5. Vektori

zadaci sa ispita

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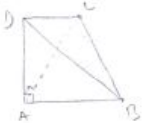
5. (10 bodova) Neka je $ABCD$ tetraedar volumena 2 takav da je brid $\overline{\mathrm{AD}}$ okomit na bridove $\overline{\mathrm{AB}}$ i $\overline{\mathrm{AC}}$. Ako je $A(1,1,2), B(2,3,3)$ i $C(2,3,5)$, nađite koordinate vrha D. Odredite sva moguća rješenja.

5. $V = 2$

$\vec{AD} \perp \vec{AB}, \quad \vec{AD} \perp \vec{AC}$



$A(1,1,2), \quad B(2,3,3), \quad C(2,3,5)$

$D = ? \quad D(x,y,z)$

$\vec{AB} = (1,2,1), \quad \vec{AC} = (1,2,3), \quad \vec{AD} = (x-1, y-1, z-2)$

I. način: Odredimo sve vektore $\vec{c}$ okomite na $\vec{AB}$ i $\vec{AC}$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 4\vec{i} - 2\vec{j} \Rightarrow \vec{c} = \lambda(4\vec{i} - 2\vec{j}), \quad \lambda \in \mathbb{R}$$

$$V = \frac{\text{površina baze} \cdot \text{visina}}{3} \Rightarrow \frac{|\vec{AB} \times \vec{AC}| \cdot |\vec{c}|}{6} = 2 \quad / \cdot 6$$

$$\Rightarrow \sqrt{4^2 + (-2)^2} \cdot \sqrt{\lambda^2 (4^2 + (-2)^2)} = 12 \Rightarrow 20|\lambda| = 12 \Rightarrow \lambda = \pm \frac{3}{5}$$

$$c_1 = \frac{12}{5}\vec{i} - \frac{6}{5}\vec{j} \Rightarrow \vec{AD} = \vec{c_1} \Rightarrow \boxed{D_1\left(\frac{17}{5}, \frac{-1}{5}, 2\right)}$$

$$c_2 = -\frac{12}{5}\vec{i} + \frac{6}{5}\vec{j} \Rightarrow \vec{AD} = \vec{c_2} \Rightarrow \boxed{D_2\left(\frac{-7}{5}, \frac{11}{5}, 2\right)}$$

II. način: Neka je $\vec{AD} = (d_1, d_2, d_3)$

$$\left.\begin{array}{l} \vec{AB} \cdot \vec{AD} = 0 \Rightarrow d_1 + 2d_2 + d_3 = 0 \\ \vec{AC} \cdot \vec{AD} = 0 \Rightarrow d_1 + 2d_2 + 3d_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} d_3 = 0 \\ d_1 = -2d_2 \end{array} \quad \vec{AD} = (-2d_2, d_2, 0)$$

$$V = \frac{|\vec{AB} \times \vec{AC}| \cdot |\vec{AD}|}{6} \Rightarrow \sqrt{20} \cdot \sqrt{4d_2^2 + d_2^2} = 12 \Rightarrow 5|d_2| = 6 \begin{cases} d_2 = \frac{6}{5} \\ d_2 = \frac{-6}{5} \end{cases}$$

$$\Rightarrow \vec{AD} = \left(-\frac{12}{5}, \frac{6}{5}, 0\right) \Rightarrow D_1\left(\frac{-7}{5}, \frac{11}{5}, 2\right)$$

ili

$$\vec{AD} = \left(\frac{12}{5}, \frac{-6}{5}, 0\right) \Rightarrow D_2\left(\frac{17}{5}, \frac{-1}{5}, 2\right)$$

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3. (10 bodova) Za vektore $\mathbf{a}$, $\mathbf{b}$ i $\mathbf{c}$ iz V^3 vrijedi

$$\|\mathbf{a}\| = 1, \qquad \|\mathbf{b}\| = 2, \qquad \|\mathbf{c}\| = 1,$$

$$\angle(\mathbf{a}, \mathbf{b}) = \frac{\pi}{3}, \qquad \angle(\mathbf{a}, \mathbf{c}) = \frac{\pi}{2}, \qquad \angle(\mathbf{b}, \mathbf{c}) = \frac{\pi}{3}.$$

Odredite vektor $\mathbf{v} \in V^3$ koji zadovoljava uvjete

$$\begin{aligned} \mathbf{v} \cdot \mathbf{a} &= 3, \\ \mathbf{v} \cdot \mathbf{b} &= 12, \\ \mathbf{v} \cdot \mathbf{c} &= 5. \end{aligned}$$

(3) $v = \alpha a + \beta b + \gamma c$

$v \cdot a = 3 \Leftrightarrow (\alpha a + \beta b + \gamma c) \cdot a = 3 \Leftrightarrow \alpha a \cdot a + \beta b \cdot a + \gamma c \cdot a = 3 \Leftrightarrow$

$\Leftrightarrow \alpha \|a\|^2 + \beta \|b\| \|a\| \cos \measuredangle (b, a) + \gamma \cdot \|c\| \|a\| \cos \measuredangle (c a) = 3$

$\Leftrightarrow \alpha \cdot 1^2 + \beta \cdot 1 \cdot 2 \cdot \cos \frac{\pi}{3} + \gamma \cdot 1 \cdot 1 \cdot \cos \frac{\pi}{2} = 3$

$\Leftrightarrow \alpha + \beta = 3$

$v \cdot b = 12 \Leftrightarrow \alpha a \cdot b + \beta b \cdot b + \gamma c \cdot b = 12 \Leftrightarrow$

$\Leftrightarrow \alpha \cdot 1 \cdot 2 \cdot \cos \frac{\pi}{3} + \beta \cdot 4 + \gamma \cdot 1 \cdot 2 \cdot \cos \frac{\pi}{3} = 12$

$\Leftrightarrow \alpha + 4\beta + \gamma = 12$

$v \cdot c = 5 \Leftrightarrow \alpha a \cdot c + \beta b \cdot c + \gamma c \cdot c = 5$

$\Leftrightarrow \alpha \cdot 1 \cdot 1 \cdot \cos \frac{\pi}{2} + \beta \cdot 2 \cdot 1 \cdot \cos \frac{\pi}{3} + \gamma \cdot 1 = 5$

$\Leftrightarrow \beta + \gamma = 5$

$$\left.\begin{array}{l} \alpha + \beta = 3 \\ \alpha + 4\beta + \gamma = 12 \end{array}\right\} \Rightarrow 3\beta + \gamma = 9 \\ \left.\begin{array}{l} \\ \beta + \gamma = 5 \end{array}\right\} \to 2\beta = 4$$

$\beta = 2, \gamma = 3, \alpha = 1$

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5. (10 bodova) Neka su $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ nekomplanarni vektori takvi da je $|\mathbf{a}| = 3$, $|\mathbf{b}| = 2$, $|\mathbf{c}| = 1$, $\angle(\mathbf{a}, \mathbf{b}) = 30°$ te neka je kut koji vektor $\mathbf{c}$ zatvara s ravninom koju razapinju vektori $\mathbf{a}$ i $\mathbf{b}$ jednak $60°$. Izračunajte

$$(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{c}) + (\mathbf{a} + \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c}) + (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} \times \mathbf{b}).$$

5. $(\vec{a}+\vec{b})\cdot(\vec{a}\times\vec{c})+(\vec{a}+\vec{c})\cdot(\vec{b}\times\vec{c})+(\vec{b}+\vec{c})\cdot(\vec{a}\times\vec{b})=$

$$= + \begin{cases} \underbrace{\vec{a}\cdot(\vec{a}\times\vec{c})}_{=0} + \vec{b}\cdot(\vec{a}\times\vec{c}) \\ \vec{a}\cdot(\vec{b}\times\vec{c}) + \underbrace{\vec{c}\cdot(\vec{b}\times\vec{c})}_{=0} \\ \underbrace{\vec{b}\cdot(\vec{a}\times\vec{b})}_{=0} + \vec{c}\cdot(\vec{a}\times\vec{b}) \end{cases}$$

$$= \underbrace{\vec{b}\cdot(\vec{a}\times\vec{c})} + \vec{a}\cdot(\vec{b}\times\vec{c}) + \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= \cancel{-\vec{a}\cdot(\vec{b}\times\vec{c})} + \cancel{\vec{a}\cdot(\vec{b}\times\vec{c})} + \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= \vec{c}\cdot(\vec{a}\times\vec{b})$$

$$= |\vec{c}|\cdot|\vec{a}\times\vec{b}|\cos\underbrace{\measuredangle(\vec{a}\times\vec{b},\vec{c})}_{90^\circ-60^\circ=30^\circ \text{ ili } 180^\circ-(90^\circ-60^\circ)=150^\circ}$$

$$= |\vec{c}|\cdot|\vec{a}|\cdot|\vec{b}|\sin\underbrace{\measuredangle(\vec{a},\vec{b})}_{30^\circ}\cdot(\pm\cos 30^\circ)$$

$$= 1\cdot 3\cdot 2\cdot\frac{1}{2}\cdot\left(\pm\frac{\sqrt{3}}{2}\right)$$

$$= \pm\frac{3\sqrt{3}}{2}$$

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5. (10 bodova)

   (a) Definirajte skalarni umnožak vektora u V^2 i V^3.

   (b) Dani su ortogonalni vektori

$$\mathbf{e} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{f} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{i} \quad \mathbf{g} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix},$$

   te vektor $\mathbf{a}$ takav da $\mathbf{a} \cdot \mathbf{e} = \mathbf{a} \cdot \mathbf{f} = \mathbf{a} \cdot \mathbf{g} = 1$. Ako je $\mathbf{a} = \alpha\mathbf{e} + \beta\mathbf{f} + \gamma\mathbf{g}$, odredite α, β i γ.

5. (a) $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos \angle(\vec{a}, \vec{b})$

(b) $1 = \vec{a} \cdot \vec{e} = \alpha \vec{e} \cdot \vec{e} + \beta \vec{f} \cdot \vec{e} + \gamma \vec{g} \cdot \vec{e} = \alpha \cdot 2 + \beta \cdot 0 + \gamma \cdot 0 \Rightarrow \alpha = \frac{1}{2}$

Slično dobivamo:

$1 = \vec{a} \cdot \vec{f} \Rightarrow \beta = \frac{1}{3}$

$1 = \vec{a} \cdot \vec{g} \Rightarrow \gamma = \frac{1}{6}$.

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3. (10 bodova) Napišite vektor $\mathbf{v} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ kao linearnu kombinaciju vektora:

$$\begin{aligned} \mathbf{u}_1 &= \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \\ \mathbf{u}_2 &= 2\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}, \\ \mathbf{u}_3 &= \mathbf{i} + 7\mathbf{j} + 8\mathbf{k}. \end{aligned}$$

3. $\vec{v} = \alpha \vec{u}_1 + \beta \vec{u}_2 + \gamma \vec{u}_3$

$$\Leftrightarrow \begin{cases} \alpha + 2\beta + \gamma = 1 \\ 2\alpha + 6\beta + 7\gamma = 3 \\ \alpha + 5\beta + 8\gamma = 2 \end{cases}$$

Rješavamo dobiveni linearni sustav:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 2 & 6 & 7 & 3 \\ 1 & 5 & 8 & 2 \end{array}\right] \begin{array}{l} |\cdot(-2) \\ + \\ + \; |\cdot(-1) \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 2 & 5 & 1 \\ 0 & 3 & 7 & 1 \end{array}\right] \begin{array}{l} \\ |\cdot(-1) \\ + \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 2 & 5 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} + \; |\cdot(-2) \\ + \\ |\cdot(-2) \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 2 & 0 \end{array}\right] \begin{array}{l} + \\ |\cdot 3 \\ |\cdot(-1) \; + \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & -2 \end{array}\right] \begin{array}{l} \Rightarrow \alpha = 4 \\ \Rightarrow \gamma = 1 \\ \Rightarrow \beta = -2 \end{array}$$

Dakle,

$$\vec{v} = 4\vec{u}_1 - 2\vec{u}_2 + \vec{u}_3$$

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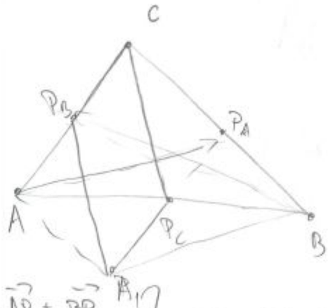
5. (10 bodova) Neka su P_A, P_B i P_C redom polovišta stranica $\overline{BC}$, $\overline{CA}$ i $\overline{AB}$ trokuta ABC.

(a) Dokažite da je

$$\overrightarrow{AP_A} + \overrightarrow{BP_B} + \overrightarrow{CP_C} = \vec{0}.$$

(b) Ako je A_1 centralnosimetrična slika točke P_A s obzirom na točku P_C, dokažite da je četverokut $A_1P_CCP_B$ paralelogram.

5)



$\overrightarrow{BP_A} = \frac{1}{2}\overrightarrow{BC}$

$\overrightarrow{AP_C} = \frac{1}{2}\overrightarrow{AB}$

$\overrightarrow{CP_B} = \frac{1}{2}\overrightarrow{CA}$

$$\left.\begin{aligned}\overrightarrow{AP_A} &= \overrightarrow{AB} + \overrightarrow{BP_A}\\ \overrightarrow{BP_B} &= \overrightarrow{BC} + \overrightarrow{CP_B}\\ \overrightarrow{CP_C} &= \overrightarrow{CA} + \overrightarrow{AP_C}\end{aligned}\right\} \quad \overrightarrow{AP_A} + \overrightarrow{BP_B} + \overrightarrow{CP_C} = \frac{3}{2}\left(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}\right) = 0$$

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b) A_1 centralno simetrična slika P_A z ozirom na P_C

$\Updownarrow$

$$\overrightarrow{P_AP_C} = \frac{1}{2}\overrightarrow{P_AA_1}$$

Dokažemo da je $\overrightarrow{P_BA_1} = \overrightarrow{CP_C}$

P_C je polovište od P_AA_1 in AB pa je AP_ABA_1 paralelogram.

$\Rightarrow \overrightarrow{A_1B} = \overrightarrow{AP_A}$

Sedaj je $\overrightarrow{P_BA_1} = -\overrightarrow{A_1B} - \overrightarrow{BP_B} = -\overrightarrow{AP_A} - \overrightarrow{BP_B} = \overrightarrow{CP_C}$

$\Rightarrow A_1P_CCP_B$ je paralelogram.

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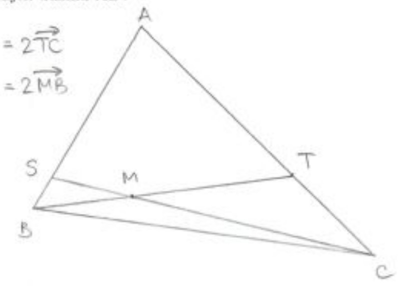
3. (10 bodova) U trokutu ABC točka T dijeli dužinu $\overline{AC}$ u omjeru $2:1$, a točka M dužinu $\overline{TB}$ u istom omjeru. Neka je točka S presjek pravca kroz točke C i M i dužine $\overline{AB}$. U kojem omjeru S dijeli dužinu $\overline{AB}$?

3. (10 bodova) U trokutu ABC točka T dijeli dužinu $\overline{AC}$ u omjeru $2:1$, a točka M dužinu $\overline{TB}$ u istom omjeru. Neka je točka S presjek pravca kroz točke C i M i dužine $\overline{AB}$. U kojem omjeru S dijeli dužinu $\overline{AB}$?

$\overrightarrow{AT} = 2\overrightarrow{TC}$

$\overrightarrow{TM} = 2\overrightarrow{MB}$



Neka su $\lambda, \mu \in \mathbb{R}$ takvi da

$$\overrightarrow{AS} = \lambda \overrightarrow{AB},$$

$$\overrightarrow{CM} = \mu \overrightarrow{CS}.$$

Imamo

$$\overrightarrow{CM} = \overrightarrow{CT} + \overrightarrow{TM} = \frac{1}{3}\overrightarrow{CA} + \frac{2}{3}\overrightarrow{TB} = \frac{1}{3}\overrightarrow{CA} + \frac{2}{3}\left(\overrightarrow{TC} + \overrightarrow{CB}\right)$$

$$= \frac{1}{3}\overrightarrow{CA} + \frac{2}{3}\left(\frac{1}{3}\overrightarrow{AC} - \overrightarrow{BC}\right) = \frac{1}{9}\overrightarrow{CA} - \frac{2}{3}\overrightarrow{BC},$$

$$\overrightarrow{CS} = \overrightarrow{CA} + \overrightarrow{AS} = \overrightarrow{CA} + \lambda\overrightarrow{AB} = \overrightarrow{CA} + \lambda\left(\overrightarrow{AC} + \overrightarrow{CB}\right)$$

$$= (1-\lambda)\overrightarrow{CA} - \lambda\overrightarrow{BC}$$

$$\Rightarrow \frac{1}{9}\overrightarrow{CA} - \frac{2}{3}\overrightarrow{BC} = \overrightarrow{CM} = \mu\overrightarrow{CS} = \mu(1-\lambda)\overrightarrow{CA} - \mu\lambda\overrightarrow{BC}.$$

Zbog linearne nezavisnosti vektora $\overrightarrow{CA}$ i $\overrightarrow{BC}$ slijedi

$$\begin{cases} \mu(1-\lambda) = \frac{1}{9} \\ \mu\lambda = \frac{2}{3} \end{cases} \Rightarrow \mu = \frac{1}{9} + \mu\lambda = \frac{7}{9}, \quad \lambda = \frac{\frac{2}{3}}{\mu} = \frac{6}{7}$$

Dakle, $\overrightarrow{AS} = \frac{6}{7}\overrightarrow{AB}$ pa točka S dijeli dužinu $\overline{AB}$ u omjeru $6:1$.

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3. (10 bodova) Dani su vektori $\mathbf{a}_1 = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{a}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

   (a) Odredite jednu ortonormiranu bazu $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ koja zadovoljava oba sljedeća uvjeta:

      i. vektori $\mathbf{a}$ i $\mathbf{a}_1$ su kolinearni,

      ii. vektori $\mathbf{b}$, $\mathbf{a}_1$ i $\mathbf{a}_2$ su komplanarni.

   (b) Koliko postoji ortonormiranih baza koje zadovoljavaju uvjete iz (a) podzadatka?

3. (10 bodova) Dani su vektori $\mathbf{a}_1 = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{a}_2 = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

(a) Odredite jednu ortonormiranu bazu $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ koja zadovoljava oba sljedeća uvjeta:

i. vektori $\mathbf{a}$ i $\mathbf{a}_1$ su kolinearni,

ii. vektori $\mathbf{b}$, $\mathbf{a}_1$ i $\mathbf{a}_2$ su komplanarni.

(b) Koliko postoji ortonormiranih baza koje zadovoljavaju uvjete iz (a) podzadatka?

(a) Iz prvog uvjeta slijedi da postoji $\lambda \in \mathbb{R}$ takav da $\vec{a} = \lambda \vec{a}_1$.

Budući da vektor $\vec{a}$ mora biti normiran, imamo

$$1 = \|\vec{a}\| = |\lambda| \cdot \|\vec{a}_1\| = |\lambda| \cdot \sqrt{4+4+1} = 3|\lambda| \Rightarrow |\lambda| = \frac{1}{3}.$$

Uzimajući, na primjer, $\lambda = \frac{1}{3}$ dobivamo $\vec{a} = \frac{1}{3}(2\vec{i} + 2\vec{j} + \vec{k})$.

Sada iz drugog uvjeta slijedi da postoje $\alpha, \beta \in \mathbb{R}$ takvi da

$$\vec{b} = \alpha \vec{a}_1 + \beta \vec{a}_2.$$

Iz ortogonalnosti vektora $\vec{a}$ i $\vec{b}$:

$$0 = \vec{a} \cdot \vec{b} = \frac{1}{3}\vec{a}_1 \cdot (\alpha \vec{a}_1 + \beta \vec{a}_2) = \frac{\alpha}{3}\|\vec{a}_1\|^2 + \frac{\beta}{3}\vec{a}_1 \cdot \vec{a}_2$$

$$= 3\alpha + \frac{\beta}{3} \cdot (2+2+1) = 3\alpha + \frac{5}{3}\beta \Rightarrow \alpha = -\frac{5}{9}\beta$$

$$\Rightarrow \vec{b} = \beta\left(-\frac{5}{9}\vec{a}_1 + \vec{a}_2\right) = \beta\left(-\frac{1}{9}\vec{i} - \frac{1}{9}\vec{j} + \frac{4}{9}\vec{k}\right).$$

Budući da i vektor $\vec{b}$ mora biti normiran:

$$1 = \|\vec{b}\| = |\beta| \cdot \frac{1}{9}\sqrt{1+1+4^2} = \frac{\sqrt{2}}{3}|\beta| \Rightarrow |\beta| = \frac{3\sqrt{2}}{2}$$

Uzimajući, na primjer, $\beta = \frac{3\sqrt{2}}{2}$ dobivamo $\vec{b} = \frac{\sqrt{2}}{6}(-\vec{i} - \vec{j} + 4\vec{k})$.

Vektor $\vec{c}$ je okomit na $\vec{a}$ i $\vec{b}$ pa je zato kolinearan s njihovim vektorskim produktom, tj. postoji $\mu \in \mathbb{R}$ takav da

$$\vec{c} = \mu(\vec{a} \times \vec{b}).$$

Budući da je i vektor $\vec{c}$ normiran,

$$1 = \|\vec{c}\| = |\mu| \cdot \|\vec{a} \times \vec{b}\| = |\mu| \cdot \underbrace{\|\vec{a}\|}_{=1} \cdot \underbrace{\|\vec{b}\|}_{=1} \underbrace{\sin \measuredangle(\vec{a}, \vec{b})}_{=\sin\frac{\pi}{2}=1} = |\mu|.$$

Uzimanjem, na primjer, $\lambda = 1$ dobivamo

$$\vec{c} = \vec{a} \times \vec{b} = \frac{\sqrt{2}}{18} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 1 \\ -1 & -1 & 4 \end{vmatrix} = \frac{\sqrt{2}}{2}(\vec{i} - \vec{j}).$$

Dakle, jedna tražena ortonormirana baza je

$$\left\{ \frac{1}{3}(2\vec{i} + 2\vec{j} + \vec{k}),\ \frac{\sqrt{2}}{6}(-\vec{i} - \vec{j} + 4\vec{k}),\ \frac{\sqrt{2}}{2}(\vec{i} - \vec{j}) \right\}.$$

(b) Uočimo da su skalari λ, β i μ (koji određuju vektore $\vec{a}$, $\vec{b}$ i $\vec{c}$) u prethodnom računu jednoznačno određeni do na predznak.
Budući da predznak za svaki od njih možemo odabrati na dva načina, ukupno postoji $2^3 = 8$ takvih baza.

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5. (10 bodova) Zadan je četverokut $ABCD$, gdje su $A(7,3,-1)$, $B(7,5,-4)$, $C(9,5,-3)$ i $D(10,4,-3)$. Neka su $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{BC}$ i $\vec{c}=\overrightarrow{CD}$. Izračunajte

$$(\vec{a}\times\vec{b})\cdot\vec{c}+\vec{b}\cdot(\vec{a}\times\vec{c})+(\vec{c}\times\vec{a})\cdot\vec{b}+(\vec{b}\times\vec{c})\cdot\vec{b}$$

Zadatak 5.

<u>RJEŠENJE</u> Stavimo $I = (\vec{a} \times \vec{b}) \cdot \vec{c} + \vec{b} \cdot (\vec{a} \times \vec{c}) + (\vec{c} \times \vec{a}) \cdot \vec{b} + (\vec{b} \times \vec{c}) \cdot \vec{b}$. Prije svega, vidimo da zadnji član iščezava, jer je $\vec{b} \times \vec{c}$ okomit na $\vec{b}$. Nadalje, drugi i treći član se poništavaju, jer je

$$\begin{aligned}
\vec{b} \cdot (\vec{a} \times \vec{c}) + (\vec{c} \times \vec{a}) \cdot \vec{b} &= (\vec{a} \times \vec{c}) \cdot \vec{b} + (\vec{c} \times \vec{a}) \cdot \vec{b} \\
&= (\vec{a} \times \vec{c}) \cdot \vec{b} + (-(\vec{a} \times \vec{c})) \cdot \vec{b} \\
&= (\vec{a} \times \vec{c}) \cdot \vec{b} - (\vec{a} \times \vec{c}) \cdot \vec{b} = 0.
\end{aligned}$$

Ostaje nam $I = (\vec{a} \times \vec{b}) \cdot \vec{c}$. Radi se o mješovitom umnošku, kojeg računamo uz pomoć determinante:

$$I = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & x_y & c_z \end{vmatrix}.$$

Sada pronalazimo vektore $\vec{a}$, $\vec{b}$ i $\vec{c}$.

$$\begin{aligned}
\vec{a} &= (7, 5, -4) - (7, 3, -1) = (0, 2, -3), \\
\vec{b} &= (9, 5 - 3) - (7, 5, -4) = (2, 0, 1), \\
\vec{c} &= (10, 4, -3) - (9, 5, -3) = (1, -1, 0).
\end{aligned}$$

Konačno, imamo

$$I = \begin{vmatrix} 0 & 2 & -3 \\ 2 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 1 \cdot \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} - (-1) \cdot \begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix} = 2 + 6 = 8.$$

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3. (10 bodova)

   U trokutu ABC, gdje su $A(-1,1)$, $B(0,3)$ i $C(2,4)$, M je polovište stranice $\overline{AC}$, a N je polovište stranice $\overline{BC}$. Odredite $\overrightarrow{AC}_{\overrightarrow{MN}}$, vektorsku projekciju $\overrightarrow{AC}$ na $\overrightarrow{MN}$.

Zadatak 3.

<u>RJEŠENJE</u> Odredimo vektore $\overrightarrow{AC}$ i $\overrightarrow{MN}$. Polovišta su

$$M = \left(\frac{1}{2}(-1+2), \frac{1}{2}(1+4)\right) = \left(\frac{1}{2}, \frac{5}{2}\right), \qquad N = \left(\frac{1}{2}(0+2), \frac{1}{2}(3+4)\right) = \left(1, \frac{7}{2}\right).$$

Dakle,

$$\overrightarrow{AC} = (2-(-1), 4-1) = (3,3), \qquad \overrightarrow{MN} = \left(\frac{1}{2}, 1\right).$$

Projekcija vektora $\overrightarrow{AC}$ na $\overrightarrow{MN}$ dana je s

$$\overrightarrow{AC}_{\overrightarrow{MN}} = |\overrightarrow{AC}| \cos(\measuredangle(\overrightarrow{AC}, \overrightarrow{MN})) \frac{\overrightarrow{MN}}{\left|\overrightarrow{MN}\right|} = \frac{\overrightarrow{AC} \cdot \overrightarrow{MN}}{\left|\overrightarrow{MN}\right|^2} \overrightarrow{MN} = \frac{\frac{3}{2}+3}{\frac{1}{4}+1} \left(\frac{1}{2}, 1\right) = \left(\frac{9}{5}, \frac{18}{5}\right).$$

$\square$

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2. (10 bodova)

   Kut između vektora $\boldsymbol{a}$ i $\boldsymbol{b}$ je $\frac{\pi}{3}$. Ako je vektor $\boldsymbol{b}$ jedinični, a vektori

$$\boldsymbol{a}+\boldsymbol{b} \qquad \text{i} \qquad \boldsymbol{a}-5\boldsymbol{b}$$

   okomiti, odredite duljinu vektora $\boldsymbol{a} \times \boldsymbol{b}$.

Zadatak 2.

RJEŠENJE Stavimo $\phi = \frac{\pi}{3}$. Uvjet ortogonalnosti nam daje

$$\begin{aligned} (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-5\boldsymbol{b}) &= 0 \\ |\boldsymbol{a}|^2 - 4\boldsymbol{a}\cdot\boldsymbol{b} - 5|\boldsymbol{b}|^2 &= 0 \\ |\boldsymbol{a}|^2 - 4|\boldsymbol{a}|\,|\boldsymbol{b}|\cos(\phi) - 5|\boldsymbol{b}|^2 &= 0 \\ |\boldsymbol{a}|^2 - 2|\boldsymbol{a}| - 5 &= 0 \implies \\ |\boldsymbol{a}| &= 1+\sqrt{6}. \end{aligned}$$

Sada imamo

$$|\boldsymbol{a}\times\boldsymbol{b}| = |\boldsymbol{a}|\,|\boldsymbol{b}|\sin(\phi) = (1+\sqrt{6})\frac{\sqrt{3}}{2}.$$