9.4. Integrali trigonometrijskih funkcija

Postupak

- ako je podintegralna f-ja neparna u sinusu, koristimo supstituciju $\boxed{t = \cos x}$
- -II- ako je neparno u kosinusu onda $\boxed{t = \sin x}$

  ↳ primjer: $\int \sin^2 x \underbrace{\cos x}_{\text{neparnog stupnja}} dx = \left| \begin{matrix} \sin x = t \\ \cos x dx = dt \end{matrix} \right|$

- ukoliko je podintegralna f-ja parna u kosinusu i sinusu koristimo formule poluričnog argumenta

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1-\sin 2x}{2}$$

  ↳ primjer: $\int \sin^2 x \, dx = \int \frac{1-\cos^2 x}{x} dx$

- u slučaju umnoška sinusa i kosinusa s različitim argum. koristimo formule pretvorbe umnoška u zbroj

  ↳ primjer: $\int \sin 3x \cos x \, dx = \frac{1}{2} \int (\sin 4x + \sin 2x) dx$

  $\rightarrow \sin x \cos y = \frac{1}{2}(\sin(x+y) \dots)$

- u ostalim slučajevima možemo uvesti univerzalnu trigonom. supstituciju $t = tg\left(\frac{x}{2}\right)$

Primjer 9.51.) $\int \sin^4 x\,dx$

$\downarrow$

parna $\longrightarrow$ $\sin^2 x = \frac{1-\cos 2x}{2}$

$$\int\left(\frac{1-\cos 2x}{2}\right)^2 dx = \int\left(\frac{1-2\cos 2x+\cos^2 2x}{4}\right)^2 dx$$

$$= \frac{1}{4}\int\left(1-2\cos 2x+\cos^2 2x\right)dx$$

$$= \frac{1}{4}\int dx - \frac{1}{4}\cdot 2\int \cos 2x\,dx + \frac{1}{4}\int \cos^2 2x\,dx$$

$$= \frac{1}{4}x - \frac{1}{2}\cdot\sin 2x + \frac{1}{4}\int \frac{1-\sin 4x}{2}dx$$

rješenja u skripti su kriva

$$= \frac{1}{4}x - \frac{1}{2}\sin 2x + \frac{1}{8}\int(1-\sin 4x)\,dx$$

$$= \frac{1}{4}x - \frac{1}{2}\sin 2x + \frac{1}{8}x + \frac{1}{8}\cos 4x = \boxed{\frac{3}{8}x - \frac{1}{2}\sin 2x + \frac{1}{8}\cos 4x}$$

Univerzalna trigonometrijska supstitucija

prethodne metode nisu primjenjive na trig fje oblika $R(\sin x, \cos x)\,dx$ gdje je R racionalna fja.

Npr: $\int \frac{dx}{2+\sin x}$ $\quad \int \frac{dx}{\sin x + \cos x} \Rightarrow$ tada koristimo supstituciju

$$t = tg\left(\frac{x}{2}\right)$$

UNIVERZALNA TRIGONOMETRIJSKA SUPSTITUCIJA

Izvod:

$t = tg\left(\frac{x}{2}\right)$, $x \in \langle -\pi, \pi \rangle$, $t \in \mathbb{R}$

$\rightarrow x = 2 arctg\, t \Rightarrow \underbrace{dx = \frac{2dt}{1+x^2}}$

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\underbrace{\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}} \cdot \cos^2\frac{x}{2}$$

$$\boxed{\frac{1}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = tg^2 x + 1 \Rightarrow \cos^2 x = \frac{1}{tg^2 x + 1}}$$

$$\sin x = 2 tg\frac{x}{2} \cdot \frac{1}{tg^2\left(\frac{x}{2}\right)+1} = \left(t = tg\left(\frac{x}{2}\right)\right) \Rightarrow \boxed{\sin x = \frac{2t}{t^2+1}}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\underbrace{\left(1 - \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\right)}_{1 - tg^2\frac{x}{2}} = \frac{1 - tg^2\frac{x}{2}}{tg^2\left(\frac{x}{2}\right)+1}$$

$$\boxed{\cos x = \frac{1-t^2}{t^2+1}}$$

Primjer: 9.53) $\int \frac{dx}{1+\sin x+\cos x}$ $\longrightarrow$ $dx = \frac{2dt}{t^2+1}$

$$= \int \frac{\frac{2dt}{t^2+1}}{1+\sin x+\cos x} = \int \frac{\frac{2dt}{t^2+1}}{1+\frac{2t}{t^2+1}+\frac{1-t^2}{t^2+1}} = \int \frac{\frac{2dt}{\cancel{t^2+1}}}{\frac{\cancel{t^2}+1+2t+1-\cancel{t^2}}{\cancel{t^2+1}}}$$

$$= \int \frac{2dt}{2t+2} = \int \frac{dt}{t+1} = \ln|t+1| = \boxed{\ln\left(\operatorname{tg}\frac{x}{2}+1\right)+C}$$

Integrali hiperboličke funkcije

- integriraju analognim metodama kao trigonomet. fje koristeći osnovne identitete

$$\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1 \qquad \operatorname{ch}^2 x = \frac{\operatorname{ch}(2x)+1}{2} \qquad \operatorname{sh}^2 x = \frac{\operatorname{sh}(2x)-1}{2}$$

ili supstitucije $t = \operatorname{sh} x$ i $t = \operatorname{ch} x$.

* nekad je najjednostavnije prijeći na eksponencijalnu fju i onda tako integrirati

Primjer: $\int \operatorname{sh}^3 x \, dx$

1. način - analogne metode

$$\int \operatorname{sh}^2 x \operatorname{sh} x \, dx = \int (\operatorname{ch}^2 x - 1) \underbrace{\operatorname{sh} x \, dx}_{dt} = \left| \begin{array}{l} \operatorname{ch} x = t \\ \operatorname{sh} x \, dx = dt \end{array} \right|$$

$$\int (t^2-1)\, dt = \int t^2 dt - \int dt = \frac{1}{3}t^3 - t + C$$

$$\boxed{= \frac{1}{3}\operatorname{ch}^3 x - \operatorname{ch} x + C}$$

2. način - eksponenc.

$$\int \operatorname{sh}^3 x \, dx = \int \left(\frac{e^x - e^{-x}}{2}\right)^3 dx = \frac{1}{8}\int \left(e^{3x} - 3e^x + 3e^{-x} - e^{-3x}\right) dx$$

$$= \frac{1}{8}\left(\int e^{3x} dx - 3\int e^x dx + 3\int e^{-x} dx - \int e^{-3x} dx\right)$$

$$\boxed{= \frac{1}{8}\left(\frac{1}{3}e^{3x} - 3e^x + 3e^{-x} + \frac{1}{3}e^{-3x}\right)}$$