2.2 Trigonometrijski Furierov red

+ ideja da se periodična f-ja zapiše pomoću Furierovog reda

$$\frac{1}{2},\ \underbrace{\cos x,\ \sin x}_{2\pi},\ \underbrace{\cos 2x,\ \sin 2x}_{\pi},\ \dots,\ \underbrace{\cos nx,\ \sin nx}_{\frac{2\pi}{n}},\ \dots$$

$$\Rightarrow \boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)}$$ Furierov red funkcije f

- $a_0, a_1, a_2, \dots, a_n, b_1, \dots b_n$ - Furierovi koeficijenti
- pribrojnici u sumi – harmonici
- f je funkcija perioda 2π

Furierov polinom
↳ kada ima neki konačni član (ne ide u ∞)

Pitanja:

① Ako je f periodična 2π, kada će postojati njen Furier. red?

② Ako postoji Furierov red, kako računati koeficijente?

najlakše, ništa razmišljati samo računati, s tim krećemo

→ računanje $(a_n$ i $b_n)$ se zasniva na ortogonalnosti trig. koeficijenata

③ U kojem smislu Furierov polinom aproksimira f-ju f?

Ortogonalnost funkcija

Funkcije $f, g: [a,b] \to \mathbb{R}$ su orto na $[a,b]$ ako vrijedi $\int_a^b f(x)g(x)\,dx = 0$

Lema: Trig sustav (*) je ortogonalan na $[-\pi, \pi]$ tj:

$$\int_{-\pi}^{\pi} \cos nx\,dx = \begin{cases} 0, & n>0 \\ 2\pi, & n=0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx\,dx = 0$$

$$\int_{-\pi}^{\pi} \sin nx \cdot \sin mx\,dx = \begin{cases} \pi, & n=m \\ 0, & n\neq m \end{cases}$$

$$\int_{-\pi}^{\pi} \cos nx \cdot \cos mx\,dx = \begin{cases} \pi, & m=n \\ 0, & m\neq n \end{cases}$$

$$\int_{-\pi}^{\pi} \sin nx \cdot \cos mx\,dx = 0$$

Računanje koeficijenata trigonometrijskog reda

▸ Pretpostavimo $f(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx) \quad / \int_{-\pi}^{\pi}$

$$\to \int_{-\pi}^{\pi} f(x)dx = \underbrace{\frac{a_0}{2}\int_{-\pi}^{\pi} \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)dx}$$

$\Downarrow$ općenito ne smije se zamijeniti sa $\sum$, ali ovdje može jer Furierov red trig jednolika

$$\int_{-\pi}^{\pi} f(x)dx = \frac{a_0}{\cancel{2}}\cdot \cancel{2}\pi + \sum_{m=1}^{\infty}\left(a_m \int_{-\pi}^{\pi} \cos mx\,dx + b_m \int_{-\pi}^{\pi} \sin mx\,dx\right)$$

$$\int_{-\pi}^{\pi} f(x)dx = a_0 \cdot \pi + \sum_{m=1}^{\infty}\left(a_m \underbrace{\int_{-\pi}^{\pi} \cos mx\,dx}_{=0 \text{ prema lemi}} + b_m \underbrace{\int_{-\pi}^{\pi} \sin mx\,dx}_{=0 \text{ (prema lemi}}\right)$$

$=0$ za $n>0$

$$\boxed{\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = a_0} \qquad \Rightarrow \int_{-\pi}^{\pi} f(x)dx = a_0 \pi$$

* ključno da je sustav orto!

$$\to \int_{-\pi}^{\pi} f(x)\cos(nx)dx = \underbrace{\frac{a_0}{2}\int_{-\pi}^{\pi} \cos mx\,dx}_{0} + \sum_{m=1}^{\infty}\left(a_m \int_{-\pi}^{\pi} \cos nx \cos mx\,dx + \underbrace{b_m \int_{-\pi}^{\pi} \cos nx \cdot \sin mx\,dx}_{0}\right)$$

$\int_{-\pi}^{\pi} \cos nx \cos mx\,dx = 0$ za $m \neq n$, $= \pi$ za $m = n$

$$\boxed{\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = a_m} \quad \Rightarrow \int_{-\pi}^{\pi} f(x)\cos(m\cdot x)dx = a_m \cdot \pi$$

$$= \begin{cases} 0, & m \neq n \\ \pi, & m = n \end{cases}$$

$$\to \int_{-\pi}^{\pi} f(x)\sin(n\cdot x)dx = \overbrace{\int_{-\pi}^{\pi} \frac{a_0}{2}\cdot \sin nx\,dx}^{=0 \text{ (lema)}} + \sum_{m=1}^{\infty}\left(\underbrace{a_m \cos mx \cdot \sin nx}_{=0 \text{ (lema)}} + b_m \sin nx \cdot \sin mx\right)$$

$$\boxed{\Rightarrow \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_m} \quad \Rightarrow \int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_m \cdot \pi$$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$, $\quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx\,dx \quad n \geq 1$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx, \quad n \geq 1 \qquad a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

* a_n ide uz cos
b_n ide uz sin

↑ tako se računaju koeficijenti (Furierovi koeficijenti)

Funkcija f ima Furierov red $S(x)$

$$S(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

→ vezu $f(x)$ i njezinog Furierovog reda S: $f(x) \sim S(x)$

Postojanje i konvergencija Furierovog reda

Dirichletovi uvjeti Kažemo da f zadovoljava D-uvjete na intervalu $[a,b]$ ako vrijedi:

1. f je po dijelovima neprekinuta i njezini prekidi su prve vrste
2. f je monotona ili nema najviše konačan broj strogih ekstrema

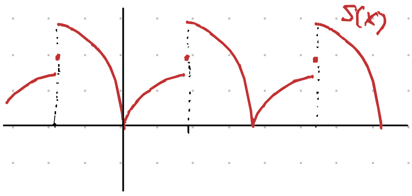
$\lim_{x\to a^+} f(x) = f(a+0)$ limes zdesna $\qquad \lim_{x\to a^-} f(x) = f(a-0)$ limes slijeva

TM osnovni za Furierov red Konvergencija Furierovog reda

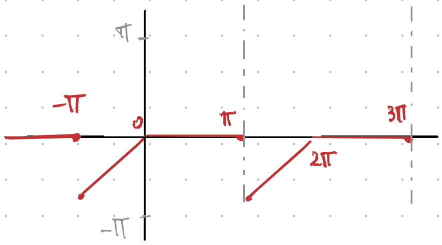
Neka je f po dijelovima glatka periodična funkcija perioda 2π koja zadovoljava Dirichletove uvjete. Tada vrijedi F. red za $x \in [-\pi,\pi]$ i vrijedi:

1) $S(x) = f(x)$, ako je f neprekinuta u x

2) $S(x) = \frac{1}{2}\left(f(x-0)+f(x+0)\right)$, ako f ima prekid u x



Primjer: Razvoj periodične f-je u Furierov red



$$f(x) = \begin{cases} x, & x\in[-\pi,0] \\ 0, & x\in[0,\pi] \end{cases} \qquad T=2\pi$$

$S(x) \overset{?}{=} f(x)$ određujemo po def. ne po računu!

↳ određimo u kojim točkama nisu jednaki

$S(x) \overset{?}{=} f(x) \quad \forall x \in \mathbb{R}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{0} x\,dx = \frac{1}{\pi}\frac{x^2}{2}\Big|_{-\pi}^{0} = \frac{-\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx = \frac{1}{\pi}\left(\frac{x}{n}\sin nx\Big|_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}\sin nx\,dx\right) = \frac{-1}{n\pi}\int_{-\pi}^{0}\sin nx\,dx = \frac{1}{n^2\pi}\cos nx\Big|_{-\pi}^{0}$$

$$a_n = \frac{1}{n^2\pi}\left(1-\cos n\pi\right)$$

možemo gledati kao: $\cos n\pi = (-1)^n = \begin{cases} 1 & \text{za parni } n \quad (2n) \\ -1 & \text{za neparni } n \quad (2n+1) \end{cases}$ $\quad n\geq 1$

$a_{2n} = 0, \quad n\geq 1 \qquad a_2 = a_4 = \dots = a_{2n} = 0$

$$\boxed{a_{2n+1} = \frac{2}{(2n+1)^2\cdot\pi}} \qquad n\geq 0 \qquad \text{npr. } a_1 = \frac{2}{\pi},\ a_3 = \frac{2}{9\pi}, \dots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{0} x\sin nx\,dx = \frac{1}{\pi}\left(-\frac{x}{n}\cos nx\Big|_{-\pi}^{0} - \int_{-\pi}^{0}\frac{1}{n}\cos nx\,dx\right) = \frac{1}{\pi}\left(\frac{\pi}{n}\cdot\cos\pi n - \frac{1}{n^2}\sin nx\Big|_{-\pi}^{0}\right)$$

$$= -\frac{\cos n\pi}{n} \quad (=(-1)^n) \quad = \frac{(-1)^{n+1}}{n} = b_n \qquad n\geq 1$$

$u = x \rightarrow du = dx$

$dv = \sin nx \rightarrow \cos nx\cdot\frac{1}{n} =$

F. red: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos nx + b_n\sin nx\right)$

$$f(x) = \frac{-\pi}{4} + \frac{2}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \dots\right) + \left(\sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right)$$

▸ $T = 2L$ period funkcije f $\qquad \frac{2\pi}{T} = \omega \to w = \frac{\pi}{L}$

$g(x) = f\left(\frac{T}{2\pi}x\right)$ Provjeri ima li g period 2π

$g(x+2\pi) = f\left(\frac{T}{2\pi}(x+2\pi)\right) = f\left(\frac{T}{2\pi}x + T\right) = f\left(\frac{T}{2\pi}x\right) = g(x)$ ✓

$g(x) = f\left(\frac{T}{2\pi}x\right) = f\left(\frac{L}{\pi}x\right)$ periodična s periodom 2π

$\cos\frac{2\pi x}{L} \qquad \sin\frac{2\pi x}{L}$

$$g\left(\frac{\pi x}{L}\right) = f(x) \longrightarrow f(x) = g\left(\frac{\pi x}{L}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n\frac{\pi x}{L}\right) + b_n \sin\left(n\frac{\pi x}{L}\right)$$

$* \hookrightarrow wx$

$a_0, a_n, b_n = ?$

$$f(x) = g\left(\frac{\pi x}{L}\right) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}$$

$T = 2L$

Računanje koeficijenata $\qquad \xi = \frac{x\pi}{L}$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\xi)\,d\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L\xi}{\pi}\right)d\xi = \left|\begin{matrix} x = \frac{L\xi}{\pi} \\ dx = \frac{L}{\pi}d\xi \end{matrix}\right| \Rightarrow a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\xi)\cdot\cos(n\xi)\,d\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\xi\right)\cos(n\xi)\,d\xi =$$

$$= \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\left(\frac{nx\pi}{L}\right)\cdot\frac{\pi}{L}\,dx \Longrightarrow a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{nx\pi}{L}\right)dx$$

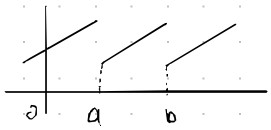
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\xi)\sin(n\xi)\,d\xi = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}\xi\right)\sin(n\xi)\,d\xi =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin\left(\frac{nx\pi}{L}\right)\frac{\pi}{L}\,dx \Longrightarrow b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{nx\pi}{L}\right)dx$$

$\frac{2\pi}{\omega} = T \Rightarrow w = \frac{2\pi}{T}$

T- formule

$T = b - a$



$$f(x) \sim \frac{a}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{T} + b_n \sin\frac{2n\pi x}{T}$$

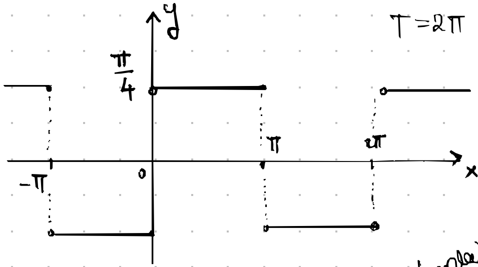
$a_0 = \frac{2}{T}\int_a^b f(x)\,dx \qquad a_n = \frac{2}{T}\int_a^b f(x)\cos\frac{2n\pi x}{T}\,dx \qquad b_n = \frac{2}{T}\int_a^b f(x)\sin\frac{2n\pi x}{T}\,dx$

f - parna ima samo kosinuse → a

$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$ → $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$, $\quad a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$

f - neparna $f(x) \sim \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \longrightarrow b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

Primjer: $f(x) = \frac{\pi}{4}$ razvij u F red na $[0, \pi]$ po sinus funkcijama
suma $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$?



$T = 2\pi$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin(nx) \quad \forall x \neq k\pi \qquad S(x) = f(x) \quad \forall x \neq k\pi$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} \frac{\pi}{4} \cdot \sin(nx)\, dx = \frac{-1}{2} \cdot \frac{1}{n} \cos(nx) \Big|_0^{\pi}$$

$$b_n = \frac{1}{2n} \left(1 - (-1)^n\right)$$

propadaju za parne

$$b_{2n} = 0 \qquad b_2 = b_4 = \dots = 0$$

$$b_{2n+1} = \frac{1}{2(2n+1)} \cdot 2 = \frac{1}{2n+1}$$

$n = 0 \quad b_n = 1$

$n = 1 \quad b_n = \frac{1}{3} \qquad f(x) \sim$

$n = 2 \quad b_n = \frac{1}{5}$

$\vdots$

$$f(x) \sim \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots$$

$$\downarrow$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{2n+1}$$

~~$x \neq k\pi$~~

$x \in \left[0, \frac{\pi}{4}\right]$ bolji način

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$?

$x = \frac{\pi}{2}$

$$x = \frac{\pi}{2} \longrightarrow \sum_{n=0}^{\infty} \frac{\sin(2n+1)\cdot\frac{\pi}{2}}{2n+1} = \sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \frac{1}{7}\sin\frac{7\pi}{2} + \dots$$

$\boxed{2\pi}$ $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \, dx$

? Kiyfu je jednaka Furierovom redu u točkama neprekinutosti

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \quad n \geq 1$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \quad n \geq 1$$

$\boxed{2L}$ $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \cos \frac{n\pi x}{L}$

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x)\,dx \qquad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L}\,dx$$

$\boxed{f\text{ - parna}}$ $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$ $\qquad a_0 = \frac{2}{L} \int_0^L f(x)\,dx \qquad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\,dx$

$\boxed{T}$ $T = 2L \qquad T = b - a$

$$f(x) \sim \frac{a_0}{2} + \sum_{n \geq 1}^{\infty} a_n \cos \frac{2n\pi x}{T} + b_n \sin \frac{2n\pi x}{T}$$

$$a_0 = \frac{2}{T} \int_a^b f(x)\,dx \qquad a_n = \frac{2}{T} \int_a^b f(x) \cos \frac{2n\pi x}{T} \qquad b_n = \frac{2}{T} \int_a^b f(x) \sin \frac{2n\pi x}{T}$$