(1)

(a) $\cdot\ \arg(z^4) = \pi \Rightarrow 4\arg(z) = \pi + 2k\pi, \quad k \in \mathbb{Z}$

$\Rightarrow \arg(z) = \frac{\pi}{4} + \frac{\pi}{2}k, \quad k \in \mathbb{Z}$

$\Rightarrow \varphi_1 = \frac{\pi}{4}, \quad \varphi_2 = \frac{3\pi}{4}, \quad \varphi_3 = \frac{5\pi}{4}, \quad \varphi_4 = \frac{7\pi}{4}$

$\cdot\ |z + i + 1| = 1 \Rightarrow (r\cos\varphi + 1)^2 + (r\sin\varphi + 1)^2 = 1$

$\Rightarrow r^2 + 2r(\cos\varphi + \sin\varphi) + 1 = 0$

$\Rightarrow$ 1°) $\varphi_1 = \frac{\pi}{4} \Rightarrow \cos\varphi_1 + \sin\varphi_1 = \sqrt{2}$

$\Rightarrow r^2 + 2r\sqrt{2} + 1 = 0$

$\Rightarrow r = -\sqrt{2} \pm 1 < 0$ ↯

2°) $\left.\begin{array}{l} \varphi_2 = \frac{3\pi}{4} \\ \varphi_4 = \frac{7\pi}{4} \end{array}\right\} \Rightarrow \cos\varphi_{2/4} + \sin\varphi_{2/4} = 0 \Rightarrow$

$\Rightarrow r^2 + 1 = 0$ ↯

3°) $\varphi_3 = \frac{5\pi}{4} \Rightarrow \cos\varphi_3 + \sin\varphi_3 = -\sqrt{2}$

$\Rightarrow r^2 - 2\sqrt{2}r + 1 = 0 \Rightarrow r_{1/2} = \sqrt{2} \pm 1$

$\Rightarrow$ 2 rješenja : $z_1 = (\sqrt{2} + 1)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$z_2 = (\sqrt{2} - 1)\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

① (b)

$$z_1 \cdot z_2 = r_1 r_2 \left(\cos\varphi_1 + i \sin\varphi_1\right)\left(\cos\varphi_2 + i \sin\varphi_2\right)$$

$$= r_1 r_2 \left[\left(\cos\varphi_1 \cos\varphi_2 - \sin\varphi_1 \sin\varphi_2\right) + i\left(\sin\varphi_1 \cos\varphi_2 + \cos\varphi_1 \sin\varphi_2\right)\right]$$

$$= r_1 r_2 \left[\cos\left(\varphi_1 + \varphi_2\right) + i \sin\left(\varphi_1 + \varphi_2\right)\right]$$

② (a)

6	5	4	3
MARA	LUCE	KATA	FRANKA

$\Rightarrow \quad 6 \cdot 5 \cdot 4 \cdot 3 = 360$

(Mara bira između 6 dečkiju za ples, Luce bira između preostalih 5 dečkiju za ples, Kata bira između preostalih 4 i Franka među preostalih 3. Rješenje je neovisno od redoslijeda djevojki koje biraju dečke.)

(b) $6 \cdot 5 \cdot 5 = 150$ mogućih zastava

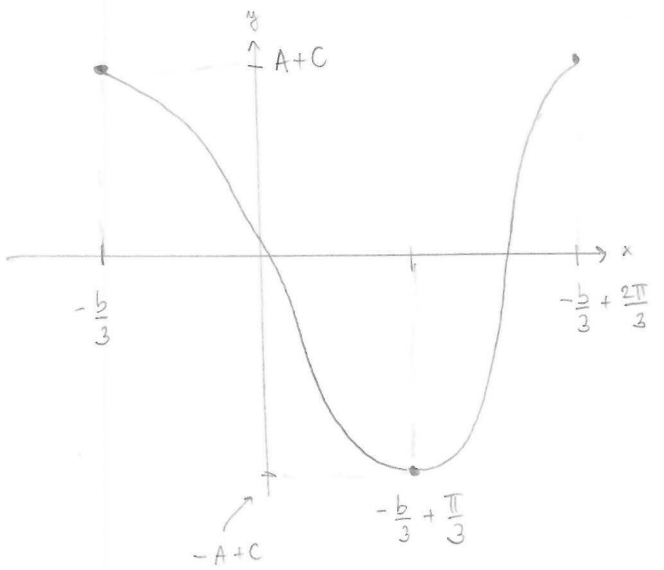
- 6: broj mogućih boja za srednju horizontalnu traku
- 5: broj mogućih boja za gornju horizontalnu traku
- 5: broj mogućih boja za donju horizontalnu traku

(C)
- neparni : 1, 3, 5, 7, 9 ; parni : 2, 4, 6, 8, 10
- postoji točno jedan peteročlani skup koji ne sadrži niti jedan neparni broj
- peteročlanih skupova koji sadrže barem jedan neparni broj

ima $\binom{10}{5} - 1$

3

(a) $f(x) = A \cos\left(3\left(x + \frac{b}{3}\right)\right) + C \quad , \quad A, b > 0$



• uvjet surjektivnosti

$$[-A + C \,,\, A + C] = [0, 4]$$

$$\Rightarrow \left.\begin{aligned} -A + C &= 0 \\ A + C &= 4 \end{aligned}\right\} A = C = 2$$

• uvjet injektivnosti

$$\left.\begin{aligned} -\frac{b}{3} &= -\frac{\pi}{6} \\ -\frac{b}{3} + \frac{\pi}{3} &= \frac{\pi}{6} \end{aligned}\right\} b = \frac{\pi}{2}$$

$$\Rightarrow f_1(x) = 2 \cdot \cos\left(3\left(x + \frac{\pi}{6}\right)\right) + 2$$

[uvjet injektivnosti smo mogli i promatrati kao

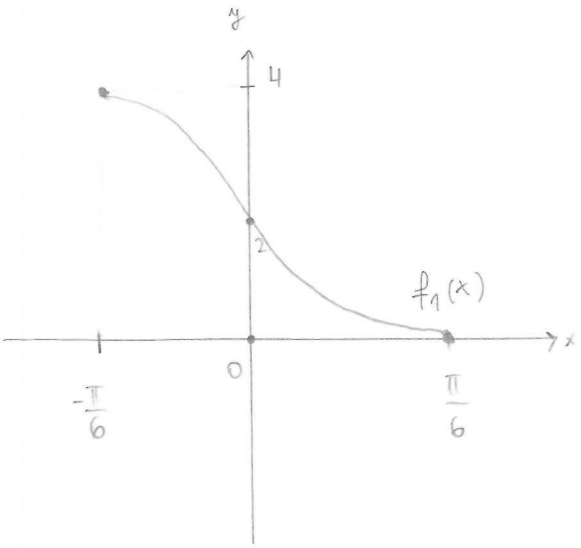
$$\left.\begin{aligned} -\frac{b}{3} + \frac{\pi}{3} &= -\frac{\pi}{6} \\ -\frac{b}{3} + \frac{2\pi}{3} &= \frac{\pi}{6} \end{aligned}\right\} b = \frac{3\pi}{2}$$

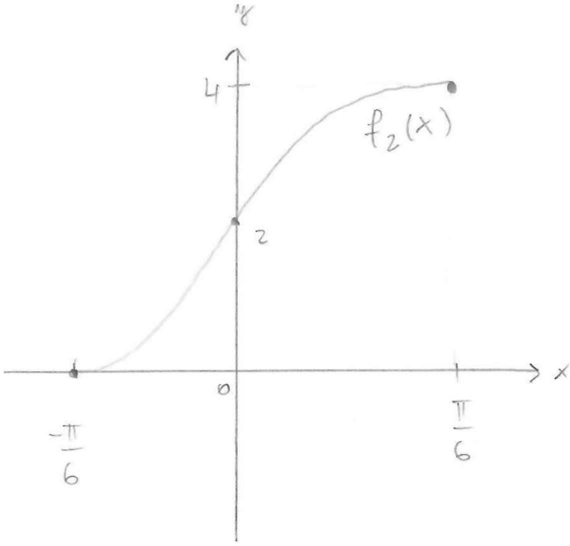
$$\Rightarrow f_2(x) = 2 \cdot \cos\left(3\left(x + \frac{\pi}{2}\right)\right) + 2$$

]

3

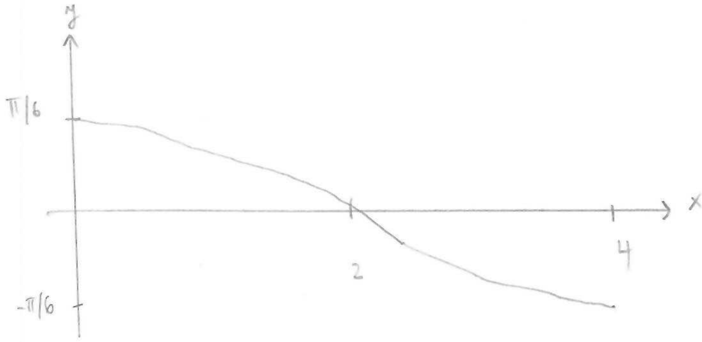
(b)





(c) $f_1^{-1}(x) = \frac{1}{3} \arccos\left(\frac{x-2}{2}\right) - \frac{\pi}{6}$

$f_1^{-1} : [0,4] \longrightarrow \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$



$f_2^{-1}(x) = \frac{1}{3} \arccos\left(\frac{x-2}{2}\right) - \frac{\pi}{2}$

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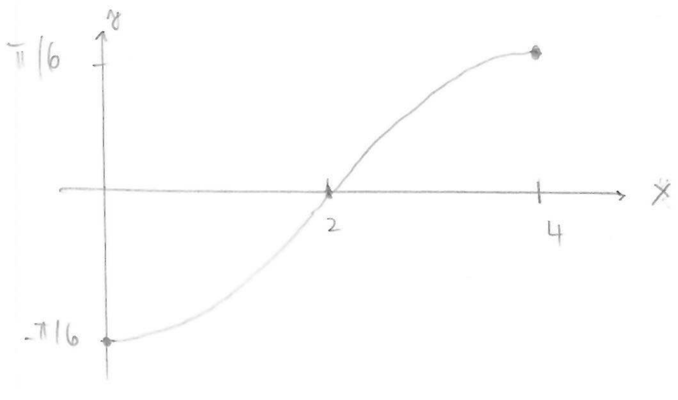


(4) (a) $(\forall \varepsilon > 0)\ (\exists n_0 \in \mathbb{N})\ (\forall n \geq n_0)\ \left(\frac{1}{n^2} < \varepsilon\right)$

$$\frac{1}{n^2} < \varepsilon \iff n^2 > \frac{1}{\varepsilon} \iff n > \frac{1}{\sqrt{\varepsilon}} \Rightarrow$$

$$n_0 = \left\lfloor \frac{1}{\sqrt{\varepsilon}} \right\rfloor + 1$$

$$\Rightarrow n_0 = 100001$$

(b) Realan broj b zovemo gomilište niza (b_n) ako se unutar svake ε-okoline broja b nalazi beskonačno mnogo članova niza (b_n).

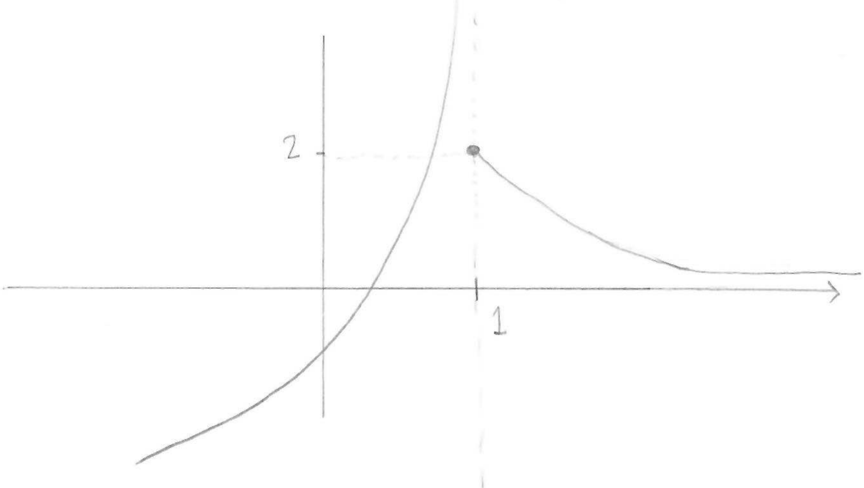
- za $n = 2k$ je $b_{2k} = (-1)^k \left(5 + \frac{1}{2k}\right)$ pa imamo dva gomilišta : -5 i 5
- za n neparan je $b_n = -2 + \frac{1}{n^2}$ pa imamo jedno gomilište : -2

$\Rightarrow$ Dakle, (b_n) ima 3 gomilišta : $-5, 5$ i -2.

(c) $$c_n = 2 \cdot (-1)^{2n-1} + \frac{1}{(2n-1)^2} = -2 + \frac{1}{(2n-1)^2},$$

pa je $\lim\limits_{n\to\infty} c_n = -2$, tj (c_n) konvergira prema -2.

⑤ (a)



(b) $\lim\limits_{x \to a} f(x) = L \iff \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^-} f(x) = L$

Funkcija iz (a) dijela zadatka nema limes u $x = 1$

(c) (c_1) $\lim\limits_{x \to +\infty} \left(\dfrac{2x+1}{2x}\right)^{3x+2} = \lim\limits_{x \to +\infty} \left[\left(1 + \dfrac{1}{2x}\right)^{2x}\right]^{\frac{3x+2}{2x}} = e^{3/2}$

(c_2) $\lim\limits_{x \to +\infty} \ln\left(\dfrac{5x^2+x}{2x^2+1}\right) \cdot \lim\limits_{x \to \infty} \underbrace{\left(\sqrt{x+1} - \sqrt{x}\right)}_{(\infty - \infty)} =$

$= \ln \dfrac{5}{2} \cdot \lim\limits_{x \to +\infty} \dfrac{\overset{\nearrow 0}{x+1-x}}{\sqrt{x+1} + \sqrt{x}} = \ln \dfrac{5}{2} \cdot 0 = 0$

6 (a) $f(x) = \ln(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\ln\left(\frac{x+h}{x}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \ln\left(1 + \frac{h}{x}\right) = \lim_{h\to 0} \ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}}$$

$$\left[\begin{array}{l} \text{Supst.} \quad n = \frac{h}{x}, \quad h = nx, \quad \frac{1}{h} = \frac{1}{nx} = \frac{1}{n}\cdot\frac{1}{x} \\ \text{Kada } \quad h \to 0 \Rightarrow n \to 0 \end{array}\right]$$

$$= \lim_{n\to 0} \ln\left(1+n\right)^{\frac{1}{n}\cdot\frac{1}{x}} = \frac{1}{x} \lim_{n\to 0} \ln(1+n)^{\frac{1}{n}}$$

$$= \frac{1}{x} \ln\left(\lim_{n\to 0} (1+n)^{\frac{1}{n}}\right) = \frac{1}{x} \ln(e) = \frac{1}{x}$$

(b) f^{-1} je diferencijabilna

$$f\left(f^{-1}(y)\right) = y \quad \Big| \frac{d}{dy}$$

$$f'\left(f^{-1}(y)\right)\left(f^{-1}\right)'(y) = 1$$

$$\left(f^{-1}\right)'(y) = \frac{1}{f'(x)} \quad \text{gdje je} \quad x = f^{-1}(y)$$

6

(c) $\underbrace{\arcsin x}_{f^{-1}(x)} = y \Rightarrow \underbrace{\sin y}_{f(y)} = x$

Prema (b) zadatku $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(y)}$

$f^{-1}(x) = \arcsin x$

$$\left(f^{-1}(x)\right)' = \frac{1}{f'(y)} = \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}$$

Znamo $\sin^2 y + \cos^2 y = 1$

$\cos^2 y = 1 - \sin^2 y$

$\cos y = \sqrt{1-\sin^2 y}$

Imamo zadano

$\sin y = x$

$\sin^2 y = x^2$