Svojstva Laplaceove transf.

① Množenje varijable konstantom

$f(t) \multimap F(s) \qquad a>0$

$$\mathcal{L}(f(at)) = \int_0^\infty e^{-st} f(at)\,dt = \left| \begin{array}{l} at = u \\ a\,dt = du \end{array} \right| = \int_0^\infty e^{-s\cdot\frac{u}{a}} f(u)\,\frac{1}{a}\,du$$

$$\mathcal{L}(f(at)) = \frac{1}{a}\cdot\int_0^\infty e^{-\frac{s}{a}u} f(u)\,du = \frac{1}{a}\,F\left(\frac{s}{a}\right)$$

② TM o prigušenju i pomaku

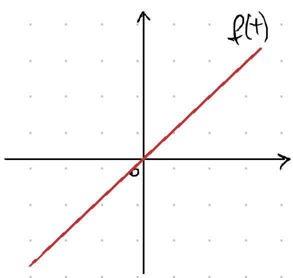
Prigušenje $e^{-at}f(t)$ prigušenje od f za $a \in \mathbb{R}$ (ili $a \in \mathbb{C}$)

$$\mathcal{L}(e^{-at}f(t)) = \int_0^\infty e^{-st}\cdot e^{-at} f(t)\,dt = \int_0^\infty e^{-t(s+a)} f(t)\,dt$$

$$\mathcal{L}(e^{-at}f(t)) = F(s+a)$$

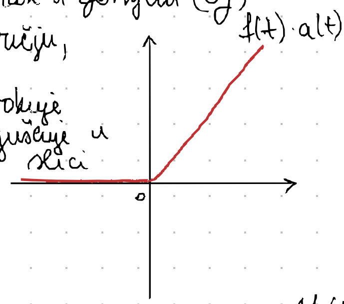
MOMAK RADIMO POMAK!

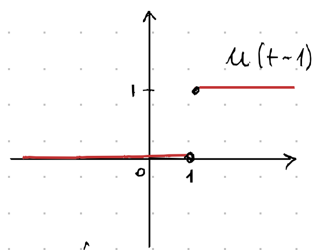
Pomak



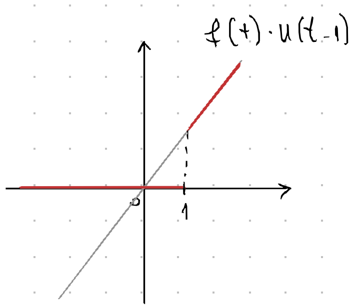
pomak u gornjem (og) području,

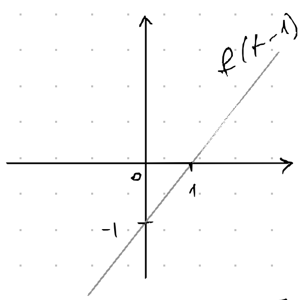
uzrokuje prigušenje u slici

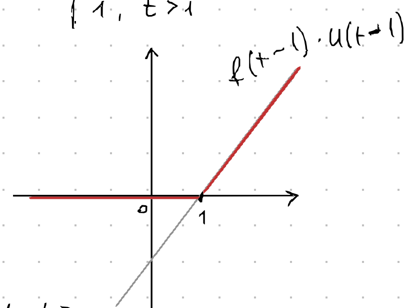




$$u(t-1) = \begin{cases} 0, & t<1 \\ 1, & t>1 \end{cases}$$







$$\mathcal{L}(f(t-a)\,u(t-a)) = \int_0^\infty e^{-st} f(t-a)\,u(t-a)\,dt \qquad u(t-a)=\begin{cases} 1, & t \geq a \\ 0, & t<a \end{cases}$$

$$0 = \int_a^\infty e^{-st} f(t-a)\,dt = \left(\begin{array}{l} u = t-a \\ du = dt \end{array} \right) = \int_0^\infty e^{-s(u+a)} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su} f(u)\,du = e^{-as}\cdot F(s)$$

$f(at) \circ\!\!-\!\!\bullet \frac{1}{a} F(\frac{s}{a})$
$e^{-at} f(t) \circ\!\!-\!\!\bullet F(s+a)$ prigušenje
$f(t-a)u(t-a) \circ\!\!-\!\!\bullet e^{-as} F(s)$ pomak
$f'(t) \circ\!\!-\!\!\bullet sF(s) - f(0)$ deriv. originala
$t f(t) \circ\!\!-\!\!\bullet -F'(s)$ deriv slike

Primjer 1)

a) $t^n e^{2t} \longrightarrow \frac{n!}{(s-2)^{n+1}}$, $t^n \longrightarrow \frac{n!}{s^{n+1}}$

b) $e^{-at} u(t) \circ\!\!-\!\!\bullet \frac{1}{s+a}$

c) $e^{2t} \cos t \circ\!\!-\!\!\bullet \frac{s-2}{(s-2)^2+1}$ $\quad$ $^{*}\cos t \circ\!\!-\!\!\bullet \frac{s}{s^2+1}$

d) $\frac{\sin t}{e^t} = e^{-t} \sin t \circ\!\!\longrightarrow \frac{1}{(s+1)^2+1}$ $\quad$ $^{*}\sin t \circ\!\!-\!\!\bullet \frac{1}{s^2+1}$

e) $e^{2t} sh(4t) \circ\!\!-\!\!\bullet \frac{4}{(s-2)^2-16}$ $\quad$ $sh(t) \bullet\!\!-\!\!\bullet \frac{4}{s^2-16}$

Primjer 2)

a) $(t-2)^2 = t^2 - 4t + 4 \circ\!\!-\!\!\bullet \frac{2}{s^3} - \frac{4}{s^2} + \frac{4}{s}$

b) $(t-2)^2 u(t-2)$ $\quad$ $t^2 \circ\!\!-\!\!\bullet \frac{2}{s^3}$, $(t-2)^2 u(t-2) \circ\!\!-\!\!\bullet e^{-2s} \cdot \frac{2}{s^3}$

c) $(t-2)^2 e^{-t} \circ\!\!-\!\!\bullet \frac{2}{(s+1)^3} - \frac{4}{(s+1)^2} + \frac{4}{(s+1)}$ (prigušeni (a))

d) $(t-2)^2 u(t-2) e^{-t} \circ\!\!-\!\!\bullet e^{-2s} \cdot \frac{2}{(s+1)^3}$ (prigušeni (b))

Primjer 3)

a) $3(t-3)u(t-3) \circ\!\!-\!\!\bullet 3 \cdot \frac{e^{-3s}}{s^2}$

b) $5u(t-2) - 2u(t-3)$: $\quad$ $u(t-2) \circ\!\!-\!\!\bullet e^{-2s} \frac{1}{s}$

$u(t-2) \cdot u(t-2)$

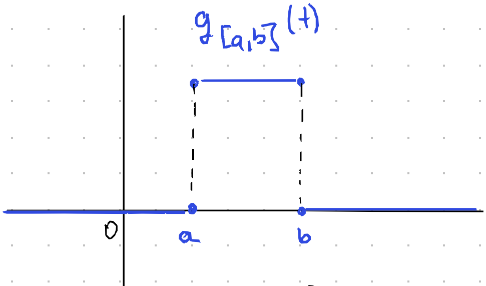
$\circ\!\!-\!\!\bullet 5e^{-2s} \cdot \frac{1}{s} - 2e^{-3s} \frac{1}{s}$

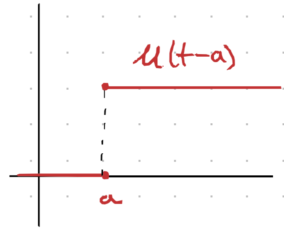
c) $3(t-1)^3 u(t-1) \circ\!\!-\!\!\bullet 3e^{-3} \cdot \frac{6}{s^4}$

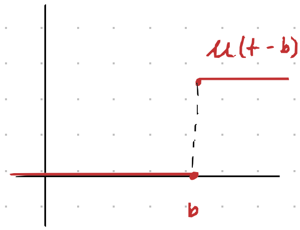
d) $(2t+1)u(t-1) = 2tu(t-1) + u(t-1) = 2(t-1)u(t-1) + \overbrace{2u(t-1) + u(t-1)}^{3u(t-1)}$

$\circ\!\!-\!\!\bullet 2e^{-s} \frac{1}{s^2} + 3 \cdot e^{-2s} \cdot \frac{1}{s}$

Gate funkcija







$$g_{[a,b]}(t) = \begin{cases} 1, & a \leq t < b \\ 0, & \text{inače} \end{cases}$$

$$\mathcal{L}\left(g_{[a,b)}(t)\right) \qquad g_{[a,b]}(t) = u(t-a) - u(t-b)$$

$$g_{[a,b]}(t) = u(t-a) - u(t-b) \circ\!\!-\!\!\bullet\ e^{-as}\cdot\frac{1}{s} - e^{-bs}\cdot\frac{1}{s}$$

Primjer:

$$f(t) = \begin{cases} 3, & 0 < t < 2 \\ -1, & 2 < t < 4 \\ 0, & t > 4 \end{cases}$$

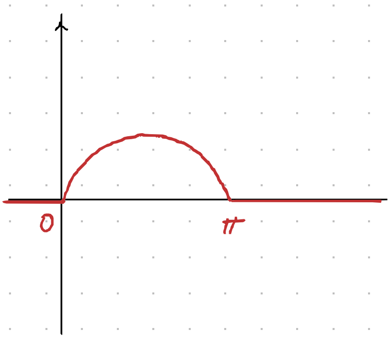
$$f(t) = 3g_{[0,2]}(t) - g_{[2,4]}(t)$$

$$f(t) = 3\left(u(t-0) - u(t-2)\right) - \left(u(t-2) - u(t-4)\right)$$

$$f(t) = 3u(t) - 3(t-2) - u(t-2) + u(t-4)$$

$$f(t) = 3u(t) - 4(t-2) + u(t-4)$$

Primjer: $f(t) = \begin{cases} \sin t, & 0 \leq t \leq \pi \\ 0, & \text{inače} \end{cases}$



$$f(t) = \sin t \cdot g_{[0,\pi]}(t) = \sin t\left(u(t) - u(t-\pi)\right)$$

$$f(t) = \sin t \cdot u(t) - \sin(t-\pi)\, u(t-\pi)$$

$$\circ\!\!-\!\!\bullet\ \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

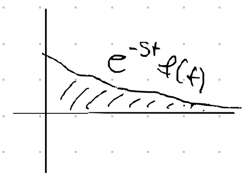
Deriviranje originala

$f(t) \circ\!\!-\!\!\!-\!\!\bullet F(s)$, f original , $\exists f'$

$$\mathcal{L}(f'(t)) = \int_0^\infty e^{-st} f'(t)\,dt = \left| \begin{matrix} u = e^{-st} & \longrightarrow du = -se^{-st}\,dt \\ dv = f'(t)\,dt & \longrightarrow v = f(t)\Big|_0^\infty \end{matrix} \right|$$

$$\mathcal{L}(f'(t)) = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)e^{-st}(-s)\,dt$$

$\lim\limits_{t\to\infty} f(t)e^{-st} = 0$ f original $\Rightarrow \exists \int_0^\infty e^{-st} f(t)\,dt$ za neki s



$\forall$ NUK $\Rightarrow \lim\limits_{t\to\infty} f(t)e^{-st} = 0$

↳ ako je lim općeg člana 0...

$$\mathcal{L}(f'(t)) = 0 - f(0) + s\boxed{\int_0^\infty f(t)e^{-st}\,dt} = F(s)$$

$$f'(t) \circ\!\!-\!\!\!-\!\!\bullet sF(s) - f(0)$$

npr:

$f''(t) \circ\!\!-\!\!\!-\!\!\bullet ?$

$$f''(t) \circ\!\!-\!\!\!-\!\!\bullet s(sF(s) - f(0)) - f'(0) = s^2F(s) - sf(0) - f'(0)$$

Primjer: $t^3 \circ\!\!-\!\!\!-\!\!\bullet \frac{3!}{s^4}$, $3t^2 \circ\!\!-\!\!\!-\!\!\bullet 3\frac{2!}{s^3}$

$(t^3)' \circ\!\!-\!\!\!-\!\!\bullet s \cdot \frac{6}{s^4}$

Derivacije slike

$? \circ\!\!-\!\!\!-\!\!\bullet F'(s)$

$$F'(s) = \frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$ derivacija po parametru

$$= \int_0^\infty \frac{\partial}{\partial s}(e^{-st}f(t))\,dt = \int_0^\infty -te^{-st}f(t)\,dt = -t\cdot f(t) \circ\!\!-\!\!\!-\!\!\bullet F'(s)$$

Primjer: $t e^{t} \sin t$

$\sin t \circ\!\!-\!\!\bullet \frac{1}{s^2+1}$

$e^{t} \sin t \circ\!\!-\!\!\bullet \frac{1}{(s-1)^2+1}$

$t e^{t} \sin t \circ\!\!-\!\!\bullet \frac{-d}{ds} \frac{1}{((s-1)^2+1)^2} \cdot 2(s-1)$

Primjer: $t f''(t) \circ\!\!-\!\!\bullet ?$ $\quad f(t) \circ\!\!-\!\!\bullet F(s)$

$t f''(t) \circ\!\!-\!\!\bullet \frac{-d}{ds}\left(s^2 F(s) - s f(0) - f'(0)\right)$

$= -\left(2 s F(s) + s^2 F'(s) - f(0)\right)$

Integriranje slike

$f(t) \circ\!\!-\!\!\bullet F(s)$, $\frac{f(t)}{t}$ original $\rightarrow$ $\boxed{\frac{f(t)}{t} \circ\!\!-\!\!\bullet \int_s^{\infty} F(s)\, ds}$

Primjer $\frac{e^{-3t} - e^{-5t}}{t}$

$e^{-3t} - e^{-5t} \circ\!\!-\!\!\bullet \frac{1}{s+3} - \frac{1}{s+5}$

$\Rightarrow \frac{e^{-3t} - e^{-5t}}{t} \circ\!\!-\!\!\bullet \int_s^{\infty} \frac{1}{s+3} - \frac{1}{s+5}\, ds = \ln(s+3) - \ln(s+5) \Big|_s^{\infty}$

$= \ln\left(\frac{s+3}{s+5}\right)\Big|_s^{\infty} = 0 - \ln\left(\frac{s+3}{s+5}\right)$

$\lim_{s\to\infty} \ln\left(\frac{s+3}{s+5}\right) = \ln \lim_{s\to\infty} \frac{s+3}{s+5} = \ln(1) = 0$

Primjer: $\int_0^{\infty} e^{-t} \frac{\sin^2 t}{t}\, dt = F(1)$

$\frac{\sin^2 t}{t} = f(t)$

Izračunati int.

$f(t) = \frac{\sin^2 t}{t}$

$F(s) = ?$

$\sin^2 t = \frac{1 - \cos 2t}{2} \circ\!\!-\!\!\bullet \frac{1}{2} \cdot \frac{1}{s} - \frac{1}{2} \cdot \frac{s}{s^2+4}$

$\frac{\sin^2 t}{t} \circ\!\!-\!\!\bullet \frac{1}{2} \int_s^{\infty} \left(\frac{1}{s} - \frac{s}{s^2+4}\right) ds = \frac{1}{4} \ln \frac{s^2+4}{s^2} \rightarrow F(1) = \frac{1}{2} \ln(5)$