10.2. Primjena L transformacije u rješavanju krugova

opći slučaj:

$$a_n \frac{d^n y_n}{dt^n} + a_{n-1} \frac{d^{n-1} y_n}{dt^{n-1}} + \dots + a_0 y_n = b_m \frac{d^m x}{dt^m} + \dots + b_0 x$$

Primjenom $\mathcal{L}$. trans:

$\mathcal{L}(y(t)) = Y(s)$

$\mathcal{L}(y'(t)) = sY(s) - y(0)$

$\mathcal{L}(y^n(t)) = s^n Y(s) - s^{n-1} y(0) - \dots - y^{n-1}(0)$

$$a_n\left(s^n Y(s) - s^{(n-1)} y(0) \dots\right) + a_{n-1}\left(s^{n-1} Y(s) - s^{n-2} y(0) \dots\right) + a_0 Y(s) = b_m\left[s^m X(s) - s^{m-1} x(0) - s^{m-2} \frac{dx}{dt}\Big|_{t=0} - \dots\right) + \dots + b_0 X(s)$$

Pretpostavka: svi početni uvjeti jednaki su nuli

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0) Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_0) X(s)$$

Formiramo funkciju $H(s)$ $\frac{Y(s)}{X(s)} = \frac{(b_m s^m + b_{m-1} s^{m-1} + \dots + b_0)}{(a_n s^n + a_{n-1} s^{n-1} + \dots + a_0)} = \frac{P(s)}{Q(s)}$

odnos odziva i poticaja

Funkcija mreže ili funkcija sistema

- odziv mreže je u potpunosti definiran sa $Y(s) = H(s) \cdot X(s) = \frac{P(s)}{Q(s)} \cdot X(s)$

→ možemo koristiti i s početnim uvjetima ali za definiciju je potrebno da kažemo da su 0

↳ ako def da su poč. uvjeti $\neq 0$ → $Y(s) = \frac{P(s)}{Q(s)} \cdot X(s) + \frac{D(s)}{Q(s)}$

odziv mreže na pobudu $x(t)$

↳ $x \to \boxed{H(s)} \to y$ $\qquad Y(s) = H(s) \cdot X(s)$

→ u vremenskoj domeni to je konvolucija

$$y(t) = \int_0^t h(\tau) x(t-\tau)\, d\tau = \int_0^t h(t-\tau) x(\tau)\, d\tau$$

(primjenjiva na sve linearne funkcije, vrijedi superpozicija)

$y(t) = h(t) * x(t)$

Diracova delta fja

→ za slučaj kada je $x(t) = \delta(t)$

↳ $X(s) = 1 \xrightarrow{\text{odziv}} Y(s) = H(s)$

– u vremenskoj domeni $y(t) = h(t)$ → jedinični impuls mreže

$$H(s) = \frac{P(s)}{Q(s)} = k \frac{(s - s_{01})(s - s_{02}) \cdots (s - s_{0n})}{(s - s_{p1})(s - s_{p2}) \cdots (s - s_{pn})} = k \frac{\prod_{i=1}^{m} (s - s_{0i})}{\prod_{j=1} (s - s_{pj})} \quad \begin{matrix}\text{(nule)} \\ \text{(polovi)}\end{matrix}$$

s_{0i} → $s_{0i} = \sigma_{0i} + j\omega_{0i}$
→ nule funkcije mreže

s_{pj} → polovi funkcije mreže
→ $s_{pj} = \sigma_{pj} + j\omega_{pj}$

} realni (sam) ili kompleksni (konjugirano kompleksni parovi) (parovi)

Odziv mreže $y(t)$ → rastav $Y(s)$ u parcijalne razlomke

1) $m > n$ → podijeli

„Pravi razlomak"

$$Y(s) = \cancel{P_1(s)} + \frac{P_2(s)}{Q(s)}$$

polinom

↳ polinom čiji je stupanj < stupnja Q

2) ostatak funkcije $H(s)$ → u parcijalne

$$\frac{P_2(s)}{Q(s)} = \frac{k_1}{s-s_{p1}} + \frac{k_2}{s-s_{p2}} + \dots + \frac{k_n}{s-s_{pn}} = \sum_{i=1}^{n} \frac{k_i}{s-s_{pi}}$$

$$\frac{k_1}{s-s_{p1}} \circ\!\!-\!\!\circ \; k_1 \cdot e^{s_{p1} t} \cdot S(t)$$

u vremenskoj domeni svakom članu sume odgovara eksp. f-ja

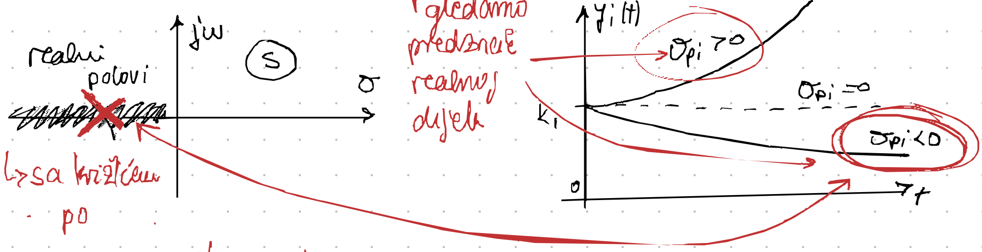
=> odziv ovisi o polovima funkcije

Primjer: suma u gornjem izrazu sadrži član oblika $Y_i(s) = \frac{k_i}{s-\sigma_{pi}}$, $s_{pi} = \sigma_{pi}$ realni pol

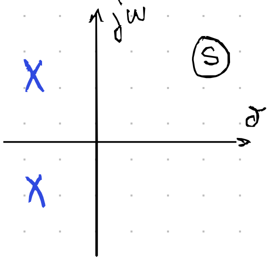
$s_{p1} = \sigma_{p1} + \cancel{j\omega_{p1}}$

$$Y_i(s) = \frac{k_i}{s-\sigma_{pi}} \;\bullet\!\!-\!\!\circ\; y(t) = \mathcal{L}^{-1}\left[\frac{k_i}{s-\sigma_{pi}}\right] = k_i \cdot e^{\sigma_{pi} t}$$

gledamo predznak realnog dijela



Primjer: slučaj da je neki pol $\boxed{s_{pi}}$ kompleksan $s_{pi} = \sigma_{pi} + j\omega_{pi}$



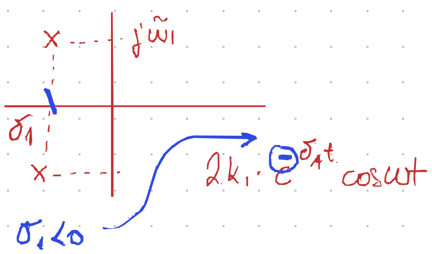
↳ uvijek postoji njegov kompleksno konjugirani par

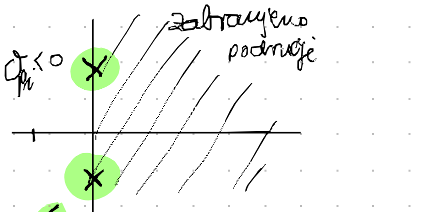
$$s_{p1}^* = \sigma_{pi} - j\omega_{pi}$$

$$\Rightarrow Y_1(s) = \frac{k_i}{s-\sigma_{pi} - j\omega_{pi}} + \frac{k_i}{s-\sigma_{pi} + j\omega_{pi}}$$

$$\dots \rightarrow Y_1(s) = 2k_i \frac{s-\sigma_{pi}}{(s-\sigma_{pi})^2 + \omega_{pi}^2}$$

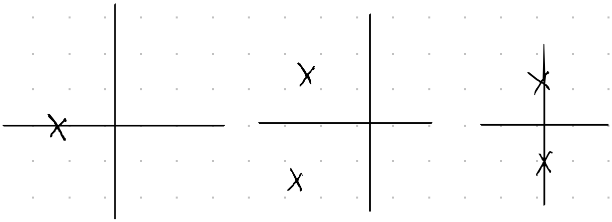
$$\circ\!\!-\; y(t) = \mathcal{L}^{-1}[Y_1(s)] = 2k_i e^{\sigma_{pi} t} \cos \boxed{\omega t_{pi} t}$$



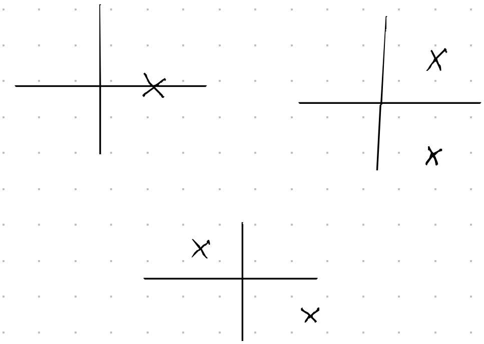


↳ $\sigma_{pi} = 0$ → nema prigušenja

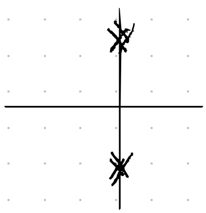
Okej polovi



NE! polovi

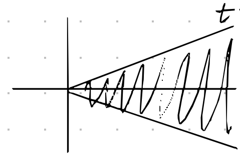


Specijalan slučaj: Višestruki pol na $j\omega$-osi

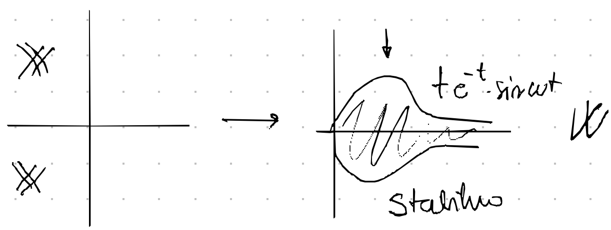


$$Y(s) = \frac{k_i}{(s-j\omega_i)^2} + \frac{k_i}{(s+j\omega_i)^2} = k_i \frac{2(s^2-\omega_{pi}^2)}{(s^2+\omega_{pi}^2)^2}$$

$\hookrightarrow y(t) = 2k_i t \cdot \cos\omega_0 t \longrightarrow$



nije stabilno

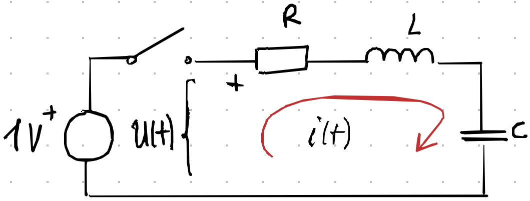


Amplituda teži u ∞
$\hookrightarrow$ nestabilan odziv $\to$ nije dozvoljeno

Ukratko:

- racionalna funkcija kompleksne varijable „s" s realnim koef.
- polovi funkcije $H(s)$ ne smiju biti u desnoj poluravnini
- polovi na $j\omega$ osi ne smiju biti višestruki

Primjer 1.



$R = 3\,\Omega$

$L = 1\,H$

$C = 0.5F$

$i_L(0) = 0$

$u_C(0) = 0$

$u(t) = S(t)$

$$u(t) = Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)d\tau) + u_C(0)$$

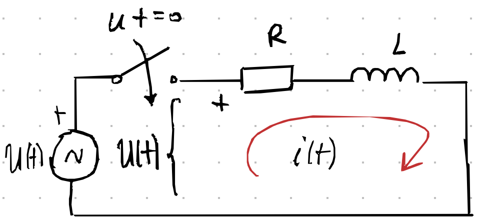
$$U(s) = RI(s) + sLI(s) - Li(0) + \frac{1}{Cs}I(s) + \frac{u_C(0)}{s}$$

$i = 0 \quad u_C = 0 \qquad U(s) = \frac{1}{s} \quad \longrightarrow \quad U(s) = I(s)\left(sL + R + \frac{1}{Cs}\right)$

$$\rightarrow I(s) = \frac{sU(s)}{s^2L + Rs + \frac{1}{C}} = \frac{1}{L}U(s)\cdot s\,\frac{1}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\frac{1}{s}\cdot s}{s^2 + 3s + 2} = \frac{1}{(s+1)(s+2)}$$

$$\Rightarrow \dots = i(t) = (e^{-t} - e^{-2t})\,S(t)$$

Primjer 2.)



$u(t) = V\sin\omega t$

$i_L(0) = 0$

$i(t) = ?$

$$u(t) = i(t)\cdot R + L\cdot\frac{di}{dt}$$

$$U(s) = RI(s) + L\cdot(sI(s) - \underbrace{\cancel{i(0)}}_{0}) \Rightarrow RI(s) + LsI(s) = U(s)$$

$$I(s) = \frac{U(s)}{R + sL} = \frac{V}{L}\left[\frac{\omega}{\omega^2 + s^2}\cdot\frac{1}{s + \frac{R}{L}}\right] = \frac{V\cdot\omega}{L}\left[\frac{As + B}{s^2 + \omega^2} + \frac{C}{s + \frac{R}{L}}\right]$$

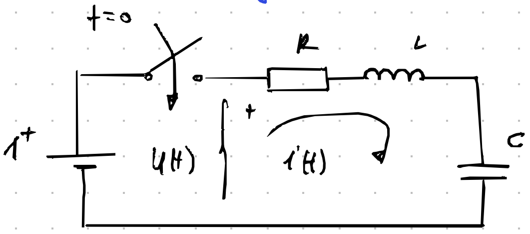
$$\dots \quad A = \frac{-L^2}{R^2 + \omega^2L^2} \qquad B = \frac{RL}{R^2 + \omega^2L^2} \qquad C = \frac{L^2}{R^2 + \omega^2L^2}$$

$$I(s) = \frac{V\omega}{R^2 + \omega^2L^2}\cdot\left[\frac{-sL}{s^2 + \omega^2} + \frac{R}{s^2 + \omega^2} + \frac{L}{s + \frac{R}{L}}\right]$$

↓ invert.

$$\hookrightarrow i(t) = \frac{V\omega}{R^2 + \omega^2L^2}\cdot\left[-L\cos(\omega t) + \frac{R}{\omega}\sin(\omega t) + e^{-\frac{R}{L}t}\right]S(t)$$

RLC - krug - karakterističan krug drugog reda



$u(t) = S(t) \rightarrow U(s) = \frac{1}{s}$

$$I(s) = \frac{\frac{1}{L} \cdot s}{s^2 + \frac{R}{L}s + \frac{1}{LC}} \cdot U(s)$$

$$= \frac{1}{L} \cdot \frac{1}{s^2 + \frac{R}{L} \cdot s + \frac{1}{LC}}$$

POLOVI:

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \longrightarrow s_{1,2} = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L} - \frac{1}{LC}}$$

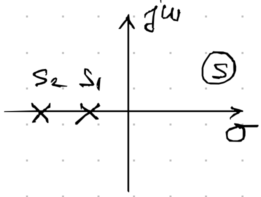
$\frac{R}{2L} = \alpha$

$\frac{1}{LC} = \omega_0^2$

1) Nadkritično prigušeni odziv

$\alpha > \omega_0 \Rightarrow R > 2\sqrt{\frac{L}{C}}$

$s_{1,2} = -\alpha \pm \sqrt{a^2 - \omega_0^2}$



s_1 i $s_2 \longrightarrow$ realni i različiti

$s_1 = -a_1$

$s_2 = -a_2 \longrightarrow I(s) = \frac{1}{L} \cdot \frac{1}{(s+a_1)(s+a_2)} = \frac{k_1}{s+a_1} + \frac{k_2}{s+a_2}$

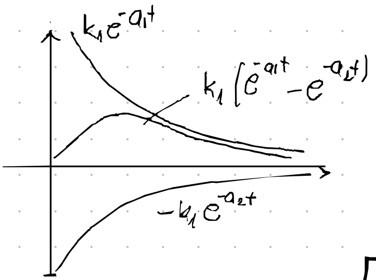
$$\frac{1}{L} = k_1(s+a_2) + k_2(s+a_1) \Rightarrow \begin{cases} k_1 + k_2 = 0 \\ k_1 a_2 + k_2 a_1 = \frac{1}{L} \end{cases}$$

$$k_1 = -k_2 = \frac{1}{L} \cdot \frac{1}{a_2 - a_1}$$

odziv:

$i(t) = k_1 e^{-a_1 t} + k_2 e^{-a_2 t}$

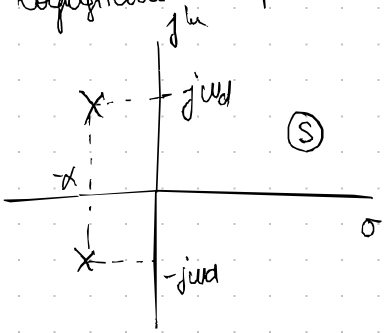
$i(t) = k_1 (e^{-a_1 t} - e^{-a_2 t})$



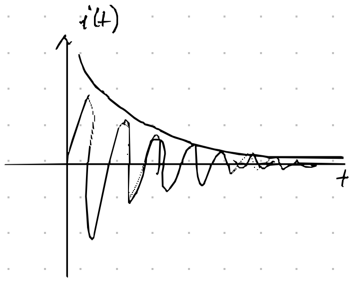
2) Kritično prigušeni odziv $a = \omega_0 \Rightarrow R = 2\sqrt{\frac{L}{C}}$

3) Podkritično prigušeni odziv $a < \omega_0 \rightarrow R < 2\sqrt{\frac{L}{C}}$

konjugirano kompleksni korijeni

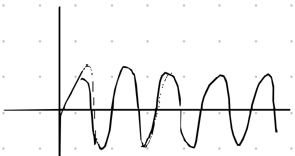


$\Rightarrow$



4) Neprigušeni odziv $\alpha = 0 \rightarrow R = 0$

imaginarni polovi $\rightarrow$



$i(t) = k \cdot \sin(\omega_0 t + \varphi)$