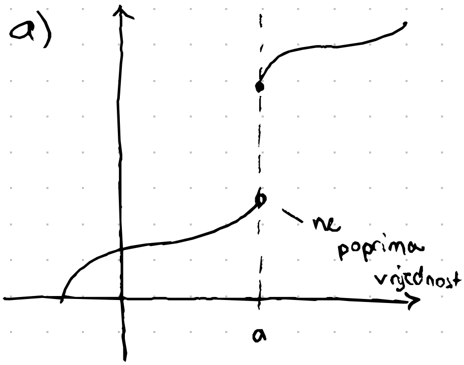
5.2. NEPREKINUTE Fje I LIMESI

Pr.) Postoji li $\lim\limits_{x\to a} f(x)$? Vrijedi li $\lim\limits_{x\to a} f(x) = f(a)$?

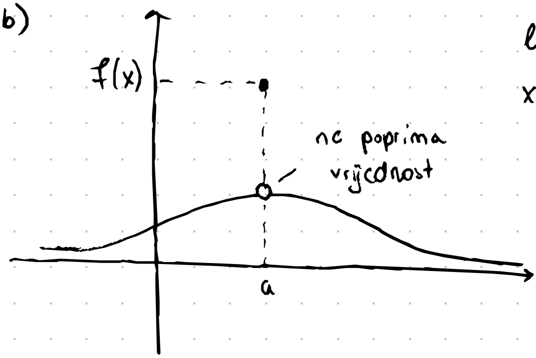
a)



$$\lim_{x\to a^+}(f(x)) \neq \lim_{x\to a^-}(f(x))$$

$\lim\limits_{x\to a}(f(x))$ ne postoji

b)



$$\lim_{x\to a^+}(f(x)) = \lim_{x\to a^-}(f(x))$$

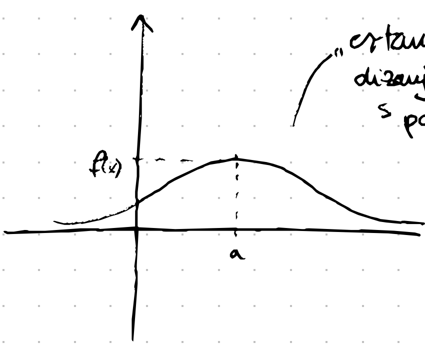
$\lim\limits_{x\to a} f(x)$ postoji

$\lim\limits_{x\to a} f(x) \neq f(a)$

jer nije neprekinuta funkcija

c)

„crtanje bez dizanja olovke s papira"



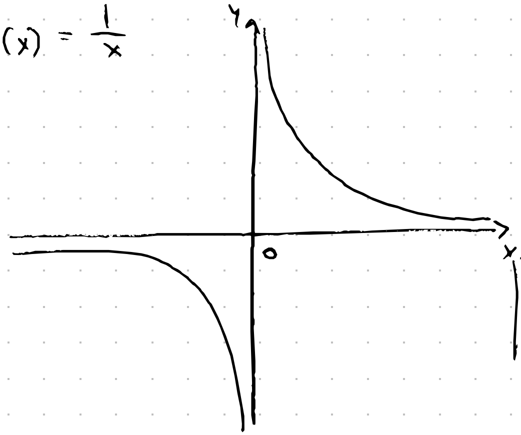
$$\lim_{x\to a^+}(f(x)) = \lim_{x\to a^-}(f(x)) = \lim_{x\to a} f(x) = f(a)$$

fja je neprekinuta na a!

5.3. LIMESI I ASIMPTOTE

5.3.1. BESKONAČNI LIMESI $(L \pm \infty)$

Pr $f(x) = \frac{1}{x}$

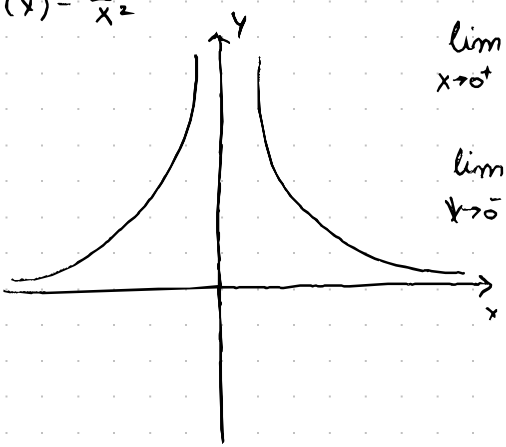


$\lim_{x \to 0^+} \frac{1}{x} = +\infty$

$\lim_{x \to 0^-} \frac{1}{x} = -\infty$

$$\boxed{\frac{1}{0^+} = +\infty \quad , \quad \frac{1}{0^-} = -\infty}$$

Pr.) $f(x) = \frac{1}{x^2}$



$\lim_{x \to 0^+} \frac{1}{x^2} = +\infty$

$\lim_{x \to 0^-} \frac{1}{x^2} = +\infty$

$\Rightarrow \lim_{x \to 0} \frac{1}{x^2} = +\infty$

DEF | Koju ne vidimo na ploču

Zad.] Izračunajte limes

$$\lim_{x\to 3^+} \frac{2x}{x-3} = \frac{6}{0^+} = +\infty$$

$$\lim_{x\to 3^-} \frac{2x}{x-3} = \frac{6}{0^-} = -\infty$$

$$\lim_{x\to 3^-} \frac{2x}{(x-3)^2} = \frac{6}{(0^-)^2} = \frac{6}{0^+} = +\infty$$

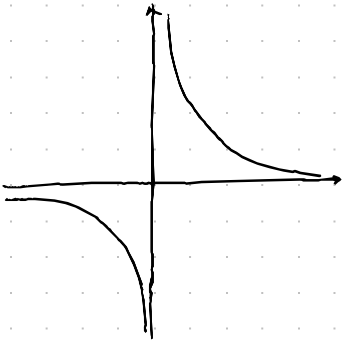
$$\lim_{x\to -3^+} \frac{2x}{x+3} = \frac{-6}{0^+} = -\infty$$

5.4.2. LIMESI U BESKONAČNOST ($x \to \pm\infty$)

Pr. $f(x) = \frac{1}{x}$

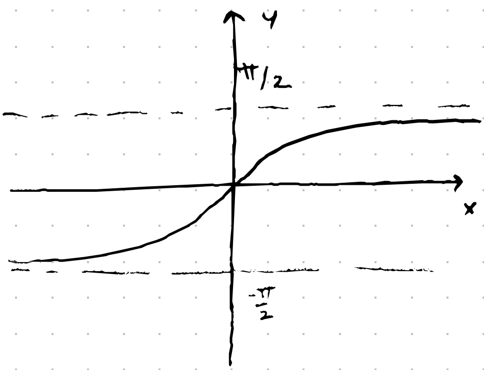
$$\lim_{x\to +\infty} \frac{1}{x} = \frac{1}{+\infty} = 0^+$$

$$\boxed{\frac{1}{+\infty} = 0^+, \quad \frac{1}{-\infty} = 0^-}$$



$$\lim_{x\to -\infty} \frac{1}{x} = \frac{1}{-\infty} = 0^-$$

Pr.) $f(x) = \text{arc tg}\, x$



$$\lim_{x\to +\infty} \text{arctg}(x) = \text{arctg}(+\infty) = \frac{\pi}{2}$$

$$\lim_{x\to -\infty} \text{arctg}(x) = \text{arc tg}(-\infty) = \frac{-\pi}{2}$$

Zad. | Izračunajte:

a) $\lim\limits_{x\to+\infty} \frac{1}{x^2+1} = \frac{1}{+\infty} = 0$

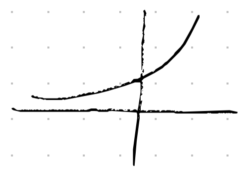
b) $\lim\limits_{x\to-\infty} \operatorname{arctg}\left(\frac{x+1}{2}\right) = \operatorname{arctg}(-\infty) = \frac{-\pi}{2}$

Zad. | Ispitajte je li $\lim\limits_{x\to+\infty} e^x = \lim\limits_{x\to-\infty} e^x$

NE! Nije isto, pazi!!

Rj: $\lim\limits_{x\to+\infty} e^x = e^{+\infty} = +\infty$

$\lim\limits_{x\to-\infty} e^x = e^{-\infty} = \frac{1}{e^{+\infty}} = \frac{1}{\infty} = 0$



5.3.3. ASIMPTOTE

DEF Ako se točka T neprekidno giba po grafu fije f tako da barem jedna od njenih koordinata teži prema $\pm\infty$ te pritom njena udaljenost od nekog pravca teži k nuli, tada taj pravac zovemo ASIMPTOTA I ; Γ_0.

1. VERTIKALNE ASIMPTOTE

DEF Neka $a \notin D(f)$. Tada pravac $x=a$ zovemo V.A. fije f ako je:

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{ili} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

Pr. $f(x) = \ln x$

$D_f = \langle 0, +\infty \rangle$

$\boxed{x = 0^+}$

$$\lim_{x \to 0^+} \ln x = -\infty$$

$x = 0$ V.A

ALGORITAM ZA TRAŽENJE V.A.:

1. Odredite D_f
2. Računati lijeve i desne limese u rubovima domene
3. Odredite V.A.

Zad. $f(x) = \dfrac{x}{x^2-9}$

$D_f = \mathbb{R} \setminus \{3, -3\}$

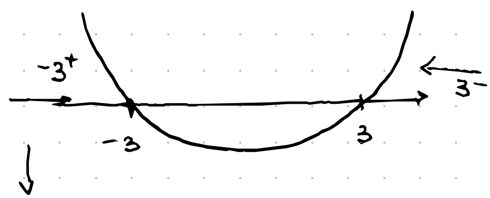
$$\lim_{x \to 3^+} \frac{x}{x^2-9} = \lim_{x\to 3^+} \frac{x}{(x-3)(x+3)} = \frac{3}{0^+} = +\infty$$

$$\lim_{x \to 3^-} \frac{x}{x^2-9} = \frac{3}{0^-} = -\infty$$

$$\lim_{x \to -3^+} \frac{x}{x^2-9} = \frac{-3}{0^-} = +\infty$$

$$\lim_{x \to -3^-} \frac{x}{x^2-9} = \frac{-3}{0^+} = -\infty$$

$\underline{x = 3}$ $\underline{x = -3}$



„Prema -3 u smjeru plusa"

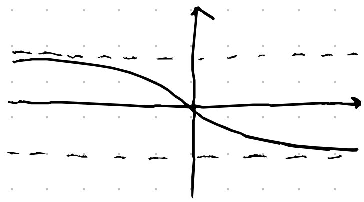
2) HORIZONTALNE ASIMPTOTE

DEF Pravac $y = L$ zovemo DESNA (LIJEVA) HOR. ASIMPT. f-je f. ako vrijedi $\lim_{x \to +\infty} f(x) = L$ $\left(\lim_{x \to -\infty} (f(x)) = L\right)$

Pr.) Odredite H.A. f-je $f(x) = \operatorname{arctg}(2-x)$

$$\lim_{x \to +\infty} (\operatorname{arctg}(2-x)) = \operatorname{arctg}(-\infty) = -\frac{\pi}{2}$$

$$\lim_{x \to -\infty} (\operatorname{arctg}(2-x)) = \operatorname{arctg}(+\infty) = \frac{\pi}{2}$$

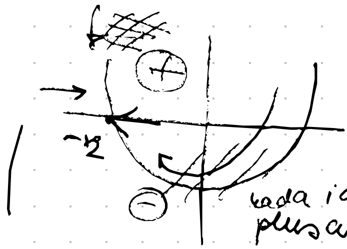


Zad.) Odredite VA i HA f-je $f(x) = \dfrac{2x^2+x-1}{x^2+x-2}$

1. VA.

$D_f = \mathbb{R} \setminus \{-2, 1\}$

$$\lim_{x \to -2^+} \frac{2x^2+x-1}{x^2+x-2} =$$



kada ide u -2 iz plusa, ide iz 0^- jer ovo

kada ide iz lijeva u -2, onda u 0^+

$$\lim_{x \to -2^+} \frac{5}{0^-} = -\infty$$

$\boxed{x = -2 \quad V.A.}$

$$\lim_{x \to -2^-} \frac{2x^2+x-1}{x^2+x-2} = \frac{5}{0^+} = +\infty$$

$$\lim_{x \to 1^+} \frac{2x^2+x-1}{(x+2)(x-1)} = \frac{2}{3 \cdot 0^+} = +\infty$$

$$\lim_{x \to 1^-} \frac{2x^2+x-1}{(x+2)(x-1)} = \frac{2}{3 \cdot 0^-} = -\infty$$

$\boxed{x = 1 \quad V.A.}$

3. H.A.

$$\lim_{x \to +\infty} \frac{2x^2+x-1}{x^2+x-2} \begin{matrix} /:x^2 \\ = \\ /:x^2 \end{matrix} \frac{2}{1} = 2$$

$$\lim_{x \to -\infty} \frac{2x^2+x-1}{x^2+x-2} \begin{matrix} /:x^2 \\ = \\ /:x^2 \end{matrix} \frac{2}{1} = 2$$

$\boxed{y = 2 \quad H.A.}$

3. KOSE ASIMPTOTE

DEF Pravac $y = kx + l$ zovemo DESNA (LIJEVA) K.A. ako vrijedi

$$\lim_{x\to+\infty} (f(x) - kx - l) = 0 \quad \left(\lim_{x\to-\infty} (f(x) - k - l) = 0\right)$$

$$\boxed{l = \lim_{x\to\pm\infty} (f(x) - kx)}$$

$k = ?$

$$\lim_{x\to+\infty} \frac{f(x) - kx - l}{x} = \frac{0}{+\infty} = 0$$

$$\lim_{x\to+\infty} \left(\frac{f(x)}{x} - k - \frac{l}{x}\right) = 0$$

$$\boxed{k = \lim_{x\to+\infty} \frac{f(x)}{x}}$$

$\hookrightarrow$ * H.A. su posebne k.A. kojima je $k = 0$.

TRAŽENJE K.A.

1. $k_{1,2} = \lim_{x\to\pm\infty} \frac{f(x)}{x}$
2. $l_{1,2} = \lim_{x\to\pm\infty} (f(x) - k_{1,2} \cdot x)$

Ako su limesi $\exists$ i konačni su $\Rightarrow$ $y = k_1 x + l_1$ D.K.A. (Desna)

$y = k_2 x + l_2$ L.K.A. (lijeva)

ZAD. Odredite sve asimptote f-ja: (a) $f(x) = \text{arctg}\left(\frac{x^2}{x+1}\right)$

b) $f(x) = e^{1/x} - x$

a) VA $D_f = \mathbb{R} \setminus \{-1\}$

$$\lim_{x\to-1^+} \text{arctg}\left(\frac{x^2}{x+1}\right) = \text{arctg}\frac{1}{0^+} = \text{arctg}(+\infty) = \frac{\pi}{2}$$

$$\lim_{x\to-1^-} \text{arctg}\left(\frac{x^2}{x+1}\right) = \text{arctg}\frac{1}{0^-} = \text{arctg}(-\infty) = -\frac{\pi}{2}$$

ako za V.A. dobijemo konačne brojeve, nema asimptote!

NEMA V.A.

KA: $k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{\text{arctg}\left(\frac{x^2}{x+1}\right)}{x} = \frac{\text{arctg}(+\infty)}{+\infty} = \frac{\frac{\pi}{2}}{+\infty} = 0$

$\hookrightarrow$ to je H.A.

$$l_1 = \lim_{x\to+\infty} (f(x) - kx) = \lim_{x\to+\infty} \text{arctg}\left(\frac{x^2}{x+1}\right) = \frac{\pi}{2}$$

$\boxed{y = \frac{\pi}{2} \text{ DHA}}$

$k_2 = 0$

$$l_1 = \lim_{x\to-\infty} (f(x) - kx) = \lim_{x\to-\infty} \text{arctg}\left(\frac{x^2}{x+1}\right) = -\frac{\pi}{2}$$

$\boxed{y = -\frac{\pi}{2} \text{ LHA}}$

$\theta)$ VA. $D_f = \mathbb{R} \setminus \{0\}$

$$\lim_{x \to 0^+} \left(e^{\frac{1}{x}} - x\right) = e^{\frac{1}{+\infty}} - 0 = e^{+\infty} = +\infty$$

$$\lim_{x \to 0^-} \left(e^{1/x} - x\right) = e^{\frac{1}{+\infty}} - 0 = e^{-\infty} = 0$$

$\boxed{\begin{array}{c} x = 0 \\ \text{D.V.A} \end{array}}$

K.A. $k_1 = \lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} \frac{e^{1/x} - x}{x} = \lim_{x \to +\infty} \left(\frac{e^{1/x}}{x} - 1\right) = \frac{e^{\frac{1}{+\infty}}}{+\infty} - 1$

$$= \frac{1}{+\infty} - 1 = \underline{-1}$$

$$l_1 = \lim_{x \to +\infty} (f(x) + x) = \lim_{x \to +\infty} \left(e^{1/x} - x + x\right) = e^{\frac{1}{+\infty}} = e^0 = 1$$

$\boxed{y = -x + 1 \quad \text{D.K.A}}$

D.Z. $y = -x + 1$ L.K.A

5.3. RAČUNANJE LIMESA

$\boxed{TM}$ $a \in \mathbb{R}$ ili $a = \pm\infty$, $\lim (f(x))$, $\lim g(x) \in \mathbb{R}$

1) $\lim\limits_{x \to a} (f(x) \pm g(x)) = \lim\limits_{x \to a} (f(x)) \pm \lim\limits_{x \to a} (g(x))$

2) $\lim\limits_{x \to a} (f(x) \cdot g(x)) = \lim\limits_{x \to a} (f(x)) \cdot \lim (g(x))$

3) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim (f(x))}{\lim (g(x))}$

4) $\lim\limits_{x \to a} (f(x))^{g(x)} = \left(\lim\limits_{x \to a} (f(x)) \right)^{\lim\limits_{x \to a} (g(x))}$

Pravila za beskonačne limese

→ zbrajanje $\infty + \infty = \infty$

$\infty + c = \infty \quad c \in \mathbb{R}$

$(\infty - \infty)$ = neodređeni oblik

→ množenje $\infty \cdot \infty = \infty$

$c \cdot \infty = \pm\infty$ (ovisno o $c > 0$ ili $c < 0$)

$0 \cdot \infty$ = neodređeni oblik

→ dijeljenje $\frac{\infty}{}$

5.4.3. Ekvivalentne neizmjerno male veličine

DEF Fje f i g su neizmjerno male veličine istog reda kad $x \to a$ ako je $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} (g(x)) = 0$ i $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = c \in \mathbb{R}^+$

Fje f i g za ekvivalentne neizmjerno male veličine ako je $c=1$, tj $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = 1$ i pišemo $f(x) \sim g(x)$, $x \to a$.

Nap.

$$\lim_{x\to a} \frac{\overset{\sim \ln(x),\ x\to a}{f(x)}}{g(x)} = \lim_{x\to a} \frac{\ln(x)}{g(x)}$$

$$\lim_{x\to a} f(x) - g(x) = \lim_{x\to a}$$