10.1. Diferencijalne jednadžbe

*bez pomoći Laplaceove transf.

- neovisni izvori (naponski / strujni) → poticaji / pobude
- Struja ili napon neke grane → odziv kruga
- postavljanjem jed. ⇒ sistem integrodiferencijalnih jednadžbi 1. reda
  - └ deriviranjem → sistem od n lin. dif. jed. 0-2. reda s n nepoznanica

→ svaki y koji zadovoljava jednadžbu: RJEŠENJE DIF. JED.

pretpostavka: → $y_n(t)$ → rješenje
provjera → uvrsti u jednadžbu

$x(t)$ – pobuda (poznata vrem. f-ja)
$y(t)$ – odziv (nepoznata vrem. f-ja)

$$a_n \frac{d^n y_n}{dt^n} + a_{n-1} \frac{d^{n-1} y_n}{dt^{n-1}} + \dots + a_0 y_n = b_m \frac{d^m x}{dt^m} + \dots + b_0 x$$

HLDJ

1) isključivanje pobude → zdesna je ∅
2) rješenje $y(t) = y_n(t)$ (rješenje homogene dj)
3) nađemo $y_p(t)$ (nehomogeno rj) → $y(t) = \underbrace{y_h(t)}_{\text{slobodni odziv}} + \underbrace{\boxed{y_p(t)}}_{\text{prisilni odziv}}$ ⇒ OPĆE RJEŠENJE JE

$y_n(t)$ → sadrži n konstanti

- rješenja HLDJ su oblika: $y(t) = Ae^{st}$

→ uvrstimo i dobijemo: $a_n A s^n e^{st} + a_{n-1} A s^{n-1} e^{st} + \dots + a_0 A e^{st} = 0 \; / : Ae^{st}$

$$\underbrace{a_n s^n + a_{n-1} s^{n-1} + \dots + a_0 = 0}_{\text{karakter. jedn. homogene dif. jed.}}$$

- rješenja karakt. jednadžbe: $s_1, s_2, s_3 \dots s_n$ → vlastite ili prirodne frekvencije kruga
  - → određuju oblik rješenja HLDJ
  - → ovise samo o elementima kruga

Vlastite ili prirodne frekvencije kruga

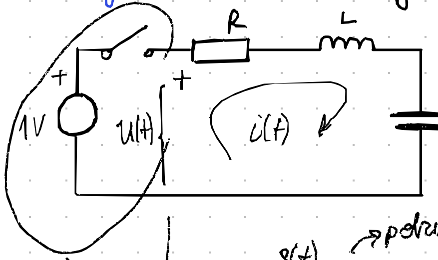
- određuju valni oblik slobodnog odziva
- korijeni karakteristične jednadžbe s_i mogu biti: realni, kompleksni, imaginarni
  - ↳ REALNI: korijeni < 0
  - KOMPLEKSNI: realni dio < 0, konjugirano kompleksni par
  - IMAGINARNI KORIJENI mogu biti samo jednostruki

- konstante $A_1, \dots A_n$ → iz općeg rješenja → iz početnih uvjeta $u_C(0)$ i $i_L(0)$ } * slobodni odziv

L ovisne o poč. uvjetima, pobudnim funkcijama i el. mreže

⇒ ako na el. mrežu spojimo izvora pobudu: DJ više nije HLDJ
- partikularni integral ili prisilni odziv → rješavanje Nehom. DJ

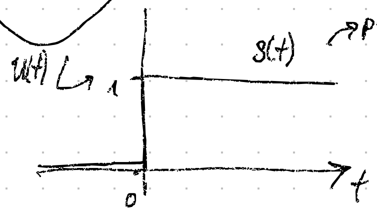
Primjer 1) RLC krug



$R = 3\,\Omega$ $\quad i_L(0) = 0$

$L = 1\,H$ $\quad u_C(0) = 0$

$C = 0{,}5\,F$ $\quad i(t) = ?$



→ pobuda je step funkcija

$u(t) = 1 \cdot S(t)$

Integrodif. jed. kruga:

$$u(t) = R \cdot i + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \Big/ \frac{d}{dt}$$

$$\frac{du}{dt} = R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i$$

HLDJ:

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i = 0$$

pretpostavka: rješenje oblika $i(t) = Ae^{st}$

$$\Rightarrow R \cdot s\,Ae^{st} + L \cdot s^2 Ae^{st} + \frac{1}{C} Ae^{st} = 0 \quad / : Ae^{st}$$

$$Rs + Ls^2 + \frac{1}{C} = 0 \quad / : L$$

$$s^2 + \underbrace{\frac{R}{L}}_{a_1} s + \underbrace{\frac{1}{LC}}_{a_0} = 0$$ karakteristična jednadžba → $$s_1, s_2 = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4 \cdot \frac{1}{LC}}}{2}$$

$$s_1, s_2 = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \rightarrow s_1 = -1,\quad s_2 = -2$$

Rješenje HLDJ

$$i_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$i_h(t) = A_1 e^{-t} + A_2 e^{-2t}$$

- rješenje Nehom. DJ = partikularno rješenje
- pretpostavka ⇒ isti oblik kao i funkcija pobude $i_p(t) = K$ (konstanta)
  ↳ zbog step funkcije
  ↳ deriv step fje je 0

→ uvrštavanje (deriv. bilo koje konstante = 0)

$$R\frac{di}{dt} + L\frac{d^2 i}{dt^2} + \frac{1}{C} i = \frac{du}{dt} \rightarrow R \cdot 0 + L \cdot 0 + \frac{1}{C} \cdot K = 0$$

$$\Rightarrow K = 0$$

→ rješenje nehom DJ → $\boxed{i_p(t) = 0}$

opće rješenje $i_h(t) + i_p(t)$ → $i(t) = A_1 e^{-t} + A_2 e^{-2t}$

Konstante A_1 i A_2 → iz početnih uvjeta

tražimo $i(t)$ — početni uvjeti $u_C(0)$, $i_L(0)$ [redacted] → trebaju nam $i(t)\big|_{t=0}$, $\frac{di}{dt}\big|_{t=0}$

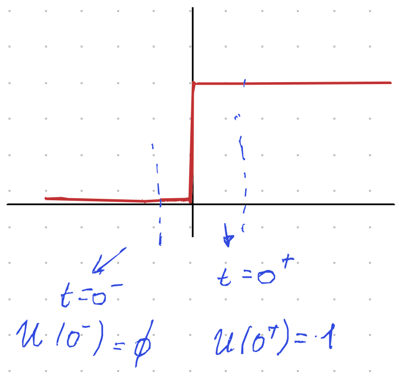
1)

$$\Rightarrow i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$t = 0 \rightarrow \boxed{i(0) = A_1 + A_2 = 0}$$

2) $$u(t) = R \cdot i(t) + L\frac{di}{dt} + u_C(t)$$

↓



$$u(t)\Big|_{t=0^+} = R \cdot i(0) + L \cdot \frac{di}{dt}\Big|_{t=0^+} + u_C(0)$$

$$u(0^+) = 1 \qquad \underbrace{i(0) = 0 \qquad u_C(0) = 0}_{\text{zadano na početku}} \Rightarrow L\frac{di}{dt}\Big|_{t=0^+} = 1 \rightarrow 1 \cdot \frac{di}{dt}\Big|_{t=0^+} = 1$$

$$\frac{di}{dt}\Big|_{t=0^+} = 1$$

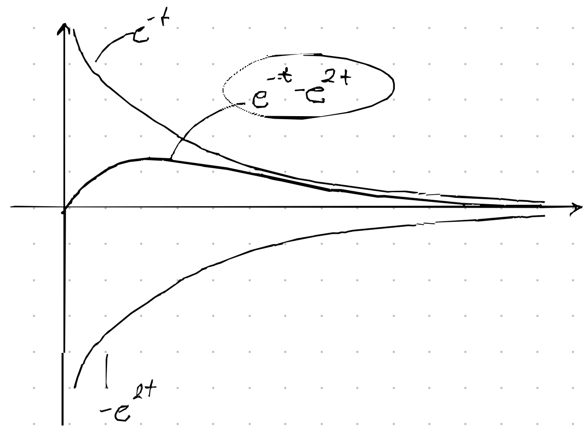
$$\frac{d}{dt}\left(A_1 e^{s_1 t} + A_2 e^{s_2 t}\right)\Big|_{t=0^+} = 1 \rightarrow s_1 A_1 + A_2 s_2 = 1$$

$$\boxed{-A_1 - 2A_2 = 1}$$

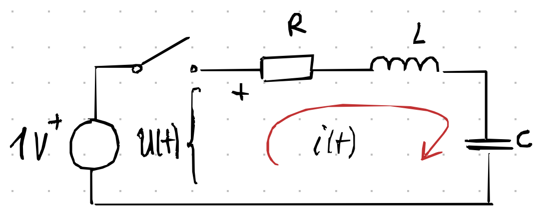
$$\Rightarrow \begin{cases} A_1 + A_2 = 0 \\ -A_1 - 2A_2 = 1 \end{cases} \qquad A_1 = 1 \quad A_2 = -1$$

opće rješenje: $\boxed{i(t) = \left(e^{-t} - e^{-2t}\right) S(t)}$

Matlab



Primjer 2.) Isti RLC sa $R = 2\,\Omega$



$R = 2\,\Omega$

$L = 1\,H$

$C = 0{,}5\,F$

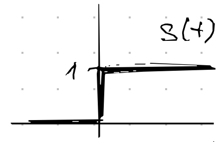
$i_L(0) = 0$

$u_C(0) = 0$

$i(t) = ?$

$$u(t) = R \cdot i(t) + L \cdot \frac{di(t)}{dt} + u_C(t)$$

$$u(t) = R \cdot i(t) + L \cdot \frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau \;\rightarrow u(t) \rightarrow$$



$\rightarrow u(t) = 1 \cdot s(t)$

$$\frac{du(t)}{dt} = R \cdot \frac{di(t)}{dt} + L \cdot \frac{d^2 i(t)}{dt} + \frac{1}{C} \cdot i(t)$$

$\frac{du}{dt} = 0$

$$R \cdot \frac{di}{dt} + L \cdot \frac{d^2 i}{dt} + \frac{1}{C} i(t) = 0 \leftarrow *$$

pretpostavka: $\boxed{i(t) = Ae^{st}}$ da je tj. oblika

$$R \cdot s A e^{st} + L \cdot s^2 A e^{st} + \frac{1}{C} A e^{st} = 0 \;/ : Ae^{st}$$

$$Rs + Ls^2 + \frac{1}{C} = 0 \;/ : L$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \longrightarrow s^2 + 2s + 2 = 0$$

$$s_1 = -1 + j \qquad s_2 = -1 - j$$

$$i_h = A_1 e^{(-1+j)t} + A_2 e^{(-1-j)t}$$

$$\boxed{i_h = e^{-t}\left(A_1 e^{jt} + A_2 e^{-jt}\right)}$$

$\hookrightarrow (1)$

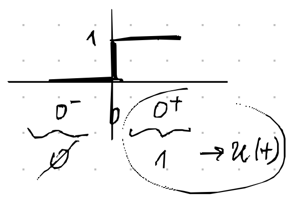
početni uvjeti:

$i_L(0) = 0$

$u_C(0) = 0$

potrebni uvjeti

$i(t),\ \frac{d}{dt} i(t) \Big|_{t = 0^+} \rightarrow$ jer



$$\rightarrow u(t) = R \cdot i(t) + L\frac{di(t)}{dt} + u_C(t) \;/\; (t = 0^+)$$

$$u(0^+) = R \cdot i(0) + L \cdot \frac{di}{dt}\Big|_{t=0^+} + u_C(0) \longrightarrow \text{zadano } u_C,\ i_L = 0$$

$i(0) = 0$ (zadano)

$i(0) = \boxed{A_1 + A_2 = 0}$

$A_1 = -A_2$

$$\rightarrow L \cdot \frac{d}{dt}\left[e^{-t}\left(A_1 e^{jt} + A_2 e^{-jt}\right)\right]\Big|_{0^+} = 1$$

$$1 \cdot \left((-1+j)A_1 + A_2(-1-j)\right) = 1$$

$$\boxed{-A_1 + jA_1 - A_2 - jA_2 = 1} \longrightarrow$$

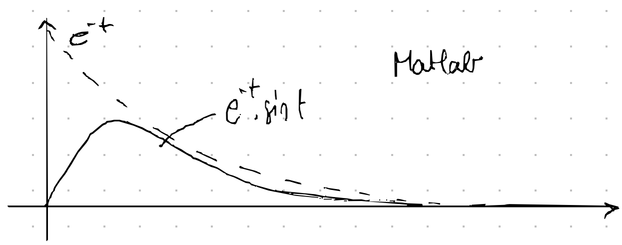
$$-\cancel{A_1} + jA_1 + \cancel{A_1} + jA_1 = 1$$

$$\boxed{A_2 = -j\,\frac{1}{2}} \rightarrow \boxed{A_1 = j\,\frac{1}{2}}$$

$$\rightarrow i(t) = i_h = e^{-t}\left[\frac{-j}{2} e^{jt} + \frac{j}{2} e^{-jt}\right]$$

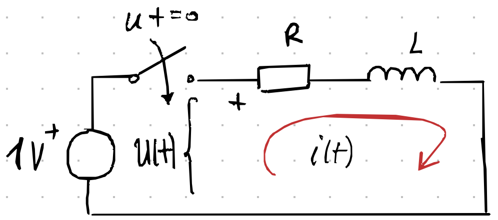
$$\rightarrow \boxed{i(t) = e^{-t} \cdot \sin t \cdot S(t)}$$

$*e^{jt} = \cos t + j\sin t \qquad e^{-jt} = \cos t - j\sin t$



Matlab

Primjer 3.) RL krug



$u(t) = V \sin \omega t$

$i_L(0) = 0$

$i(t) = ?$

$u(t) = R \cdot i(t) + L \cdot \frac{di}{dt}$, $^{*}: \; i = A e^{st}$

$\rightarrow \frac{du}{dt} \neq 0$ jer $u(t) = V \cdot \sin \omega t$

1) Homogen.:

$R \cdot i(t) + L \cdot \frac{di}{dt} = 0 \longrightarrow R \, A e^{st} + L \cdot s A e^{st} = 0$

$R + Ls = 0 \rightarrow S_1 = -\frac{R}{L} \longrightarrow \boxed{i_h = A_1 e^{-\frac{R}{L} t}}$

2) Partikularno rj:

* pretpostavka: ima isti oblik kao i funkcija pobude $i_p = k \cdot \sin(\omega t + \varphi)$

$i_p = k(\sin \omega t \cdot \cos \varphi + \cos \omega t \cdot \sin \varphi) \rightarrow i_p = k_1 \sin \omega t + k_2 \cos \omega t$

$\hookrightarrow$ uvrstimo u $u(t) = R i(t) + L \cdot \frac{di}{dt}$

$V \cdot \sin \omega t = R \cdot (k_1 \sin \omega t + k_2 \cos \omega t) + L(k_1 \omega \cos \omega t - k_2 \omega \sin \omega t)$

$V \sin \omega t = R k_1 \sin \omega t - L k_2 \omega \sin \omega t + R k_2 \cos \omega t + L k_1 \omega \cos \omega t$

$V = R_1 k_1 - L k_2 \cdot \omega$ $\qquad 0 = R k_2 + L k_1 \omega$

$\hookrightarrow k_2 = -\frac{L k_1 \omega}{R}$

$\rightarrow V = R_1 k_1 + L^2 \frac{\omega^2}{R} k_1$

$k_1 = \frac{RV}{R^2 + \omega^2 L^2}$ $\rightarrow$ da bismo dobili amplitudu k, trebamo MODUL sinusne i kosinusne komponente

$k = \sqrt{k_1^2 + k_2^2} \Rightarrow \left\{ k = \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \right\}$

$\varphi = \operatorname{arctg}\left(\frac{Im\{Z\}}{Re\{Z\}}\right) \rightarrow \varphi = \operatorname{arctg} \frac{\omega L}{R} \xrightarrow[\text{za naponom}]{\text{struja kasni}} \left\{ \varphi = -\operatorname{arctg} \frac{\omega L}{R} \right\}$

$\rightarrow i(t) = i_h(t) + i_p(t) \rightarrow \boxed{i(t) = A_1 e^{-\frac{R}{L} t} + \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \cdot \sin(\omega t + \varphi)}$

u trenutku $t = 0$ — $t = 0^-$ $\rightarrow u = \emptyset$ $\hookrightarrow i(0) = 0$ (zadano)

— $t = 0^+$

$i(0^+) = 0 = A_1 + \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \cdot \sin \varphi \rightarrow \boxed{A_1 = \frac{-V}{\sqrt{R^2 + \omega^2 L^2}} \sin \varphi}$

$$\boxed{Rj: \; i(t) = \frac{-V}{\sqrt{R^2 + \omega^2 L^2}} \cdot \sin \varphi + \frac{V}{\sqrt{R^2 + \omega^2 L^2}} \cdot \sin(\omega t + \varphi)}$$