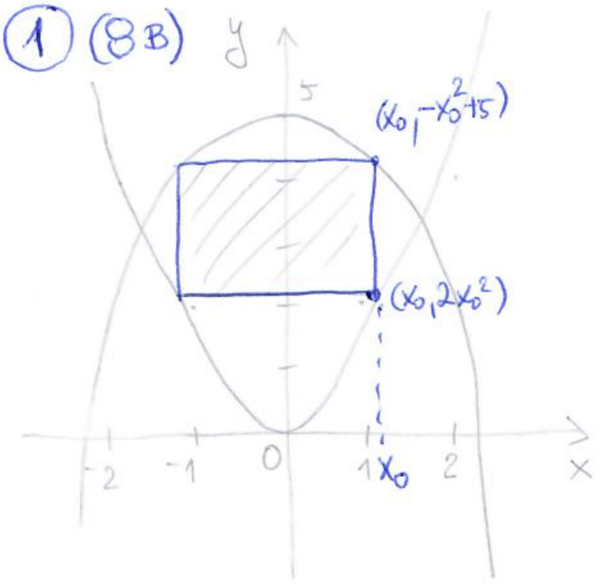
RJEŠENJA = ZAVRŠNI ISPIT, MATAN 1, 04. veljače 2019.

① (8 B)



$y = 2x^2$

$y = -x^2 + 5$

$P = a \cdot b = 2x_0(-x_0^2+5-2x_0^2) = 2x_0(-3x_0^2+5)$

$\boxed{P(x) = 10x - 6x^3,}$ $x \in \langle 0, 2\rangle$

$P'(x) = 10 - 18x^2 = 0$

$x^2 = \frac{5}{9} \Rightarrow x_{1,2} = \pm\frac{\sqrt5}{3}, x>0$

$\Rightarrow \boxed{x = \frac{\sqrt5}{3}}$

	0	$\frac{\sqrt5}{3}$	$+\infty$
f'	+	−	
f	↗	↘	

MAX

$P\left(\frac{\sqrt5}{3}\right) = 10 \cdot \frac{\sqrt5}{3} - 6\left(\frac{\sqrt5}{3}\right)^3 = \frac{10\sqrt5}{3} - \overset{2}{\cancel{6}} \cdot \frac{5\sqrt5}{\cancel{27}_9} = \frac{30-10}{9}\sqrt5 = \frac{20\sqrt5}{9}$

② (7 BOD)

(T1) (3b) Ⓣ $f'(x) > 0 \Rightarrow f$ strogo raste

Dokaz: Odaberemo $x_1, x_2 \in \langle a, b\rangle$ takve da $x_1 < x_2$.

$f(x_2) - f(x_1) \underset{\substack{\downarrow \\ \text{LTSV} \\ \exists c \in \langle x_1, x_2\rangle}}{=} \underbrace{f'(c)}_{>0}\underbrace{(x_2 - x_1)}_{>0} > 0 \Rightarrow f(x_2) > f(x_1)$ za proizvoljne x_1, x_2 td $x_1 < x_2$

$\Rightarrow \boxed{f \text{ strogo raste na } \langle a, b\rangle}$

(T2) (2b) Ⓕ Protuprimjer: $\boxed{f : \langle -1, 1\rangle \to \mathbb{R},\ f(x) = -x^3}$ je strogo padajuća $\not\Rightarrow f'(x) < 0$ na $\langle -1, 1\rangle$ jer je $f'(0) = 0$

(T3) (2b) Ⓣ Dokaz: primijenimo LTSV na fji $f(x) = \sin x$ na intervalu $\langle 0, x\rangle$, $x \in \mathbb{R}$.

$\Rightarrow \exists c \in \langle 0, x\rangle$ td $\sin x - \sin 0 = \cos(c)(x - 0)$

$\Rightarrow \frac{\sin x}{x} = \cos(c)$

③ (8 BOD) $f(x)=\frac{1}{x}e^{\frac{x^2}{2}}$

$D(f)=\mathbb{R}\setminus\{0\}$

V.A. $\lim\limits_{x\to0^+}\frac{1}{x}e^{\frac{x^2}{2}}=+\infty\cdot1=+\infty$ $\boxed{x=0 \text{ V.A.}}$

$\lim\limits_{x\to0^-}\frac{1}{x}e^{\frac{x^2}{2}}=-\infty\cdot1=-\infty$

K.A. $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to\infty}\frac{\frac{1}{x}e^{\frac{x^2}{2}}}{x}=\lim\limits_{x\to+\infty}\frac{e^{\frac{x^2}{2}}}{x^2}=\left(\frac{\infty}{\infty}\right)\overset{L'H}{=}\lim\limits_{x\to+\infty}\frac{e^{\frac{x^2}{2}}\cdot\cancel{x}}{2\cancel{x}}=+\infty$

analogno $\lim\limits_{x\to-\infty}\frac{e^{\frac{x^2}{2}}}{x^2}=+\infty$ $\boxed{\text{NEMA KOSIH ASIM.}}$

LOK. EKSTREMI

$$f'(x)=-\frac{1}{x^2}e^{\frac{x^2}{2}}+\frac{1}{\cancel{x}}e^{\frac{x^2}{2}}\cdot\cancel{x}=e^{\frac{x^2}{2}}\left[1-\frac{1}{x^2}\right]=\underbrace{e^{\frac{x^2}{2}}}_{\neq0}\cdot\frac{x^2-1}{\underbrace{x^2}_{x=0\notin D(f)}}=0$$

$\Rightarrow x^2-1=0$

$x_{1,2}=\pm1$

	$-\infty$	-1	0	1	$+\infty$
f'	+	−	−	+	
f	↗	↘	↘	↗	
		MAX		MIN	

$\boxed{T_{MAX}(-1,-\sqrt{e})}$
$\boxed{T_{MIN}(1,\sqrt{e})}$

KONV/KONK TOČKE INF

$$f''(x)=e^{\frac{x^2}{2}}\cdot x\left(1-\frac{1}{x^2}\right)+e^{\frac{x^2}{2}}\cdot\left(\frac{2}{x^3}\right)=e^{\frac{x^2}{2}}\left[x-\frac{1}{x}+\frac{2}{x^3}\right]=$$

$$=e^{\frac{x^2}{2}}\frac{\underbrace{x^4-x^2+2}_{>0}}{x^3}$$

$x^4-x^2+2=0$

$t^2-t+2=0$

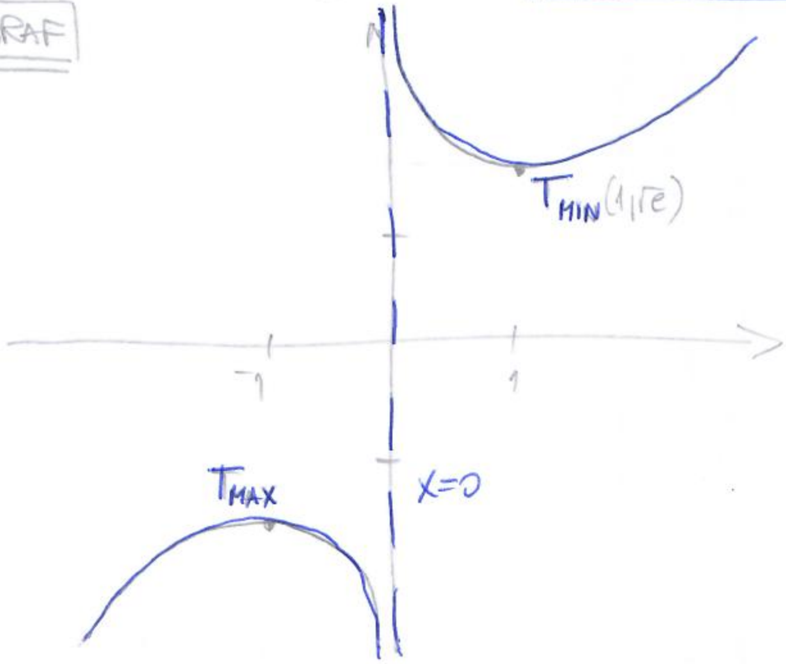
$t_{1,2}=\frac{1\pm\sqrt{1-8<0}}{2}$

NEMA REAL NULT $\Rightarrow x^4-x^2+2>0, \forall x\in\mathbb{R}$

$\boxed{\text{NEMA TOČAKA INFLEKSIJE}}$

	$-\infty$	0	$+\infty$
f''	−	+	
f	∩	∪	

GRAF



$\sqrt{e}\approx1.6$

④ (11 bod)

(a) (2b) $$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Dokaz: Pokazat ćemo da je desna strana prim. fja od $f(x)g'(x)$:

$$\left(f(x)g(x) - \int f'(x)g(x)\,dx\right)' = \cancel{f'(x)g(x)} + f(x)g'(x) - \cancel{f'(x)g(x)} =$$
$$= f(x)g'(x) \quad \text{što je dokaz tvrdnje}$$

(b) (4b) $$\underbrace{\int e^{-2x}\sin(3x)\,dx}_{I} = \begin{pmatrix} \overset{\text{parc.int.}}{e^{-2x} = u} & \sin(3x)dx = dv \\ -2e^{-2x}dx = du & v = -\frac{1}{3}\cos(3x) \end{pmatrix} =$$

$$= -\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{3}\int e^{-2x}\cos(3x)\,dx = \begin{pmatrix} \overset{\text{P.I.}}{e^{-2x} = u} & \cos(3x)dx = dv \\ -2e^{-2x}dx = du & v = \frac{1}{3}\sin(3x) \end{pmatrix} =$$

$$= -\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{3}\left[\frac{1}{3}e^{-2x}\sin(3x) + \frac{2}{3}\underbrace{\int e^{-2x}\sin(3x)\,dx}_{I}\right]$$

$$I = -\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{9}e^{-2x}\sin(3x) - \frac{4}{9}I$$

$$\frac{13}{9}I = -\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{9}e^{-2x}\sin(3x)$$

$$I = \frac{9}{13}\left(-\frac{1}{3}e^{-2x}\cos(3x) - \frac{2}{9}e^{-2x}\sin(3x)\right) + C$$

$$\boxed{I = -\frac{3}{13}e^{-2x}\cos(3x) - \frac{2}{13}e^{-2x}\sin(3x) + C}$$

(c) (5b) $$\int_0^{\pi/2} \frac{\sin x\,dx}{(\cos x + 1)(\underbrace{\sin^2 x}_{1-\cos^2 x} - 2)} = \begin{pmatrix} \text{supst} & x=0 \;\; t=1 \\ \cos x = t & x=\pi/2 \;\; t=0 \\ -\sin x\,dx = dt & \end{pmatrix} = -\int_1^0 \frac{dt}{(t+1)(1-t^2-2)} =$$

$$= \int_0^1 \frac{dt}{(t+1)(-1-t^2)} = -\int_0^1 \frac{dt}{(t+1)(t^2+1)} =$$

$$\left[\begin{array}{l} \text{PARC. RAZL.} \\ \frac{1}{(t+1)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1} = \frac{At^2+A+Bt^2+Bt+Ct+C}{(t+1)(t^2+1)} = \frac{(A+B)t^2+(B+C)t+A+C}{(t+1)(t^2+1)} \\ \left.\begin{array}{l} A+B=0 \\ B+C=0 \\ A+C=1 \end{array}\right\} \Rightarrow \begin{array}{l} A=C=\frac{1}{2} \\ B=-\frac{1}{2} \end{array} \qquad \frac{1}{(t+1)(t^2+1)} = \frac{1}{2}\cdot\frac{1}{t+1} - \frac{1}{2}\,\frac{t-1}{t^2+1} \end{array}\right]$$

$$= -\frac{1}{2}\left(\int_0^1 \frac{dt}{t+1} - \int_0^1 \frac{t-1}{t^2+1}dt\right) = -\frac{1}{2}\left(\int_0^1 \frac{dt}{t+1} - \int_0^1 \frac{t}{t^2+1}dt + \int_0^1 \frac{dt}{t^2+1}\right) =$$

$$= -\frac{1}{2}\left[\ln|t+1|\Big|_0^1 - \frac{1}{2}\ln|t^2+1|\Big|_0^1 + \operatorname{arctg} t\Big|_0^1\right] = -\frac{1}{2}\left[\ln 2 - \underset{0}{\ln 1} - \frac{1}{2}(\ln 2 - \underset{0}{\ln 1}) + \operatorname{arctg} 1 - \underset{0}{\operatorname{arctg} 0}\right]$$

$$= \boxed{-\frac{1}{4}\ln 2 - \frac{\pi}{8}}$$

⑤ (8 bod)

(a) (3b) $\Phi(x)=\int_a^x f(t)\,dt \Rightarrow \Phi'(x)=f(x)$

Dokaz:

$$\Phi'(x)=\lim_{\Delta x\to 0}\frac{\Phi(x+\Delta x)-\Phi(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(t)\,dt-\int_a^{x} f(t)\,dt}{\Delta x}=$$

$$=\lim_{\Delta x\to 0}\frac{\int_x^{x+\Delta x} f(t)\,dt}{\Delta x}\underset{\substack{\text{T.S.V. integr. računa}\\ \exists c\in\langle x,x+\Delta x\rangle}}{=}\lim_{\Delta x\to 0}\frac{f(c)\cdot(\cancel{x}+\cancel{\Delta x}-\cancel{x})}{\cancel{\Delta x}}\underset{\boxed{\substack{\Delta x\to 0\\ c\to x}}}{=}f(x)$$

(b) (3b) $\int_a^b f(x)\,dx=F(b)-F(a)$

Dokaz:

F = neka prim. fja od f

$\Phi(x)=\int_a^x f(t)\,dt$ prim. fja od f

Dvije prim. fje se razlikuju do na konst: $\boxed{\Phi(x)=F(x)+C}$

$\boxed{x=a}$ uvrstimo u

$\Rightarrow \underset{\substack{\parallel\\ 0}}{\Phi(a)}=F(a)+C \rightarrow \boxed{C=-F(a)}$

$\int_a^a f(t)\,dt=0$

$\boxed{\Phi(x)=F(x)-F(a)}$

$\boxed{x=b}$ uvrstimo u

$\Phi(b)=F(b)-F(a) \Longrightarrow \boxed{\int_a^b f(x)\,dx=F(b)-F(a)}$

(c) (2b) $G(x)=\int_{2x}^{1} f(t)\,dt=-\int_1^{2x} f(t)\,dt$

1. način:

$$G(x)=\begin{pmatrix}\text{supst } t=2u\Rightarrow u=\frac{t}{2}\\ dt=2du\\ t=1\Rightarrow u=\frac{1}{2}\\ t=2x\Rightarrow u=x\end{pmatrix}=-2\int_{\frac{1}{2}}^{x} f(2u)\,du \Rightarrow G'(x)=-2f(2x)$$

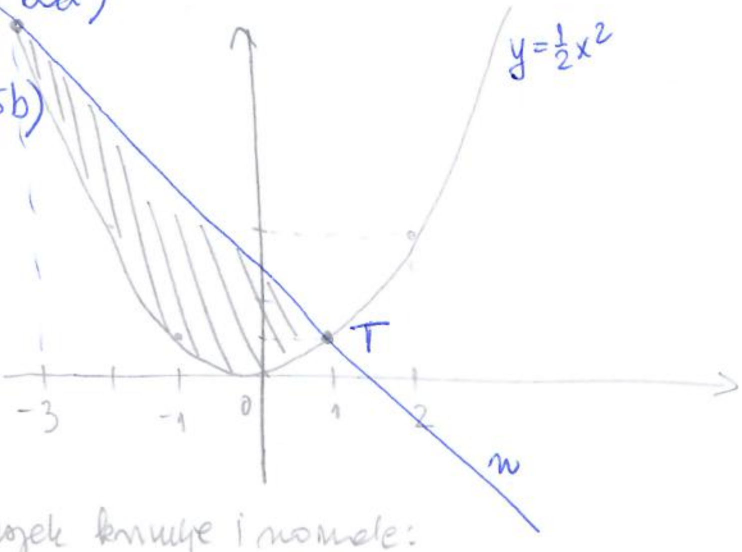
2. način: F = prim. fje od f

$$G(x)=-\int_1^{2x} f(t)\,dt=-\left(F(2x)-F(1)\right)$$

$$G'(x)=-F'(2x)\cdot 2=-2f(2x)$$

⑥ (8 bod)

(a) (5b)



$y=\frac{1}{2}x^2$

normala u $T(1,\frac{1}{2})$:

$$y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$$

$x_0=1,\ y_0=\frac{1}{2}$

$f'(x)=x \qquad f'(1)=1$

$$y-\frac{1}{2}=-1(x-1)$$

$$\boxed{n\ldots \quad y=-x+\frac{3}{2}}$$

presjek krivulje i normale:

$$\frac{1}{2}x^2=-x+\frac{3}{2} \,\Big|\cdot 2$$

$$x^2+2x-3=0$$

$$x_{1,2}=\frac{-2\pm 4}{2} \quad x_1=1, \quad x_2=-3$$

gornja fja : $f(x)=-x+\frac{3}{2}$

donja fja : $g(x)=\frac{1}{2}x^2$

$$P=\int_{-3}^{1}(f(x)-g(x))\,dx$$

$$\boxed{P=\int_{-3}^{1}\left(-x+\frac{3}{2}-\frac{1}{2}x^2\right)dx}=\left(\frac{3}{2}x-\frac{1}{2}x^2-\frac{1}{6}x^3\right)\Big|_{-3}^{1}=\left(\frac{3}{2}-\frac{1}{2}-\frac{1}{6}-\left(-\frac{9}{2}-\cancel{\frac{9}{2}}+\cancel{\frac{9}{2}}\right)\right)$$

$$=\left(1-\frac{1}{6}+\frac{9}{2}\right)=\frac{16}{3}$$

(b) (3b)

$$\boxed{V=\pi\int_{-3}^{1}\left(\left(-x+\frac{3}{2}\right)^2-\left(\frac{1}{2}x^2\right)^2\right)dx}=\pi\int_{-3}^{1}\left(x^2-3x+\frac{9}{4}-\frac{1}{4}x^4\right)dx=$$

$$=\pi\left(\frac{x^3}{3}-\frac{3}{2}x^2+\frac{9}{4}x-\frac{1}{20}x^5\right)\Big|_{-3}^{1}=\pi\left(\frac{1}{3}-\frac{3}{2}+\frac{9}{4}-\frac{1}{20}-\left(-9-\frac{27}{2}-\frac{27}{4}+\frac{3^5}{20}\right)\right)=$$

$$=\pi\left(\frac{1}{3}+9+12+\underbrace{\frac{9}{4}+\frac{27}{4}}_{9}-\frac{1}{20}-\frac{3^5}{20}\right)=\pi\left(\frac{1}{3}+30-\frac{3^5+1}{20}\right)$$

$$\boxed{V=\pi\int_{-3}^{1}\left(f^2(x)-g^2(x)\right)dx}$$

Volumen tijela je razlika volumena tijela koje se dobiju vrtnjom svakog grafa