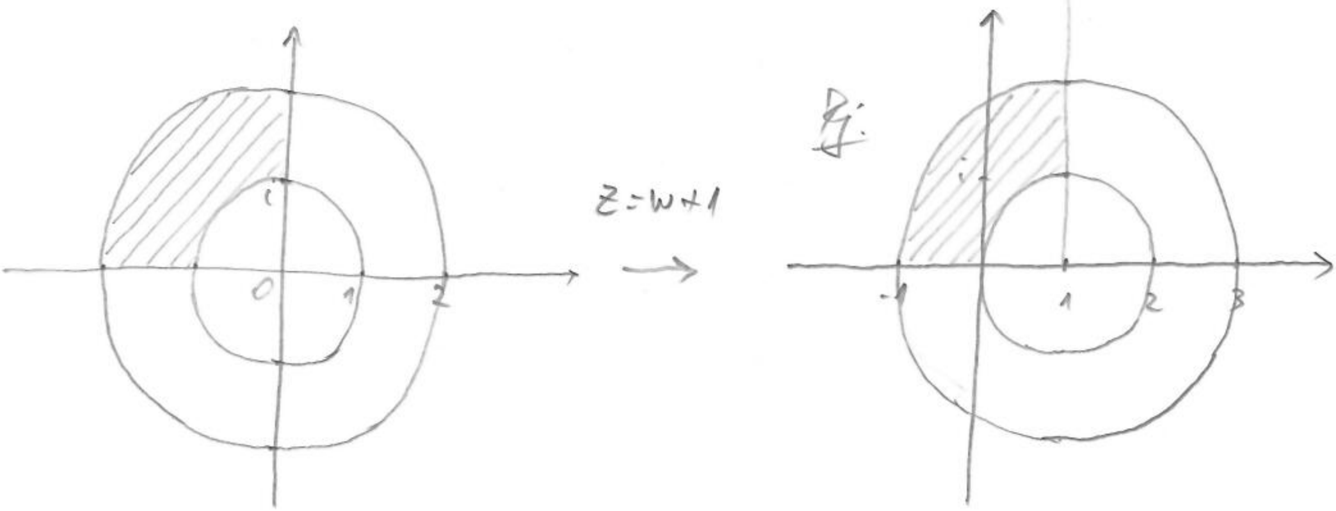
MATEMATIČKA ANALIZA 1 - DIR rješenja

1. a) $w = z-1 \Rightarrow 1 \leq |w| \leq 2$ & $\arg(w) \in [\frac{\pi}{2}, \pi]$



b) $z^3 = (1-i)\bar{z}$

- $|z^3| = |1-i||\bar{z}|$

  $r^3 = \sqrt{2}\, r$

  $r^3(r^2 - \sqrt{2}) = 0$

  $r = 0 \quad \vee \quad r^2 = \sqrt{2}$

  $r_1 = 0 \qquad r_2 = \sqrt[4]{2} \quad \cancel{r_3 = -\sqrt[4]{2} < 0}$

- $\varphi = \arg(z)$

  $3\varphi = \arg(1-i) + \arg(\bar{z}) + 2k\pi$

  $\rightarrow 1-i = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \Rightarrow \arg(1-i) = \frac{7\pi}{4}$

  $\rightarrow \arg(\bar{z}) = -\varphi$

  $3\varphi = \frac{7\pi}{4} - \varphi + 2k\pi \Rightarrow 4\varphi = \frac{7\pi}{4} + 2k\pi \Rightarrow \varphi = \frac{7\pi}{16} + \frac{k\pi}{2}, \ k = 0,1,2,3$

Rj: $z_k = \sqrt[4]{2}\left(\cos\left(\frac{7\pi}{16} + \frac{k\pi}{2}\right) + i\sin\left(\frac{7\pi}{16} + \frac{k\pi}{2}\right)\right), \ k = 0,1,2,3$

$z_4 = 0$ (zbog $r_1 = 0$)

2\. a) • ρ je simetrična

$$10\rho 20 \Rightarrow 20\rho 10$$

• ρ nije tranzitivna

$$20\rho 10 \wedge 10\rho 20, \text{ ali } 20 \not\rho 20$$

Budući da nije tranzitivna, ρ nije relacija ekvivalencija (a nije ni refleksivna jer $20 \not\rho 20$)

b) Iz (a) dijela vidimo da je u relaciju ρ dovoljno dodati uređeni par $(20,20)$ da dobijemo tranzitivnost i refleksivnost (a i simetričnost će biti sačuvana)

$\tilde{\rho} = \{(10,10),(10,20),(20,10),(20,20),(30,30)\}$

Klase ekvivalencije relacije $\tilde{\rho}$ su:

$[10] = \{10, 20\}$

$[30] = \{30\}$

Dakle, pripadni koeficijentni skup je $\{[10],[30]\}$

Particija od A generirana s $\tilde{\rho}$ je:

$A = \{10, 20\} \cup \{30\} \quad (\{10,20\} \cap \{30\} = \emptyset)$

3. a) Teorem 6.6.1.

b) $a_1 = \frac{1}{5}$

$a_2 = \frac{\frac{2}{5}+1}{5} = \frac{7}{25}$

$a_3 = \frac{\frac{14}{25}+1}{5}$

- TV: Niz $(a_n)_n$ je rastući, tj. za svaki $n \in \mathbb{N}$ vrijedi $a_n \leq a_{n+1}$

Dokaz indukcijom:

1° BAZA $(n=1)$

$$a_1 = \frac{1}{5} < \frac{7}{25} = a_2$$

2° PRETPOSTAVKA:

za neki $n \in \mathbb{N}$ vrijedi $a_n \leq a_{n+1}$

3° KORAK

$$a_n \leq a_{n+1} \quad / \cdot 2$$

$$2a_n \leq 2a_{n+1} \quad / +1$$

$$2a_n + 1 \leq 2a_{n+1} + 1 \quad / : 5$$

$$\frac{2a_n+1}{5} \leq \frac{2a_{n+1}+1}{5}$$

- TV: Niz $(a_n)_n$ je omeđen odozgo s 1, tj. za svaki $n \in \mathbb{N}$ vrijedi $a_n \leq 1$

Dokaz indukcijom

1° BAZA $(n=1)$

$$a_1 = \frac{1}{5} < 1$$

2° PRETPOSTAVKA

za neki $n \in \mathbb{N}$ vrijedi $a_n \leq 1$

3° KORAK

$$a_{n+1} = \frac{2a_n + 1}{5} \overset{\text{pretp.ind.}}{\leq} \frac{2+1}{5} \leq 1$$

Dakle, niz $(a_n)_n$ je rastući i omeđen odozgo pa po (a) dijelu zadatka zaključujemo da je niz $(a_n)_n$ konvergentan tj. postoji $L \in \mathbb{R}$ t.d.

$$L = \lim_{n \to \infty} a_n$$

$$a_{n+1} = \frac{2a_n + 1}{5} \quad \Big/ \lim_{n \to \infty}$$

$$L = \frac{2L+1}{5} \Rightarrow 5L = 2L + 1$$

$$\Rightarrow 3L = 1$$

$$\Rightarrow L = \frac{1}{3}$$

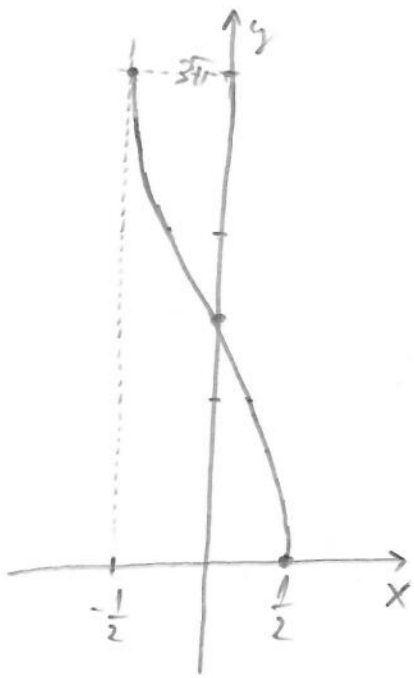
4. $f(x) = 3\arccos(2x)$

a) $\mathcal{D}(\arccos) = [-1, 1] \Rightarrow -1 \leq 2x \leq 1$

$\Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2}$

$\Rightarrow \mathcal{D}(f) = \left[-\frac{1}{2}, \frac{1}{2}\right]$

$\mathrm{Im}(\arccos) = [0, \pi] \Rightarrow \mathrm{Im}(f) = [0, 3\pi]$



b) $y = 3\arccos(2x) \quad /:3$

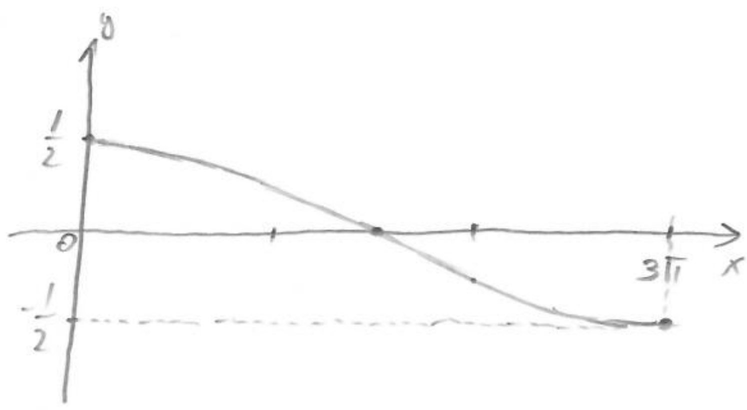
$\frac{y}{3} = \arccos(2x) \quad / \cos$

$\cos\left(\frac{y}{3}\right) = 2x \quad /:2$

$x = \frac{1}{2}\cos\left(\frac{y}{3}\right)$

$\Rightarrow f^{-1}(x) = \frac{1}{2}\cos\left(\frac{x}{3}\right)$

$\mathcal{D}(f^{-1}) = \mathrm{Im}(f) = [0, 3\pi]$



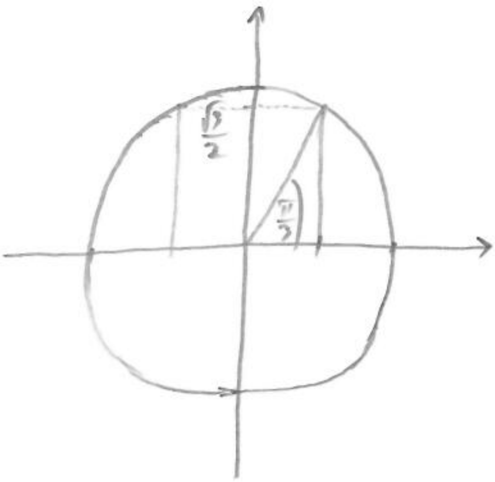
c) $f^{-1}(x) = \frac{1}{2}\cos\left(\frac{x}{3}\right) \quad , x\in[0, 3\pi]$

$$\frac{d}{dx} f^{-1}(x) = -\frac{1}{2}\sin\left(\frac{x}{3}\right)\cdot\frac{1}{3} = -\frac{1}{6}\sin\left(\frac{x}{3}\right)$$

$$-\frac{1}{6}\sin\left(\frac{x}{3}\right) = \frac{-\sqrt{3}}{12} \quad /\cdot(-6)$$

$$\sin\left(\frac{x}{3}\right) = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \frac{x}{3} = \frac{\pi}{3} + 2k\pi,\ k\in\mathbb{Z}$$

$$\frac{x}{3} = \frac{2\pi}{3} + 2k\pi,\ k\in\mathbb{Z}$$

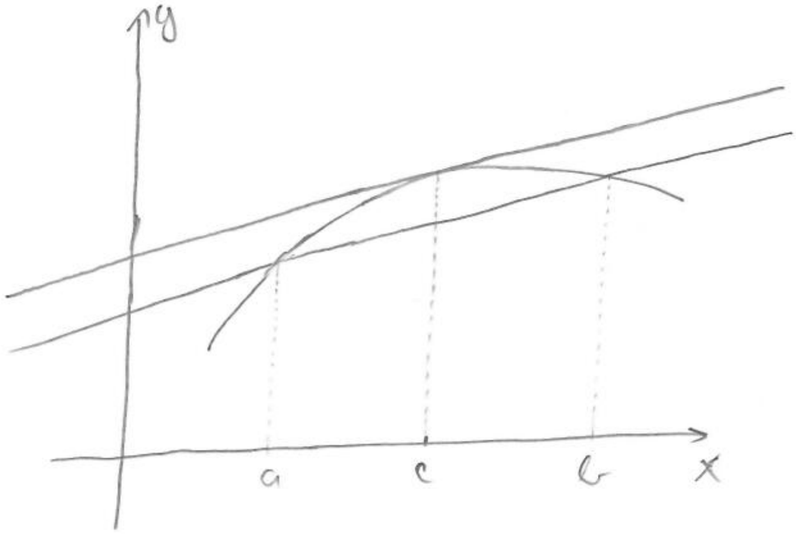


Budući da tražimo rješenja na segmentu $[0, 3\pi]$ (tj. u domeni f-je f^{-1}), imamo samo 2 rješenja:

$x = \pi$ & $x = 2\pi$

5. a) Teorem 9.2.3

b) Postoji tangenta koja je paralelna sa sekantom



c) Korolar 9.2.4 (i)

6. $f(x) = (x^2-3)e^x$

- $D(f) = \mathbb{R}$

- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x^2-3)e^x = \begin{bmatrix} x=-t \\ t\to+\infty \end{bmatrix} =$

$$= \lim_{t\to+\infty} (t^2-3)e^{-t} = \lim_{t\to+\infty} \frac{t^2-3}{e^t} = 0$$

U $-\infty$ f ima horizontalnu asimptotu $y=0$

$$\lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{(x^2-3)e^x}{x} = +\infty$$

U $+\infty$ f raste brže od linearne funkcije $\Rightarrow$ nema asimptote u $+\infty$

- $f'(x) = 2xe^x + (x^2-3)e^x = e^x(x^2+2x-3) = e^x(x+3)(x-1)$

$f'(x) = 0 \iff x=-3 \vee x=1$

- $f''(x) = e^x(x^2+2x-3) + e^x(2x+2) = e^x(x^2+4x-1)$

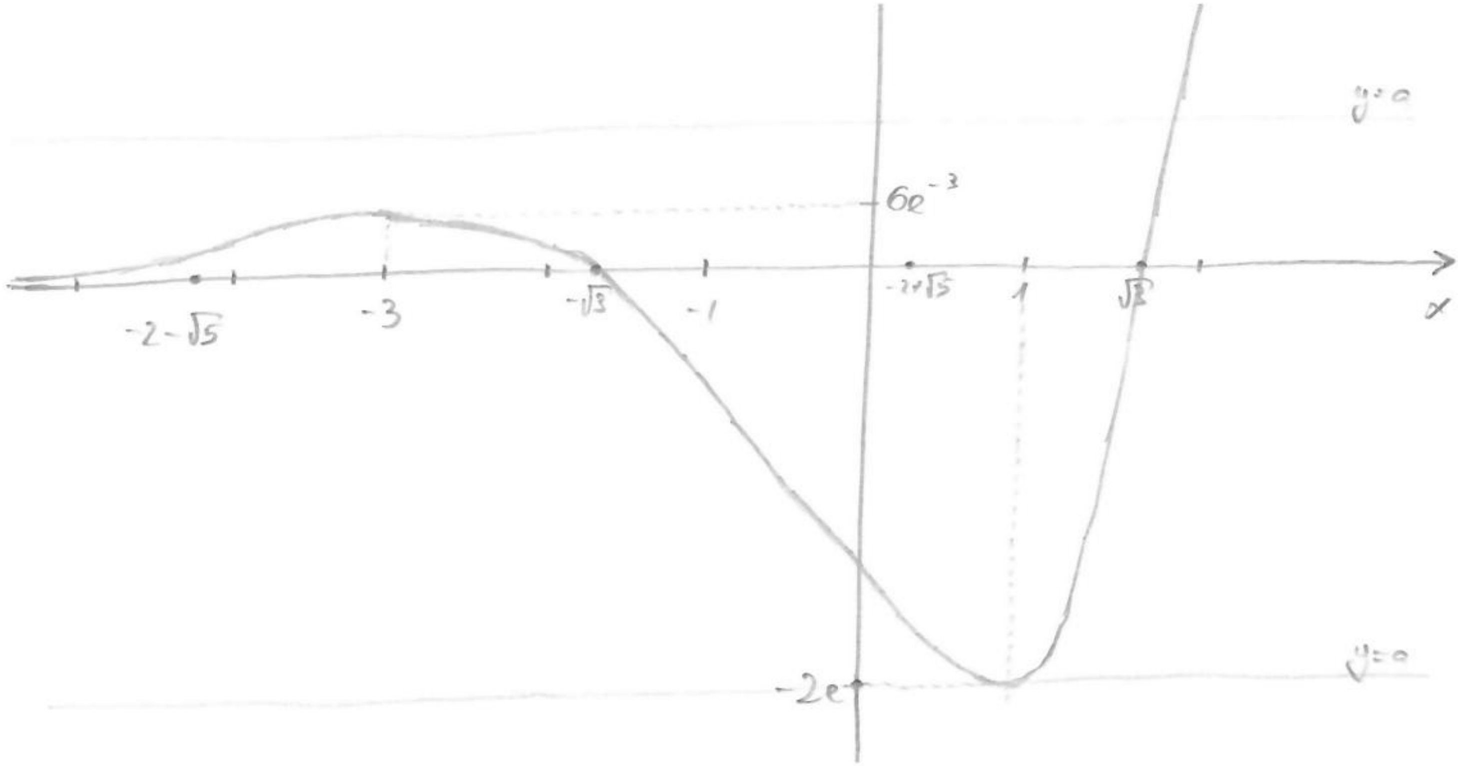
$f''(x) = 0 \iff x^2+4x-1=0$

$$x_{1,2} = \frac{-4 \pm \sqrt{16-4\cdot1\cdot(-1)}}{2} = \frac{-4\pm2\sqrt{5}}{2} = -2\pm\sqrt{5}$$

	$-\infty$ … $-2-\sqrt{5}$	$-2-\sqrt{5}$ … -3	-3 … $-2+\sqrt{5}$	$-2+\sqrt{5}$ … 1	1 … $+\infty$
f'	$+$	$+$	$-$	$-$	$+$
f''	$+$	$-$	$-$	$+$	$+$
f	$\nearrow$ $\cup$	$\nearrow$ $\cap$	$\searrow$ $\cap$	$\searrow$ $\cup$	$\nearrow$ $\cup$

Lok. max u $x=-3$ i iznosi $f(-3) = 6e^{-3}$

Lok. min u $x=1$ i iznosi $f(1) = -2e$



b) Sa slike vidimo da za $a = -2e$ i za $a > 6e^{-3}$

jednadžba $(x^2-3)e^x = a$ ima točno jedno rješenje

7. a) Definicija 11.2.1.

b) $$\int_0^1 \frac{x^3+1}{x^3+2x^2+x+2}\,dx$$

- $(x^3 \qquad +1) : (x^3+2x^2+x+2) = 1$

$$\frac{-x^3 \pm 2x^2 \pm x \pm 2}{-2x^2-x-1}$$

$$\Rightarrow \frac{x^3+1}{x^3+2x^2+x+2} = 1 - \frac{2x^2+x+1}{x^3+2x^2+x+2}$$

- $x^3+2x^2+x+2 = x^2(x+2) + (x+2) = (x+2)(x^2+1)$

$$\frac{2x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \quad / \cdot (x+2)(x^2+1) \qquad (*)$$

$$2x^2+x+1 = Ax^2+A+Bx^2+2Bx+Cx+2C$$

$$2x^2+x+1 = (A+B)x^2 + (2B+C)x + (A+2C)$$

$$\begin{aligned} A+B \qquad &= 2 \\ 2B+C &= 1 \\ A \qquad +2C &= 1 \end{aligned} \qquad \left(\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \end{array}\right) /\cdot(-1) \sim$$

$$\sim \left(\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & -1 & 2 & -1 \end{array}\right) /:2 \sim \left(\begin{array}{ccc:c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & -1 & 2 & -1 \end{array}\right) /\cdot(-1) \sim$$

$$\sim \left(\begin{array}{ccc:c} 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 5/2 & -1/2 \end{array}\right) /\cdot\frac{2}{5} \sim \left(\begin{array}{ccc:c} 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & 1/2 & 1/2 \\ 0 & 0 & 1 & -1/5 \end{array}\right) /\cdot(-1/2) \; /\cdot\frac{1}{2} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{7}{5} \\ 0 & 1 & 0 & \frac{3}{5} \\ 0 & 0 & 1 & -\frac{1}{5} \end{array}\right) \Rightarrow \begin{array}{l} A = \frac{7}{5} \\ B = \frac{3}{5} \\ C = -\frac{1}{5} \end{array}$$

$$\Rightarrow \frac{2x^2+x+1}{x^3+2x^2+x+2} = \frac{7/5}{x+2} + \frac{3/5 \cdot x - 1/5}{x^2+1} = \frac{7}{5(x+2)} + \frac{3x-1}{5(x^2+1)}$$

$$\int_0^1 \frac{x^3+1}{x^3+2x^2+x+2}\,dx = \int_0^1 dx - \int_0^1 \left(\frac{7}{5(x+2)} + \frac{3x-1}{5(x^2+1)}\right) dx = (*)$$

$$\int_0^1 dx = x\Big|_0^1 = 1$$

$$\int_0^1 \frac{7}{5(x+2)}\,dx = \frac{7}{5}\ln|x+2|\Big|_0^1 = \frac{7}{5}(\ln 3 - \ln 2) = \frac{7}{5}\ln\left(\frac{3}{2}\right)$$

$$\int_0^1 \frac{3x-1}{5(x^2+1)}\,dx = \frac{3}{5}\int_0^1 \frac{x}{x^2+1}\,dx - \frac{1}{5}\int_0^1 \frac{1}{x^2+1}\,dx = \left[\begin{array}{ll} t = x^2+1 & 0 \mapsto 1 \\ dt = 2x\,dx & 1 \mapsto 2 \end{array}\right] =$$

$$= \frac{3}{5}\int_1^2 \frac{1}{t}\,\frac{dt}{2} - \frac{1}{5}\operatorname{arctg} x\Big|_0^1 =$$

$$= \frac{3}{10}\ln|t|\Big|_1^2 - \frac{1}{5}\cdot\frac{\pi}{4} = \frac{3}{10}\ln 2 - \frac{\pi}{20}$$

$$(*) = 1 - \left(\frac{7}{5}\ln\left(\frac{3}{2}\right) + \frac{3}{10}\ln 2 - \frac{\pi}{20}\right) =$$

$$= 1 + \frac{\pi}{20} - \frac{7}{5}\ln\left(\frac{3}{2}\right) + \frac{3}{10}\ln 2$$

c) $$\int \frac{\cos x \sin^4 x}{\sin^{10} x + 1}\, dx = \left[\begin{array}{l} t = \sin x \\ dt = \cos x\, dx \end{array} \right] =$$

$$= \int \frac{t^4}{t^{10}+1}\, dt = \left[\begin{array}{l} u = t^5 \\ du = 5t^4 dt \Rightarrow t^4 dt = \frac{du}{5} \end{array} \right] =$$

$$= \frac{1}{5} \int \frac{du}{u^2+1} = \frac{1}{5} \operatorname{arctg} u + C = \frac{1}{5} \operatorname{arctg}(t^5) + C =$$

$$= \frac{1}{5} \operatorname{arctg}(\sin^5 x) + C, \quad C \in \mathbb{R}$$

(*) Napomena: Parcijalni razlomak u b) se može riješiti i uvrštavanjem

$$\frac{2x^2+x+1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1} \quad \Big| \cdot (x+2)(x^2+1)$$

$$2x^2+x+1 = A(x^2+1) + (Bx+C)(x+2)$$

$$\underline{x=-2} \rightsquigarrow \quad 7 = 5A \Rightarrow \boxed{A = 7/5}$$

$$\Rightarrow 2x^2+x+1 = \frac{7}{5}x^2 + \frac{7}{5} + Bx^2 + (2B+C)x + 2C$$

$$[x^2]: \quad 2 = \frac{7}{5} + B \Rightarrow \boxed{B = 3/5}$$

$$[x]: \quad 1 = 2B + C = \frac{6}{5} + C \Rightarrow \boxed{C = -1/5}$$

$$[1]: \quad 1 = 7/5 + 2C \checkmark$$

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8. a) Definicija 13.2.1

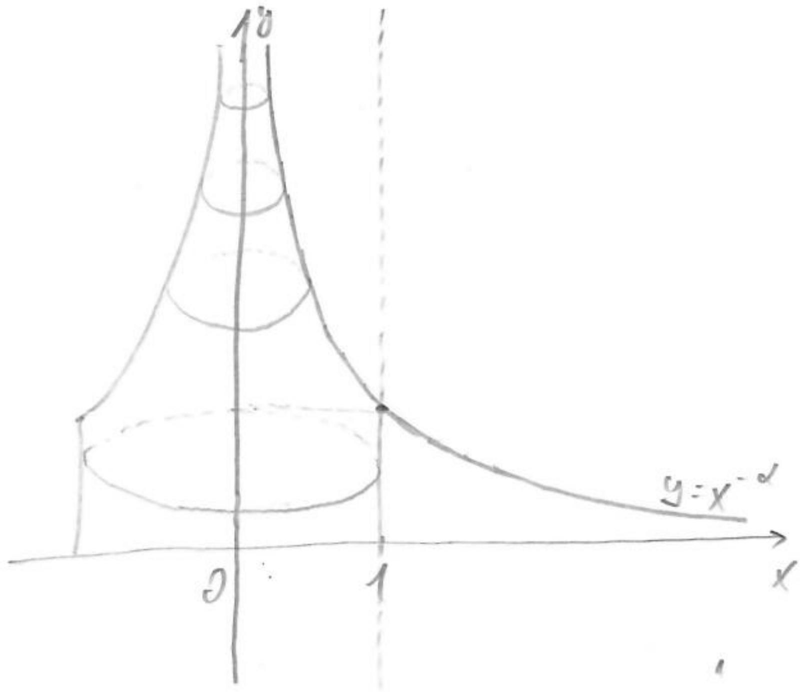
b) $$\int_1^{+\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to+\infty} \int_1^{b} \frac{x}{x^2+1}\,dx = \begin{bmatrix} t=x^2+1 & 1\mapsto 2 \\ dt=2x\,dx & b\mapsto b^2+1 \end{bmatrix}$$

$$= \lim_{b\to\infty} \frac{1}{2}\int_2^{b^2+1} \frac{dt}{t} = \frac{1}{2} \lim_{b\to+\infty} \ln|t| \Big|_2^{b^2+1} =$$

$$= \frac{1}{2} \lim_{b\to+\infty} \left(\ln(b^2+1) - \ln 2\right) = +\infty$$

$\Rightarrow$ Integral divergira u $+\infty$

c)



$$y = x^{-\alpha} \quad / ^{(-1/\alpha)}$$

$$y^{-1/\alpha} = x$$

$$V_\alpha = \int_0^1 1^2\cdot\pi\,dy + \int_1^{+\infty} (y^{-1/\alpha})^2\cdot\pi\,dy$$

$$O_\alpha = \int_0^1 2\cdot 1\cdot\pi\,dy + \int_1^{+\infty} 2\cdot(y^{-1/\alpha})\cdot\pi\,dy$$

$$V_\alpha < +\infty \iff \int_1^{+\infty} y^{-2/\alpha}\,dy < +\infty$$

$$O_\alpha = +\infty \iff \int_1^{+\infty} y^{-1/\alpha}\,dy = +\infty$$

- $\int_1^{+\infty} y^{-2/\alpha}\,dy = \int_1^{+\infty} \frac{1}{y^{2/\alpha}}\,dy < +\infty \iff \frac{2}{\alpha} > 1 \iff \alpha < 2$
- $\int_1^{+\infty} y^{-1/\alpha}\,dy = \int_1^{+\infty} \frac{1}{y^{1/\alpha}}\,dy = +\infty \iff \frac{1}{\alpha} \leq 1 \iff \alpha \geq 1$

$\Rightarrow$ Za $\alpha \in [1, 2\rangle$ imamo konačan volumen i beskonačno oplošje