*3.1. na auditornima u petak

3.2. Taylor (za 2 var)

TM Taylor za 2 var. oko $T_0(x_0,y_0)$

$$f(x,y)=f(x_0,y_0)+\frac{\partial f}{\partial x}(T_0)(x-x_0)+\frac{\partial f}{\partial y}(T_0)(y-y_0)+$$
$$+\frac{1}{2!}\left[f''_{xx}(T_0)(x-x_0)^2+2f''_{xy}(T_0)(x-x_0)(y-y_0)+f''_{y}(T_0)(y-y_0)^2\right]+$$

često zaboravljaju

$$+\dots+\frac{1}{n!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^n f(x_0,y_0)+\dots$$

$$+(\text{Lagrangeov ostatak}):\ \frac{1}{(n+1)!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^{n+1} f(T_c),$$

gdje je T_c točka na spojnici $T(x,y)$ i $T_0(x_0,y_0)$.

$$\boxed{f(x,y)=T_n(x,y)+R_n(x,y)}$$

LJIR-20-2 $f(x,y)=e^{x^2}+\ln\frac{1}{xy}$ oko $T(1,1)$

1. $$\frac{\partial f}{\partial x}=e^{x^2}\cdot 2x+\frac{1}{\frac{1}{xy}}\cdot\left(\frac{-1}{x^2y}\right)=e^{x^2}\cdot 2x-\frac{1}{x}\longrightarrow 2e-1$$

2. uvrstiti točku 1,1 u sve

$$\frac{\partial f}{\partial y}=\frac{1}{\frac{1}{xy}}\cdot\left(\frac{-1}{xy^2}\right)=-\frac{1}{y}\longrightarrow -1$$

$$f''_{xx}=e^{x^2}\cdot 4x^2+e^{x^2}\cdot 2+\frac{1}{x^2}\longrightarrow 4e+2e+1=6e+1$$

$$f''_{yy}=\frac{1}{y^2}\to 1\qquad f''_{xy}=0\Rightarrow 0$$

3. uvrstimo u fju

$$T_2(x,y)=e+(2e-1)(x-1)+(-1)(y-1)+\frac{1}{2!}\left[(6e+1)(x-1)^2+2\cdot 0+1(y-1)^2\right]$$

$$\boxed{T_2(x,y)=\underbrace{e+(2e-1)(x-1)-1(y-1)}_{\text{prvi diferencijal}}+\frac{1}{2}(6e+1)(x-1)^2+\frac{1}{2}(y-1)^2}$$

*ne piše da treba ostatak R_n

4. aproksimirati

$f(1{,}02,\ 0{,}9)\approx T_2\ \to\ T_2(1.02,\ 0.9)=\overbrace{e+(2e-1)\cdot 0{,}02-0{,}1}^{\text{prvi (a) dio zad}}+$

$\Delta x=0{,}02$

$\Delta y=0{,}1$

$$+\underbrace{\frac{1}{2}(6e+1)(0{,}02)^2+\frac{1}{2}(0{,}1)^2}_{\text{drugi (b) dio zadatka}}$$

$\Rightarrow$ a) 2.907013 Točno: 2,91591

b) 2.915475

DZ → 3. slučaj

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$f(x,y) = (1-x^2)(y-2)$ oko $T(0,0)$

b) $T_3(x,y) = ?$ $R_3(x,y) = ?$ $\longrightarrow$ DZ sve izderivirati

$f(x,y) = y - 2 - x^2y + 2x^2$

$\underbrace{= \underbrace{2}_{f(0,0)}x^2 - x^2y + y - 2}$ $= T_3(x,y)$ oko $0,0$

jer trebamo dobiti potencije $(x-0)(y-0)$

$\rightarrow$ već zadana fja je polinom

$R_3(x,y) = 0$ jer nema greške kad je fja polinom, tj. ta fja je sama svoj polinom

c) $I(\alpha) = \int_{-\alpha^2}^{3\alpha} \frac{e^{\alpha x}}{x} dx$ odrediti $\frac{dI}{d\alpha}$

$\hookrightarrow$ ne postoji integral; ne može se

$I(\alpha) = \int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\, dx$ integral ovisan o parametru

deriv. gornje granice   deriv. donje granice

$$\frac{d}{d\alpha} I(\alpha) = \psi'(\alpha) \cdot f(\psi(\alpha),\alpha) - \varphi'(\alpha) f(\varphi(\alpha),\alpha) + \int_{\varphi}^{\psi} \frac{\partial f}{\partial \alpha} dx$$

$$\frac{d}{d\alpha} I(\alpha) = 3 \cdot f(3\alpha, \alpha) + 2\alpha \cdot f(-\alpha^2, \alpha) + \int_{-\alpha^2}^{3\alpha} \frac{\partial f}{\partial \alpha} dx$$

$$= \cancel{3} \cdot \frac{e^{3\alpha^2}}{\cancel{3}\alpha} + 2\cancel{\alpha} \cdot \frac{e^{-\alpha^3}}{-\alpha^{\cancel{2}}} + \int_{-\alpha^2}^{3\alpha} \frac{e^{\alpha x}}{\cancel{x}} \cdot \cancel{x}\, dx$$

$$= \frac{e^{3\alpha^2}}{\alpha} - \frac{2e^{-\alpha^3}}{\alpha} -$$

$\hookrightarrow \cancel{3\alpha} \frac{e^{3\alpha^2}}{\cancel{3\alpha}} - \cancel{(-\alpha^2)} \frac{e^{-\alpha^3}}{\cancel{-\alpha^2}}$

$e^{3\alpha^2} - e^{-\alpha^3}$

3.3 KVADRATNE FORME

DEF Kvadratna forma dviju varijabli je homogena kvadratna funkcija $Q : \mathbb{R}^2 \to \mathbb{R}$ oblika $Q(h,k) = ah^2 + 2bhk + ck^2$,
$a,b,c \in \mathbb{R}$

$\Rightarrow$ svakoj formi je pridružena <u>simetrična</u> matrica $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

$$Q = A \cdot \begin{bmatrix} h \\ k \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$$

DEF Za kvadratnu formu $Q(h,k)$ kažemo

a) pozitivno definitna ako je $Q(h,k) > 0$

b) negativno —||— $Q(h,k) < 0$

c) indefinitna ako $Q(h,k)$ mijenja predznak
(tj. za neke (h,k) je >0; a za neke (h,k) <0)

$*$ Q može biti ništa od navedenog

P_1: $Q(h,k) = 2h^2 - 2hk + 4k^2$
$= h^2 + (h-k)^2 + 3k^2 > 0$
$\Rightarrow$ poz. def forma

TM Sylvesterov tm

Neka je $Q(h,k)$ kvad. forma s matricom A

a) Ako je $\det A > 0$ i $a > 0 \Rightarrow Q$ je poz. def.

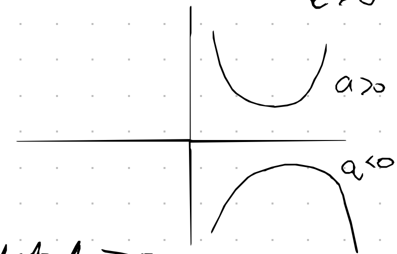
b) Ako je $\det A > 0$ i $a < 0 \Rightarrow Q$ je neg. def

c) Ako je $\det A < 0 \longrightarrow Q$ je indefinitna

regularne mat

$*$ Ako je $\det A = 0 \Rightarrow$ nema inverza $\Rightarrow$ nema odluke

DOKAZ: $Q(h,k) = ah^2 + 2bhk + ck^2 < k^2\left(a\left(\frac{h}{k}\right)^2 + 2b\frac{h}{k} + c\right)$ $\quad k \neq 0$



$t = \frac{h}{k} \to p(t) = at^2 + 2bt + c$

a) $p(t) > 0 \longrightarrow D = 4b^2 - 4ac < 0 \quad /:4$

$$b^2 - ac < 0$$

$$ac - b^2 > 0 \longrightarrow \underline{\underline{\det A > 0}}$$

b) $a < 0 \to$ neg. def.

c) $D > 0 \Rightarrow \det A < 0$

Pr: $Q(h,k) = 2h^2 - \underbrace{2hk}_{b=-1} + 4k^2$

$\Rightarrow A = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \longrightarrow \det A = 7 > 0,$

3.4. LOKALNI EKSTREMI

DEF

a) $f(x,y)$ ima lokalni min. u $T_0(x_0,y_0)$ ako postoji otvoreni krug (okolina) $K_\varepsilon(T_0)$ t.d. $f(x,y) \geq f(x_0,y_0)$, $\forall (x,y) \in K_\varepsilon$.

b) $f(x,y)$ ima lokalni MAX u T_0 ako postoji otvoreni krug $K_\varepsilon(T_0)$ t.d. $f(x,y) \leq f(x_0,y_0)$, $\forall (x,y) \in K_\varepsilon$.

TM Fermatov teorem = nužan uvjet za lok. ekstreme

Ako dif. $f(x,y)$, ima lok. ekstreme u T_0, tada $\nabla f(T_0) = \vec{0}$.

(tj. $\frac{\partial f}{\partial x} = 0$, $\frac{\partial f}{\partial y} = 0$)

DOKAZ:

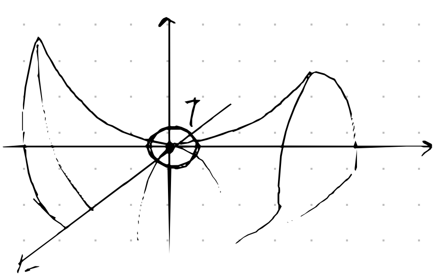
Definirajmo $f: \mathbb{R} \to \mathbb{R}$ kao restrikciju funkcije na $y = y_0$ (fiksirali smo y_0 u f-ji) kroz T_0, tj. $f_1(x) = f(x, y_0)$. Po pretpostavci $f_1(x)$ ima lok. ekstrem u x_0, pa mogu koristiti Fermatov teorem za jednu varijablu $\longrightarrow f_1'(x_0) = 0 \Rightarrow \frac{\partial f}{\partial x}(x_0, y_0) = 0$.

*Nap. IMPLIKACIJA: obrat ne vrijedi

ako je $\nabla f = \vec{0}$ u T_0, T_0 ne mora biti ekstrem.

Odnosno $\nabla f(T) = \vec{0}$ daje kandidate za ekstreme (stacionarne točke)

Pr. Sedlo $\rightarrow z = x^2 - y^2$, $\nabla z = (2x, -2y) = \vec{0} \Rightarrow T(0,0)$ stacionarna točka, ali nije ekstrem

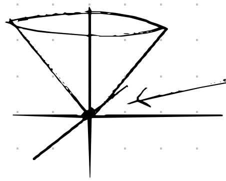


$\Rightarrow$ sedlasta točka

2. NAP: Ako f nije diferencijabilna (∇f ne postoji)

$\Longrightarrow$ ali možemo imati ekstrem.

Pr. $z = \sqrt{x^2+y^2}$



nije dif, ali ima lok min!

Drugi diferencijal f-je $f(x,y)$ je kvad. forma:

$$d^2 f = \underbrace{f''_{xx}}_{a} (dx)^2 + 2 \underbrace{f''_{xy}}_{b} \, dx dy + \underbrace{f''_{yy}}_{c} (dy)^2 \Rightarrow H_f = \begin{bmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{bmatrix}$$

$\longrightarrow$ tu pridruženu matricu nazivamo Hesseova matrica