1.6 RJEŠAVANJE DIFERENCIJALNIH I INTEGRALNIH JEDNADŽBI

$x''(t) + a_1 x'(t) + a_0 x(t) = f(t) \longrightarrow$ Mat 2: $y'' + a_1 y' + a_0 y = f(x)$

ako f nije pre problematičan, znamo od prošle godine

POSTUPAK:

① homogena * $\quad x''(t) + a_1 x'(t) + a_0 x(t) = 0$

② karakteristični polinom $\quad \lambda^2 + a_1 \lambda + a_0 = 0$

⋮

x_h = rješenje homogene — zaključi iz karakt. pol.

$x(t) = \underbrace{x_h(t)}_{\text{homogeno}} + x_p(t)$ — partikularno

linearna dif. jednadžba drugog reda s konstantnim koeficijentima

*ako je f fja gate ili step, onda prošlogodišnji način ne može

Laplaceova metoda

$x(0) = x_0$

$x'(0) = x_1$

$x(t) \circ\!\!-\!\!\bullet\ X(s)$

$x'(t) \circ\!\!-\!\!\bullet\ s \cdot X(s) - x(0)$

$(x'(t))' = x''(t) \circ\!\!-\!\!\bullet\ s \cdot (s \cdot X(s) - x(0)) - x'(0)$

$x''(t) + a_1 x'(t) + a_0 x(t) = f(t) \quad / \mathcal{L}$

$s(s \cdot X(s) - X(0)) - x'(0) + a_1 (sX(s) - x(0)) + a_0 X(s) = F(s)$

$s^2 X(s) - s x_0 - x_1 + a_1 s X(s) - a_1 x_0 + a_0 X(s) = F(s)$

$X(s)\underbrace{\boxed{s^2 + a_1 s + a_0}}_{P(s)\ \text{karakt. pol.}} \underbrace{- s x_0 - x_1 + a_1 x_0}_{G(s)} = F(s)$

$X(s) \cdot P(s) + G(s) = F(s) \longrightarrow \boxed{X(s) = \dfrac{F(s) - G(s)}{P(s)}}$ — ako ima realne nultočke onda se rastavi $s - \alpha$

$\Rightarrow x(t) = \mathcal{L}^{-1}(X(s)) = \mathcal{L}^{-1}\left(\dfrac{F(s) - G(s)}{P(s)}\right)$

$x(0)=2$
$x'(0)=1$

Primjer: $x''(t)+4x(t)=e^{t} \,/\, \mathcal{L}$

$$s\cdot(sX(s)-x(0))-x'(0)+4(sX(s)-x(0))=\frac{1}{s-1}$$

$$s^2X(s)-s\cdot 2-1+4sX(s)-4\cdot 2=\frac{1}{s-1}$$

$$X(s)(s^2+4s)\quad -2s-9=\frac{1}{s-1}$$

$$X(s)(s^2+4s)=\ldots ?$$

PITAJ NEKOG

$$\Longrightarrow\quad X(s)=\frac{1}{(s^2+4s)(s-1)}+\frac{2s+1}{s^2+4}\qquad \frac{As+B}{s^2+4s}+\frac{C}{s-1}$$

$$1=(As+B)(s-1)+C(s^2+4s)$$

$s=1 \longrightarrow 1=C(1+4) \Rightarrow \boxed{C=\frac{1}{5}}$

$s=0 \longrightarrow 1=As^2+B\cdot s \qquad ?\ ?$

$$\Rightarrow X(s)=\frac{1}{5}\,\frac{1}{s-1}-\frac{1}{5}\,\frac{s+1}{s^2+4}+\frac{2s+1}{s^2+4}$$

$$=\frac{1}{5}\cdot\frac{1}{s-1}-\frac{1}{5}\,\frac{s}{s^2+4}-\frac{1}{5}\,\frac{1}{s^2+4}+2\,\frac{s}{s^2+4}+\frac{1}{s^2+4}$$

$$=\frac{1}{5}\,\frac{1}{s-1}-\frac{9}{5}\cdot\frac{s}{s^2+4}-\frac{4}{5}\cdot\frac{1}{s^2+4}\ \circ\!\!-\!\!\bullet\ \boxed{\frac{1}{5}e^{t}-\frac{9}{5}\cos(2t)-\frac{1}{5}\sin(2t)}$$

Primjer: $x'(t)-x(t)=f(t) \,/\, \mathcal{L}$

$x(0)=1 \qquad f(t)=g_{[0,1]}(t)=u(t)-u(t-1)$

$$sX(s)-x(0)-X(s)=F(s)\longrightarrow F(s)=\frac{1}{s}-e^{-s}\frac{1}{s}$$

$$sX(s)-X(s)-1=\frac{1}{s}-e^{-s}\cdot\frac{1}{s}\Rightarrow 1+\frac{1}{s}-\frac{1}{s}e^{-s}$$

PITAJ NEKOG

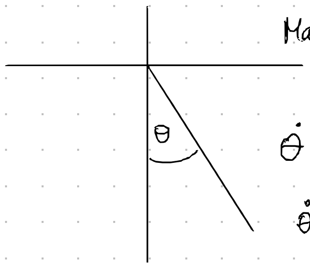
$$X(s)(s-1)=1+\frac{1}{s}-\frac{1}{s}e^{-s}$$

$$X(s)=\frac{1}{s-1}+\frac{1}{s(s-1)}-\frac{1}{s(s-1)}e^{-s}=$$

Sustavi diferencijalnih jednadžbi

$$x''(t) + a_1 x'(t) + a_0 x(t) = 0$$

Nelinearna: $\ddot{\theta} + \sin\theta = 0$



Matematičko njihalo bez otpora zraka

$$\dot{\theta} = \frac{d\theta}{dt}$$

$$\dot{\theta} = v$$

Za vježbu zapišite jednadžbu kao sustav:

Uvedi $x'(t) = y(t)$

$$y'(t) + a_1 y(t) + a_0 x(t) = 0$$

$$y'(t) = -a_0 x(t) - a_1 y(t)$$

$x' = y \qquad y' = -a_0 x - a_1 y$

Ubit ću se

$$\begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}$$

Naći svojstveni (vlastiti, karakt.) polinom matrice A

Primjer: $y' = -4y - 4z \;/\mathcal{L}$

$z' = -2y - 6z \;/\mathcal{L}$

$y(0) = 3$

$z(0) = 15$

$y(t) \to Y(s)$

$z(t) \to Z(s)$

Linearni sustavi

$$sY(s) - Y(0) = -4Y(s) - 4Z(s)$$

$$sZ(s) - Z(0) = -2Y(s) - 6Z(s)$$

$$(s+4)Y(s) + 4Z(s) = 3$$

$$2Y(s) + (s+6)Z(s) = 15$$

$$D = \begin{vmatrix} s+4 & 4 \\ 2 & s+6 \end{vmatrix}$$

$$D = (s+2)(s+8)$$

$$D_z = \begin{vmatrix} s+4 & 3 \\ 2 & s+6 \end{vmatrix} \qquad D_y = \begin{vmatrix} 3 & 4 \\ 15 & s+6 \end{vmatrix}$$

$$D_z = 15s + 54 \qquad D_y = 3s - 42$$

$$Y = \frac{D_y}{D} = \frac{3s-42}{(s+2)(s+8)} = \frac{-8}{s+2} + \frac{11}{s+8}$$

$$Z = \frac{D_z}{D} = \frac{15s+54}{(s+2)(s+8)} = \frac{4}{s+2} - \frac{11}{s+8}$$

$$Y \multimap -8e^{-2t} + 11e^{-8t}$$

$$Z \multimap 4e^{-2t} - 11e^{-8t}$$

Integralne jednadžbe konvolucijskog tipa

Primjer:

$$y(t) = at + \underbrace{\int_0^t \sin(t-\tau)\, y(\tau)\, d\tau}_{\sin t * y(t)}$$

$$Y(s) = \frac{a}{s^2} + \frac{1}{1+s^2}\cdot Y(s)$$

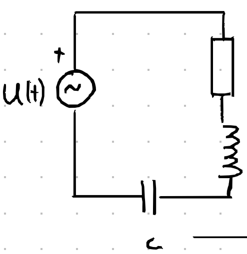
$$Y(s)\left(1-\frac{1}{1+s^2}\right) = \frac{a}{s^2}$$

$$Y(s)\cdot\frac{s^2}{1+s^2} = \frac{a}{s^2}$$

$$Y(s) = \frac{a}{s^2}\cdot\frac{1+s^2}{s^2} = a\left(\frac{1}{s^4}+\frac{1}{s^2}\right)$$

$$u(t) = a\left(\frac{t^3}{6}+t\right)$$

Primjer: električni krugovi $\quad i(t) = \frac{dq(t)}{t}$



$R \to R\cdot i(t) = R\cdot\frac{dq(t)}{t}$

$L \to L\cdot\frac{di(t)}{dt} = L\cdot\frac{d^2q(t)}{dt^2}$

$c \longrightarrow u(t) = \frac{q}{c}$

$* \; e(t) = u(t)$

dif. jed. po q: $\quad L\frac{d^2q(t)}{dt} + R\frac{dq(t)}{dt} + \frac{q}{c} = e(t)$

$/L/$

dif. integ. jed $\quad L\cdot\frac{di(t)}{dt} + R\cdot i(t) + \frac{1}{c}\int_0^t q(t)\,dt = e(t)$ $\quad$ ove ćemo više koristiti

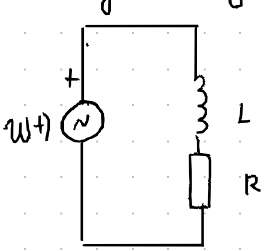
Laplace?

$e_R(t) = R\cdot i(t) \circ\!\!-\!\!\bullet R\cdot I(s) = R\cdot I(s)$ $\quad$ simbolički otpori

$e_L(t) = L\cdot i'(t) \circ\!\!-\!\!\bullet L\cdot(s\cdot I(s) - I(0)) = L\cdot s\, I(s) - L\cdot I(0)$

$e_c(t) = \frac{1}{c}\int_0^t q(t)\,dt \circ\!\!-\!\!\bullet \frac{1}{c}\,\frac{I(s)}{s} = \frac{1}{c}\cdot\frac{1}{s}\, I(s)$

Primjer: serijski LR krug



zadano $R, L = 1$

$I(s) = \frac{E(s)}{Z(s)}$, $i(t) = ?$

$E(s) \circ\!\!-\!\!\bullet e(t) = u(t)$

$\frac{1}{s}$ "

$Z(s) = R + Ls$

$$I(s) = \frac{\frac{1}{s}}{R+Ls} = \frac{1}{s(1+s)}$$

$$I(s) = \frac{1}{s} + \frac{1}{s+1}$$

$$I(s) \circ\!\!-\!\!\bullet u(t) - e^{-t}u(t) = \boxed{(1-e^{-t})\,u(t)}$$

Primjer: R-C serijski $e(t)=u(t)$ $i(t)=?$

$$E(s)=\frac{1}{s} \quad Z(s)=R+\frac{1}{cs} \qquad I(s)=\frac{E(s)}{Z(s)}=\frac{1}{s(R+\frac{1}{cs})}=\frac{1}{Rs+\frac{1}{c}}$$

$$I(s)=\frac{1}{R}\cdot\frac{1}{s+\frac{1}{RC}}$$ 2?

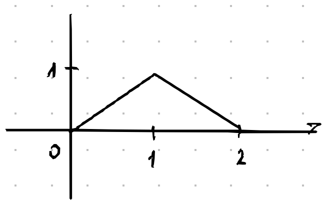
Kako se napon ponaša za velike t? $\lim_{t\to\infty}\frac{1}{R}e^{-\frac{t}{RC}}=0$

Primjer: L-S krug serijski

$$E(s)=\frac{1}{s} \quad Z(s)=Ls+\frac{1}{cs} \qquad I(s)=\frac{\frac{1}{s}}{Ls+\frac{1}{cs}}=\frac{1}{L}\cdot\frac{1}{s^2+\frac{1}{cL}}$$

$$I(s)=\frac{1}{L}\cdot\frac{1\cdot\frac{1}{\sqrt{cL}}}{s^2+\frac{1}{cL}}\cdot\frac{1}{\sqrt{cL}} \circ\!\!-\!\!\bullet \boxed{\frac{1}{L}\cdot\frac{1}{\sqrt{cL}}\sin\left(\frac{t}{cL}\right)}$$

Primjer



$$e(t)=t\,g_{[0,1]}+(2-t)\,g_{[1,2]}$$

$$e(t)=t\,(u(t)-u(t-1))+(2-t)(u(t-1)-u(t-1))$$

$$e(t)=tu(t)-\cancel{tu(t-1)}+2u(t-1)\cancel{tu(t-1)}$$

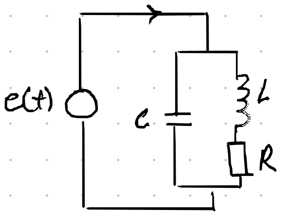
Podsjetnik: R, LS, $\frac{1}{CS}$ simbolički otpori

$Z(S)$ = otpor

$I(s) \circ\!\!-\!\!\bullet\ i(t)$

$E(S) \circ\!\!-\!\!\bullet\ e(t)$

Primjer:



$C = L = 1 \qquad R = 2$

$i(t) = 1 \qquad I(S) = \frac{1}{S}$

$e(t) = ?$

$$Z(S) = \frac{\frac{1}{CS}\cdot(LS+R)}{\frac{1}{CS} + LS + R} = \frac{\frac{1}{S}(S+2)}{\frac{1}{S} + S + 2}\cdot\frac{S}{S} = \frac{S+2}{1+S^2+2S} = \frac{S+2}{(S+1)^2}$$

$$E(S) = I(S)\cdot Z(S) = \frac{1}{S}\cdot\frac{S+2}{(S+1)^2} = \frac{S+2}{S(S+1)^2}$$

$$\longrightarrow \frac{S+2}{S(S+1)^2} = \frac{A}{S} + \frac{B}{S+1} + \frac{C}{(S+1)^2} \quad /\cdot \text{nazivnik}$$

$$S+2 = A(S+1)^2 + B\,S(S+1) + C\cdot S = A(S^2+2S+1) + B(S^2+S) + CS$$

$$= AS^2 + BS^2 + 2AS + BS + CS + A$$

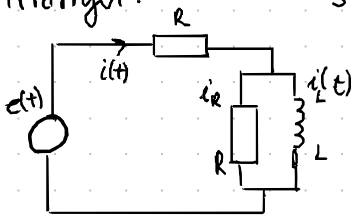
$A+B=0 \qquad 2A+B+C=1 \qquad A=2 \qquad B=-2 \qquad C=-1$

$$E(S) = \frac{2}{S} - \frac{2}{S+1} - \frac{1}{(S+1)^2}$$

$$\frac{2}{S} \circ\!\!-\!\!\bullet\ 2u(t) - 2e^{-t} - e^{-t}\cdot t\cdot u(t)$$

$2e^{-t}\cdot u(t)$ često smo sve množili sa $u(t)$?

Primjer: $E(S) = \frac{1}{S}$



$e(t) = u(t)$

$R = L = 1$

a) $i(t) = ?$

b) $i_L(t) = ?$

a) $$Z(s) = R + \left(\frac{1}{R} + \frac{1}{LS}\right)^{-1} = R + \frac{LS\cdot R}{R+LS}$$

$$Z(S) = 1 + \frac{S}{1+S} = \frac{1+S+S}{1+S}$$

$$Z(S) = \frac{1+2S}{1+S}$$

$$E(S) = \frac{1}{S} = I(S)\cdot Z(S)$$

$$\hookrightarrow I(S) = \frac{E(S)}{Z(S)} = \frac{1}{S}\,\frac{1+S}{1+2S} = \frac{1+S}{S(1+2S)} \overset{*}{=} \frac{A}{S} + \frac{B}{1+2S}$$

$A + AS + BS = 1+S \qquad A = 1,\ B = -1 \quad \rightarrow$

$$= \frac{1}{S} - \frac{1}{1+2S} = \frac{1}{S} - \frac{1}{2}\,\frac{1}{\frac{1}{2}+S}$$

$$I(S) \circ\!\!-\!\!\bullet\ i(t) = u(t) - e^{-\frac{1}{2}t}\cdot u(t)\cdot\frac{1}{2}$$

$$\boxed{i(t) = \left(1 - \frac{1}{2}e^{-\frac{t}{2}}\right)u(t)}$$

b) $$I(S) = I_R(s) + I_L(S)$$

$$R\cdot I_R(S) = L\cdot S\cdot I_L(S), \qquad I_R(S) = I_L(S)\cdot S$$

$$I(S) = I_L(S)\cdot S + I_L(S) = (S+1)\,I_L(S)$$

$$I_L(S) = \frac{I(S)}{S+1} = \frac{1}{S(1+2S)} = \frac{1}{S} - \frac{1}{S+\frac{1}{2}} \circ\!\!-\!\!\bullet\ i_L(t) = u(t) - e^{-\frac{t}{2}}u(t)$$

$$\boxed{i_L(t) = u(t)\left(1 - e^{-\frac{t}{2}}\right)}$$