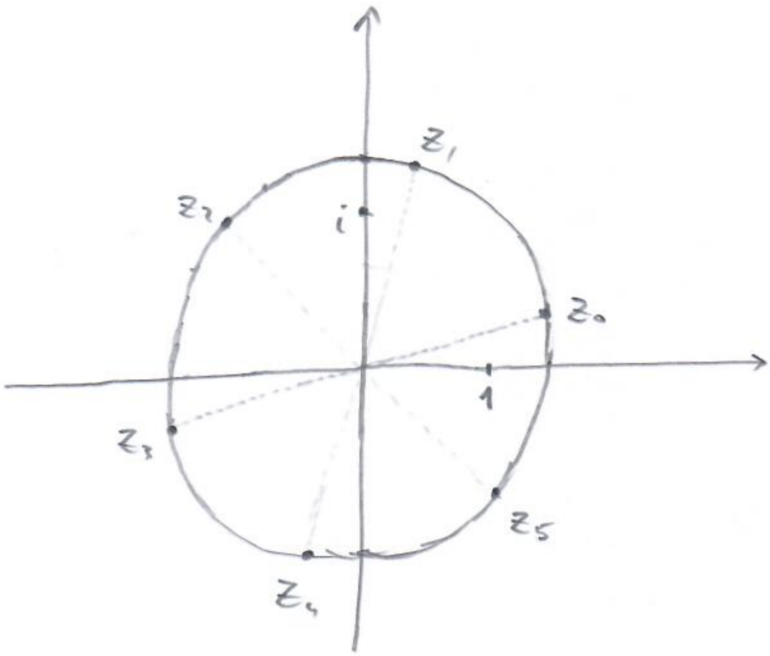
MATAN1 - JIR (4.9.2019.)

1. a) $z^6 = (2+2i)^2 = 4+8i-4 = 8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$$z_i = \sqrt[6]{8}\left(\cos\left(\frac{\frac{\pi}{2}+2k\pi}{6}\right) + i\sin\left(\frac{\frac{\pi}{2}+2k\pi}{6}\right)\right) \quad i=0,1,2,3,4,5$$

$$z_i = \sqrt{2}\left(\cos\left(\frac{\pi}{12}+\frac{k\pi}{3}\right) + i\sin\left(\frac{\pi}{12}+\frac{k\pi}{3}\right)\right) \quad i=0,1,2,3,4,5$$



b) $\begin{cases} |z+3i| = 3|z| \\ \arg z = \frac{\pi}{4} \end{cases}$

$z = x+iy$

$\rightarrow \arg z = \frac{\pi}{4} \Rightarrow x = y > 0$

$\rightarrow |z+3i| = 3|z| \Rightarrow \sqrt{x^2+(y+3)^2} = 3\sqrt{x^2+y^2} \;/^2$

$x^2+y^2+6y+9 = 9x^2+9y^2$

$18x^2 - 2x^2 - 6x - 9 = 0$

$16x^2 - 6x - 9 = 0$

$x_{1,2} = \frac{6 \pm \sqrt{36+4\cdot 9\cdot 16}}{32} = \frac{3 \pm 3\sqrt{17}}{16}$

$\cancel{x_1 = \frac{3-3\sqrt{17}}{16} < 0} \quad x_2 = \frac{3+3\sqrt{17}}{16}$

$\Downarrow$

Jedino rješenje:

$z = \frac{3+3\sqrt{17}}{16} + \frac{3+3\sqrt{17}}{16}i$

(2) $A = \{a_1, a_2, \dots, a_k\}$ $\quad k \le n$

$B = \{b_1, b_2, \dots, b_n\}$

a) $|B^A| = |B|^{|A|} = n^k$

Svakom elementu iz A pridružujemo bilo koji od n elemenata iz B.

b) $f: A \to B$ je injekcija ako

$x_1, x_2 \in A,\ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$\#$ inj. s A u B $= n(n-1)(n-2)\cdots(n-k+1)$

c) $\#$ bij. s A u A $= k!$

$\#$ bij. s A u A uz $f(a_1) = a_k$ & $f(a_k) = a_1$ $= (k-2)!$

d) (i) 7^5

(ii) $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3$

3. a) Funkcija f je neprekinuta slijeva u tački $x=0$ ako vrijedi

$$\lim_{x\to 0^-} f(x) = f(0)$$

$$f(0) = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(1+\frac{\sin(ax)}{x}\right) = 1 + a\lim_{x\to 0^-}\frac{\sin(ax)}{ax}$$

$$= 1+a$$

b) Funkcija f je neprekinuta u tački $x=0$ ako vrijedi

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{e^{bx}-1}{x} = b\lim_{x\to 0^+}\frac{e^{bx}-1}{bx} = b$$

$\Rightarrow$ da bi f bila nepr. u $x=0$ mora vrijediti

$$1+a = b \quad \& \quad f(0) = 1+a = b$$

c) Funkcija f ima prekid u $x=0$ ako vrijedi

$$1+a \neq b$$

ili

$$1+a = b, \text{ ali } f(0) \neq 1+a = b$$

d) $a=0 \Rightarrow f(x)=\begin{cases} 1 & , x<0 \\ \dfrac{e^{bx}-1}{x} & , x>0 \end{cases}$

Funkcija f je diferencijabilna u točki $x=0$ ako je neprekidna u $x=0$ i ako vrijedi $f'(0^-)=f'(0^+)$.

$$f'(x)=\begin{cases} 0 & , x<0 \\ \dfrac{be^{bx}\cdot x-e^{bx}+1}{x^2}=\dfrac{e^{bx}(bx-1)+1}{x^2} & , x>0 \end{cases}$$

$f'(0^-)=0$

$$f'(0^+)=\lim_{x\to 0^+}\frac{e^{bx}(bx-1)+1}{x^2}=\lim_{x\to 0^+}\frac{bxe^{bx}-e^{bx}+1}{x^2}\overset{LH}{=}$$

$$=\lim_{x\to 0^+}\frac{\cancel{be^{bx}}+b^2xe^{bx}-\cancel{be^{bx}}}{2x}=\frac{b^2}{2}$$

$\left.\begin{array}{l} \cdot\ \text{neprekinutost} \Rightarrow f(0)=1+a=b \overset{a=0}{\Longrightarrow} b=1 \\ \cdot\ f'(0^-)=f'(0^+) \Rightarrow 0=\dfrac{b^2}{2} \Rightarrow b=0 \end{array}\right\} \lightning$

$\Rightarrow$ ne postoji $b\in\mathbb{R}$ t.d. je f diferencijabilna u $x=0$.

(4) a) $v(t) = \lim_{\Delta t \to 0} \frac{s(t+\Delta t) - s(t)}{\Delta t}$

$$a(t) = \lim_{\Delta t \to 0} \frac{v(t+\Delta t) - v(t)}{\Delta t}$$

b) $s(t) = t^3 + at^2 + bt$

$s(1) = 5 \Rightarrow 1 + a + b = 5 \quad \Rightarrow \quad a + b = 4$

$s'(t) = 3t^2 + 2at + b$

$s'(1) = 0 \Rightarrow 3 + 2a + b = 0 \quad \Rightarrow \quad 2a + b = -3$

$$\begin{cases} a + b = 4 \\ 2a + b = -3 \end{cases} \Bigg] -$$

$-a = 7 \Rightarrow a = -7$

$-7 + b = 4 \Rightarrow b = 11$

$s''(t) = 6t + 2a = 6t - 14$

$s''(1) = 6 - 14 = -8 \ cm/s^2$

5. a) ISTINA (Tm 9.2.1 - skripta)

b) ISTINA

Proizvoljna primitivna funkcija od f na I je oblika

$F(x)+C, C\in\mathbb{R}$ gdje je $F'(x)=f(x), \forall x\in I$.

Budući da je $f(a)=0$ imamo

$F'(a)=f(a)=0 \Rightarrow a$ je stacionarna točka funkcije F

Budući da je f dfb. i strogo rastuća imamo

$F''(x)=f'(x)\geq 0 \quad \forall x\in I$

Budući da je $f'(a)\neq 0$ imamo

$F''(a)=f'(a)\neq 0 \underset{f'(a)\geq 0}{\Rightarrow} F''(a)>0$

$\Rightarrow a$ je točka lokalnog minimuma funkcije $F(x)+C$

c) LAŽ

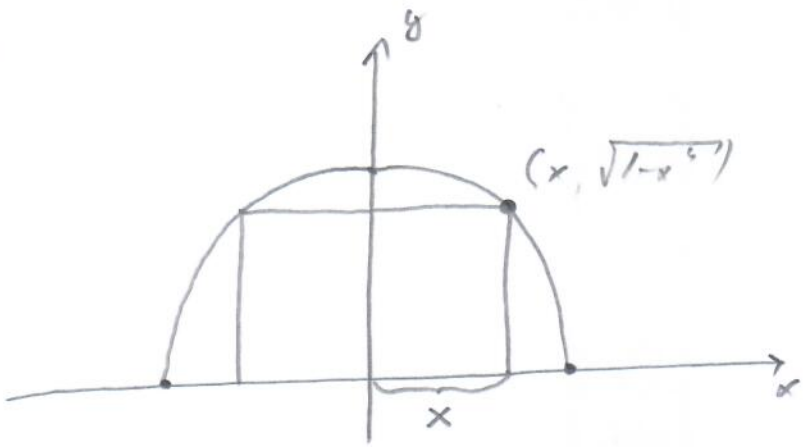
TV: $f(x)\geq 0 \ \forall x\in I$

Prema Taylorovoj formuli $\forall x\in I \ \exists a_x\in I$ t.d.

$$f(x)=f(a)+f'(a)(x-a)+\frac{f''(a_x)}{2}(x-a)^2 \underset{f(a)=f'(a)=0}{=} \frac{f''(a_x)}{2}(x-a)^2 \geq 0 \ \forall x\in I$$

f konvekna i dva puta dfb. $\Rightarrow f''(x)\geq 0 \ \forall x\in I$

6.



$$P(x) = 2x\sqrt{1-x^6}$$

Tražimo maksimum funkcije P na $(0,1)$!

$$P(x) = 2\sqrt{x^2 - x^6}$$

Budući da je $x \mapsto \sqrt{x}$ rastuća funkcija, dovoljno je naći u kojoj točki funkcija $f(x) = x^2 - x^6$ poprima maksimum

$$f(x) = x^2 - x^6$$

$$f'(x) = 2x - 6x^5 \qquad f'(x) = 0 \Leftrightarrow 2x - 6x^5 = 0$$
$$\Leftrightarrow 2x(1 - 3x^4) = 0$$
$$\Leftrightarrow x = 0 \lor x = \frac{1}{\sqrt[4]{3}}$$

$$f''(x) = 2 - 30x^4$$

$$f''\left(\frac{1}{\sqrt[4]{3}}\right) = 2 - 30 \cdot \frac{1}{3} = -8 < 0$$

$\Rightarrow f$ poprima max u $x = \frac{1}{\sqrt[4]{3}}$; $f\left(\frac{1}{\sqrt[4]{3}}\right) = \frac{1}{\sqrt{3}} - \frac{1}{3}\frac{1}{\sqrt{3}} = \frac{2}{3} \cdot \frac{1}{\sqrt{3}}$

$$P_{max} = 2\sqrt{f_{max}} = 2\sqrt{\frac{2}{3} \cdot \frac{1}{\sqrt{3}}} = \frac{2}{\sqrt[4]{3}}\sqrt{\frac{2}{3}}$$

(7) a) Teorem 12.2.1 - skripta

b) $$\int e^{-x}\cos\frac{x}{2}\,dx = \begin{bmatrix} u = \cos\frac{x}{2} & dv = e^{-x}\,dx \\ du = -\frac{1}{2}\sin\frac{x}{2}\,dx & v = -e^{-x} \end{bmatrix} =$$

$$= -e^{-x}\cos\frac{x}{2} - \frac{1}{2}\int e^{-x}\sin\frac{x}{2}\,dx = \begin{bmatrix} u = \sin\frac{x}{2} & dv = e^{-x}\,dx \\ du = \frac{1}{2}\cos\frac{x}{2}\,dx & v = -e^{-x} \end{bmatrix} =$$

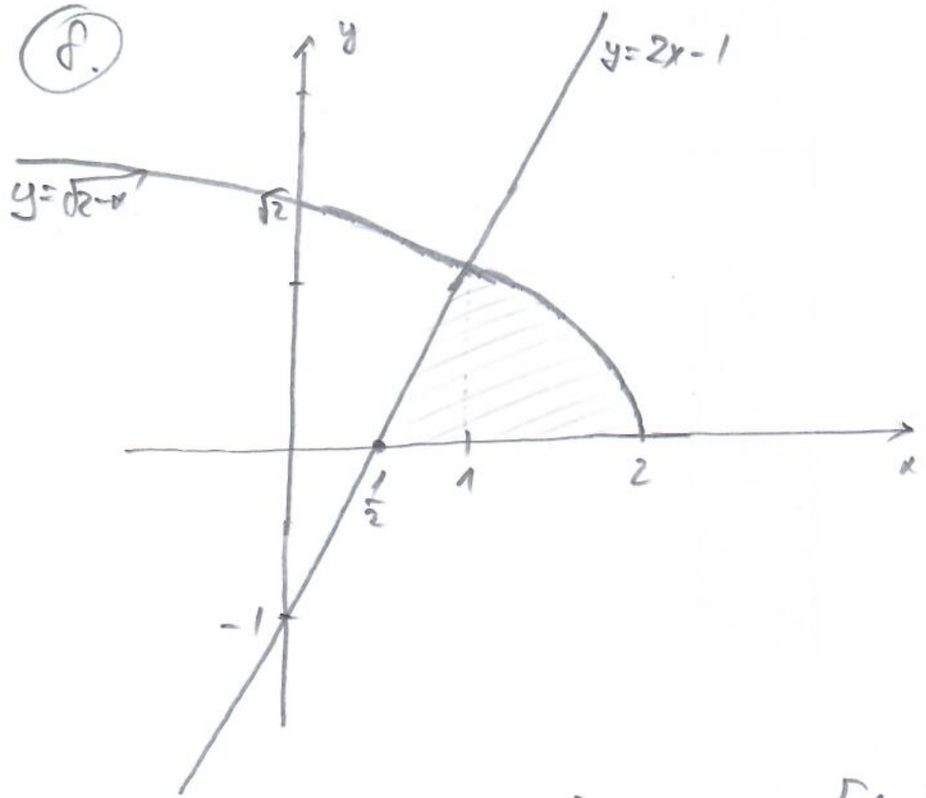
$$= -e^{-x}\cos\frac{x}{2} - \frac{1}{2}\left[-e^{-x}\sin\frac{x}{2} + \frac{1}{2}\underbrace{\int e^{-x}\cos\frac{x}{2}\,dx}_{=:I}\right]$$

$$I = -e^{-x}\cos\frac{x}{2} + \frac{1}{2}e^{-x}\sin\frac{x}{2} - \frac{1}{4}I$$

$$\frac{5}{4}I = e^{-x}\left(\frac{1}{2}\sin\frac{x}{2} - \cos\frac{x}{2}\right) \quad / \cdot \frac{4}{5}$$

$$I = \frac{4}{5}e^{-x}\left(\frac{1}{2}\sin\frac{x}{2} - \cos\frac{x}{2}\right) + C$$

8.



a) $P = \int_{\frac{1}{2}}^{1} (2x-1)\,dx + \int_{1}^{2} \sqrt{2-x}\,dx = \begin{bmatrix} t = 2-x & 1 \mapsto 1 \\ dt = -dx & 2 \mapsto 0 \end{bmatrix} =$

$= (x^2 - x)\Big|_{\frac{1}{2}}^{1} + \int_0^1 t^{1/2}\,dt = (1-1) - \left(\frac{1}{4} - \frac{1}{2}\right) + \frac{t^{3/2}}{\frac{3}{2}}\Big|_0^1 =$

$= \frac{1}{4} + \frac{2}{3} = \frac{11}{12}$

b) $V = \pi \int_{\frac{1}{2}}^{1} (2x-1)^2\,dx + \pi \int_1^2 (\sqrt{2-x})^2\,dx =$

$= \pi \int_{\frac{1}{2}}^{1} (4x^2 - 4x + 1)\,dx + \pi \int_1^2 (2-x)\,dx =$

$= \pi \left[\left(\frac{4x^3}{3} - 2x^2 + x \right)\Big|_{\frac{1}{2}}^{1} + \left(2x - \frac{x^2}{2}\right)\Big|_1^2 \right] =$

$= \pi \left[\left(\frac{4}{3} - 2 + 1 - \frac{1}{6} + \frac{1}{2} - \frac{1}{2} \right) + \left(4 - 2 - 2 + \frac{1}{2} \right) \right]$

$= \pi \left(\frac{4}{3} - 1 - \frac{1}{6} + \frac{1}{2} \right) = \frac{8-6-1+3}{6}\pi = \frac{4}{6}\pi = \frac{2\pi}{3}$