3.4. LOKALNI EKSTREMI

DEF

a) $f(x,y)$ ima lokalni min. u $T_0(x_0,y_0)$ ako postoji otvoreni krug (okolina) $K_\varepsilon(T_0)$ t.d. $f(x,y) \geq f(x_0,y_0)$, $\forall (x,y) \in K_\varepsilon$.

b) $f(x,y)$ ima lokalni MAX u T_0 ako postoji otvoreni krug $K_\varepsilon(T_0)$ t.d. $f(x,y) \leq f(x_0,y_0)$, $\forall (x,y) \in K_\varepsilon$.

TM Fermatov teorem = nužan uvjet za lok. ekstreme

Ako dif. $f(x,y)$, ima lok. ekstreme u T_0, tada $\nabla f(T_0) = \vec{0}$.

(tj. $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$)

DOKAZ:

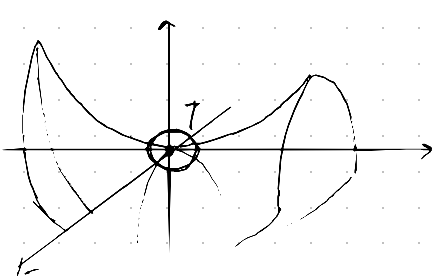
Definirajmo $f: \mathbb{R} \to \mathbb{R}$ kao restrikciju funkcije na $y = y_0$ (fiksirali smo y_0 u f-ji) kroz T_0, tj. $f_1(x) = f(x, y_0)$. Po pretpostavci $f_1(x)$ ima lok. ekstrem u x_0, pa mogu koristiti Fermatov teorem za jednu varijablu $\longrightarrow f_1'(x_0) = 0 \Rightarrow \frac{\partial f}{\partial x}(x_0, y_0) = 0$.

*Nap. IMPLIKACIJA: obrat ne vrijedi

ako je $\nabla f = \vec{0}$ u T_0, T_0 ne mora biti ekstrem.

Odnosno $\nabla f(T) = \vec{0}$ daje kandidate za ekstreme (stacionarne točke)

Pr. Sedlo $\rightarrow z = x^2 - y^2$, $\nabla z = (2x, -2y) = \vec{0} \Rightarrow T(0,0)$ stacionarna točka, ali nije ekstrem

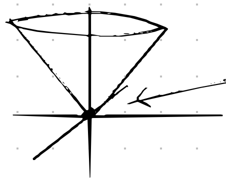


$\Rightarrow$ sedlasta točka

2\. NAP: Ako f nije diferencijabilna (∇f ne postoji)

$\Longrightarrow$ ali možemo imati ekstrem.

Pr. $z = \sqrt{x^2+y^2}$



nije dif, ali ima lok min!

Drugi diferencijal fije $f(x,y)$ je kvad. forma:

$$d^2 f = \underbrace{f''_{xx}}_{a}(dx)^2 + 2\underbrace{f''_{xy}}_{b}\,dx\,dy + \underbrace{f''_{yy}}_{c}(dy)^2 \Rightarrow H_f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix}$$

$\longrightarrow$ tu pridruženu matricu nazivamo Hesseova matrica

$\boxed{TM}$ Dovoljan uvjet za lokalni ekstrem funkcije dvije varijable.

Neka je f dvaput neprekinuto diferenc. te neka je T_0 stacionarna točka ($\nabla f = \vec{0}$). Tada:

a) ako je $d^2 f(T_0) > 0$ (kvad. forma poz definitivna) tada je T_0 strogi lok. min.

b) ako je $d^2 f < 0$, tada je T_0 strogi lok. max

c) ako je $d^2 f$ $>/<$, indefinitna kvad. forma $\Rightarrow$ SEDLO, nije točka ekstrema

* ovo su dovoljni uvjeti - stac. točka može biti ekstrem ili sedlasta točka bez ispunjavanja gornjih uvjeta

DOKAZ: Koristimo Taylorovu formulu oko T_0 $\longrightarrow$ $T_0(x_0, y_0)$

* (prvog stupnja jer je ostatak drugog stupnja) : $f(x,y) = T_1(x,y) + R_1(x,y)$

$$f(x,y) = f(x_0,y_0) + \left[\left(\frac{\partial f}{\partial x}\right)\Big|_{T_0} \underbrace{(x-x_0)}_{\Delta x} + \left(\frac{\partial f}{\partial y}\right)\Big|_{T_0} \underbrace{(y-y_0)}_{\Delta y} \right] +$$

$$+ \frac{1}{2!}\left[f''_{xx}(T_c)(x-x_0)^2 + 2 f''_{xy}(T_c)(x-x_0)(y-y_0) + f''_{yy}(T_c)(y-y_0)^2 \right]$$

T_c se nalazi na spojnici točaka $T_0(x_0,y_0)$ i $T(x,y)$

je isto što i $df(T_0)(\Delta x, \Delta y)$

je isto što i $d^2 f(\Delta x, \Delta y)$

Budući da je $T_0(x_0, y_0)$ stacionarna $\longrightarrow$ $\frac{\partial f}{\partial x}\Big|_{T_0} = (f'_x)_0 = 0$

$\frac{\partial f}{\partial y}\Big|_{T_0} = (f'_y)_0 = 0$ pa prema tome taj dio možemo izbrisati

$$\boxed{\Rightarrow f(x,y) = f(x_0,y_0) + \frac{1}{2!} \cdot d^2 f(\Delta x, \Delta y)}$$

Ako je $d^2 f > 0$ $\Rightarrow$ $f(x,y) > f(T_0)$ $\Rightarrow$ prema tome, to je lokalni minimum

Zbog uvjeta neprekinutosti predznak drugog dif. je isti u T_0 i T_c.

Obrat ne vrijede, a $d^2 f = 0$ ne daje odluku

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Pr.) c) $z = 3x^2 - 2y^2$

$\nabla z\ (6x, -4y)$

$6x = 0 \qquad -4y = 0$

$x = 0 \qquad y = 0$

1. stac. točka je $T_1(0,0)$

2. sve druge parc.:

$$\frac{\partial^2 f}{\partial x^2} = 6 \qquad \frac{\partial^2 f}{\partial y^2} = -4 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

formula za $d^2 f(x,y) = \frac{\partial^2 f}{\partial x^2}(\Delta x)^2 + 2\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2}(\Delta y)^2$

$\Downarrow$ 0

$\rightarrow d^2 f(0,0) = 6 \cdot (dx)^2 + 0 - 4 \cdot (dy)^2$

$\Rightarrow$ indiferentna kvadratna forma jer kad je $(dx, dy) = (0,1)$ onda je < 0, a kad je $(dx, dy) = (1,0)$ je > 0.

3. $\Rightarrow$ Točka $T(0,0)$ nije točka ekstrema nego sedlasta.

Pr.)

a) $z = 2 + 3x^2 + 4y^2$

1. $\nabla z = (6x, 8y) = (0,0)$

$\rightarrow x = 0,\ y = 0 \rightarrow T(0,0)$

2. sve druge parc.

$$\frac{\partial^2 f}{\partial x^2} = 6 \qquad \frac{\partial^2 f}{\partial x \partial y} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = 8$$

$\Rightarrow d^2 f(0,0) = 6 \cdot (dx)^2 + 8(dy)^2$

$\hookrightarrow d^2 f(0,0)$ je uvijek veći od 0

$\hookrightarrow$ Lokalni minimum

3. u kojoj točki je lok. min

$z = 2 + 3 \cdot 0 + 4 \cdot 0 = 2$

$\Downarrow$

$M = (0,0,2)$

b) $z = 2x^2 - 3y^2$ $\qquad \frac{\partial^2 f}{\partial x^2} = 4 \quad \frac{\partial^2 f}{\partial y^2} = -6 \quad \frac{\partial^2 f}{\partial x \partial y} = 0$

$\nabla z = (4x, -6y) \rightarrow T(0,0)$

$\hookrightarrow d^2 f(0,0) = 4(dx)^2 - 6(dy)^2 < 0$ za $(dx, dy) = (0,1)$

> 0 za $(dx, dy) = (1,0)$

$\hookrightarrow$ indefinitna, sedlasta točka

$\boxed{\text{TM}}$ Sylvesterov dovoljan uvjet za kvadratne forme

*(Dovoljan uvjet za ekstrem pomoću Hesseove matrice)

Neka je $T_0(x_0, y_0)$ stacionarna točka, te neka je H_f Hesseova mat. od f u T_0:

$$H_f(T_0) = \begin{bmatrix} f''_{xx}(T_0) & f''_{xy}(T_0) \\ f''_{yx}(T_0) & f''_{yy}(T_0) \end{bmatrix}$$

Tada:

a) $(f''_{xx})_0 > 0$, $\det H_f(T_0) > 0 \rightarrow f$ u T_0 ima **strogi lok. min**

b) $(f''_{xx})_0 < 0$, $\det H_f(T_0) > 0 \rightarrow f$ u T_0 ima **strogi lok. max**

c) $\det H_f(T_0) < 0 \Rightarrow$ **sedlasta točka** od f (nije točka ekstrema)

*ako je $\det H_f(T_0) = 0 \rightarrow$ ovaj teorem ne daje odluku

DOKAZ: spoji prethodna dva TM

M1-19

4. b) $f(x,y) = x^2y + 2xy^2 + \frac{1}{2}xy$

$\nabla f = \left(2xy + 2y^2 + \frac{1}{2}y,\ x^2 + 4xy + \frac{1}{2}x\right)$

$2xy + 2y^2 + \frac{1}{2}y = 0 \qquad x^2 + 4xy + \frac{1}{2}x = 0$

$y\left(2x + 2y + \frac{1}{2}\right) = 0$

$y = 0 \qquad 2x + 2y + \frac{1}{2} = 0$

$2y = -2x - \frac{1}{2}$

$y = -x - \frac{1}{4}$

$x^2 + 4x\left(-x - \frac{1}{4}\right) + \frac{1}{2}x = 0$

$x^2 - 4x^2 - x + \frac{1}{2}x = 0$

$-3x^2 - \frac{1}{2}x = 0$

$-x\left(3x + \frac{1}{2}\right) = 0$

$x = 0 \qquad x = \frac{-1}{6}$

$y = -\frac{1}{4} \qquad y = \frac{-1}{12}$

$T_3\left(0, \frac{-1}{4}\right) \qquad T_4\left(\frac{-1}{6}, \frac{-1}{12}\right)$

ako je $y = 0$

$x^2 + \frac{1}{2}x = 0$

$x = 0 \qquad x = -\frac{1}{2}$

$y = 0 \qquad y = 0$

$T_1(0,0) \qquad T_2\left(-\frac{1}{2}, 0\right)$

$f''_{xx} = 2y$

$f''_{yy} = 4x$

$f''_{xy} = 2x + 4y + \frac{1}{2}$

$H_f = \begin{bmatrix} 2y & 2x+4y+\frac{1}{2} \\ 2x+4y+\frac{1}{2} & 4x \end{bmatrix} \rightarrow \det H_f = 8xy - \left(2x + 4y + \frac{1}{2}\right)^2$

$\det H_f(T_1) = \begin{vmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4} < 0 \Rightarrow$ SEDLO

$\det H_f(T_2) = \begin{vmatrix} 0 & -1+\frac{1}{2} \\ -1+\frac{1}{2} & -2 \end{vmatrix} = \frac{1}{4} < 0$ sedlo

$H_f(T_3) = \begin{vmatrix} -\frac{1}{2} & -1+\frac{1}{2} \\ -1+\frac{1}{2} & 0 \end{vmatrix} = -\frac{1}{4}$ sedlo

$\det H_f(T_4) = \begin{vmatrix} -\frac{1}{6} & \frac{-1}{3} - \frac{1}{3} + \frac{1}{2} \\ \frac{-2}{3} + \frac{1}{2} & \frac{-2}{3} \end{vmatrix} = \frac{2}{9} - \frac{1}{36} > 0$

$f''_{xx} = \frac{-1}{6} < 0 \longrightarrow$ lokalni max

Pr.) a) $z = 1 - y^2$ cilinder

stac. točka $(x_0, 0)$ $x \in \mathbb{R}$

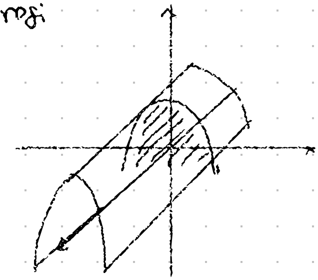
$\Downarrow$

lokalni max, ali ne strogi

$\frac{\partial z}{\partial y} = -2y = 0$ $\frac{\partial z}{\partial x} = 0$

$\downarrow$

$\hookrightarrow \underline{y=0}$



$$\text{Hesse} = \begin{bmatrix} 0 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \det = 0$$

$$d^2 f = f''_{xx}(dx)^2 + 2 \cdot f''_{xy}(dx)(dy) + f''_{yy}(dy)^2$$

$$d^2 f = 0 + 0 - 2(dy)^2 \Rightarrow -2(dy)^2 \leq 0$$

$\hookrightarrow$ inače definiramo za sve osim $(0,0)$

b) $z = x^4 + y^2$

$T(0,0)$

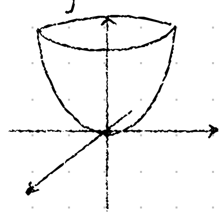
$\frac{\partial z}{\partial x} = 4x^3 = 0$ $\underline{\underline{x=0}}$

$\frac{\partial z}{\partial y} = 2y = 0$ $\underline{\underline{y=0}}$

$$\text{Hesse} = \begin{bmatrix} 12x^2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow H_z = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \det = 0$$

$\Downarrow$

nema odluke



Slučaj 3 varijable

Neka je $u = u(x,y,z)$ Drugi diferencijal je kvadratna forma oblika

$$d^2 u = u''_{xx}(dx)^2 + u''_{yy}(dy)^2 + u''_{zz}(dz)^2 + 2u''_{xy}(dx)(dy) + 2u''_{xz}(dx)(dz) + 2u''_{yz}(dy)(dz)$$

Pripada Hesseova matrica:

$$H_u = \begin{bmatrix} u''_{xx} & u''_{xy} & u''_{xz} \\ u''_{xy} & u''_{yy} & u''_{yz} \\ u''_{xz} & u''_{yz} & u''_{zz} \end{bmatrix}$$

TM Sylvesterov za 3 var   ne treba znat dokazat ovaj, ali treba za 2 var

a) Ako su sve glavne minore (determinante manjeg reda) od Hf pozitivne u T_0 $\Rightarrow$ <u>strogi lokalni minimum</u>

b) Ako glavne minore mijenjaju predznak na način da je $u_{xx} < 0$,

$$\begin{vmatrix} u_{xx} & u_{xy} \\ u_{yx} & u_{yy} \end{vmatrix} > 0, \quad \det Hf < 0 \Rightarrow \text{strogi lokalni max}$$

d) Ako je $\det Hf \neq 0$ i predznaci su različiti od a) i b) $\rightarrow$ <u>sedlo</u>

* det = 0, ne možemo ništa zaključiti

a m v
+ + + = min
− + − = max
ostalo = sedlo

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$$u(x,y,z) = x^3 + 2y^3 + z^2 + 3x^2y + 3xy^2 - 3x - 6y - 4z$$

4 b)

$$\frac{\partial u}{\partial x} = 3x^2 + 6xy - 3 = 0$$

$$\frac{\partial u}{\partial y} = 6y^2 + 3x^2 + 6xy - 6 = 0$$

$$\frac{\partial u}{\partial z} = 2z - 4 = 0$$

$$z = 2$$

$$3x^2 + 6xy - 3 = 0$$
$$3x^2 + 6xy - 6 + 6y^2 = 0 \quad / -$$

$$0 + 0 + 6 - 6y^2 = 0$$

$$y^2 = 1 \quad /\sqrt{\ }$$

$$y = \pm 1$$

$y = 1$

$$\cancel{6} + 3x^2 + 6x - \cancel{6} = 0$$

$$x^2 + 2x = 0$$

$$x(x+2) = 0$$

$$x = 0 \qquad x = -2$$

$y = -1$

$$\cancel{6} + 3x^2 - 6x - \cancel{6} = 0$$

$$x^2 - 2x = 0$$

$$x = 0 \qquad x = 2$$

$$T_3(0,-1,2) \qquad T_4(2,-1,2)$$

$$T_1(0,1,2) \qquad T_2(-2,1,2)$$

$f''_{xx} = 6x + 6y$ $\qquad f''_{xy} = 6x + 6y, \rightarrow$

$f''_{yy} = 12y + 6x$ $\qquad f''_{xz} = 0$

$f''_{zz} = 2$ $\qquad f''_{yz} = 0$

$$Hf = \begin{bmatrix} f''_{xx} & f''_{xy} & f'_{xz} \\ f'_{xy} & f''_{yy} & f''_{yz} \\ f''_{zx} & f_{yz} & f''_{zz} \end{bmatrix}$$

$$Hf = \begin{bmatrix} 6x+6y & 6x+6y & 0 \\ 6x+6y & 12y+6x & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$T_1(0, 1, 2) \qquad T_2(-2, 1, 2)$

$T_3(0, -1, 2) \qquad T_4(2, -1, 2)$

$$H_f(T_1) = \begin{bmatrix} 6 & 6 & 0 \\ 6 & 12 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \cancel{6}\cdot 12 \cdot \cancel{2} - \cancel{6}\cdot 6 \cdot \cancel{2} > 0$$

$6 > 0$

$12 - 6 > 0$

$6 > 0$ ✓ T_1 je lokalni minimum

a>0 m v

$+ \quad + \quad +$ ✓

$$H_f(T_2) = \begin{bmatrix} -6 & -6 & 0 \\ -6 & -12 & 0 \\ 0 & 0 & 2 \end{bmatrix} \Rightarrow \cancel{-6}(-12)\cancel{2} - \cancel{(-6)}(-6)\cancel{2}$$

$-6 < 0$

$-12 + 6 < 0$

T_2 je lok. max ✓

a m v

$- \quad - \quad -$

$$H_f(T_3) = \begin{bmatrix} -6 & -6 & 0 \\ -6 & -12 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$ — isto ko preda $\rightarrow$

a m v

$+ \quad - \quad -$

$$H_f(T_4) = \begin{bmatrix} 6 & 6 & 0 \\ 6 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

a: $6 > 0$

mala: $-36 < 0$

velika: $2(0 - 36) < 0$

sedlasta