ZIR202021 - rješenja

1. (a)
$$0 = (z+i)^2 + 1 = (z+i)^2 - i^2 = (z+i+i)(z+i-i) = z(z+2i)$$
$\Rightarrow$ Skup rješenja je $\{0, -2i\}$.

   (b)
$$\arg(1+i\sqrt{3}) = \frac{\pi}{3}$$
$$\Rightarrow \arg(z^4(1+i\sqrt{3})) = 4\arg z + \frac{\pi}{3} + 2k\pi$$
$$\arg(\pi z^2) = \arg(z^2) = 2\arg z + 2k\pi$$
$$\Rightarrow 4\arg z + \frac{\pi}{3} + 2k\pi = 2\arg z$$
$$\Rightarrow \arg z = -\frac{\pi}{6} - k\pi$$
$$\Rightarrow \arg z \in \left\{\frac{5\pi}{6}, \frac{11\pi}{6}\right\}$$

2. (a) Definicija iz skripte.

   (b)(b1) DA. $a_n := (-1)^n$

   (b2) DA. $a_n := (-1)^n \frac{1}{n}$

   (b3) DA. $a_n := \sin \frac{2n\pi}{3}$

   (c) (c1)
$$-\frac{1}{n} \leq \frac{\sin(3n^2)}{n} \leq \frac{1}{n}$$
$$\lim_{n\to\infty} -\frac{1}{n} = 0, \lim_{n\to\infty} \frac{1}{n} = 0$$
Po Teoremu o sendviču dobivamo:
$$\lim_{n\to\infty} \frac{\sin(3n^2)}{n} = 0$$

   (c2)
$$\lim_{n\to\infty} \left(\frac{n+1}{n+2}\right)^{2n+3} = \lim_{n\to\infty} \left(\left(1 - \frac{1}{n+2}\right)^{n+2}\right)^{\frac{2n+3}{n+2}} =$$
$$= \left(e^{-1}\right)^2 = e^{-2}$$

3. (a) Neprekidnost nužno daje:

$$0 = f(1) = \lim_{x\to 1-} f(x) = \lim_{x\to 1-} ax^2 + bx + b = a + 2b$$

Diferencijabilnost nužno daje:

$$f'(1) = f'(1+) = \ln'(1) = 1$$

$$1 = f'(1) = f'(1-) = (ax^2 + bx + b)' \mid_{x=1} = (2ax + b) \mid_{x=1} = 2a + b$$

$$\Rightarrow a = \frac{2}{3}, b = -\frac{1}{3}$$

(b)

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} =$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = (\text{za } x > 0) = \frac{1}{2\sqrt{x}}$$

(c)

$$g(x) = \sqrt{x} \Rightarrow g'(x) = \frac{1}{2\sqrt{x}}$$

$$h(x) = e^{-x} \Rightarrow h'(x) = -e^{-x}$$

$$d(x) = \sin(2x) \Rightarrow d'(x) = 2\cos(2x)$$

$$\Rightarrow f = g \circ h \circ d$$

$$\Rightarrow f'(0) = g'(h(d(0)))h'(d(0))d'(0) = g'(1)h'(0)d'(0) = \frac{1}{2}\cdot(-1)\cdot 2 = -1$$

4. Tvrdnja **(T2)** je uvijek istinita, odnosno to je tvrdnja **Rolle-ovog teorema**. Dokaz u skripti.
Protuprimjer za tvrdnju **(T1)**:

$$a = -1, b = 1, f(x) := x^2$$

Protuprimjer za tvrdnju **(T3)**:

$$a = -1, b = 1, f(x) := x^2 + 1$$

5. Primijetimo da su obje tangente simetrične s obzirom na $y-$os. Također, primijetimo da je problem isti ako smo krenuli od $x_0 \in (0, +\infty)$, odnosno $x-$koordinate točke dirališta tangente u desnoj poluravnini. Označimo li u tom slučaju s y^* i x^* odsječke na koordinatnim osima, vidimo da je ukupna površina jednaka x^*y^*. Izrazimo y^* i x^* pomoću x_0:
Jednadžba tangente dana je s:

$$y - y_0 = f'(x_0)(x - x_0), \text{ gdje je } f(x) = 1 - x^2 \text{i } y_0 = f(x_0) = 1 - x_0^2$$

$$\Rightarrow y^* - y_0 = -2x_0(0 - x_0) \Rightarrow y^* = 1 + x_0^2$$

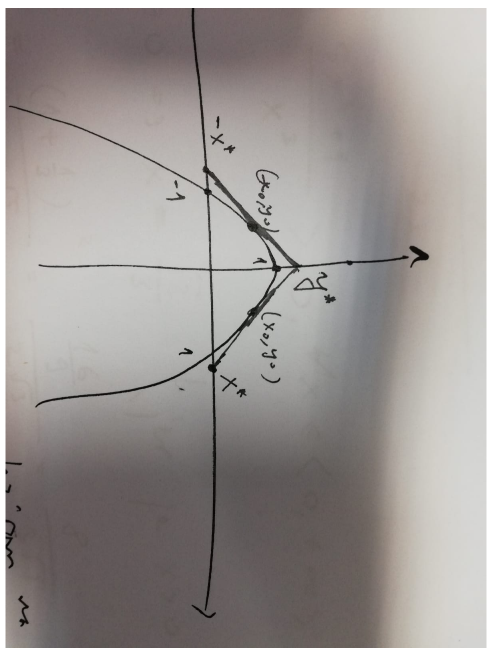
$$\Rightarrow 0 - y_0 = -2x_0(x^* - x_0) \Rightarrow x^* = \frac{1 + x_0^2}{2x_0}$$

Površina u ovisnosti o x_0 je dana s $P(x_0) = \frac{(1+x_0^2)^2}{2x_0}$.

$$\Rightarrow P'(x) = \frac{(3x^2 - 1)(x^2 + 1)}{2x^2}, P''(x) = \frac{3x^4 + 1}{x^3}$$

Nužan uvjet za minimum ($P'(x) = 0$) daje $x_0 = \frac{1}{\sqrt{3}}$, a zbog $P''(x) > 0, \forall x \in (0, +\infty)$ znamo da je onda ta stacionarna točka doista minimum. Konačno, tražena minimalna površina iznosi:

$$P\left(\frac{1}{\sqrt{3}}\right) = \frac{8}{3\sqrt{3}}$$



Slika 1: Zadatak 5.

6. (a) Dokaz iz skripte.
   (b) Po teoremu iz (a) imamo

$$\exists c \in (1, 3), \int_1^3 \sin(x)dx = \sin(c)(3 - 1)$$

$$\Rightarrow \left| \int_1^3 \sin(x)dx \right| = 2\left|\sin(c)\right| \leq 2 \cdot 1 = 2$$

7. (a) Iskaz iz skripte.

   (b)

$$\int \frac{dx}{e^x+1} = \left[t = e^x, dx = \frac{dt}{t} \right] = \int \frac{1}{t(t+1)}dt = \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt =$$

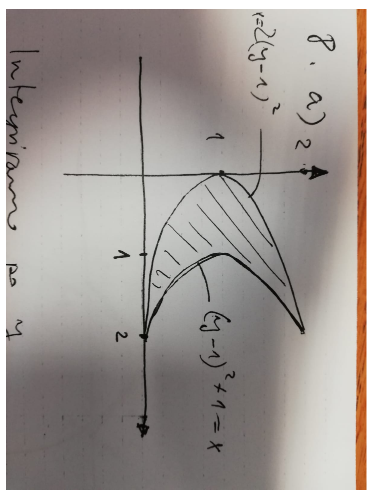
$$= \ln(|t|) - \ln(|t+1|) + C = x - \ln(e^x+1) + C$$

   (c)

$$\int_0^{\frac{\pi}{2}} e^{\sin(x)} \sin(2x)dx = 2\int_0^{\frac{\pi}{2}} e^{\sin(x)} \sin(x)\cos(x)dx = \left[\begin{smallmatrix} y=\sin(x) \\ dy=\cos(x)dx \end{smallmatrix}\right] =$$

$$2\int_0^1 ye^y dy = \left[\begin{smallmatrix} u=y, du=dy \\ dv=e^y dy, v=e^y \end{smallmatrix}\right] = 2e^y\Big|_0^1 - 2\int_0^1 e^y dy = 2e^y(y-1)\Big|_0^1 = 2$$

8. (a) Integriramo po y površinu između grafova funkcija $x(y) = 2(y-1)^2$ i $x(y) = (y-1)^2 + 1$.



Slika 2: Zadatak 8.

$$P = \int_0^2 \left(\left[(y-1)^2 + 1 \right] - 2(y-1)^2 \right) dy = \int_0^2 \left(1 - (y-1)^2 \right) dy =$$

$$= \int_0^2 \left(2y - y^2 \right) dy = \left(y^2 - \frac{1}{3}y^3 \right)\Big|_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

(b) Rotiramo oko $y-$osi pa volumen računamo po formuli:

$$V = 2\pi \int_0^{\frac{\pi}{2}} x f(x) dx = 2\pi \int_0^{\frac{\pi}{2}} x \sin(x) dx = \left[\substack{u=x, du=dx \\ dv=\sin(x)dx, v=-\cos(x)} \right] =$$

$$= -2\pi x \cos(x) \Big|_0^{\frac{\pi}{2}} + 2\pi \int_0^{\frac{\pi}{2}} \cos(x) dx = 0 + 2\pi \sin(x) \Big|_0^{\frac{\pi}{2}} = 2\pi$$