Primjeri i zadaci

25. str.

Primjeri

44.)

$$\int \frac{x^3+x^2-4}{x^2-2x}\,dx$$

$$\begin{array}{l}(x^3+x^2-4):(x^2-2x) = x+3 \\ \underline{-x^3+2x^2+0} \\ 0 \quad +3x^2-4 \\ \quad \underline{-3x^2+6x} \\ \quad 0 \quad +6x-4 \to R(x)\end{array}$$

$\downarrow S_{n-m}(x)$

$$\frac{x^3+x^2-4}{x^2-2x} = (x+3) + \frac{6x-4}{x^2-2x}$$

$$\Rightarrow \int\left(x+3+\frac{6x-4}{x^2-2x}\right)dx \longrightarrow \int\left(x+3+\frac{2}{x}+\frac{4}{x-2}\right)dx$$

$$\frac{6x-4}{(x^2-2x)} = \frac{2(3x-2)}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2}$$

$$2(3x-2) = A(x-2) + Bx$$

$$6x-4 = Ax-2A+Bx$$

$$6x = Ax+Bx \qquad -4 = -2A$$

$$6 = A+B \qquad \boxed{A=2}$$

$$\boxed{B=4}$$

$$= \underbrace{\int x\,dx}_{\frac{1}{2}x^2} + \underbrace{\int 3\,dx}_{3x} + 2\int\frac{dx}{x} + 4\int\frac{dx}{x-2}$$

$$\boxed{= \frac{1}{2}x^2 + 3x + 2\ln|x| + 4\ln|x-2| + C}$$

45.) $\displaystyle\int \frac{x^2+2}{x^3-3x+2}\,dx$ stupanj brojnik < stupanj nazivnik ✓

*naći djelitelje slobodnog člana

$\pm 1, \pm 2 \rightarrow -2$ & 1

-2 $\quad -8+6+2=0$ jedna nultočka

1 $\quad 1-3+2=0$ druga nultočka

uvrstimo jednu nultočku

$$(x^3-3x+2):(x+2) = x^2-2x+1$$

$$-x^3-2x^2$$

$$0-2x^2-3x+2$$

$$2x^2+4x$$

$$0+x+2$$

uvrstimo drugu nultočku u naš izraz

$$x^2-x-x+1$$

$$x(x-1)-(x-1)$$

$$(x-1)(x-1)$$

$$\Rightarrow x^3-3x+2=(x-1)^2(x+2)$$

$$\frac{x^2+2}{(x+2)(x-1)^2} = \frac{A}{x+2} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \Big/ \cdot (x+2)(x-1)^2$$

$$x^2+2 = A(x-1)^2 + B(x+2)(x-1) + C(x+2)$$

$x=1$

$$1+2 = \cancel{A\cdot 0} + \cancel{B\cdot 0} + 3C$$

$$\boxed{C=1}$$

$x=-2$

$$4+2 = A(-3)^2 + B\cdot 0 + C\cdot 0$$

$$6 = A\cdot 9 \,/ :9$$

$$\boxed{A=\frac{2}{3}}$$

$$A(x-1)^2 \rightarrow Ax^2$$

$$B(x+2)(x-1) \Rightarrow Bx^2$$

$$C(x+2) = 0x^2$$

$$A+B=1$$

$$B = 1-\frac{2}{3} \Rightarrow \boxed{B=\frac{1}{3}}$$

$$\Rightarrow \int \frac{x^2+2}{x^3-3x+2}\,dx = \int \left(\frac{\frac{2}{3}}{x+2} + \frac{\frac{1}{3}}{x-1} + \frac{1}{(x-1)^2} \right) dx$$

$$= \frac{2}{3}\int \frac{dx}{x+2} + \frac{1}{3}\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2} = \left| \begin{array}{l} t = x-1 \\ dx = dt \end{array} \right|$$

$$= \frac{2}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + \int \frac{dt}{t^2} \qquad \left(\frac{1}{t^2} \rightsquigarrow \frac{-1}{t} \right.$$

$$-t^{-1} = -1 \cdot (-1) \cdot t^{-1-1} = t^{-2} \text{ ok}$$

$$= \frac{2}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| - \frac{1}{t} + C$$

$$\boxed{= \frac{2}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| - \frac{1}{x-1} + C}$$

$$46.)\ \int \frac{x^3+x-2}{x^3+x^2+x}dx$$ stupnjevi su isti

$$(x^3+x-2):(x^3+x^2+x)=1$$
$$\underline{-x^3-x^2-x}$$
$$0\ -x^2+0-2$$

$$\Rightarrow \frac{x^3+x-2}{x^3+x^2+x}=1+\frac{-x^2-2}{x^3+x^2+x}$$

$$\frac{-x^2-2}{x^3+x^2+x}=\frac{-x^2-2}{x(x^2+x+1)}\Rightarrow \frac{A}{x}+\frac{Bx+C}{x^2+x+1}$$

$$-x^2-2=A(x^2+x+1)+(Bx+C)(x)$$
$$-x^2-2=Ax^2+\cancel{Ax}+A+Bx^2+\cancel{Cx}$$

$$-1=A+B \qquad Ax+Cx=0x$$
$$-2=A \rightarrow \boxed{B=1} \qquad A+C=0$$
$$C=2$$

$$\int\left(1+\frac{-2}{x}+\frac{x+2}{x^2+x+1}\right)dx$$

$$\int dx+\int\frac{-2}{x}dx+\int\frac{x+2}{x^2+x+1}dx=x-2\ln|x|+\int\frac{x+2}{(x+\frac{1}{2})^2+\frac{3}{4}}dx$$
$$\downarrow \qquad \downarrow \qquad 2\cdot\frac{1}{2}$$
$$x \qquad -2\ln|x|$$

$$=\begin{vmatrix} t=x+\frac{1}{2} \\ dx=dt \end{vmatrix}=x-2\ln|x|+\int\frac{t+\frac{3}{2}}{t^2+\frac{3}{4}}dt \rightarrow \int\frac{t}{t^2+\frac{3}{4}}dt+\frac{3}{2}\int\ldots$$

$$=x-2\ln|x|+\frac{3}{2}\cdot\frac{1}{\left(\frac{\sqrt3}{2}\right)}\text{arctg}\frac{t}{\frac{\sqrt3}{2}}+\int\frac{t}{t^2+\frac{3}{4}}dt=\begin{vmatrix} t^2+\frac{3}{4}=u \\ 2t\,dt=du \\ dt=\frac{1}{2}du \end{vmatrix}$$

$$=x-2\ln|x|+\frac{3}{\sqrt3}\text{arctg}\frac{2(x+\frac{1}{2})}{\sqrt3}+\int\frac{1}{u}\cdot\frac{1}{2}du$$

$$=x-2\ln|x|+\frac{3}{\sqrt3}\text{arctg}\frac{2x+1}{\sqrt3}+\frac{1}{2}\ln|u|+C$$

$$=x-2\ln|x|+\frac{3}{\sqrt3}\text{arctg}\frac{2x+1}{\sqrt3}+\frac{1}{2}\ln\left|(x+\tfrac{1}{2})^2+\tfrac{3}{4}\right|+C$$

$$\boxed{=x-2\ln|x|+\frac{3}{\sqrt3}\text{arctg}\frac{2x+1}{\sqrt3}+\frac{1}{2}\ln|x^2+x+1|+C}$$

47) $\int \frac{dx}{(x^2+1)^2} = \int \frac{1}{(x^2+1)^2}\,dx$

$$\frac{1}{(x^2+1)^2} = \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

$1 = (Bx+C)(x^2+1) + Dx + E$

$1 = \underline{Bx^3} + Bx + Cx^2 + C + Dx + E$

$B = 0$ $\quad B + D = 0 \quad C + E = 1$

$C = 0$ $\quad 0 + D = 0 \quad 0 + E = 1$

$D = 0 \quad E = 1$

ostaje isti razlomak

→ nije rastavljiva na parcijalne razlomke

→ dodajemo i oduzimamo odgovarajuće članove u brojniku

$\int u\,dv = uv - \int v\,du$

$$\int \frac{x^2+1-x^2}{(x^2+1)^2}\,dx = \int\left(\cancel{\frac{x^2+1}{(x^2+1)^2}} - \frac{x^2}{(x^2+1)^2}\right)dx = \int \frac{1}{x^2+1}\,dx - \int x\,\frac{x}{(x^2+1)^2} \ast$$

$$= \frac{1}{1}\,\mathrm{arctg}\,\frac{x}{1} - \int x\,\frac{x}{(x^2+1)^2}\,dx = \left| \begin{array}{ll} u = x & dv = \frac{x}{(x^2+1)^2}\,dx \\ du = dx & v = -\frac{1}{2}\cdot\frac{1}{x^2+1} \end{array} \right|$$

$\ast$
$$\int \frac{x}{(x^2+1)^2}\,dx = \left| \begin{array}{l} x^2+1 = t \\ 2x\,\cancel{dx} = dt \\ dx = \frac{1}{2}\,dt \end{array} \right| = \int \frac{1}{t^2}\cdot\frac{1}{2}\,dt = \frac{1}{2}\left(-\frac{1}{t}\right) = \underline{\frac{-1}{2}\cdot\frac{1}{x^2+1}}$$

arctg

$$= \mathrm{arctg}\,x + x\cdot\frac{1}{2}\cdot\frac{1}{x^2+1} + \int\left(-\frac{1}{2}\right)\frac{1}{x^2+1}\,dx$$

$$= \mathrm{arctg}\,x + \frac{x}{2}\cdot\frac{1}{x^2+1} - \frac{1}{2}\,\mathrm{arctg}\,x = \boxed{\frac{1}{2}\,\mathrm{arctg}\,x + \frac{1}{2}\,\frac{x}{x^2+1}}$$