5.2. TROSTRUKI INTEGRALI

5.2.1. Uvod, svojstva, računanje

$$\underbrace{\iint}_{2D} \underbrace{f(x,y)}_{\text{graf u 3D}}\,dx\,dy \longrightarrow \text{interpretacija je volumen}$$

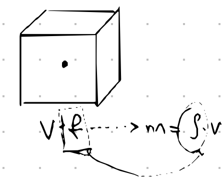
$\int$ - površina ispod krivulje

$\iint$ - volumen ispod plohe

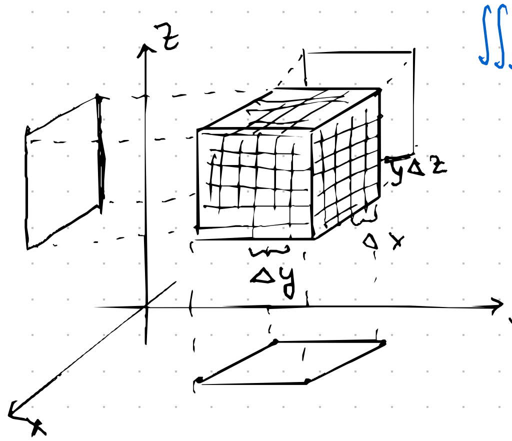
$\iiint$ - interpretiramo fizikalno

$$\underbrace{\iiint}_{\text{tijelo u 3D}} \underbrace{f(x,y,z)}_{\text{graf u 4D}}\,dx\,dy\,dz$$

↳ masa tijela V s gustoćom $f(x,y,z)$



DEF Definicija na kvadru



$$\iiint_V f(x,y,z)\,dx\,dy\,dz = \lim_{m,n,l\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} f(x_{ijk}, y_{ijk}, z_{ijk})\,\Delta x\Delta y\Delta z$$

$$\Longrightarrow \boxed{\iiint_V f(x,y,z)\,dV}$$

Fubinijev TM: uzastopno integriranje

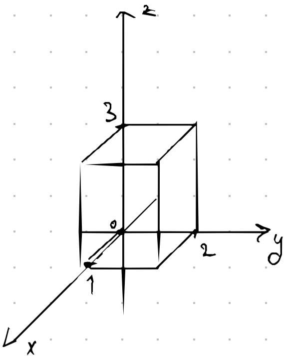
→ nije bitan redoslijed

Integracija po kvadru $V = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$ se svodi na 3 iterirana integrala u bilo kojem poretku

→ za n višestruki integral ima $n!$ poredaka

Primjer: $\int_0^1 dx \int_0^2 dy \int_0^3 \overbrace{(x^2+y^2+z^2)}^{\text{gustoća}} dz$

* sada gledamo tijela koja nemaju svuda jednaku gustoću



$$= \int_0^1 dx \int_0^2 \left(x^2 z + y^2 z + \frac{z^3}{3}\right)\Big|_{z=0}^{z=3} dy$$

$$= \int_0^1 dx \int_0^2 (3x^2 + 3y^2 + 9)\, dy$$

$$= \int_0^1 (3x^2 y + y^3 + 9y)\Big|_{y=0}^{y=2} dx$$

$$= \int_0^1 (6x^2 + 8 + 18)\, dx = (2x^3 + 26x)\Big|_0^1 = \underline{28}$$

Općenito:

$$\int_a^b dx \int_{y_1(x)}^{y_2(x)} dy \int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz$$

$y_2(x)$ (gornja) $\quad z_2(x,y)$ (gornja PLOHA!) → uvrstimo z koji ovisi o x i y

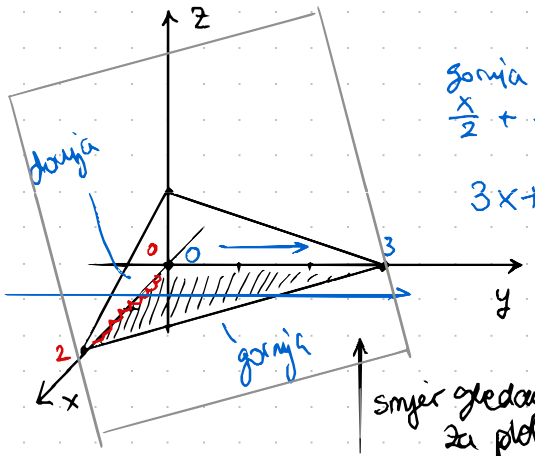
$y_1(x)$ (donja) → uvrstimo y koji ovisi o x $\quad z_1(x,y)$ (donja PLOHA!)

a, b konstante

postavljamo kao za dvostruki integral

=> projekcija u x-y ravnini

ZAD: Postavite granice za tetraedar s vrhovima (0,0,0), (2,0,0), (0,3,0), (0,0,1)



gornja

$\frac{x}{2} + \frac{y}{3} = 1$

$3x + 2y = 6$

$y = 3 - \frac{3}{2}x$

$$\int_0^2 dx \int_0^{3-\frac{3}{2}x} dy \int_{z=0}^{z=1-\frac{x}{2}-\frac{y}{3}} f(x,y,z)\, dz$$

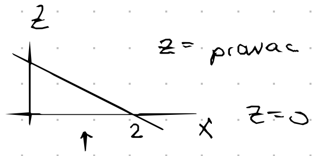
$z = 1 - \frac{x}{2} - \frac{y}{3}$

$\frac{x}{2} + \frac{y}{3} + \frac{z}{1} = 1$

→ donja ploha je $z = 0$

→ gornja je kosi trokut → jednadžba ravnine 3 točke

drugačiji poredak: x-z ravnina gledamo →



$$\int_0^1 dz \int_0^{2-2z} dx \int_0^{3-3/2x-3z} dy$$

Svojstva trostrukog integrala -slajd 8 moodle ppt

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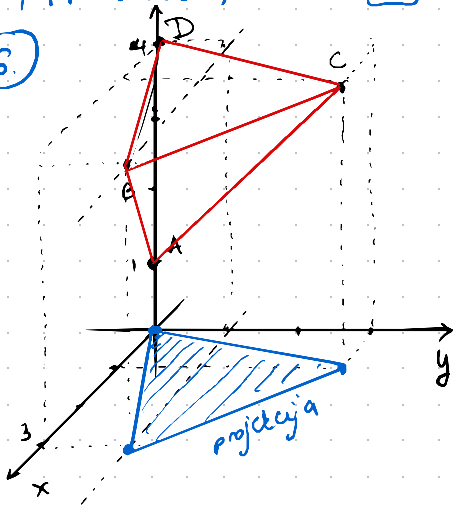
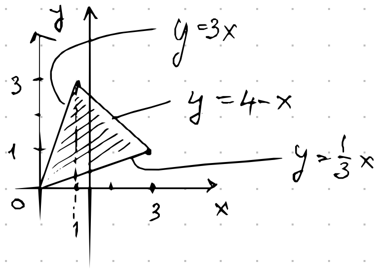
$A(\underbrace{0,0},1)$ $B(\underbrace{3,1},4)$ $C(\underbrace{1,3},4)$ $D(\underbrace{0,0},4)$

6.



$$\int dx \int dy \int f(x,y,z)\,dz$$



$$= \int_0^3 dx \int_{1/3 x}^{3x} dy \int_{\frac{3}{4}x+\frac{3}{4}y+1}^{4} f\,dz + \int_0^3 dx \int_{1/3 x}^{4-x} dy \int_{\frac{3}{4}x+\frac{3}{4}y+1}^{4} f\,dz$$

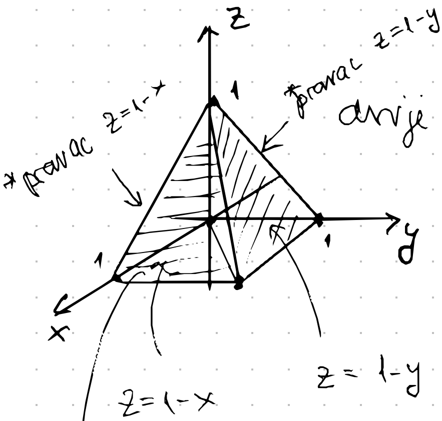
$$\vec{n} = \vec{AC} \times \vec{AB} = \begin{vmatrix} i & j & k \\ 1 & 3 & 3 \\ 3 & 1 & 3 \end{vmatrix} = (6,6,-8) \xrightarrow{:2} (3,3,-4)$$

točka A: $\pi \ldots 3(x-0) + 3(y-0) - 4(z-1) = 0$

$\pi \ldots 3x + 3y - 4z + 4 = 0$

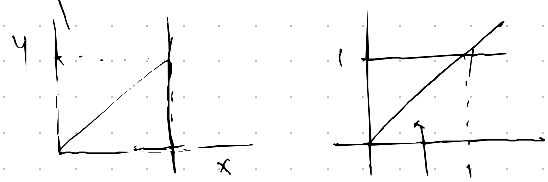
$z = \frac{3}{4}x + \frac{3}{4}y + 1$

Zadatak: Piramida s vrhovima $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$



dvije plohe → dva trostruka int:

$$\int_0^1 dx \int_0^x dy \int_0^{1-x} f\,dz + \int_0^1 dx \underbrace{\int_x^1 dy}_{\text{ili } \int_0^1 dy \int_0^y dx} \int_0^{1-y} f\,dz$$



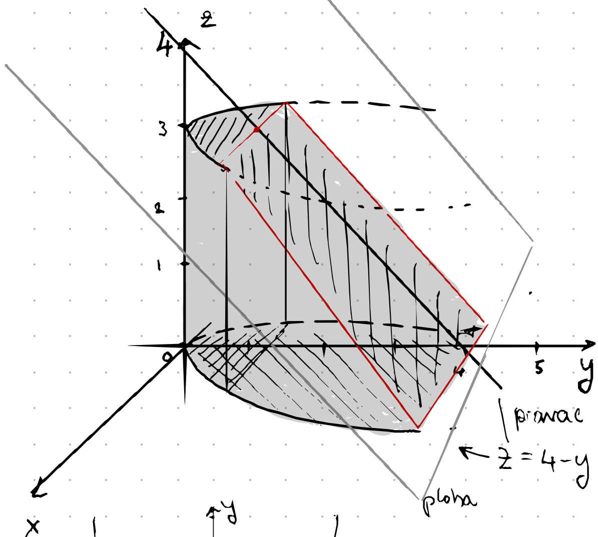
ILI: jedan trostruki

$$\int_0^1 dy \int_0^{1-y} dz \int_0^{1-z} f\,dx$$

ILI

$$\int_0^1 dz \int_0^{1-z} dx \int_0^{1-z} f\,dy$$

Pr. 5.) Izračunajte volumen tijela

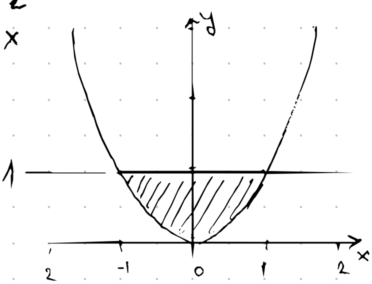


$z = 4 - y$

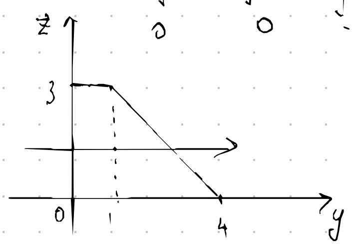
$$V = \iiint_V dV$$

krivulje

$$V = \int_{-1}^{1} dx \int_{x^2}^{1} dy \int_0^3 dz + \int_1^4 dy \int_{-\sqrt{y}}^{\sqrt{y}} dx \int_0^{4-y} dz$$



|Z|: $\displaystyle\int_0^3 dz \int_0^{4-z} dy \int_{-\sqrt{y}}^{\sqrt{y}} dx$



TM Srednje vrijednosti

Postoji točka takva da vrijedi $\rightarrow \displaystyle\iiint_V f(x,y,z)\,dV = f(x_0,y_0,z_0)\,\mu(V)$

- Ako f predstavlja gustoću tijela V, onda postoji točka tijela u kojoj se gustoća podudara s prosječnom gustoćom

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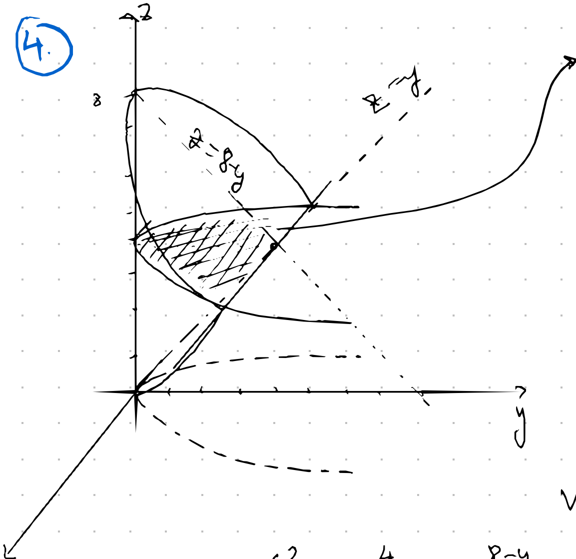
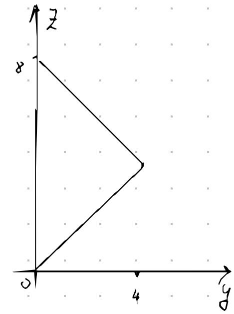
4.



(c) sve te točke, kojih je beskonačno (na ravnini $z=4$ unutar parabole) imaju presječnu gustoću



a)

$$V = \int_0^4 dy \int_y^{8-y} dz \int_{-\sqrt{y}}^{\sqrt{y}} dx = \frac{512}{15}$$

ILI (a): $\int_{-2}^{2} dx \int_{x^2}^{4} dy \int_y^{8-y} dz$

b) $\int_{-2}^{2} dx \int_{x^2}^{4} dy \int_y^{8-y} z\, dz = \frac{2048}{15}$