6.2.2. Taylorovi redovi

Podsjetimo se Taylora za f(x) (matan 1)

$$f(x) = T_N(x) + R_N(x)$$

↳ Taylorov polinom n-tog stupnja $\quad T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$

- f-ja će imat Taylorov red ako $R \to 0$ (napadnemo limesom $\lim_{N\to\infty}$)

$$R_N(x) = \frac{f^{(n+1)}(c)}{(N+1)!}(x-x_0)^{N+1}$$

(c je iz okoline x_0)

DEF | Red oblika $\sum_{n=0}^{\infty} \frac{f^{n}(x_0)}{n!}(x-x_0)^n$ zovemo Taylorov red funkcije $f(x)$ oko točke x_0.

NUŽAN I DOVOLJAN UVJET:

Taylor red je jednak $f(x)$ akko $\boxed{\lim_{N\to\infty} R_N(x) = 0}$

*Nap.) Ako je $x_0 = 0$, red se zove Maclaurinov.

Primjer: $f(x) = e^x$ — sve njene deriv. su jednake = beskonačno diferencijabilna

$f^{n}(x) = e^x, \quad x_0 = 0, \quad f^{(n)}(0) = 1$

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

područje konv je $\forall x \in \mathbb{R}$

$$\lim_{n\to\infty} R_N(x) = \lim_{n\to\infty} \frac{e^c}{(n+1)!}x^{n+1} = e^c \lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$$

jer faktorijele brže rastu od polinoma pa i da piše $(\text{milijun})^n$, $n!$ raste brže

Izr. sumu:

a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = e^{-1} = \frac{1}{e}$

b) $\sum_{n=2}^{\infty} \frac{1}{2^n n!} = \underbrace{e^{\frac{1}{2}} - \overset{n=0}{1} - \overset{n=1}{\frac{1}{2}}} = \sqrt{e} - \frac{3}{2}$

$x = \frac{1}{2}$

to vrijedi kada ide od 0

oduzmemo prva dva člana

Zadatak: Proračunaj n-tu derivaciju i uvrstiti $x_0=0$

$f(x)=\sin x \to 0$

$f'(x)=\cos x \to 1$

$f''(x)=-\sin x \to 0$

$f'''(x)=-\cos x \to -1$

$f^{IV}(x)=\sin x \to$

$\vdots$

$\Rightarrow \sin x = 0 + x + 0 - \frac{1}{3!}x^3 + 0 + \frac{1}{5!}x^5 + \dots$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

↳ sinus je neparan

$$\lim_{n\to\infty} R_n(x) = \frac{\sin^{(n+1)}(c)}{(n+1)!} x^{n+1} = 0$$

uvijek $\in [-1,1]$

jer faktorijele brže teže u ∞

$\boxed{\forall x \in \mathbb{R} \text{ tj. } R=\infty}$

→ ovaj red najbrže konvergira od svih funkcija

Zadaci za DZ:

a) $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$

b) $\operatorname{sh} x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

b) $\operatorname{ch} x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$, svi oni konvergiraju $\forall x \in \mathbb{R}$, $R=\infty$

* tg nema lijepe zatvorene formule

Zadatak: $f(x) = \cancel{\ln x}$ → ne postoji McLaurinov razvoj oko 0 jer su deriv od $\ln x = \frac{1}{x}, -\frac{1}{x}$

Zato razvijamo $\ln(1+x)$ i sad možemo oko 0

$f(x) = \ln(1+x)$

$f'(x) = \frac{1}{1+x} \rightarrow f'(0) = 1$ → koeficijenti su alternirajuće fakt!

$$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$$

$f''(x) = \frac{-1}{(1+x)^2} \rightarrow f''(0) = -1$

$f'''(x) = \frac{2}{(1+x)^3} \rightarrow f'''(0) = 2$

$f^{IV}(x) = \frac{-6}{(1+x)^{-4}} \rightarrow f^{IV}(0) = -6$

$f^{V}(x) = \frac{24}{(1+x)^{-5}} \rightarrow f^{V}(0) = 24 \implies f^{(n)}(0) = (-1)^{n+1}(n-1)!$

$$\ln(1+x) = 0 + 1x - \frac{1x^2}{2!} + \frac{2}{3!} + \frac{\cancel{2}}{\cancel{3!}_{3}}x^3 + \dots + \frac{(-1)^{n+1}\,\cancel{(n-1)!}}{\cancel{n!}_{n}}x^n + \dots$$

→ jedan od malo zapisa koji u sebi nemaju fakt! jer se pokrate

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \dots = \boxed{\sum_{n=0}^{\infty} (-1)^{(n+1)} \frac{x^n}{n}}$$

Područje konvergencije:

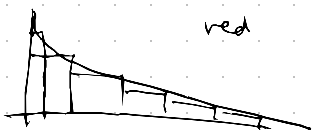
$$\lim_{n\to\infty} R_n(x) = \frac{\frac{(-1)^{n+2}\,\cancel{n!}}{(1+c)^{n+1}}}{\cancel{(n+1)!}_{n+1}} x^{n+1} = \lim_{n\to\infty} \frac{(-1)^n x^{n+1}}{(1+c)^{n+1}(n+1)} = 0$$

x mora < 1, $\left(\frac{1}{2}\right)^n \to 0$; ovo je jednako 0 samo ako je apsolutno od x $(|x|) < 1$,

$\implies \boxed{R = 1}$ $\quad x \in \langle -1, 1 \rangle$

Rubovi $x = -1$:

$\sum_{n=1}^{\infty} -\frac{1}{n}$ div → to je harmonijski red { to je ona priča da $\log_n$ opisuje harmonijski red



$x = 1$

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ po Leibnizu konvergira

ZAKLJUČAK:
$x \in \langle -1, 1]$

Suma $S = \ln 2$ $= 0{,}69$
za $x=1$ $\quad 1+1=2$

$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n$$ Binomni red

Dokazuje se jednako:

$f'(x) = \alpha(1+x)^{\alpha-1}$

$f''(x) = \alpha(\alpha-1)(1+x)^{\alpha-2}$

$f'''(x) = \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}$

$\vdots$

$f^{n}(x) = \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n}$

$\hat{f}^{n}(0) = \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} =$ formula za binomni koeficijent. $\binom{\alpha}{n}$

* $\binom{n}{k}$ - kombinacije bez ponavljanja

▶ $\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n$

$\binom{\frac{1}{2}}{2} = \dfrac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2!} = -\dfrac{1}{8}$ $\qquad \binom{\frac{1}{2}}{3} = \dfrac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{\cancel{3!}\ 2} = \dfrac{1}{16}$

$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$

nama za matan 2 treba samo: $\binom{\alpha}{n} = \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$

▶ $\alpha = -1$ $\qquad (1+x)^{-1} = \dfrac{1}{1+x} = \sum_{n=1}^{\infty} \binom{-1}{n} x^n = \underline{\sum_{n=1}^{\infty} (-1)^n x^n}$

$\binom{-1}{n} = \dfrac{-1(-2)(-3)\cdots(-n)}{n!} = (-1)^n \qquad = 1 - x + x^2 - x^3 + x^4 + x^5 + \cdots$

ako uzmemo: $\dfrac{1}{1-x} = \sum_{n=1}^{\infty} (-1)^n (-x)^n = \sum_{n=1}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$

u zadacima smijemo koristiti područje konv. pomoću poznatih redova

Primjer) Razvij oko $x_0 = 0$

a) $e^{-2x} = \sum \frac{(-2x)^n}{n!} = \sum \frac{(-2)^n}{n!} x^n, \quad \forall x \in \mathbb{R}$

b) $\sin\left(\frac{\pi}{2}x^2\right) = \sum (-1)^n \left(\frac{\pi}{2}\right)^{2n+1} \frac{(x^2)^{2n+1}}{(2n+1)!}$

c) $\frac{1}{2-x^3} = \frac{1}{2} \cdot \frac{1}{1-\frac{x^3}{2}} = \sum_{n=0}^{\infty} \frac{x^{3n}}{2^{n+1}}, \quad \left|\frac{x^3}{2}\right| < 1 \quad R = \sqrt[3]{2}$

$|x| < \sqrt[3]{2}$

d) $\ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right)$

$= \ln 3 + \ln\left(1+\left|\frac{x}{3}\right|\right) = \ln 3 + \underbrace{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{3^n \cdot n}}$

Zad: Razvije e^{3x} do $x_0 = 2$. ako razvijamo ko nekog drugog broja moramo: $e^{3x} = e^{3(\cancel{x-2}+2)}$ ← i moramo dodat taj isti

$*(x_0)$

$\sum a_n (x-2)^n$

$= e^{3(x-2)+6} = e^6 \cdot e^{3(x-2)}$

$$\boxed{= e^6 \sum_{n=0}^{\infty} \frac{3^n (x-2)^n}{n!}}$$

Zadatak WIR-23 b)

$\cos(3x) = \cos(3(x-\pi+\pi))$

$= \cos[3(x-\pi)+3\pi] = \cos(3(x-\pi)) \cdot \underbrace{\cos(3\pi)}_{-1} - \cancel{\sin[3(x-\pi)] \cdot \sin(3\pi)}$

$= -\cos[3(x-\pi)]$

$= -\sum_{n=0}^{\infty} (-1)^n \frac{3^{2n}(x-\pi)^{2n}}{(2n)!}$

21-23-2) $\frac{1}{x^2+2x+8}$

$$= \frac{1}{(x-2)(x+4)} = \frac{A = \frac{1}{6}}{x-2} + \frac{B = \frac{-1}{6}}{x+4}$$

a) $\frac{\frac{1}{6}}{x-2} = \frac{-1}{12} \cdot \frac{1}{1-\frac{x}{2}} = \frac{-1}{2} \sum_{n=0}^{\infty} \frac{x^n}{2^n}$

$\left|\frac{x}{2}\right| < 1 \rightarrow |x| < 2$

$\cap \left|\frac{x}{4}\right| < 1 \rightarrow |x| < 4$

$\frac{\frac{-1}{6}}{x+4} = \frac{-1}{24} \cdot \frac{1}{1+\frac{x}{4}} = \frac{-1}{4} \sum (-1) \frac{x^n}{4^n}$

$\hookrightarrow$ presjek $\Rightarrow |x| < 2$, $R = 2$

b) dio $x_0 = -1 \longrightarrow (x+1)^n$

$$\frac{1}{x+4} = \frac{1}{(x+1)+3} = \frac{1}{3} \frac{1}{1+\frac{x+1}{3}} = \frac{1}{3} \sum (-1)^n \frac{(x+1)^n}{3^n}$$

$$\frac{1}{x-2} = \frac{1}{x+1-3} = \frac{-1}{3} \frac{1}{-1-\frac{x+1}{3}}$$