5.1.4. ZAMJENA VARIJABLI

Matan 1: $\int_a^b f(x)dx = \begin{vmatrix} x=\varphi(t) \\ dx = \varphi'(t)dt \end{vmatrix} = \int_\alpha^\beta f(\varphi(t))\varphi'(t)\,dt$

Što je s 2D int?

$$\iint_D f(x,y)\,dx\,dy = \begin{vmatrix} x = x(u,v) \\ y = y(u,v) \end{vmatrix} = \iint_D g(u,v) \; ?? \; du\,dv$$

|det J| → koristimo J

Prisjetimo se Jacobijeve matrice: $J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$

↳ det J = Jakobijan

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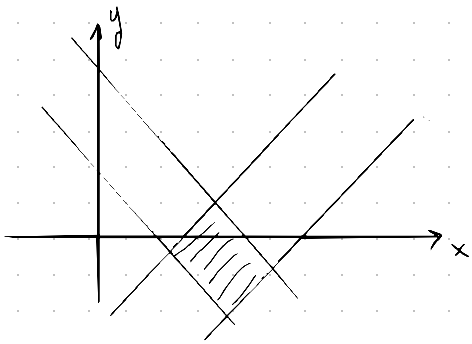
$$\iint_P \frac{\ln(3x+y)}{9x^2 - y^2}\,dx\,dy$$

$3x+y=3$

$3x+y=9$

$3x-y=3$

$3x-y=9$



$$\iint_P \frac{\ln(\overbrace{3x+y}^{u})}{\underbrace{(3x+y)}_{u}\underbrace{(3x-y)}_{v}}\,dx\,dy$$

$\begin{vmatrix} u = 3x+y \\ v = 3x-y \end{vmatrix} /+ \rightarrow u+v = 6x$

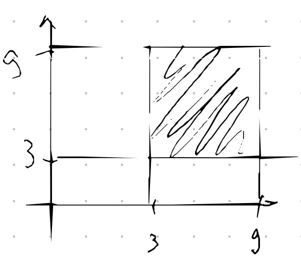
trebamo Jakobijan $\rightarrow x = \frac{1}{6}u + \frac{1}{6}v$

$y = \frac{1}{2}u - \frac{1}{2}v$

$$\rightarrow \iint_P \frac{\ln(u)}{u \cdot v}\,du\,dv$$

$$\det J = \begin{vmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{12} - \frac{1}{12} = \left|\frac{-1}{6}\right| = \frac{1}{6}$$

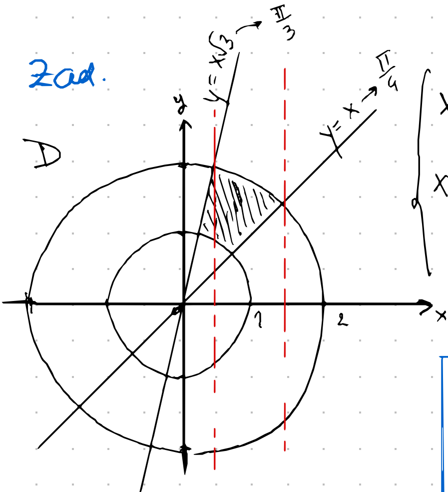
$$\frac{1}{6}\int_3^9 du \int_3^9 \frac{\ln(u)}{u(v)}\,dv \rightarrow \ln v \Big|_3^9$$



$$= \frac{1}{6}\int_3^9 \frac{\ln(u)}{u}\left(\ln|v|\Big|_3^9\right)du = \begin{vmatrix} \ln(u) = t \\ \frac{1}{u} = dt \end{vmatrix}$$

5.1.5. POLARNE KOORDINATE

Zad.



$$\begin{cases} x^2+y^2 \leq 1 \\ x^2+y^2 \leq 4 \\ x \leq y \leq x\sqrt{3} \end{cases}$$

$$\boxed{\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \end{aligned}}$$

$$\boxed{\iint\limits_{D} f(x,y)\,dx\,dy = \iint\limits_{D'} f(r\cos\varphi, r\sin\varphi)\,r\,dr\,d\varphi}$$

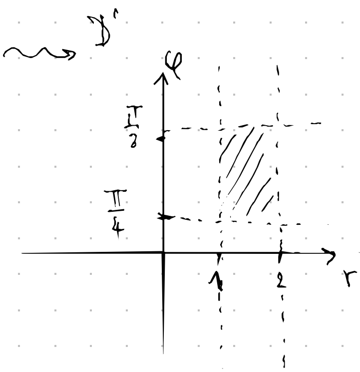
$$\Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix}$$

$= r\cos^2\varphi + r\sin^2\varphi$

$= \underline{\underline{r}} \rightarrow$ uvijek ≥ 0 !

(ne treba aps. vrijednost)

$\rightsquigarrow D'$

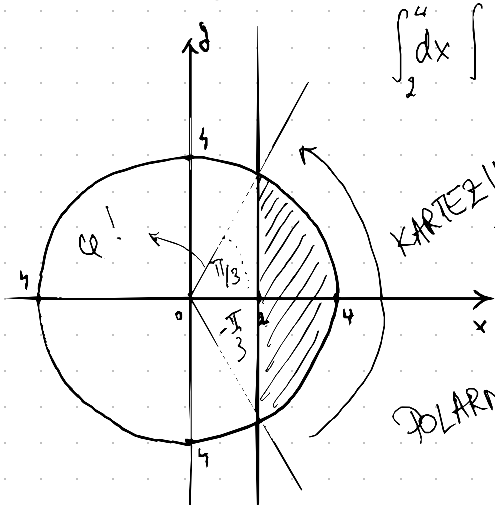


$$\int_{\pi/4}^{\pi/3} d\varphi \int_{1}^{2} f(\varphi, r)\, r\, dr$$

→ prelaskom na polarne koordinate dobivamo □

Zad) $\iint\limits_D x\,dx\,dy \qquad D \ldots\ x^2+y^2 \le 16,\quad x \ge 2$

$\int_2^4 dx \int \qquad y = \pm\sqrt{16-x^2}$



KARTEZIJEVE: $\int_2^4 dx \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} x\,dy$

POLARNE: $\varphi \in \left\langle -\frac{\pi}{3}, \frac{\pi}{3} \right\rangle \qquad \int d\varphi \int r\cos\varphi\, r\,dr$

$\rightarrow \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\varphi \int^{4} r\cos\varphi\, r\,dr$

r ovisi o φ

ovaj primjer daje $x=2$, znači $r\cos\varphi = 2$

$r = \frac{2}{\cos\varphi}$

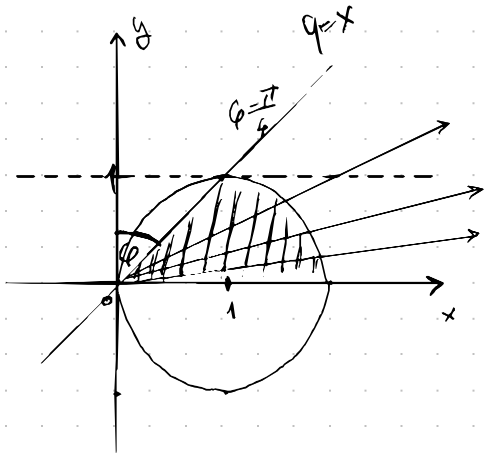
$\cos\frac{\pi}{3} = \frac{1}{2}$

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} d\varphi \int_{\frac{2}{\cos\varphi}}^{4} r^2\cos\varphi\,dr = \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \cos\varphi \cdot \frac{r^3}{3}\Bigg|_{\frac{2}{\cos\varphi}}^{4} d\varphi = \frac{1}{3}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \left(64\cos\varphi - \frac{8}{\cos^3\varphi}\cdot\cos\varphi\right) d\varphi$$

$$\frac{1}{3}\left(\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} 64\cos\varphi\,d\varphi - \int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{8}{\cos^2\varphi}\,d\varphi\right) = \frac{1}{3}\left(64\sin\varphi - 8\operatorname{tg}\varphi \Big|_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\right)$$

$$= \frac{1}{3}\left(64\cdot\frac{\sqrt{3}}{2} - 8\sqrt{3} + 64\cdot\frac{\sqrt{3}}{2} - 8\sqrt{3}\right) = \frac{1}{3}\left(64\sqrt{3} - 16\sqrt{3}\right) = \underline{\underline{\frac{\sqrt{3}}{3}\cdot 48}}$$

MI-19-5)



$$\int_0^1 dy \int_y^{1+\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx$$

1. $y = x$

2. $x = 1+\sqrt{1-y^2}$ granice crtamo!

$$x-1 = \sqrt{1-y^2} \quad /^2$$

$$(x-1)^2 = 1-y^2$$

$$\boxed{(x-1)^2+y^2=1}$$ translatirana kružnica

$$\int_0^{\pi/4} d\varphi \int_0 \sqrt{r^2\cos^2\varphi + r^2\sin^2\varphi}\cdot r\, dr$$

→ r se opet mijenja u ovisnosti o kutu

$$(r\cos\varphi-1)^2 + r^2\sin^2\varphi = 1$$

$$r^2\cos^2\varphi - 2r\cos\varphi + \cancel{1} + r^2\sin^2\varphi = \cancel{1}$$

$$r^{\cancel{2}} = 2\cancel{r}\cos\varphi$$ (možemo kratiti jer je $r>0$)

$$r = 2\cos\varphi$$

$$= \int_0^{\pi/4} d\varphi \int_0^{2\cos\varphi} \sqrt{r^2}\cdot r\, dr = \int_0^{\frac{\pi}{4}} d\varphi \int_0^{2\cos\varphi} r^2\, dr$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{4}} r^3\Big|_0^{2\cos\varphi} d\varphi = \frac{1}{3}\int_0^{\frac{\pi}{4}} 8\cos^3\varphi\, d\varphi = \left| \begin{matrix} \sin\varphi = t \\ \cos\varphi\, d\varphi = dt \end{matrix} \right|$$

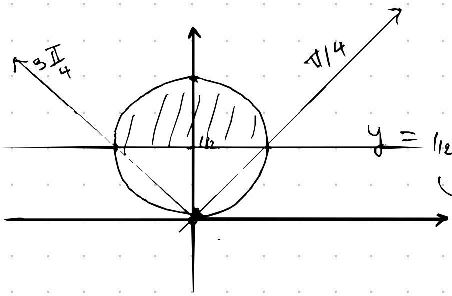
$$= \frac{8}{3}\int_0^{\frac{\pi}{4}} (1-\sin^2\varphi)\cos\varphi\, d\varphi = \frac{8}{3}\int_0 (1-t^2)\, dt$$

$$= \frac{8}{3}\left(\sin\varphi - \frac{1}{3}\sin^3\varphi \Big|_0^{\pi/4}\right)$$

Zad.) $\iint_D y\,dx\,dy$ $\quad x^2+y^2 \le y \quad y \ge \frac{1}{2}$

$x^2+y^2-y=0$

$x^2+\left(y-\frac{1}{2}\right)^2=\frac{1}{4} \to r=\frac{1}{2}$



$r\sin\varphi = 1/2$

$r=\frac{1}{2\sin\varphi}$

donja

$x^2+y^2=y$

$r^{\cancel{2}}=\cancel{r}\sin\varphi$

$r=\sin\varphi$

gornja

$$\int_{\pi/4}^{3\pi/4} d\varphi \int_{\frac{1}{2\sin\varphi}}^{\sin\varphi} r\sin\varphi\, r\,dr$$

$$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} d\varphi \int_{\frac{1}{2\sin\varphi}}^{\sin\varphi} r^2\sin\varphi\, dr = \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin\varphi \cdot r^3 \Big|_{\frac{1}{2\sin\varphi}}^{\sin\varphi} d\varphi = \frac{1}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^4\varphi - \frac{1}{8\sin^2\varphi}\, d\varphi$$

2. način: pomaknute polarne koordinate

$$\boxed{\begin{aligned} x-x_0&=r\cos\varphi \\ y-y_0&=r\sin\varphi \end{aligned}}$$

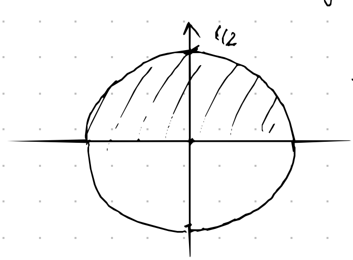
treba vratiti pomaknuti!

$\hookrightarrow x=r\cos\varphi+x_0$

$y=r\sin\varphi+y_0$

$x=r\cos\varphi$

$y-\frac{1}{2}=r\sin\varphi$ $\to$



$$\to \int_0^{\pi} d\varphi \int_0^{1/2} (r\sin\varphi+1/2)\,r\,dr$$

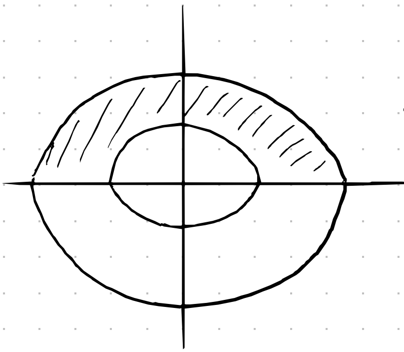
$$= \int_0^{\pi} d\varphi \int_0^{1/2} \left(r^2\sin\varphi + \frac{1}{2}r\right)dr$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{3}r^3\sin\varphi + \frac{r^2}{4}\Big|_0^{1/2} d\varphi = \int_0^{\frac{\pi}{4}} \left(\frac{1}{24}\sin\varphi + \frac{1}{16}\right) d\varphi = -\frac{1}{24}\cos\varphi + \frac{1}{16}\varphi \Big|_0^{\pi} = \frac{1}{12} + \frac{\pi}{16}$$

ELIPTIČKE KOORD.

$$\boxed{\begin{aligned} x &= ar\cos\varphi \\ y &= br\sin\varphi \end{aligned}} \longrightarrow J = abr$$

M1-23-5) $\frac{x^2}{9}+y^2=1$, $\frac{x^2}{81}+\frac{y^2}{9}=1$ $\quad y\geq 0$



$$\iint_D \sqrt{9-\frac{x^2}{9}-y^2}\,dx\,dy$$

$\longrightarrow a=3,\quad b=1$

$x = 3r\cos\varphi$

$y = r\sin\varphi$

$\boxed{J = 3r}$

$$\iint_D \sqrt{9-\frac{9r^2\cos^2\varphi - 9r^2\sin^2\varphi}{9}}$$

$$= \iint_D \sqrt{9-r^2(\cos^2\varphi-\sin^2\varphi)} = \iint_D \sqrt{9-r^2(2\cos^2\varphi-1)}$$

krivo posao

$-1+\cos\varphi$

$\Longrightarrow$