1 a) $z^2(1-i) = \bar{z}^3(\sqrt{3}+i) \quad \Big| \; |\cdot| \quad \Big| \arg$

$(1-i) = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$(\sqrt{3}+i) = 2\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$

$|z^2(1-i)| = |\bar{z}^3(\sqrt{3}+i)|$

$|z|^2 \cdot \sqrt{2} = |z|^3 \cdot 2$

$\Rightarrow |z| = 0$ ili $\frac{\sqrt{2}}{2} = |z|$

$\boxed{z_5 = 0}$ //

$2\arg z + \left(-\frac{\pi}{4}\right) = -3\arg z + \frac{\pi}{6} + 2k\pi$, $k \in \mathbb{Z}$

$5\arg z = \frac{5\pi}{12} + 2k\pi$

$\boxed{\arg z = \frac{\pi}{12} + \frac{2k\pi}{5}, \; k \in \mathbb{Z}}$

$$\boxed{z_k = \frac{\sqrt{2}}{2}\left[\cos\left(\frac{\pi}{12} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{12} + \frac{2k\pi}{5}\right)\right] \text{ za } k = 0,1,2,3,4}$$ //

Ukupno 6 rješenja.

b) (S1) LAŽ jer za $z = i$ imamo $z^2 = i^2 = -1 \cancel{\geq} 0$!

(S2) ISTINA jer za $z = \frac{-i + \sqrt{(i)^2+4}}{2} = \underbrace{\frac{\sqrt{3}}{2} - \frac{1}{2}i}_{\in \mathbb{C}}$ vrijedi

$z^2 = \frac{3}{4} - \frac{1}{4} - \frac{\sqrt{3}}{2}i = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$iz = \frac{\sqrt{3}}{2}i + \frac{1}{2}$

$\Rightarrow z^2 + iz - 1 = \frac{1}{2} - \cancel{\frac{\sqrt{3}}{2}i} + \cancel{\frac{\sqrt{3}}{2}i} + \frac{1}{2} - 1 = 0$ ✓

2 A = rasporedi u kojima dedečati iz prve države sjede zajedno

B = —||— druge —||—

C = —||— treće —||—

Tražimo broj rasporeda koji su u $\bar{A} \cap \bar{B} \cap \bar{C}$

$|\bar{A} \cap \bar{B} \cap \bar{C}| = |\underset{\uparrow \text{ svi rasporedi}}{X} \setminus (A \cup B \cup C)| = |X| - |A| - |B| - |C| + |A \cap B| + |A \cap C| + |B \cap C| - |A \cap B \cap C|$

$|X| = 8!$

$|A| = |B| = |C| = \underset{\uparrow \text{ 7 blokova}}{6!} \cdot \underset{\text{permutacije u bloku duljine 3}}{3!}$

$|A \cap B| = |A \cap C| = |B \cap C| = \underset{\uparrow \text{ 5 blokova}}{4!} \cdot (3!)^2$

$|A \cap B \cap C| = \underset{\uparrow \text{ 3 bloka}}{2!} \cdot (3!)^3$ △

$= 8! - 3 \cdot 6! \cdot 3! + 3 \cdot 4! \cdot (3!)^2 - 2! \cdot (3!)^3$

$(= \underline{29520})$

[3] (a) (T1) ISTINA

Dem. Neka je $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. Uzmimo $\varepsilon = 1$. Tada postoji $n_0 \in \mathbb{N}$ td $|a_n - L| < 1$ za sve $n \geq n_0$.

Uzmimo $\max\{a_1, a_2, \dots, a_{n_0-1}, L+1\} = M$

$\min\{a_1, a_2, \dots, a_{n_0-1}, L-1\} = m$

Za $1 \leq n < n_0$ vrijedi $\underline{m \leq a_n \leq M}$ po definiciji

Za $n \geq n_0$ vrijedi $|a_n - L| < 1$

$\Rightarrow -1 < a_n - L < 1$

$m \leq L - 1 < a_n < L + 1 \leq M$

U svakom slučaju $\underline{m \leq a_n \leq M, \ \forall n \in \mathbb{N}}$, pa je (a_n) omeđen. ✓

(T2) LAŽ

Kontraprimjer $a_n = n$, $b_n = -n$

$a_n + b_n = 0$, $\forall n \in \mathbb{N}$

uočimo $\lim_n (a_n + b_n) = 0$

No ni $\lim_n a_n$ ni $\lim_n b_n$ $\underline{\text{ne postoje}}$

jer su i (a_n) i (b_n) NEOMEĐENI.

(b) $$\lim_{n\to\infty} \left(\frac{3\sqrt{n}+1}{3\sqrt{n}}\right)^{\sqrt{n-1}} = \lim_n \left(1 + \frac{1}{3\sqrt{n}}\right)^{\frac{3\sqrt{n}}{3\sqrt{n}} \cdot \sqrt{n-1}} = \lim_n \left[\underbrace{\left(1 + \frac{1}{3\sqrt{n}}\right)^{3\sqrt{n}}}_{e}\right]^{\frac{\sqrt{n-1}}{3\sqrt{n}} \to \frac{1}{3}} = \sqrt[3]{e}$$

4 $f(x)=\begin{cases} x^a \sin\left(\frac{1}{x}\right), & x\neq 0 \\ 0, & x=0\end{cases}$ $\quad a\in\mathbb{R}$

(a) Da bi f bila neprekinuta u x_0 mora vrijediti

$$\lim_{x\to x_0} f(x) = f(x_0),$$

f je neprekinuta u svakoj točki $x\in\mathbb{R}\setminus\{0\}$ jer je $x^a\sin\left(\frac{1}{x}\right)$ sastavljena od elementarnih funkcija i ima domenu $\mathbb{R}\setminus\{0\}$.

Da bi f bila neprekinuta u $x_0=0$ mora vrijediti

$$\lim_{x\to 0} f(x) = f(0) = 0$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^a \sin\left(\frac{1}{x}\right) = \begin{cases} 0, & a>0 \\ \text{ne postoji}, & a\leq 0\end{cases}$$

Za $a>0$: $0\leq \left|x^a\sin\left(\frac{1}{x}\right)\right| \leq |x^a| \quad \Big| \lim_{x\to 0}$

$\downarrow 0 \qquad \downarrow 0 \qquad$ tm o sendviču

Za $a\leq 0$: uzmimo $x_k=\frac{1}{\frac{\pi}{2}+2k\pi}$, $k\in\mathbb{N}$, $x_k\to 0$ ali $x_k^a\sin\left(\frac{1}{x_k}\right)=x_k^a \to \begin{cases}1, & a=0\\ \infty, & a<0\end{cases}$

$y_k=\frac{1}{\frac{3\pi}{2}+2k\pi}$, $k\in\mathbb{N}$, $y_k\to 0$ ali $y_k^a\sin\left(\frac{1}{y_k}\right)=-y_k^a \to \begin{cases}-1, & a=0\\ -\infty, & a<0\end{cases}$

Dakle limes ne postoji

(b)

Za $x\in\mathbb{R}\setminus\{0\}$:

$$f'(x) = \left(x^a\sin\left(\frac{1}{x}\right)\right)' = a\cdot x^{a-1}\cdot\sin\left(\frac{1}{x}\right) + x^a\cdot\cos\left(\frac{1}{x}\right)\cdot\left(-\frac{1}{x^2}\right)$$

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^a\sin\left(\frac{1}{h}\right)-0}{h} = \lim_{h\to 0} h^{a-1}\cdot\sin\left(\frac{1}{h}\right) = \begin{cases} 0, & \underbrace{a-1>0}_{a>1} \\ \text{ne postoji}, & a\leq 1\end{cases}$$

kao i gore

Za $a>1$ f je diferencijabilna u svakoj točki $x\in\mathbb{R}$

i vrijedi $f'(x)=\begin{cases} a x^{a-1}\cdot\sin\left(\frac{1}{x}\right) + x^a\cos\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right), & x\neq 0 \\ 0, & x=0\end{cases}$

Za $a\leq 1$ f nije dif. u 0 (pa ni na cijelom $\mathbb{R}$)

5

$$f(x) = \frac{x}{\sqrt[3]{x^2-1}} \qquad D_f = \{x \in \mathbb{R} : x^2 - 1 \neq 0\} = \{x \in \mathbb{R} : x \neq \pm 1\} = \mathbb{R} \setminus \{\pm 1\}$$

Vertikalne asimptote:

$$\lim_{x \to 1+} f(x) = \frac{1}{0+} = +\infty \qquad \lim_{x \to 1-} f(x) = \frac{1}{0-} = -\infty \quad - - \quad x = 1$$

$$\lim_{x \to -1+} f(x) = \frac{-1}{0-} = +\infty \qquad \lim_{x \to -1-} f(x) = \frac{-1}{0+} = -\infty \quad - - \quad x = -1$$

vert. asimp.

Kose asimptote:

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{1}{\sqrt[3]{x^2-1}} = \frac{1}{\pm\infty} = 0 = k$$

$$l = \lim_{x \to \pm\infty} f(x) - 0x = \lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \sqrt[3]{\frac{x^3}{x^2-1}} = \lim_{x \to \pm\infty} \sqrt[3]{\frac{x}{1-\frac{1}{x^2}}} = \boxed{\pm\infty}$$

nema kosih asimptota.

$$f'(x) = \frac{\sqrt[3]{x^2-1} - x \cdot \frac{1}{3}(x^2-1)^{-2/3} \cdot 2x}{(x^2-1)^{2/3}} = \frac{(x^2-1) - \frac{2x^2}{3}}{(x^2-1)^{4/3}} = \frac{x^2-3}{3(x^2-1)^{4/3}} = \frac{(x-\sqrt{3})(x+\sqrt{3})}{3(x^2-1)^{4/3}}$$

$$f''(x) = \frac{1}{3} \cdot \frac{2x(x^2-1)^{4/3} - (x^2-3) \cdot \frac{4}{3}(x^2-1)^{1/3} \cdot 2x}{(x^2-1)^{8/3}} = \frac{2x(x^2-1)^{1/3}\left[3x^2 - 3 - 4x^2 + 12\right]}{9(x^2-1)^{8/3}}$$

$$= \frac{2x(9-x^2)}{9 \cdot (x^2-1)^{7/3}}$$

	$-\infty$	-3		$-\sqrt{3}$		-1		0		1		$\sqrt{3}$		3	$+\infty$
f	$-$		$-$		$-$	$\times$	$+$	0	$-$	$\times$	$+$		$+$		$+$
f'	$+$		$+$	0	$-$	$\times$	$-$		$-$	$\times$	$-$	0	$+$		$+$
f''	$+$	0	$-$		$-$	$\times$	$+$	0	$-$	$\times$	$+$		$+$	0	$-$
f	↗		↗		↘		↘		↘		↘		↗		↗

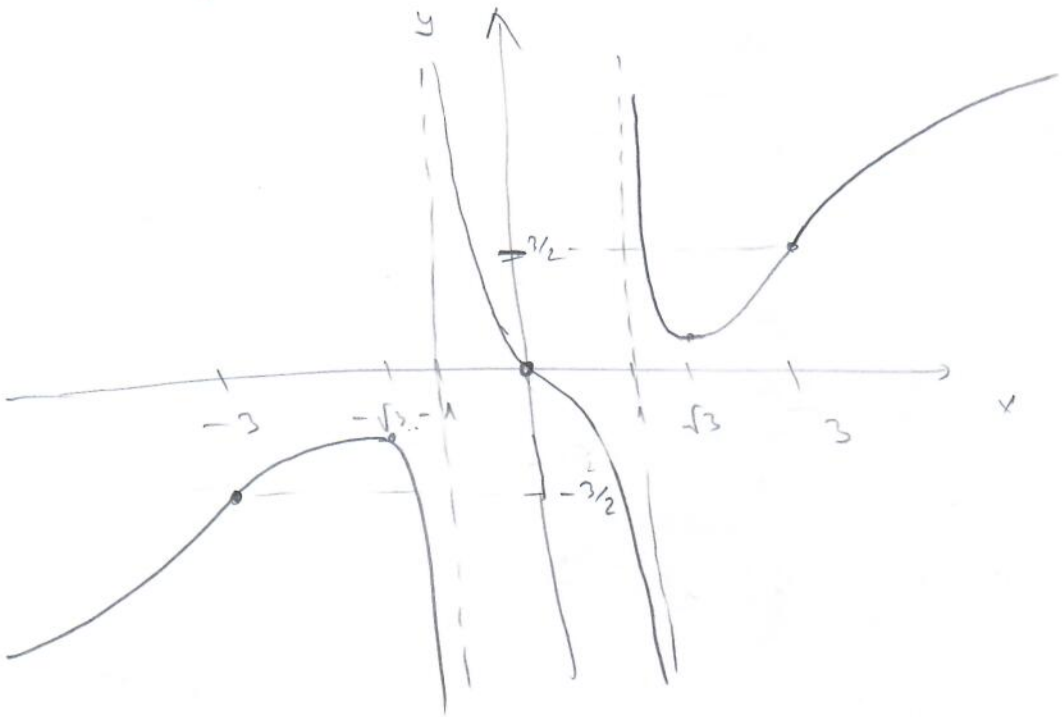
Točke infleksije: $x = \pm 3$ i $x = 0$

Lokalni ekstremi: $x = -\sqrt{3}$ (lok. max) $\quad x = \sqrt{3}$ (lok. min)

Intervali monotonosti: $\langle -\infty, -\sqrt{3} \rangle$ ↗, $\langle -\sqrt{3}, -1 \rangle$ ↘, $\langle -1, 1 \rangle$ ↘, $\langle 1, \sqrt{3} \rangle$ ↘, $\langle \sqrt{3}, +\infty \rangle$ ↗

Intervali konv/konk.: $\langle -\infty, -3 \rangle$ konveks., $\langle -3, -1 \rangle$ konkav., $\langle -1, 0 \rangle$ konveks, $\langle 0, 1 \rangle$ konkav., $\langle 1, 3 \rangle$ konveks, $\langle 3, +\infty \rangle$ konkav.

5 (nastavak)



6 (a) $f: \mathbb{R} \to \mathbb{R}$ neprekinuta

$$F(x) = \int_0^x f(t)\,dt$$

Neka $x \in \mathbb{R}$:

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left[\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt\right] = \lim_{h \to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Po tm o srednjoj vrijednosti za integrale $\exists\, c \in \langle x, x+h \rangle$ ili $\langle x+h, x \rangle$ td $\int_x^{x+h} f(t)\,dt = h \cdot f(c)$

$$= \lim_{h \to 0} \frac{1}{\cancel{h}} \cdot \cancel{h} \cdot f(c) = f(x) \quad \blacksquare$$

$\boxed{\text{Kada } h \to 0 \text{ onda } c \to x}$

(b) $\Phi(x) = \int_0^{x^2} f(t)\,dt = F(x)\Big|_0^{x^2}$ Stavimo $F(x) = \int_0^x f(t)\,dt$

F je primitivna od f

$$\Rightarrow \Phi(x) = F(x^2) - F(0) \quad \Big| \frac{d}{dx}$$

$$\Phi'(x) = F'(x^2) \cdot 2x - 0 = \underline{\underline{f(x^2) \cdot 2x}}$$

$\boxed{7}$ (a) $\displaystyle\int_1^3 \frac{dx}{(\sqrt{x}+1)(\sqrt{x}+2)} = \begin{bmatrix} \sqrt{x} = t \\ x = t^2 \\ dx = 2t\,dt \end{bmatrix} = \int_1^{\sqrt{3}} \frac{2t\,dt}{(t+1)(t+2)} = \int_1^{\sqrt{3}} \left(\frac{-2}{t+1} + \frac{4}{t+2}\right) dt$

$$= \left(-2\ln|t+1| + 4\ln|t+2|\right)\Big|_1^{\sqrt{3}}$$

$$= \ln\left|\frac{(t+2)^4}{(t+1)^2}\right|\Bigg|_1^{\sqrt{3}}$$

$$= \ln\frac{(2+\sqrt{3})^4}{(\sqrt{3}+1)^2} - 0$$

$$= \ln\frac{(2+\sqrt{3})^4}{(1+\sqrt{3})^2}$$

$$\frac{2t}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2} \quad \Big|\cdot(t+1)(t+2)$$

$$2t = A(t+2) + B(t+1)$$

$$\left.\begin{matrix} 2 = A+B \\ 0 = 2A+B \end{matrix}\right\} \Rightarrow \boxed{\begin{matrix} A = -2 \\ B = 4 \end{matrix}}$$

(b) $\displaystyle\int_0^1 \frac{\ln(1+x)}{\sqrt[3]{x^4}}\,dx = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 \frac{\ln(1+x)}{x^{4/3}}\,dx$

$$\int_0^1 \frac{1}{x^p} = \begin{cases} \text{konv.}, & p<1 \\ \text{diverg.}, & p \geq 1 \end{cases}$$

ovo znamo s predavanja $(p = 1/3 < 1)$

$\ln(1+x) \sim x$ za $x \to 0$

$\dfrac{\ln(1+x)}{x^{4/3}} \sim \dfrac{x}{x^{4/3}} = \dfrac{1}{x^{1/3}}$ za $x \to 0$

$\left(\text{Jer } \displaystyle\int_0^1 \frac{1}{x^{1/3}}\,dx = \left(\frac{3}{2}x^{2/3}\right)\Big|_0^1 = \frac{3}{2} < +\infty\right)$

Po usporednom kriteriju konvergira i integral $\displaystyle\int_0^1 \frac{\ln(1+x)}{\sqrt[3]{x^4}}\,dx$.

8 a)

$\boxed{a>0}$

$$\mathrm{Vol}(a) = \int_a^{a+2} y^2(x)\pi \, dx = \pi \int_a^{a+2} x e^{-x} \, dx = \begin{bmatrix} u = x & dv = e^{-x} dx \\ du = dx & v = -e^{-x} \end{bmatrix}$$

$$= \pi \left[\left(-x e^{-x}\right)\Big|_a^{a+2} + \int_a^{a+2} e^{-x} dx \right] = \pi \left[a e^{-a} - (a+2) e^{-(a+2)} + \left(-e^{-x}\right)\Big|_a^{a+2} \right]$$

$$= \pi \left[a e^{-a} - (a+2) e^{-a} \cdot e^{-2} + e^{-a} - e^{-(a+2)} \right] =$$

$$= \pi e^{-a} \left[a - \frac{a+2}{e^2} + 1 - \frac{1}{e^2} \right] = \pi e^{-a} \left[(a+1) - \frac{a+3}{e^2} \right]$$

(b) $$V'(a) = \pi(-1) e^{-a} \left[(a+1) - \frac{(a+3)}{e^2} \right] + \pi e^{-a} \left[1 - \frac{1}{e^2} \right] = 0 \;\Big|\; : \pi e^{-a} \neq 0$$

$$\Rightarrow \quad -a - \cancel{1} + \frac{a+3}{e^2} = \frac{1}{e^2} - \cancel{1}$$

$$\Rightarrow \quad a = \frac{a+2}{e^2} \Rightarrow a(e^2-1) = 2 \Rightarrow \boxed{a = \frac{2}{e^2-1}}$$

Za ovaj a se postiže maksimum