DETERMINANTE

- skalar pridružen kvadratnoj matrici → det(A) ili |A|

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\boxed{\det A = \sum_{j=1}^{n} a_{ij} A_{ij}}$$

Veza s linearnim sustavima

$$ax + by = e \quad / \cdot (d)$$
$$cx + dy = f \quad / \cdot (-b)$$

$$adx + \cancel{bdy} =$$
$$-bcx - \cancel{bdy} = -bf \quad /+$$

$$adx - bcx = ed - bf$$
$$x(ad - bc) = de - bf$$

$$x = \frac{ed - bf}{ad - bc} \qquad x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$\cancel{acx} + cby = ce$$
$$-\cancel{acx} - ady = -af \quad /+$$

$$cby - ady = ce - af$$
$$y(cb - ad) = ce - af \quad / \cdot (-1)$$
$$y(ad - bc) = af - ce$$

$$y = \frac{af - ce}{ad - bc} \qquad y = \frac{\begin{vmatrix} a & c \\ e & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

Minore

· Laplaceov razvoj determinante – M_{ij} minora elementa a_{ij}

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{31} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad M_{11} := \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} \qquad M_{12} := \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$M_{13} := \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Pomoću minora definiramo algebarske komplemente: $\Longrightarrow$ $A_{ij} = (-1)^{i+j} \cdot M_{ij}$

$$A_{11} := +M_{11} \qquad A_{12} := -M_{12} \qquad A_{13} := +M_{13}$$

SVOJSTVO DETERMINANATA

Binet-Cauchyjev teorem $\det(AB) = \det A \cdot \det B$

① Ako mat. A ima redak sastavljen od samih 0 $\Rightarrow \det A = 0$

② Determinanta trokutaste matrice jednaka je umnošku elemenata na istoj dijagonali.

#2 DOKAZ:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & 0 & & \vdots \\ \vdots & & & \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

$$\Rightarrow \det A = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

③ Ako matrica ima dva jednaka retka; $\det A = 0$

#3 DOKAZ: indukcijom

Baza: $n=2$ $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$

Pretpostavka: tvrdnja vrijedi za sve det. reda n

Korak: $(n+1)$

det. mat. reda n koja je po indukciji jednaka 0

$$\det A = \sum_{\ell=1}^{n} (-1)^{k+\ell} a_{k\ell} M_{k\ell}$$

$$\det A = \sum_{\ell=1}^{n} (-1)^{k+\ell} a_{k\ell} \cdot 0 \Rightarrow \det A = 0$$

④ Transponiranjem mat vrijednost det. se ne mijenja: $\det A = \det A^T$

#4 DOKAZ: indukcijom

Baza: $n=2$ $A = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ $A^T = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$\det A = ad - bc$
$\det A^T = ad - bc$ $\equiv$ jednako

K: $(n+1)$ $A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$ $A^T = \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & i \end{vmatrix}$

$\det A = a(ei - fh) - b(di - fg) + c(dh - eg)$
$\det A^T = a(ei - fh) - b(di - fg) + c(dh - eg)$ $\Big\}$ $\det A = \det A^T$

(5.) Determinanta se množi skalarom tako da se jedan (bilo koji) njezin redak množi tim skalarom.

#5 DOKAZ: A i A' (A kojem je i-ti redak pomnožen s λ)

$$\det A' = \sum_{j=1}^{n} (\lambda a_{ij})A_{ij} = \lambda \underbrace{\sum_{j=1}^{n} a_{ij} A_{ij}}_{} = \lambda \cdot \det(A)$$

(6.) Rastave li se svi elementi nekog retka matrice na zbroj dvaju el., onda je ta det jednaka zbroju dviju odgovarajućih determinanata

razdvojimo elemente a_i retka na a_i' i a_i'' te razvijemo $\det A$ na i

$$\det A = \sum_{j=1}^{n} (a'_{ij} + a''_{ij})A_{ij} = \sum_{j=1}^{n} a'_{ij} A_{ij} + \sum_{j=1}^{n} a''_{ij} A_{ij} = \det(A') + \det(A'')$$

(7.) Ako zamijenimo dva retka matrice, determinanta mijenja predznak

$$\underset{\text{zato } 0}{0} = \begin{vmatrix} \vdots \\ a_i + a_j \\ \vdots \\ a_i + a_j \\ \vdots \end{vmatrix} = \overset{=0}{\begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_i \\ \vdots \end{vmatrix}} + \begin{vmatrix} \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \end{vmatrix} + \overset{=0}{\begin{vmatrix} \vdots \\ a_j \\ \vdots \\ a_j \\ \vdots \end{vmatrix}} \Rightarrow \begin{vmatrix} \vdots \\ a_j \\ \vdots \\ a_i \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ a_i \\ \vdots \\ a_j \\ \vdots \end{vmatrix} = 0$$

dva ista retka

mijenjamo li i-ti i j-ti redak pomoću prošlog svojstva, u novoj mat i-ti i j-ti redak su zbroj ta dva retka

(8.) Ako nekom retku mat. dodamo neki drugi redak pomnožen skalarom, vrijednost determinante se neće promijeniti.

$$\begin{vmatrix} a_1 \\ \lambda a_1 + a_2 \\ \vdots \end{vmatrix} = \begin{vmatrix} a_1 \\ \lambda a_1 \\ \vdots \end{vmatrix} + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} = \lambda \underbrace{\begin{vmatrix} a_1 \\ a_1 \\ \vdots \end{vmatrix}}_{0} + \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix} = \begin{vmatrix} a_1 \\ a_2 \\ \vdots \end{vmatrix}$$