# 6.2.2. Taylorovi redovi

Podsjetimo se Taylora za $f(x)$ (matan 1)

$$f(x) = T_N(x) + R_N(x)$$

↳ Taylorov polinom n-tog stupnja $T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

- fja će imati Taylorov red ako $R \to 0$ (napadnemo limesom $\lim_{N\to\infty}$)

$$R_N(x) = \frac{f^{(n+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$$

($c$ je iz okoline $x_0$)

**DEF** Red oblika $\sum_{n=0}^{\infty} \frac{f^n(x_0)}{n!}(x-x_0)^n$ zovemo Taylorov red funkcije $f(x)$ oko točke $x_0$.

NUŽAN I DOVOLJAN UVJET:

Taylor red je jednak $f(x)$ akko $\boxed{\lim_{N\to\infty} R_N(x) = 0}$

*Nap.) Ako je $x_0 = 0$, red se zove Maclaurinov.

Primjer: $f(x) = e^x$ — sve njene deriv. su jednake = beskonačno diferencijabilna

$f^n(x) = e^x$, $x_0 = 0$, $f^{(n)}(0) = 1$

$$f(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

područje konv je $\forall x \in \mathbb{R}$

$$\lim_{n\to\infty} R_N(x) = \lim_{n\to\infty} \frac{e^c}{(n+1)!}x^{n+1} = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

jer faktorijele brže rastu od polinoma pa i da piše $(\text{miljun})^n$, $n!$ raste brže

Izr. sumu:

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}$

b) $\sum_{n=2}^{\infty} \frac{1}{2^n n!} = e^{\frac{1}{2}} - 1 - \frac{1}{2} = \sqrt{e} - \frac{3}{2}$

$n=0 \quad n=1$

$x = \frac{1}{2}$

to vrijedi kada ide od 0

oduzmemo prva dva člana

## Zadatak: Provjeri n-tu derivaciju i uvrstiti $x_0 = 0$

$f(x) = \sin x \rightarrow 0$

$f'(x) = \cos x \rightarrow 1$

$f''(x) = -\sin x \rightarrow 0$

$f'''(x) = -\cos x \rightarrow -1$

$f^{IV}(x) = \sin x \rightarrow$

$\vdots$

$\Rightarrow \sin x = 0 + 1 + 0 - \frac{1}{3!} x^3 + 0 + \frac{1}{5!} x^5 + \dots$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$\hookrightarrow$ sinus je neparan

$$\lim_{n\to\infty} R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1} = 0$$

uvijek $\in [-1, 1]$

jer faktorijele brže teže u $\infty$

$\boxed{\forall x \in \mathbb{R} \text{ tj. } R = \infty}$

$\rightarrow$ ovaj red najbrže konvergira od svih funkcija

## Zadatci za DZ:

a) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

b) $\operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

b) $\operatorname{ch} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, svi oni konvergiraju $\forall x \in \mathbb{R}$, $R = \infty$

* tg nema lijepe zatvorene formule

**Zadatak:** $\cancel{f(x) = \ln x}$ → ne postoji McLaurinov razvoj oko 0 jer su deriv od $\ln x = \frac{1}{x}, -\frac{1}{x}$

Zato razvijamo $\ln(1+x)$ i sad možemo oko 0

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$ → koeficijenti su alternirajuće fakt!

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

$f''(x) = -\frac{1}{(1+x)^2} \rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \rightarrow f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^{-4}} \rightarrow f^{IV}(0) = -6$

$f^{V}(x) = \frac{24}{(1+x)^{-5}} \rightarrow f^{V}(0) = 24$

$\Longrightarrow f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$$\ln(1+x) = 0 + 1x - \frac{1x^2}{2!} + \frac{2}{3!} + \frac{\cancel{2}}{\cancel{3!}_{3}}x^3 + \dots + \frac{(-1)^{n+1}\cancel{(n-1)!}}{\cancel{n!}_{n}}x^n + \dots$$

→ jedan od malo zapisa koji u sebi nemaju fakt! jer se pokrate

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots = \boxed{\sum_{n=0}^{\infty}(-1)^{(n+1)}\frac{x^n}{n}}$$

Područje konvergencije:

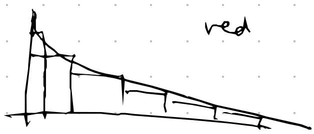
$$\lim_{n\to+\infty} R_n(x) = \frac{\frac{(-1)^{n+2}\cancel{n!}}{(1+c)^{n+1}}}{\cancel{(n+1)!}_{n+1}}x^{n+1} = \lim_{n\to\infty}\frac{(-1)^n x^{n+1}}{(1+c)^{n+1}(n+1)} = 0$$

$x$ mora $< 1$, $\left(\frac{1}{2}\right)^n \to 0$; ovo je jednako 0 samo ako je apsolutno od $x$ $(|x|) < 1$,

$\Longrightarrow \boxed{R = 1}$ $\quad x \in \langle -1, 1\rangle$

Rubovi $x = -1$:

$\sum_{n=1}^{\infty} -\frac{1}{n}$ div → to je harmonijski red { to je ona priča da $\log_n$ opisuje harmonijski red



$x = 1$

$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}$ po Leibnizu konvergira

ZAKLJUČAK:
$x \in \langle -1, 1]$

Suma za $x=1$ $S = \ln 2 = 0,69$ $\quad 1+1=2$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$ Binomni red

Dokazuje se jednako:

$f'(x) = \alpha(1+x)^{\alpha-1}$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$

$\vdots$

$f^{n}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

$\frac{f^{n}(0)}{n!} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} =$ formula za binomni koeficijent $\binom{\alpha}{n}$

* $\binom{n}{k}$ - kombinacije bez ponavljanja

▶ $\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$

$\binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}$ $\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\cancel{3!}\ 2} = \frac{1}{16}$

$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

nama za matan 2 treba samo: $\binom{\alpha}{n} = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$

▶ $\alpha = -1$ $(1+x)^{-1} = \frac{1}{1+x} = \sum_{n=0}^{\infty}\binom{-1}{n}x^n = \sum_{n=0}^{\infty}(-1)^n x^n$

$\binom{-1}{n} = \frac{-1(-2)(-3)\cdots(-n)}{n!} = (-1)^n$ $= 1 - x + x^2 - x^3 + x^4 + x^5 + \cdots$

ako uzmemo: $\frac{1}{1-x} = \sum_{n=0}^{\infty}(-1)^n(-x)^n = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

u zadacima smijemo koristiti područje konv. pomoću poznatih redova

**Primjer) Razvij oko $x_0=0$**

a) $$e^{-2x} = \sum \frac{(-2x)^n}{n!} = \sum \frac{(-2)^n}{n!} x^n, \quad \forall x \in \mathbb{R}$$

b) $$\sin\left(\frac{\pi}{2}x^2\right) = \sum (-1)^n \left(\frac{\pi}{2}\right)^{2n+1} \frac{(x^2)^{2n+1}}{(2n+1)!}$$

c) $$\frac{1}{2-x^3} = \frac{1}{2} \cdot \frac{1}{1-\frac{x^3}{2}} = \underbrace{\frac{1}{2}\sum_{n=0}^{\infty} \frac{x^{3n}}{2^{n+1}}}, \quad \left|\frac{x^3}{2}\right| < 1 \quad R = \sqrt[3]{2}$$

$$|x| < \sqrt[3]{2}$$

d) $$\ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right)$$

$$= \ln 3 + \ln\left(1+\left|\frac{x}{3}\right|\right) = \underbrace{\ln 3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{3^n \cdot n}}$$

**Zad: Razvijte $e^{5x}$ do $x_0=2$.** ako razvijamo ko nekog drugog broja moramo: $e^{3x} = e^{3(\boxed{x-2}+2)}$ i moramo dodat taj isti

$*(x_0)$

$$\sum a_n(x-2)^n$$

$$= e^{3(x-2)+6} = e^6 \cdot e^{3(x-2)}$$

$$\boxed{= e^6 \sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n!}}$$

**Zadatak WIR-23 b)**

$$\cos(3x) = \cos(3(x-\pi+\pi))$$

$$= \cos[3(x-\pi)+3\pi] = \cos(3(x-\pi)) \cdot \underbrace{\cos(3\pi)}_{-1} - \cancel{\sin[3(x-\pi)] \cdot \sin(3\pi)}$$

$$= -\cos[3(x-\pi)]$$

$$= -\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}(x-\pi)^{2n}}{(2n)!}$$

21-23-2) $\frac{1}{x^2+2x+8}$

$$= \frac{1}{(x-2)(x+4)} = \frac{A}{x-2} + \frac{B}{x+4} \qquad A = \frac{1}{6}, \quad B = \frac{-1}{6}$$

a) $\frac{\frac{1}{6}}{x-2} = \frac{-1}{12} \cdot \frac{1}{1-\frac{x}{2}} = \frac{-1}{2}\sum_{n=0}^{\infty} \frac{x^n}{2^n}$

$\left|\frac{x}{2}\right| < 1 \rightarrow |x| < 2$

$\cap \left|\frac{x}{4}\right| < 1 \rightarrow |x| < 4$

$\frac{\frac{-1}{6}}{x+4} = \frac{-1}{24} \cdot \frac{1}{1+\frac{x}{4}} = \frac{-1}{4} \sum (-1) \frac{x^n}{4^n}$

↳ presjek $\Rightarrow |x| < 2$, $R = 2$

b) dio $x_0 = -1 \longrightarrow (x+1)^n$

$$\frac{1}{x+4} = \frac{1}{(x+1)+3} = \frac{1}{3} \frac{1}{1+\frac{x+1}{3}} = \frac{1}{3} \sum (-1)^n \frac{(x+1)^n}{3^n}$$

$$\frac{1}{x-2} = \frac{1}{x+1-3} = \frac{-1}{3} \frac{1}{-1-\frac{x+1}{3}}$$