Kompleksni brojevi

Zad. 1).

a) $z = 1+i$

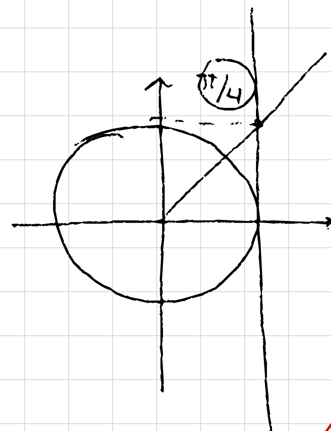
$z = r(\cos\varphi + i\sin\varphi)$

$r = |z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\operatorname{tg}\varphi = \dfrac{y}{x} = \dfrac{1}{1} = 1$

$\varphi = \dfrac{\pi}{4} \qquad z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$$\boxed{z = \sqrt{2}\operatorname{cis}\frac{\pi}{4}}$$



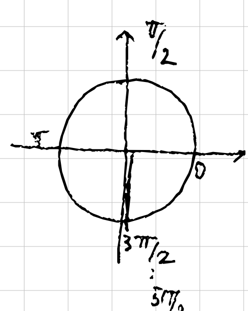
$x = y = r$

zajednička oznaka

$z = r + ri = r(1+i)$

b) $\mathrm{Re}\,(1+i)^{10}$

$r = \sqrt{2} \qquad \varphi = \dfrac{\pi}{4}$



$(1+i)^{10} = \left(\sqrt{2}\operatorname{cis}\frac{\pi}{4}\right)^{10} = \left(\sqrt{2}^{10}\operatorname{cis}\frac{\pi}{4}\cdot 10\right)$

$= 2^5 \cdot \operatorname{cis}\dfrac{5\pi}{2} = 32\cdot\operatorname{cis}\dfrac{5\pi}{2} = 32\left(\underbrace{\cos\frac{5\pi}{2}}_{0} + i\underbrace{\sin\frac{5\pi}{2}}_{1}\right)$

$= 32i \qquad \boxed{\mathrm{Re} = 0}$

Tahina zbirka:

1.

$$\left(\frac{1+i}{1-i}\right)^n = z^n$$

$$z = \frac{i+1}{1-i} \cdot \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+2i-1}{1+1} = \frac{+2i}{2} = i$$

$i^n = i, -i, 1, -1$; ovisno je li n djeljiv sa 4
(4k, 4k+1, +2, +3..)

Zad sa sata)

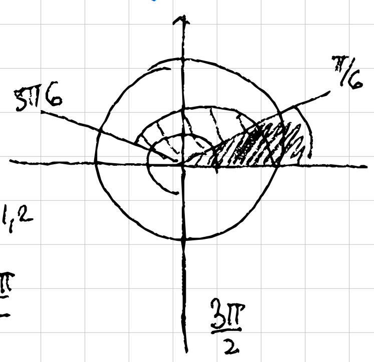
Skicirajte $\arg z^3 = \pi/2$, skicirajte sve $z \in \mathbb{C}$; $|z+2|=1$.
Nađite sve z ~~za~~ $\in \mathbb{C}$ t.d. $\arg(z^3) = \pi/2$ i $|z+2|=1$.

$\arg(z^3) = \pi/2 + 2k\pi$

$3\varphi = \frac{\pi}{2} + 2k\pi \quad / \cdot 3$

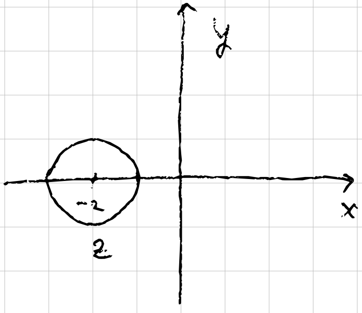
$\varphi = \frac{\pi}{6} + \frac{2}{3}k\pi = k_1 = 0,1,2$

$\frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6} = \frac{3\pi}{2}$



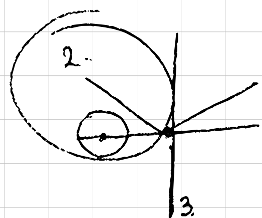
$|z+2| = 1 \longrightarrow$ kružnica

$S(-2,0)$, $r=1$



*Nađite sve $z \in \mathbb{C}$ td $\arg(z^3) = \frac{\pi}{2}$ i $|z+2| = 1$

$\varphi = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{9\pi}{6}$



1. promjer je moguć samo u prvom slučaju $\frac{5\pi}{6}$

$|z+2| = 1$

$|\underbrace{r\cos\varphi}_{\mathbb{R}} + \underbrace{i \cdot r \cdot \sin\varphi}_{\mathrm{Im}} + 2| = 1$

$\sqrt{(r\cdot\cos\varphi+2)^2 + (r\cdot\sin\varphi)^2} = 1 \quad /^2$

$r\cdot\cos^2\varphi + 4r\cos\varphi + 4 + r\sin^2\varphi = 0$

$r(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1}) + 3 + 4r\cos\varphi = 0$

$r + 3 + 4r\cos\varphi = 0$

$r + 3 + 4r\cdot\cos\frac{5\pi}{6} = 0$

$r + 3 + 4r\cdot\left(-\frac{\sqrt{3}}{2}\right) = 0$

$r\left(1 + \cancel{4}^{2}\cdot\frac{\sqrt{3}}{\cancel{2}}\right) + 3 = 0$

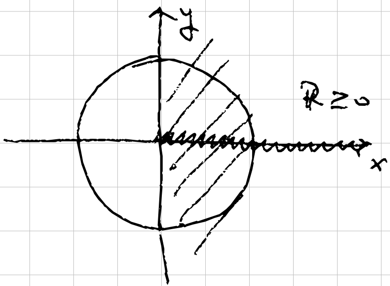
$r - 2\sqrt{3}\cdot r + 3 = 0$

$r_{1,2} = \frac{+2\sqrt{3} \pm \sqrt{4\cdot 3 - 4\cdot 3}}{2} = \frac{+\cancel{2}\sqrt{3} \pm 0}{\cancel{2}}$

$r_1 = \sqrt{3} > 0$ ✓

$\boxed{z = \sqrt{3}\,\mathrm{cis}\,\frac{5\pi}{6}}$

Odredite kompl. br. za koje je $z^2+\frac{1}{z^2}=-1$ i $\mathrm{Re}\, z \geq 0$



$$z^2+\frac{1}{z^2}=-1 \quad / \cdot z^2$$

$$z^4+1+z^2=0$$

$$t=z^2$$

$$t^2+t+1=0$$

$$t_{1,2}=\frac{-1\pm\sqrt{1-4}}{2}=\frac{-1\pm\sqrt{3}i}{2}$$

$$t_1=\frac{-1+\sqrt{3}i}{2}=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$$

$$t_2=\frac{-1-\sqrt{3}i}{2}=\frac{-1}{2}-\frac{\sqrt{3}}{2}i$$

$$z^2=\frac{-1}{2}+\frac{\sqrt{3}}{2}i$$

$$r=\sqrt{\frac{1}{4}+\frac{3}{4}}=\underline{\underline{1}}$$

$$\mathrm{tg}\,\varphi=\frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}=-\sqrt{3}$$

$$\underline{\underline{\varphi=\frac{2\pi}{3}}} \qquad z^2=\mathrm{cis}\,\frac{2\pi}{3}$$

$$z=\cos\frac{\frac{2\pi}{3}+2k\pi}{2}+i\sin\frac{\frac{2\pi}{3}+2k\pi}{2} \qquad k=0,1$$

$$\boxed{\begin{aligned} z_1&=\mathrm{cis}\,\frac{\pi}{3}\\ z_2&=\mathrm{cis}\,\frac{4\pi}{3}\end{aligned}}$$

$$z^2=-\frac{1}{2}-\frac{\sqrt{3}}{2}i \qquad r=1$$

$$\mathrm{tg}\,\varphi=\frac{-\frac{\sqrt{3}}{2}}{\frac{-1}{2}}=\sqrt{3}\rightarrow\frac{\pi}{3}$$

$$z^2=\mathrm{cis}\,\frac{\pi}{3} \quad /\sqrt{\ }$$

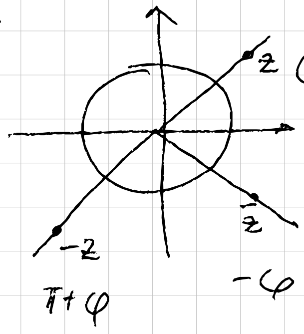
$$k=0,1 \quad z=\cos\frac{\frac{\pi}{3}+2k\pi}{2}+i\sin\frac{\frac{\pi}{3}+2k\pi}{2}$$

$$\boxed{\begin{aligned} z_1&=\mathrm{cis}\,\frac{\pi}{6}\\ z_2&=\mathrm{cis}\,\frac{7\pi}{6}\end{aligned}}$$

$\mathrm{Re}\, z\geq 0$ a to je I. i IV. kvadrant

Zad: Riješite jednakost kompleksnog broja: $z^6 = \bar{z}(1-i)$

$$\begin{cases} |z_1| = |z_2| \\ \varphi_1 = \varphi_2 + 2k\pi \quad (\arg z_1 = \arg z_2) \end{cases} \longrightarrow \text{vrijedi}$$



$$|z^6| = |\bar{z}(1-i)|$$

$$6\varphi = \underbrace{\arg(\bar{z})}_{-\varphi} + \underbrace{\arg(1-i)}_{\frac{-1}{1} = -\ \frac{7\pi}{4}} + 2k\pi$$

1. $r = |z|$

$$|z^6| = |\bar{z}| \cdot \underbrace{|1-i|}_{\sqrt{1+1}}$$

$$|z|^6 = |\bar{z}| \cdot \sqrt{2}$$

$$r^6 = r\sqrt{2}$$

$$r^6 - r\sqrt{2} = 0$$

$$r(r^5 - \sqrt{2}) = 0$$

$$r_1 = 0 \qquad r^5 = \sqrt{2} \ / \sqrt[5]{\ }$$

$$r = \sqrt[10]{2}$$

2. $6\varphi = -\varphi + \frac{7\pi}{4} + 2k\pi$

$$7\varphi = \frac{7\pi}{4} + 2k\pi \ / :7$$

$$\varphi = \frac{\pi}{4} + \frac{2}{7}k\pi, \ k = 0, 1 \ldots 6$$

$$\boxed{\begin{array}{l} z_1 = 0 \\ z_{2,3,4,5,6,7} = \sqrt[10]{2}\ \mathrm{cis}\left(\frac{\pi}{4} + \frac{2}{7}k\pi\right) \\ \qquad\qquad k = 0, 1, 2 \ldots 6 \end{array}}$$

Jednakost: $z_1 = z_2 \Rightarrow \begin{cases} x_1 = x_2 \\ y_1 = y_2 \end{cases}$

Zbrajanje: $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

Mnoezenje: $z_1 \cdot z_2 = x_1 \cdot x_2 - y_1 \cdot y_2 + i(x_1 y_1 + x_2 \cdot y_1)$

Dijeljenje: $\frac{z_1}{z_2} = \frac{x_1 + i y_1}{x_2 + i y_2} \cdot \frac{x_2 - i y_2}{x_2 - i y_2}$

MODUL: $|z| = r = \sqrt{x^2 + y^2}$

$$\bar{z} = x - iy$$

$$z \cdot \bar{z} = |z|^2 \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \left(\frac{\bar{z}_1}{\bar{z}_2}\right)$$

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2 \qquad |z^n| = |z|^n$$

$$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2 \qquad |z_1 + z_2| \leq |z_1| + |z_2|$$

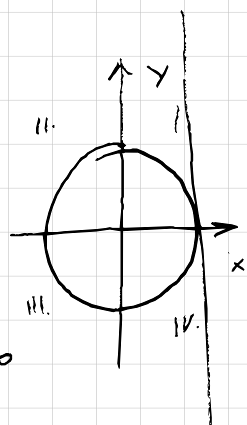
TRIGONOMETRIJSKI ZAPIS:

$$z = |z|(\cos\varphi + i\sin\varphi) = r(\cos\varphi + i\sin\varphi)$$

$$\boxed{z = r \cdot \text{cis}(\varphi)}$$

$r = |z|$

$\text{tg}\,\varphi = \frac{y}{x}$



$\arg z = \{\varphi + 2k\pi,\ k \in \mathbb{Z}\}$

$\varphi = [0, 2\pi]$

*paziti u kojem smo kvadrantu

Jednakost: $z_1 = z_2 \Leftrightarrow \begin{cases} r_1 = r_2 \\ \varphi_1 = \varphi_2 \end{cases}$

Množenje: $z_1 \cdot z_2 = r_1 \cdot r_2 (\cos(\varphi_1 + \varphi_2) + i\sin(\varphi_1 + \varphi_2))$

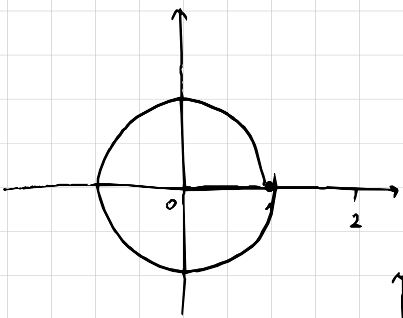
Dijeljenje: $\frac{z_1}{z_2} = \frac{r_1}{r_2}(\cos(\varphi_1 - \varphi_2) + i\sin(\varphi_1 - \varphi_2))$

Potenciranje: $z_1^n = r_1^n (\cos(n \cdot \varphi_1) + i\sin(n \cdot \varphi_1))$

Korjenovanje: $\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi + 2k\pi}{n} + i\sin\frac{\varphi + 2k\pi}{n}\right)$

1.11.)

a) $|z|=1 \quad r=1$

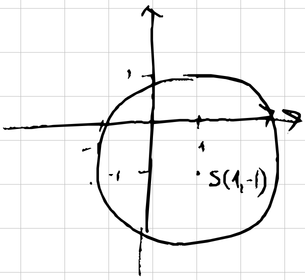


$(1) \quad (-1)$

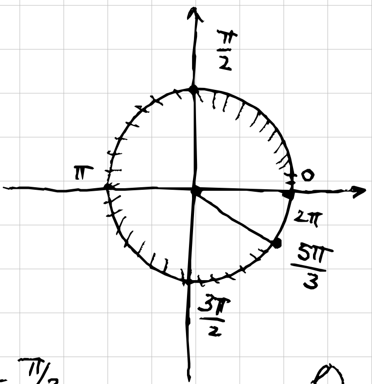
$-1 \quad 1$

b) $|z-1+i| = 2$

$r=2 \quad S(1,-1)$

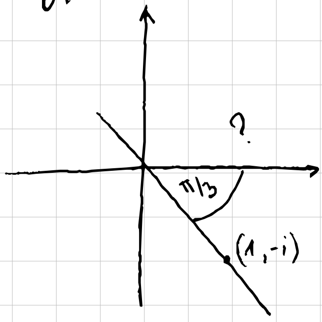


d) $\arg z = \frac{5\pi}{3}$



$1 \quad -1$

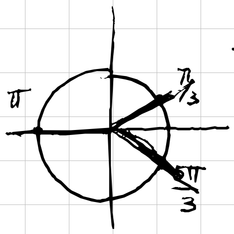
e) $\arg(z-1+i) = \pi/3$



f) $\arg(z^3) = \pi$

$\arg(z) = \frac{\pi + 2k\pi}{3}$

$\frac{\pi}{3}, \ \pi \quad \frac{5\pi}{3}$



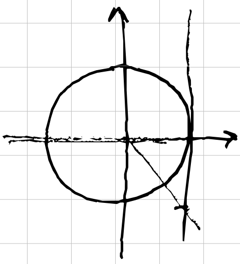
1.15.)

a) $i^{157} = i^{39\cdot 4+1} \quad i^1 = \underline{\underline{i}}$

b) $(1+i)^{859} = |z|^{859}(\cos 859\cdot\varphi + i\sin 859\cdot\varphi)$

$r=\sqrt{2} \quad tg\,\varphi = \frac{1}{1} = 1 \quad \varphi = \frac{\pi}{4}$

$= (\sqrt{2})^{859}\left(\cos\left(859\cdot\frac{\pi}{4}\right) + i\sin\left(859\cdot\frac{\pi}{4}\right)\right)$



c) $(\sqrt{3}-i)^6 = r^6(\cos 6\cdot\varphi + i\sin 6\varphi)$

$r=\sqrt{1+3} = 2$

$tg\,\varphi \ \frac{-1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\sqrt{3}}{3} \qquad \frac{-\pi}{6} = \frac{11\pi}{6}$

$\boxed{(\sqrt{3}-i)^6 = 64(\cos 11\pi + i\cdot \sin 11\pi)}$

1.16)

$z^6 = (1+i)^2$ $\qquad r = \sqrt{2} \quad tg\varphi = \frac{1}{1} = 1 = \frac{\pi}{4}$

$z = ?$

$w = z^6$

$w = \left(\sqrt{2}\, cis \frac{\pi}{4}\right)^2 = 2\, cis \frac{\pi}{2}$

$w = z^6$

$z = \sqrt[6]{w}$ $\boxed{= \sqrt[6]{2}\, cis \frac{\frac{\pi}{2} + 2k\pi}{6} \qquad k = 0, 1, \dots 5}$

$z_0 = \sqrt[6]{2}\, cis \frac{\pi}{12}$ $\qquad z_1 = \sqrt[6]{2}\, cis \frac{5\pi}{12}$ ...

1.17)

$z^4 + z^2 + 1 = 0$ $\qquad t_{1,2} = \frac{-1 \pm \sqrt{1 - 4 \cdot 1}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$

$t = z^2$ $\qquad t_1 = \frac{-1}{2} + \frac{i\sqrt{3}}{2} \qquad t_2 = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

$t^2 + t + 1 = 0$

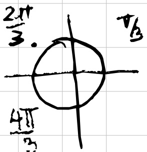
$z^2 = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$ $\qquad z^2 = 1\, cis \frac{4\pi}{3}$

$r = \frac{1}{4} + \frac{3}{4} = 1$

$z^2 = 1\, cis \frac{2\pi}{3}$

$tg = \frac{\frac{\sqrt{3}}{2}}{\frac{-1}{2}}$

$z_{1,2} = cis \frac{\frac{2\pi}{3} + 2k\pi}{2}$



$\boxed{z_1 = cis \frac{2\pi}{6} \qquad z_2 = cis \frac{4\pi}{3}}$

$\boxed{z_3 = cis \frac{2\pi}{3} \qquad z_4 = cis \frac{5\pi}{3}}$