f(x,y) dx \right) dy$

Dvostruki integral na omeđenom skupu D



pomoćna funkcija $F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in P \setminus D \end{cases}$

→ tamo gdje nije D stavimo 0

⇒ tada dvostruki integral definiramo:

$$\iint_D f(x,y) dx dy = \iint_P F(x,y) dx dy$$

→ gledamo projekciju

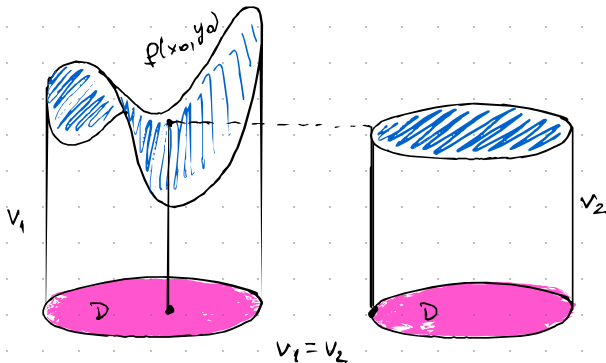
Računajmo: $\iint_D f(x,y) dx dy = \iint_P F(x,y) dx dy = \int_a^b dx \int_c^d F(x,y) dy \Rightarrow \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y) dy$

TM TM srednje vrijednosti za 2 dimenzije

Neka je f neprekidna na zatvorenom području D . Tada postoji točka $(x_0, y_0) \in D$ takva da je $\iint_D f(x, y) dx dy = f(x_0, y_0) \cdot \mu(D)$ gdje je $\mu(D)$ površina od D .

↳ Geometrijska interpretacija

- postoji točka $(x_0, y_0) \in D$ za koju je volumen valjka visine $f(x_0, y_0)$ i baze D jednak volumenu plohe $z = f(x, y)$



DOKAZ:

Neka je $m = \min_D f(x, y)$, $M = \max_D f(x, y)$

↳ Tada je $\underbrace{m \iint_D dx dy}_{\mu_D} \leq \underbrace{\iint_D f(x, y) dx dy}_{\text{volumen}} \leq \underbrace{M \iint_D dx dy}_{\mu_D}$ / μ_D

$$m \leq \frac{1}{\mu(D)} \iint_D f(x, y) dx dy \leq M$$

Zbog neprekidnosti funkcije $f(x, y)$ sigurno postoji $(x_0, y_0) \in D$ takva da vrijedi

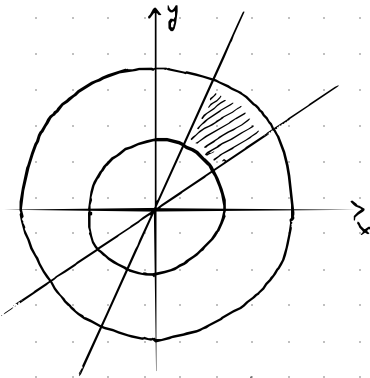
$$f(x_0, y_0) = \frac{1}{\mu(D)} \iint_D f(x, y) dx dy$$

5.1.4. Zamjena varijabli

$$\iint_D f(x,y) dx dy = \left| \begin{matrix} (x,y) = \vec{g}(u,v) \\ dx, dy = (??) du dv \end{matrix} \right| = \iint f(g(u,v)) \boxed{J} du dv$$

→ Jakobijan $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

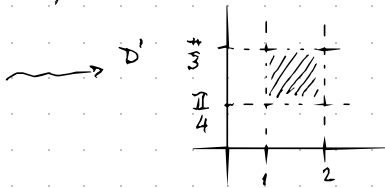
► POLARNE KOORDINATE



$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \quad \iint_D f(x,y) dx dy = \iint_D f(r \cos \varphi, r \sin \varphi) \boxed{r} dr d\varphi$$

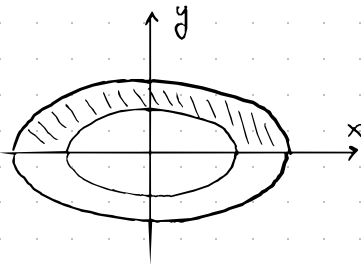
$$J \rightarrow \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial y}{\partial r} \\ \frac{\partial x}{\partial \varphi} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{vmatrix} = r$$

- Jakobijan polarnih je r



prelaskom na polarne koordinate dobivamo \square

► ELIPTIČKE KOORDINATE



$$\begin{cases} x = a \cos \varphi \\ y = b \sin \varphi \end{cases} \rightarrow J = abr$$

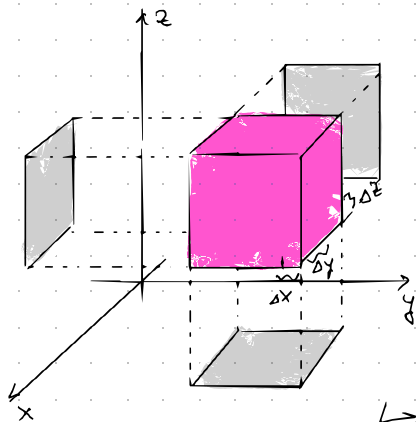
$$\iint_D f(a \cos \varphi, b \sin \varphi) abr dr d\varphi$$

5.2. TROSTRUKI INTEGRALI

$$\underbrace{\iiint_V f(x,y,z) dx dy dz}_{\substack{\text{tijelo} \\ \text{u 3D}}} \quad \underbrace{\quad}_{\text{graf u 4D}}$$

interpretiramo fizikalno \Rightarrow GUSTOĆA
 \Rightarrow masa tijela V s gustoćom $f(x,y,z)$

DEF na kvadratu:



$$\iiint_V f(x,y,z) dx dy dz = \lim_{m,n \rightarrow \infty} \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l f(x_{ijk}, y_{ijk}, z_{ijk}) \Delta x \Delta y \Delta z$$

$$\Rightarrow \iiint_V f(x,y,z) dV$$

FM Fubinijev TM - uzastopno integriranje
 - nije litan redoslijed

\hookrightarrow integracija po kvadratu $V = [a,b] \times [c,d] \times [e,f]$

se može na 3 identična integrala u bilo kojem poretku

Općenito:

postavljamo kao za 2D

$$\underbrace{\int_a^b dx}_{\text{konstante}} \underbrace{\int_{y_1(x)}^{y_2(x)} dy}_{\substack{\downarrow \\ \text{neprekidna funkcija}}} \underbrace{\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z) dz}_{\substack{\downarrow \\ \text{neprekidna ploha}}}$$

TM TM Srednje vrijednosti

Postoji takva točka da vrijedi $\iiint_V f(x,y,z) dV = f(x_0, y_0, z_0) \cdot \mu(V)$

\rightarrow ako f predstavlja gustoću tijela V , onda postoji točka tijela u kojoj se gustoća podudara s prosječnom gustoćom

5.22. Zamenjavanje varijabli

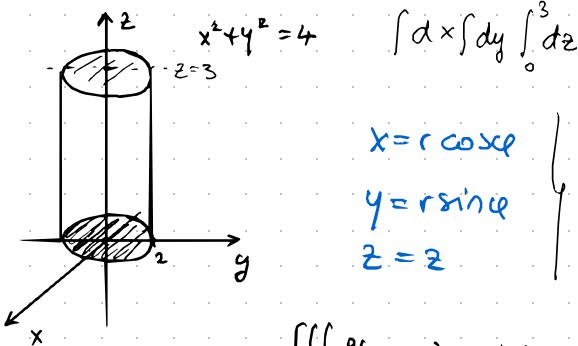
$$x = x(u, v, w)$$

$$y = y(u, v, w)$$

$$z = z(u, v, w)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \Rightarrow \iiint_V f(x, y, z) dx dy dz = \iiint_V f(u, v, w) |J| du dv dw$$

CILINDRIČNE KOORDINATE



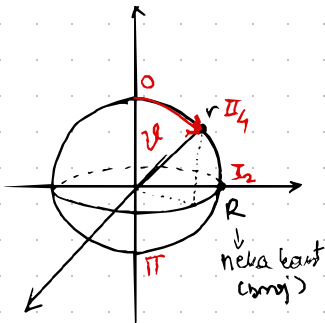
$$J = \begin{vmatrix} \cos \varphi & -r \sin \varphi & 0 \\ \sin \varphi & r \cos \varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$J = 1 (r \cos^2 \varphi + r \sin^2 \varphi)$$

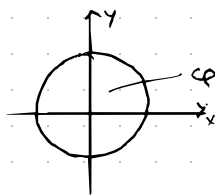
$$J = r$$

$$\iiint_V f(x, y, z) dx dy dz = \iiint_V f(r \cos \varphi, r \sin \varphi, z) r dr d\varphi dz$$

SFERNE KOORDINATE

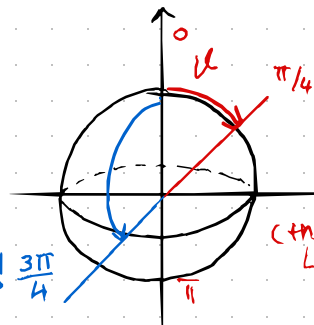


$$\varphi \in [0, 2\pi]$$



$$(x, y, z) \rightarrow (\varphi, \vartheta, r)$$

$r > 0$, trodimenzionalna udaljenost od ishodišta



ϑ - kut s pozitivnim dijelom z osi (prema dolje raste)

$$\begin{aligned} x &= r \cos \varphi \sin \vartheta \\ y &= r \sin \varphi \sin \vartheta \\ z &= r \cos \vartheta \end{aligned}$$

(metoda)
→ kucka idemo gore i dolje po z

→ NE MOŽE BITI VEĆI od π , niti manji od 0