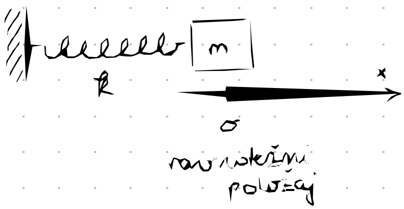
TITRANJE

Jednostavno harmonijsko titranje



Kada je tijelo pomaknuto iz $x=0$, na njega djeluje $F_x = -kx$

* za razliku od mat. njihala v_{max} je u amplitudi

U - potencijalna en.

$U = \frac{1}{2} k x^2$

jed. gib: $m\ddot{x} = -kx \Rightarrow \ddot{x} + \underbrace{\frac{k}{m}}_{\omega_0^2} x = 0$

$\Rightarrow \boxed{\ddot{x} + \omega_0^2 x = 0}$

opće rj. jednadžbe:

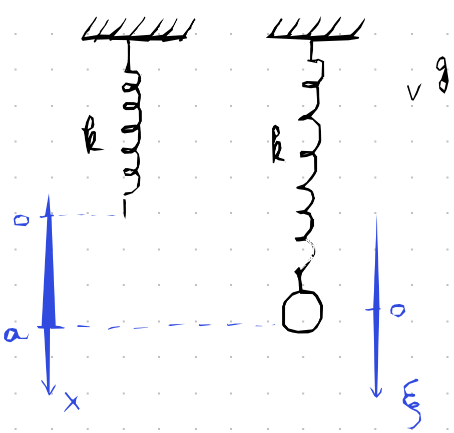
$x(t) = A\cos[\omega_0 t + \varphi]$

amplituda   faza

faza u $t=0$ = početna faza

harmonijsko titranje jer kutna frekv. titranja ne ovisi o amplitudi titranja (ω_0)

Uteg na opruzi i položaj ravnoteže



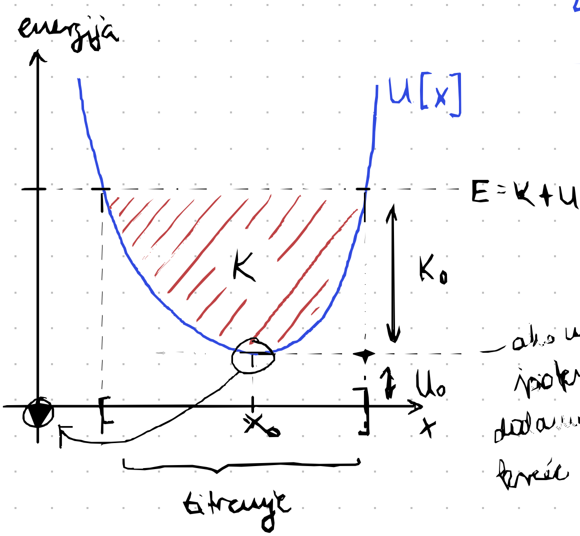
ravnotežni položaj kada uteg miruje na opruzi

$a = \frac{mg}{k}$ u odnosu na neopterećenu duljinu

$m\ddot{\xi} = mg - k(a+\xi) = \cancel{mg} - \cancel{ka} - k\xi$

$m\ddot{\xi} = -k\xi \Rightarrow \ddot{\xi} + \omega_0^2 \xi = 0$

Energija pri harmonijskom titranju



- ako u toj točki minimuma potencijalne en. dodamo K_i krene titranje →

ne uđe u zabranjena područja nego se vraća nazad

-> ako prebacimo točku minimuma u ishodište koord. sustava => $x_0 = 0$, $U_0 = 0$

$$U[x] = \frac{1}{2} k x^2$$

$$F_x[x] = -kx$$

Primjer: masa na opruzi

jednadžba gibanja:

$$F = F_x \rightarrow m \cdot a = -kx \; /:m$$

$$\ddot{x} = \left(-\frac{k}{m}\right) x \qquad \omega^2$$

$$\Rightarrow \boxed{\ddot{x} + \omega^2 x = 0} \Rightarrow x = A \cos(\omega_0 t + \phi)$$

amplituda faza

Energija oscilatora $E = K + U = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$

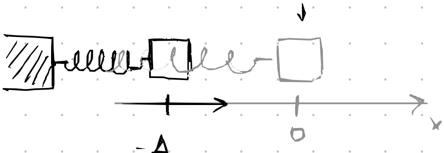
E = konst.

-> najjednostavniji primjer - točka u kojoj je $v_i = 0$

$\dot{x} = 0$ (ovisi x u krajnjim otklonjenim položajima)

$$E = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$$

ravnoteža



$\dot{x} = 0$; $x = A$ => $E = \frac{1}{2} k A^2$

$x = 0$; $\dot{x} = \pm v_{max}$ => $E = \frac{1}{2} m v_{max}^2$

Srednja vrijednost K i U

* međusobno jednake

$$\to \langle U\rangle = \frac{1}{T}\int_0^T U[t']\,dt' = \frac{1}{T}\int_0^T \frac{1}{2}kx^2[t]\,dt \qquad * \; x = A\cos(\omega t)$$

$$\langle U\rangle = \frac{1}{T}\cdot\frac{1}{2}k\int_0^T A^2\cdot\cos^2(\omega t)\,dt = \frac{1}{T}\,\frac{1}{2}kA^2\int_0^T \frac{1}{2}(1+\cos 2\omega t)\,dt$$

$$\langle U\rangle = \frac{1}{T}\,\frac{1}{2}kA^2\left(\frac{1}{2}\int_0^T dt + \cancel{\frac{1}{2}\int_0^T \cos(2\omega t)\,dt}\right)$$ pokrate se periodi pa je 0

$$\langle U\rangle = \frac{1}{4}\cdot\frac{kA^2}{\cancel{T}}\cdot\cancel{T} \Rightarrow \boxed{\langle U\rangle = \frac{1}{4}kA^2} = \frac{E}{2} = \langle K\rangle$$

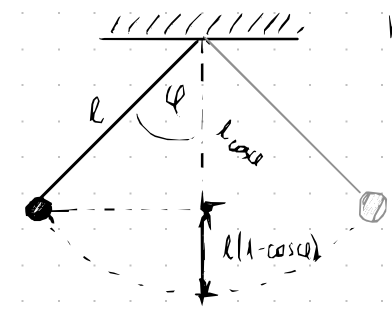
* srednja vrijednost kvadrata (bilo sin ili cos) preko 1 ili više punih perioda je uvijek 1/2

Nalaženje jednadžbe gibanja korištenjem očuvanja meh. en.

$E = K + U$ je očuvana

$$\to \frac{dE}{dt} = 0 \Rightarrow \text{j.g.}$$

Primjer: matematičko njihalo (idealizirano)



$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l\frac{d\varphi}{dt}\right)^2 = \frac{1}{2}m(l\dot\varphi)^2$$

$$U = mgh = mgl(1-\cos\varphi)$$

$$E = K + U = \frac{1}{2}ml^2\cdot\dot\varphi^2 + mgl(1-\cos\varphi)$$

$$0 = \frac{dE}{dt} = ml^2\cdot\dot\varphi\,\ddot\varphi + mgl\sin\varphi\,\dot\varphi$$

$$0 = ml^2\dot\varphi\left(\ddot\varphi + \frac{g}{l}\sin\varphi\right)$$

$$\omega = 2\pi f = \sqrt{\frac{g}{l}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{g}{l}}$$

1. slučaj: $mgl^2\dot\varphi = 0$
- stanje trajnog mirovanja (ravnotežni položaj)

$\dot\varphi = 0$
$\to$ ne zanimljivo

2. slučaj: $\boxed{\ddot\varphi + \frac{g}{l}\sin\varphi = 0}$

prepoznajemo: za $\varphi \lll$
$\sin\varphi \simeq \varphi$

$$\omega^2 = \frac{g}{l}$$

$$\Rightarrow \boxed{\ddot\varphi + \frac{g}{l}\varphi = 0}$$ jednadžba lin. harm. oscilatora

► drugi pristup: $E=\frac{1}{2}ml^2\dot{\varphi}^2 + \underbrace{mgl(1-\cos\varphi)}$

nije oblika $U=\frac{1}{2}kx^2$!
$\hookleftarrow \varphi$

$\varphi <<<$

ALI: $\cos\varphi \simeq 1-\frac{\varphi^2}{2}+\dots$

$\Rightarrow U = mgl\left(\cancel{1}-\cancel{1}+\frac{\varphi^2}{2}\right) = \boxed{mgl\frac{\varphi^2}{2}}$

e sad pazi da smo to odmah napravili, brže bismo došli do jed. harm. osc.

OPĆENITO: $\ddot{x} + \omega_0^2[x]x = 0$

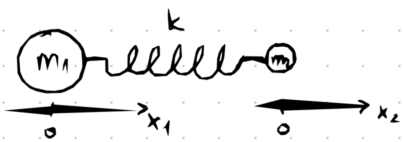
- ako ovisi o x - nije linearno $\Rightarrow \ddot{x} + \omega_0^2 x = 0$

ali za $\omega_0^2 = \lim_{x\to 0}\omega_0^2(x)$

njihalo: $\ddot{\varphi} + \frac{g}{l}\sin\varphi = 0$

$\ddot{\varphi} + \frac{g}{l}\underbrace{\frac{\sin\varphi}{\varphi}}_{1}\varphi = 0 \Rightarrow \ddot{\varphi} + \frac{g}{l}\varphi = 0$

Primjer: frekv. titranja sustava



U - isključivo potenc. en. od opruge

$U=\frac{1}{2}(x_2-x_1)^2k$

* da se oba tijela pomaknu za 10 cm →, duljina opruge se ne bi promijenila $\Rightarrow 10-10=0$

svako se pomaknuo u svoju stranu pa je $-(-)=+$

$K=\frac{m_1}{2}\dot{x}_1^2+\frac{m_2}{2}\dot{x}_2^2$

* početno stanje ⇒ čestice u ravnotežnom (0) položaju

ZOKG: $\boxed{p_x^* = m_1\dot{x}_1+m_2\dot{x}_2 = 0}$ želimo da središte mase miruje

$m_1\frac{dx_1}{dt}+m_2\frac{dx_2}{dt}=0 \;/dt$

$\dot{x}_2 = -\frac{m_1\dot{x}_1}{m_2}$ $\quad \dot{x}_1 = -\frac{m_2\dot{x}_2}{m_1}$

$m_1dx_1+m_2dx_2=0 \;/\int$

$m_1\int_{poč}^{kon}dx_1 + m_2\int_0^{x_2}dx_2 = 0$

$\boxed{m_1x_1+m_2x_2=0}$

Energija titranja sustava

$$E=K+U=\frac{m_1}{2}\dot{x}_1^2+\frac{m_2}{2}\cdot\left(-\frac{m_1\dot{x}_1}{m_2}\right)^2+\frac{1}{2}\left(-\frac{m_1x_1}{m_2}-x_1\right)^2k$$

$$E=\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}\frac{m_1^2\dot{x}_1^2}{m_2}+\frac{1}{2}kx_1^2\left(1+\frac{m_1}{m_2}\right)^2$$

$$E=\frac{1}{2}m_1\dot{x}_1^2\left(1+\frac{m_1}{m_2}\right)+\frac{1}{2}kx_1^2\left(1+\frac{m_1}{m_2}\right)^2=\left(1+\frac{m_1}{m_2}\right)\left[\frac{1}{2}m_1\dot{x}_1^2+\frac{1}{2}kx_1^2\left(1+\frac{m_1}{m_2}\right)\right]$$

$$0=\frac{dE}{dt}=\left(1+\frac{m_1}{m_2}\right)\left[m_1\dot{x}_1\ddot{x}_1+kx_1\dot{x}_1\left(1+\frac{m_1}{m_2}\right)\right]$$

$$0=\frac{dE}{dt}=\underbrace{\left(1+\frac{m_1}{m_2}\right)m_1\dot{x}_1}_{\Rightarrow 0\ \text{samo kada sustav miruje}}\Bigg(\underbrace{\ddot{x}_1+\frac{k}{m_1}x_1\left(1+\frac{m_1}{m_2}\right)}_{\text{mora biti } 0}\Bigg)$$

$$\Rightarrow \ddot{x}_1+k\left(\frac{1}{m_1}+\frac{1}{m_2}\right)x_1=0 \longrightarrow \ddot{x}_1+\overbrace{\boxed{k\left(\frac{m_1+m_2}{m_1m_2}\right)}}^{\omega^2}x_1=0$$

$$\Rightarrow \omega_0=\sqrt{k\left(\frac{m_1+m_2}{m_1m_2}\right)}$$

Slaganje titranja

Općenito u 1D

$$x[t] = A_1\cos[\omega_1 t + \varphi_1] + A_2\cos[\omega_2 t + \varphi_2] + \dots + A_n\cos[\omega_n t + \varphi_n]$$

slučaj: $A_1 = A_2 = A$, $\omega_1 \approx \omega_2$, $\varphi_1 = \varphi_2 = 0$

$$x[t] = A\cos[\omega_1 t] + A\cos[\omega_2 t]$$

novo: $\bar{\omega} = \frac{\omega_1 + \omega_2}{2}$, $\delta = \frac{\omega_1 - \omega_2}{2}$ $\qquad \omega_{1,2} = \bar{\omega} \pm \delta$

srednja vrijednost $\qquad$ razlika

$e^{i\varphi} + e^{-i\varphi} = \cos\varphi + i\cancel{\sin\varphi} + \cos(-\varphi) + i\cancel{\sin(-\varphi)}$

$= 2\cos\varphi$

$$\Rightarrow x[t] = A\cos[(\bar{\omega} + \delta)t] + A\cos[(\bar{\omega} - \delta)t]$$

Eulerova formula: $e^{i\varphi} = \cos\varphi + i\sin\varphi$

$$x[t] = Re\left[Ae^{i(\bar{\omega}+\delta)t} + Ae^{i(\bar{\omega}-\delta)t}\right] = Re\left[Ae^{i\bar{\omega}t}\underbrace{\left(e^{i\delta t} + e^{-i\delta t}\right)}_{=2\cos\delta t}\right]$$

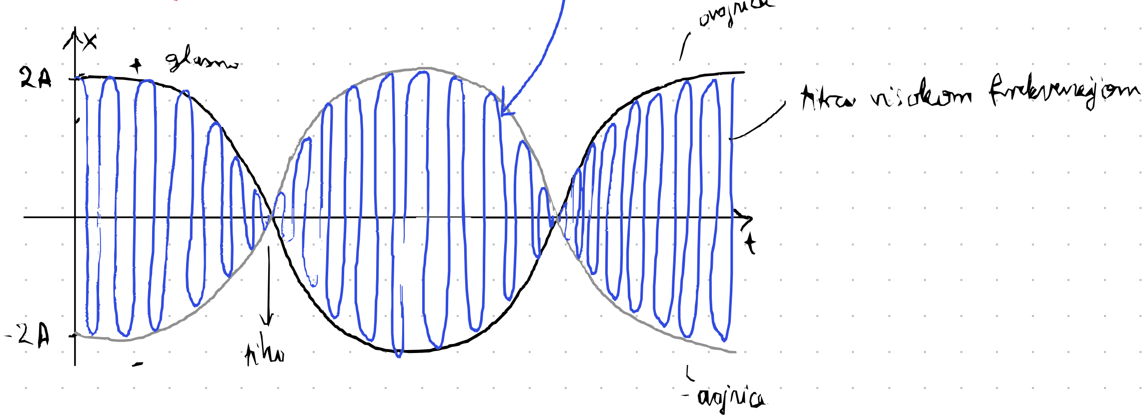
$$\Rightarrow \quad x[t] = 2A\cos\delta t \; Re\left[e^{i\bar{\omega}t}\right]$$

$$x[t] = 2A\cos(\delta t)\boxed{\cos(\bar{\omega}t)}$$

sporo titranje
OVOJNICA

brzo titranje

→ ovojnicu pomnožimo sa cos velike $\omega \cdot t$

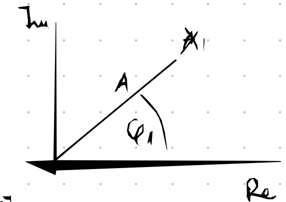


Slučaj: $N \geq 2$, $\omega_1 = \omega_2 = \dots = \omega_3 = \omega$

$$x[t] = A_1 \cos[\omega t + \varphi_1] + A_2 \cos[\omega t + \varphi_2] + \dots$$

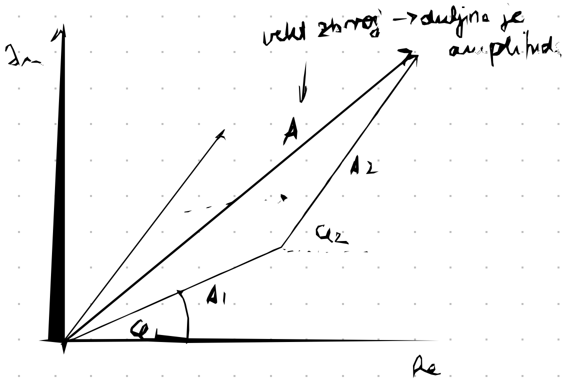
$$x[t] = \mathrm{Re}\left[A_1 e^{i(\omega t + \varphi_1)} + A_2 e^{i(\omega t + \varphi_2)} + \dots\right]$$

$$= \mathrm{Re}\left[\left(A_1 e^{i\varphi_1} + A_2 e^{i\varphi_2} + \dots\right) e^{i\omega t}\right] \rightarrow$$



možemo shvatiti kao kompleksne

$$\Rightarrow \underbrace{\boxed{A e^{i\varphi}}}_{\text{amplituda}} \Rightarrow A \cos[\omega t + \varphi]$$



A ako npr.:

$$x(t) = a \cos(\omega t) + a \cos(\omega t + \pi)$$

$$= \mathrm{Re}\left[\left(a + a e^{i\pi}\right) e^{i\omega t}\right]$$

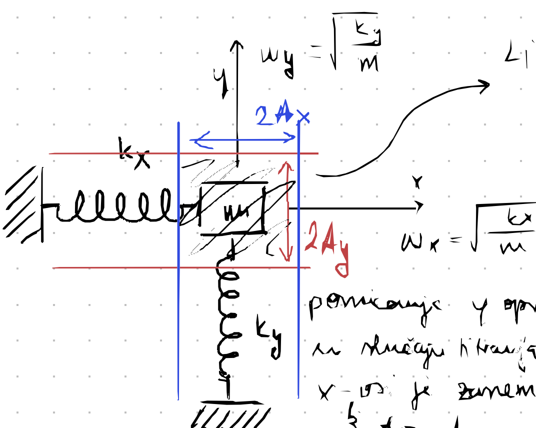
$$\mathrm{Re}\left(\left(a - a\right) e^{i\omega t}\right)$$

$$\Rightarrow x[t] = 0$$

mogu se poništiti

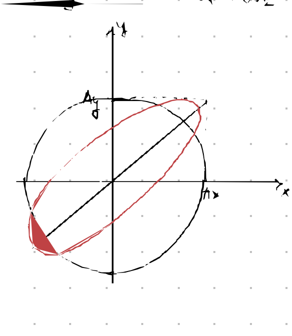
Slaganje titranja u 2D

$$x[t] = A \cos[\omega_x t + \varphi_0]$$

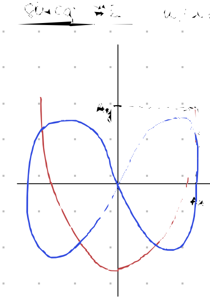


Lissajousove krivulje

slučaj #1 $\omega_1 : \omega_2 = 1:1$



slučaj #2

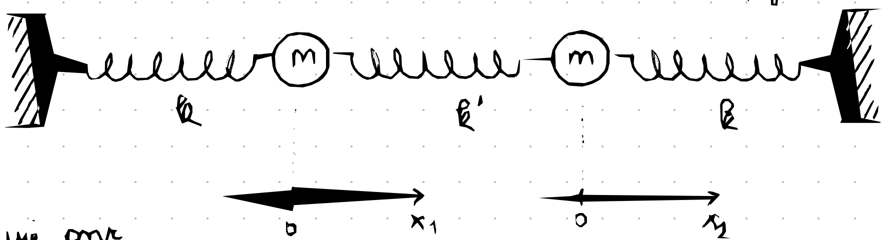


pomicanje y opruge u slučaju titranja x-osi je zanemarivo

} [illegible]

Vezani oscilatori

– 2 stupnja slobode

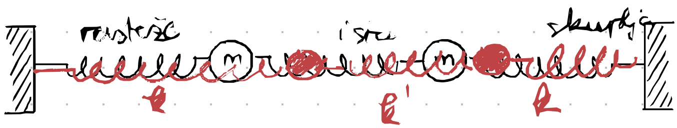


– gibanje prve čestice utječe na gibanje druge

Normalni modovi titranja „Svi stupnjevi slobode titraju istom f"

→ Br. stupnji slobode = br. modova

U FAZI 1. mod

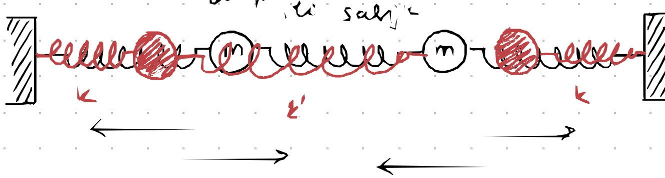


nema promjene u k' ⟹ kao da je nema $\omega_A = \sqrt{\frac{k}{m}}$

$x_1 = x_2 = A_A \cos[\omega t + \varphi_A]$

duplo se rasteže ili sabija

U PROTUFAZI 2. mod



$\omega_B = \sqrt{\frac{k + 2k'}{m}}$

$x_1 = -x_2 = A_B \cdot \cos(\omega_B t + \varphi_B)$

općenito titranje: $x_1 = x_1^{(1)} + x_1^{(2)} = A_A \cos(\omega_A t + \varphi_A) + A_B \cos[\omega_B t + \varphi_B]$

ako je srednja opruga (k') puno mekša od druge onje (k' ≪ k)

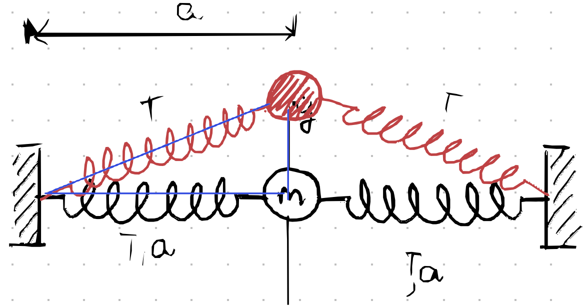
onda su $\omega_A \simeq \omega_B$

tako $A_A = A_B = A$, $\varphi_A = \varphi_B$ ⟹ pojava UDARA

Transverzalno titranje

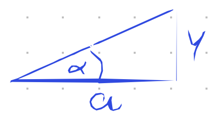
! utječe napetost !

ali ne i rastezanje



jednadžba gibanja:

$\lim_{x \to 0}$: ? $\sin\alpha \approx tg(\alpha)$



$tg\,\alpha = \frac{y}{a}$

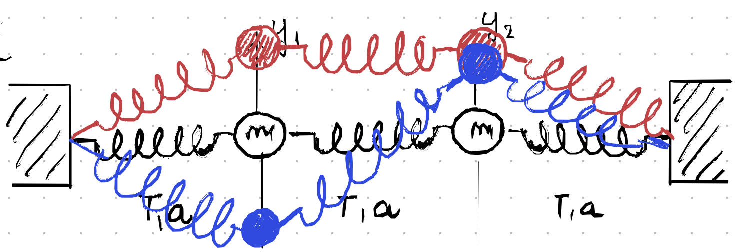
$tg\,\alpha \cdot a = y \approx$

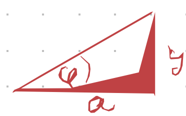
$$ma = T \cdot \sin\alpha$$

$$m\ddot{y} = -2T \cdot \frac{y}{a} \quad /:m$$

$$\ddot{y} + \boxed{\frac{2T}{ma}}\, y = 0 \quad \longrightarrow \omega_0^2$$

2. slučaj





$\rightarrow tg\,\alpha = \frac{y}{a}$

u fazi – središnja opruga ostaje ista $\quad \omega_1^2 = \frac{T}{ma}$

$$y_1 = y_2 = y$$

$$ma = -2T \cdot \sin\varphi \Rightarrow -2T \cdot \frac{y}{a}$$

$$\Rightarrow m\ddot{y} = -2T\,\frac{y}{a} \longrightarrow \ddot{y} + \frac{2T}{ma}\,y = 0 \longrightarrow \omega_0^2 = \frac{2T}{ma}$$

u protufazi $\quad y_1 = -y_2$

$$m\ddot{y}_1 = -T\,\frac{y_1}{a} - T \cdot \frac{2y_1}{a} = \frac{-3y_1}{a}\,T \quad /:m$$

$$\ddot{y}_1 + \frac{3T}{ma}\,y = 0 \longrightarrow \omega_0^2 = \frac{3T}{ma}$$