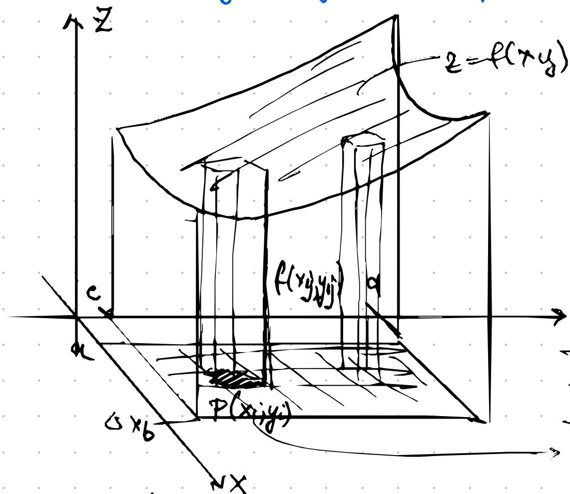
5.1. DVOSTRUKI INTEGRALI

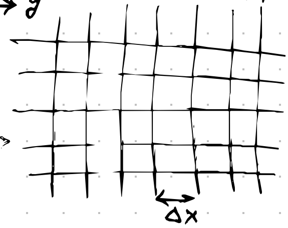
Matan 1: $\int_a^b f(x,y)\,dxdy$

DEF dvostrukog integrala na pravokutniku P



$$V = \iint_P f(x,y)\,dx,dy$$

$$V_i = f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$



Znači: $\sum_{i=1}^{n}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y \approx V$

DEF: Dvostruki integral

Dvostruki integral funkcije $f(x,y)$ po pravokutniku P definiramo s

$$\iint_P f(x,y)\,dxdy = \lim_{m,n \to \infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

ukoliko taj limes postoji i ne ovisi o izboru točaka $(x_{ij}^*, y_{ij}^*) \in P_{ij}$.

Tada kažemo da je $f(x,y)$ na P.

→ Dakle dvostruki integral po pravokutniku P nenegativne funkcije $f(x,y)$ predstavlja volumen tijela iznad P, a omeđeno s gornje strane plohom $z = f(x,y)$.

TM Ako je f neprekinuta na pravokutniku P, onda je integrabilna na P

TM Ako je f omeđena na pravokutniku P i neprekinuta na P osim na konačno mnogo glatkih krivulja, onda je integrabilna na P.

FUBINIJEV TEOREM

Ako postoji dvostruki integral funkcije f na pravokutniku

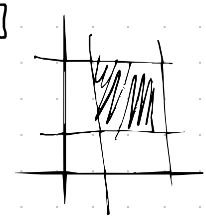
$$P=[a,b]\times[c,d] \xrightarrow{\text{tada}} \iint_P f(x,y)\,dx\,dy=\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

*Rješenje dvostrukih integrala je uvijek <u>BROJ</u>

Zapisujemo:

$$\iint_P f(x,y)\,dx\,dy=\int_a^b dx\int_c^d f(x,y)\,dy=\int_c^d dy\int_a^b f(x,y)\,dx \longleftarrow \text{češće}$$

Primjer: $\iint_P (y^2+xy)\,dP$ po pravokutniku $P=[0,2]\times[1,3]$

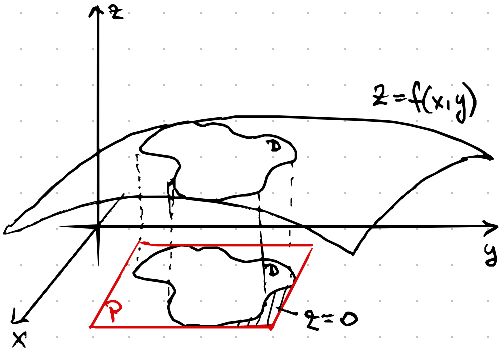


$$\rightarrow \int_0^2 dx\int_1^3 (y^2+xy)\,dy=\int_0^2\left(\frac{y^3}{3}+\frac{y^2}{2}x\right)\Bigg|_1^3 dx$$

$$=\int_0^2\left(9+\frac{9}{2}x-\frac{1}{3}-\frac{x}{2}\right)dx=\frac{76}{3}$$

drugi način: $\int_1^3 dy\int_0^2 (y^2+xy)\,dx=\int_1^3\left(y^2x+\frac{1}{2}x^2y\right)\Big|_0^2 dy=\frac{76}{3}$

Dvostruki integral na omeđenom skupu D



$$\iint_D f(x,y)\,dx\,dy = ?$$

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in D \\ 0, & (x,y) \in P \setminus D \end{cases}$$

↳ di nema D, stavimo visinu 0 da nemamo volumen

DEF: $\iint_D f(x,y)\,dx\,dy = \iint_P F(x,y)\,dx\,dy$

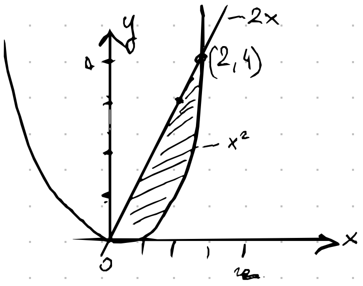
$\boxed{\text{TM}}$ računanje po omeđenom području D (DOKAZ)

$$\underbrace{\iint_D f(x,y)\,dx\,dy = \iint_P F(x,y)\,dx\,dy}_{\text{def.}} \underbrace{=}_{\text{Fubini za F}} \int_a^b dx \int_c^d F(x,y)\,dy = \int_a^b dx \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy$$

neprekinute funkcije

ANALOGNO ZA DRUGI POREDAK:

$$\iint_D f(x,y)\,dx\,dy = \int_c^d dy \int_{h_1(x)}^{h_2(x)} f(x,y)\,dx$$



ZAD: $\iint (2x+y)\,dx\,dy = D \ldots \quad y = 2x, \; y = x^2$

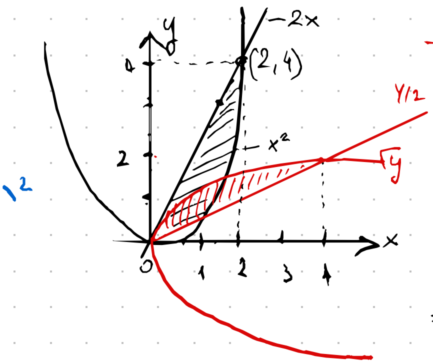
FIKSIRAMO VARIJABLU x

1) $\int_0^2 dx \int_{x^2}^{2x} (2x+y)\,dy = \int_0^2 \left(2xy + \frac{y^2}{2}\right)\Big|_{x^2}^{2x} dx = \int_0^2 \left(4x^2 + 2x^2 - 2x^3 - \frac{x^4}{2}\right) dx$

$= \int_0^2 \left(6x^2 - 2x^3 - \frac{1}{2}x^4\right) dx = \left(2x^3 - \frac{1}{2}x^4 - \frac{1}{10}x^5\right)\Big|_0^2 = 16 - 8 - \frac{32}{10} = \frac{80}{10} - \frac{32}{10} = \frac{24}{5}$

→ 1. način : normalno gledamo graf:  (kao što je gore nacrtano)

2) Drugi poredak : inverzna funkcija gornjeg grafa



→ $y = x^2 \rightarrow x = \sqrt{y}$
$y = 2x \rightarrow x = \frac{y}{2}$
} FIKSIRAMO VARIJABLU y

$$\int_0^4 dy \int_{y/2}^{\sqrt{y}} (2x+y)\,dx = \int_0^4 \left(x^2 + yx\right)\Big|_{\frac{y}{2}}^{\sqrt{y}} dy$$

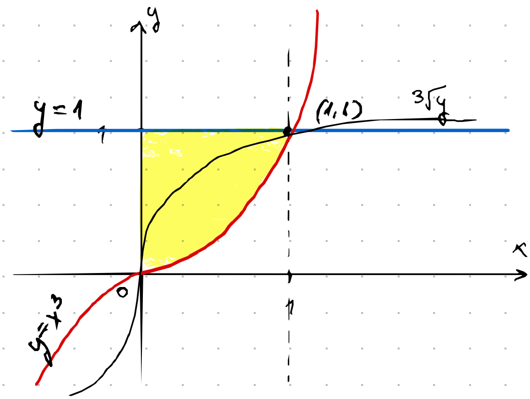
$$= \int_0^4 \left(y + y\sqrt{y} - \frac{y^2}{4} - \frac{y^2}{2}\right) dy = \frac{24}{5} \text{ isto}$$

Ponekad se može dogoditi da je jedan poredak integracije puno jednostavniji i pogodniji od drugog pa radimo, logično, po lakšem.

Primjer :) $\int_0^1 x^2\,dx \int_{x^3}^{1} e^{y^2}\,dy$

$y=1$

$y=x^3$



unutarnji integ. nije elementarno integrabilan!

↪ zamijenimo

$y=1 \rightarrow x=0$

$y=x^3 \rightarrow x=\sqrt[3]{y}$

$$\int_0^1 e^{y^2}dy\int_0^{\sqrt[3]{y}} x^2\,dx = \int_0^1 e^{y^2}\left(\frac{1}{3}x^3\Big|_0^{\sqrt[3]{y}}\right)dy$$

$$= \int_0^1 e^{y^2}\cdot\frac{1}{3}y\,dy = \frac{1}{3}\int_0^1 y e^{y^2}\,dy = \left|\begin{matrix} t=y^2 \\ dt=2y\,dy\end{matrix}\right|$$

$$= \frac{1}{3}\int_0^1 \frac{1}{2}e^t\,dt = \frac{1}{6}\int_0^1 e^t\,dt = \frac{1}{6}\cdot(e^t)\Big|_0^1 = \frac{1}{6}(e^{y^2})\Big|_0^1 = \underline{\underline{\frac{1}{6}(e-1)}}$$

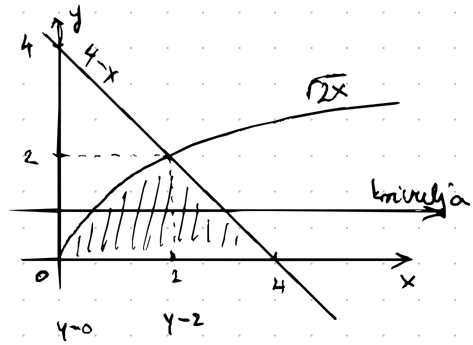
* puno lakše nego da smo išli kako je zadano

M1Z1-2020-4)

$y=\sqrt{2x}$

$$\iint_D (x+y)\,dx\,dy$$

$y=4-x$, os x

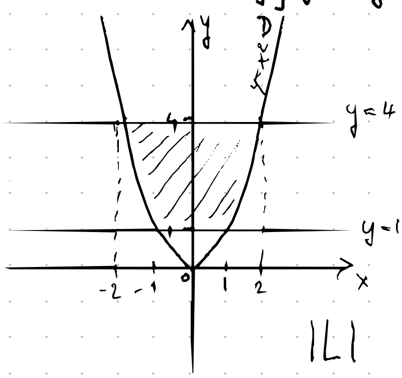


$$\int_0^2 dx\int_0^{\sqrt{2x}}(x+y)\,dy+\int_2^4 dx\int_0^{4-x}(x+y)\,dy$$

ILI $\int dy\int(x+y)\,dx$ (y; krivulje) $\longrightarrow \int_0^2 dy\int_{\frac{y^2}{2}}^{4-y}(x+y)\,dx=\int_0^2\left(\frac{1}{2}x^2+yx\right)\Big|_{\frac{y^2}{2}}^{4-y}dy=\frac{178}{15}$

ako integriramo po x, granice su y jer smo izrazili x

M1-2021-5) $\iint_D y\,dx\,dy$



fiksirani x; krivulje

$$\int_{-2}^{-1}dx\int_{x^2}^{4}y\,dy+\int_{-1}^{1}dx\int_{1}^{4}y\,dy+\int_{1}^{2}dx\int_{x^2}^{4}y\,dy$$

! ne koristimo površine! → ne koristiti simetriju

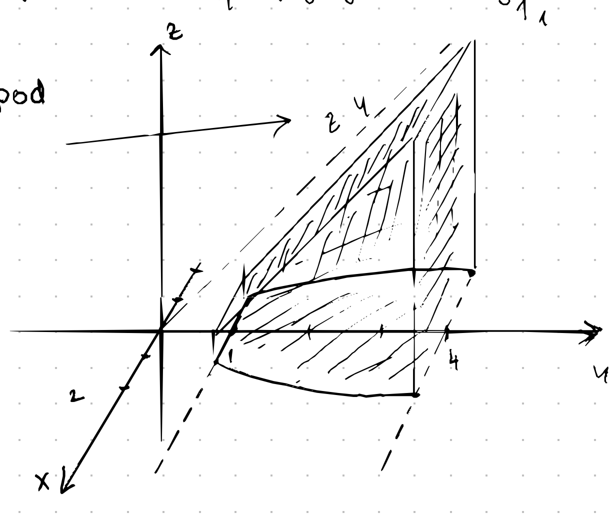
↳ ne znamo kakva je ta ploha iznad

ILI $\int_1^4 dy\int_{-\sqrt{y}}^{\sqrt{y}}y\,dx=\int_1^4 xy\Big|_{-\sqrt{y}}^{\sqrt{y}}dy$

inverz: $y=x^2$
$x=\pm\sqrt{y}$

$$=\int_1^4 2y\sqrt{y}\,dy=2\left(y^{5/2}\cdot\frac{2}{5}\right)\Big|_1^4=\frac{4}{5}\cdot(32-1)=31\frac{4}{5}$$

volumen ispod
$z=y$

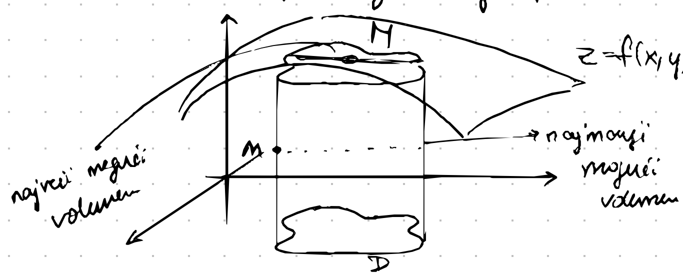


TM (Teorem srednje vrijednosti za 2 dimenzije)

Neka je f neprekinuta na zatvorenom području D. Tada postoji točka $(x_0,y_0)\in D$ takva da je $\boxed{\iint_D f(x,y)\,dx\,dy = f(x_0,y_0)\cdot\mu(D)}$

gdje je $\mu(D)$ površina područja D

→ Geom. interpretacija: volumen ispod plohe $z=f(x,y)$ jednak je volumenu cilindra s bazom D i visinom $f(x_0,y_0)$



DOKAZ: Neka je $m=\min_D f(x,y)$, $M=\max_D f(x,y)$ (pretpostavimo da postoje)

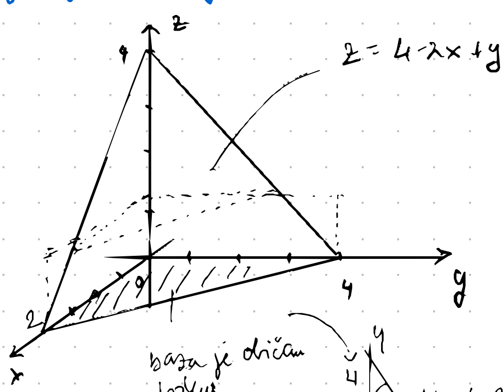
Tada $m\underbrace{\iint_D dx\,dy}_{\mu(D)} \le \underbrace{\iint_D f(x,y)\,dx\,dy}_{\text{volumen}} \le M\iint_D dx\,dy \cdot \mu(D)$ / $:\mu(D)$

$$m \le \boxed{\frac{1}{\mu(D)}\iint_D f(x,y)\,dx\,dy} \le M$$

– Zbog neprekinutosti f-je sigurno $f(x,y)$ postoji $(x_0,y_0)\in D$ t.d.j.

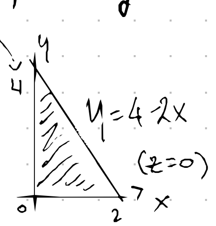
$\to = f(x_0,y_0)$

Zad.) Odredite srednju vrijednost funkcije $f(x,y) = 4-2x-y$ na njenoj projekciji u xOy ravnini i geometrijski interpretirajte.



$$f(x_0,y_0) = \frac{1}{\mu(D)}\iint_D (4-2x-y)\,dx\,dy$$

$$\mu(D) = \frac{1}{\cancel{2}}\cdot\cancel{2}\cdot 4 = 4$$



$$\int_0^2 dx\int_0^{4-2x}(4-2x-y)\,dy$$

$$=\int_0^2 4(4-2x)-2x(4-2x)-\frac{1}{2}(4-2x)^2\,dx=$$

$$=\frac{1}{3}Bv=\frac{1}{3}\cdot 4\cdot 4=\frac{16}{3}$$

$$f(x_0,y_0)=\frac{1}{4}\cdot\frac{16}{3}=\frac{4}{3}$$