4.6. EGZAKTNE DJ

Pr.) $(3x^2+y)dx + (2y+x)dy = 0$

- ne možemo separirati (ne možemo izlučiti x i y)
- nije linearna (samo y)
- nije Bernoullijeva ni homogena

$\ast \ (3x^2+y) + (2y+x)\dfrac{dy}{dx} = 0$ ($\dfrac{dy}{dx} \to y'$)

$\to (3x^2+y)dx + (2y+x)dy = 0 \to$ pun diferencijal

$\uparrow \dfrac{\partial u}{\partial x}$ $\uparrow \dfrac{\partial u}{\partial y}$ $du(x,y) = 0$

$\longrightarrow u(x,y) = x^3 + xy + y^2 + \underline{\underline{C}}$ (mora odgovarati $\dfrac{\partial u}{\partial x}$! $\dfrac{\partial u}{\partial y}$!)

rješenje ćemo dobiti kada izjednačimo s 0:

Rj: $x^3 + xy + y^2 = C$

DEF DJ $P(x,y)dx + Q(x,y)dy = 0$ je EGZAKTNA ako postoji $u(x,y)$ t.d. je $du(x,y) = P(x,y)dx + Q(x,y)dy$, tj. ako je $\dfrac{\partial u}{\partial x} = P$, $\dfrac{\partial u}{\partial y} = Q$. Tada je opće rješenje $u(x,y) = C$.

TM Nužan uvjet egzaktnosti

Ako je DJ egzaktna, tada $\boxed{\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}}$.

Dokaz: Po pretpostavci $\dfrac{\partial u}{\partial x} = P$, $\dfrac{\partial u}{\partial y} = Q$.

$\dfrac{\partial u}{\partial x} = P \Big/ \dfrac{\partial u}{\partial y}$ $\dfrac{\partial u}{\partial y} = Q \Big/ \dfrac{\partial u}{\partial x}$

$$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial P}{\partial y} \quad = \quad \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

prema Schwarzovom teoremu $\boxtimes$

$\to$ vrijedi obrat?

$\boxed{\text{TM}}$ (dovoljan uvjet): Ako je $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, tada je DJ egzaktna, tj. tada postoji $u(x,y)$ koji se računa po formuli $u(x,y) = \int_{x_0}^{x} P(x,y)dx + \int_{y_0}^{y} Q(x_0,y)dy + C$ gdje je (x_0,y_0) proizvoljno odabrana točka iz domene funkcija.

Nap. $u(x,y)$ se naziva POTENCIJAL.

Dokaz: $\frac{\partial u}{\partial x} = P(x,y) \Big/ \int_{x_0}^{x} dx$

$$u(x,y) = \underbrace{\int_{x_0}^{x} P(x,y)dx}_{} + C(y) \Big/ \frac{\partial}{\partial y}$$

jer derivacijom po x, y tretiramo kao konstantu, odnosno funkciju od konstante što možemo zapisati kao $C(y)$ ALI TO NIJE KONSTANTA!

$$\frac{\partial u}{\partial y} = Q(x,y) = \int_{x_0}^{x} \underbrace{\frac{\partial P(x,y)}{\partial y}}_{} dx + C'(y)$$

→ koristili smo integ. zad. par (3.1)

Po pretpostavci $= \frac{\partial Q(x,y)}{\partial x}$.

$$\cancel{Q(x,y)} \Rightarrow \int_{x_0}^{x} \frac{\partial Q(x,y)}{\partial x} dx + C'(y) = \cancel{Q(x,y)} - Q(x_0,y) + C'(y)$$

$$C'(y) = Q(x_0,y) \Big/ \int_{y_0}^{y} dy \Rightarrow \underline{C(y) = \int_{y_0}^{y} Q(x_0,y)\,dy.}$$

Analogno: $u(x,y) = \int_{x_0}^{x} P(x,y_0)\,dx + \int_{y_0}^{y} Q(x,y)\,dy + C$ } DZ

$\boxed{\text{MI-2020}}$ 17. $(2x + y^2\cos(xy^2))dx + (2xy\cos(xy^2) + 3y^2)dy = 0$

1. provjeriti je li egzaktna

$$\frac{\partial P}{\partial y} = 2y\cos(xy^2) + y^2(-\sin(xy^2))\cdot(2xy)$$

$$\parallel$$

$$\frac{\partial Q}{\partial x} = 2y\cos(xy^2) + 2xy(-\sin(xy^2))\cdot y^2$$

2. što nam je lakše prvo:

granice su proizvoljne, ali u 99% slučajeva možemo uzeti 0

$$u(x,y) = \int_0^x (2x + y^2\cos(xy^2))dx + \int_0^x (0 + 3y^2)dy$$

$$= \left(x^2 + \cancel{y^2}\frac{\sin(xy^2)}{\cancel{y^2}}\right)\Big|_0^x + y^3\Big|_0^y = x^2 + \sin(xy^2) + y^3 + C = \text{POTENCIJAL}$$

3. Rješenje: $\boxed{x^2 + \sin(xy^2) + y^3 = C}$

$\boxed{\text{JIR-20-7}}$ 7. Odredite parametar $\alpha \in \mathbb{R}$

$$\left(\frac{\sin^2 x}{y^2}\right)dx + \left(\frac{\alpha(x - \sin x\cdot\cos x)}{y^3} + \cos y\right)dy = 0$$

$$\frac{\partial P}{\partial y} = \frac{-2\sin^2 x}{y^3} \qquad \frac{\partial Q}{\partial x} = \alpha\frac{1-\cos^2 x + \sin^2 x}{y^3} = \frac{2\sin^2 x}{y^3}\cdot\alpha$$

$$\boxed{\alpha = -1}$$

$x_0 = 0$

$$u(x,y) = \int_0^x \left(\frac{\sin^2 x}{y^2}\right)dx + \int_0^x \left(\frac{\sin x\cos x - x}{y^3} + \cos y\right)dy = 0$$

$\sin y - \sin 1$

$$= \int_0^x \frac{\sin^2 x}{y^2}dx + \int_0^x (0 + \cos y)\,dy = 0$$

$$= \frac{1}{y^2}\int_0^x \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx + \sin y\Big|_1^y = \frac{1}{y^2}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right)\Big|_0^x + \sin y - \sin 1$$

$$\boxed{\rightarrow \frac{1}{y^2}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) + \sin y = C}$$

egz. $(x^2+y)dx + (y^2+x)dy = 0$

nije egz. $\left(x+\frac{y}{x}\right)dx + \left(\frac{y^2}{x}+1\right)dy = 0 \ / \cdot x$ ← eulerov multiplikator

→ množenje jednadžbe s fjom → mijenja egzaktnost DJ

DEF Eulerov multiplikator

Funkciju $\mu(x,y)$ s kojom treba pomnožiti DJ da postane egzaktna nazivamo EULEROV MULTIPLIKATOR.

Postupak traženja:

$$P(x,y)dx + Q(x,y)dy = 0 \ \Big/ \cdot \mu(x,y)$$

$$\mu(x,y)\cdot P(x,y)\,dx + \mu(x,y)\,Q(x,y)\,dy = 0$$

uvjet: $\frac{\partial \mu P}{\partial y} = \frac{\partial \mu Q}{\partial x} \Rightarrow \mu_y' \cdot P + \mu P_y' = \mu_x' Q + \mu Q_x'$, $\mu = ?$

ZI-2019-4

a) Izvesti formulu: $\mu(y) \Rightarrow \mu_y' P + \mu P_y' = 0 + \mu Q_x'$

$$\frac{d\mu P}{dy} = \mu(Q_x' - P_y') \ \Big/ \frac{dy}{\mu P}$$

$$\int \frac{d\mu}{\mu} = \int \underbrace{\frac{(Q_x'-P_y')}{P}}\,dy \Rightarrow \boxed{\ln|\mu| = \int \frac{Q_x'-P_y'}{P}\,dy}$$

uvjet: ovo je funkcija ovisna SAMO o y !! — ovisi samo o y

b) Riješiti Cauchyjev korem

$$\begin{cases} (\cos x + y)dx + \left(3x + \frac{2}{y}\sin x\right)dy = 0 \\ y(\pi) = 1 \end{cases}$$

$\frac{\partial P}{\partial y} = 1 \qquad \frac{\partial Q}{\partial x} = 3 + \frac{2}{y}\cos x$ nije egz.

$$\rightarrow \ln|\mu| = \int \frac{Q_x' - P_y'}{P}\,dy = \int \frac{3+\frac{2}{y}\cos x - 1}{\cos x + y}\,dy = \int \frac{2(\cos x + y)}{y(\cos x + y)}\,dy$$

$\ln|\mu| = 2\ln y \Rightarrow \boxed{\mu = y^2}$ (C koji god želimo)

$\rightarrow (y^2\cos x + y^3)\,dx + (3xy + 2y\sin x)dy = 0$

$\frac{\partial P}{\partial y} = 2y\cos x + y^2(-\sin x) + 3y^2 = \frac{\partial Q}{\partial x} = \quad T(0,0)$ ↯

$$u(x,y) = \int_0^x (y^2\cos x + y^3)dx + \int_0^y (0+0)\,dy$$

$$= (y^2\sin x + y^3 x)\Big|_0^x \rightarrow$$