2.2. PARCIJALNE DERIVACIJE

- gledamo samo 1 varijablu

gledamo promjenu po jednoj tako da drugu fiksiramo

- neka je $f(x,y)$ neprekinuta $T_0(x_0,y_0)$

y_0 je fiksiran

DEF $$\left(\frac{\partial f}{\partial x}\right)_{T_0} = \frac{\partial f}{\partial x}(x_0,y_0) = \lim_{\Delta x\to 0}\frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

$$\left(\frac{\partial f}{\partial y}\right)_{T_0} = \frac{\partial f}{\partial y}(x_0,y_0) = \lim_{\Delta y\to 0}\frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

→ brzina promjene u samo jednom smjeru varijable

LJIR 22) pomoću $\frac{\partial f}{\partial y}$ za $f(x,y)=\frac{x}{y}$

$$\cancel{\left(\frac{\partial f}{\partial y}\right)_{T_0}} = \frac{\partial f}{\partial y} = \lim_{\Delta y\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y} = \lim_{\Delta y\to 0}\frac{\frac{x}{y+\Delta y}-\frac{x}{y}}{\Delta y}$$

$$\lim_{\Delta y\to 0}\frac{\frac{xy - x(y+\Delta y)}{y(y+\Delta y)}}{\Delta y} = \lim_{\Delta y\to 0}\frac{\frac{\cancel{xy}-\cancel{xy}-x\cancel{\Delta y}}{y(y+\Delta y)}}{\cancel{\Delta y}} = \left(\frac{-x}{y(y+\underbrace{\Delta y}_{0})}\right) = \frac{-x}{y^2}$$

MI 22 $f(x,y)=\frac{e^{x+2y}-1}{y}$

$$\frac{\partial f}{\partial x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\frac{e^{x+\Delta x+2y}-1}{y}-\frac{e^{x+2y}-1}{y}}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{e^{x+\Delta x+2y}-e^{x+2y}}{y\cdot\Delta x} = \lim_{\Delta x\to 0}\frac{e^{x+2y}}{y}\underbrace{\boxed{\frac{e^{\Delta x}-1}{\Delta x}}}_{1} = \frac{e^{x+2y}}{y}$$

ako deriviram po x, y tretiramo kao konstantu

$\hookrightarrow (4y^2)' = 0$

Dakle: $f(x,y) = x^3 + 4y^2 + 2x^3y^4 + \pi$

$$\frac{\partial f}{\partial x} = 3x^2 + 0 + 6x^2 \cdot y^4 + 0$$

$$\frac{\partial f}{\partial y} = 0 + 8y + 8y^3x^3 + 0$$

Pr :) $f(x,y) = \dfrac{e^{x+2y} - 1}{y}$

$$\frac{\partial f}{\partial y} = \frac{(e^{x+2y}-1)'y - (e^{x+2y}-1)y'}{y^2} = \frac{(e^{x+2y}\cdot 2 - 0)y - e^{x+2y} + 1}{y^2}$$

$$= \frac{e^{x+2y}\cdot 2y - e^{x+2y} + 1}{y^2}$$

Parcijalne derivacije višeg reda

ZAD: $f(x,y) = 4x^2e^{3y}$

$\frac{\partial f}{\partial x} = 8xe^{3y}$ — po x (po x, po y), po y (po x, po y)

$\frac{\partial f}{\partial y} = 12x^2e^{3y}$ — po x, po y ...

... 2^n

I. deriv. → 2 II. deriv → 4 III. deriv → 8

$$\left(\frac{\partial f}{\partial x}\right)\left(\frac{\partial}{\partial x}\right) = \frac{\partial^2 f}{(\partial x)^2} = \frac{\partial^2 f}{\partial x^2} = 8e^{3y}$$

operator

prvi put i drugi put po x. i po x

mješovite parcijalne derivacije

$$\frac{\partial^2 f}{\partial y \partial x} = 24xe^{3y}$$

po x i po y → f''_{yx}

$$\frac{\partial^2 f}{\partial x \partial y} = 24xe^{3y}$$

po y i po x

→ sve mješ. parc. deriv. su uvijek iste

⇒ NIJE BITAN REDOSLIJED DERIVIRANJA

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = 36x^2e^{3y}$$

po y i po y

TM Schwarzov teorem – nije bitan redoslijed deriviranja

* nije jednostavno za dokazat

· neka postoje parc. der. $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial x \partial y}$, $\frac{\partial f}{\partial y \partial x}$

i neka su mješovite neprekinute

Tada: $\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}$ (vrijedi i općenitije)

→ za dokaz: upiši PMF

Primjer: $f'''_{xxy} = 24e^{3y} = f'''_{yxx} = f'''_{xyx}$

Primjer) $u(x,y,z) = x^z + y\sin z$, nađite sve deriv. drugog reda

$\frac{\partial u}{\partial x} = z \cdot x^{z-1}$ $\qquad \frac{\partial u}{\partial y} = \sin z$ $\qquad \frac{\partial u}{\partial z} = x^z \cdot \ln x + y\cos z$

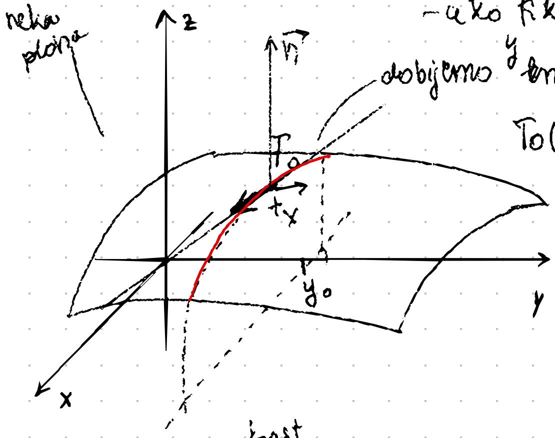
$u''_{xx} = z(z-1)\,x^{z-2}$ $\qquad u''_{xy} = u''_{yx}$

$u''_{xz} = \underset{(\text{po } z)}{x^{z-1} + z \cdot x^{z-1} \cdot \ln x} = \underset{(\text{po } x)}{z\,x^{z-1}\ln x + x^z \cdot \frac{1}{x}}$ ✓

$u''_{yz} = \cos z = 0 + \cos z$ ✓

$u''_{zz} = x^z \cdot \ln^2 x - y\sin z$

Geometrijska interpretacija



– ako fiksiramo y – presječemo plohu sa ravninom y

dobijemo krivulju na presjeku $y = y_0$

$T_0(x_0, y_0, z_0)$

– tangenta t_x se dobije presjekom s y_0 ?

→ presjecanjem po y_0 dobijemo

$z = f(x, y_0)$

$\vec{r}_x[x] = (x, y_0, f(x, y_0))$

$\vec{t}_x = \vec{r}_x{}'[x_0] = (1, 0, \frac{\partial f}{\partial x}(x_0, y_0))$

po x raste

Parcijalna der. po x je z komponenta tangente na presječnicu?

– linija na presjeku $x = x_0$ → $\vec{r}_y[y] = (x_0, y, f(x_0, y))$

$\vec{t}_y = r_y'[y] = (0, 1, \frac{\partial f}{\partial y}(x_0, y_0))$

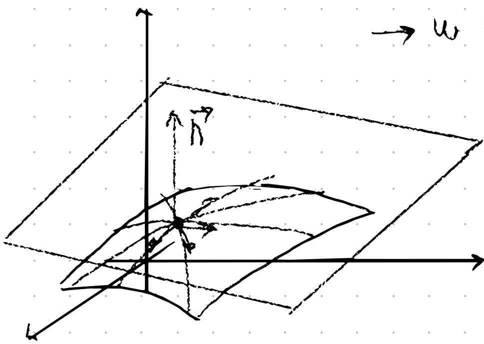
⟹ da bismo dobili jednadžbu ravnine trebamo $\vec{n}$

$$\vec{n} = \vec{t}_x \times \vec{t}_y = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & \frac{\partial f}{\partial x}(x_0, y_0) \\ 0 & 1 & \frac{\partial f}{\partial y}(x_0, y_0) \end{vmatrix} = \Big(\underset{A}{-\frac{\partial f}{\partial x}}, \underset{B}{-\frac{\partial f}{\partial y}}, \underset{C}{1} \Big) \,/\cdot(-1)$$

jednadžba ravnine: $A(x - x_0) + B(y - y_0) - (z - z_0) = 0$

$$\Longrightarrow z - z_0 = \frac{\partial f}{\partial x}(T_0)(x - x_0) + \frac{\partial f}{\partial x}(T_0)(y - y_0)$$

$\boxed{\text{DEF}}$ Tangencijalna ravnina na plohu S u točki T_0 je ravnina u kojoj leže tangente u točki T_0 svih glatkih krivulja koji leže na plohi i prolaze točkom T_0.



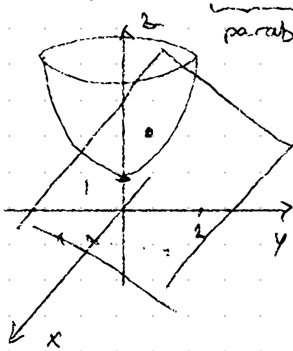
→ u tang. ravnini leže tangente svih glatkih krivulja na plohu u T_0

Jednadžba tangencijalne ravnine (AKO POSTOJI)!

$$z - z_0 = \left(\frac{\partial f}{\partial x}\right)_{T_0}(x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{T_0}(y - y_0)$$

$$n \ldots \quad \frac{x - x_0}{\left(\frac{\partial f}{\partial x}\right)_{T_0}} = \frac{y - y_0}{\left(\frac{\partial f}{\partial y}\right)_{T_0}} = \frac{z - z_0}{-1}$$

Zad.) $z = \underbrace{1+3x^2}_{\text{parabola}}\overbrace{+2y^2}^{\text{parabola}}$ u $T(\overset{x_0}{1},\overset{y_0}{2})$



$\Rightarrow$ paraboloid

$\rightarrow$ elipsa $\longrightarrow$ eliptični par.

$$\frac{\partial f}{\partial x} = 6\underset{x_0}{x} \qquad \frac{\partial f}{\partial y} = 4\underset{y_0}{y} \qquad z_0 = 12$$

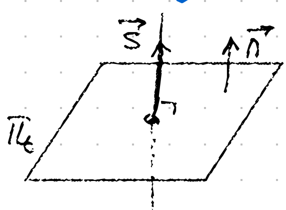
$$z - 12 = 6(x - \overset{x_0}{1}) + 8(y - \overset{y_0}{2})$$

$$z = 6x + 8y - 10$$

$$\vec{n} = (6, 8, -1)$$

WIR-23-1c) a i b Sami rješiti

$$z = x^2 y$$



$$\vec{n} = \lambda \vec{s} = \lambda(-6, 8, -2)$$ ono u $z = t$

$$(2xy,\ x^2,\ \underline{-1})$$

$$\frac{\partial f}{\partial x} = 2xy \qquad \frac{\partial f}{\partial y} = x^2$$

$$-1 = -2\lambda \qquad \lambda = \frac{1}{2}$$

$$2xy = -3 \rightarrow xy = \frac{-3}{2}$$

$$x^2 = 4$$

$$x = \pm 2$$

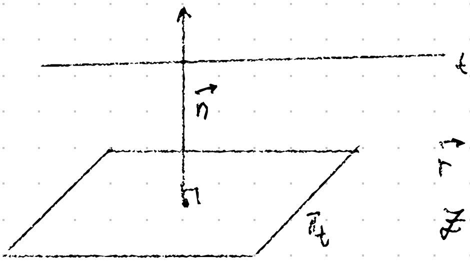
$$y = \frac{-3}{4},\ x = +2 \qquad y = \frac{3}{4},\ x = -2$$

$T_1\left(2, \frac{-3}{4}\right)$ $z_1 = -3$ $\qquad$ $T_2\left(-2, \frac{3}{4}\right)$ $z_2 = 3$

$$\pi_{t_1} \ldots\ z - (-3) = (-3)(x - 2) + 4\left(y + \frac{3}{4}\right)$$

$$\pi_{t_2} \ldots\ z - 3 = -3(x + 2) + 4\left(y - \frac{3}{4}\right)$$

MI -23 - 1b) → možda ove godine



$\vec{n} \perp \vec{S}_t$

$\vec{n} \cdot \vec{S}_t = 0$

$\vec{r}\,[t] = (t^2, 3\ln t, 8t)$

$z = x^2 - 2y^2$ u $T(1,1,-1)$

$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = -4y \qquad \overset{z}{-1} \rightarrow \vec{n} = (2,-4,-1)$

$t \ldots \vec{r}\,'[t] = \left(2t, \frac{3}{t}, 8\right) = \vec{S}_t$

$\Rightarrow \quad 4t - \frac{12}{t} - 8 = 0 \quad / \cdot t$

$4t^2 - 12 - 8t = 0$

$t_{1,2} = \frac{8 \pm 16}{8}$ → $t_1 = 3$, $\cancel{t_2 = -1}$

ovo ne možemo uvrstiti u ln

LJIR-21-1c) $z = f(x,y) \quad T(\overset{x}{0}, \overset{y}{0}, \overset{z}{4})$

$z = f(x,y) = \frac{1}{x} + \frac{1}{y}$

$\frac{\partial f}{\partial x} = -\frac{1}{x^2} \qquad \frac{\partial f}{\partial y} = \frac{-1}{y^2}$

$z - z_0 = \frac{\partial f}{\partial x}(T_0)(x - x_0) + \frac{\partial f}{\partial x}(T_0)(y - y_0)$

$z - z_0 = \frac{-1}{x_0^2}(x - x_0) - \frac{1}{y_0^2}(y - y_0)$

$4 - z_0 = \underbrace{\frac{1}{x_0} + \frac{1}{y_0}}_{z_0}$

$4 = 2z_0 \rightarrow \boxed{z_0 = 2}$

ta ravnina je $\underline{\underline{z_0 = 2}}$