# KOMPLEKSNI BROJEVI

$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$

$z \in \mathbb{C} \qquad z = x + iy$

$x = \mathrm{Re}\, z \qquad y = \mathrm{Im}\, z \qquad i^2 = -1$

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$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

1. Jednakost kompl. br.

$$z_1 = z_2 \iff \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$$

2. Zbrajanje: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

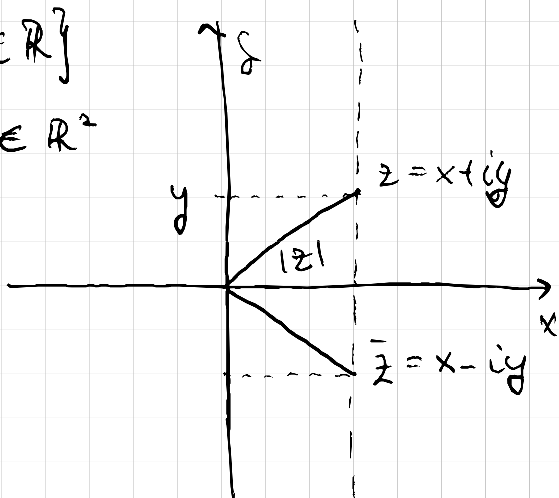
3. Množenje: $z_1 z_2 = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$

4. Dijeljenje: $\frac{z_1}{z_2} = \frac{x_1 + y_1}{x_2 + y_2} \cdot \frac{x_2 - y_2}{x_2 - y_2} = \frac{(x_1 + y_1)(x_2 - y_2)}{x_2^2 - y_2^2}$

Kompleksna Gaussova ravnina

$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$

$z = x + iy \longmapsto T(x, y) \in \mathbb{R}^2$




Modul kompl. broja

$|z| = \sqrt{x^2 + y^2}$

## Konjugirano kompl. broj broja $z = x + iy$

$$\boxed{\bar{z} = x - iy}$$

VRIJEDI:

1. $z \cdot \bar{z} = |z|^2$
2. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
3. $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
4. $\overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\bar{z}_1}{\bar{z}_2}\right)$

1. $z \cdot \bar{z} = |z|^2$

Dokaz: $(x+iy)(x-iy) = x^2 + y^2 = |z|^2$

2. $|z_1 z_2| = |z_1| \, |z_2|$

Dokaz: DZ

3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ Dokaz DZ.

## 4. $|z^n| = |z|^n$ za $\forall n \in \mathbb{N}$

Dokaz: mat. ind.

1. BAZA $\quad n=1 \quad |z| = |z|$ ✓

2. PRETP. za neki $n \in \mathbb{N}$ vrijedi $|z^n| = |z|^n$

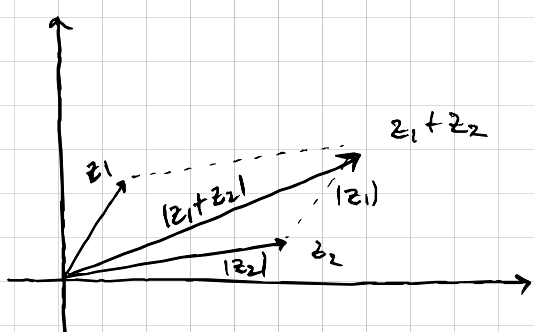
3. KORAK $\quad T: |z^{n+1}| = |z|^{n+1}$

$$|z^{n+1}| = |z^n \cdot z| \overset{2.\text{modul}}{=} |z^n| \cdot |z| = |z|^n \cdot |z| = |z|^{n+1} \checkmark$$

## 5. $|z_1 + z_2| \leq |z_1| + |z_2|$

Zbrajamo po pravilu paralelograma vektorima

Dokaz




Tvrdnja vrijedi iz nejednakosti trokuta?

P5)

Odredite $\left|\frac{z_1}{z_2}\right|$ ako je $z_1 = 2-3i$

$z_2 = 1+2i$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{\sqrt{4+9}}{\sqrt{1+4}} = \sqrt{\frac{13}{5}}$$

P6)

Odredite $Re\, z$ i $|z|$ ako je $z = (1-2i)^2$

$z = 1-4i+4i^2 = -3-4i$

$Re\, z = -3$

$|z| = \sqrt{9+16} = \underline{5}$

DZ:

② $|z_1 z_2| = |z_1|\,|z_2|$

Dokaz: DZ

2.

$|z_1 z_2| = |(x_1 + y_1 i)(x_2 + y_2 i)|$

$= |(x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i|$

$= \sqrt{(x_1 x_2 - y_1 y_2)^2 + (x_1 y_2 + x_2 y_1)^2}$

$= \sqrt{x_1^2 x_2^2 \cancel{- 2 x_1 x_2 y_1 y_2} + y_1^2 y_2^2 + x_1^2 y_2^2 \cancel{+ 2 x_1 y_2 x_2 y_1} + x_2^2 y_1^2}$

$= \sqrt{x_1^2 (x_2^2 + y_2^2) + y_1^2 (x_2^2 + y_2^2)}$

$= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{x_1^2 + y_1^2} \cdot \sqrt{x_2^2 + y_2^2}$

$\boxed{= |z_1| \cdot |z_2|}$ ✓

3. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ Dokaz Dz.

$$\searrow \quad \frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}}$$

$$\left|\frac{z_1}{z_2}\right| = |w|$$

$$w = \frac{z_1}{z_2} = \frac{x_1+iy_1}{x_2+iy_2} \cdot \frac{x_2-iy_2}{x_2-iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{x_2^2-(iy_2)^2}$$

$-1 \searrow +$

$$= \frac{x_1x_2 - i\cdot y_2\cdot x_1 + i y_1\cdot x_2 + y_1y_2}{x_2^2+y_2^2}$$

$$= \frac{x_1x_2+y_1^2+(-x_1y_1+x_2y_1)i}{x_2^2+y_1^2}$$

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pokušaj unazad.

$$\cancel{\frac{\sqrt{x_1^2+y_1^2}}{\sqrt{x_2^2+y_2^2}} = \sqrt{\frac{x_1^2+y_1^2}{x_2+y_2^2}}} =$$

ne znam

6.10. Petak

$|z|$ = udaljenost komp. br. $z$ od ishodišta

$|z - z_0|$ = udaljenost komp. br. $z$ i $z_0$

$\boxed{|z - z_0| = r}$

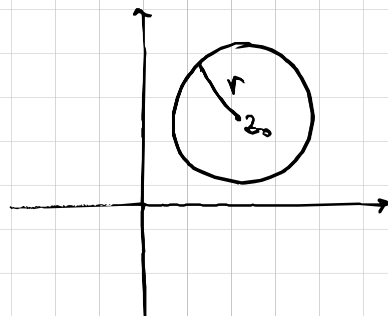
$z_0$ stalan $\quad$ $r$ stalan

$z$ varijabilan

KRUŽNICA

$S(x_0, y_0), r$

$z_0 = x_0 + iy_0$




$$z - z_0 = (x - x_0) + i(y - y_0)$$

$$|z - z_0| = \sqrt{(x - x_0)^2 + i(y - y_0)^2}$$

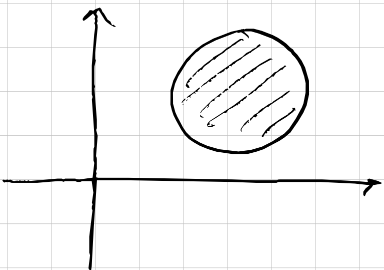
$$|z - z_0| = r$$

$$\sqrt{(x - x_0)^2 + i(y - y_0)^2} = r \quad /^2$$

$$\Downarrow$$

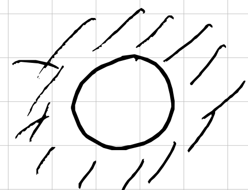
$$(x - x_0)^2 + i(y - y_0)^2 = r^2$$

$|z - z_0| < r$ unutar kružnice bez kružnice



 sa kružnicom

$|z - z_0| > r$



van kružnice

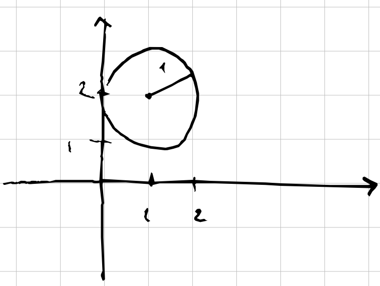
ZAD)

Odredite i skicirajte skup $\mathbb{C}$ koji zadovoljavaju uvjete:

a) $|z-1-2i|=1$

$|z-(1+2i)|=1$

$S(1,2)$ $r=1$




b) $\begin{cases} 1<|z-2+3i|<2 \\ \mathrm{Re}\, z \geq 2 \end{cases}$

$z-(2-3i)$

$r_1 = 1$

$S(2,-3)$ $r_2 = 2$