4. LINEARNI SUSTAVI

Linearni sustavi : m jednadžbi, n nepoznanica

$$a_{11} x_1 + a_{12} \cdot x_2 + \dots + a_{1n} \cdot x_n = b_1$$

$$a_{21} \cdot x_1 + a_{22} \cdot x_2 + \dots + a_{2n} \cdot x_n = b_2$$

$$\vdots$$

$$a_{m1} \cdot x_1 + a_{m2} \cdot x_2 + \dots a_{mn} \cdot x_n = b_m$$

Matrični zapis: $Ax = b$, A mat sustava, x vektor nepoznanica

$$x_1 \cdot \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

b vektor desne str

$A \in M_{mn}$, $x \in V^n$, $b \in V^n$

Gaussova metoda eliminacije

Ideja: el. trans. nad retcima mat. sustava lin. sust. svodimo na njem ekviv. sustav

Ekvivalent: Dva sustava nazivamo ekvivalentnim ukoliko imaju isti br. nepoznanica i isti skup rj.

Element. transf. - zamjena dvaju redaka
- množenje redaka skalarom razl. od $\emptyset$
- dodavanje nekom retku drugi redak pomnožen skalarom

Pr. 1.)

$$\underbrace{\left[\begin{array}{ccc} 1 & 2 & 3 \\ 2 & -1 & -1 \\ 1 & 3 & 4 \end{array}\right.}_{A}\underbrace{\left.\begin{array}{|c} 5 \\ 1 \\ 6 \end{array}\right]}_{b} \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -5 & -7 & -7 \\ 0 & 1 & 1 & 1 \end{array}\right] 5 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 5 & 7 & 9 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] :2 \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & \boxed{1} & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$x_1 = 1 \quad x_2 = -1 \quad x_3 = 2$

$x_2 + x_3 = 1$

$\to x_2 = 1 - x_3 = 1 - 2 = -1$

$x_1 + 2x + 3x_3 = 5$

$\Rightarrow x_1 = 5 - 2x_2 \; 3x_3 = 5 - 2(-1) - 3 \cdot (2) = 5 + 2 - 6 \; \underline{= 1}$

Rješenje je jednoznačno. $r(A) = 3 \implies$ mat. sust. je regularna!

Homogeni sustavi

Sustavi kojima je vektor desne strane jednak nul vektoru

$$a_{11}x_1 + a_{12}x_2 + \dots a_{1n} \cdot x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \dots a_{n1}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 \cdots a_{mn}x_n = 0$$

Mat. zapis = $Ax = 0$; $A \in M_{mn}$, $x \in V^n$

$\boxed{\text{TEOREM 14.}}$ (knj. 3. Rang i inverz mat.)

$A \in M_n$, jednadžba $Ax=0$ ima jedinstveno rj $x=0 \iff A$ je reg.

$$A\vec{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots x_n \begin{bmatrix} a_{m1} \\ a_{m2} \\ \vdots \\ a_{mn} \end{bmatrix}$$

$$A\vec{x} = \vec{0} \iff \underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \dots x_n\vec{a}_n}_{\text{linearna kombinacija stupaca mat. A sust.}} = \vec{0} \quad (*)$$

$\boxed{\Rightarrow}$ Pretp da sust. $A\vec{x} = \vec{0}$ ima jedinstv. rj. $\vec{x} = \vec{0}$

Iz $(*)$ i pretp. slijedi da lin. kombin. $(*)$ iščezava samo na trivijalan način; znači da su stupci mat. A linearno nezavisni. Pa je $r(A) = n$, odnosno A je reg. mat.

$\boxed{\Leftarrow}$ Pretp. da je A reg. mat slijedi $r(A) = n$, slijedi da su svi stupci lin. nezav. Sada iz $(*)$ slijedi da je

$x_1 = x_2 = x_3 \dots = x_n = 0$. jedino rj. sustava $A\vec{x} = \vec{0}$.

Pr.1)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \overset{(+1)}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline 1. & 1 & 0 & 1 \\ 2. & 0 & 1 & -1 \\ 3. & 0 & 0 & 0 \end{array}$$

$$\Rightarrow \left.\begin{matrix} x_1 + x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix}\right\} \Rightarrow \begin{matrix} x_1 = -x_3 \\ x_2 = x_3 \end{matrix} \qquad x_3 =: t \in \mathbb{R}$$

$r(A) = 2 < 3 \Rightarrow A$ je singul.

- x_3 slobodna varijabla
- x_1, x_2 vezane varijable

Pr.2)

$$\begin{bmatrix} \boxed{1} & 2 & 1 & 2 & 3 \\ 2 & 4 & 3 & 3 & 7 \\ 2 & 4 & 1 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 2 & 3 \\ 0 & 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & -1 & 1 & -1 \end{bmatrix} \sim \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \\ \hline \boxed{1} & 2 & 0 & 3 & 2 \\ 0 & 0 & \boxed{1} & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

- vezani stupci
- slobodni stupci

$r(A) = 2$

$d = n - r$ $\qquad$ x_1, x_3 vezane varijable

$d = 5 - 2 = 3$ $\qquad$ x_2, x_4, x_5 slobodne varijable

$x_1 = -2x_2 - 3x_4 - 2x_5$ $\qquad$ $x_2 = \alpha \quad x_4 = \beta \quad x_5 = \gamma$

$x_3 = x_4 - x_5$ $\qquad$ $\alpha, \beta, \gamma \in \mathbb{R}$

$x_1 = -2\alpha - 3\beta - 2\gamma$

$x_3 = \beta - \gamma$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2\alpha - 3\beta - 2\gamma \\ \alpha \\ \beta - \gamma \\ \beta \\ \gamma \end{bmatrix}$$

$$= \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad \alpha, \beta, \gamma \in \mathbb{R}$$

L(nesto)

$$\begin{array}{c} \begin{array}{cc} A & b \end{array} \\ \left[\begin{array}{cc:c} 2 & 3 & 4 \\ 4 & 6 & 7 \end{array}\right] \\ \begin{array}{cc} \downarrow & \downarrow \\ x & y \end{array} \end{array} \sim \left[\begin{array}{cc:c} 2 & 3 & 4 \\ 0 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 3/2 & 2 \\ 0 & 0 & -1 \end{array}\right]$$

$\Rightarrow 0x + 0y = -1$

sustav nema rješenja $\boxed{0 = -1}$

A_p

$1 \neq \boxed{r(A) \neq r(A_p)} = 2$

sustav nema rj.

Teorem Kronecker-Capelli

Sustav $Ax = b$ ima rješenje onda i samo onda ako je $r(A) = r(A_p)$.

Zad. 1.)

$$\begin{aligned} 3x_1 - x_2 + 5x_3 - x_4 &= 3 \\ x_1 - 2x_2 + 3x_3 + 2x_4 &= 1 \\ 2x_1 + x_2 + 2x_3 - 3x_4 &= 4 \end{aligned}$$

$\rightarrow$ sustav čiju mat. pišemo

$$\left[\begin{array}{cccc:c} 3 & -1 & 5 & -1 & 3 \\ 1 & -2 & 3 & 2 & 1 \\ 2 & 1 & 2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{cccc:c} \boxed{1} & -2 & 3 & 2 & 1 \\ 3 & -1 & 5 & -1 & 3 \\ 2 & 1 & 2 & -3 & 4 \end{array}\right] \sim \left[\begin{array}{cccc:c} 1 & -2 & 3 & 2 & 1 \\ 0 & 5 & -4 & -7 & 0 \\ 0 & 5 & -4 & -7 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc:c} 1 & -2 & 3 & 2 & 1 \\ 0 & 5 & -4 & -7 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$r(A) = 2$

$r(A_p) = 3$

$r(A) \neq r(A_p) \Rightarrow$ nema rješenja

Zad. 2.) $*\ \vec{u}$ i $\vec{v}$ su dva različita rješenja lin. sust. $Ax = b$.

$A\vec{u} = \vec{b}$

$A\vec{v} = \vec{b}$

$\Longrightarrow$ $\alpha\vec{u} + \beta\vec{v}$ lin. komb. vektora $\vec{u}$ i $\vec{v}$,

$\alpha, \beta \in \mathbb{R}$

$\vec{b} = A(\alpha\vec{u} + \beta\vec{v}) =$

$= A(\alpha\vec{v}) + A(\beta\vec{v}) = \alpha(\underline{A\vec{u}}) + \beta(\underline{A\vec{v}}) = \alpha\cdot\vec{b} + \beta\cdot\vec{b}$

$= \underline{\vec{b}(\alpha+\beta)}$

$\vec{b} \neq 0 \Longrightarrow \alpha+\beta = 1 \Longrightarrow \beta = 1-\alpha \Longrightarrow \underline{\alpha\vec{u} + (1-\alpha)\vec{v},\ \alpha\in\mathbb{R}}$

$\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ $\quad b_i = (\alpha+\beta)\, b_i$

Struktura općeg rješenja nehomogenog linearnog sustava

Pr. 1.)

$$x_1 - 2x_2 + x_3 = 1$$
$$2x_1 - x_2 + x_3 = 2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 2 & -1 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 3 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 1 & -1/3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 1 \\ 0 & 1 & -1/3 & 0 \end{array}\right]$$

$r(A) = r(A_p) = 2$

$d = n - r = \underset{3-2}{\underline{1}}$

$$\left.\begin{array}{l} x_1 + \frac{1}{3}x_3 = 1 \\ x_2 - \frac{1}{3}x_3 = 0 \end{array}\right\} \rightarrow \begin{array}{l} x_1 = 1 - \frac{1}{3}x_3 \\ x_2 = \frac{1}{3}x_3 \end{array} \qquad x_3 = \alpha \in \mathbb{R}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 - \frac{1}{3}\alpha \\ \frac{1}{3}\alpha \\ \alpha \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p} + \alpha \underbrace{\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 1 \end{bmatrix}}_{\vec{x}_H}$$

TEOREM

Opće rješenje linearnog sustava $Ax = b$ ima oblik $x = x_p + x_h$, gdje je x_h opće rješenje pripadnog homogenog sustava $Ax = 0$, a x_p jedno (partikularno) rješenje nehomogenog sust.

Treba dokazati:

Ako je $\vec{x}$ opće rj. da je zadanog oblika, i ako je $\vec{x}$ zadanog oblika, da je onda opće rj. sustava.

$\boxed{\Rightarrow}$ Neka je $\vec{x}$ rj. nehomogenog sustava $A\vec{x} = \vec{b}$ i neka je $\vec{x}_p$ partikularno rješenje tog sustava.

imamo:

$$\begin{array}{l} A\vec{x} = \vec{b} \\ A\vec{x}_p = \vec{b} \end{array} \Big/ -$$

$$A\vec{x} - A\vec{x}_p = \vec{b} - \vec{b}$$
$$A(\vec{x} - \vec{x}_p) = 0$$

$$\vec{x} - \vec{x}_p = \vec{x}_H$$
$$\vec{x} = \vec{x}_H + \vec{x}_p$$

$\boxed{\Leftarrow}$ Obratno neka je $\vec{x} = \vec{x}_p + \vec{x}_H$. Pokažimo da je $A\vec{x} = \vec{b}$

$$A\vec{x} = A(\vec{x}_p + \vec{x}_H) = A\vec{x}_p + A\vec{x}_H = \vec{b} + \vec{0} = \vec{b}$$

Alg za rj. nehom. sust.

1. $Ax = b$ napišemo u mat. obliku $[A \vdots b]$

2. Mat A svedemo na reducirani A_R.

↳ Ekvival. sust. $[A_R \vdots b']$. Ako je $rang(A) < rang(A \vdots b)$, zaustavimo se.

$\Rightarrow$ sustav nema rj.

DZ. 4.B iz knjižice

Zad. 4.) Za koje vrijednosti $\lambda \in \mathbb{R}$ sustav

$$\begin{aligned} x_1 + 3x_2 + 3x_3 &= 1 \\ -x_1 + \lambda x_2 - 2x_3 - 3x_4 &= 2 \\ -x_1 - 3x_2 + \lambda x_3 - 2x_4 &= 1 \\ 2x_1 + 6x_2 + 7x_3 + \lambda x_4 &= 4 \end{aligned}$$

a) jedinstveno rj.

b) beskonačno rj. → odredite opće rj.

c) nema rj.

$$\left[\begin{array}{cccc|c} \boxed{1} & 3 & 3 & 0 & 1 \\ -1 & \lambda & -2 & -3 & 2 \\ -1 & -3 & \lambda & -2 & 1 \\ 2 & 6 & 7 & \lambda & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & -2 & -3 & 3 \\ 0 & 0 & \lambda+3 & -2 & 2 \\ 0 & 0 & 1 & \lambda & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & \boxed{1} & \lambda & 2 \\ 0 & 0 & \lambda+3 & -2 & 2 \end{array}\right] \cdot(-\lambda-3) \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & -3\lambda & 5 \\ 0 & \lambda+3 & 0 & -(\lambda+3) & 1 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & -(\lambda+1)(\lambda+2) & -2(\lambda+2) \end{array}\right]$$

Kako smo dobili?

$\lambda(-\lambda-3)-2 = -\lambda^2-3\lambda-2 =$
$= -(\lambda^2+2\lambda+\lambda+2)$
$= -(\lambda(\lambda+2)+(\lambda+2))$
$= -(\lambda+1)(\lambda+2)$

1. slučaj $\lambda = -1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 3 & 5 \\ 0 & 2 & 0 & -2 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & -2 \end{array}\right]$$

sustav nema rj.

$r(A) = 3$

$r(A_p) = 4$

2. slučaj $\lambda = -2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & -5 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 0 & 9 & -8 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

$x_1 + 9x_4 = -8$

$x_2 - x_4 = 1$

$x_3 - 2x_4 = 2$

$x_4 = t \in \mathbb{R}$ → nevezani (slobodni) sve izrazimo preko njega

$x_1 = -8 - 9x_4$

$x_2 = 1 + x_4$

$x_3 = 2 + 2x_4$

$x_4 = t \in \mathbb{R}$

$r(A) = r(A_p) = 3$

$d = n - r = 4 - 3 = 1$

Beskonačno mnogo rj.

$$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -8-9t \\ 1+t \\ 2+2t \\ t \end{bmatrix} = \begin{bmatrix} -8 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -9 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

3. slučaj $\lambda = -3$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 9 & -9 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & -2 & 2 \end{array}\right]$$

nema rješenja

4. slučaj $\lambda \in \mathbb{R} \setminus \{-1,-2,-3\}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & -3\lambda & -9 \\ 0 & \lambda+3 & 0 & -(\lambda+3) & 1 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & -(\lambda+1)(\lambda+2) & -2(\lambda+2) \end{array}\right] \begin{array}{l} \\ /\cdot\frac{1}{\lambda+3} \\ \\ /\cdot\frac{-1}{(\lambda+1)(\lambda+2)} \end{array} \sim \left[\begin{array}{cccc|c} \boxed{1} & 3 & 0 & -3\lambda & -9 \\ 0 & \boxed{1} & 0 & -1 & \frac{1}{\lambda+3} \\ 0 & 0 & \boxed{1} & \lambda & 2 \\ 0 & 0 & 0 & \boxed{1} & \frac{2}{\lambda+1} \end{array}\right]$$

$r(A) = r(A_p) = 4$ $d = 4 - 4 = 0$ sustav ima jedinstveno rj.

a) $\lambda \in \mathbb{R} \setminus \{-1,-2,-3\}$ b) $\lambda = -2$ $\vec{X} = \begin{bmatrix} -8 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} -9 \\ 1 \\ 2 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$

c) $\lambda = -1$, $\lambda = -3$

Pr.) $A\vec{x}=\vec{b}$, $A \in M_n$ singularna. $\vec{b} \in V^n$, $\vec{b} \neq \vec{0}$

i neka ovaj sustav ima rj. $\vec{x} \in V^n$

Vidjeli smo da je $\exists \vec{z} \neq \vec{0}$ t.d. $A\vec{z}=\vec{0}$ (hom. sust.)

Za $\alpha \in \mathbb{R}$ imamo $A(\vec{x}-\alpha\vec{z}) = A\vec{x} + A(\alpha\vec{z}) = A\vec{x} - \alpha(A\vec{z})$

$$= \vec{b} + \alpha + \vec{0} = \vec{b} + \vec{0} = \vec{b}$$

Zaključujemo da je i lin. komb. $\vec{x}+\alpha\vec{z}$ također rj. sust. $A\vec{x}=\vec{b}$ za proizvoljni $\alpha \in \mathbb{R}$.

Jedinstvenost rješenja kvad. sust.

TEOREM Neka je $A \in M_n$. Za po volji odabran vektor $b \in V^n$ sustav $Ax=b$ ima jedinstveno rj. onda i samo onda kad je A regularna mat.

DOKAZ:

$\boxed{\Leftarrow}$ Pretp. da je A reg. mat. tada je $r(A)=n=r(A_p)$ pa prema K-C teoremu sustav ima rj.

Dokažimo da je rj. jedinstveno. Opće rj. nehomog. sustava ima oblik $\vec{x}=\vec{x}_p+\vec{x}_H$, a $\vec{x}_H=\vec{0}$ je jedno rješenje pripadnog hom. sustava $A\vec{x}=\vec{0}$. Budući da je $\boxed{A \text{ reg.}}$ po pretpostavci

(vidi teorem o hom. sust.)

$\boxed{\Rightarrow}$ Pretpostavimo da sustav $A\vec{x}=\vec{b}$ ima jedinstveno rj. $\forall \vec{b} \in V^n$. Tada za $\vec{b}=\vec{0}$ sustav $A\vec{x}=\vec{0}$ ima jedinstveno rj. $\vec{x}=\vec{0}$, pa iz teorema (vidi hom. sust.) slijedi da je

<u>A reg. matrica.</u>