1.4. INVERZNA LAPLACEOVA TRANSFORMACIJA

Primjer:

a) $\frac{1}{s-2} \;\bullet\!-\!\circ\; e^{2t}$

$* \; \frac{1}{s^3} \;\bullet\!-\!\circ\; \frac{t^2}{2}$

b) $\frac{1}{(s-2)^3} \;\bullet\!-\!\circ\; \frac{t^2}{2} \cdot e^{2t}$

Primjer a) $\frac{s-3}{(s-1)(s-2)}$ b) $\frac{1}{s^2+5}$ c) $\frac{1}{s^2+4s+6}$ d) $\frac{2s-2}{s^2+4s+6}$

a) $\frac{s-3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2} \;\Big/ (s-1)(s-2)$

$s-3 = A(s-2) + B(s-1) = As + Bs - 2A - B$

$A+B=1 \qquad -2A - B = -3 \rightarrow B = 3-2A$

$A + 3 - 2A = 1 \longrightarrow \boxed{A=2} \qquad \boxed{B=-1}$

$\frac{s-3}{(s-1)(s-2)} = \frac{2}{s-1} - \frac{1}{s-2} \;\bullet\!-\!\circ\; \boxed{2e^{t} - e^{2t}}$

b) $\frac{1}{\sqrt{5}} \cdot \frac{\sqrt{5}}{s^2+5} \;\bullet\!-\!\circ\; \boxed{\frac{1}{\sqrt{5}} \cdot \sin\sqrt{5}t}$

c) $\frac{1}{s^2+4s+4+2} = \frac{1}{(s+2)^2+2} \;\bullet\!-\!\circ\; \boxed{\frac{1}{\sqrt{2}} \sin(\sqrt{2}t) \cdot e^{-2t}}$

d) $\frac{2s-2}{s^2+4s+6} = \frac{2(s-1)}{(s+2)^2+2} = 2\frac{(s+2)}{(s+2)^2+2} - \frac{6}{(s+2)^2+2} = 2\frac{s+2}{(s+2)^2+2} - \frac{1}{\sqrt{2}}\frac{6\sqrt{2}}{(s+2)^2+2}$

$\rightarrow 2\frac{s+2}{(s+2)^2+2} - \frac{1}{\sqrt{2}} \cdot \frac{6\sqrt{2}}{(s+2)^2+2} \;\bullet\!-\!\circ\; \boxed{2\cos(\sqrt{2}t) \cdot e^{-2t} - \frac{6}{\sqrt{2}} \sin(\sqrt{2}t) e^{-2t}}$

Napomena: $\frac{1}{(x^2+1)^3 (x-1)^2 x}$

$= \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} + \frac{Ex+F}{(x^2+1)^3} + \frac{G}{x-1} + \frac{H}{(x-1)^2} + \frac{I}{x}$

1.5 KONVOLUCIJA FUNKCIJA

$f_1(t) \circ\!\!-\!\!\bullet F_1(s)$

$f_2(t) \circ\!\!-\!\!\bullet F_2(s)$

$? \circ\!\!-\!\!\bullet F_1(s) \cdot F_2(s)$

DEF: $$\boxed{(f_1 * f_2)(t) = \int_0^{\infty} f_1(t) f_2(t-\tau)\, d\tau}$$

vrijedi komutativnost i asocijativnost !!

f_1, f_2 originali

$f_1(\tau) = 0$ za $\tau < 0$

$f_2(t-\tau) = 0$ za $t - \tau < 0$

$\tau > t$

$$(f_1 * f_2)(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, dt$$

$$\mathcal{L}((f_1 * f_2)(t)) = \int_{-\infty}^{\infty} e^{-st} \left(\int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau \right) dt$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-st} e^{-s\tau} e^{s\tau} f_1(\tau) f_2(t-\tau)\, dt\, d\tau$$

$$\int_{-\infty}^{\infty} e^{-st} f_1(t)\, d\tau \int_{-\infty}^{\infty} e^{-s(t-\tau)} f_2(t-\tau)\, dt = \left| \begin{matrix} u = t - \tau \\ du = dt \end{matrix} \right)$$

$$\boxed{= \int_{-\infty}^{\infty} e^{-st} f_1(t)\, d\tau \cdot \int_{-\infty}^{\infty} e^{-su} f_2(u)\, dt}$$

$\forall\, (f_1 * f_2)(t) \circ\!\!-\!\!\bullet F_1(s) \cdot F_2(s)$

Primjer: $\frac{1}{s(s^2+1)}$

$F_1(s) = \frac{1}{s^2+1} \bullet\!\!-\!\!\circ \sin t = f_1(t)$

$F_2(s) = \frac{1}{s} \;-\!\!\circ\; 1 = f_2(t) = u(t)$

$(f_1 * f_2)(t) = \int_0^{\tau} \sin \tau \cdot 1\, d\tau$

$\Downarrow$

$(\sin t * u(t))$

$\longrightarrow = -\cos \tau \Big|_0^t = -(\cos t - 1) = (1 - \cos t)\, u(t)$

Primjer: $\frac{1}{(s^2+1)^2} = \frac{1}{s^2+1} \cdot \frac{1}{s^2+1} \longrightarrow \sin(t) * \sin(t)$

$$\boxed{\sin(mx) \cdot \sin(ux) = \frac{1}{2}\left(\cos(m-u)x - \cos(m+u)x\right)} \; !$$

$\longleftrightarrow \sin t * \sin t = \int_0^t \sin \tau \cdot \sin(t-\tau)\, d\tau = \frac{1}{2} \int_0^t \cos(\tau - t + \tau) - \cos(\tau + t - \tau)\, d\tau$

$= \frac{1}{2} \int_0^t \left(\cos(2\tau - t) - \underbrace{\cos t}\right) d\tau$

to je cos od konstante jer integriramo po τ !

$= \frac{1}{2}\left(\frac{1}{2} \sin(2\tau - 1) - \tau \cdot \cos t \right) \Big|_0^t$

DEF: $F(s)=\int_0^\infty e^{-st} f(t)\,dt$, $\forall s$ za koji vrijedi

$1 \circ\!-\!\bullet \frac{1}{s}$ $\quad t \circ\!-\!\bullet \frac{1}{s^2}$ $\quad t^n \circ\!-\!\bullet \frac{n!}{s^{n+1}}$ $\quad e^{\alpha t} \circ\!-\!\bullet \frac{1}{s-\alpha}$

$\sin \omega t \circ\!-\!\bullet \frac{\omega}{s^2+\omega^2}$ $\quad \cos \omega t \circ\!-\!\bullet \frac{s}{s^2+\omega^2}$

SVOJSTVA:

$f(at) \circ\!-\!\bullet \frac{1}{a} F\left(\frac{s}{a}\right)$

$e^{-at} f(t) \circ\!-\!\bullet F(s+a)$ prigušenje

$f(t-a)\,u(t-a) \circ\!-\!\bullet e^{-as}F(s)$ pomak

$f'(t) \circ\!-\!\bullet sF(s)-f(0)$ derivacija u gornjem p.

$t f(t) \circ\!-\!\bullet -F'(s)$ deriv. u donjem području

$\frac{f(t)}{t} \circ\!-\!\bullet \int_s^\infty F(s)\,ds$ integriranje slike

$(-t^n)\, f(t) \circ\!-\!\bullet F^{n'}(s)$

Primjer: Izračunaj integral $\int_0^\infty e^{-2t}\, t\cos t\, dt = F(2)$

$f(t) = t\cos t$, $F(s) = ?$

$\cos t \to \frac{s}{s^2+1}$, $t\cos t \circ\!-\!\bullet -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\frac{s'(s^2+1)-s(s^2+1)'}{(s^2+1)^2}$

$$= -\frac{1-s^2}{(s^2+1)^2}$$

$$F(2) = \frac{4-1}{(4+1)^2} = \boxed{\frac{3}{25}}$$

Primjer: Izračunaj integral $\int_0^\infty e^{-ax}\frac{\sin x}{x}\,dx$

Primjer: $(-t^n) f(t) \circ\!-\!\bullet\ ?$

$$F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right) dt = \int_0^\infty e^{-st}(-t) f(t)\,dt$$

$$F''(s) = \frac{d}{ds}\int_0^\infty e^{-st}(-t)\, f(t)\,dt = \int_0^\infty e^{-st}(-t)(-t)\, f(t)\,dt = \int_0^\infty e^{-st}(-t)^2 f(t)\,dt$$

$$\vdots$$

$$F^n(s) = \frac{d^n}{ds^n}\int_0^\infty e^{-st} f(t)\,dt = \underline{\underline{F^{n'}(s)}}$$