12.3. IZRAČUNAVANJE DULJINE LUKA RAVNINSKE KRIVULJE

Krivulja ℓ zadana je paramet. jedn. $x = x(t)$, $y = y(t)$, $t \in [a, b]$

- čije $x(t)$ i $y(t)$ imaju neprekinute derivacije na $[a, b]$

$$a = t_0 < t_1 < \dots < t_{n-1} < t_n = b$$

Lagrangeov TSV.; za svaki $i = 1, \dots, n$ postoje ξ_i i η_i takvi da vrijedi:

$$\Delta x_i = \dot{x}(\xi_i)\Delta t_i$$

$$\Delta y_i = \dot{y}(\eta_i)\Delta t_i$$

pri čemu je $\Delta x_i = x(t_i) - x(t_{i-1})$

$\Delta y_i = y(t_i) - y(t_{i-1})$

$\Delta t = t_i - t_{i-1}$

Za približnu duljinu luka dobivamo

$$s \approx \sum_{i=1}^{n} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{n} \sqrt{(\dot{x}(\xi_i))^2 + (\dot{y}(\eta_i))^2} \cdot \Delta t_i$$

prijelazom na limes $(\Delta t \to 0)$ dobivamo:

$$s = \lim_{\Delta t \to 0} \sum_{i=1}^{n} \sqrt{(\dot{x}(\xi_i))^2 + (\dot{y}(\eta_i))^2} \cdot \Delta t_i = \boxed{\int_a^b \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt}$$

diferencijal luka krivulje $ds = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\,dt$

→ ako je krivulja zadana eksplicitnom jednadžbom $y = y(x)$, $x \in [x_A, x_B]$, ulogu parametra t preuzima varijabla x → $x_A \Rightarrow t = a$

$x_B \Rightarrow t = b$

$$ds = \sqrt{1 + y'(x)^2}\,dx$$

Duljinu luka krivulje sada računamo:

$$s = \int_{x_A}^{x_B} \sqrt{1 + y'(x)^2}\,dx$$

Polarne koordinate $r = r(\varphi)$, $\varphi \in [\varphi_A, \varphi_B]$

→ ulogu parametra t preuzima varijabla φ

$$\left.\begin{aligned} x(\varphi) &= r(\varphi)\cos\varphi \\ y(\varphi) &= r(\varphi)\sin\varphi \end{aligned}\right\} \quad \begin{aligned} ds &= \sqrt{x'(\varphi)^2 + y'(\varphi)^2}\,d\varphi \\ ds &= \sqrt{\left[(r(\varphi)\cos\varphi)'\right]^2 + \left[(r(\varphi)\sin\varphi)'\right]^2} \end{aligned}$$

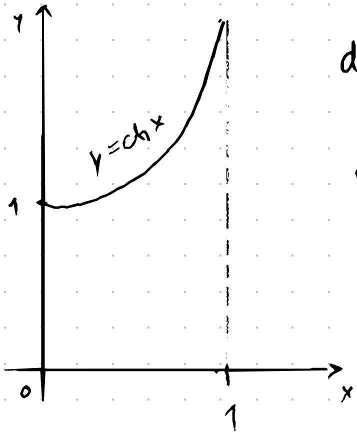
$$ds = \sqrt{r'(\varphi)\cdot\cos\varphi - r(\varphi)\sin\varphi + (r'(\varphi)\sin\varphi + r(\varphi)\cos\varphi)^2}\,d\varphi$$

$$ds = \sqrt{r(\varphi)^2 + r'(\varphi)^2}\,d\varphi$$

Znači, u polarnim koord: $\boxed{s = \int_{\varphi_A}^{\varphi_B} \sqrt{r(\varphi)^2 + r'(\varphi)^2}\,d\varphi}$

Primjer: Duljina luka dijela lančanice

$y = \operatorname{ch} x$, interval $x=0$ do $x=1$



$$ds = \sqrt{1 + (y'(x))^2}\,dx = \underbrace{\sqrt{1 + \operatorname{sh}^2 x}}_{\operatorname{ch} x}\,dx$$

$$ds = \operatorname{ch} x\,dx$$

$$s = \int_0^1 \operatorname{ch} x\,dx = \operatorname{sh} x \Big|_0^1 = \operatorname{sh} 1 = \boxed{\frac{e^2-1}{2e}}$$