4.2.3. Divergencija i rotacija vektorskog polja

Divergencija vektorskog polja

$\vec{v}(x,y,z) = v_1(x,y,z)i + v_2(x,y,z)j + v_3(x,y,z)k$ je skalarno polje koje označavamo sa div $\vec{v}$ $(\nabla\cdot\vec{v})$

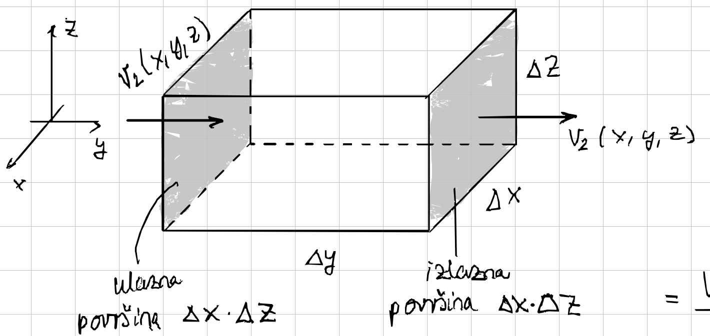
$$\Rightarrow \operatorname{div}\vec{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Primjer: $\vec{v} = x^2 y\,\vec{i} - z e^y\,\vec{j} + xyz\,\vec{k}$, $T(1,0,-1)$

$$\vec{\nabla}\cdot\vec{v} = (2xy,\ -ze^y,\ xy) \xrightarrow{T} \vec{\nabla}\vec{v}\,\Big|_T = 0 + 1 + 0$$

$$\boxed{\vec{\nabla}\vec{v}\,(T) = 1}$$

Fizikalna interpretacija → gibanje fluida



Razlika volumena fluida koji izlaze kroz desnu stranu kvadra

$$[v_2(x,y+\Delta y,z) - v_2(x,y,z)]\,\Delta x\cdot\Delta z$$

$$= \frac{v_2(x,y+\Delta y,z) - v_2(x,y,z)}{\Delta y}\,\Delta V$$

volumen kvadra

ako se gibanje odvija u y-smjeru promjena volumena fluida po jedinici volumena u jedinici vremena $\Big\}$ $-\left(\dfrac{v_2(x,y+\Delta y,z) - v_2(x,y,z)}{\Delta y}\right)$

$$\lim_{\Delta x\to 0} \frac{v_2(x,y+\Delta y,z) - v_2(x,y,z)}{\Delta y} = \frac{\partial v_2}{\partial y}(x,y,z)$$

analogno $\frac{\partial v_1}{\partial x}(x,y,z)$, $\frac{\partial v_3}{\partial z}(x,y,z)$

→ zbrojimo li doprinose volumena s obzirom na x, y, z

$$\Rightarrow \boxed{-\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right)}$$

formula za računanje divergencije vektorskog polja $\vec{v}$

i) $\vec{\nabla}\cdot\vec{v} > 0$ oko te točke je izvor fluida

ii) $\vec{\nabla}\cdot\vec{v} < 0$ oko te točke je ponor fluida

iii) $\vec{\nabla}\cdot\vec{v} = 0$ nema izvora ni ponora

DEF Kažemo da je vektorsko polje solenoidalno : $\vec{\nabla}\cdot\vec{v}=0$

$\operatorname{div}\vec{v}=\vec{\nabla}\cdot\vec{v}$, ako φ - skalarno polje, $\vec{v}$ - vektorska

$$\vec{\nabla}(\varphi\vec{v}) = \vec{v}\cdot(\vec{\nabla}\cdot\varphi) + \varphi(\vec{\nabla}\cdot\vec{v}) = \vec{v}\cdot\operatorname{grad}\varphi + \varphi\cdot\operatorname{div}\vec{v}$$

* operator divergencije je linearan operator: $\operatorname{div}(\lambda\vec{v}+\mu\vec{s}) = \lambda\operatorname{div}\vec{v}+\mu\operatorname{div}\vec{s}$

$$\operatorname{div}\vec{r}=3 \quad = \nabla\cdot\vec{r}=3$$

$$\operatorname{div}(F(r))\vec{r} = \nabla\cdot(F(r)\vec{r}) = \vec{r}(\nabla F(r)) + F(r)(\nabla\cdot\vec{r})$$

$$= \vec{r}\cdot F'(r)\cdot\vec{r}_0 + 3F(r) = r F'(r) + 3F(r)$$

Primjer:

$$\operatorname{div}\left(\frac{\vec{r}_0}{r^2}\right) = ?$$

$$\frac{\vec{r}}{|\vec{r}|}\cdot\frac{1}{r} = \vec{r}_0 \longrightarrow \operatorname{div}\left(\frac{\vec{r}_0}{r^2}\right) = \vec{\nabla}\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\underbrace{(\vec{\nabla}\vec{r})}_{3} + \vec{r}\cdot\left(\frac{1}{r^3}\nabla\right)$$

$$\operatorname{div}\left(\frac{\vec{r}_0}{r^2}\right) = \frac{3}{r^3} - \frac{3}{r^4}\cdot\vec{r} = \frac{3}{r^3} - \frac{3}{r^3}\cdot\underbrace{\vec{r}_0}_{1} \rightarrow \boxed{\operatorname{div}\left(\frac{\vec{r}_0}{r^2}\right) = 0}$$

Rotacija vektorskog polja

$$\vec{v}(x,y,z) = V_1(x,y,z)i + V_2(x,y,z)j + V_3(x,y,z)$$

! vektorska veličina

$$\operatorname{rot}\vec{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)k \rightarrow \vec{\nabla}\times\vec{v}$$

vektorsko polje: $\vec{v} = v_1 i + v_2 j + v_3 k$, rotacija: $\operatorname{rot}\vec{v} = \nabla\times\vec{v}$

$$\operatorname{rot}\vec{v} = \nabla\times\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

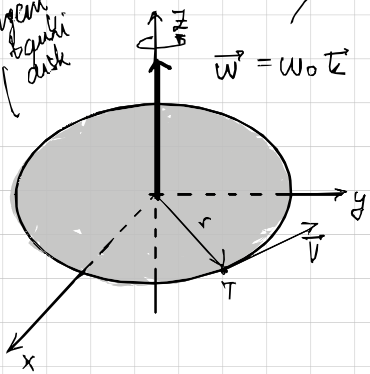
Primjer: $\vec{v}(x,y,z) = xz^4 i - 4xz j + 2yz^2 k$, $P(1,-2,1)$ $\vec{\nabla}\times\vec{v}$

$$\operatorname{rot}\vec{v} = \vec{\nabla}\times\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^4 & -4xz & 2yz \end{vmatrix} = i(2z+4x) - j(0-4xz^3) + k(-4z-0)$$

$$\operatorname{rot}\vec{v} = (2z+4x)i + 4xz^3 j - 4zk \overset{P}{\Longrightarrow} \boxed{\operatorname{rot}\vec{v} = (6,\ 4,\ -4)}$$

Primjer :

Homogeni tanki disk



$\vec{\omega} = \omega_3 \vec{k}$ oko osi z

vektor brzine je određen $\vec{v} = \vec{\omega} \times \vec{r}$

$$\vec{v} = \vec{\omega} \times \vec{r} = \begin{vmatrix} i & j & k \\ 0 & 0 & \omega_3 \\ x & y & 0 \end{vmatrix} = i(-\omega_3) + j\omega_3 \cdot x$$

$$\boxed{\vec{v} = (-\omega_3 y, \ \omega_3 \cdot x)}$$

brzina materijalne tačke T je vrtložno polje

$$\text{rot } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega_3 y & \omega_3 x & 0 \end{vmatrix}$$

$$\text{rot } \vec{v} = i \cdot 0 - j(0) + k(\omega_3 + \omega_3)$$

$$\text{rot } \vec{v} = \vec{\nabla} \times \vec{v} = (0, 0, 2\omega)$$

$2\omega \neq 0$, njegova rotacija ne iščezava

- rotacija $\varphi \cdot \vec{v} \Rightarrow \text{rot}(\varphi \vec{v}) = \vec{\nabla} \times (\varphi \vec{v})$

$$\vec{\nabla} \times (\varphi \vec{v}) = v \times (\vec{\nabla} \varphi) + \varphi \times (\vec{\nabla} \vec{v})$$

- Operator rotacije je linearan operator $\rightarrow \text{rot}(\lambda v + \mu s) = \nabla \times (\lambda v + \mu s)$

$$\text{rot}(\lambda \vec{v} + \mu \vec{s}) = \lambda \nabla \times \vec{v} + \mu \nabla \times \vec{s}$$

vekt produkt sa samim sobom jednak je nulvektoru

* za sve radijalne funkcije $f(r)$

$\longrightarrow \text{rot}(f(r)) = 0$

$\hookrightarrow$ polja oblika $f(r) \cdot \vec{r} \Rightarrow$ nužno bezvrtložna

4.2.4. Neka specijalna vektorska polja

Propozicija: Za glatko vektorsko polje $\vec{v}: \mathbb{R}^3 \to \mathbb{R}^3$ vrijedi $\begin{cases} \vec{v} = \text{grad } p \\ \text{rot } \vec{v} = 0 \end{cases}$

· potencijalna polja su nužno vrtložna

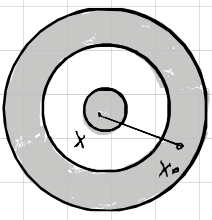
Primjer: vekt polje $\vec{v}$ je potencijalno $\vec{v} = \text{grad } p \to \vec{v} = (\partial_x p, \partial_y p, \partial_z p)$

pokažimo da je $\text{rot } \vec{v} = 0$

$$\text{rot } \vec{v} = \vec{\nabla} \times \vec{v} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \partial_x p & \partial_y p & \partial_z p \end{vmatrix}$$

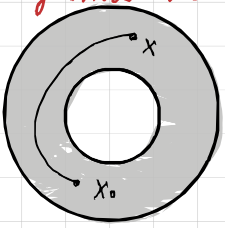
$$\vec{\nabla} \times \vec{v} = i(\partial_{yz} p - \partial_{zy} p) - j(\partial_{xz} p - \partial_{xz} p) + k(\partial_{xy} p - \partial_{xp} p) = \boxed{0}$$

$\vec{v} = \text{grad } p \implies \text{rot } \vec{v} = 0$ ALi obrat vrijedi samo ako je područje Ω jednostavno povezano!



x, x_0

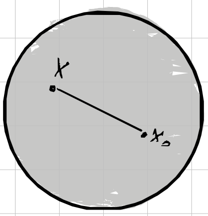
XX nije povezano



x, x_0

V povezano

X nije jednostavno povezano



x, x_0

W jednostavno povezano

Primjer 2.24.) Je li zadano polje potencijalno? Koliki je potencijal?

$$\vec{a}(x,y,z) = e^y i + (xe^y + \sin z) j + y \cos z\, k$$

$\text{rot } \vec{a} = 0$ ako je potencijalno: $\longrightarrow$ $\vec{\nabla} \times \vec{a} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & xe^y + \sin z & y\cos z \end{vmatrix}$

$$\text{rot } \vec{a} = i(\cos z - \cos z) - j(0 - 0) + k(e^y - e^y) = 0 \; W$$

$\to$ postoji f-ja p takva da je $\vec{a} = \text{grad } p$

$$\frac{\partial p}{\partial x} = e^y \qquad \frac{\partial p}{\partial y} = xe^y + \sin z \qquad \frac{\partial p}{\partial z} = y \cos z$$

$\int$

$$p = xe^y + \varphi(y,z) \longrightarrow \frac{\partial \varphi}{\partial y} = \sin z \; / \int$$

$$\implies \varphi = y \sin z + \psi(z)$$

$$\underline{\underline{p = xe^y + y \sin z + \psi(z)}}$$

$$\psi(z) = \varphi - y \sin z$$

3. uvjet: $\frac{\partial p}{\partial z} = y \cos z \to \boxed{\psi'(z) = 0}$, $\psi(z) = \text{const.}$

$$\to \boxed{p(x,y,z) = xe^y + y \sin z + C}, \quad C \in \mathbb{R}$$

Vektorski potencijal solenoidalnog polja

- $\vec{V}$ solendioalno ako $\operatorname{div} \vec{V} = 0$

- vekt polje $\vec{u}$ → potencijal vekt polja $\vec{V}$ ako $\vec{V} = \operatorname{rot} \vec{u}$

$$\operatorname{div}(\operatorname{rot} \vec{u}) = \frac{\partial}{\partial x}\left(\frac{\partial u_3}{\partial z} - \frac{\partial u_2}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) = 0$$

→ **polje koje se može zapisati kao rotacija nekog vekt polja je nužno solenoidalno!**