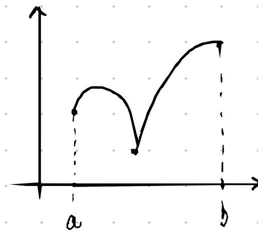
3.5. GLOBALNI EKSTREMI

Matam 1: $f(x)$ na $[a,b] \Rightarrow$



kandidati: stac. točke
gdje funkcija dif

$\boxed{\text{TM}}$ Nepr. f-ja $f: \mathbb{R}^n \to \mathbb{R}$ na omeđenom i zatvorenom skupu D uvijek ima min i max.
Točke u kojima se glob. ekst. poprimaju su kritične točke u f.
(→ stac. točke gdje . gdje f nije dif. ili rub od D).
unutar D

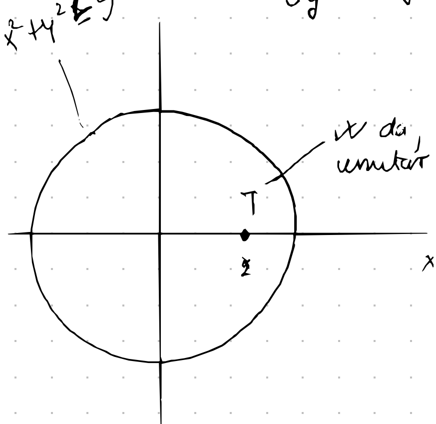
Zad: Glob. ext. $f(x,y) = x^2 - 4x + y^2$ na skupu $x^2+y^2 \leq 9$.

kandidati:

1. stac. točka: $\frac{\partial f}{\partial x} = 2x - 4 = 0$

* parc. deriv. trebaju biti 0 da bi prva deriv. bila 0.

$\frac{\partial f}{\partial y} = 2y = 0$



$(2x-4,\ 2y) = (0,0)$

$\hookrightarrow x = 2 \quad y = 0$ $\Big\}$ $T(2,0)$

Jebiga, ne vidim na ploču

2. drugih kritičnih točaka nema

3. rub: 1. način $y = \pm\sqrt{9-x^2}$

2. način $x = 3\cos t$, $y = 3\sin t$ *polarne

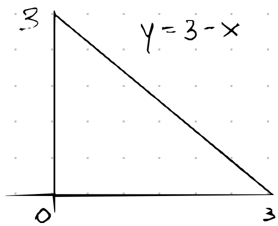
$f(t) = 9\sin^2 t - 12\cos t + 9\cos^2 t$

$f(t) = 9 - 12\cos t \to t=0 \to f(t) = -3$

$\to t=\pi \to f(t) = 21$

LWIR-23-2

c) $f(x,y) = x^2 + xy - y^2 - 3x + 3y$



1 stac točka: $\frac{\partial f}{\partial x} = 2x + y - 3 = 0$

$\frac{\partial f}{\partial y} = x - 2y + 3 = 0$

$\Big\} \; T\left(\frac{3}{5}, \frac{9}{5}\right)$

2. + analiza rubaje matem 1

a: $x = 0$, $y = t \rightarrow f(t) = -t^2 + 3t \rightarrow t \in (0, 3)$

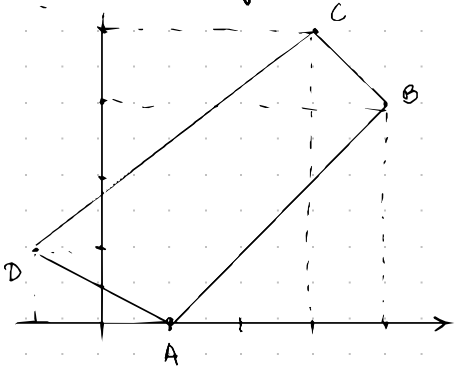
$t \in (0, 3]$, $f'(t) = -2t + 3 = 0$

$\hookrightarrow t = \frac{3}{2}$

$T_3(0,0)$ $T_4(0,3)$ $T_2 = 0, \frac{3}{2})$

b: $y = 0$ $x = t$ $f(t) = t^2 - 3t \rightarrow t \in [0, 3)$

Zad.) $f(x,y) = 3 - x - 2y$ na segmentu $A(1,0)$, $B(4,3)$, $C(3,4)$, $D(-1,1)$



1. stac. $\frac{\partial f}{\partial x} = -1$ $\quad \frac{\partial f}{\partial y} = -2 \neq 0$

nema stac točaka

2. rubovi AB: $y \Rightarrow$ pravac $\Big|$ $x = t$

$f(t) = 3 - t - 2t$

ništa ne vidim

<u>KOROLAR</u>: Afina funkcija $f(\vec{x}) = \vec{a}\vec{x} + \vec{b}$ na omeđenom i zatvorenom skupu D poprima glob. min i MAX na rubu od D (jer je $\nabla f = \vec{a} \neq \vec{0}$).

Dodatno ako je rub

3.6. UVJETNI EKSTREMI

· tražimo općenito ekstreme na skupu S koji je zadan uvjetom $\varphi(\vec{x}) = 0$.

DEF $\vec{a} \in S$ zovemo uvjetni lok. ekstrem od f na S ako postoji okolina $K \subseteq (\vec{a})$ t.d. je $f(\vec{a}) \leq f(\vec{x})$, $\forall \vec{x} \in K_\varepsilon(\vec{a}) \cap S$ Analogno za lok. max.

TM (nužan uvjet za uvjetni ekstrem)

Neki je $U \subset \mathbb{R}^n$ otvoreni skup te neka je $f: U \to \mathbb{R}$ nepr. dif. te neka je S zadana uvjetom $\varphi(\vec{x}) = 0$ (pretp. $\nabla\varphi \neq 0$) Ako je točka $\vec{a}$ uvjetni lok. extr. od f na S, tada postoji $\lambda \in \mathbb{R}$ t.d. je

$$\boxed{\nabla f(\vec{a}) + \lambda \nabla \varphi(\vec{a}) = \vec{0}}$$

Dokaz u nekom drugom obliku $\nabla f = -\lambda \cdot \nabla \varphi$

λ nazivamo Lagrangeov multiplikator, te definiramo fju

$$\boxed{L(\vec{x}, \lambda) = f(\vec{x}) + \lambda \varphi(x)}$$

→ uvjetni ekstremi od $f(x,y)$ uz uvjet $\varphi(x,y) = 0$

1. uvedemo Lang. fju ⇒ $L(x,y,\lambda) = f(x,y) + \lambda \varphi(x,y)$
2. $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial \lambda} = \varphi(x,y) = 0$ ↪ 3x3 sustav

   *ali stac. točke još moraju zadovoljiti $\varphi(x,y) = 0$
3. Provjeriti definitnost drugog dif. (d^2L)

$$\Rightarrow d^2L = L''_{xx}(dx)^2 + L''_{yy}(dy)^2 + L''_{\lambda\lambda}(d\lambda)^2$$

$$+ 2L''_{xy}(dx)(dy) + \underbrace{2L''_{x\lambda}(dx)(d\lambda) + 2 \cdot L''_{y\lambda}(dy)(d\lambda)}_{2(\varphi'_x dx + \varphi'_y dy) d\lambda}$$

$$2(\varphi'_x dx + \varphi'_y dy) d\lambda \quad \text{*prvi diferencijal}$$

$$\Downarrow$$

$$0 \text{ jer } \varphi(x,y) = 0$$

$$\Rightarrow \boxed{d^2L = L''_{xx}(dx)^2 + L''_{yy}(dy)^2 + L''_{\lambda\lambda}(d\lambda)^2 + 2L''_{xy}(dx)(dy)}$$

→ definitnost drugog dif d^2L uz diferencijal uvjeta!

· uvrstimo $\varphi'_x(dx) + \varphi'_y(dy) = 0$

Zad:) Metodom Lagrang. Multipl. (MLM)

$f(x,y) = x^2+y^2-2x-4y$ uz uvjet $xy-y-2x+1=0$

$L(x,y,\lambda) = x^2+y^2-2x-4y+\lambda(xy-y-2x+1)$

$\frac{\partial L}{\partial x} = 2x-2+\lambda(y-2) = 0$

$\frac{\partial L}{\partial y} = 2y-4+\lambda(x-1) = 0$

$\lambda = -\frac{2x-2}{y-2} = \frac{2y-4}{x-1}$

$(x-1)^2 = (y-2)^2$

$x-1 = y-2$ $\quad$ $x-1 = 2-y$

$y = x+1$ $\quad$ $y = 3-x$

$\frac{\partial L}{\partial \lambda} = xy-y-2x+1 = 0$

$x(x+1)-x-1-2x+1=0$

$x^2-2x=0$

$x(x-2)=0$

$x_1=0$ $\quad$ $x_2=2$ $\quad$ nema 0

$y_1=1$ $\quad$ $y_2=3$

$T_1(0,1), \lambda_1 = -2$

$T_2(2,3), \lambda_2 = -2$

$L_{xx}=2, \quad L_{yy}=2, \quad L_{xy}=-2$

$d^2L = 2(dx)^2+2(dy)^2-4\,dx\,dy$

dif uvjeta: $(y-2)dx-(x-1)dy=0$

$dy = \frac{-y-2}{x-1}dx = -dx$

$d^2L(T_{1,2}) = 2(dx)^2+2(dx)^2+4dx^2 = 8(dx)^2 > 0$

$\to T_1, T_2$ su uvjetni min.

MI-24-4

$\boxed{\text{MI-2016-8}}$ $f(x,y,z) = x^2 - 3xy + y^2 + z^2$ uz uvjet $x+y+z=1$

$$L(x,y,z,\lambda) = x^2 - 3xy + y^2 + z^2 + \lambda(x+y+z-1)$$

$$\frac{\partial L}{\partial x} = 2x + 3y + \lambda = 0$$

$$\frac{\partial L}{\partial y} = -3x + 2y + \lambda = 0$$

$$\frac{\partial L}{\partial z} = 2z + \lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = x+y+z-1 = 0$$

$$\left[\begin{array}{cccc|c} 2 & -3 & 0 & 1 & 0 \\ -3 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right]$$

$\rightsquigarrow$ Gauss: $x = \frac{2}{3}$ $y = \frac{2}{3}$, $z = \frac{1}{3}$, $\lambda = \frac{2}{3}$

$$d^2L = L''_{xx}(dx)^2 + L''_{yy}(dy)^2 + L_{zz}(dz)^2 + 2L''_{xy}(dx)(dy) + 2x''z(dx)(dz)$$
$$+ 2y''z(dy)(dz) - 6\,dx\,dy.$$

$\hookrightarrow$ dif. uvjeta: $1dx + 1dy + 1dz = 0$

$$d^2L = 2(dx)^2 + 2(dy)^2 + 2(dx)^2 + 2(dy)^2 + 4dx\,dy + 6dx\,dy$$

$$d^2L(T) = 3(dx)^2 + 3(dy)^2 + (dx-dy)^2 > 0$$

$\forall (dx, dy, dz) \neq (0,0,0)$

MLM

$\boxed{\text{LJIR-20-3}}$ $f(x,y,z) = xy + y^3 - z^2$ uz uvjete $y - z = 1$

* ima 3 ovakva zad u skripti $y - x = 5$

$$L(x,y,z,\lambda,\mu) = xy + y^3 - z^2 + \lambda(y - z - 1) + \mu(y - x - 5)$$

$$\frac{\partial L}{\partial x} = y - \mu = 0 \qquad \frac{\partial L}{\partial z} = -2z - \lambda = 0$$

$$\frac{\partial L}{\partial y} = x + 3y^2 + \lambda - \mu = 0 \qquad \frac{\partial L}{\partial \lambda} = y - z - 1 = 0 \qquad \frac{\partial L}{\partial \mu} = y - x - 5 = 0$$

$y = \mu$

$\mu = x + 3y^2 + \lambda$

$2z + \lambda = 0 \to \lambda = -2z$

$y = z + 1 \to z = y - 1$

$y = x + 5 \to x = y - 5$

$y = x + 3y^2 - 2z$

$\downarrow$

$\cancel{y} = \cancel{y} - 5 + 3y^2 - 2(y-1)$

$\cancel{0 = 3y^2 - 2y - 3}$

nešto sam fulala, trebala sam dobit

$3y^2 = 3$

$y^2 = 1$

$T_1(-4,1,0)$ $\quad$ $T_2(-6,-1,-2)$

$\lambda_1 = 0$ $\quad$ $\lambda_2 = 4$

$\mu_1 = 1$ $\quad$ $\mu_2 = -1$

$L_{xx} = 0 \quad L_{xy} = 1 \quad L_{xz} = 0 \quad L_{yy} = 6y \quad L_{yz} = 0 \quad L_{zz} = -2$

$$d^2L = 2dxdy + 6y(dy)^2 - 2(dz)^2$$

$$d^2L(T_1) = 2dxdy + 6(dy)^2 - 2(dz)^2 = 6(dx)^2 > 0$$

$$\left.\begin{aligned} \to dy - dz &= 0 \\ dy - dx &= 0 \end{aligned}\right\} dz = dy = dx \quad \to \quad d^2L(T_2) = -6(dx)^2 < 0$$

Zad.) odrediti glob. extr. $f(x,y) = x - \sqrt{3}y$ na krugu $x^2+y^2 \leq 1$

$\frac{\partial f}{\partial x} = 1 \neq 0 \quad \frac{\partial f}{\partial y} = -\sqrt{3} \neq 0 \rightarrow$ nema stac. točaka $\Rightarrow$ ekstremi su na rubu $x^2+y^2 \leq 1$

1. način: parametrizacija ruba

$$\left.\begin{array}{l} x = \cos t \\ y = \sin t \end{array}\right\} f(t) = \cos t - \sqrt{3}\sin t$$

2. način: Lagrangeov multiplikator $\rightarrow$ moramo ako ne možemo parametrizaciju

$$L = x - \sqrt{3}y + \lambda(x^2+y^2-1)$$

$$\frac{\partial L}{\partial x} = 1 + 2x\lambda = 0 \qquad \frac{\partial L}{\partial y} = \sqrt{3} + 2y\lambda = 0 \qquad \frac{\partial L}{\partial \lambda} = x^2+y^2-1 = 0$$

$$\Downarrow$$

$$x = \frac{-1}{2\lambda} \qquad y = \frac{-\sqrt{3}}{2\lambda} \rightarrow \frac{1}{4\lambda^2} + \frac{3}{4\lambda^2} - 1 = 0$$

$$\Downarrow$$

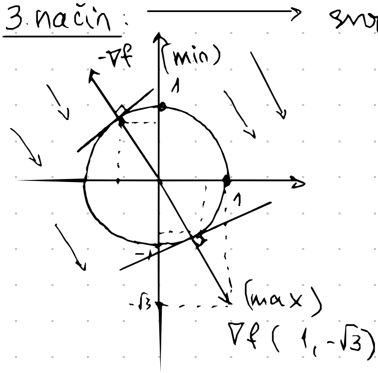
$$\frac{1}{\lambda^2} = 1$$

$$\lambda^2 = 1 \quad \begin{cases} \lambda = 1 \\ \lambda = -1 \end{cases}$$

$x = \frac{-1}{2} \quad y = \frac{-\sqrt{3}}{2} \quad T_1\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right) \rightarrow f(T_1) = -2$ min

$x = \frac{1}{2} \quad y = \frac{\sqrt{3}}{2} \quad T_2\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \rightarrow f(T_2) = 2$ max

3. način: $\longrightarrow$ smjer gradijenta



$$\nabla f = (1, -\sqrt{3})$$

* možemo i računski dobiti

$$y = -\sqrt{3}x$$

$$x^2+y^2 = 1$$

$$4x^2 = 1$$

$$\hookrightarrow x = \pm\frac{1}{2}$$

3.7. PRIMJENE MLM

MI-22-4 $z = x^2 + y^2$, nači točku najbližu točki $T(1,1,0)$

tražimo minimum udaljenosti

$$d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2} \Longrightarrow \underline{\min}$$

$T_1(1,1,0)$

$T_2(x,y,z)$

*korijen je monotono rastuća f-ja, znači one ne mijenjaju ekstrem $\longrightarrow$ možemo uzeti samo ono pod korijenom

$$\Longrightarrow f(x,y,z) = (x-1)^2 + (y-1)^2 + z^2$$

$$L(x,y,z) = (x-1)^2 + (y-1)^2 + z^2 + \lambda(x^2+y^2-z)$$

$\longrightarrow$ uvjet je da je točka na toj plohi!

$\underbrace{x^2+y^2-z^2=0}$

$$\frac{\partial L}{\partial x} = 2(x-1) + 2x\lambda = 0$$

$$\frac{\partial L}{\partial y} = 2(y-1) + 2y\lambda = 0$$

$$\frac{\partial L}{\partial z} = 2z - \lambda = 0 \longrightarrow \underline{z = \frac{\lambda}{2}}$$

$$\frac{\partial L}{\partial \lambda} = x^2+y^2-z = 0$$

$$\lambda^3 + 2\lambda^2 + \lambda - 4 = 0$$

$\pm 1, \pm 2, \pm 4$

$\underline{\lambda_1 = 1}$

$\Longrightarrow T\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), z = 1$

$$d^2L = (2+2\lambda)(dx)^2 + (2+2\lambda)(dy)^2 + 2(dz)^2$$

$$d^2L = 4(dx)^2 + 4(dy)^2 + 2(dz)^2 > 0$$

ne ovisi o dif. uvjeta

$\longrightarrow$ stogo poz $\longrightarrow$ d je lok. min (uvjetni strogi)

JIR-21-3 $x^3 - y^2 - 3x + 4y^2 + z^2 + z = 8$

1. način: implicitno deriviranje

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = -\frac{3x^2-3}{2z+1} = 0 \quad , \quad H_f$$

$\cancel{\frac{\partial z}{\partial y} = 0}$ NE!

implicitno f!

$\longrightarrow d^2 f \Rightarrow$ deriv. kvocijenta (z je funkcija)

2. način: MLM (bez implicitnih derivacija)

$\rightarrow$ funkcija je $z = z(x,y)$

a uvjet ova jednadžba jer se na „tome" nalazi točka

$$L(x,y,z,\lambda) = z + \lambda(x^3 - y^2 - 3x + 4y^2 + z^2 + z) = 8$$

$$\frac{\partial L}{\partial x} = \lambda(3x^2 - 3) = 0 \qquad \lambda \neq 0!$$

DZ

$L_{xx} = 6x\lambda$, L_{yy}, L_{zz} $\Rightarrow$ d^2L

$L_{xy} = 0$

$L_{xz} = 0$, $L_{yz} = 0$