7.3. HLDJ n-tog REDA S KONST. KOEF.

- gledamo: $y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0$ $\quad a_i \in \mathbb{R}$

znamo da je rj: $y_h = C_1 y_1 + \dots + C_n y_n$

Svakoj HLDJ s kk je pridružen karakteristični polinom

$P_n(r) = r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0$

Primijeti da vrijedi $L(e^{rx}) = 0 \iff P(r) = 0$

jer $L(e^{rx}) = r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \dots + a_1 r e^{rx} + a_0 e^{rx} = 0$

$$0 \neq e^{rx} \underbrace{(r^n + a_{n-1} r^{n-1} + \dots + a_1 r + a_0)}_{0} = 0$$

- jedina dif jed bez integriranja

1. slučaj: nultočke su realne i različite $r_1, \dots r_n$

$\Rightarrow$ tada su $e^{r_1 x}, \dots e^{r_n x}$ rješenja HLDJ (to su lin. nez. $\Rightarrow$ prošli put dokazali)

pa je rješenje $y_h = C_1 e^{r_1 x} + \dots + C_n e^{r_n x}$

Zadatak)

$y''' - 2y'' - 3y' = 0$

$r^3 - 2r^2 - 3r = 0$

$r(r^2 - 2r - 3) = 0$

$r_1 = 0 \quad r_2 = -1 \quad r_3 = 3$

$y_h = C_1 e^{0x} + C_2 e^{-x} + C_3 e^{3x}$

$$\boxed{y_h = C_1 + C_2 e^{-x} + C_3 e^{3x}}$$

2. slučaj: ako je r_1 nultočka kratnosti k (r_1 je k puta nultočka od $P_n(r)$)

$\Rightarrow$ tada je $y_h = C_1 e^{r_1 x} + C_2 x e^{r_1 x} + \dots + C_k x^{k-1} e^{r_1 x}$

Zadatak)

$y'' + 4y' + 4y = 0$

$r^2 + 4r + 4 = 0$

$r_{1,2} = -2$

$$\boxed{y_h = C_1 e^{-2x} + C_2 e^{-2x}}$$

//

3. slučaj. Ako je y kompleksno rješenje od $Ly=0$:

$L(Re\,y + i\,Im\,y) = \underbrace{L(Re_y)}_{=0} + i\underbrace{L(Im_y)}_{=0} = 0$ Dakle ako je $r_{1,2} = \alpha \pm i\beta$

nultočke polinoma tada $e^{(\alpha \pm i\beta)x} = e^{\alpha x} \cdot e^{\pm i\beta x} = e^{\alpha x}[\cos(\pm\beta x) + i\sin(\pm\beta x)]$

$\Longrightarrow e^{(\alpha \pm i\beta)x} = \underbrace{e^{\alpha x}\cos(\beta x)} \pm i\underbrace{e^{\alpha x}\sin(\beta x)}$

$\Rightarrow \boxed{y_h = c_1 e^{\alpha x}\cos\beta x + c_2 e^{\alpha x}\sin\beta x}$

Zadatak)

$y'' + 2y' + 3y = 0$

$r^2 + 2r + 3 = 0$

$r_{1,2} = \frac{-2 \pm 2\sqrt{2}i}{2}$

$r_{1,2} = \underbrace{-1}_{\alpha} \pm \underbrace{\sqrt{2}}_{\beta} i$

$\boxed{y_h = C_1 e^{-x}\cos\sqrt{2}x + C_2 e^{-x}\sin\sqrt{2}x}$

Zadatak) $y^{(IV)} - 16y = 0$

$r^4 - 16 = 0$ $\quad r_1 = 2 \quad r_2 = -2 \quad r_{3,4} = \pm 2i$

$(r-2)(r+2)(r^2+4) = 0$

$\boxed{y_h = C_1 e^{2x} + C_2 e^{-2x} + C_3\cos 2x + C_4\sin 2x}$

Zadatak: 21-2021-5) $y''' + 3y'' + 3y' + y = 0$

$y(0) = 3$

$y'(0) = 1$

$y''(0) = 4$

$r^3 + 3r^2 + 3r + 1 = 0$

$(r+1)^3 = 0 \longrightarrow r_1 = -1$ 3 puta

nultočka kratnosti 3

$\Longrightarrow y_h = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}$ opće rješenje

$3 = C1$

$y_h' = -c_1 e^{-x} + c_2(e^{-x} - xe^{-x}) + c_3(2xe^{-x} - x^2 e^{-x})$

$1 = -3 + c_2 + 0 \Rightarrow \boxed{c_2 = 4}$

$y_h'' = c_1 e^{-x} + c_2(-e^{x} - (\cdots$ $\quad c_3 = \frac{9}{2}$

Zadatak ZI-19-6)

a) $y_1(x)=1$ $\quad y_2(x)=x$, $\quad y_3(x)=\cos(3x)$ $\quad y_4=\sin(3x)$

$$W=\begin{vmatrix} 1 & x & \sin 3x & \cos 3x \\ 0 & 1 & 3\cos 3x & -3\sin 3x \\ 0 & 0 & -9\sin 3x & -9\cos 3x \\ 0 & 0 & -27\cos 3x & 27\sin 3x \end{vmatrix} = -9\cdot 27 \neq 0$$

Funkcije su lin nezavisne

b) * imamo, rješenje ali želimo dobiti HLDJ

- trebamo nultočke

$y_1=1$ $\quad y_2=x\,e^{rx}$ $\quad rx\to 0$ $\quad \to r_{1,2}=0$, kratnost 2

$\sin 3x, \cos 3x \longrightarrow \alpha=0,\ \beta=3 \qquad r_{3,4}=\pm 3i$

$\Rightarrow (r-\cancel{r_1}^{\,0})(r-\cancel{r_2}^{\,0})(r-r_3)(r-r_4)=0$

$r^2(r-3i)(r+3i)=0 \longrightarrow r^2(r^2+9)=0$

$r^4+r^2 9=0$

$\Rightarrow$ HLDJ $\Rightarrow$ $\boxed{y^{IV}+y''9=0}$

Zadatak JIR-2021-8)

$y_1=xe^{3x}$ $\qquad y_2=\sin 3x$ $\qquad$ uvijek dolaze u paru $\Rightarrow$ drugo rješenje je $\cos 3x$

dobivamo se nešto ponavlja

$y_3=e^{3x}$

$r_{1,2}=3$ $\qquad r_{3,4}=\pm 3i$

opće rješenje:

$y=C_1e^{2x}+C_2xe^{3x}+C_3\sin 3x+C_4\cos 3x$

$\Rightarrow (r-3)^2(r^2+9) \to r^4-6r^3+18r^2-54r+81\,0$

*nije isto!

$\longrightarrow \boxed{y^{IV}-6y'''+18y''-54y'+81y=0}$

VIR 2021 - 8 - b)

Dokažite da je bilokoja lin. kombinacija $a_1 y_1 + a_2 y_2$ dvaju rješenja jednadžbe y_1 i y_2

b) $Ly = 0$

$$L(\alpha_1 y_1 + \alpha_2 y_2) = L(\alpha_1 y_1) + L(\alpha_2 y_2)$$

$$= \alpha_1 L y_1 + \alpha_2 L y_2$$