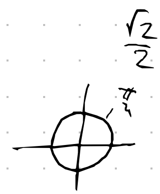
10.3. METODE INTEGRIRANJA U ODREĐENOM INTEGRALU

- koristimo iste tehnike kao za neodređeni
- koristimo istu tablicu integrala, ali bez konstante C

↳ u odgovarajućem primitivnom obliku uvrštavat ćemo granice integracije u skladu s Newton-Leibniz formulom

Primjer 10.15.)

$$\int_0^{\frac{\pi}{4}} \left(2\sin x + \frac{3}{\cos^2 x}\right)dx = \int_0^{\frac{\pi}{4}} (2\sin x)dx + \int_0^{\frac{\pi}{4}} \left(\frac{3}{\cos^2 x}\right)dx$$



$$= 2\int_0^{\frac{\pi}{4}} \sin x\,dx + 3\int_0^{\pi/4} \frac{1}{\cos^2 x}\,dx$$

$$= 2\left(-\cos x \Big|_0^{\frac{\pi}{4}}\right) + 3\left(\operatorname{tg} x \Big|_0^{\pi/4}\right) = 2\left(-\cos\left(\frac{\pi}{4}\right) + \cos(0)\right) + 3\left(\operatorname{tg}\left(\frac{\pi}{4}\right) - \operatorname{tg}(0)\right)$$

$$= 2\left(-\frac{\sqrt{2}}{2} + 1\right) + 3(1-0) = -\sqrt{2} + 2 + 3 = \boxed{5-\sqrt{2}}$$

$\boxed{\text{TM}}$ Metoda supstitucije u određenom integralu

Neka je $f : [\alpha, \beta] \to \mathbb{R}$ neprekinuta,

a $\varphi : [a, b] \to [\alpha, \beta]$ neprekinuto diferencijabilna

i $\varphi([a,b]) \subseteq [\alpha, \beta]$. Tada iz supstitucije $t = \varphi(x)$ vrijedi

$$\boxed{\int_a^b f(\varphi(x))\,\varphi'(x)\,dx = \int_{\varphi(a)}^{\varphi(b)} f(t)\,dt}$$

DOKAZ:

F je prim fija od f

$$\frac{d}{dx} F(\varphi(x)) = F'(\varphi(x)) \cdot \varphi'(x) = f(\varphi(x)) \cdot \varphi'(x)$$

$\longrightarrow$ $F(\varphi(x))$ je primitivna fija od $f(\varphi(x)) \cdot \varphi'(x)$

$$\Rightarrow \int_a^b f(\varphi(x)) \cdot \varphi'(x)\,dx = \int_a^b F(\varphi(x)) = F(\varphi(x))\Big|_a^b$$

$$= \underbrace{F(\varphi(b))}_{A} - \underbrace{F(\varphi(a))}_{B} \overset{*}{=} \int_A^B f(x)\,dx^{*}$$

$$= \int_{\varphi(b)}^{\varphi(a)} f(t)\,dt$$

Primjer (10.17.)

$$\int_0^1 \frac{x\,dx}{\sqrt{4-x^4}} = \left| \begin{array}{l} x^2 = t \\ 2x\,dx = dt \\ \begin{array}{c|c|c} x & 0 & 1 \\ \hline t & 0 & 1 \end{array} \end{array} \right| = \int_0^1 \frac{1}{2} \cdot \frac{dt}{\sqrt{4-t^2}} = \frac{1}{2} \int_0^1 \frac{dt}{\sqrt{4-t^2}}$$

$$= \frac{1}{2} \left(\arcsin \frac{t^2}{2} \Big|_0^1 \right) = \frac{1}{2} \left(\arcsin \frac{1}{2} - \arcsin 0 \begin{array}{l} 0 \\ \pi \end{array} \right)$$

$$= \frac{1}{2} \arcsin \frac{1}{2} = \frac{1}{2} \cdot \frac{\pi}{6} = \frac{\pi}{12}$$

Primjer 19.)

$$\int_1^2 x^3 \sqrt{5-x^2}\,dx = \left| \begin{array}{ll} t = 5-x^2 & \longrightarrow x^2 = 5-t \\ dt = -2x\,dx & \begin{array}{c|c|c} x & 1 & 2 \\ \hline t & 4 & 1 \end{array} \\ -\frac{dt}{2} = x\,dx & \end{array} \right|$$

$$\int_4^1 \sqrt{t} \cdot (5-t) \cdot \left(-\frac{1}{2}\right) dt = \frac{-1}{2} \int_4^1 (5-t)\sqrt{t}\,dt = \frac{1}{2} \int_1^4 (5-t)\sqrt{t}\,dt$$

$$= \frac{1}{2} \int_1^4 (5^{1/1} - t^{1/1}) t^{1/2}\,dt = \frac{1}{2} \int_1^4 \left(5t^{1/2} - t^{3/2}\right) dt \qquad \begin{array}{l} t^{1/2 + 2/2} = t^{3/2} \\ t^{3/2 + \frac{2}{2}} = t^{5/2} \end{array}$$

$$= \frac{1}{2} \left(\frac{2}{3} \cdot 5 \cdot t^{3/2} - \frac{2}{5} \cdot t^{5/2} \right) \Bigg|_1^4 = \frac{1}{\cancel{2}} \cdot \cancel{2} \left(\frac{5 \cdot t^{3/2}}{3} - \frac{t^{5/2}}{5} \right) \Bigg|_1^4$$

$$= -\left(\frac{5}{3} \cdot 1 - \frac{1}{5} \right) + \left(\frac{5}{3} \sqrt{4^3} - \frac{1}{5} \sqrt{4^5} \right) = -\frac{22}{15} + \left(\frac{40}{3} - \frac{32}{5} \right) = \boxed{\frac{82}{15}}$$

2^3

$\frac{104}{15}$

$\boxed{\text{TM}}$ Metoda parcijalne integracije za određeni integral

Neka su f i g neprekinuto diferencijabilne funkcije na [a,b].

$$\boxed{\int_a^b f(x)g'(x)\,dx = f(x)\,g(x)\Big|_a^b - \int_a^b f'(x)\,g(x)\,dx}$$

Primjer 25.)

$$\int_0^1 x e^{-x}\,dx = \left| \begin{array}{l} u = x \longrightarrow du = dx \\ dv = e^{-x}dx \longrightarrow v = -e^{-x} \end{array} \right|$$

$$= x\cdot(-e^{-x})\Big|_0^1 - \int_0^1 (-e^{-x})\,dx = -xe^{-x}\Big|_0^1 + \int_0^1 e^{-x}\,dx$$

$$= (-1\cdot e^{-1} + 0) + (-e^{-x})\Big|_0^1 = -e^{-1} - (e^{-1} + 1) = -e^{-1} - e^{-1} + 1 = \boxed{1 - \frac{2}{e}}$$

$x^{-2} = -x^{-1}$

Primjer 26.)

$$\int_1^e \frac{\ln|x|}{x^2}\,dx = \int_1^e \ln|x|\cdot\frac{1}{x^2}\,dx = \left| \begin{array}{l} u = \ln|x| \longrightarrow du = \frac{1}{x}dx \\ dv = \frac{1}{x^2} \longrightarrow v = -\frac{1}{x} \end{array} \right|$$

$$= -\ln|x|\cdot\frac{1}{x}\Big|_1^e - \int_1^e \left(-\frac{1}{x}\right)\cdot\frac{1}{x}\,du = -\ln|x|\cdot\frac{1}{x}\Big|_1^e + \int_1^e \frac{1}{x^2}\,du$$

$$= -\ln|x|\cdot\frac{1}{x}\Big|_1^e - \frac{1}{x}\Big|_1^e = \Big(-\underbrace{\ln(e)}_{1}\cdot\frac{1}{e} + \underbrace{\ln(1)}_{0}\cdot 1\Big) - \left(\frac{1}{e} - 1\right)$$

$$= -\frac{1}{e} + 0 - \frac{1}{e} + 1$$

$$\boxed{= 1 - \frac{2}{e}}$$