4.6. EGZAKTNE DJ

Pr.) $(3x^2+y)dx + (2y+x)dy = 0$

- ne možemo separirati (ne možemo izlučiti x i y)
- nije linearna (samo y)
- nije Bernoullijeva ni homogena

$* \; (3x^2+y) + (2y+x)\frac{dy}{dx} = 0 \quad \frac{dy}{dx} \to y'$

$\to (3x^2+y)dx + (2y+x)dy = 0 \to$ pravi diferencijal

$(3x^2+y)$ je $\frac{\partial u}{\partial x}$, $(2y+x)$ je $\frac{\partial u}{\partial y}$

$du(x,y) = 0$

$\to u(x,y) = x^3 + xy + y^2 + \underline{C}$ (mora odgovarati $\frac{\partial u}{\partial x}$! i $\frac{\partial u}{\partial y}$!)

rješenje ćemo dobiti kada izjednačimo s 0:

Rj: $x^3 + xy + y^2 = C$

$\boxed{\text{DEF}}$ DJ $P(x,y)dx + Q(x,y)dy = 0$ je EGZAKTNA ako postoji $u(x,y)$ t.d. je $du(x,y) = P(x,y)dx + Q(x,y)dy$, tj. ako je $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = Q$. $\boxed{\text{Tada je opće rješenje } u(x,y) = C.}$

$\boxed{\text{TM}}$ Nužan uvjet egzaktnosti

Ako je DJ egzaktna, tada $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.}$

Dokaz: Po pretpostavci $\frac{\partial u}{\partial x} = P$, $\frac{\partial u}{\partial y} = Q$.

$\frac{\partial u}{\partial x} = P \Big/ \frac{\partial u}{\partial y} \qquad \frac{\partial u}{\partial y} = Q \Big/ \frac{\partial u}{\partial x}$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial P}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

prema Schwarzovom teoremu $\boxtimes$

$\to$ vrijedi obrat?

$\boxed{\text{TM}}$ (dovoljan uvjet): Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, tada je DJ egzaktna, tj. tada postoji $u(x,y)$ koji se računa po formuli $u(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy + C$ gdje je (x_0,y_0) proizvoljno odabrana točka iz domene funkcija.

Nap. $u(x,y)$ se naziva POTENCIJAL.

Dokaz: $\frac{\partial u}{\partial x} = P(x,y) \Big/ \int_{x_0}^{x} dx$

$$u(x,y) = \underbrace{\int_{x_0}^{x} P(x,y)dx}\; + C(y) \Big/ \frac{\partial}{\partial y}$$

jer derivacijom po x, y tretiramo kao konstantu, odnosno funkciju od konstante što možemo zapisati kao $C(y)$ ALI TO NIJE KONSTANTA!

$$\frac{\partial u}{\partial y} = Q(x,y) = \int_{x_0}^{x} \underbrace{\frac{\partial P(x,y)}{\partial y}}\, dx + C'(y)$$

→ koristili smo integ. zad. par (3.1)

Po pretpostavci $= \frac{\partial Q(x,y)}{\partial x}$.

$$\cancel{Q(x,y)} \Rightarrow \int_{x_0}^{x} \frac{\partial Q(x,y)}{\partial x} dx + C'(y) = \cancel{Q(x,y)} - Q(x_0,y) + C'(y)$$

$$C'(y) = Q(x_0,y) \Big/ \int_{y_0}^{y} dy \implies \underline{C(y) = \int_{y_0}^{y} Q(x_0,y)\,dy.}$$

$$\underbrace{\text{Analogno: } u(x,y) = \int_{x_0}^{x} P(x,y_0)\,dx + \int_{y_0}^{y} Q(x,y)\,dy + C}_{\text{DZ}}$$

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$$(2x + y^2\cos(xy^2))dx + (2xy\cos(xy^2) + 3y^2)dy = 0$$

1. provjeriti je li egzaktna

$$\frac{\partial P}{\partial y} = 2y\cos(xy^2) + y^2(-\sin(xy^2)) \cdot (2xy)$$

$$\parallel$$

$$\frac{\partial Q}{\partial x} = 2y\cos(xy^2) + 2xy(-\sin(xy^2)) \cdot y^2$$

2. što nam je lakše prvo:

granice su proizvoljne, ali u 99% slučajeva možemo uzeti 0

$$u(x,y) = \int_0^x (2x + y^2\cos(xy^2))dx + \int_0^y (0 + 3y^2)dy$$

$$= \left(x^2 + \cancel{y^2}\frac{\sin(xy^2)}{\cancel{y^2}}\right)\Big|_0^x + y^3\Big|_0^y = x^2 + \sin(xy^2) + y^3 + C = \text{POTENCIJAL}$$

3. Rješenje: $\boxed{x^2 + \sin(xy^2) + y^3 = C}$

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$$\left(\frac{\sin^2 x}{y^2}\right)dx + \left(\frac{\alpha(x - \sin x \cdot \cos x)}{y^3} + \cos y\right)dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{-2\sin^2 x}{y^3} \qquad \frac{\partial Q}{\partial x} = \alpha\frac{(1 - \cos^2 x) + \sin^2 x}{y^3} = \frac{2\sin^2 x}{y^3} \cdot \alpha$$

$$\boxed{\alpha = -1}$$

$x_0 = 0$

$$u(x,y) = \int_0^x \left(\frac{\sin^2 x}{y^2}\right)dx + \int_0^x \left(\frac{\sin x\cos x - x}{y^3} + \cos y\right)dy = 0$$

$\sin y - \sin 1$

$$= \int_0^x \frac{\sin^2 x}{y^2}dx + \int_0^x (0 + \cos y)\,dy = 0$$

$$= \frac{1}{y^2}\int_0^x \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx + \sin y\Big|_1^y = \frac{1}{y^2}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^x + \sin y - \sin 1$$

$$\boxed{\rightarrow \frac{1}{y^2}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) + \sin y = C}$$

egz. $(x^2+y)dx + (y^2+x)dy = 0$

nije egz. $\left(x+\frac{y}{x}\right)dx + \left(\frac{y^2}{x}+1\right)dy = 0 \;/\cdot x$ ← eulerov multiplikator

→ množenje jednadžbe s fjom → mijenja egzaktnost DJ

$\boxed{\text{DEF}}$ Eulerov multiplikator

Funkciju $\mu(x,y)$ s kojom treba pomnožiti DJ da postane egzaktna nazivamo EULEROV MULTIPLIKATOR.

Postupak traženja:

$$P(x,y)dx + Q(x,y)dy = 0 \;/\cdot \mu(x,y)$$

$$\mu(x,y)\cdot P(x,y)\,dx + \mu(x,y)Q(x,y)dy = 0$$

uvjet: $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x} \Rightarrow \mu_y'\cdot P + \mu P_y' = \mu_x' Q + \mu Q_x'$, $\mu = ?$

$\boxed{\text{ZI-2019-4}}$

a) Izvesti formulu: $\mu(y) \Rightarrow \mu_y' P + \mu P_y' = 0 + \mu Q_x'$

$$\frac{d\mu P}{dy} = \mu(Q_x' - P_y') \;/\; \frac{dy}{\mu P}$$

$$\int \frac{d\mu}{\mu} = \int \underbrace{\frac{(Q_x' - P_y')}{P}}\,dy \Rightarrow \boxed{\ln|\mu| = \int \frac{Q_x' - P_y'}{P}\,dy}$$

uvjet: ovo je funkcija ovisna SAMO o y !!

ovisi samo o y

b) Riješiti Cauchyjev korem

$$\begin{cases} (\cos x + y)dx + \left(3x + \frac{2}{y}\sin x\right)dy = 0 \\ y(\pi) = 1 \end{cases}$$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 3 + \frac{2}{y}\cos x$ nije egz.

$$\rightarrow \ln|\mu| = \int \frac{Q_x' - P_y'}{P}\,dy = \int \frac{3 + \frac{2}{y}\cos x - 1}{\cos x + y}\,dy = \int \frac{2(\cos x + y)}{y(\cos x + y)}\,dy$$

$\ln|\mu| = 2\ln y \Rightarrow \boxed{\mu = y^2}$ (C koji god želimo)

$\rightarrow (y^2\cos x + y^3)\,dx + (3xy + 2y\sin x)dy = 0$

$\frac{\partial P}{\partial y} = 2y\cos x + y^2(-\sin x) + 3y^2 = \frac{\partial Q}{\partial x} = \quad T(0,0)$ ✓

$$u(x,y) = \int_0^x (y^2\cos x + y^3)dx + \int_0^y (0+0)dy$$

$$= (y^2\sin x + y^3 x)\Big|_0^x \rightarrow$$

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7\. a) Izvesti formulu za Eulerov multiplikator oblika $\mu=\mu(x)$ dif. jed. $P(x,y)\,dx+Q(x,y)\,dy=0 \;/\cdot\mu(x)$ → eulerov mult pl. je fja s kojom trebamo pomnožiti DJ da bismo dobili egzaktnu DJ

$$\mu(x)P(x,y)\,dx+\mu(x)Q(x,y)\,dy=0$$

uvjet:

$$\frac{\partial}{\partial y}\left[\mu(x)P(x,y)\right]=\frac{\partial}{\partial x}\left[\mu(x)\cdot Q(x,y)\right]$$

$$0\cdot P(x,y)+\mu(x)\cdot P_y'(x,y)=\mu'(x)\cdot Q(x,y)+\mu(x)\cdot Q_x'(x,y)$$

$$\frac{d\mu}{dx}\cdot Q(x,y)=\mu(x)\left[P_y'(x,y)-Q_x'(x,y)\right]\Big/\cdot\frac{dx}{\mu(x)\cdot Q}\qquad \mu\neq 0$$

$$\int\frac{d\mu}{\mu(x)}=\int\frac{P_y'-Q_x'}{Q(x,y)}\,dx$$

$$\Rightarrow \boxed{\ln|\mu(x)|=\int\underbrace{\frac{P_y'-Q_x'}{Q(x,y)}}_{}\,dx}$$

uvjet: ovo je funkcija od x

Postupak je isti za bilo koji od multipl. koji je zadan

b) pripadni $\mu(x)$ i Cauchyjev teorem

$$(x^2+y^2+x)\,dx+y\,dy=0 \qquad y(0)=1$$

$$\frac{\partial P}{\partial y}=2y \qquad \frac{\partial Q}{\partial x}=0 \rightarrow \text{nije egzaktna} \Rightarrow \ln|\mu(x)|=\int\frac{2y-0}{y}\,dx=\int 2\,dx$$

ovisi samo o x

$$=2x+C$$

ne utječe na egzaktnost (možemo $c=0$)

$$\Rightarrow \ln\mu(x)=2x \rightarrow \boxed{\mu(x)=e^{2x}}$$

→ pomnožimo s multiplikatorom:

$$e^{2x}(x^2+y^2+x)\,dx+y\cdot e^{2x}\,dy=0$$

$$\frac{\partial P}{\partial y}=e^{2x}\cdot 2y \qquad \frac{\partial Q}{\partial x}=e^{2x}\cdot 2\cdot y \quad \text{sada je egzaktna}$$

→ možemo rješavati egzaktnost:

proizvoljna točka $T(0,0)$ → u domeni je

$$u(x,y)=\int_0^y y e^{2x}\,dy+\int_0^x (x^2+x)e^{2x}\,dx=\frac{y^2}{2}e^{2x}\Big|_0^y+$$

↳ dupla parc. integracija (jer e^{2x})

$$+\left[\frac{1}{2}e^{2x}(x^2+x)-\frac{1}{4}(2x+1)e^{2x}+\frac{1}{4}e^{2x}\right]\Big|_0^x$$

$$u(x,y)=\frac{y^2}{2}e^{2x}+\frac{1}{2}e^{2x}(x^2+x)-\frac{1}{4}(2x+1)e^{2x}+\frac{1}{4}e^{2x}+\cancel{\left[0-\frac{1}{4}+\frac{1}{4}\right]}+C$$

$$u(x,y)=\frac{y^2}{2}e^{2x}+\frac{x^2}{2}e^{2x}+\cancel{\frac{x}{2}e^{2x}}-\cancel{\frac{x}{2}e^{2x}}-\cancel{\frac{1}{4}e^{2x}}+\cancel{\frac{1}{4}e^{2x}}+C$$

$$u(x,y)=\frac{y^2}{2}e^{2x}+\frac{x^2}{2}e^{2x}+C \rightarrow \text{opće rješenje: } \frac{1}{2}y^2e^{2x}+\frac{1}{2}x^2e^{2x}=C$$

$$\boxed{e^{2x}(y^2+x^2)=C}$$

Napomena: DJ sa sep. varijablama su egzaktne DJ

→ separacije varijable su egzaktne DJ!

↳ sep: $Q(y)dy = P(x)dx \rightarrow \frac{\partial Q}{\partial x} = 0 \quad \frac{\partial P}{\partial y} = 0$ ✓

↳ egzaktne: $u(x,y) = \int_{x_0}^{x} P(x)dx + \int_{y_0}^{y} Q(y)dy + C = 0$

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$P(x, y_0) + Q(x, y)$ ILI $P(x, y) + Q(x_0, y)$ to je rješenje sep.

a) $\underbrace{\frac{x+y^2}{x^2}}_{P} dx - \underbrace{\frac{2y}{x}}_{Q} dy = 0 \qquad \frac{\partial P}{\partial y} = \frac{2y}{x^2} \qquad \frac{\partial Q}{\partial x} = \frac{-2y}{x^2}$ ✓

proizvoljna točka je $T(1,0)$ jer $x \neq 0$
(pr)

$u(x,y) = \int_1^x \frac{x+0}{\cancel{x^2}} dx + \int_0^y -\frac{2y}{x} dy = \ln|x| \Big|_1^x - \frac{y^2}{x}\Big|_0^y + C$ } ovo nije RJ.!

OPĆE RJ: $\boxed{\ln|x| - \frac{y^2}{x} = C}$

$\frac{1}{x} + \frac{y^2}{x^2} - \frac{2y}{x} y' = 0 \Big/ \cdot \left(\frac{-x}{2y}\right)$

Bernoulli način

$y' - \frac{1}{2y} - \frac{y}{2x} = 0$

$y' - \frac{y}{2x} = \frac{1}{2y}$