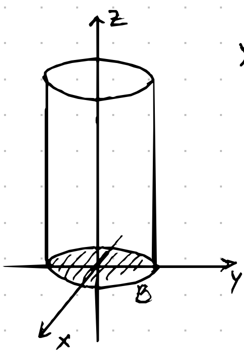
5.2.2 ZAMJENA VARIJABLI

$$x = x(u,v,w)$$
$$y = y(u,v,w)$$
$$z = z(u,v,w)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\Rightarrow \boxed{\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V^*} f^*(u,v,w)\,|J|\,du\,dv\,dw}$$

Cilindrične koordinate



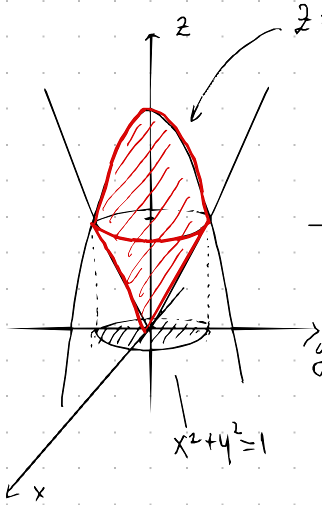
$$x^2+y^2=4$$
$$y = \pm\sqrt{4-x^2}$$

$$\underbrace{\int dx \int dy}_{\text{polarne koord}} \int_0^3 dz$$

$$\Rightarrow \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases} \qquad (x,y,z) \to (r,\varphi,z)$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1(r\cos^2\varphi + r\sin^2\varphi) = r$$

LJIR-2020)



$$z = 2 - x^2 - y^2$$

$$z = \sqrt{x^2+y^2}$$

$$z^2 = x^2+y^2 \quad +$$

$$z + z^2 = 2$$

$$z = 1 \qquad \cancel{z = -2}$$

jer to je donja polovica stošca (ne nacrtana)

iz Jacobijana

$$V = \iiint_V dV = \int_0^{2\pi} d\varphi \int_0^1 r\,dr \int_r^{2-r^2} dz$$

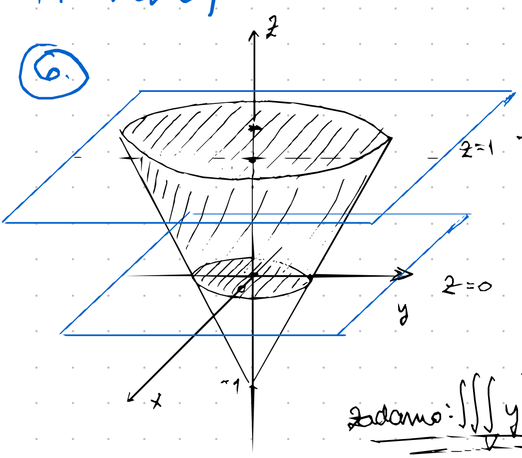
jer je cijeli krug u x-y polju

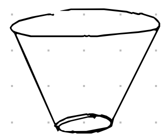
$$V = 2\pi \int_0^1 r(2-r^2-r)\,dr$$

$$\boxed{V = \frac{5\pi}{6}}$$

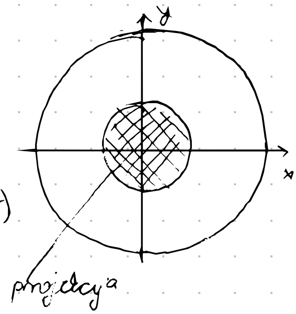
MI-2020)

6.





puna kružnica $(0, 2\pi)$

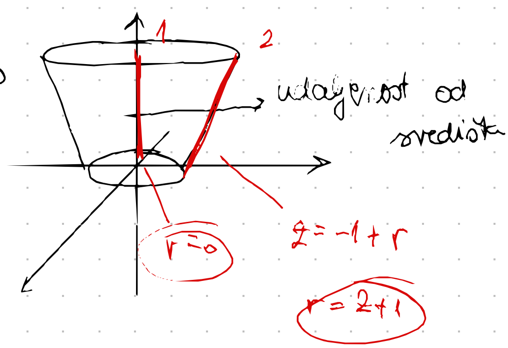


$$\text{Zadamo: } \iiint_V \overbrace{y^2}^{r^2\sin^2\varphi}\,dV = \int_0^{2\pi} d\varphi \int_0^1 r\,dr \int_0^1 r^2\sin^2\varphi\,dz$$

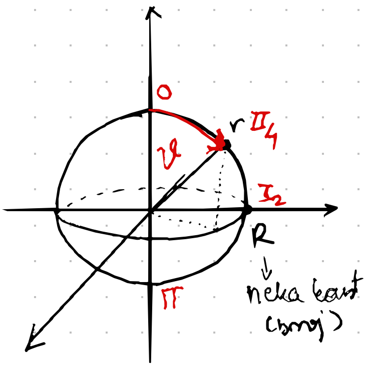
$$+ \int_0^{2\pi} d\varphi \int_1^2 r\,dr \int_{-1+r}^{1} r^2\sin^2\varphi\,dz$$

$$\text{ALI: } \int_0^{2\pi} d\varphi \int_0^1 dz \int_0^{z+1} r^2\sin^2\varphi$$

→ r gledamo →



Sferne koordinate



↓ neka kut (broj)

$x^2+y^2+z^2=R^2$

- cilindrične koordinate:

PROBLEM:

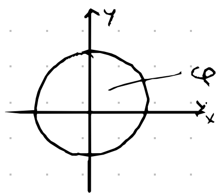
$$\int_0^{2\pi} d\varphi \int_0^R r\,dr \int f\,dz \longrightarrow z = \pm\sqrt{R^2-x^2-y^2}$$

→ da bi izbjegli granicu za z trebamo transformirati

$$\boxed{\begin{aligned} x &= r\cos\varphi\sin\vartheta \\ y &= r\sin\varphi\sin\vartheta \\ z &= r\cos\vartheta \end{aligned}}$$

$\varphi \in [0, 2\pi]$

* ko u cil. plohama je r u 2D

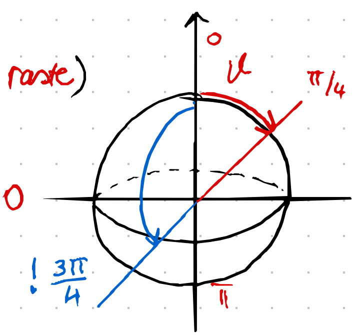


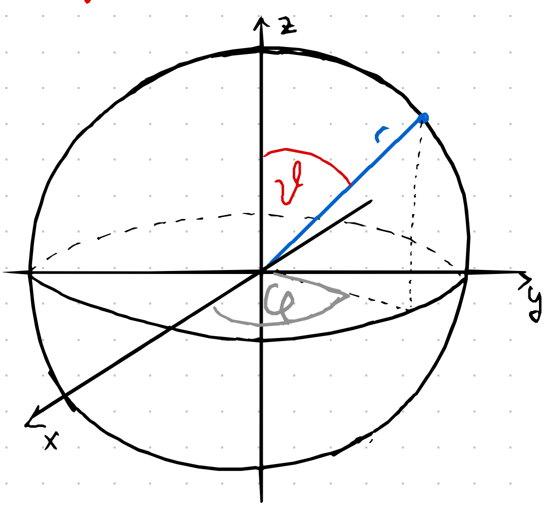
$(x,y,z) \longrightarrow (\varphi, \vartheta, r)$

$r > 0$, trodimenzionalna udaljenost od ishodišta

ϑ - kut s pozitivnim dijelom z osi (prema dolje raste)
(theta)
↳ kuda idemo gore i dolje po z

→ ! NE MOŽE BITI VEĆI od π, niti manji od 0

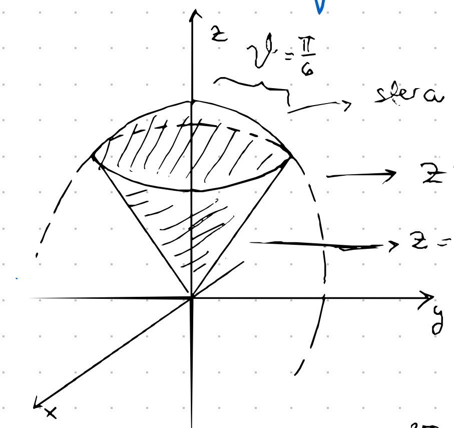




DIR-22-4) $\iiint_V y\,dV,$ $z \geq \sqrt{3x^2+3y^2}$ – stožac

$x^2+y^2+z^2 \leq 8$ – sfera

$\Downarrow r = 2\sqrt{2}$



sfera oko stošca

$z = \sqrt{8-x^2y^2}$

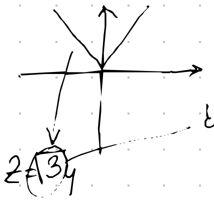
$z = \sqrt{3x^2+3y^2}$

$\iiint r\sin\varphi \sin\vartheta \, |J| \, dr$

$$\int_0^{2\pi} d\varphi \int_0^{\pi/6} d\vartheta \int_0^{2\sqrt{2}} \underbrace{\overbrace{r\sin\varphi\cos\varphi\sin\vartheta}^{y}}_{k} \cdot \underbrace{\overbrace{r^2\sin\vartheta}^{J}}_{k} \, dr$$

$z = \sqrt{3x^2+3y^2}$

↳ projekcija: maknemo $3x^2$



koeficijent smjera → kut ⇒ tg kut ϑ ⇒ $\frac{\pi}{6}$

$8 \cdot 2\sqrt{2}$

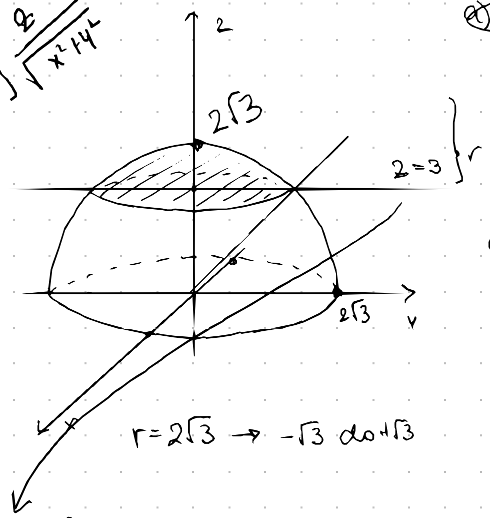
$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{6}} \underbrace{\frac{1}{3} r^3 \sin\varphi}_{\text{konst}} \cdot \sin^2\vartheta \Big|_0^{2\sqrt{2}} d\vartheta = \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{6}} \frac{16\sqrt{2}}{3} \cdot \sin\varphi \cdot \sin^2\vartheta \, d = \underline{\underline{0}}$$

MI-22-6) $x^2+y^2+z^2 \leq 12$ $z \geq 3$

sfera

*ne treba nam cijela sfera jer gledamo samo $z \geq 3$

$$\iiint \frac{z}{\sqrt{x^2+y^2}} dV$$



a) u kartezijevima:

$$\int_{-\sqrt{3}}^{\sqrt{3}} dx \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} dy \int_3^{\sqrt{12-x^2-y^2}} \frac{z}{\sqrt{x^2+y^2}} dz$$

b) Sferne

$$\int_0^{2\pi} d\varphi \int_0^{\pi/6} d\vartheta \int \frac{z}{\sqrt{x^2+y^2}} \cdot r^2\sin\vartheta \, dr$$

$$= \int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{6}} d\vartheta \int \frac{r\cos\vartheta}{\cancel{r^2\sin\vartheta}} \cdot \cancel{r^2\sin\vartheta} \, dr$$

$r = 2\sqrt{3}$ → $-\sqrt{3}$ do $+\sqrt{3}$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/6} d\vartheta \int_{\frac{3}{\cos\vartheta}}^{2\sqrt{3}} r\cos\vartheta \, dr$$

$z = 3$

$r\cos\vartheta = 3$

$r = \frac{3}{\cos\vartheta}$

ILI cilindrične:

$$\int_0^{2\pi} d\varphi \int_0^{\sqrt{3}} r\,dr \int_3^{\sqrt{12-r^2}} \frac{z}{r} dz = \underline{2\sqrt{3}\pi}$$

sfera

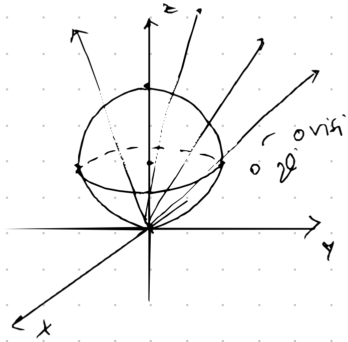
↓

2D

ZAD: $\iiint_V (x^2+y^2+z^2)\,dV$, $\quad V \ldots x^2+y^2+z^2 \le 2z$

$(x^2+y^2+z^2) \hookrightarrow r^2$

$\underbrace{x^2+y^2+(z-1)^2 \le 1}$ pomaknuta sfera

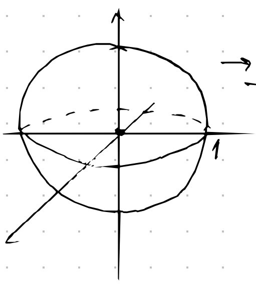


$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{2\cos\vartheta} r^2 \underbrace{r^2 \sin\vartheta}_{J}\, dr = \ldots = 2\pi \cdot \frac{32}{5}$$

$r^{\cancel{2}} = 2\cancel{r}\cos\vartheta$

$r = 2\cos\vartheta$

ILI POMAKNUTE!



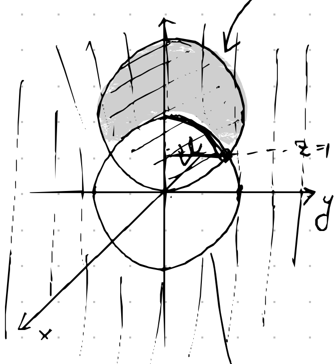
$\rightarrow \underline{z-1 = r\cos\vartheta}$

$$\int_0^{2\pi} d\varphi \int_0^{\frac{\pi}{2}} d\vartheta \int_0^{1} \left(r^2\cos^2\varphi\sin^2\vartheta + r^2\sin^2\varphi\sin^2\vartheta + (r\cos\vartheta+1)^2\right) \cdot r^2 \sin\vartheta\, dr$$

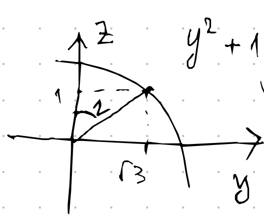
$$\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta \int_0^{1} \left[r^2\sin^2\vartheta + (r\cos\vartheta+1)^2\right] r^2 \sin\vartheta\, dr$$

MI-19-6)

$x^2+y^2+(z-2)^2=4$



$$V = \int_0^{2\pi} d\varphi \int d\vartheta \int 1 \cdot r^2 \sin\vartheta\, dr$$



$y^2+1=4$

$y=\sqrt{3}$

$\cos\vartheta = \frac{1}{2} \rightarrow \frac{\pi}{3}$

* uvrstimo / gornju sferu

$x^2+y^2+z^2=4$

$x^2+y^2+(z-2)^2=4$

$z=1$ tu se sijeku

$r^2 =$

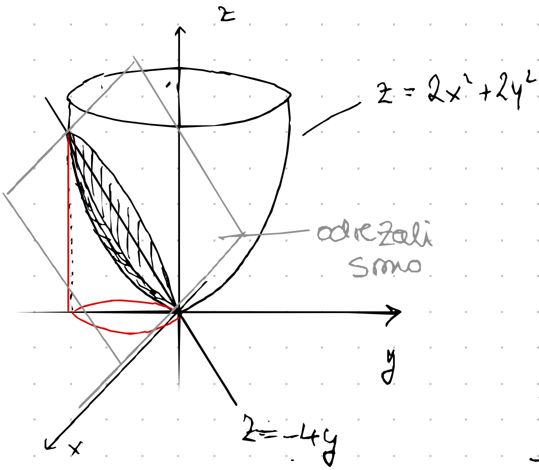
$$V = \int_0^{2\pi} d\varphi \int_0^{\pi/3} d\vartheta \int_2^{4\cos\vartheta} r^2 \sin\vartheta\, dr$$

od polumjera manje i sredina

POMAKNUTE NE

CILINDRIČNE/polarne $\rightarrow$ dva dijela

M1-23-6b)



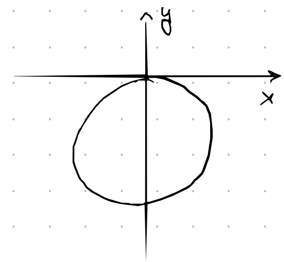
Projekcija se dobije izjednačavanjem $z = 2x^2+2y^2$ i $z=-4y$

- pomoću cilindričnih (pomaknutih)

$x = r\cos\varphi$

$y+1 = r\sin\varphi$ → (crvena projekcija)

$z = z$



$-4y = 2x^2+2y^2$

$2x^2+2y^2+4y = 0$

$2x^2+2(y+1)^2=1$

$x^2+(y+1)^2=1$ - jednadžba projekcije