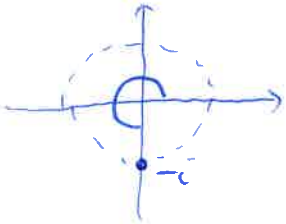
1°) a) $$i z^2 + (1+i) z + 1 = 0$$

kvadratna jednadžba: $$z_{1,2} = \frac{-(1+i) \pm \sqrt{(1+i)^2 - 4i}}{2i}$$

$$\sqrt{(1+i)^2 - 4i} = \sqrt{1 + 2i - 1 - 4i} = \sqrt{-2i} = \sqrt{2}\sqrt{-i}$$

$-i =$  $= \cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}$

$$\sqrt{-i} = \cos\left(\frac{3\pi}{4} + k\pi\right) + i \sin\left(\frac{3\pi}{4} + k\pi\right), \quad k = 0, 1$$

$k=0$ ... $\underbrace{\sqrt{-i}}_{\tilde{z}_0} = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$, $k=1$ ... $\underbrace{\sqrt{-i}}_{\tilde{z}_1} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i$

$$\tilde{z}_0 = -\tilde{z}_1$$

Imamo dakle, dva rješenja:

$$z_1 = \frac{-(1+i) + \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)}{2i} = \cancel{\left(-\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} + \frac{1}{2\sqrt{2}}\right) i}$$

$$z_2 = \frac{-(1+i) - \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)}{2i} = \cancel{\left(-\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) + \left(\frac{1}{2} - \frac{1}{2\sqrt{2}}\right) i}$$

$$= -i, \quad 1$$

1) b)

$$z = r(\cos(\varphi) + i\sin(\varphi))$$

Tvrdnja vrijedi za $n=1$.

Pretpostavimo da tvrdnja vrijedi za neki $n \in \mathbb{N}$:

$$z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$$

$$z^{n+1} = z \cdot z^n$$

$$= r(\cos(\varphi) + i\sin(\varphi)) \cdot r^n \cdot (\cos(n\varphi) + i\sin(n\varphi))$$

$$= r^{n+1} \cdot (\cos(\varphi) \cdot \cos(n\varphi) + i\sin(\varphi)\cos(n\varphi) + i(\cos\varphi)(\sin(n\varphi)) - \sin(\varphi)\sin(n\varphi))$$

$$= r^{n+1}(\cos(\varphi) \cdot \cos(n\varphi) - \sin(\varphi)\sin(n\varphi) + i[\sin(\varphi)\cos(n\varphi) + \cos(\varphi)\sin(n\varphi)])$$

$$= r^{n+1}(\cos(\varphi + n\varphi) + i\sin(\varphi + n\varphi))$$

$$= r^{n+1}(\cos((n+1)\varphi) + i\sin((n+1)\varphi))$$

Dakle, tvrdnja vrijedi i za $n+1 \in \mathbb{N}$.

Po principu matematičke indukcije, tvrdnja vrijedi $\forall n \in \mathbb{N}$.

2) $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$

a) Neparna fja: $f(-x) = -f(x)$

$f(-x) = \ln\left(-x + \sqrt{x^2+1}\right)$

$= \ln\left(-x + \sqrt{x^2+1} \cdot \frac{x+\sqrt{x^2+1}}{x+\sqrt{x^2+1}}\right) = \ln\left(\frac{\cancel{x^2}+1-\cancel{x^2}}{x+\sqrt{x^2+1}}\right) = \ln\left(\left(x+\sqrt{x^2+1}\right)^{-1}\right)$

$= -\ln\left(x+\sqrt{x^2+1}\right)$

$= -f(x).$

b) $y = f(x) = \ln\left(x + \sqrt{x^2+1}\right)$

$e^y = x + \sqrt{x^2+1}$ (1)

Neparnost: $-y = -f(x) = f(-x) = \ln\left(-x + \sqrt{x^2+1}\right)$

$\Rightarrow e^{-y} = -x + \sqrt{x^2+1}$ (2)

(1) $-$ (2)

$\Rightarrow e^y - e^{-y} = 2x$

$\Rightarrow x = \frac{e^y - e^{-y}}{2}$

$\boxed{f^{-1}(y) = \frac{e^y - e^{-y}}{2} = sh(y)}$

c) f neparna, $y = f(x)$, $x = f^{-1}(y)$

$f^{-1}(-y) = f^{-1}(-f(x)) = f^{-1}(f(-x)) = -x = -f^{-1}(y).$

3) $\begin{cases} a_1 = 0 \\ a_{n+1} = \frac{1}{2}\sqrt{a_n^2+1} & n \geq 2 \end{cases}$

1°) a_n je rastući niz

Baza: $a_2 = \frac{1}{2}\sqrt{0+1} = \frac{1}{2} > 0 = a_1$

Korak: Pretp. $a_n \leq a_{n+1}$ za neki $n \in \mathbb{N}$.

$$a_{n+1} = \frac{1}{2}\sqrt{a_n^2+1} \leq \frac{1}{2}\sqrt{a_{n+1}^2+1} = a_{n+2}$$

2°) $a_n \leq 1$, $\forall n \in \mathbb{N}$

Baza: $a_1 = 0 \leq 1$

Korak: Pretpostavimo $a_n \leq 1$ za neki $n \in \mathbb{N}$.

$$a_{n+1} = \frac{1}{2}\sqrt{a_n^2+1} \leq \frac{1}{2}\sqrt{1+2} = \frac{\sqrt{2}}{2} < 1.$$

$\Rightarrow$ a_n je rastući niz i omeđen odozgo $\Rightarrow$ (a_n) je konvergentan.

$$L = \lim_{n\to\infty} a_n$$

Imamo:

$$a_{n+1} = \frac{1}{2}\sqrt{a_n^2+1} \quad / \lim$$

$$L = \frac{1}{2}\sqrt{L^2+1} \Rightarrow L_1 = \frac{1}{\sqrt{3}},\ L_2 = \cancel{-\frac{1}{\sqrt{3}}}$$ ← ovo nije jer $a_n \geq 0$ $\forall n$.

b) skripta TEOREM 6.6.1.

4) a) Teorem 7.1.3

b) Definicija 7.2.1

c) $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+} \frac{x-1}{\sqrt{x^2-1}} = \lim\limits_{x\to 1^+} \sqrt{\frac{x-1}{x+1}} = 0$

$\lim\limits_{x\to 1^-} f(x) = 0$ (jer je tamo $f \equiv 0$)

$0 = f(1)$ $\Rightarrow$ fja neprekidna.

$$\lim_{x\to 1^+} \frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+} \frac{\frac{x-1}{\sqrt{x^2-1}} - 0}{x-1} = \lim_{x\to 1^+} \frac{1}{\sqrt{x^2-1}} = +\infty$$

$\Rightarrow$ f nije diferencijabilna u $x=1$

5) $x^y + \arctan(2x-y) - 1 = 0 \quad / \frac{d}{dx}$

$$\frac{d}{dx}\left(\arctan(2x-y)\right) = \frac{1}{1+(2x-y)^2} \cdot (2-y')$$

$$\frac{d}{dx}\left(x^y\right) = \frac{d}{dx}\left(e^{y\ln x}\right) = e^{y\ln x} \cdot \left(y'\ln x + \frac{y}{x}\right)$$
$$= x^y\left(y'\ln x + \frac{y}{x}\right)$$

Vrijedi:

$$\boxed{x^y\left(y'\ln x + \frac{y}{x}\right) + \frac{1}{1+(2x-y)^2} \cdot (2-y') = 0} \quad (*)$$

Tangenta u $T(1,2)$:

$$y - \overset{2}{\overset{\parallel}{y(1)}} = y'(1)(x-1)$$

$y'(1)$ dobivamo uvrštavanjem u jednadžbu $(*)$

$$1^2\left(\cancel{y'(1)\cdot\ln(1)} + \frac{2}{1}\right) + \frac{1}{1+\cancel{(2\cdot1-2)^2}} \cdot (2-y'(1)) = 0$$

$$2 + 2 - y'(1) = 0$$
$$y'(1) = 4$$

Tangenta ...

$$y = 4x - 2$$

6) $f(x) = x + \frac{\ln(x)}{x}$

Domena: $D_f = \langle 0, +\infty \rangle$

$x=0$ je na rubu područja, potencijalna vertikalna asimptota

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \Big(\underbrace{x}_{\to 0} + \underbrace{\ln x}_{\to -\infty} \cdot \underbrace{\frac{1}{x}}_{\to +\infty}\Big) = -\infty$$

$\Rightarrow$ $x=0$ je vertikalna asimptota.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x + \frac{\ln x}{x} = \lim_{x\to\infty} \frac{x^2 + \ln x}{x} =$$

$$= L'H = \lim_{x\to\infty} \frac{2x + \frac{1}{x}}{1} = \lim_{x\to\infty} \frac{2x^2+1}{x} = +\infty$$

nema horizontalnih asimptota

$$k = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \left(1 + \frac{\ln x}{x^2}\right) = L'H = 1 + \lim_{x\to+\infty} \frac{\frac{1}{x}}{2x} = 1$$

$$l = \lim_{x\to+\infty} (f(x) - 1\cdot x) = \lim_{x\to+\infty} \frac{\ln x}{x} = \dots = 0$$

$y = kx + l = x$ je kosa asimptota od f.

7) a) knjižica

b) Traži se $b>0$ takav da je

$$4 = \frac{\int_0^b 2+6x-3x^2\,dx}{b-0} = \frac{2x+3x^2-x^3\Big/_0^b}{b}$$

$$= \frac{2b+3b^2-b^3}{b} = 2+3b-b^2$$

Rješimo:

$$b^2-3b-2=0$$

$$b_{1,2} = \frac{3 \pm \sqrt{9-8}}{2}$$

$$= \frac{3\pm 1}{2}$$

$$\boxed{b_1 = 2,\ b_2 = 1}$$

8) $\int \frac{x+2}{x^3-1}\,dx$

Rastav na parcijalne razlomke:

$$\frac{x+2}{x^3-1} = \frac{x+2}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

(x^2+x+1 nema realnih nultočaka)

$$\Rightarrow \quad x+2 = (x^2+x+1)A + (x-1)(Bx+C)$$

$$= Ax^2 + Ax + A + Bx^2 - Bx + Cx - C$$

$$\Rightarrow \quad \begin{aligned} 2 &= A - C \\ 1 &= A - B + C \\ 0 &= A + B \end{aligned} \qquad 3A = 3 \; \left\{ \begin{aligned} \Rightarrow A &= 1 \\ B &= -1 \\ C &= -1 \end{aligned} \right.$$

$$\Rightarrow \int \frac{x+2}{x^3-1} = \int \frac{1}{x-1} - \int \frac{x+1}{x^2+x+1} = I_1 - I_2$$

$$I_1 = \int \frac{1}{x-1} = \ln|x-1| + C$$

$$I_2 = \frac{1}{2}\int \frac{2(x+\frac{1}{2})}{(x+\frac{1}{2})^2+\frac{3}{4}} + \int \frac{\frac{1}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{1}{2}\ln\left|(x+\tfrac{1}{2})^2+\tfrac{3}{4}\right| + \frac{1}{2}\cdot\frac{1}{\frac{\sqrt{3}}{2}}\,\mathrm{arctg}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{1}{2}\ln\left|(x+\tfrac{1}{2})^2+\tfrac{3}{4}\right| + \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + C$$

$$\Rightarrow \int \frac{x+2}{x^3-1} = -\frac{1}{2}\ln|x^2+x+1| - \frac{1}{\sqrt{3}}\,\mathrm{arctg}\left(\frac{2x+1}{\sqrt{3}}\right) + \ln|x-1| + C$$

Ime i prezime:

JMBAG: