1. (a) $z = r(\cos\varphi + i\sin\varphi)$

$|z| = 1 \Rightarrow r = 1$

$\Rightarrow Re(z^2) = \cos(2\varphi)$

$Re(z^4) = \cos(4\varphi)$

$\cos(2\varphi) = \cos(4\varphi)$

$\Leftrightarrow \cos(2\varphi) = \cos^2(2\varphi) - \sin^2(2\varphi)$

$\Leftrightarrow \cos(2\varphi) = 2\cos^2(2\varphi) - 1$

$\Leftrightarrow 2\cos^2(2\varphi) - \cos(2\varphi) - 1 = 0$

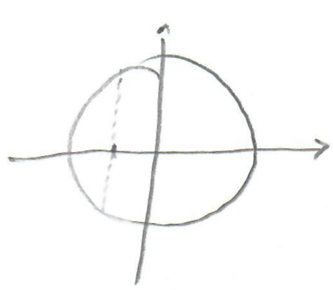
$t = \cos(2\varphi)$

$2t^2 - t - 1 = 0$

$$t_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}$$

$t_1 = -\frac{1}{2} \quad t_2 = 1$

$1^\circ \quad \cos(2\varphi) = -\frac{1}{2} \Rightarrow \begin{cases} 2\varphi = \frac{2\pi}{3} + 2k\pi, \ k \in \mathbb{Z} \\ 2\varphi = -\frac{2\pi}{3} + 2k\pi, \ k \in \mathbb{Z} \end{cases}$



$\Rightarrow \begin{cases} \varphi = \frac{\pi}{3} + k\pi, \ k \in \mathbb{Z} \\ \varphi = -\frac{\pi}{3} + k\pi, \ k \in \mathbb{Z} \end{cases}$

$$\Rightarrow z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$z_3 = \cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2}$$

$$z_4 = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

2° $\cos(2\varphi) = 0$

$$\Rightarrow 2\varphi = 2k\pi, \ k \in \mathbb{Z}$$

$$\varphi = k\pi, \ k \in \mathbb{Z}$$

$$\Rightarrow z_5 = \cos(0) + i\sin(0) = 1$$

$$z_6 = \cos(\pi) + i\sin(\pi) = -1$$

(b) $|z| = a \Rightarrow r = a$

$$Re(z^2) = a^2\cos(2\varphi)$$

$$Re(z^4) = a^4\cos(4\varphi) = a^4(2\cos^2(2\varphi) - 1)$$

$$Re(z^2) = Re(z^4) \Leftrightarrow a^2\cos(2\varphi) = 2a^4\cos^2(2\varphi) - a^4$$

$$\Leftrightarrow 2a^4\cos^2(2\varphi) - a^2\cos(2\varphi) - a^4 = 0$$

$$t = \cos(2\varphi)$$

$$2a^4t^2 - a^2t - a^4 = 0$$

$$t_{1,2} = \frac{a^2 \pm \sqrt{a^4 + 8a^8}}{4a^4} = \frac{a^2 \pm \sqrt{a^4(1+8a^4)}}{4a^4}$$

$$= \frac{a^2 \pm a^2\sqrt{1+8a^4}}{4a^4} = \frac{1 \pm \sqrt{1+8a^4}}{4a^2}$$

$$t_1 = \frac{1+\sqrt{1+8a^4}}{4a^2} \qquad t_2 = \frac{1-\sqrt{1+8a^4}}{4a^2}$$

1° Uočimo: $t_1 > 0 \quad \forall a > 0$

$$t_1 \le 1 \iff \frac{1+\sqrt{1+8a^4}}{4a^2} \le 1$$

$$\iff 1+\sqrt{1+8a^4} \le 4a^2$$

$$\iff \sqrt{1+8a^4} \le 4a^2 - 1 \quad /^2$$

$$\iff 1+8a^4 \le 16a^4 - 8a^2 + 1$$

$$\iff 8a^4 - 8a^2 \ge 0$$

$$\iff 8a^2(a^2-1) \ge 0$$

$$\iff a^2 - 1 \ge 0$$

$$\iff a \le -1 \text{ ili } a \ge 1$$

Budući da imamo $a > 0 \Rightarrow$
$t_1 > 1$ za $a < 1$
$t_1 = 1$ za $a = 1$
$t_1 < 1$ za $a > 1$

Dakle, za $a < 1$ imamo $\cos(2\varphi) = t_1 > 1$ što ima 0 rješenja

$a = 1$ $\cos(2\varphi) = t_1 = 1$ što ima 2 rješenja

$a > 1$ $\cos(2\varphi) = t_1 \in (0,1)$ što ima 4 rješenja

2° Uočimo $t_2 < 0$ $\forall a > 0$

$$t_2 \geq -1 \iff \frac{1-\sqrt{1+8a^4}}{4a^2} \geq -1$$

$$\iff 1-\sqrt{1+8a^4} \geq -4a^2$$

$$\iff 1+4a^2 \geq \sqrt{1+8a^4} \; /^2$$

$$\iff 1+8a^2+16a^4 \geq 1+8a^4$$

$$\iff 8a^4+8a^2 \geq 0$$

Za svaki $a > 0$ $8a^4+8a^2 > 0 \Rightarrow \forall a > 0$ $t_2 \in (-1,0)$

$\Rightarrow \cos(2\varphi) = t_2$ ima 4 rješenja

$1^\circ, 2^\circ \Rightarrow$ sustav jednadžbi

$|z| = a$, $Re(z^4) = Re(z^2)$ ima

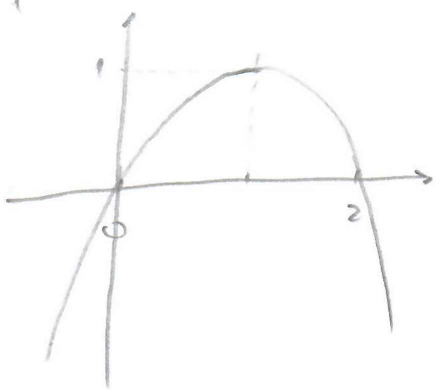
$$\begin{cases} 8 \text{ rješenja za } a > 1 \\ 6 \text{ rješenja za } a = 1 \\ 4 \text{ rješenja za } a \in (0,1) \end{cases}$$

2. (a) $f(x) = \mathrm{sh}(2x - x^2)$

$f_1(x) = 2x - x^2$

$f_2(x) = \mathrm{sh}(x)$

$2x - x^2 = 0 \iff x(2-x) = 0$



$\forall x \in \mathbb{R} \quad -\infty < f_1(x) \le 1$

$\Rightarrow -\infty < f_2(f_1(x)) \le f_2(1)$

$f_2 = \mathrm{sh}$ rastuća funkcija

$\Rightarrow -\infty < \mathrm{sh}(2x - x^2) \le \mathrm{sh}(1)$

$\Rightarrow -\infty < f(x) \le \mathrm{sh}(1)$

$\Rightarrow \mathrm{Im}(f) = \langle -\infty, \mathrm{sh}(1)]$

(b) $f'(x) = \mathrm{ch}(2x - x^2) \cdot (2 - 2x)$

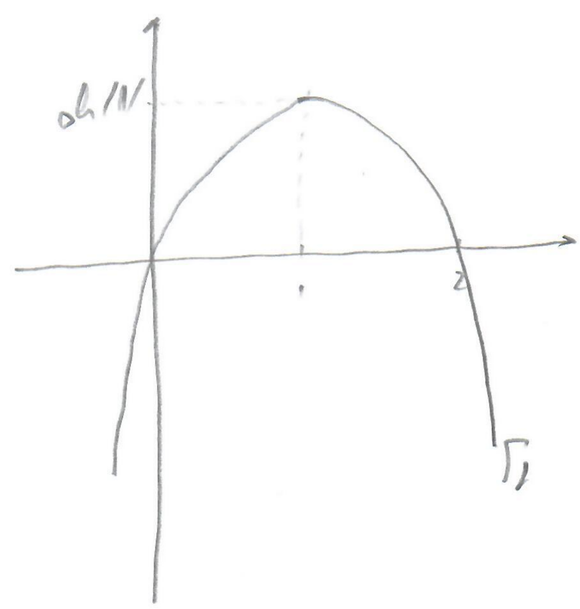
$f'(x) = 0 \iff x = 1$

$f'(x) > 0 \quad \forall x < 1$

$f'(x) < 0 \quad \forall x > 1$

$f(0) = f(2) = \mathrm{sh}(0) = 0$

$f(1) = \mathrm{sh}(1)$



f je strogo rastuća na $\langle -\infty, 1]$

$\Rightarrow f$ je injekcija na $\langle -\infty, 1]$

$m = 1 \Rightarrow I_m = \langle -\infty, 1]$

(c) $f|_{I_m} : I_m \to f(I_m)$

$f|_{I_m} : I_m \to \langle -\infty, \operatorname{sh}(1)]$

$f(x) = \operatorname{sh}(2x - x^2)$

$y = \operatorname{sh}(2x - x^2) \quad / \operatorname{Arsh}$

$\operatorname{Arsh}(y) = 2x - x^2$

$x^2 - 2x + \operatorname{Arsh}(y) = 0$

$$x_{1,2} = \frac{2 \pm \sqrt{4 - 4\operatorname{Arsh}(y)}}{2} = 1 \pm \sqrt{1 - \operatorname{Arsh}(y)}$$

$$\Rightarrow \left(f|_{I_m}\right)^{-1}(x) = 1 - \sqrt{1 - \operatorname{Arsh}(x)}$$

3. (a) Skripta, Teorem 6.6.1

(b) $a_n = \sqrt{6+a_{n-1}}$, $a_1 = \sqrt{6}$

1° Tvrdimo: $(a_n)_n$ je monotono rastući

BAZA: $a_1 = \sqrt{6} < \sqrt{6+\sqrt{6}} = a_2$

PRETPOSTAVKA: Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi $a_n < a_{n+1}$

KORAK: Po pretpostavci indukcije

$$a_n < a_{n+1} \quad / +6$$

$$6+a_n < 6+a_{n+1} \quad / \sqrt{\cdot}$$

$$\sqrt{6+a_n} < \sqrt{6+a_{n+1}}$$

$$a_{n+1} < a_{n+2}$$

Po principu matematičke indukcije, $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$

2° Tvrdimo: $(a_n)_n$ je odozgo omeđen (sa 6)

BAZA: $a_1 = \sqrt{6} \leq 6$

PRETPOSTAVKA: Pretpostavimo da za neki $n \in \mathbb{N}$ vrijedi $a_n \leq 6$

KORAK: Po pretpostavci indukcije

$$a_n \leq 6 \quad / +6$$

$$6 + a_n \leq 12 \quad / \sqrt{\cdot}$$

$$\sqrt{6 + a_n} \leq \sqrt{12}$$

$$a_{n+1} \leq \sqrt{12} \leq \sqrt{36} = 6$$

Po principu matematičke indukcije, $a_n \leq 6 \ \forall n \in \mathbb{N}$

$1^\circ, 2^\circ \xrightarrow{\text{(a) dio zadatka}}$ $(a_n)_n$ je konvergentan, tj. postoji $L \in \mathbb{R}$ takav da je $L = \lim\limits_{n \to \infty} a_n$

$$a_{n+1} = \sqrt{6 + a_n} \quad / \lim_{n \to \infty}$$

$$L = \sqrt{6 + L} \quad /^2$$

$$L^2 = 6 + L$$

$$L^2 - L - 6 = 0$$

$$L_{1,2} = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2}$$

$L_1 = -2 \quad L_2 = 3$

Budući da je $a_1 = \sqrt{6} > 0$ & $(a_n)_n$ je rastući $\Rightarrow L = 3$

4. $f : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$

$$f(x) = \frac{e^{\frac{2}{x}} + e^{\frac{1}{x}} + 3}{e^{\frac{2}{x}} + a}$$

f je neprekidna derivabilna na cijelom $\mathbb{R} \setminus \{0\}$ pa je potrebno promatrati samo točku $x = 0$

(a) $$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{\frac{2}{x}} + e^{\frac{1}{x}} + 3}{e^{\frac{2}{x}} + a} = \frac{e^{-\infty} + e^{-\infty} + 3}{e^{-\infty} + a} = \frac{3}{a}$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{\frac{2}{x}} + e^{\frac{1}{x}} + 3}{e^{\frac{2}{x}} + a} \quad \frac{: e^{\frac{2}{x}}}{: e^{\frac{2}{x}}}$$

$$= \lim_{x \to 0^+} \frac{1 + e^{-\frac{1}{x}} - 3e^{-\frac{2}{x}}}{1 + a e^{-\frac{2}{x}}} = 1$$

f je neprekidna u 0 ako vrijedi

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$$

$$\Rightarrow \frac{3}{a} = 1 \Rightarrow a = 3$$

Za $a = 3$ f se može proširiti do neprekidne funkcije na $\mathbb{R}$

$(f(0) = 1)$

(b) f je derivabilna $\forall x \in \mathbb{R} \setminus \{0\}$ i imamo

$$f'(x) = \frac{\left(e^{\frac{2}{x}}\cdot\left(-\frac{2}{x^2}\right) + e^{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)\right)\left(e^{\frac{2}{x}}+3\right) - \left(e^{\frac{2}{x}}+e^{\frac{1}{x}}+3\right)e^{\frac{2}{x}}\cdot\left(-\frac{2}{x^2}\right)}{\left(e^{\frac{2}{x}}+3\right)^2}$$

$$= \frac{\left(-2e^{\frac{2}{x}} - e^{\frac{1}{x}}\right)\left(e^{\frac{2}{x}}+3\right) + 2e^{\frac{2}{x}}\left(e^{\frac{2}{x}}+e^{\frac{1}{x}}+3\right)}{x^2\left(e^{\frac{2}{x}}+3\right)^2}$$

$$= \frac{\cancel{-2e^{\frac{4}{x}}} - e^{\frac{3}{x}} \cancel{- 6e^{\frac{2}{x}}} - 3e^{\frac{1}{x}} + \cancel{2e^{\frac{4}{x}}} + 2e^{\frac{3}{x}} \cancel{+ 6e^{\frac{2}{x}}}}{x^2\left(e^{\frac{2}{x}}+3\right)^2}$$

$$= \frac{e^{\frac{3}{x}} - 3e^{\frac{1}{x}}}{x^2\left(e^{\frac{2}{x}}+3\right)^2} = \frac{e^{\frac{1}{x}}\left(e^{\frac{2}{x}}-3\right)}{x^2\left(e^{\frac{2}{x}}+3\right)^2}$$

$$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{\frac{e^{\frac{2}{x}}+e^{\frac{1}{x}}+3}{e^{\frac{2}{x}}+3} - 1}{x}$$

$$\overset{L'H}{=} \lim_{x\to 0^-} \frac{e^{\frac{1}{x}}\left(e^{\frac{2}{x}}-3\right)}{x^2\left(e^{\frac{2}{x}}+3\right)^2} = 0 \quad \text{jer je} \quad \lim_{t\to -\infty} \frac{e^t}{\frac{1}{t^2}} = 0$$

$$\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{\frac{e^{\frac{2}{x}}+e^{\frac{1}{x}}+3}{e^{\frac{2}{x}}+3} - 1}{x}$$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{e^{\frac{1}{x}}\left(e^{\frac{2}{x}}-3\right)}{x^2\left(e^{\frac{2}{x}}+3\right)^2} \quad : e^{\frac{2}{x}}$$

$$= \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}\left(1-3e^{-\frac{2}{x}}\right)}{x^2\left(1+3e^{-\frac{2}{x}}\right)} = 0 \quad \text{ker je} \lim_{t\to +\infty} \frac{e^{-t}}{\frac{1}{t^2}} = 0$$

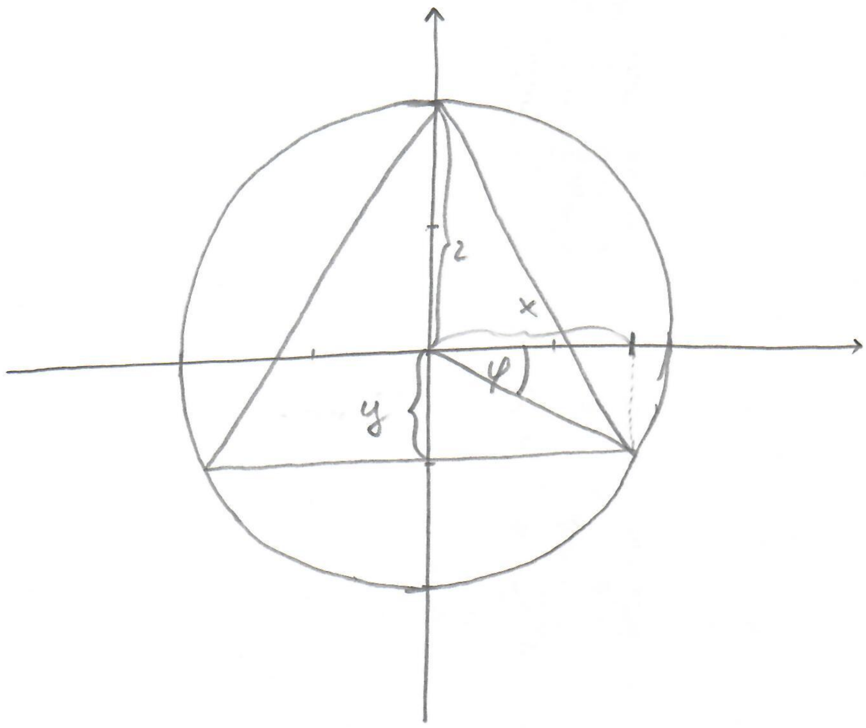
$\Rightarrow$ f je derivabilna v 0 & $f'(0)=0$

(c) Vodnar da smo v (a) dijelu pokazali

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) = 0$$

$\Rightarrow$ dakle za $a=3$ f se može prosiriti do neprekidno derivabilne funkcije na $\mathbb{R}$

5.



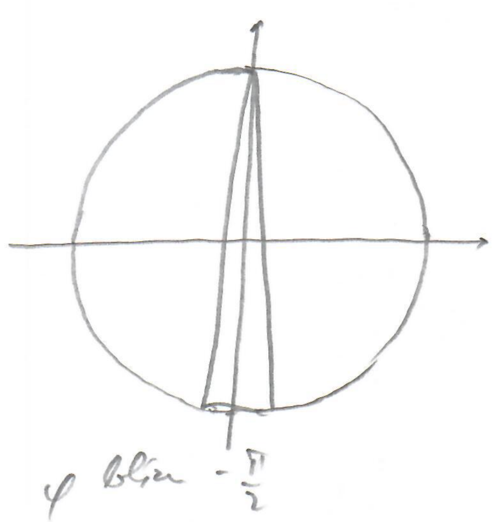
$$x = 2\cos\varphi \Rightarrow a = 2x = 4\cos\varphi$$

$$y = -2\sin\varphi \Rightarrow v = 2 - 2\sin\varphi$$

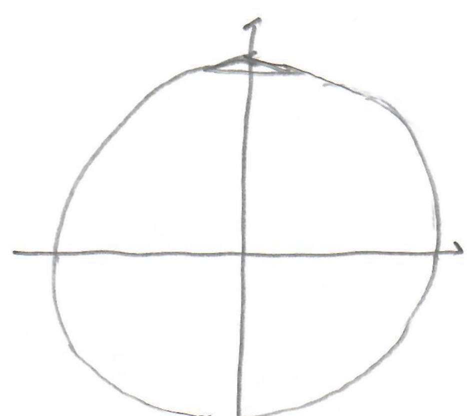
$\sin\varphi$ sa slike je negativan
a y je duljina dijela visine

$$P = \frac{1}{2} a v = \frac{1}{2} 4\cos\varphi\,(2 - 2\sin\varphi) = 4\cos\varphi - 4\sin\varphi\cos\varphi$$

$$\Rightarrow P(\varphi) = 4\cos\varphi - 2\sin(2\varphi), \quad \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$



φ blizu $-\frac{\pi}{2}$



φ blizu $\frac{\pi}{2}$

$$P(\varphi) = 4\cos\varphi - 2\sin(2\varphi) \quad , \varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$P'(\varphi) = -4\sin\varphi - 2\cos(2\varphi)\cdot 2$$
$$= -4\sin\varphi - 4\cos(2\varphi)$$

$$P'(\varphi) = 0 \iff \sin\varphi = -\cos(2\varphi)$$
$$\iff \sin\varphi = -\cos^2\varphi + \sin^2\varphi$$
$$\iff \sin\varphi = -1 + \sin^2\varphi + \sin^2\varphi$$
$$\iff 2\sin^2\varphi - \sin\varphi - 1 = 0$$

$$t = \sin\varphi$$

$$2t^2 - t - 1 = 0$$
$$t_{1,2} = \frac{1 \pm \sqrt{1+8}}{4} = \frac{1 \pm 3}{4}$$

$$t_1 = -\frac{1}{2} \qquad t_2 = 1$$

$$\sin\varphi = -\frac{1}{2} \qquad \sin\varphi = 1$$

$$\varphi = \frac{\pi}{2} + 2k\pi$$

$$\varphi = -\frac{\pi}{6} + 2k\pi$$

$$\varphi = -\frac{5\pi}{6} + 2k\pi$$

Jediný $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ je $\varphi = -\frac{\pi}{6}$

$$P''(\varphi) = -4\cos\varphi + 8\sin(2\varphi)$$

$$P''\left(-\frac{\pi}{6}\right) = -4\cos\left(-\frac{\pi}{6}\right) + 8\sin\left(-\frac{\pi}{3}\right) = -4\cdot\frac{\sqrt{3}}{2} + 8\cdot\left(-\frac{\sqrt{3}}{2}\right) = -6\sqrt{3} < 0$$

$\Rightarrow$ P poprima maksimum u $\varphi = -\frac{\pi}{6}$ i taj maksimum iznosi

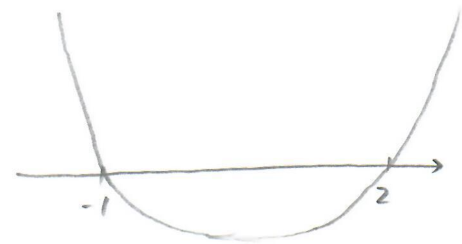
$$P\left(-\frac{\pi}{6}\right) = 4\cos\left(-\frac{\pi}{6}\right) - 2\sin\left(-\frac{\pi}{3}\right) = 4\cdot\frac{\sqrt{3}}{2} - 2\cdot\left(-\frac{\sqrt{3}}{2}\right) = 3\sqrt{3}$$

6. $f(x) = e^{\frac{1}{x^2-x-2}}$

• UVJETI: $x^2-x-2 \neq 0$

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}$$

$$x_1 = -1, \quad x_2 = 2$$



$\Rightarrow D(f) = \mathbb{R} \setminus \{-1, 2\}$

• ASIMPTOTE:

$\lim\limits_{x \to -1^-} f(x) = e^{+\infty} = +\infty \Rightarrow x = -1$ je vertikalna asimptota

$\lim\limits_{x \to -1^+} f(x) = e^{-\infty} = 0$

$\lim\limits_{x \to 2^-} f(x) = e^{-\infty} = 0$

$\lim\limits_{x \to 2^+} f(x) = e^{+\infty} = +\infty \Rightarrow x = 2$ je vertikalna asimptota

$\lim\limits_{x \to \pm\infty} f(x) = e^{\frac{1}{+\infty}} = e^0 = 1 \Rightarrow y = 1$ je horizontalna asimptota u plus i u minus beskonačno

$$f'(x) = e^{\frac{1}{x^2-x-2}} \cdot \frac{(-1)}{(x^2-x-2)^2} \cdot (2x-1)$$

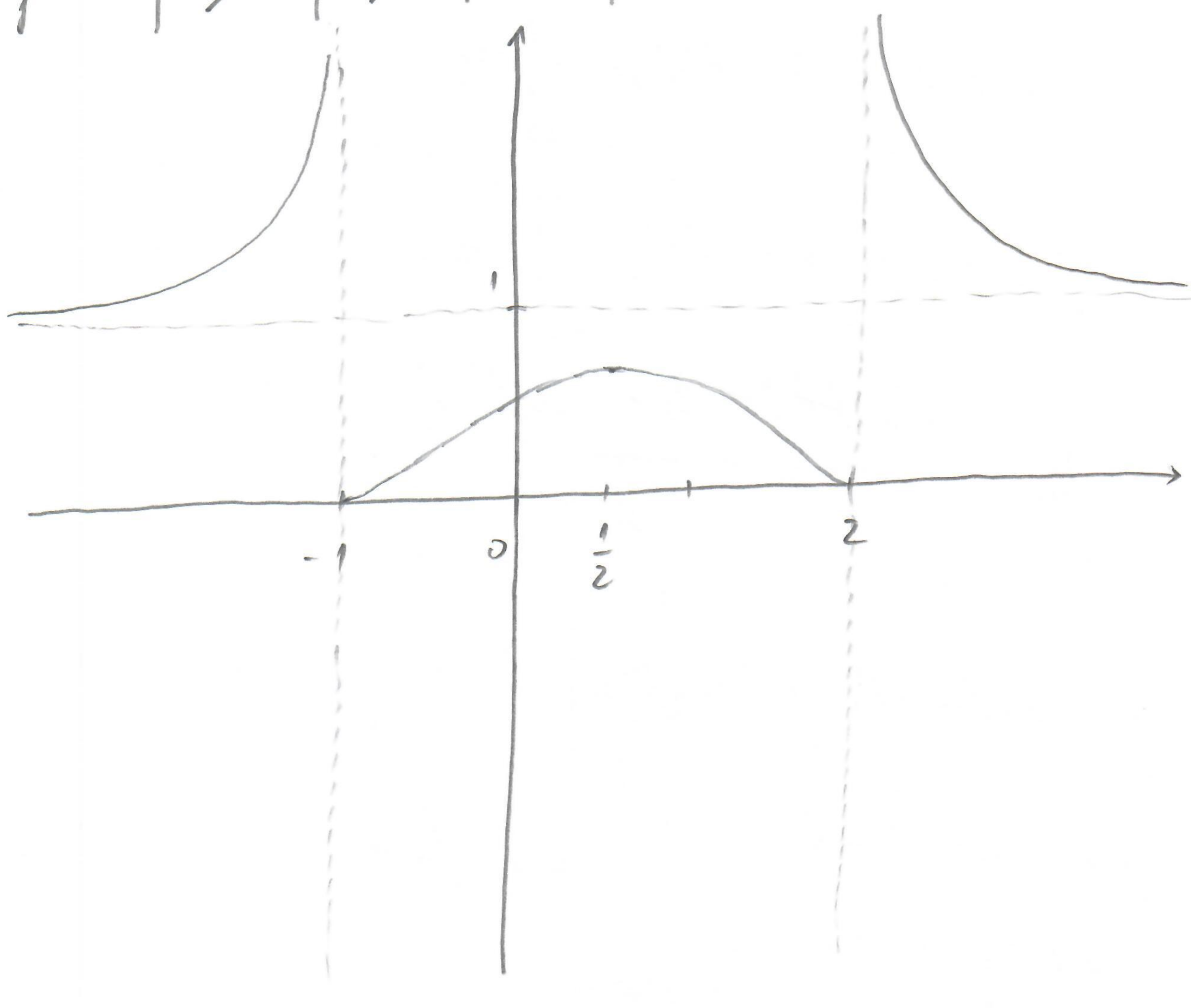
$$= \frac{1-2x}{(x^2-x-2)^2} \, e^{\frac{1}{x^2-x-2}}$$

$$f'(x) = 0 \iff 1-2x = 0 \iff x = \frac{1}{2}$$

x	$-\infty$		-1		$\frac{1}{2}$		2		$+\infty$
f'		$+$		$+$		$-$		$-$	
f		$\nearrow$		$\nearrow$		$\searrow$		$\searrow$	

$$f\left(\tfrac{1}{2}\right) = e^{\frac{1}{\frac{1}{4}-\frac{1}{2}-2}}$$

$$= e^{\frac{1}{-\frac{9}{4}}} = \frac{1}{e^{\frac{4}{9}}} < 1$$



7. (a) Skripta, Teorem 11.3.1

(b) $$\lim_{h\to 0}\frac{\Phi(x+h)-\Phi(x)}{h} = \lim_{h\to 0}\frac{\int_{-x-h}^{2x+2h}\cos(t^2)dt - \int_{-x}^{2x}\cos(t^2)dt}{h}$$

$$= \lim_{h\to 0}\frac{\int_{-x-h}^{0}\cos(t^2)dt - \int_{-x}^{0}\cos(t^2)dt + \int_{0}^{2x+2h}\cos(t^2)dt - \int_{0}^{2x}\cos(t^2)dt}{h}$$

$$= \lim_{h\to 0}\frac{\int_{-x-h}^{-x}\cos(t^2)dt + \int_{2x}^{2x+2h}\cos(t^2)dt}{h} = (*)$$

Po (a) dijelu zadatka postoje

- $\hat{c}_x \in \langle -x-h, -x\rangle$ t.d. je $\int_{-x-h}^{-x}\cos(t^2)dt = \cos(\hat{c}_x^{\,2})\cdot h$
- $\tilde{c}_x \in \langle 2x, 2x+2h\rangle$ t.d. je $\int_{2x}^{2x+2h}\cos(t^2)dt = \cos(\tilde{c}_x^{\,2})\cdot 2h$

$$(*) = \lim_{h\to 0}\frac{\cos(\hat{c}_x^{\,2})\cdot h + \cos(\tilde{c}_x^{\,2})\cdot 2h}{h}$$

$$= \cos((-x)^2) + 2\cos((2x)^2) = \cos(x^2) + 2\cos(4x^2)$$

$\hat{c}_x \to -x$ (kada $h\to 0$)

$\tilde{c}_x \to 2x$ (kada $h \to 0$)

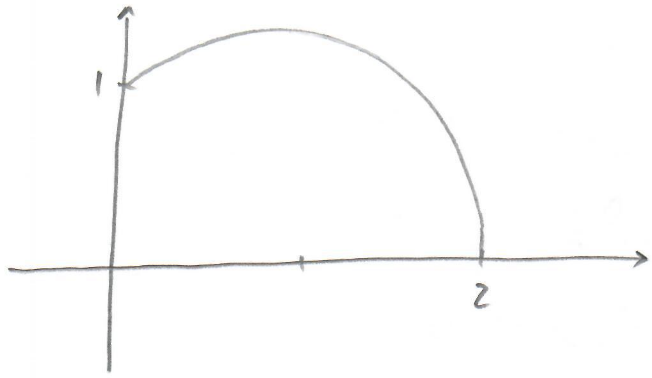
$t \mapsto \cos(t^2)$ je neprekinuta

$$\Rightarrow \Phi'(x) = \cos(x^2) + 2\cos(4x^2)$$

8. $r = 1+\cos\varphi$, $\varphi \in [0, \frac{\pi}{2}]$

$\varphi = 0 \Rightarrow r = 2$

$\varphi = \frac{\pi}{2} \Rightarrow r = 1$



$$x(\varphi) = r(\varphi)\cdot\cos\varphi = (1+\cos\varphi)\cos\varphi$$

$$y(\varphi) = r(\varphi)\cdot\sin\varphi = (1+\cos\varphi)\cdot\sin\varphi$$

$$V = \int_0^{\frac{\pi}{2}} y^2(\varphi)\cdot\pi\cdot\underbrace{\dot{x}(\varphi)}_{\text{ak je } \dot{x}(\varphi) \text{ negatívna, uvažujeme } -\dot{x}(\varphi)}\,d\varphi$$

$$x(\varphi) = \cos\varphi + \cos^2\varphi$$

$$\Rightarrow \dot{x}(\varphi) = -\sin\varphi + 2\cos\varphi\cdot(-\sin\varphi) = -\sin\varphi - \sin(2\varphi)$$

$$V = \pi\int_0^{\frac{\pi}{2}} \sin^2\varphi(1+\cos\varphi)^2(\sin\varphi + \overbrace{\sin 2\varphi}^{=2\sin\varphi\cos\varphi})\,d\varphi$$

$$= \pi\int_0^{\frac{\pi}{2}} \sin^3\varphi(1+\cos\varphi)^2\,d\varphi + 2\pi\int_0^{\frac{\pi}{2}} \sin^3\varphi\cos\varphi(1+\cos\varphi)^2\,d\varphi$$

$$= \underbrace{\pi\int_0^{\frac{\pi}{2}}(1-\cos^2\varphi)(1+\cos\varphi)^2\sin\varphi\,d\varphi}_{=:I_1} + \underbrace{2\pi\int_0^{\frac{\pi}{2}}\cos\varphi(1-\cos^2\varphi)(1+\cos\varphi)^2\sin\varphi\,d\varphi}_{=:I_2}$$

$$I_1 = \begin{bmatrix} t = \cos\varphi & 0 \longmapsto 1 \\ dt = -\sin\varphi \, d\varphi & \frac{\pi}{2} \longmapsto 0 \end{bmatrix}$$

$$= \int_0^1 (1-t^2)(1+t)^2 dt = \int_0^1 (1-t^2)(1+2t+t^2) dt$$

$$= \int_0^1 (1 - \cancel{t^2} + 2t - 2t^3 + \cancel{t^2} - t^4) dt$$

$$= \int_0^1 (1 + 2t - 2t^3 - t^4) dt$$

$$= \left(t + t^2 - \frac{2t^4}{4} - \frac{t^5}{5} \right) \Big|_0^1$$

$$= 1 + 1 - \frac{1}{2} - \frac{1}{5} = \frac{10+10-5-2}{10} = \frac{13}{10}$$

$$I_2 = \begin{bmatrix} t = \cos\varphi & 0 \longmapsto 1 \\ dt = -\sin\varphi \, d\varphi & \frac{\pi}{2} \longmapsto 0 \end{bmatrix}$$

$$= \int_0^1 t(1-t^2)(1+t)^2 dt = \int_0^1 (t + 2t^2 - 2t^4 - t^5)$$

$$= \left(\frac{t^2}{2} + \frac{2t^3}{3} - \frac{2t^5}{5} - \frac{t^6}{6} \right) \Big|_0^1$$

$$= \frac{1}{2} + \frac{2}{3} - \frac{2}{5} - \frac{1}{6} = \frac{15+20-12-5}{30} = \frac{18}{30} = \frac{3}{5}$$

$$V = \pi \cdot I_1 + 2\pi \cdot I_2$$

$$= \pi \cdot \frac{13}{10} + 2\pi \cdot \frac{3}{5} = \frac{13\pi}{10} + \frac{6\pi}{5} = \frac{13+12}{10}\pi$$

$$= \frac{25}{10}\pi = \frac{5}{2}\pi$$