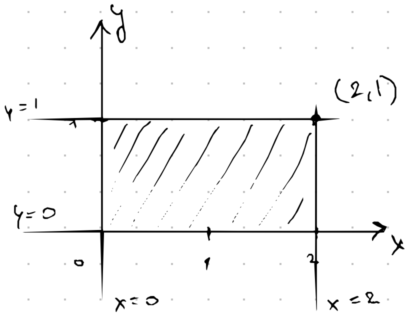
DVOSTRUKI INTEGRALI

Zadaci za vježbu

1. $\iint_D (x^4+x^2y^2+y^4)\,dx\,dy$, D je područje omeđeno pravcima $x=2$, $y=1$, te s koordinatnim osima.

$$= \int_0^2 dx\int_0^1 (x^4+x^2y^2+y^4)\,dy = \int_0^2\left[\left(yx^4+\frac13y^3x^2+\frac15y^5\right)\Big|_0^1\right]dx$$

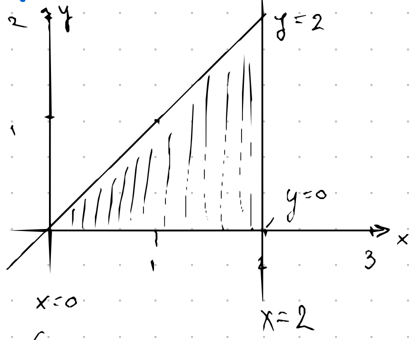


$$= \int_0^2\left(x^4+\frac13x^2+\frac15\right)dx = \left(\frac15x^5+\frac19x^3+\frac15x\right)\Big|_0^2$$

$$= \frac{32}{5}+\frac89+\frac25 = \frac{34}{5}+\frac89 = 9\,\frac{34}{45}+\frac{40}{45}$$

ne da mi se zbrajat

2. Postavite granice u oba poretka te potom izračunajte integral $\iint_D x^2ye^{xy}\,dx\,dy$ gdje je D područje omeđeno pravcima $y=x$, $x=2$ i osi x.



$$= \int_0^2 dx\int_0^2 (x^2ye^{xy})\,dy \Rightarrow x^2\int_0^2 ye^{xy}\,dy$$

konstanta: x

$$\left|\begin{array}{ll} u=y \to du=1 \\ dv=e^{xy} \to v=e^{xy}\cdot\frac1x \end{array}\right| = x^2\left(\frac yx e^{xy}\Big|_0^2-\frac1x\int_0^2 e^{xy}\,dy\right)$$

$$= x^2\left(\frac yx e^{xy}\Big|_0^2-\frac1x\left(\frac1x e^{xy}\right)\Big|_0^2\right)\Bigg|_0^2 = x\left(ye^{xy}\right)\Big|_0^2 - x\left(\frac1x e^{xy}\right)\Big|_0^2 =$$

$$= \left(xye^{xy}-e^{xy}\right)\Big|_0^2 = 2xe^{2x}-e^{2x}+1$$

$$e^{2x}\cdot\frac12\Big|_0^2 \qquad x\Big|_0^2$$

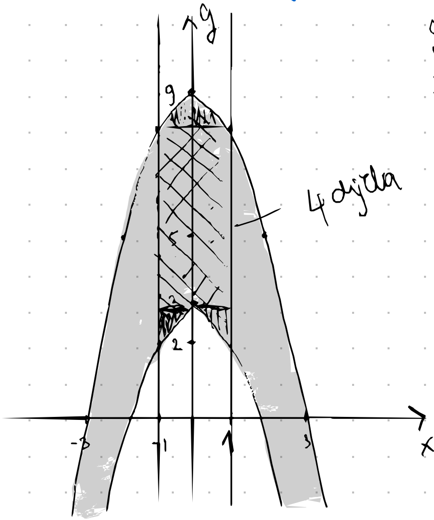
$$= \int_0^2(2xe^{2x}-e^{2x}+1)\,dx = 2\int_0^2 xe^{2x}\,dx-\int_0^2 e^{2x}\,dx+\int_0^2 dx$$

$$\left|\begin{array}{ll} u=x \to du=1 \\ dv=e^{2x} \to v=e^{2x}\cdot\frac12 \end{array}\right| = \frac x2e^{2x}\Big|_0^2-\frac12\int_0^2 e^{2x}\,dx = \frac x2e^{2x}\Big|_0^2-\frac12\cdot\frac12\left(e^{2x}\right)\Big|_0^2$$

$$= e^4-\frac14(e^4-1) = \frac34e^4+\frac14 \to 2\cdot\left(\frac34e^4+\frac14\right) = \frac32e^4+\frac12$$

$$= \frac32e^4+\frac12-\frac12(e^4-1)+2 = \frac32e^4+\frac12-\frac12e^4+1+2 = e^4+\frac72$$

Zad. 3) Promijenite poredak integracije u integralu $\int_{-1}^{1} dx \int_{3-x^2}^{9-x^2} f(x,y)dy$



$y = 9 - x^2$

$x^2 = 9 - y$

$x = \pm\sqrt{9-y}$

$y = 3 - x^2$

$x = \pm\sqrt{3-y}$

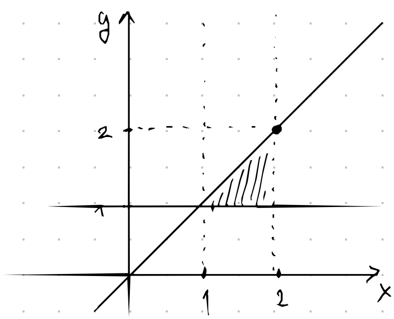
$-\sqrt{9-y} < -\sqrt{3-y}$

$\sqrt{3-y} < \sqrt{9-y}$

$$\int_2^3 dy \int_{-1}^{-\sqrt{3-y}} f(x,y)dx + \int_2^3 dy \int_{\sqrt{3-y}}^{1} f(x,y)dx$$

$$+ \int_3^8 dy \int_{-1}^{1} f(x,y)dx + \int_8^9 dy \int_{-\sqrt{9-y}}^{\sqrt{9-y}} f(x,y)dx$$

Zad. 4.) Promjenom poretka integracije izračunajte $\int_1^2 x dx \int_1^x \sqrt{x^2-y^2}\, dy$



$$= \int_1^2 \sqrt{x^2-y^2}\, dy \int_1^y x dx$$

$\frac{x^2}{2}$

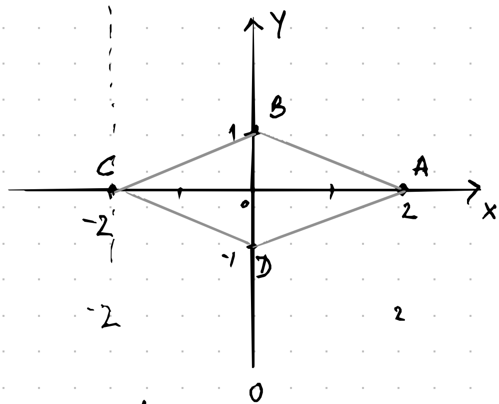
$$= \int_1^2 \sqrt{x^2-y^2} \cdot \frac{x^2}{2}\Big|_1^y dy$$

$$= \int_1^2 \sqrt{x^2-y^2} \cdot \left(\frac{y^2}{2} - \frac{1}{2}\right) dy$$

$$= \frac{1}{2}\left(\int_1^2 y^2\sqrt{x^2-y^2}\,dy - \int_1^2 \sqrt{x^2-y^2}\, dy\right) =$$

Zad. 5.) Neka je D četverokut s vrhovima $A(2,0)$, $B(0,1)$, $C(-2,0)$ i $D(0,-1)$. Izračunajte integral $\iint_D f(x,y)\,dx\,dy$ za

a) $f(x,y) = x$



$CB \dots \frac{x}{-2} + \frac{y}{1} = 1$

$\hookrightarrow y = 1 + \frac{1}{2}x$

$CD \dots \frac{x}{-2} + \frac{y}{-1} = 1$

$y = -\frac{x}{2} - 1$

$BA \dots \frac{x}{2} + y = 1 \rightarrow y = 1 - \frac{1}{2}x$

$DA \dots y = \frac{x}{2} - 1$

$$\Rightarrow \int_{-2}^{0} dx \int_{0}^{1+\frac{1}{2}x} x\,dy + \int_{-2}^{0} dx \int_{-\frac{x}{2}-1}^{0} x\,dy + \int_{0}^{2} dx \int_{0}^{1-\frac{x}{2}} x\,dy + \int_{0}^{2} dx \int_{\frac{x}{2}-1}^{0} x\,dy$$

$$= \int_{-2}^{0} xy \Big|_{0}^{y=1+\frac{1}{2}x} dx + \int_{-2}^{0} xy \Big|_{-\frac{x}{2}-1}^{y=0} dx + \int_{0}^{2} xy \Big|_{0}^{y=1-\frac{x}{2}} dx + \int_{0}^{2} xy \Big|_{\frac{x}{2}-1}^{y=0} dx$$

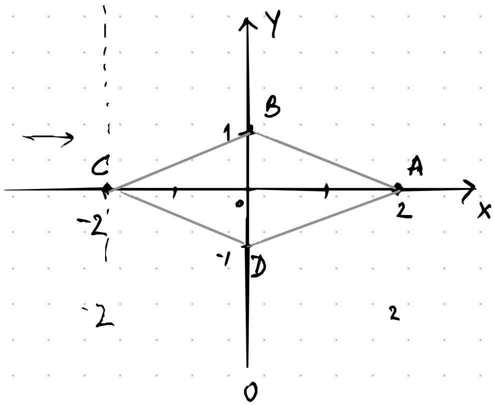
$$= \int_{-2}^{0} x\left(1+\frac{1}{2}x\right) dx + \int_{-2}^{0} -x\left(-\frac{x}{2}-1\right) dx + \int_{0}^{2} x\left(1-\frac{x}{2}\right) dx + \int_{0}^{2} -x\left(x/2 - 1\right) dx$$

$x + \frac{1}{2}x^2 \quad = \quad \frac{x^2}{2} + x \qquad x - \frac{x^2}{2} \qquad -\frac{x}{2} + x$

$$= 2\left(\frac{1}{2}x^2 + \frac{1}{6}x^3\right)\Big|_{-2}^{0} + 2\left(\frac{1}{2}x^2 - \frac{1}{6}x^3\right)\Big|_{0}^{2} = \left(x^2 + \frac{1}{3}x^3\right)\Big|_{-2}^{0} + \left(x^2 - \frac{1}{3}x^3\right)\Big|_{0}^{2}$$

$$= 0 - 4 + \frac{8}{3} + 4 - \frac{1}{3}8 \quad \boxed{= 0}$$

b) $f(x,y) = y$



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$y = -\frac{x}{2} - 1$

BA... $\frac{x}{2} + y = 1 \rightarrow y = 1 - \frac{1}{2}x$

DA... $y = \frac{x}{2} - 1$

$\int dx \int f(x,y)\,dy$

$$= 2\left(\int_{-2}^{0} dx \int_{1+\frac{1}{2}x}^{0} y\,dy + \int_{0}^{2} dx \int_{0}^{1-\frac{1}{2}x} y\,dy \right)$$

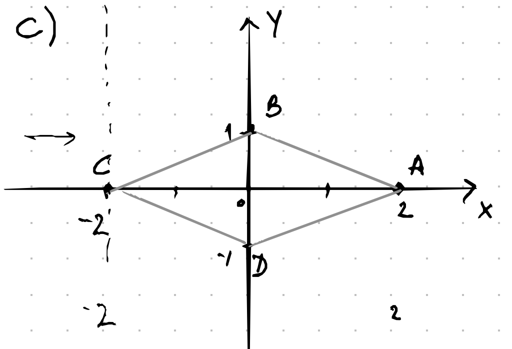
$$= \cancel{2}\left(\int_{-2}^{0} \frac{1}{\cancel{2}} y^2 \Big|_{y=1+\frac{1}{2}x}^{y=0} dx + \int_{0}^{2} \frac{1}{\cancel{2}} y^2 \Big|_{0}^{y=1-\frac{1}{2}x} dx \right)$$

$$= \int_{-2}^{0} -\left(1+\frac{1}{2}x\right)^2 dx + \int_{0}^{2} \left(1-\frac{1}{2}x\right)^2 dx = \int_{-2}^{2} -\left(1+\frac{1}{2}x\right)^2 + \left(1-\frac{1}{2}x\right)^2 dx$$

$$= \int_{-2}^{2} \cancel{-1} - x - \cancel{\frac{1}{4}x^2} \cancel{+1} - x + \cancel{\frac{1}{4}x^2}\, dx = -2\int_{-2}^{2} x\,dx = -\cancel{2}\left(\frac{1}{\cancel{2}}x^2\right)\Big|_{-2}^{2}$$

$$= -4 + 4 = 0$$

c)



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$BA \ldots \frac{x}{2} + y = 1 \rightarrow y = 1 - \frac{1}{2}x$

$DA \ldots y = \frac{x}{2} - 1$

$f(x,y) = e^{x+y}$

$$\Rightarrow \int_{-2}^{0} dx \int_{1+\frac{1}{2}x}^{0} e^{x+y}\, dy + \int_{-2}^{0} dx \int_{\frac{-x}{2}-1}^{0} e^{x+y}\, dy + \int_{0}^{2} dx \int_{0}^{1-\frac{x}{2}} e^{x+y}\, dy + \int_{0}^{2} dx \int_{0}^{\frac{x}{2}-1} e^{x+y}\, dy$$

$$\rightarrow \int_{-2}^{2} dx \int_{1+\frac{1}{2}x}^{1-\frac{1}{2}x} e^{x+y}\, dy + \int_{-2}^{2} dx \int_{-\frac{x}{2}-1}^{\frac{x}{2}-1} e^{x+y}$$

$$\Rightarrow \int_{-2}^{2} e^{x+y} \cdot \frac{1}{2}y^2 \Big|_{1+\frac{1}{2}x}^{1-\frac{1}{2}x} dx = \int_{-2}^{2} e^{x+1-\frac{1}{2}x} \cdot \frac{1}{2}\left(1-\frac{1}{2}x\right)^2 - e^{x+\frac{1}{2}x+1} \cdot \frac{1}{2}\left(1+\frac{1}{2}x\right)^2 dx$$

$$= \frac{1}{2}\int_{-2}^{2} e\, e^{\frac{1}{2}x}\left(1-\frac{1}{2}x\right)^2 - e\, e^{\frac{3}{2}x}\left(1+\frac{1}{2}x\right)^2 dx$$

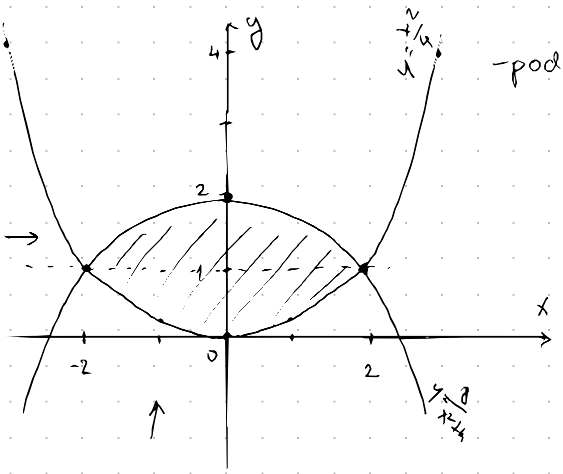
$$= \frac{1}{2}e\int_{-2}^{2} e^{\frac{1}{2}x} - x e^{\frac{1}{2}x} + \frac{1}{4}x^2 \cdot e^{\frac{1}{2}x} - e^{\frac{3}{2}x} - x e^{\frac{3}{2}x} - \frac{1}{4}x^2 e^{\frac{3}{2}x}\, dx$$

(1) $-\int_{-2}^{2} x e^{\frac{1}{2}x} = \left| \begin{array}{l} u = x \rightarrow du = dx \\ e^{\frac{1}{2}x} \longrightarrow \end{array} \right| = \frac{1}{4}x^3 e^{\frac{1}{2}x} - \int e^{\frac{1}{2}} dx = -\left(\frac{1}{4}x^3 e^{\frac{1}{2}x} - \frac{1}{4}x^2 e^{\frac{1}{2}x}\Big|_{-2}^{2}\right)$

(2) $\frac{1}{4}\int x^2 e^{\frac{1}{2}x} = \left| \begin{array}{ll} u = x^2 & du = 2x \\ e^{\frac{1}{2}x} \longrightarrow & \frac{1}{4}x^2 e^{\frac{1}{2}x} \end{array} \right| = \frac{1}{4}x^4 e^{\frac{1}{2}x} - \frac{1}{2}\int x^3 e^{\frac{1}{2}x} = \Big| \; u = x^3$

previšc integriranja.

6. Površina lika omeđenog krivuljama $y=\frac{x^2}{4}$ i $y=\frac{8}{x^2+4}$. Skica!



- podijelimo na 4 integrala

$$\int_0^2 dy \int \; dx$$

Gracia $\int_{-2}^{2} dx \int_{\frac{x^2}{4}}^{\frac{8}{x^2+4}} dy$

obrnuti:

$$2\cdot\int_0^2 dy \int_{2\sqrt{y}}^{\sqrt{\frac{8}{y}-4}} dx$$

$y=\frac{x^2}{4} \rightarrow x=2\sqrt{y}$

$y=\frac{8}{x^2+4} \rightarrow x=\sqrt{\frac{8}{y}-4}$

$$=2\cdot\int_0^2\left(\sqrt{\frac{8}{y}-4}-2\sqrt{y}\right)dy = 2\cdot\int_0^2\left(2\sqrt{\frac{2}{y}-1}-2\sqrt{y}\right)dy$$

$$=4\int_0^2\left(\sqrt{\frac{2}{y}-1}-2\sqrt{y}\right)dy \quad \text{zeznuto}$$

$-x^4-x^2+32$

Gracia: $$\int_{-2}^{2} dx\int_{\frac{x^2}{4}}^{\frac{8}{x^2+4}} dy = \int_{-2}^{2}\left(\frac{8}{x^2+4}-\frac{x^2}{4}\right)dx = \int_{-2}^{2}\frac{32-x^2(x^2+4)}{4(x^2+4)}dx$$

$$=\frac{1}{4}\left(\int_{-2}^{2}\frac{32\,dx}{x^2+4}-\int_{-2}^{2}\frac{x^4}{x^2+4}dx+\int_{-2}^{2}\frac{4x^2}{x^2+4}dx\right)$$

1 $32\cdot\frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)\Big|_{-2}^{2}$

2 $x^4-16=(x^2+4)(x^2-4)$

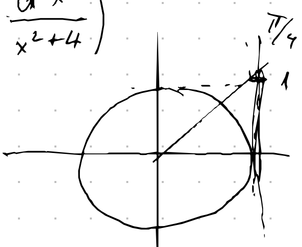
$$\frac{x^4}{x^2+4}=\frac{x^4-16}{x^2+4}+\frac{16}{x^2+4}=\frac{x^2-4}{1}+\frac{16}{x^2+4}$$

$$\int_{-2}^{2} x^2-4+\frac{16}{x^2+4}\,dx = \frac{1}{3}x^3\Big|_{-2}^{2}-4x\Big|_{-2}^{2}+16\cdot\frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)\Big|_{-2}^{2}$$

3 $$4\int_{-2}^{2}\frac{x^2+4}{x^2+4}+\frac{-4}{x^2+4}\,dx = 4\left(\int_{-2}^{2}dx-4\int_{-2}^{2}\frac{dx}{x^2+4}\right)$$

$$=4x\Big|_{-2}^{2}-16\,\frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)\Big|_{-2}^{2}$$

1

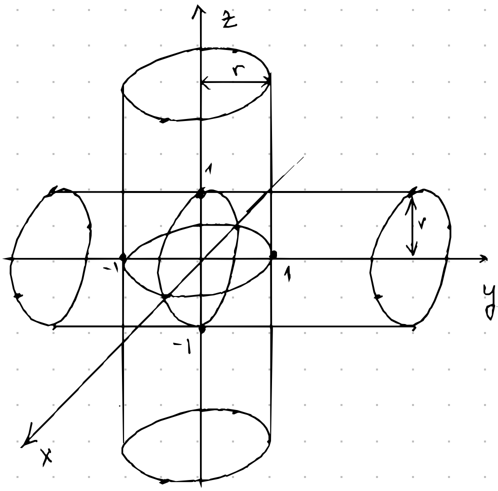


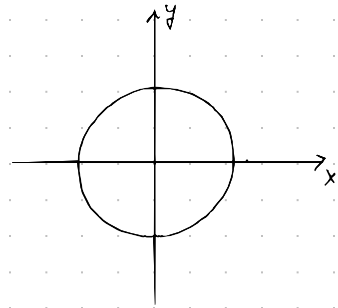
$$\frac{1}{4}\left(16\left(\frac{\pi}{4}+\frac{\pi}{4}\right)-\left(\frac{8}{3}-8+8\,\frac{\pi}{2}-\left(\frac{-8}{3}+8-\frac{8\pi}{2}\right)\right)+\left(8-8\cdot\frac{\pi}{2}-\left(-8+8\,\frac{\pi}{2}\right)\right)\right)=$$

$$\frac{1}{4}\left(16\,\frac{\pi}{2}-\left(\frac{16}{3}\cancel{-16}\cancel{+16\frac{\pi}{2}}\right)+\left(\cancel{16}\cancel{-16\frac{\pi}{2}}\right)\right)=$$

$$\frac{1}{4}\left(8\pi-\frac{16}{3}\right)=\boxed{2\pi-4/3}$$

7. $x^2+y^2=1$ $\quad x^2+z^2=1$





$$V=\int_{-1}^{1} dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy=\int_{-1}^{1} 2\sqrt{1-x^2}\,dx$$

osim što ovo ne znam integrirati, zašto sam trebala postaviti ovako:

$$\frac{1}{2}V=\int_{-1}^{1} dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy$$

→ kao da sam napisala $\int dx\int y\,dy$?

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mislim da je to jer je $z=f(x,y)$

↳ $z=\sqrt{1-x^2}$

$y^2=1-x^2$

$y=\pm\sqrt{1-x^2}$

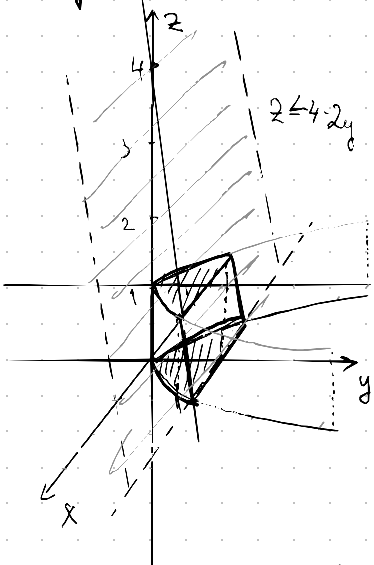
$$V=\int_{-1}^{1} dx\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}\,dy$$

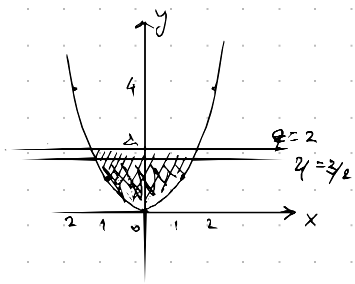
$$V=\int_{-1}^{1} y\sqrt{1-x^2}\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx=\int_{-1}^{1}(1-x^2)+(1-x^2)\,dx=\int_{-1}^{1}2(1-x^2)\,dx$$

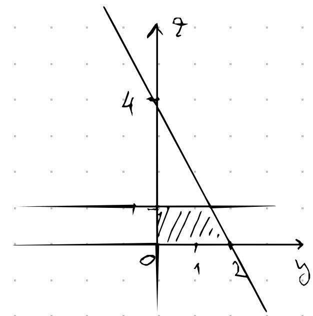
$$V=2\int_{-1}^{1}1-x^2\,dx=2\left(x-\frac{1}{3}x^3\Big|_{-1}^{1}\right)=2\left(1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)\right)=2\left(1-\frac{1}{3}+1-\frac{1}{3}\right)$$

$$=2\left(2-\frac{2}{3}\right)=\frac{8}{3}$$

8. $y \ge x^2$ $\quad z \le 1$ $\quad z \le 4-2y$ $\quad z \ge 0$



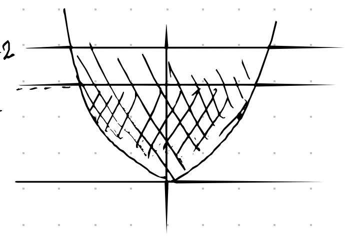




$f(x,y) = 4-2y$

$$\int_{3/2}^{2} dy \int_{-\sqrt{y}}^{\sqrt{y}} 4-2y \, dx$$

zašto $[\frac{3}{2}, 2]$?



- jer je manji komadić na $z=1$ do $3/2$ | kada se poklopi

↳ $z=1 \to 1 = 4-2y \to y = 3/2$

$$* + \int_0^{3/2} dy \int_{-\sqrt{y}}^{\sqrt{y}} 4-2y \, dx$$

1. $$\int_{3/2}^{2} dy \int_{-\sqrt{y}}^{\sqrt{y}} 4-2y \, dx = \int_{\frac{3}{2}}^{2} (4x-2xy)\Big|_{-\sqrt{y}}^{\sqrt{y}} dy = \int_{\frac{3}{2}}^{2} 4\sqrt{y} - 2y\sqrt{y} + 4\sqrt{y} - 2y\sqrt{y} \, dy$$

$$= \int_{\frac{3}{2}}^{2} 8\sqrt{y} - 4y\sqrt{y}\, dy = 8\int_{\frac{3}{2}}^{2} \sqrt{y}\, dy - 4\int_{\frac{3}{2}}^{2} y^{\frac{3}{2}} dy = 8 \cdot \left(\frac{2}{3} y^{3/2}\Big|_{3/2}^{2} - 4\left(\frac{2}{5} y^{5/2}\right)\Big|_{3/2}^{2}\right.$$

$$= \frac{16}{3}\left(\sqrt{8} - \overbrace{\sqrt{\frac{27}{4}}}^{3\sqrt{3}}\right) - \frac{8}{5}\left(\sqrt{32} - \sqrt{\frac{27\cdot 9}{32}}\right) = \frac{32}{3}\sqrt{2} - 8\sqrt{3} - \frac{32}{5}\sqrt{2} - \frac{9\sqrt{3}}{4\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \sqrt{2}\left(\frac{160-96}{15}\right) - \sqrt{3}\left(8 + \frac{9\sqrt{2}}{8}\right)$$

$$= \frac{64\sqrt{2}}{15} - \frac{64\sqrt{3} + 9\sqrt{6}}{8}$$

2. $$\int_0^{3/2} 8\sqrt{y} - 4y\sqrt{y}\, dy = 8\left(\frac{2}{3} y^{3/2}\Big|_0^{3/2}\right) - 4 \cdot \frac{2}{5} y^{5/2}\Big|_0^{3/2}$$

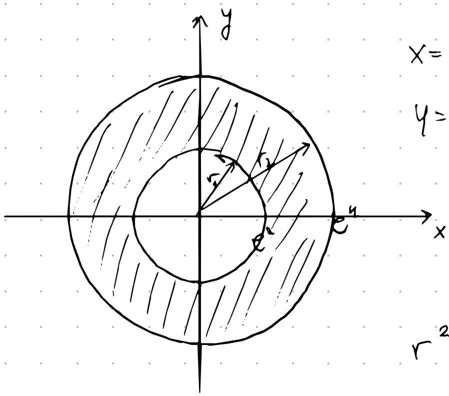
$$= \frac{16}{3}\sqrt{\frac{27}{4}} - \frac{8}{5}\sqrt{\frac{27\cdot 9}{32}} = \frac{24\sqrt{3}}{3} - \frac{72\sqrt{3}}{20\sqrt{2}}$$

nema da to je to

9. $\iint_D \ln(x^2+y^2)\,dxdy$, D je kružni vijenac $e^2 \le x^2+y^2 \le e^4$

$e^2 = 7{,}39$

$e^4 = 54{,}6$



$x = r\cos\varphi$

$y = r\sin\varphi$

$$\iint_D \ln(x^2+y^2)\,dxdy$$

$$= \iint_{D'} f(\underbrace{r\cos\varphi, r\sin\varphi}_{\ln(r^2)})\, r\,dr\,d\varphi$$

$r^2 \ge e^2 \qquad r^2 \le e^4$

$r \ge e \qquad r \le e^2$

$$\int_0^{2\pi} d\varphi \int_e^{e^2} \ln(r^2)\, r\,dr$$

$$= \left| \begin{matrix} t = r^2 \\ dt = 2r\,dr \end{matrix} \right| \to \left| \begin{matrix} r\,dr = \frac{dt}{2} \\ r = e \to t = e^2 \\ r = e^2 \to t = e^4 \end{matrix} \right| = \int_0^{2\pi} d\varphi \int_{e^2}^{e^4} \frac{1}{2}\ln(t)\,dt$$

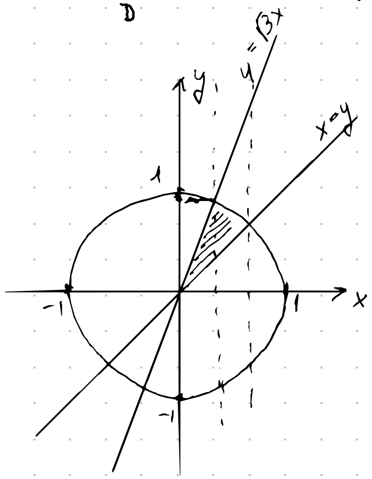
$$\rightsquigarrow \frac{1}{2}\int_{e^2}^{e^4} \ln(t)\,dt \qquad \begin{matrix} u = \ln(t) \longrightarrow du = \frac{1}{t}\,dt \\ dv = dt \longrightarrow v = t \end{matrix} \Rightarrow \frac{1}{2}\left(t\ln(t)\Big|_{e^2}^{e^4} - \int_{e^2}^{e^4} dt \right)$$

$$= \frac{1}{2}\left(e^4\cdot 4 - e^2\cdot 2 - e^4 + e^2\right) = \frac{1}{2}\left(3e^4 - e^2\right) = \frac{1}{2}e^2(3e^2-1)$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{2}e^2(3e^2-1)\,d\varphi = \frac{1}{2}\int_0^{2\pi} 3e^4 - e^2\,d\varphi = \frac{1}{2}\left(3e^4\varphi - e^2\varphi \Big|_0^{2\pi}\right)$$

$$= \frac{1}{2}\left(3e^4\cdot 2\pi - e^2\cdot 2\pi\right) = 3e^4\pi - e^2\pi = \boxed{\pi e^2(3e^2-1)}$$

10. $\iint_D \sqrt{1-x^2-y^2}\,dx\,dy$ D je kružni isječak: $x^2+y^2\le 1,\ y\ge x,\ y\le\sqrt{3}x$



$x=r\cos\varphi \quad y=r\sin\varphi$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^1 \sqrt{1-(r^2)}\, r\, dr =$$

$$\left| \begin{array}{l} t=1-r^2 \\ dt=-2r\,dr \end{array} \right| = -\frac{1}{2}\sqrt{t}\,dt = -\frac{1}{2}$$

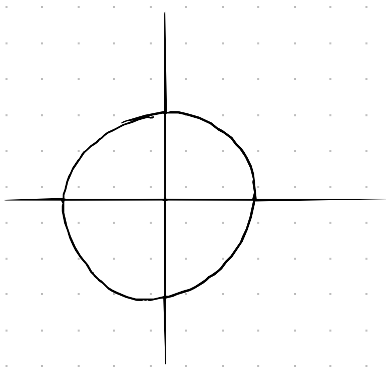
$r=1 \to t=0,\ r=0 \to t=1$ $\qquad \frac{4\pi-3\pi}{12}$

$1/2$

$$-\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_1^0 \sqrt{t}\,dt = -\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} -\frac{2}{3}\cdot t^{3/2}\Big|_1^0 d\varphi = +\frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi = \frac{1}{3}\left(\frac{\pi}{3}-\frac{\pi}{4}\right) = \frac{\pi}{36}$$

11. Površina lika unutar kružnice $r=3\cos\varphi$, a izvan kardioide $r=1+\cos\varphi$

?

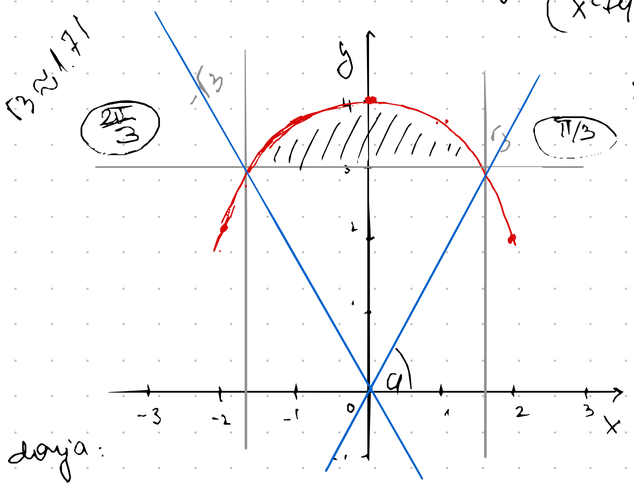


12. Prijelazom na polarne koordinate:

a) $\int_{-\sqrt{3}}^{\sqrt{3}} dx \int_{3}^{2+\sqrt{4-x^2}} \frac{dy}{(x^2+y^2)^{3/2}}$

$x = r\cos\varphi \quad y = r\sin\varphi$

$f(x,y) = \frac{1}{(x^2+y^2)^{3/2}} = \frac{1}{r^{2\cdot\frac{3}{2}}} = \frac{1}{r^3}$



gornja: $2+\sqrt{4-x^2} = y$

$2+\sqrt{4-r^2\cos^2\varphi} = r\sin\varphi$

$\sqrt{4-r^2\cos^2\varphi} = r\sin\varphi - 2 \;/^2$

$\cancel{4} - r^2\cos^2\varphi = r^2\sin^2\varphi - 4r\sin\varphi + \cancel{4}$

$4r\sin\varphi = r^2(\underbrace{\cos^2+\sin^2\varphi}_{1})$

$r = 4\sin\varphi$

donja:

$3 = r\sin\varphi$

$r = \frac{3}{\sin\varphi}$

$\operatorname{tg}\varphi = \frac{3}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \sqrt{3} \to \frac{\pi}{3}$

$r^{-2} = \frac{r^{-1}}{-2+1}$

$\int_{\frac{\pi}{3}}^{2\pi/3} d\varphi \int_{\frac{3}{\sin\varphi}}^{4\sin\varphi} f(x,y)\, r\, dr = \int_{\frac{\pi}{3}}^{2\pi/3} d\varphi \int_{\frac{3}{\sin\varphi}}^{4\sin\varphi} \frac{1}{r^{\cancel{3}2}} \cdot \cancel{r}\, dr = \int_{\frac{\pi}{3}}^{2\pi/3} -\frac{1}{1}\frac{1}{r}\Big|_{\frac{3}{\sin\varphi}}^{4\sin\varphi} d\varphi = -\frac{1}{1}\int_{\frac{\pi}{3}}^{2\pi/3} \frac{1}{4\sin\varphi} - \frac{\sin\varphi}{3}\, d\varphi$

$= -\frac{1}{1}\left(\frac{1}{4}\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \frac{d\varphi}{\sin\varphi} - \frac{1}{3}\int_{\frac{\pi}{3}}^{2\pi/3} \sin\varphi\, d\varphi\right) = \frac{-1}{1}\left(\frac{1}{4}\ln\left|\operatorname{tg}\frac{\varphi}{2}\right|\Big|_{\frac{\pi}{3}}^{2\pi/3} + \frac{1}{3}\cos\varphi\Big|_{\frac{\pi}{3}}^{2\pi/3}\right)$

$= \frac{-1}{1}\left(\frac{1}{4}\left(\ln\left(\operatorname{tg}\frac{\pi}{3}\right) - \ln\left(\operatorname{tg}\frac{\pi}{6}\right)\right) + \frac{1}{3}\left(\underbrace{\cos\frac{2\pi}{3}}_{-\frac{1}{2}} - \underbrace{\cos\frac{\pi}{3}}_{-\frac{1}{2}}\right)\right)$

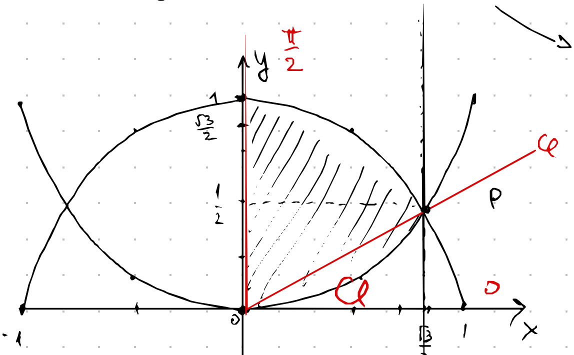
$= -\left(\frac{1}{4}\left(\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right) - \frac{1}{12}\right) = \frac{1}{3} - \frac{1}{4}\ln\left(\left|\frac{\sqrt{3}}{\frac{1}{\sqrt{3}}}\right|\right) = \frac{1}{3} - \frac{1}{4}\ln\left(\frac{1}{3}\right)$ P

Korisno jesmo li radili s pomaknutim koord!! = isto što i $\frac{1}{3} - \frac{1}{2}\ln\left(\frac{\sqrt{3}}{3}\right)$

b) $\int_0^{\frac{\sqrt3}{2}} dx \int_{1-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy$

$y=\sqrt{1-x^2}$

x	y
0	1
$\frac{1}{2}$	$\frac{\sqrt3}{2}$
1	0



$r = 1$

$x = r\cos\varphi$

$y = r\sin\varphi$

$\frac{\sqrt3}{2} \approx 0.87$

$r\sin\varphi = \sqrt{1-r^2\cos^2\varphi} \;/^2$

$r^2\sin^2\varphi = 1 - r^2\cos^2\varphi$

$r^2 = 1 \longrightarrow r = 1$

$r\sin\varphi = 1 - \sqrt{1-r^2\cos^2\varphi}$

$\sqrt{1-r^2\cos^2\varphi} = 1 - r\sin\varphi \;/^2$

$\cancel{1} - r^2\cos^2\varphi = \cancel{1} - 2r\sin\varphi + r^2\sin^2\varphi$

$2\cancel{r}\sin\varphi = r^{\cancel{2}} \longrightarrow r = 2\sin\varphi$

$\varphi = \left(\operatorname{tg}\frac{\frac{1}{2}}{\frac{\sqrt3}{2}}\right)^{-1} \Rightarrow \frac{\pi}{6}$

$y = \sqrt{1-\left(\frac{\sqrt3}{2}\right)^2} = \frac{1}{2}$

jednadžba pravca p: $y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$y - 0 = \frac{\frac{1}{2}-0}{\frac{\sqrt3}{2}-0}(x-0)$

$y = \frac{1}{\sqrt3}x$

1. integral $\int_0^{\pi/6} d\varphi \int r^2 dr$

donja: 0

gornja $y = 1-\sqrt{1-x^2} \longrightarrow r = 2\sin\varphi$

1. $\int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\sin\varphi} r^2 dr$

2. integral $\int_{\pi/6}^{\frac{\pi}{2}} d\varphi \int_0^1 r^2 dr$

zašto r ide od 0?

$\Rightarrow$ 1. + 2. $\Rightarrow \int_0^{\frac{\pi}{6}} d\varphi \int_0^{2\sin\varphi} r^2 dr + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} d\varphi \int_0^1 r^2 dr$

$= \int_0^{\pi/6} \frac{1}{3} 8\sin^3\varphi\, d\varphi + \int_{\frac{\pi}{6}}^{\pi/2} \frac{1}{3} 1\, d\varphi = \frac{1}{3}\left(8\int_0^{\pi/6} \sin^3 d\varphi + \frac{\pi}{2} - \frac{\pi}{6}\right) =$

$t = \cos\varphi$

$\sin^2\varphi = 1-\cos^2\varphi$

$\sin^3\varphi = (1-t^2)dt$

$= \frac{1}{3}\left(8\int_0 (1-t^2)dt + \frac{\pi}{3}\right)$

$= \frac{\pi}{9} - \frac{8}{3}\left(\int_0^{\pi/6} dt - \int_0^{\pi/6} t^2 dt\right) = \frac{\pi}{9} - \frac{8}{3}\left(t\Big|_0^{\frac{\pi}{6}} - \frac{1}{3}t^3\Big|_0^{\frac{\pi}{6}}\right)$

$= \frac{\pi}{9} - \frac{8}{3}\left(\cos\frac{\pi}{6} - \cos 0 - \frac{1}{3}\left(\cos^3\left(\frac{\pi}{6}\right) - \cos^3(0)\right)\right)$

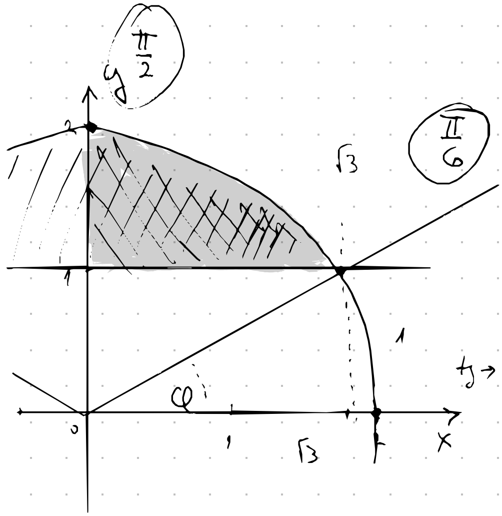
$= \frac{\pi}{9} - \frac{8}{3}\left(\frac{\sqrt3}{2} - 1 - \frac{1}{3}\left(\frac{3\sqrt3}{8} - 1\right)\right)$

$= \frac{\pi}{9} - \frac{8}{3}\left(\frac{\sqrt3}{2} - 1 - \frac{\sqrt3}{8} + \frac{1}{3}\right) = \frac{\pi}{9} - \frac{8}{3}\left(\frac{3\sqrt3}{8} - \frac{2}{3}\right) = \boxed{\frac{\pi}{9} - \sqrt3 + \frac{16}{9}}$

e) $\int_0^1 dx \int_1^{\sqrt{4-x^2}} \frac{dy}{(x^2+y^2)^{5/2}}$

$-x^2 = y^2 - 4$

$x^2 =$



$r\sin\varphi = 1$

$r = \frac{1}{\sin\varphi}$

$r\sin\varphi = \sqrt{4 - r^2\cos^2\varphi} \;/^2$

$r^2\sin^2\varphi = 4 - r^2\cos^2\varphi$

$r^2 = 4$

$r = 2$

$f(x,y) = \frac{1}{(x^2+y^2)^{5/2}} = \frac{1}{(r^2)^{5/2}} \cdot r\,dr$

$\frac{1}{r^5}\cdot r = \frac{1}{r^4}$

$tg \to \frac{1}{\sqrt3} \Rightarrow \frac{\sqrt3}{3}$

$\varphi = \frac{\pi}{6}$

$r^{-4} \Rightarrow r^{-3} \cdot \frac{1}{-3}$

$$\Rightarrow \int_{\pi/6}^{\pi/2} d\varphi \int_{\frac{1}{\sin\varphi}}^{2} \frac{dr}{r^4} = \int_{\pi/6}^{\pi/2} -\frac13 r^{-3}\Big|_{\frac{1}{\sin\varphi}}^{2} d\varphi = -\frac13 \int_{\pi/6}^{\pi/2} \frac18 - \sin^3\varphi \, d\varphi =$$

$t = \cos\varphi$

$\sin(1-\cos^2\varphi)$

$dt \quad (1-t)$

$$= -\frac13\left(\frac18\left(\frac\pi2 - \frac\pi6\right) - \int_{\pi/6}^{\pi/2}(1-t^2)\,dt\right) = -\frac13\left(\frac{\pi}{24} - t\Big|_{\pi/6}^{\pi/2 = 0} + \frac13 t^3\Big|_{\pi/6}^{\pi/2 = 0}\right)$$

$\frac{\pi}{3}$

$$= -\frac13\left[\frac{\pi}{24} + \cos\frac\pi6 + \frac13\left(-\cos^3\left(\frac\pi6\right)\right)\right] = -\frac{\pi}{72} + \frac{\sqrt3}{2}\cdot\left(-\frac13\right) + \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}\sqrt3}{8} =$$

$$= -\frac{\pi}{72} - \frac{\sqrt3}{6} + \frac{\sqrt3}{24}$$

nešto smo i ja i Nikola Pulaki

jer kada se u kalk.

ubaci $\int_{\pi/6}^{\pi/2}\left(-\frac{1}{24} + \frac{\sin^3(\varphi)}{3}\right)d\varphi \to -0{,}04$

?

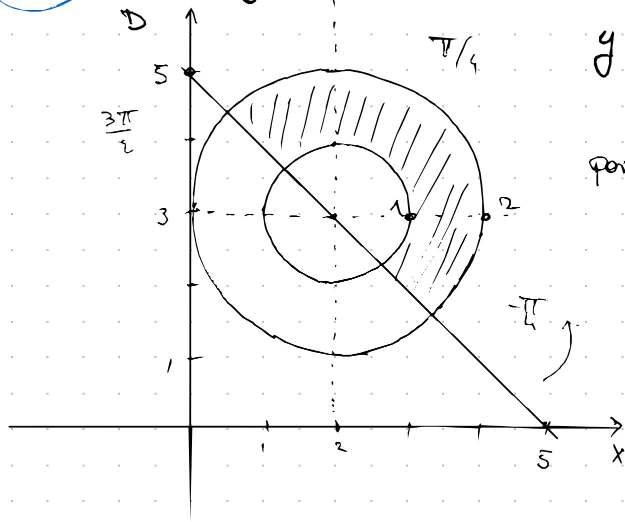
njihovo je $\frac{-\pi}{72} - \frac{\sqrt3}{8} + 5$

pogreška $0{,}08$??

$-0{,}0218$

13) $\iint_D x\,dx\,dy$ $\quad D \to 1 \le (x-2)^2 + (y-3)^2 \le 4$

$y \ge -x+5$



pomaknute:

$$\begin{aligned} x - x_0 &= r\cos\varphi \\ y - y_0 &= r\sin\varphi \end{aligned} \quad \begin{cases} x = r\cos\varphi + 2 \\ y = r\sin\varphi + 3 \end{cases}$$

$y = -x+5$ je $x = y$ pomaknut za $5 \to \frac{\pi}{4}$

$$\int_{-\frac{\pi}{4}}^{3\pi/4} d\varphi \int_1^2 (r\cos\varphi + 2)\, r\, dr = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_1^2 r^2\cos\varphi + 2r\, dr$$

$$= \int_{-\frac{\pi}{4}}^{3\pi/4} \cos\varphi \cdot \frac{1}{3}r^3 + \frac{1}{2}r^2 \Big|_1^2 d\varphi = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \cos\varphi\left(\frac{8}{3} - \frac{1}{3}\right) + 3\, d\varphi = \int_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{7}{3}\cos\varphi + 3\, d\varphi$$

$$= \frac{7}{3}\sin\varphi \Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}} + 3\left(\frac{3\pi}{4} + \frac{\pi}{4}\right) = \frac{7}{3}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) + 3\pi \boxed{= \frac{7\sqrt{2}}{3} + 3\pi}$$

$\cancel{\frac{2\sqrt{2}}{2}}$

14 ELIPTIČE

$$\int_{-1}^{1} dx \int_{\sqrt{2}}^{\sqrt{4-2x^2}} \frac{dy}{(2x^2+y^2)^2}$$

$x = ar\cos\varphi$

$y = br\sin\varphi$

$$* \iint_D f(x,y)\,dx\,dy = \iint_{D'} f(ar\cos\varphi,\ br\sin\varphi)\ abr\ dr\,d\varphi$$

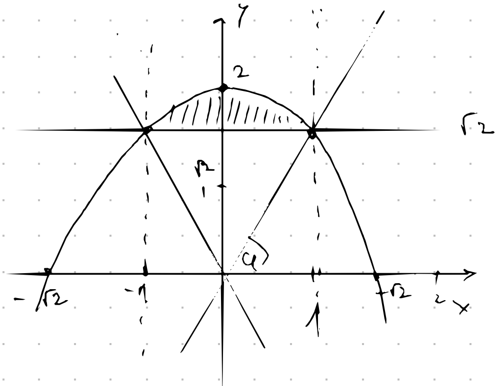
$y^2 = 4-2x^2$

$y^2 - 4 = -2x^2$

$4 - y^2 = 2x^2$

$2 - \frac{y^2}{2} = x^2$

$x = \sqrt{2}$



$y = \sqrt{4-2x^2}$

$(\sqrt{2})^2 = 4-2x^2$

$2 - 4 = -2x^2$

$-2 = -2x^2$

$x^2 = 1$

$\operatorname{tg}\varphi \to \frac{\sqrt{2}}{1}$

Kako je Nikola dobio kut $\frac{\pi}{4}$?

↳ provjerena na Geogebri da nije $\frac{\pi}{4}$

$y = \sqrt{4-2x^2}$

$y^2 = 4-2x^2$

$2x^2 + y^2 = 4 \ /:4$

$\frac{x^2}{2} + \frac{y^2}{4} = 1$

$a = \sqrt{2} \quad b = 2$

$\Longrightarrow x = \sqrt{2}\, r\cos\varphi$

$y = 2r\sin\varphi$

$$\frac{1}{(2\cdot 2\cdot r^2\cos\varphi + 4r^2\sin\varphi)^2} = \frac{1}{(4r^2\cos^2\varphi + 4r^2\sin^2\varphi)^2} = \frac{1}{16r^4}$$

$$\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int \frac{1}{\cancel{16}_{8}\, \cancel{r^4}_{3}} \cdot \cancel{2}\sqrt{2}\, r\, dr = \frac{\sqrt{2}}{8} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_{\frac{\sqrt{2}}{2}\cos\varphi}^{1} \frac{1}{r^3}\, dr = \frac{\sqrt{2}}{8} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} -\frac{1}{2}\cdot\frac{1}{r^2}\Big|_{\frac{\sqrt{2}}{2}\cos\varphi}^{1} d\varphi$$

$1 = \sqrt{2}\, r\cos\varphi$

$* \ r = \frac{1}{\sqrt{2}\cos\varphi}$

$2r\sin\varphi = \sqrt{4 - 2\cdot 2r^2\cos^2\varphi} \ /^2$

$4r^2\sin^2\varphi = 4 - 4r^2\cos^2\varphi$

$4r^2 = 4$

$r^2 = 1 \to r = 1$

$$\Longrightarrow \frac{-\sqrt{2}}{16} \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} 1 - \frac{2}{\sqrt{2}\cos\varphi}\, d\varphi$$

$= 0{,}05759$ * krivo zbog granica kuta

→ ako umjesto $\frac{\pi}{4}$ i $\frac{3\pi}{4}$ uvrstim 54° dobije se: 0,417

↳ still WRONG BRUH.