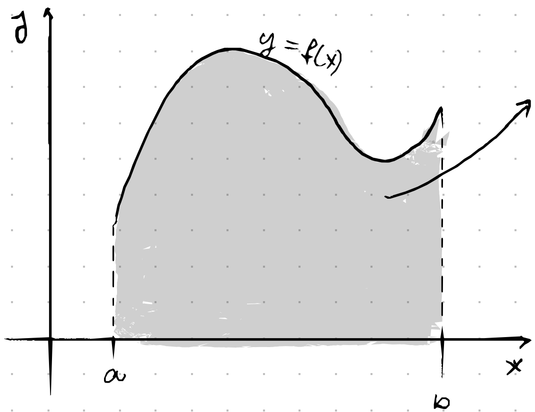
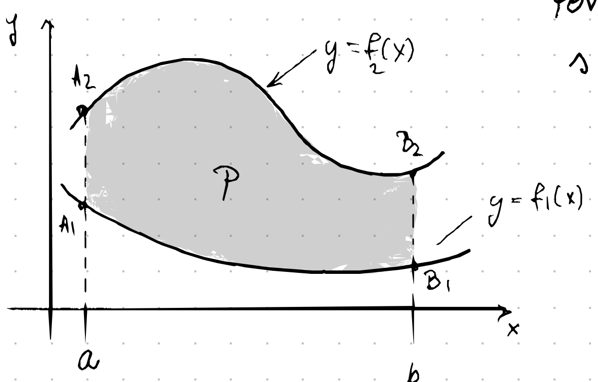
12.1. IZRAČUNAVANJE POVRŠINE RAVNINSKOG LIKA



$\int_a^b f(x)\,dx$ predstavlja površinu:

- omeđenu krivuljom $y=f(x)$, $f(x)\geq 0$, $x\in[a,b]$
- omeđenu pravcima $x=a$ i $x=b$ i osi x

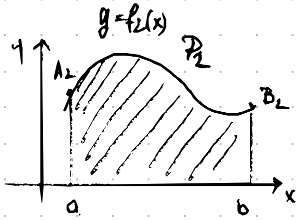
Primjer: Imamo lik:

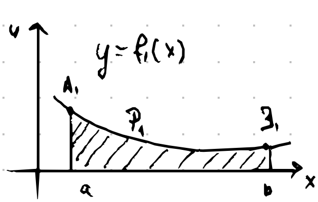


Površina ovog lika omeđena je s $y=f_2(x)$ i $y=f_1(x)$, $f_2(x)\geq f_1(x)$

te pravcima $x=a$, $x=b$

$\longrightarrow$ površina lika jednaka je razlici površina



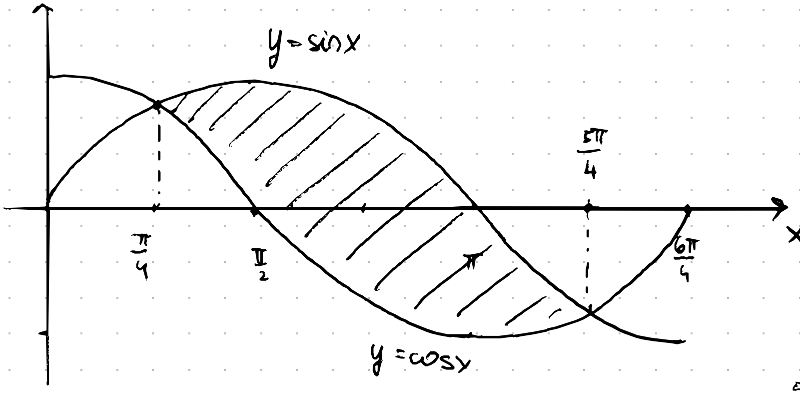


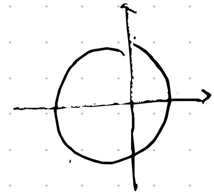
$$P = P_2 - P_1$$

$$\Longrightarrow \quad P=\int_a^b f_2(x)\,dx-\int_a^b f_1(x)\,dx$$

$$P=\int_a^b \big(f_2(x)-f_1(x)\big)\,dx$$

Primjer 12.2.) Izračunajte površinu lika:



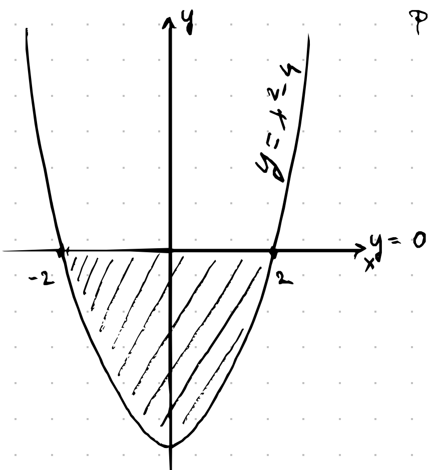


$$P = \int_{\pi/4}^{\frac{5\pi}{4}} (\sin x - \cos x)\, dx = \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin x\, dx - \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \cos x\, dx$$

$$= -\cos x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} - \sin x \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{4}} = \left(-\cos\left(\tfrac{5\pi}{4}\right) + \cos\left(\tfrac{\pi}{4}\right)\right) - \left(\sin\tfrac{5\pi}{4} - \sin\tfrac{\pi}{4}\right)$$

$$= +\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} + \left(+\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) = \frac{4\sqrt{2}}{2} = \boxed{2\sqrt{2}}$$

Primjer 12.3.)



$$P = \int_{-2}^{2} (0 - x^2 + 4) = \int_{-2}^{2} (4 - x^2)\, dx$$

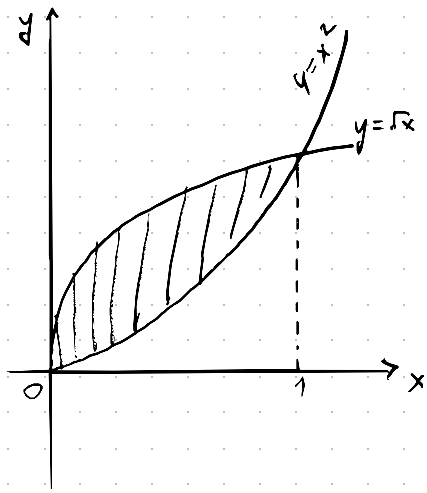
$$= \int 4\, dx - \int x^2 dx = \left(4x - \frac{1}{3}x^3\right)\Big|_{-2}^{2} =$$

$$= \left(4 \cdot 2 - \frac{1}{3} 2^3\right) - \left(4 \cdot (-2) - \frac{1}{3}(-2)^3\right)$$

$$= 8 - \frac{8}{3} - \left(-8 + \frac{8}{3}\right) = 16 - \frac{16}{3}$$

$$= \frac{48}{3} - \frac{16}{3} = \boxed{\frac{32}{3}}$$

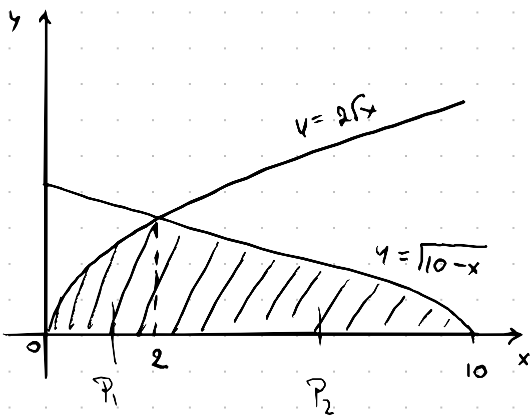
Primjer 12.4.)



$$P = \int_0^1 (\sqrt{x} - x^2)\,dx = \left(\frac{2}{3}x^{3/2} - \frac{1}{3}x^3\right)\Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

Primjer 5.)



kada je „odsječena" površina sa V.A, onda računamo određ. int. od te fje.

$$P = P_1 + P_2 \qquad \underbrace{\text{gleda}}\ \text{prvi dio}$$

$$P_1 = \int_0^2 (2\sqrt{x})\,dx + \int_2^{10} \sqrt{10-x}\,dx$$

$$P_1 = 2\cdot\frac{2}{3}x^{3/2}\Big|_0^2 + \frac{2}{3}\sqrt{10-x}^{\,3}\Big|_0^2$$

$$P_1 = \frac{4}{3}\cdot\sqrt{8} + \frac{2}{3}\sqrt{(10-2)^3}$$

$$P_1 = \frac{8\sqrt{2}}{3} + \frac{2}{3}\sqrt{64\cdot 8} = \frac{4}{3}\sqrt{8} + \frac{16}{3}\sqrt{8}$$

$$P_1 = \frac{4}{3}\left(\sqrt{8} + 4\sqrt{8}\right) = \frac{40\sqrt{8}}{3}$$

Izračunavanje površine lika omeđenog krivuljom zadanom parametarskim jednadžbama

Skup svih točaka (x, y) za koje vrijedi

da je $\begin{array}{l} x = x(t) \\ y = y(t),\ t \in I \end{array}$ je parametarski zadana krivulja.

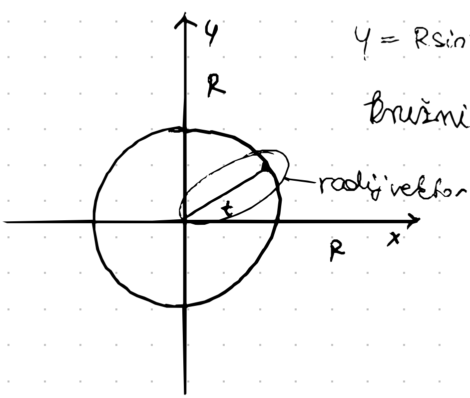
- varijabla x i varijabla y su funkcije varijable t

Primjer 10.)

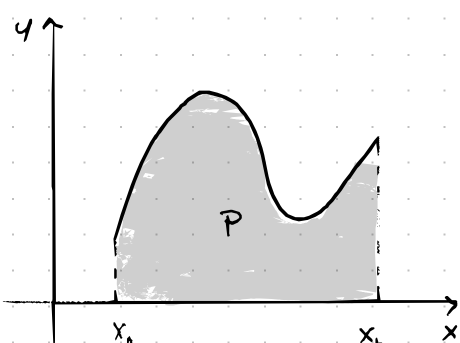
$$\left.\begin{array}{l} x = R\cos t \\ y = R\sin t \end{array}\right\} \boxed{t} \in [0, 2\pi\rangle$$

parametar t je kut

kružnica polumjera R sa središtem u ishodištu



Zamislimo da je $x = x(t)$, $y = y(t)$, t $[a, b]$ i da je dana krivulja C ; $x_a = x(a)$ i $x_b = x(b)$



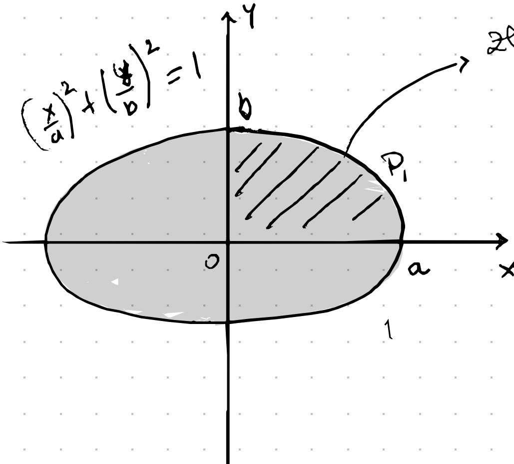
time dobivamo da je površina jednaka $P = \int_{x_a}^{x_b} y(x)dx$

$$= \int_{x_a}^{x_b} y\,dx = \int_a^b y(t)\,\dot{x}(t)\,dt$$

integ po t var.

* U slučaju da je $x_b < x_a$ imali bismo $P = \int^{x_a} y\,dx$

Primjer 11.) Izračunajmo površinu elipse s poluosima a i b



zbog simetrije u odnosu na osi $P=4P_1$

$$\left(\frac{y}{b}\right)^2 = 1-\left(\frac{x}{a}\right)^2$$

$$\frac{y}{b} = \sqrt{1-\left(\frac{x}{a}\right)^2} \quad / \cdot b$$

$$\underline{\underline{y = b\sqrt{1-\left(\frac{x}{a}\right)^2}}}$$

$\sin 1 = \frac{\pi}{2}$

$$P = 4\int_0^a y\,dx = 4\int_0^a b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx = \left| \begin{array}{l} x = a\cdot\sin t \\ dx = a\cdot\cos t \end{array} \right|$$

$$P = 4b\int_0^{\sin(a)} \sqrt{1-\left(\frac{\cancel{a}\sin t}{\cancel{a}}\right)^2}\cdot a\cdot\cos t\,dt = 4ab\int_0^{\pi/2}\sqrt{\cos^{\cancel{2}} t}\,\cos t\,dt$$

$$P = 4ab\int_0^{\pi/2}\cos^2 t\,dt = 4ab\int_0^{\pi/2}\frac{\cos 2t+1}{2} = 2ab\int_0^{\pi/2}(\cos 2t+1)$$

$$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - 1 + \cos^2 t$$

$$\cos 2t = 2\cos^2 t - 1$$

$$\frac{\cos 2t+1}{2} = \cos^2 t$$

$$2ab\cdot\left(\tfrac{1}{2}\sin 2t + x\right)\Big|_0^{\pi/2}$$

$$= 2ab\left(\frac{1}{2}\underbrace{\sin\pi}_{0} + \frac{\pi}{2}\right) = ab\,(\underbrace{\sin\pi}_{0} + \pi)$$

$$\boxed{P = ab\pi}$$

Izračunavanje površine lika u polarnim koordinatama

× kod kompleksnih brojeva : r = modul
φ = kut

→ jednadžbama $x = \cos\varphi$, $y = \cos\varphi$ se uvode polarne koordinate (r, φ)

· jednadžbom $r = r(\varphi)$ je dana krivulja u polarnim koord.

Primjer 13.)

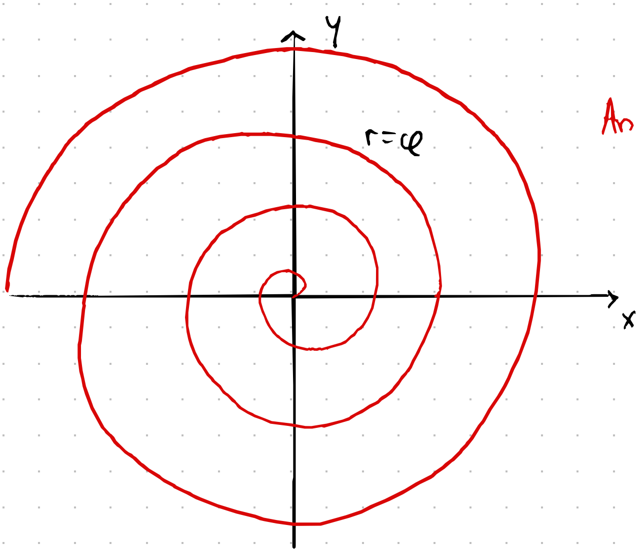
Zadana je krivulja $r = a\varphi$ ($a > 0$, $\varphi \geq 0$) → $a\varphi \geq 0 \Rightarrow r \geq 0$

- r je udaljenost točke od ishodišta
- φ je kut koji radij vektor te točke zatvara s apscisom

→ ako je $\varphi = 0$, onda je $r = 0$ ⇒ točka ishodišta $(0,0)$

· porastom varijable φ linearno raste r (udaljenost od ish.)

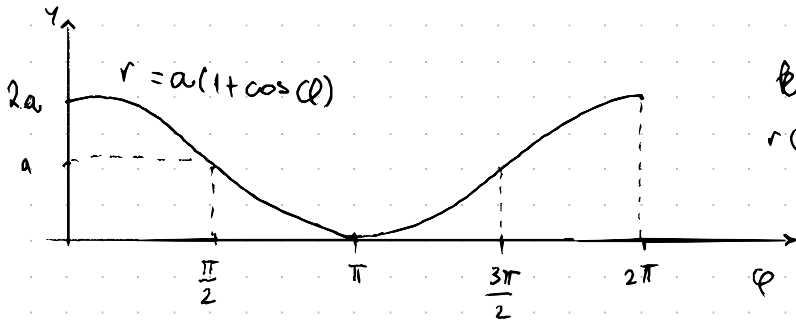
⇒ za $\varphi \in [0, 2\pi]$ se dođe puni krug, za 2π ponovno imamo točku na poz x



Arhimedova spirala

Primjer 14.) Zadana je krivulja $r = a(1+\cos\varphi)$, $a > 0$

- $a(1+\cos\varphi) \geq 0$

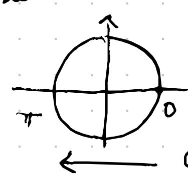


krivulja u $rO\varphi$ sustavu:

$\varphi = 0 \to r$ $\quad r = a(1+1) = 2a > 0 \quad (x,y) = (2a, 0)$

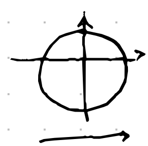
$\varphi \in [0, \pi]$ r pada

krivulja se približava ishodištu



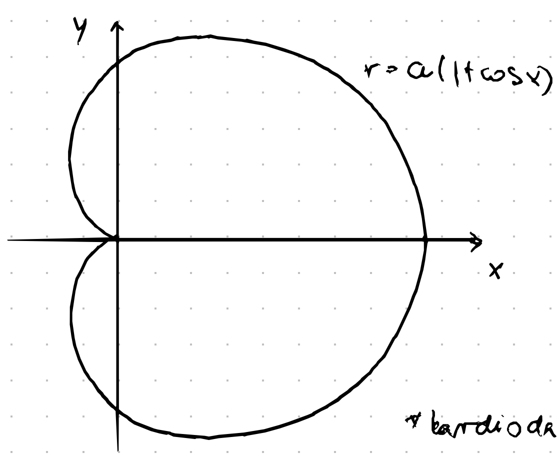
cos se smanjuje

$\varphi \in [\pi, 2\pi]$ r raste



krivulja simetrična na os x

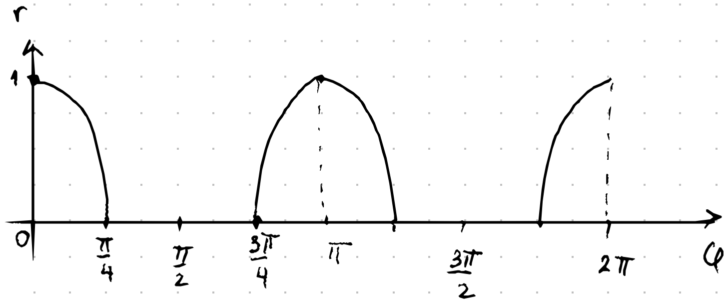
simetrična s obzirom na os x

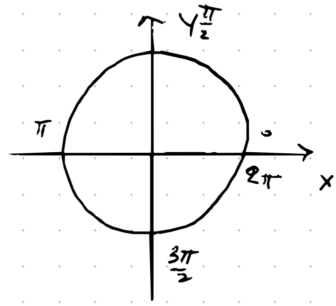


* kardioda ♡

Primjer 15.) $r^2=\cos(2\varphi)$

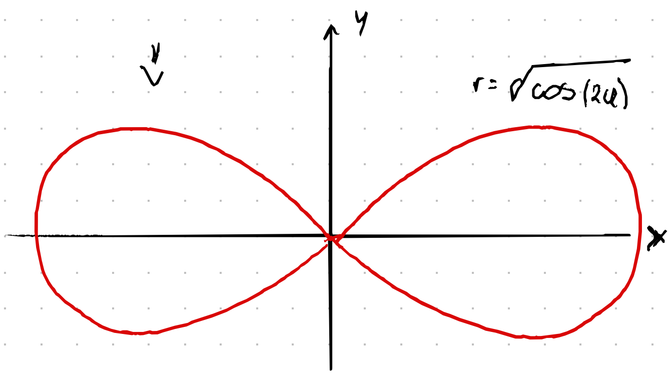
$r=\sqrt{\cos(2\varphi)}$





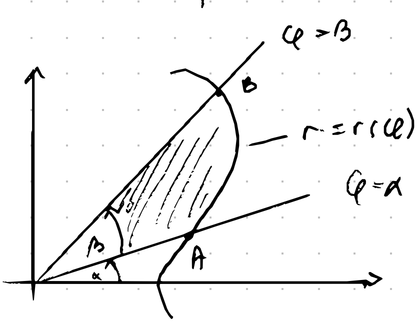
$r=\sqrt{\cos(0)}=1$ $\quad$ $r=\sqrt{\cos\left(2\cdot\frac{3\pi}{4}\right)}$

$r=\sqrt{\cos(\pi)}=\cancel{\sqrt{-1}}$ $\quad$ $r=\sqrt{\cos(2\cdot\pi)}$



$r=\sqrt{\cos(2\varphi)}$ $\longrightarrow$ trebam izračunati površinu ovog lika

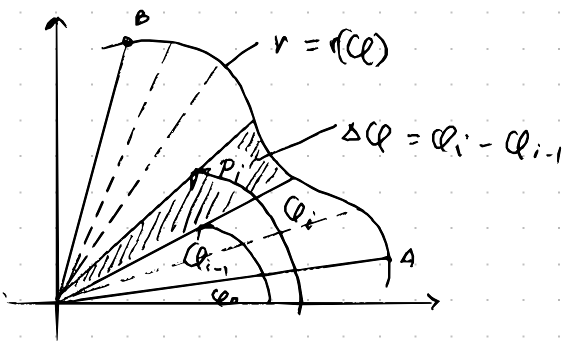
računamo površinu lika $r=r(\varphi)$, $\varphi=\alpha$ i $\varphi=\beta$.



Interval $[\alpha,\beta]$ podijelimo na dijelove

$a=\varphi_0<\varphi_1<\dots<\varphi_{n-1}<\varphi_n=\beta$

/ Aproksimirajmo površinu svakog od dobivenih dijelova površinom P_i kružnog isječka polumjera $r(\varphi_i)$ i kuta $\Delta\varphi_i = \varphi_i - \varphi_{i-1}$



$\Rightarrow$ Tada je površina lika P približno jednaka zbroju n površina odgovarajućih kružnih isječaka

$$P \approx \sum_{i=1}^{n} P_i$$

P_i = površina kružnog isječka

$$P_i = r^2\pi \cdot \frac{\alpha}{360^\circ} = r^2 \cancel{\pi} \cdot \frac{\varphi}{2\cancel{\pi}}$$

$$P_i = \frac{1}{2} r^2 \varphi$$

$\Delta\varphi$

$r = r(\varphi)$

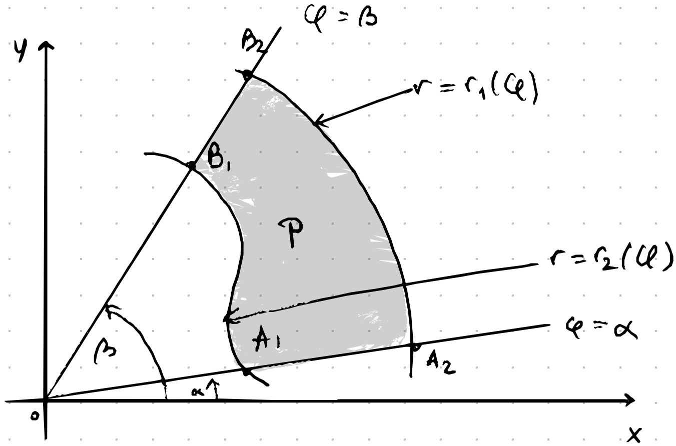
$$\sum_{i=1}^{n} \frac{1}{2}(r(\varphi_i))^2 \cdot \Delta\varphi$$

$$\boxed{= \frac{1}{2} \sum_{i=1}^{n} r^2(\varphi_i) \cdot \Delta\varphi}$$

kada graf razdijelimo na beskonačno mnogo komadića, odnosno kada je $n \to \infty$

$$\lim_{n\to\infty}\left(\frac{1}{2}\sum_{i=1}^{n} r^2(\varphi_i) \cdot \Delta\varphi\right) = \boxed{\frac{1}{2}\int_{\alpha}^{\beta} r^2(\varphi)\, d\varphi}$$

Općenitije, neka je lik omeđen krivuljama $r=r_1(\varphi)$ i $r=r_2(\varphi)$, te polupravcima $\varphi=\alpha$ i $\varphi=\beta$.



Površina zadanog lika dana je izrazom

$$P=\frac{1}{2}\int_{\alpha}^{\beta}\left(r_2^2(\varphi)-r_1^2(\varphi)\right)d\varphi$$

Primjer)

→ onaj primjer sa kardiodom

$[0,2\pi]$

$a>0$

Izračunajmo površinu P lika određenog krivuljom $r=a(1+\cos\varphi)$.

$$P=\frac{1}{2}\int_0^{2\pi} r^2(\varphi)\,d\varphi=\frac{1}{2}\int_0^{2\pi}\overset{\text{konst.}}{a^2}(1+\cos\varphi)^2\,d\varphi=\frac{a^2}{2}\int_0^{2\pi}(1+2\cos\varphi+\cos^2\varphi)\,d\varphi$$

$$=\frac{a^2}{2}\int_0^{2\pi}\left(1+2\cos\varphi+\frac{\cos 2\varphi+1}{2}\right)d\varphi=\frac{a^2}{2}\int_0^{2\pi}\left(\frac{3}{2}+2\cos\varphi+\frac{\cos 2\varphi}{2}\right)$$

$$=\frac{a^2}{4}\int_0^{2\pi}(3+4\cos\varphi+\cos 2\varphi)=\frac{a^2}{4}\left(3\varphi+4\sin\varphi+\frac{1}{2}\sin 2\varphi\right)\Big|_0^{2\pi}$$

$$=\frac{a^2}{4}\left(6\pi+0+\frac{1}{2}\cdot 0\right)=\frac{a^2}{4}\cdot 6\pi=\boxed{\frac{3\pi a^2}{2}}$$

$\cos^2\varphi-\sin^2\varphi=\cos 2\varphi$

$\cos^2\varphi=\cos 2\varphi+\sin^2\varphi$

$\cos^2\varphi=\cos 2\varphi+(1-\cos^2\pi)$

$\cos^2\varphi=\cos 2\varphi+1-\cos^2\pi$

$\cos^2\varphi=\frac{\cos 2\varphi+1}{2}$