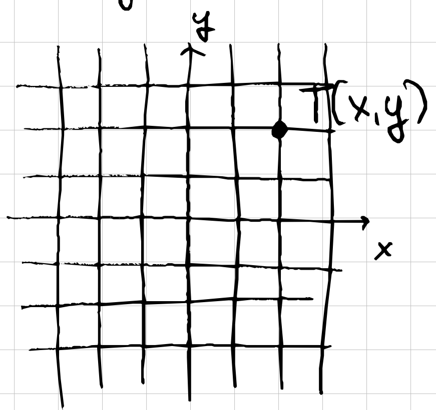
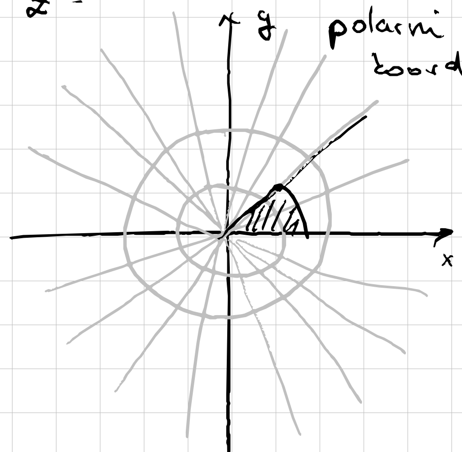
TRIGONOMETRIJSKI ZAPIS KOMPL. BROJA

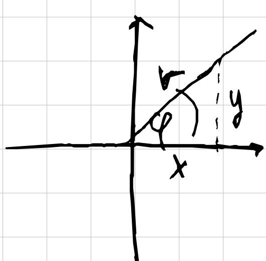
$z = x + iy$



prav. koord. sustav

$z =$





$z = x + iy$ $\quad |z|$

$x = r \cos\varphi$

$y = r \cdot \sin\varphi$

$r = |z|$

$r = \sqrt{x^2 + y^2}$

$r \geq 0$
$\forall x \in \mathbb{R}$

$\operatorname{tg}\varphi = \frac{y}{x}$

φ je $\arg z$ $(+2k\pi)$

* paziti u kojem smo kvadrantu dobili rj.

↓

predznaci određuju kvadrant

$\operatorname{Arg} z = \{\varphi + 2k\pi,\ k \in \mathbb{Z}\}$

$\varphi = [0, 2\pi]$

$\boxed{\text{TRIG. ZAPIS} \quad z = r\cos\varphi + i \cdot r \cdot \sin\varphi}$

$z = r(\cos\varphi + i \cdot \sin\varphi)$

$z = r \cdot \underbrace{\operatorname{cis}\varphi}$ cosinus i sinus od φ

ZAD.) Slj. kompl. br. prikažite u trig. obliku

a) $z_1 = 1 + i$

$r = \sqrt{1+1} = \sqrt{2}$

$\operatorname{tg}\varphi = \frac{y}{x} = \frac{1}{1} = 1$

$\hookrightarrow$ 1. kv.

$\varphi = \frac{\pi}{4} + k\pi, \ k \in \mathbb{Z}$

$\hookrightarrow \boxed{\varphi = \frac{\pi}{4}}$

$z_1 = \sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

b) $z_2 = \frac{-1}{2} - \frac{\sqrt{3}}{2} i$

$r = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

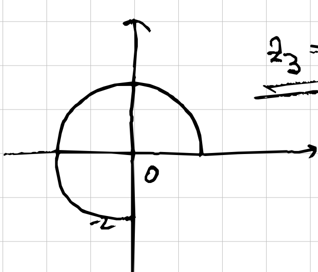
$\operatorname{tg}\varphi = \frac{-\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \sqrt{3}$

$\hookrightarrow$ 3 kv.

$\varphi_1 = \frac{\pi}{3} \longrightarrow \boxed{\varphi = \frac{4\pi}{3}}$

$z_2 = 1 \cdot \operatorname{cis}\left(\frac{4\pi}{3}\right)$

c) $z_3 = -2i$



$z_3 = 2\operatorname{cis}\left(\frac{\pi}{2}\right)$

d) $z_4 = 5$

$z_1 = 5 \operatorname{cis}(0)$

$$z_1 = r_1 (\cos\varphi_1 + i \sin\varphi_1) \qquad z_2 = r_2 (\cos\varphi_2 + i \sin\varphi_2)$$

1. Jednakost kompl. br.

$$z_1 = z_2 \iff \begin{cases} r_1 = r_2 \\ \varphi_1 = \varphi_2 + 2k\pi \end{cases}$$

2. Množenje

$$z_1 \cdot z_2 = r_1 \cdot r_2 \left(\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2) \right)$$

$$|z_1 z_2| = |z_1| |z_2|$$

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$$

Dokaz:

$$z_1 \cdot z_2 = r_1 (\cos\varphi_1 + i \sin\varphi_1) \cdot r_2 (\cos\varphi_2 + i \sin\varphi_2)$$

$$= r_1 r_2 \left(\cos{}_1 \cdot \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2 + i (\cos\varphi_1 \sin\varphi_2 + \cos\varphi_2 \cdot \sin\varphi_1) \right)$$

$$= r_1 \cdot r_2 \left(\cos(\varphi_1 + \varphi_2) + i \sin(\varphi_1 + \varphi_2) \right)$$

$\hookrightarrow$ adicijske formule

3. Dijeljenje

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\varphi_1 - \varphi_2) + i \cdot \sin(\varphi_1 - \varphi_2)\right)$$

$$\Rightarrow \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2k\pi$$

Dokaz:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\varphi_1 + i\sin\varphi_1)}{r_2(\cos\varphi_2 + i\sin\varphi_2)} \cdot \frac{\cos\varphi_2 - i\sin\varphi_2}{\cos\varphi_2 - i\sin\varphi_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\text{.......}}{\cos^2\varphi_2 + \sin^2\varphi_2 = 1}$$

4. Potenciranje (Moivreova formula)

$$z^n = r^n\left(\cos(n\varphi) + i\sin(n\varphi)\right) \Longleftrightarrow \begin{array}{l} |z^n| = |z|^n \\ \arg(z^n) = n \cdot \arg z + 2k\pi \end{array}$$

Dokaz: mat. ind

1. BAZA $n=2$ $z^2 = r^2(\cos 2\varphi + i\sin(2\varphi))$ ✓
   ↳ slijedi iz z
2. PRETP. za neki $n \in \mathbb{N}$ $z^n = r^n(\cos(n\varphi) + i\sin(n\varphi))$
3. KORAK: T: $z^{n+1} = r^{n+1}(\cos(n+1)\varphi + i\sin(n+1)\varphi)$

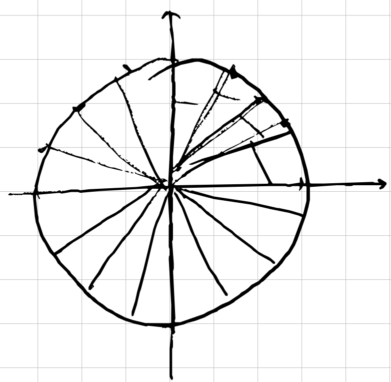
$$z^{n+1} = z \cdot z^n = r(\cos\varphi + i\sin\varphi)\, r^n(\cos n\varphi + i\sin n\varphi)$$

$$z^{n+1} = r^{n+1}(\ldots \text{ pomnožimo svaki sa svakim})$$

$$\boxed{= r^{n+1}(\cos(n+1)\varphi + i\sin(n+1)\varphi)}$$ ✓

5. Korjenovanje

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad \underbrace{k=0,1,\dots,n-1}_{n \text{ korijena}}$$



Kompl. br. $\sqrt[n]{z}$ su vrhovi pravilnog n-terokuta upisanog u kružnicu radijusa $\sqrt[n]{r}$ sa središtem u ishodištu.

ZAD 1.) Izračunajte

a) $\mathrm{Re}\,(1+i)^{10}$

b) $\arg\left(\frac{z_1}{z_2}\right)$
$z_1 = 1+i$
$z_2 = \sqrt{3}-i$

c) $\arg z$ ako je
$z = -\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}$

a) $1+i = \sqrt{2}\,\mathrm{cis}\frac{\pi}{4}$

$$(1+i)^{10} = (\sqrt{2})^{10}\left(\cos\frac{\pi}{4}\cdot 10 + i\sin\frac{\pi}{4}\cdot 10\right)$$

$$= 32\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = 32\left(\underset{0}{\underset{\shortparallel}{\cos\frac{\pi}{2}}} + i\underset{1}{\underset{\shortparallel}{\sin\frac{\pi}{2}}}\right)$$

$$= 32i \qquad \boxed{\mathrm{Re}\,(1+i)^{10} = 0}$$

b) $1+i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$

$\sqrt{3}-i = 2 \operatorname{cis} \frac{11\pi}{6}$

$$\arg\left(\frac{1+i}{\sqrt{3}-i}\right) = \arg(1+i) - \arg(\sqrt{3}-i) + 2k\pi$$

$$= \frac{\pi}{4} - \frac{11\pi}{6} + 2k\pi = \frac{19\pi}{12} + 2k\pi$$

$$= \frac{5\pi}{12} + 2k\pi \;?$$

c) $z = \underbrace{-\cos\frac{7\pi}{6}}_{-\frac{\sqrt{3}}{2}} + i\underbrace{\sin\frac{7\pi}{6}}_{-\frac{1}{2}} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$ II. kv.

IV.

$r = \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$

$\operatorname{tg}\varphi = \dfrac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \dfrac{-1}{\sqrt{3}} = \dfrac{-\sqrt{3}}{3}$

Zad. Riješite jednadžbu

a) $z^{10} + 3z^5 - 4 = 0$

b) $z^7 - z^5 + z^2 - 1 = 0$

c) $(z-1)^6 + 64 = 0$

d) $z^4 + \left(\sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right)^8 = 0$

a) $z^{10} + 3z^5 - 4 = 0$

supstitucija $t = z^5$

$t^2 + 3t - 4 = 0$

$t^2 + 4t - t - 4 = 0$

$t(t-1) + 4(t-1) = 0$

$(t-1)(t+4) = 0$

$t = 1$

$z^5 = 1$

$z = \sqrt[5]{1}$

$z = \sqrt[5]{1 \cdot \text{cis } 0}$

$$\boxed{= \sqrt[5]{1} \cdot \left(\cos \frac{0 + 2k\pi}{5} + i \sin \frac{0 + 2k\pi}{5} \right) \quad k = 0, 1, 2, 3, 4}$$

$t = -4$

$z^5 = -4$

$z = \sqrt[5]{-4}$

$$z = \sqrt[5]{4 \text{ cis } \pi} = \boxed{\sqrt[5]{4} \cdot \left(\cos \frac{\pi + 2k\pi}{5} + i \sin \frac{\pi + 2k\pi}{5} \right) \quad k = 0, 1, 2, 3, 4}$$

b) $z^7 - z^5 + z^2 - 1 = 0$

$z^5(z^2-1) + (z^2-1) = 0$

$(z^2-1)(z^5+1) = 0$

$z^2 - 1 = 0$

$z^2 = 1 \quad / \sqrt{\ }$

$z = \sqrt{1}$

$(z+1)(z-1) = 0$

$\hookrightarrow z_{1,2} = \pm 1$

$z^5 + 1 = 0$

$z^5 = -1$

$z = \sqrt[5]{-1} = \sqrt[5]{1 \text{ cis } \pi}$

$= 1 \left(\text{cis } \frac{\pi + 2k\pi}{5} \right)$

$k = 0, 1, 2, 3, 4$

c) $(z-i)^6 + 64 = 0$

$(z-i)^6 = -64 \quad / \sqrt[6]{\ }$

$(z-i) = \sqrt[6]{-64} = \sqrt[6]{64} \text{ cis } \pi = 2 \cdot \text{cis } \frac{\pi + 2k\pi}{6},$

$\sqrt[6]{64} = 2$

$k = 0, 1, 2, 3, 4, 5$

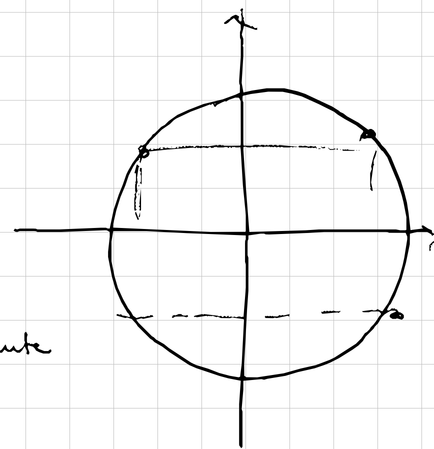
$$\boxed{z_{1,2,3,4,5,6} = 2 \cdot \text{cis } \frac{\pi + 2k\pi}{6} + i, \quad k = 0, 1, 2, 3, 4, 5}$$

d) $z^4 + \left(\sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right)^8 = 0$

$z^4 = -\left(\sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)\right)^8$

$z^4 = -16\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)^8$

$z^4 = -16\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$



I. način – kružnica

minus će samo prebaciti kvadrante

$z^4 = 16\left(-\cos\frac{4\pi}{5} - i\sin\frac{4\pi}{5}\right)$

$z^4 = 16 \operatorname{cis}\frac{9\pi}{5}$

II. način -1 pretvoriti u trig zapis $-1 \operatorname{cis}\pi$

i pomnožit sa