4. DIFERENCIJALNE JEDNADŽBE PRVOG REDA

1. Separacija varijabli $y' = f(x,y)$, $y(x) = ?$

↳ integriramo, želimo y' izolirati na jednu stranu

$$y' = f(x,y) = f_1(x) \cdot f_2(y)$$

*Cauchyjev problem: $\left\{ \begin{array}{l} y' = f(x,y) \\ y(x) = y_0 \end{array} \right\}$ tražimo konkretna rješenja

$$xy' + x^2 y^3 = 0$$

$$y' = \frac{-x^2 y^3}{x} \rightarrow \frac{dy}{dx} = -xy^3 \Big/ \cdot \frac{dx}{y^3}$$

$$\frac{dy}{y^3} = -x\,dx \quad \Big/ \int$$

$$\frac{1}{y^2} = x^2 + C$$

↳ direktna integracija

$$\Downarrow$$

opće rješenje sadržava $\underline{C}$ - konstantu $\Rightarrow$ $y^2 = \dfrac{1}{x^2+C}$

↳ ako gledamo za određeni = PARTIKULARNO

2. Linearna DJ $y' + f(x)\,y = g(x)$ MVK

(1.) $y' + f(x)y = 0$ * HOMOGENA

$$y' = -f(x)y$$

$$\frac{dy}{y} = -f(x)\,dx \Big/ \int$$

$$\ln|y| = -\int f(x)\,dx + c \Big/ e$$

$$|y| = e^{-\int f(x)dx} \cdot e^{c}$$ nova konstanta $\Rightarrow$ $y = C \cdot e^{-\int f(x)dx}$

(2.) Variramo konstantu

$$y = C(x) \cdot e^{-\int f(x)dx} \xrightarrow[\text{početnu}]{\text{uvrstiti u}} y' + f(x)y = g(x)$$

$$\left(C(x)\,e^{-\int f(x)dx}\right)' + f(x)\left(C(x) \cdot e^{-\int f(x)dx}\right) = g(x)$$

$$C'(x) \cdot e^{-\int f(x)dx} \cancel{- f(x) \cdot C(x)\,e^{-\int f(x)dx}} + \cancel{f(x)\,C(x) \cdot e^{-\int f(x)dx}} = g(x)$$

$$g(x) = C'(x)\,e^{-\int f(x)dx} \longrightarrow C'(x) = g(x) \cdot e^{\int f(x)dx} \Big/ \int$$

$$C(x) = \int g(x)\,e^{\int f(x)dx}\,dx + C$$

$\Rightarrow$ uvrstimo sve u $y = C\,e^{-\int f(x)dx}$

... → $$\boxed{y = e^{-\int f(x)dx}\left(\int g(x)\,e^{\int f(x)dx} + C\right)}$$

3. Bernoullijeva $y' + f(x)y = g(x)y^{\alpha}$

$\alpha \in \mathbb{R} \setminus \{0, 1\}$

$\downarrow$ LDJ $\quad \downarrow$ separacija

SUPSTITUCIJA: $z = y^{1-\alpha}$

$\hookrightarrow z' = (1-\alpha) y^{-\alpha} \cdot y'$

- dobije se LDJ sa z i na kraju samo treba vratiti iz supstitucije

4. Homogena $f(tx, ty) = t^{\alpha} \cdot f(x,y)$

moraju biti istog stupnja s lijeve i desne strane

$\Rightarrow P(x,y)\,dx + Q(x,y)\,dy = 0$

SUPSTITUCIJA: $z = \frac{y}{x}$

- transformacija homogene

$y' = f\left(\frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}\right) \Rightarrow$ uvedemo supstituciju $\begin{array}{l} u = x - x_0 \\ v = y - y_0 \end{array}$ (x_0, y_0) je rj. jedinstvenog sustava

Pr: $y' = \frac{x + y - 3}{x - y - 1}$

(1) $x_0 = 2 \quad y_0 = 1$

(2) supstitucija

$u = x - 2$

$v = y - 1 \rightarrow y = v - 1$

$y' = v'$

$\Rightarrow v' = \frac{u + v \,/ : u}{u - v \,/ : u}$

(3) $v' = \frac{1 + \frac{v}{u}}{1 - \frac{v}{u}}$ $\quad z = \frac{v}{u}$

$v' = z'u + z$

$z'u + z = \frac{1+z}{1-z}$

... dalje na separaciju

5. Egzaktna $P(x,y)\,dx + Q(x,y)\,dy = 0$

ako postoji $u(x,y) \longrightarrow du(x,y) = P(x,y)dx + Q(x,y)$

$\frac{\partial u}{\partial x} = P \quad \frac{\partial u}{\partial y} = Q \Rightarrow \underline{u(x,y) = c}$

- nužan uvjet egzaktnosti: $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

$\frac{\partial u}{\partial x} = P \Big/ \frac{\partial}{\partial y} \qquad \frac{\partial u}{\partial y} = Q \Big/ \frac{\partial}{\partial x}$

$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial P}{\partial y} \qquad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$ $\xrightarrow{\text{zbog Schwarzovog TM}}$ $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$

$\boxed{\text{TM}}$ Dovoljan uvjet ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, tada postoji $u(x,y)$ koji se računa po formuli: $u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x,y)\,dy + C$

POTENCIJAL:

$\frac{\partial u}{\partial x} = P(x,y) \Big/ \int_{x_0}^{x} dx \qquad \frac{\partial u}{\partial y} = Q(x,y)$

$u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + C(y) \Big/ \frac{\partial}{\partial y}$

$Q(x,y) = \int_{x_0}^{x} \frac{\partial P(x,y)}{\partial y}\,dx + C'(y)$

$\Rightarrow Q(x,y) = \int_{x_0}^{x} \frac{\partial Q}{\partial x}\,dx + C'(y)$

$\cancel{Q(x,y)} = \cancel{Q(x,y)} - Q(x_0,y) + c'(y)$

$c'(y) = Q(x_0, y) \Big/ \int_{y_0}^{y} dy$

$\Rightarrow c(y) = \int_{y_0}^{y} Q(x_0,y)\,dy + C$

$$\Rightarrow \boxed{u(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x,y)\,dy + C}$$

Eulerov multiplikator $\mu(x)$

$P(x,y)dx + Q(x,y)dy = 0 \;/ \cdot \mu(x)$

$\underbrace{\mu(x) \cdot P(x,y)}_{P} dx + \underbrace{\mu(x) Q(x,y)}_{Q} dy = 0 \longrightarrow$ znam da: $\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$

$0 \cdot \cancel{P(x,y)} + \mu(x) \cdot P_y' = \mu_x'(x) \cdot Q(x,y) + \mu(x) \cdot Q_x'$

$\mu(x)(P_y' - Q_x') = \mu'_x(x) Q(x,y) \;/ \cdot \frac{1}{\mu(x) \cdot Q(x,y)}$

$\frac{P_y' - Q_x'}{Q(x,y)} = \frac{d\mu}{dx} \frac{1}{\mu(x)} \;/ \cdot dx \;, \;/ \int$

$\int \frac{P_y' - Q_x'}{Q} dx = \int \frac{d\mu}{\mu(x)} \Rightarrow \ln|\mu(x)| = \int \frac{P_y' - Q_x'}{Q} dx$

6. Ortogonalne trajektorije

Dvije krivulje $y_1(x)$ i $y_2(x)$ su međusobno ORTOGONALNE ako u svakoj točki u kojoj se sijeku su im pripadne tangente $\perp$.

$\Rightarrow \forall x_0 \in \mathbb{R}, \; y_1(x_0) = y_2(x_0) \longrightarrow y_1'(x_0) = \frac{1}{y_2'(x_0)}$

- familija krivulja $C_2: \Phi_2(x,y,C_2) = 0$ je ortogonalna familija danoj familiji krivulja $C_1: \Phi_1(x,y,C_1) = 0$ $(C_1, C_2 \in \mathbb{R})$ ako su svake dvije krivulje c_1 i C_2 međusobno ortogonalne.

Traženje ORT.

1. Deriviramo $\Phi_1(x,y,C_1) = 0$, eliminiramo C_1, $\rightarrow F(x,y,y') = 0$
2. uvrštavamo u drugu $F_2(x,y,y') = F_1(x,y,\frac{-1}{y'}) = 0$
3. rješavamo novu jed. i njeno opće rj. je $\Phi_2(x,y,C_2) = 0$

7. Egzistencija i jedinstvenost rj.

Cauchyjev problem $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$

- rješenje je DJ $y(x)$ s neprekinutom derivacijom $y'(x)$
  - def. na nekom otvorenom intervalu: $\langle x_0-h, x_0+h\rangle$

TM Peanov teorem

$f: \mathbb{R}^2 \to \mathbb{R}$ je neprekinuta na $\square$ oko točke (x_0, y_0)

$x \in \exists$ $y \in \exists$

$D = \{(x,y) \in \mathbb{R}^2 : |x-x_0| < a, |y-y_0| < b\} \Rightarrow \langle x_0-a, x_0+a\rangle \times \langle y_0-b, y_0+b\rangle$

→ tada postoji interval na kojem CP ima barem 1 rješenje

TM Picardov teorem $f: \mathbb{R}^2 \to \mathbb{R}$ na $\square$ $D = \{(x,y) \in \mathbb{R}^2 : |x-x_0| < a, |y-y_0| < b\}$

- oko točke (x_0, y_0) i ima neka svojstva
- f je neprekinuta na D
- $\frac{\partial f}{\partial y}$ je omeđena fja na D

$\Rightarrow$ onda postoji interval $\langle x_0-h, x_0+h\rangle$ na kojem CP ima jedinstveno rješenje

DEF

1. $y(x)$ je REGULARNO rješenje ako za $\forall x_0 \in \mathbb{R}$ CP ima jedinstveno rj.
   - ako postoji $\exists x_0 \in \mathbb{R}$ u kojem CP nema rj $\Rightarrow$ NE-REGULARNO
2. $y(x)$ je SINGULARNO rješenje ako $\forall x_0 \in \mathbb{R}$ CP nema jedinstveno rj.

8. Lagrangeova DJ

– kada je y' unutar neke fje

1. SUPSTITUCIJA: $y' = p(x)$
2. derivacija po x $\left(\frac{d}{dx}\right)$

$y = \varphi(y') + \psi(y')$

- ima na ROKOVIMA!

Clairautova DJ

$y = y' \cdot x + \psi(y')$ (spec. slučaj)

SUPSTITUCIJA: $y' = p$