4.2. LINEARNA DIF. JEDNADŽBA

Linearna:

$\Rightarrow y' + f(x)y = g(x)$

* moramo prepoznati oblik da bismo ga znali riješiti

Rješavamo metodom varijacije konstanti (MVK)

(1) $y' + f(x)y = 0$ (homogena DJ) $\longrightarrow$ to je separacija lijevog dijela

$\frac{dy}{dx} = -f(x)y \;/ : y \neq 0$ ($y=0$ nije rj. linearne jd.)

$\frac{dy}{y} = -f(x)dx \;/ \int \longrightarrow \int \frac{dy}{y} = -\int f(x)dx$

$\ln|y| = -\int f(x)dx + c \;/ e$

$|y| = e^{-\int f(x)dx + c} = -e^{-\int f(x)dx}\,e^{c}$

opet neka konstanta pa možemo samo označiti sa c

$|y| = c \cdot e^{-\int f(x)dx}$

$c^c > 0$ jer je to uvijek tako (restrikcija da je samo pozitivno)

$y = c \cdot e^{-\int f(x)dx}$

budući da je apsolutna, c postaje $\in \mathbb{R}$

$\Rightarrow$ maknuta restrikcija

(2) variramo konstantu – u konačnom rješenju c nije konst broj nego neka funkcija od x

$c \longrightarrow c(x)$

$\Rightarrow$ opće rj.: $y = c \cdot e^{-\int f(x)dx}$: uvrstimo u $y' + f(x)\cdot y = g(x)$

$\left(c(x) \cdot e^{-\int f(x)dx}\right)' = c'(x) \cdot e^{-\int f(x)dx} + \cancel{c(x) \cdot e^{-\int f(x)dx} \cdot (-f(x))} +$

$\cancel{f(x) \cdot c(x) \cdot e^{-\int f(x)dx}} = g(x).$

poništi se

$\rightarrow c'(x) = g(x) \cdot e^{\int f(x)dx} \;/ \int$

$c(x) = \int g(x)\, e^{\int f(x)dx}\, dx + D$

opće rješenje LDJ 1. reda: $y = e^{-\int f(x)dx}\left[\int g(x) e^{\int f(x)dx}\, dx + c\right]$

$c \in \mathbb{R}$

Zad.) $y' + \frac{2y}{x} = \ln x$ konstéi MVK

(1.) sep. $\frac{dy}{dx} + \frac{2y}{x} = 0 \quad / \cdot \frac{dx}{y}$

$\frac{dy}{y} + 2\frac{dx}{x} = 0 \quad /\int \longrightarrow \int \frac{dy}{y} + 2\int \frac{dx}{x} = 0$

$\ln|y| = -2\ln|x| + \cancel{c}$ $\ln c$ $\Longrightarrow \ln|y| = \ln|x|^{-2} + \ln c \quad /e$

$\Downarrow$

$\ln(x^{-2} \cdot c)$

$|y| = |x^{-2} \cdot c|$

$y_h = c \cdot x^{-2} = \frac{c}{x^2}$ $\longrightarrow$ $(y_h)'$ $\downarrow$ $y' + \frac{2y}{x} = \ln x$ $\nearrow y$

(2.) $y = \frac{c(x)}{x^2} = c(x) \cdot x^{-2}$

$c'(x) \cdot x^{-2} + c(x) \cdot (-2) \cdot x^{-3} + \frac{2 \cdot c(x)x^{-2}}{x} = \ln x \quad / \cdot x^2$

$c'(x) \cancel{- 2c(x) \cdot x^{-1}} + \cancel{\frac{2c(x)}{x}} = \ln(x) \cdot x^2$

$c'(x) = \ln(x) \cdot x^2 \quad /\int$

$c(x) = \int \ln(x) \cdot x^2 \, dx \longrightarrow c(x) = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + c$

$\longrightarrow y = \frac{c(x)}{x^2} \longrightarrow c(x) = y \cdot x^2$

$y \cdot x^2 = \frac{x^3}{3}\ln x - \frac{1}{9}x^3 + c \Longrightarrow y = \frac{\frac{x^3}{3}\ln x - \frac{1}{9}x^3 + c}{x^2}$

$\int \ln(x) \cdot x^2 \, dx = \left| \begin{array}{ll} u = \ln|x| \longrightarrow du = \frac{1}{x} \\ dv = x^2 \longrightarrow v = \frac{1}{3}x^3 \end{array} \right| \quad * \int u\,dv = uv - \int v\,du$

$= \frac{1}{3}x^3 \cdot \ln|x| - \int \frac{1}{3}x^{\cancel{3}\,2} \cdot \frac{1}{\cancel{x}} dx = \frac{x^3}{3}\ln(x) - \frac{1}{3} \cdot \frac{1}{3}x^3 = \frac{x^3}{3}\ln(x) - \frac{1}{9}x^3$

Zad.) $y'\cos x = 1 - y\sin x \;/ : \cos x \qquad y(0)=\pi$

! prebaciti u stand oblik

$$y' - \frac{1}{\cos x} + y\,tg\,x = 0$$

$$y' + y\cdot tg\,x = \frac{1}{\cos x}$$

$$* \; y = e^{-\int f(x)dx}\left[\int g(x)e^{\int f(x)dx}dx + C\right]$$

možemo tek kada izvedemo koristiti

$$y = e^{-\int tg\,x\cdot dx}\left[\int \frac{1}{\cos x}\cdot e^{\int tg\,x\,dx}dx + C\right] \longrightarrow = -\ln|\cos x|$$

$\cos x = t$

$$y = e^{\ln|\cos x|}\left[\int \frac{1}{\cos x} e^{-\ln|\cos x|}dx + C\right]$$

$$\int \frac{\sin x}{\cos x} \rightarrow \begin{matrix} t = \cos x \\ dt = -\sin x \end{matrix} \rightarrow \int \frac{-dt}{t} \rightarrow -\ln|t|$$

$$-\ln|\cos x|$$

$$\Rightarrow y = |\cos x|\left[\int \frac{1}{\cos x}\cdot |\cos x|^{-1}dx + C\right]$$

$$y = |\cos x|\left[\int \frac{1}{\cos^2 x}dx + C\right] \Rightarrow \cos x\,(tg\,x + C)$$

$$\boxed{\Rightarrow y = \sin x + C\cdot\cos x}$$

ono što znamo:

$$y'(x) = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \Rightarrow \frac{1}{x'(y)}$$

Zad.) $3xy' + y'e^y = 1$

$$\frac{3x}{x'} + \frac{e^y}{x'} = 1 \quad / \cdot x'$$

$$3x + e^y = x'$$

$$\underbrace{x'}_{y'} \underbrace{\overset{f(x)}{-3}}_{f(x)y} x = \underbrace{e^y}_{g(x)}$$

formula: $x = e^{-\int f(y)dy}\left[\int g(y)\, e^{\int g(y)dy} dy + C\right]$

$$x = e^{-\int(-3)dy}\left[\int e^y \cdot e^{\int(-3)dy} dy + c\right]$$

$$x = e^{3y}\left[\int e^y \cdot e^{-3y} dy + c\right]$$

$$x = e^{3y}\left[\frac{e^{-2y}}{-2} + c\right]$$

$$\Rightarrow x = -\frac{1}{2}e^y + C \cdot e^{3y}$$

Rj: $y(x)$ zadan implicitnom jedn.

6.3. BERNOULLIJEVA DJ

$\Rightarrow y' + f(x)y = g(x)y^{\alpha}, \ \alpha \in \mathbb{R} \setminus \{0,1\}$ $\to$ supst. $z = y^{1-\alpha}$

$y=0$ je rješenje za $\alpha > 0$

$\alpha = 0 \to$ LDJ, $\alpha = 1 \to$ sep.

dobivamo LDJ $(z(x))$

ZAD: $y' - \frac{2}{x} y = x^2 y^4$, supst. $z = y^{-3} \Big| \frac{d}{dx}$

$z' = -3y^{-4} \cdot y'$

$\to$ čim je potencija $\neq 0, 1$ znamo je da je Bernoullijeva DJ

$$y' - \frac{2}{x} y = g(x) y^4 \ \Big| \cdot (-3y^{-4})$$

$$\underbrace{-3y^{-4} \cdot y'}_{} - \frac{2}{x} y \cdot (-3y^{-4}) = -3x^2$$

$$z + \frac{6}{x} \cdot y^{-3} = -3x^2 \ , \ \text{LDJ}$$

LDJ. sep

$$z = e^{-\int \frac{6}{x} dx} \left[\int (-3x^2) \cdot e^{\int \frac{6}{x} dx} dx + c \right]$$

$$z = e^{-6 \ln|x|} \left[-\int 3x^2 e^{6 \ln|x|} dx + c \right] = e^{-6 \ln|x|} \left[-3 \int x^2 \underbrace{e^{\ln|x|^6}}_{x^6} dx + c \right]$$

$$z = e^{-6 \ln|x|} \left[-3 \int \underbrace{x^2 \cdot x^6}_{x^8} dx + c \right] = e^{-6 \ln|x|} \left[-\cancel{3} \cdot \frac{1}{\cancel{9}} \cdot x^9 + c \right]$$

$$\boxed{z = e^{-6 \ln|x|} \left[-\frac{1}{3} x^9 + c \right]} \to y^{-3} = \frac{-x^3}{3} + \frac{c}{x^6}$$

$$\boxed{\Rightarrow y = \frac{1}{\sqrt[3]{-\frac{x^3}{3} + \frac{c}{x^6}}}}$$

LJIR-21-6 $\quad xy' + y - x\sqrt{y} = 0$ — nije linearna $\quad / : x$

a) $\longrightarrow$ Bernoully sa $y^{\frac{1}{2}}$

$$y' + \frac{1}{x}y = y^{\frac{1}{2}} \quad = \alpha \quad / \cdot \frac{1}{2}y^{-\frac{1}{2}}$$

supstitucija: $z = y^{\frac{1}{2}} \; /'$

$$z' = \frac{1}{2}y^{-\frac{1}{2}} \cdot y'$$

$$z' + \underbrace{\frac{1}{2x}}_{f(x)} y^{\frac{1}{2}} = \underbrace{\frac{1}{2}}_{g(x)}$$

$$z = e^{-\int \frac{1}{2x}dx}\left[\int \frac{1}{2}\cdot e^{\int \frac{1}{2x}dx}dx + c\right]$$

$$z = e^{-\int \frac{1}{2x}dx}\left[\int \frac{1}{2} e^{\frac{1}{2}\cdot \ln|x|}dx + c\right]$$

$$z = e^{-\frac{1}{2}\ln|x|}\left[\frac{1}{2}\int x^{1/2}dx + c\right] = x^{-1/2}\left[\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot x^{3/2} + c\right]$$

$$z = \frac{1}{\sqrt{x}}\cdot\left[\frac{x\sqrt{x}}{3} + c\right] = \frac{x}{3} + \frac{c}{\sqrt{x}} \quad \Rightarrow y = \left(\frac{x}{3} + \frac{c}{\sqrt{x}}\right)^2$$

+ opće rj $y = 0$.

b) Koje rješenje zadovoljava $y(1) = 0$?

$$0 = \left(\frac{1}{3} + \frac{c}{1}\right)^2 \longrightarrow \frac{1}{3} + c = 0 \rightarrow c = -\frac{1}{3} \Rightarrow \boxed{y = \frac{x}{3} - \frac{1}{3\sqrt{x}}}$$

također $y = 0$