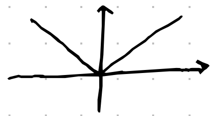
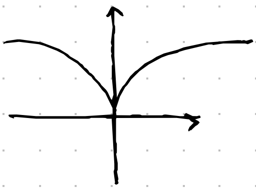
Pr.) F-je koje jesu neprekinute, ali nisu diferencijalne u $x_0 = 0$:



① $f(x) = |x|$

• f nepr u $x_0 = 0$: $\lim\limits_{x\to 0^+} |x| = \lim\limits_{x \to 0^-} |x| = 0 = f(0)$

• f difer. u $x_0 = 0$: $f'_+(0) = \lim\limits_{h\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^+} \frac{|h|}{h}$



$= \lim\limits_{h\to 0^+} \frac{\cancel{h}}{\cancel{h}} = 1$

$f'_-(0) = \lim\limits_{h\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{h\to 0^-} \frac{|h|}{h} = \lim \frac{-h}{h} = -1$

$\to$ desna derivacija $\neq$ lijeve $\rightsquigarrow$ nema difer. jed. ?

$f'_+(0) \neq f'_-(0) \Rightarrow \nexists f'(0)$

② $f(x) = \sqrt[3]{x^2}$: • f nepr u $x_0 = 0$: $\lim\limits_{x\to 0^+} \sqrt[3]{x^2} = \lim\limits_{x\to 0^-} \sqrt[3]{x^2} = f(0) = 0$ ✓

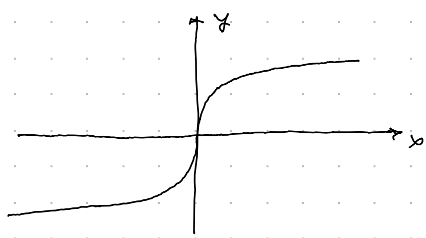
• f nije difer. u $x_0 = 0$:

$f'_+(0) = \lim\limits_{x\to 0^+} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x\to 0^+} \frac{\sqrt[3]{h^2}}{h} = \lim\limits_{x\to 0^+} \frac{1}{h^{1/3}} \left(\frac{1}{0^-}\right) = +\infty$

$f'_-(0) = \lim\limits_{x\to 0^-} \frac{f(0+h) - f(0)}{h} = \lim\limits_{x\to 0^-} \frac{\sqrt[3]{h^2}}{h} = \lim\limits_{h\to 0^-} \frac{1}{h^{1/3}} \left(\frac{1}{0^-}\right) = -\infty$

③ neprekinuta, nije difer., ima tangentu

$f(x) = \sqrt[3]{x}$



• f nepr u $x_0 = 0$: $\lim\limits_{x\to 0^+} \sqrt[3]{x} = 0$

$f(0) = 0$

$\lim\limits_{x\to 0^-} \sqrt[3]{x} = 0$ ✓

• ?

6.4. PRAVILA DERIVIRANJA

$\boxed{\text{TM}}$ Ako su f, g diferencijalne, tada vrijedi:

(1) $\left(f(x) \pm g(x)\right)' = f'(x) \pm g'(x)$

(2) $\left(f(x) \cdot g(x)\right)' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

(3) $\left(\dfrac{f(x)}{g(x)}\right)' = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{g^2(x)}, \; g \neq 0$

(4) $\left(c \cdot f(x)\right)' = c f'(x), \; c \in \mathbb{R}$

$\Big\}$ DZ

DOKAZ:

(1) $$\left(f(x) + g(x)\right)' = \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h}$$

$$= \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} \right) = \boxed{f'(x) + g'(x) \text{ ✓}}$$

(2) $$\left(f(x) \cdot g(x)\right)' = \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - g(x) \cdot f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x+h) + f(x) \cdot g(x+h) - f(x)\, g(x)}{h}$$

$$= \lim_{h \to 0} \left(g(x+h) \frac{f(x+h) - f(x)}{h} + f(x) \frac{g(x+h) - g(x)}{h} \right)$$

$$= \lim_{h \to 0} g(x+h) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = g(x) \cdot f'(x) + f(x) \cdot g'(x)$$

ZADACI Izračunajte derivacije fja:

a) $f(x) = 5 + x + 2\sqrt{x} + x^4$

$f(x) = c$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim \frac{c - c}{h} = \frac{0}{h}$

$= 0$

$f'(x) = (5')+(x)' + (2\sqrt{x})' + (x^4)'$

$= 0 + 1 + 2 \cdot \frac{1}{2\sqrt{x}} + 4x^3$

b) $f(x) = x^2 \ln(x)$

$f'(x) = 2x\ln(x) + x^2 \cdot \frac{1}{x} = 2x\ln(x) + x$

c) $f(x) = \frac{x^2+1}{2x-1}$

$f'(x) = \frac{(x^2+1)'(2x-1) - (x^2+1)(2x-1)'}{(2x-1)^2} = \frac{2x(2x-1) - (x^2+1)2}{(2x-1)^2}$

d) $f(x) = \sqrt{2} + x\sqrt{x} + \frac{1}{x^3} + \frac{\sin x}{x}$

$f'(x) = 0 + (x^{\frac{3}{2}})' + (x^{-3})' + \left(\frac{\sin x}{x}\right)$

$= \frac{3}{2}x^{\frac{1}{2}} - 3x^{-4} + \frac{(\sin x)' x - x'(\sin x)}{x^2}$

$= \frac{3}{2}x^{\frac{1}{2}} - \frac{3}{x^4} + \frac{\cos x \cdot x - \sin x}{x^2}$

VIŠE DERIVACIJA

$$f', \; f'' = (f')' = \frac{d^2 f}{dx^2}, \quad f'''(f'')' = \frac{d^3 f}{dx^3}$$

$$\boxed{\ldots \; f^{(n)} = \left(f^{(n-1)}\right)' = \frac{d^n f}{dx^n} =}$$ n-ta derivacija

Zad Izračunajte $f^{IV}(x)$ ako je $f(x) = x^7 + 3x^4 + \sqrt{x} + 2$

Rj:

$$f'(x) = 7x^6 + 12x^3 + \frac{1}{2}x^{-\frac{1}{2}} + 0$$

$$f''(x) = 42x^5 + 36x^2 + \frac{1}{2}\cdot\left(\frac{-1}{2}\right)x^{-\frac{3}{2}}$$

$$f'''(x) = 210x^4 + 72x - \frac{1}{4}\cdot\left(\frac{-3}{2}\right)x^{-\frac{5}{2}}$$

$$f^{IV}(x) = 840x^3 + 72 + \frac{3}{8}\cdot\left(\frac{-5}{2}\right)x^{-\frac{7}{2}}$$

6.5. DERIVACIJA SLOŽENE F-IJE

Kako definirati kompoziciju $f \circ g$?

TM

Neka je $f \circ g$ definirano u x te je g difer. u x i f difer u $g(x)$. Tada je $f \circ g$ difer u x i $(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$

Primjer) Izračunajte (a) $f_1(x) = \sin(x^2)$

vanjska, unutarnja

a) $f_1'(x) = (\sin(x^2))' \cdot \cos(x^2) \cdot 2x$

vanjska

b) $f_2(x) = \sin^2 x$ $\qquad f(x) = x^2$

unutarnja

Derivacije inverzne funkcije

Promatramo vezu između f' i $(f^{-1})'$!

Izvod formule: $(f \circ f^{-1})(y) = y \quad \Big| \ \frac{d}{dy}$

$$f'(f^{-1}(y)) = (f^{-1}(y))' = 1$$

$$(f^{-1}(y))' = \frac{1}{f'(f^{-1}(y))} = \frac{1}{f'(x)}$$

FORMULA:

$$\boxed{(f^{-1})(y) = \frac{1}{f'(x)}}$$

- Drugi zapis formule

$$y = y(x) \iff x = x(y)$$

$$y' = \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{x'} \longrightarrow y' = \frac{1}{x'} \qquad y'(x) = \frac{1}{x'(y)}$$

zgodno za primjenu u zad.

ZAD.)

Koristeći formulu za derv. inv. funkcije nađite derv. sljed. f-ja.

a) $f(x) = 3\sqrt{x}$ b) $f(x) = \arcsin x$

1. formula
2. izračunati deriv te inverzne f-je
3. vratiti x

a) $y = 3\sqrt{x} \Rightarrow x = y^3 \Rightarrow x' = (y^3)' = 3y^2$

$$y' = \frac{1}{x'} = \frac{1}{3y^2} = \frac{1}{3 \cdot (3\sqrt{x})^2}$$

moramo vratiti x

b) $y = \arcsin x \Rightarrow x = \sin y \Rightarrow x' = \cos y$

$$y' = \frac{1}{x'} = \frac{1}{\cos y} = \frac{1}{\cos(\arcsin x)} =$$

u x *malo zeznuto

$$\sin^2 x + \cos^2 x = 1$$
$$\cos^2 x = 1 - \sin^2 x \ \Big| \sqrt{\ }$$
$$\cos x = \pm\sqrt{1 - \sin^2 x}$$

$$= \frac{1}{\pm\sqrt{1 - \sin^2(\arcsin x)}} = \frac{1}{\pm\sqrt{1 - x^2}}$$

$\arcsin x$ leži na $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\hookrightarrow \arcsin x \geq 0$

trebamo $+$

$$\boxed{\Rightarrow \arcsin x \left(\frac{1}{\sqrt{1 - x^2}} \right)}$$

Matematičko modeliranje nečemu raditi