4.4. HOMOGENE JEDNADŽBE

DEF Funkcija $f(x,y)$ je hom. stupnja α ako $f(tx, ty) = t^{\alpha} f(x,y)$

$\Rightarrow$ DJ je homogena (hom stupnja) ako se može prikazati u obliku $P(x,y)dx + Q(x,y)dy = 0$ gdje su P i Q funkcije istog hom stupnja.

Tada se DJ rješava : $\boxed{z = \frac{y}{x}}$ $\longrightarrow$ ovom supstitucijom se uvijek svodi na separaciju

Pr.) $(y+x)y' = y - x$

$(y+x)dy = (y-x)dx \longrightarrow \underbrace{(y-x)}_{P(x,y)}dx - \underbrace{(y+x)}_{Q(x,y)}dy = 0$

$\longrightarrow P(tx, ty) = t^1(y-x) \Longrightarrow$ hom stupnja 1

$Q(tx, ty) = t^1(y+x)$

2. način : $(y+x)y' = (y-x) \,/ : x$

$\left(\frac{y}{x} + 1\right) y' = \left(\frac{y}{x} - 1\right)$ z

supstitucija $z = \frac{y}{x}$

$y = z \cdot x \;/\,'$

$y' = z' \cdot x + z$

uvrštavamo supstituciju:

$\Rightarrow (z+1)(z'x + z) = (z - 1)$

separacija:

$\Rightarrow (z+1)z'x + z^2 + \cancel{z} = \cancel{z} - 1$

$(z+1)\frac{dz}{dx} \cdot x = -(z^2+1) \Big/ \frac{dx}{x(z^2+1)}$

$\int \frac{z+1}{z^2+1} dz = -\int \frac{dx}{x} \Longrightarrow \frac{1}{2}\ln|z^2+1| + \operatorname{arctg} z = -\ln|x| + C$

$\frac{1}{2}\ln\left|\frac{y^2}{x^2} + 1\right| + \operatorname{arctg}\left(\frac{y}{x}\right) = -\ln|x| + C$

$\ln(y^2+x^2) - \cancel{\ln|x|} + \operatorname{arctg}\left(\frac{y}{x}\right) = \cancel{-\ln|x|} + C$

$\ln(y^2+x^2) + \operatorname{arctg}\left(\frac{y}{x}\right) = C$

JIR-21-6 $x\,dy = (y + (xy)^{\beta})\,dx$

$$x\,dy = (y + (xy)^{\beta})\,dx \quad /:dx$$

$$\underbrace{(y + (xy)^{\beta})}_{P(x,y)}dx - \underbrace{x}_{Q(x,y)}dy = 0$$

$$P(tx, ty) = ty + (tx\,ty)^{\beta} \qquad Q(tx, ty) = t_x$$

$$P(tx, ty) = ty + (t^2 xy)^{\beta} \qquad \Rightarrow \text{stupanj hom} = 1$$

$$= t\,(y + (xy)^{1/2})$$

$$\Rightarrow \beta = \frac{1}{2} \Rightarrow (ty + \sqrt{t^2}\cdot(xy)^{1/2})$$

$$xy' = y + \sqrt{xy} \quad /:x$$

supstitucija

$$y = z\cdot x \quad /'$$

$$\underline{y' = z'\cdot x + z}$$

$$y' = \frac{y}{x} + \sqrt{\frac{y}{x}}$$

$$z'x + \cancel{z} = \cancel{z} + \sqrt{z}$$

$$\frac{dz}{dx}\cdot x = \sqrt{z} \quad /\cdot\frac{dx}{x\cdot\sqrt{z}}$$

$$\sqrt{z} = 0 \rightarrow z = 0 \rightarrow \boxed{y = 0} \quad \text{dod. rj:}$$

$$\int\frac{dz}{\sqrt{z}} = \int\frac{dx}{x} \Rightarrow 2\sqrt{z} = \ln|x| + c \quad /:2$$

$$\underline{\sqrt{\frac{y}{x}} = \frac{1}{2}\ln|x| + C} \quad /^2 \quad \text{opće rješenje}$$

$$T(1,1) \Rightarrow 1 = 0 + C \rightarrow \underline{C = 1}$$

$$\underline{\text{Konačno rj: } y = x\left(\frac{1}{2}\ln|x| + 1\right)^2}$$

*nap. rj. $y=0$ ne prolazi kroz T

b) $xy' = y + x^{\beta}y^{\beta} \quad /:x$

$$\underline{y' = \frac{y}{x} + x^{\beta-1}\cdot y^{\beta-1}}$$

$$\beta \in \mathbb{R}\setminus\{0,1\}$$

Transformacija na homogene

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right) \Rightarrow$$ uvedemo supstituciju $$\boxed{\begin{aligned} u &= x - x_0 \\ v &= y - y_0 \end{aligned}}$$

gdje je (x_0, y_0) rj. nehom. (jedinstvenog) sustava

Zad.)

$$y' = \frac{x+y-3}{x-y-1}$$

1.
$$\begin{cases} x + y = 3 \\ x - y = 1 \end{cases}$$

$$x_0 = 2 \quad y_0 = 1$$

2. supstitucija

$u = x - 2$

$v = y - 1 \,/\, \frac{dy}{dx}$

$$\Rightarrow v' = \frac{u+v}{u-v} \quad /:u$$

3. uvrsti

$$v' = \frac{1 + \frac{v}{u}}{1 - \frac{v}{u}} \qquad z = \frac{v}{u}$$

$$v' = z'u + z$$

$$z'u + z = \frac{1+z}{1-z}$$

$$\frac{1+z}{1-z} - z = z'u$$

$$\frac{1+z^2}{1-z} = \frac{dz}{du} \cdot u \,/ \cdot \frac{du \cdot (1-z)}{u \;\; (1+z^2)}$$

$$\Rightarrow \int \frac{1-z}{1+z^2}\,dz = \int \frac{du}{u}$$

$$\left| \begin{aligned} t &= 1 + z^2 \\ dt &= 2z\,dz \end{aligned} \right|$$

$$\frac{dt}{2} = z\,dz$$

$$\int \frac{1}{1+z^2}\,dz - \int \frac{z}{1+z^2}\,dz = \ln|u| + c$$

$$\operatorname{arctg} z - \frac{1}{2}\int \frac{dt}{t} = \ln|u| + c$$

$$\operatorname{arctg} z - \frac{1}{2}\ln|t| = \ln|u| + c$$

$$\operatorname{arctg} z - \frac{1}{2}\ln|1+z^2| = \ln|u| + c$$

$$z = \frac{v}{u} \qquad \begin{aligned} v &= y - 1 \\ u &= x - 2 \end{aligned} \rightarrow z = \frac{y-1}{x-2}$$

$$\operatorname{arctg}\left(\frac{y-1}{x-2}\right) - \frac{1}{2}\ln\left|1 + \left(\frac{y-1}{x-2}\right)^2\right| = \ln|x-2| + c$$

Zad.)

$$y' = \frac{2x-y+3}{4x-2y+1}$$ } sustav nema rj. $\rightarrow$ ne možemo svest na homogenu

član koji se ponavlja

$$y' = \frac{2x-y+3}{2(2x-y)+1} \longrightarrow \underline{z = 2x - y}$$

$$z' = 2 - y'$$

$$2 - z' = \frac{z+3}{2z+1}$$

$$z' = 2 - \frac{z+3}{2z+1} \longrightarrow z' = \frac{4z+2-z-3}{2z+1} = \underline{\frac{3z-1}{2z+1}}$$

$$\frac{dz}{dx} = \frac{3z-1}{2z+1} \rightarrow \int dx = \int \frac{2z+1}{3z-1}\, dz$$

dodatno rješenje:

$3z-1=0$

$3(2x-y)-1=0$

$6x-3y=1$

$$\boxed{y = 2x - 3}$$

$$\int \left(\frac{2}{3} + \frac{\frac{5}{3}}{3z-1}\right) dz = x + C$$

$$\frac{2}{3}z + \frac{5}{9}\ln|3z-1| = x + C$$

$$\boxed{\frac{2}{3}(2x-y) + \frac{5}{9}\ln|3(2x-y)-1| = x + C}$$

Zad.) $xy' - y\ln y = y(1-\ln x)$, $y(1)=e$

$\rightarrow$ nije homogenog stupnja jer ne možemo izlučit t (iz ln)

$$xy' - y\ln(y) = y - y\ln(x)$$

$$xy' - y\ln\left(\frac{y}{x}\right) = y \,/ : x$$

$$y' - \frac{y}{x}\ln\left(\frac{y}{x}\right) = \frac{y}{x}$$

jednadžba je hom. stupnja čim možemo sve izraziti preko $\frac{y}{x}$

* ne moramo pokazivati da je hom. ako ne piše u zadatku „pokaži/dokaži"

$$z = \frac{y}{x}$$

$$zx = y \longrightarrow \underline{y' = z'x + z}$$

$$z'x + \cancel{z} - z\ln z = \cancel{z}$$

$$z'x = z\ln z$$

$$\frac{dz}{dx}x = z\ln z$$

$$\int \frac{dz}{z\ln z} = \int \frac{dx}{x}$$

$$\left| \begin{array}{l} \ln z = t \\ \frac{dz}{z} = dt \end{array} \right| \Rightarrow \int \frac{dt}{t} = \ln|x| + C$$

$$\Rightarrow \ln|t| = \ln|x| + C$$

$$\ln|\ln z| = \ln|x| + \ln|c|$$

$$\ln|\ln|z|| = \ln|x \cdot c| \,/^{e}$$

$$\ln|z| = x \cdot c \,/^{e}$$

$$z = e^{cx}$$

opće rješenje

$$\frac{y}{x} = e^{cx} \Rightarrow \boxed{y = xe^{cx}}$$

dodatno rješenje:

$$z\ln z = 0$$

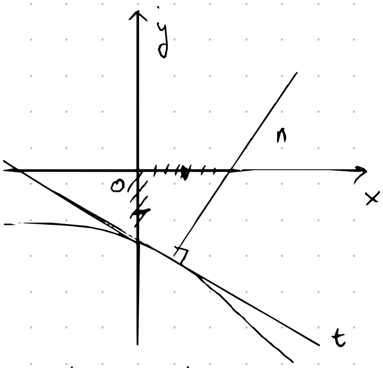
$z = 0$ $\quad$ $\ln z = 0 \,/^{e}$

$y = 0$ $\quad$ $z = 1$

$\downarrow$ $\quad$ $\boxed{y = x}$ dod.

~~$\ln(0) =$~~

Z1-2022-3



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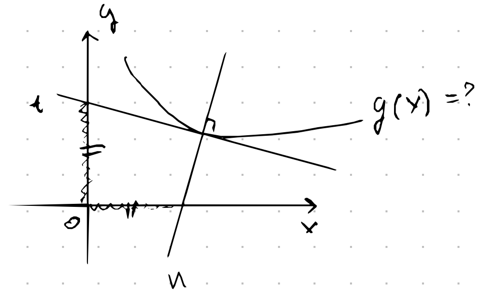

sjecište s x osi: $y=0$

n... $y - y_0 = \frac{-1}{f'(x_0)} \cdot (x_S - x_0)$

$+y_0 \cdot y'(x_0) = x_S - x_0$

$x_S = x_0 + y_0 \cdot y'(x_0)$

t... $y - y_0 = y'(x_0)(x - x_0)$

sjecište s y-osi: $x=0$

$y_S - y_0 = y'(x_0)(-x_0)$

$y_S = y_0 - x_0 \cdot y'(x_0)$

$\downarrow$

$\longrightarrow$ $y_0 - x_0 \cdot y'(x_0) = x_0 + x_0 y'(x_0), \ \forall x_0, y_0$

$\Rightarrow y - x y' = x + y \cdot y'$

$y - x = y'(y + x)$

$y' = \frac{y - x}{y + x}$ hom. stupnja 1

Danas smo riješili $\frac{y}{x} = z$