$\int_a^{+\infty} f(x)dx$ - nepravi integral jer je gornja granica $+\infty$

$= \lim\limits_{b \to +\infty} \int_a^b f(x)dx$ ako $\int_a^b f(x)dx$ postoji na intervalu $[a,b]$

Newton-Leib. $=$ $\underline{\lim\limits_{b \to +\infty} (F(b) - F(a))}$ pitanje postojanja integrala svodi se na pitanje postoji li horizontalni test

na sličan način:

$\int_{-\infty}^{b} f(x)dx := \lim\limits_{a \to -\infty} \int_a^b f(x)dx \in \mathbb{R}$

ako limes postoji, nepravi integral <u>konvergira</u>

u suprotnom <u>divergira</u>

$\int_{-\infty}^{+\infty} f(x)dx = \lim\limits_{\substack{a \to -\infty \\ b \to +\infty}} \int_a^b f(x)dx \in \mathbb{R}$ — cijeli $\mathbb{R}$ je domena

ako limes vrijedi za $a \to -\infty$ neovisno od b, onda vrijedi i kada uzmemo da je $a = -b$

$\int_{-\infty}^{+\infty} f(x)dx \neq \boxed{\int_{-a}^{a} f(x)dx}$

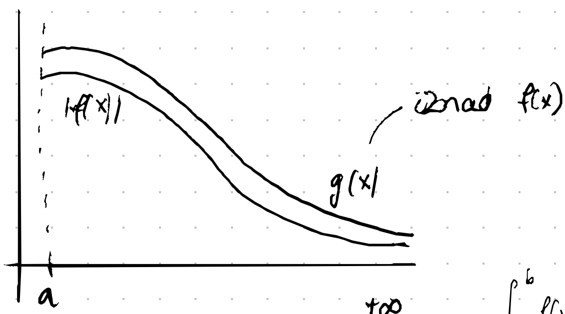
$f(x) = \frac{1}{2\sqrt{\pi}} e^{-\frac{x^2}{2}}$ = $\underbrace{\frac{1}{2\sqrt{\pi}}}_{\text{konstanta}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1$

gustoća vjerojatnosti

$\boxed{\text{TM}}$ Neka su $f, g : [a, +\infty\rangle \to \mathbb{R}$

1.) Ako je $|f(x)| \leq g(x)$, $\forall x \in [a, +\infty\rangle$ i ako $\int_a^{+\infty} g(x)$ konvergira onda i konvergira i $\int_a^{+\infty} f(x)dx$.

2.) Ako je $g(x) \geq f(x) \geq 0$ $\forall x \in [a, +\infty\rangle$ i ako $\int_a^{+\infty} f(x)dx$ divergira onda i integral $\int_a^{+\infty} g(x)dx$ divergira.



Dokaz: 1.)

$b \in [a, +\infty\rangle$ proizvoljan

$$\int_a^b f(x)dx \leq \left| \int_a^b f(x)dx \right|$$

*bilo koji br je $\leq$ svojoj aps. vrijed.

$$\int_a^b f(x)dx \leq \left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$$

*primjenjujemo $|x+y| \leq |x| + |y|$

$$\left| \sum x_i \right| \leq \sum |x_i|$$

$$\int_a^b f(x)dx \leq \left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx \leq \underbrace{\int_a^b g(x)dx}_{\text{pretpostavka teorema}} \leq \int_a^{+\infty} g(x)dx$$

$< +\infty$ po pretpostavci

$\hookrightarrow$ pretpostavka do one konvergira

Primjer: Ispitajte konvergenciju integrala:

$$\int_0^{+\infty} \frac{\cos x}{1+x^2}\,dx$$

po apsolutnoj vrijednosti dominira na $g(x)$

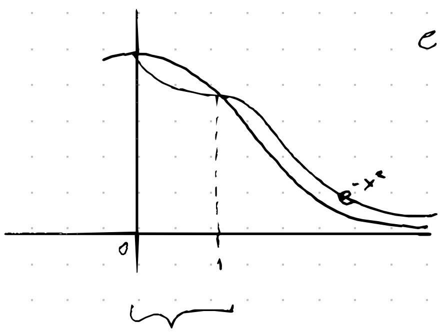
$$\cdot\ |\cos x| \le 1 \Rightarrow \overbrace{\left|\frac{\cos x}{1+x^2}\right|}^{f(x)} \le \overbrace{\frac{1}{1+x^2}}^{g(x)} \qquad \forall x \in [0,+\infty\rangle$$

otprije znamo da je $\int_0^{+\infty} \frac{dx}{1+x^2}$ konvergentno

$$\int_0^{+\infty} \frac{\cos x}{1+x^2}\,dx = \lim_{b\to+\infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to+\infty} (\operatorname{arctg} x)\Big|_0^b = \frac{\pi}{2}$$

po usporednom kriteriju slijedi da $\Rightarrow \int_0^{+\infty} \frac{\cos x}{1+x^2}\,dx$ konvergira

Primjer: $\int_0^{+\infty} e^{-x^2}\,dx \rightarrow x \ge 0$ nas zanima



$$(1)\ /\ x \le x^2, \quad \forall x \ge 1$$

$$e\ /\ -x \ge -x^2, \quad \forall x \ge 1$$

$$e^{-x} \ge e^{-x^2} \quad \forall x \ge 1$$

dominirana o e^{-x} za $x \ge 1$

$$\int_0^{+\infty} e^{-x}\,dx = \lim_{b\to+\infty} \int_0^b e^{-x} = \lim_{b\to+\infty} (e^{-b}+1)$$

$$\int_0^{+\infty} e^{-x}\,dx = 1$$

što se događa na početnom intervalu nebitno je za pitanje konvergencije

$$\underbrace{\int_0^{+\infty} e^{-x^2}\,dx}_{\text{konvergira}} \overset{\text{akko}}{\Longleftrightarrow} \int_0^1 e^{-x^2}\,dx + \underbrace{\int_1^{+\infty} e^{-x^2}\,dx}_{\text{ovaj konvergira}} \le \int_1^{+\infty} e^{-x}\,dx < +\infty$$

po usporednom kriteriju konvergira

ovo ne možemo primijeniti za funkcije koje bi oko 0 imale vertikalnu asimptotu

TM Neka su $f, g : [a, +\infty\rangle \to \mathbb{R}$

Ako su $f(x), g(x) \geq 0$ $\forall x \in [a, +\infty\rangle$ i $\lim\limits_{x \to +\infty} \frac{f(x)}{g(x)} = L \neq 0$, $L \in \mathbb{R}$

Onda integrali $\int_a^{+\infty} f(x)dx$ i $\int_a^{+\infty} g(x)dx$ imaju isti tip konvergencije

$\Rightarrow$ ili da konvergiraju ili da divergiraju

Pr :) $\int_1^{+\infty} \frac{dx}{\sqrt{x^3+1}}$; $f(x) = \frac{1}{\sqrt{x^3+1}}$; $g(x) = \frac{1}{\sqrt{x^3}} = \frac{1}{x^{3/2}}$

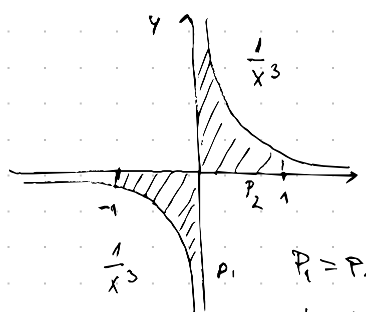
↳ odredimo neki $g(x)$ za koji znamo već konv.

$$\lim_{x \to +\infty} \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{\frac{1}{\sqrt{x^3+1}}}{\frac{1}{\sqrt{x^3}}} = 1 \neq 0 \checkmark$$

$$f(x) \sim g(x) \text{ za } x \to +\infty \longrightarrow \int_1^{+\infty} f(x)dx \sim \int_1^{+\infty} g(x)dx = \int_1^{+\infty} \frac{dx}{x^{3/2}}$$

$\frac{dx}{x^{3/2}}$ konvergira! $\Longleftarrow$ $\frac{1}{x^p}$ je konv. kada je $p > 1$

Pr: $\int_{-1}^{1} \frac{dx}{x^3}$



$P_1 = P_2$ pokrate se

$$\int \frac{dx}{x^3} = -\frac{1}{2} \cdot \frac{1}{x^2} + C$$

$$\int_{-1}^{1} \frac{dx}{x^3} \underset{N-L}{=} -\frac{1}{2x^2}\Big|_{-1}^{1} = -\frac{1}{2} + \frac{1}{2}$$

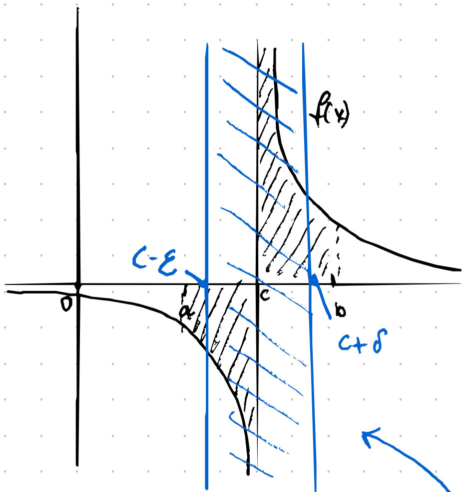
ali ovo nije točno

jer $\int \frac{dx}{x^3} \neq -\frac{1}{2x^2} + C$ jer je to samo na nekom intervalu $I \subset \mathcal{D}(\frac{1}{x^3}) = \mathbb{R} \setminus \{0\}$

- ne možemo primijeniti N-L formulu na $[a,b]$ koji sadrži 0
- $\frac{1}{x^3}$ nije omeđena na $\langle -1, 1\rangle$, $x = 0$ je V.A.

Neka je $f : [a,b] \setminus \{c\} \to \mathbb{R}$ $\quad (c \in [a,b])$ t.d. $\lim\limits_{x \to c} f(x) = \pm\infty$

tj. $x = c$ je V.A. f-je $f(x)$.



$$\int_a^b f(x)\,dx$$

na $[a, c\rangle$ f-ja je neomeđena

=> da bismo osigurali omeđenost

$[a, c-\varepsilon]$, $\varepsilon > 0$

updejt → $\int_a^b f(x)\,dx + \boxed{?} \int_{c+\delta}^b f(x)\,dx$

Neka je f integrabilna na $[a, c-\varepsilon]$, $\forall \varepsilon > 0$

na $[c+\delta, b]$, $\forall \delta > 0$

$$\to \int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{c+\delta}^b f(x)\,dx$$

nisu u nikakvoj vezi

→ ova dva limesa međusobno ne ovise ∄

(ε, δ ne ovise jedan o drugom)

ako postoji limes 1. integrala

i ako postoji limes 2. integrala } njihov zbroj definira konvergentan integral

↳ ako jedan limes ne postoji

taj integral divergira

Pr.: $$\int_{-1}^{1}\frac{dx}{x^3} = \lim_{\varepsilon\to 0^+}\underbrace{\int_{-1}^{-\varepsilon}\frac{dx}{x^3}}_{\text{za bilo koji } \varepsilon>0, \text{ ne sadrži } 0} + \lim_{\delta\to 0^+}\int_{\delta}^{1}\frac{dx}{x^3} =$$

$\Big\{$ sad možemo primijeniti N-L na $[-1,-\varepsilon]$ i $[\delta,1]$

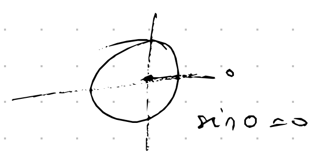
$$= \lim_{\varepsilon\to 0^+} -\frac{1}{2x^2}\Big|_{-1}^{-\varepsilon} + \lim_{\delta\to 0^+} -\frac{1}{2x^2}\Big|_{\delta}^{1}$$

$$= -\lim_{\varepsilon\to 0^+}\left(\frac{1}{2(-\varepsilon)^2} - \frac{1}{2\cdot 1}\right) - \lim_{\delta\to 0^+}\left(\frac{1}{2} - \frac{1}{2\cdot\delta^2}\right)$$

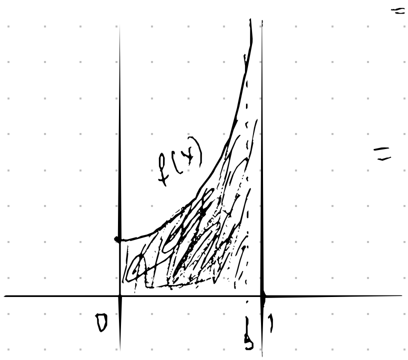
$$= -\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon^2} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \lim_{\delta\to 0^+}\frac{1}{2\delta^2}$$

$$= \underbrace{-\lim_{\varepsilon\to 0^+}\frac{1}{2\varepsilon^2}}_{-\infty} + \underbrace{\lim_{\delta\to 0^+}\frac{1}{2\delta^2}}_{+\infty}$$

divergira



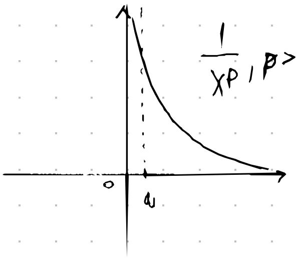
Pr.: $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ $x=1$ je V.A.

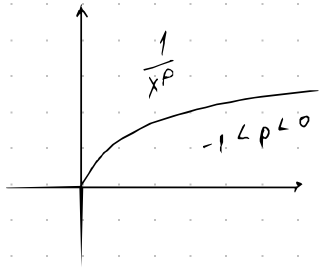


$$= \lim_{b\to 1^-}\int_0^b \frac{dx}{\sqrt{1-x^2}} = \lim_{b\to 1^-}(\arcsin x)\Big|_0^b$$

$$= \lim_{b\to 1^-}\arcsin b = \frac{\pi}{2}$$

Primjer: $\int_0^b \frac{dx}{x^p}$ (b > 0 zadan broj)





$$\int_0^b \frac{dx}{x^p} = \lim_{a\to 0^+} \int_a^b \frac{dx}{x^p} \overset{p\neq 1}{=} \lim_{a\to 0^+} \left.\frac{x^{-p+1}}{-p+1}\right|_a^b = \lim_{a\to 0^+}\left(\frac{b^{1-p}}{1-p} - \frac{a^{1-p}}{1-p}\right)$$

$$= \frac{b^{1-p}}{1-p} - \lim_{a\to 0^+}\frac{a^{1-p}}{1-p} = \begin{cases} +\infty, & p>1 \\ 0, & p<1 \end{cases}$$

$\ln|0| =$

$p=1$

$$\int_0^b \frac{dx}{x} = \lim_{a\to 0^+} \ln|x|\Big|_a^b = \lim_{a\to 0^+}\left(\ln|b| - \ln|a|\right) = \ln|b| - \underbrace{\lim_{a\to 0^+} \ln|a|}_{-\infty}$$

$= +\infty$

Neka je $\int_a^b f(x)dx$ nepravi integral neomeđene fje $f(x)$ oko točke $c \in [a,b]$.

Ako je $\lim_{x\to c} \frac{f(x)}{g(x)} = L \neq 0$ onda $\int_a^b f(x)dx$ i $\int_a^b g(x)dx$ imaju „istu konvergenciju"

DZ: ispitajte konv. integ $\int_0^1 \frac{\ln(1+x)}{\sqrt{x^5}}dx$