Zadaci i primjeri za vježbu

13. str.

Metoda supstitucije

Podsjeti se:

primitivna funkcija je $F'(x) = f(x)$

ako je $F_1(x)$ prim fja od $f(x)$, onda je $F_2(x)$ (određena na istom intervalu) također primitivna funkcija od f

takva da je $F_2(x) = F_1(x) + C \rightarrow F_2'(x) = F_1'(x) + 0$

$F_2'(x) = f(x)$

* kod neodređenih integrala

kada deriviramo desnu stranu ona je jednaka lijevoj:

$$\int f'(x)\,dx = f(x) + C$$

$(f(x)+C)' = f'(x) + 0 = f(x) \Rightarrow$ podintegralna fj ...

supstitucija

npr.

varijabla postaje t

$$\int e^{-2x}\,dx = \left| \begin{array}{l} -2x = t \\ -2\,dx = dt \\ dx = -\frac{1}{2}\,dt \end{array} \right|$$

$\rightarrow$ derivacija $(-2x)\,dx = -2\,dx$

$\rightarrow$ vraćamo u normalno

Parcijalna

$$\int u\,dv = uv - \int v\,du$$

$u = f(x)$ $\quad dv = g'(x)\,dx$

$\downarrow$

ne pojednostavljuje deriv.

$\rightarrow$ izraz koji se ne komplicira integr.

Primjeri:

17) $$\int \frac{dx}{x\ln x} = \int \left(\overbrace{\frac{1}{x}}^{(\ln x)'} \cdot \underbrace{\frac{1}{\ln x}}_{\text{ovo ne mogu}}\right) dx = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x} dx = dt \end{array} \right| \rightarrow \text{brig' uz } x$$

$$= \int \frac{dt}{t} = \left(\int \underbrace{\frac{1}{t}}_{(\ln t)'} dt \right) = \ln|t| + C \rightarrow \boxed{\ln(\ln x) + C}$$

$$\int \frac{dx}{x} = \ln|x| \ *$$

18.) $$\int \frac{\cos x}{\sin^3 x} dx = \int \left(\cos x \cdot \frac{1}{\sin^2 x} \cdot \frac{1}{\sin x} \right) dx = \left| \begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right|$$

$$\int \frac{dt}{t^3} = \frac{t^{-2}}{-2} = \underline{-2 \cdot t^{-3} = \frac{-2}{t^3} = \frac{1}{\cancel{-2}} \cdot \left(\frac{\cancel{-2}}{t^3}\right)}$$

$$\Downarrow$$

$$\int \frac{dt}{t^3} = \frac{t^{-2}}{-2} = -\frac{1}{2} \cdot \frac{1}{t^2} = \boxed{-\frac{1}{2} \cdot \frac{1}{\sin^2 x} + C}$$

19) $$\int \frac{\arcsin^2 x}{\sqrt{1-x^2}} dx = \left| \begin{array}{l} \arcsin x = t \\ \sqrt{1-x^2}\, dx = dt \end{array} \right| = \int t^2 dt = \frac{1}{3} t^3 + C$$

$$\boxed{= \frac{1}{3} \arcsin^3 x + C}$$

20.) $$\int \frac{e^x}{e^{2x}+1} dx = \left| \begin{array}{l} e^x = t \\ e^x dx = dt \end{array} \right| = \int \frac{dt}{t^2+1} = \frac{1}{1} \operatorname{arctg} \frac{t}{1} + C$$

$$t^2 + 1^2$$

$$= \operatorname{arctg}(t) + C \rightarrow \boxed{\operatorname{arctg}(e^x) + C}$$

2.1.)

$$\int x e^{-x^2} dx = \left| \begin{array}{l} -x^2 = t \\ -2x dx = dt \\ x dx = -\frac{1}{2} dt \end{array} \right| = \int (e^t)\left(-\frac{1}{2} dt\right) = -\frac{1}{2} e^t + C$$

$$\boxed{= -\frac{1}{2} e^{-x^2} + C}$$

22.)

$$\int x^2 \cos(2x^3+5) dx = \left| \begin{array}{l} 2x^3+5 = t \\ 6x\, dx = dt \\ x^2 dx = \frac{1}{6} dt \end{array} \right| = \int \cos(t)\left(\frac{1}{6} dt\right) = \frac{\sin t}{6} + C$$

$$\boxed{= \frac{1}{6} \sin(2x^3+5) + C}$$

23.)

$$\int \frac{\operatorname{arctg} x + x}{x^2+1} dx = \int \frac{\operatorname{arctg} x}{x^2+1} dx + \int \frac{x}{x^2+1} dx = \left| \begin{array}{ll} \operatorname{arctg} x = t & x^2+1 = u \\ \frac{1}{x^2+1} dx = dt & 2x\,dx = du \end{array} \right|$$

$$\int t\,dt + \int \frac{1}{2} \cdot \frac{du}{u} = \int t\,dt + \frac{1}{2}\int \frac{du}{u} = \frac{1}{2} t^2 + \frac{1}{2} \ln|u| + C =$$

$$\boxed{= \frac{1}{2} \operatorname{arctg}^2 x + \frac{1}{2} \ln|x^2+1| + C}$$

24.)

$$\int x \sqrt{3x-2}\, dx = \left| \begin{array}{l} 3x-2 = t \\ x = \frac{1}{3}(t+2) \\ dx = \frac{1}{3} dt \end{array} \right|$$

ako uvođenjem supstitucije i računanjem diferencijala u integralu i dalje ostane varijabla x, potrebno ju je izraziti preko nove varijable iz uvedene zamjene

$$\int \frac{1}{3}(t+2)\sqrt{t} \cdot \frac{1}{3} dt = \frac{1}{9} \int (t+2)\sqrt{t}\, dt = \frac{1}{9} \int \left(t^{3/2} + 2t^{\frac{1}{2}}\right) dt =$$

$$= \frac{1}{9} \left(\frac{t^{5/2}}{\frac{5}{2}} + 2 \cdot \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) = \frac{1}{9} \left(\frac{t^{\frac{5}{2}}}{\frac{5}{2}} + 2 \frac{t^{\frac{3}{2}}}{\frac{3}{2}} \right) + C$$

$$\boxed{= \frac{1}{9} \left(\frac{(3x-2)^{\frac{5}{2}}}{5/2} + 2 \cdot \frac{(3x-2)^{3/2}}{3/2} \right) + C}$$

Parcijalna integracija

$$\int u\,dv = uv - \int v\,du$$

→ kada je ispod $\int$ umnožak polinoma i trigonomet. ili ekspon. f-je

→ kada ispod $\int$ ne možemo direktno integrirati, ali možemo derivirati

u = funkcija koja se pojednostavljuje deriviranjem

dv = biramo izraz koji se ne komplicira integriranjem

Primjeri (17. str.)

31)
$$\int \ln x\,dx = \left| \begin{matrix} u=\ln x & dv=dx \\ du=\frac{1}{x}dx & v=x \end{matrix} \right| = x\ln x - \int \cancel{x}\cdot\frac{1}{\cancel{x}}dx$$

$$= x\ln x - \int dx = \boxed{x\ln x - x + C}$$

32.)
$$\int \arcsin x\,dx = \left| \begin{matrix} u=\arcsin x & dv=dx \\ du=\frac{1}{\sqrt{1-x^2}} & v=x \end{matrix} \right| = x\cdot\arcsin x - \int x\cdot\frac{1}{\sqrt{1-x^2}}dx$$

$$= \left| \begin{matrix} 1-x^2=t \\ -2x\,dx=dt \\ dx=\frac{-1}{2}dt \end{matrix} \right| = x\cdot\arcsin x - \int \frac{1}{\sqrt{t}}\left(\frac{-1}{2}dt\right)$$

$$\sqrt{t} \rightarrow 2\cdot\frac{1}{2\sqrt{t}}$$

$$= x\cdot\arcsin x - \cancel{2}\sqrt{t}\cdot\left(\frac{-1}{\cancel{2}}\right) = \boxed{x\cdot\arcsin x + \sqrt{t} + C}$$

33.)
$$\int e^x \sin x\,dx = \left| \begin{matrix} u=e^x & dv=\sin x\,dx \\ du=e^x dx & v=-\cos x \end{matrix} \right| = e^x(-\cos x) - \int(-\cos x)\,e^x dx$$

$$= -e^x\cos x + \int e^x\cos x\,dx = \left| \begin{matrix} u=e^x & dv=\cos x\,dx \\ du=e^x dx & v=\sin x \end{matrix} \right|$$

$$= -e^x\cos x + e^x\sin x - \int \sin x\cdot e^x dx$$ vratili smo se na početni, prebacujemo na lijevu stranu

$$2\int e^x \sin x\,dx = e^x\sin x - e^x\cos x \;/ :2$$

$$\boxed{\int e^x \sin x\,dx = \frac{e^x}{2}(\sin x - \cos x) + C}$$

Vježba 10.)

a) $\int \arccos x \, dx = x \arccos x - \sqrt{1-x^2} + C$ DOKAZATI!

$$\int \arccos x \, dx = \left| \begin{matrix} u = \arccos x & dv = dx \\ du = -\frac{1}{\sqrt{1-x^2}} & v = x \end{matrix} \right| = x \arccos x - \int x \left(-\frac{1}{\sqrt{1-x^2}} \right)$$

$$= x \arccos x + \int x \frac{1}{\sqrt{1-x^2}} = \left| \begin{matrix} 1-x^2 = t \\ -2x\,dx = dt \\ dx = \frac{-1}{2} dt \end{matrix} \right|$$

$$= x \arccos x + \int \frac{1}{\sqrt{t}} \cdot \left(\frac{-1}{2} dt \right) = x \arccos x + \cancel{2}\sqrt{t} \cdot \left(\frac{-1}{\cancel{2}} \right)$$

$$= x \arccos x - \sqrt{t} + C = \boxed{x \arccos x - \sqrt{1-x^2} + C}$$

b) $\int \operatorname{arctg} x \, dx = x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + c$ DOKAZATI!

$$\int \operatorname{arctg} x \, dx = \left| \begin{matrix} u = \operatorname{arctg} x & dv = dx \\ du = \frac{1}{1+x^2} & v = x \end{matrix} \right| = x \operatorname{arctg} x - \int x \frac{1}{1+x^2} dx =$$

$$= \left| \begin{matrix} 1+x^2 = t \\ 2x\,dx = dt \\ dx = \frac{1}{2} dt \end{matrix} \right| = x \operatorname{arctg} x - \int \frac{1}{t} \cdot \left(\frac{1}{2} dt \right) = x \operatorname{arctg} x - \frac{1}{2} \int \frac{dt}{t}$$

$$= x \operatorname{arctg} x - \frac{1}{2} \ln|t| + c = \boxed{x \operatorname{arctg} x - \frac{1}{2} \ln(1+x^2) + c}$$