7.4. Računanje limesa neodređenih oblika pomoću L'Hospitalovog pravila

Radi za:

Neodređeni oblik $\left(\frac{0}{0}\right)$, $\left(\frac{\infty}{\infty}\right)$

$\boxed{\text{TM}}$ - L'Hospitalovo pravilo $\left(\frac{0}{0}\right)$

Neka su f, g diferencijabilne fje na $S = \langle a, x_0 \rangle \cup \langle x_0, b \rangle$ i $g'(x) \neq 0$ na S. Ako vrijedi

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad \text{i} \quad \text{postoji} \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$$

onda je:

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

$\boxed{\text{TM}}$ - L'Hospitalovo pravilo $\left(\frac{\infty}{\infty}\right)$

Neka su f, g diferencijabilne fje na $S = \langle a, x_0 \rangle \cup \langle x_0, b \rangle$ i $g'(x) \neq 0$ na S. Ako vrijedi

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \infty \quad \text{i} \quad \text{postoji} \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$$

onda je:

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

Zad.)

a) $\lim_{x\to\infty} \frac{\ln^2 x}{x^2} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{2\ln x \cdot \frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{\ln x}{x^2} = \left(\frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{\underset{\downarrow\\ 0}{2x^2}} = 0$

b) $\lim_{x\to\infty} \frac{e^x}{x^3} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{3x^2} = \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{6x} = \left(\frac{\infty}{\infty}\right)$

$\overset{L'H}{=} \lim_{x\to\infty} \frac{e^x}{6} = \infty$

Neodređeni oblici: $(0\cdot\infty), (\infty-\infty), (0^0), (\infty^0), (1^\infty)$

1. $(\infty\cdot 0)$

$$\lim_{x\to a} f(x)\cdot g(x) = 0\cdot\infty = \begin{cases} \lim\limits_{x\to a} \dfrac{f(x)}{\frac{1}{g(x)}} = \left(\dfrac{0}{0}\right) \\[2ex] \lim\limits_{x\to a} \dfrac{g(x)}{\frac{1}{f(x)}} = \left(\dfrac{\infty}{\infty}\right) \end{cases}$$

Zad $\lim\limits_{x\to 0^+} x\cdot \ln x = (0\cdot\infty) = \lim\limits_{x\to 0^+} \dfrac{\ln x}{\frac{1}{x}} = \left(\dfrac{\infty}{\infty}\right) \overset{L'H}{=} \lim\limits_{x\to 0^+} \dfrac{\frac{1}{x}}{\frac{-1}{x^2}}$

$= \lim\limits_{x\to 0^+} \dfrac{x}{-1} = \lim\limits_{x\to 0^+}(-x) = \infty$

Zad. $\lim\limits_{x\to\infty} x\cdot \operatorname{arctg}(2x) = (\infty\cdot 0) \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{\operatorname{arctg}(2x)}{\frac{1}{x}} = \left(\dfrac{0}{0}\right) \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{-\frac{1}{1+4x^2}\cdot 2}{-\frac{1}{x^2}}$

$= \lim\limits_{x\to\infty} \dfrac{2x^2}{4x^2+1} \;/:x^2 = \lim\limits_{x\to\infty} \dfrac{2}{4+\underset{\infty}{\frac{1}{x^2}}} = \dfrac{1}{2}$

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2. $(\infty-\infty)$

$$\lim_{x\to a}(f(x)-g(x)) = (\infty-\infty) = \lim_{x\to a}\left(1-\frac{g(x)}{f(x)}\right)f(x) = \begin{cases} (0\cdot\infty), & \lim \dfrac{g(x)}{f(x)} = 1 \\[2ex] \text{određeni oblik} & \lim \dfrac{g(x)}{f(x)} \neq 1 \end{cases}$$

Zad. $\lim\limits_{x\to\infty}\left(x\cdot e^{-\frac{1}{x^2}} - x\right) = (\infty-\infty) = \lim\limits_{x\to\infty} x\cdot\left(e^{-\frac{1}{x^2}}-1\right) \to (\infty\cdot\infty)$

$= \lim\limits_{x\to\infty} \dfrac{e^{-\frac{1}{x^2}}-1}{\frac{1}{x}} = \left(\dfrac{0}{0}\right) \overset{L'H}{=} \lim\limits_{x\to\infty} \dfrac{e^{-\frac{1}{x^2}}\cdot\frac{2}{x^3}}{-\frac{1}{x^2}} = \lim\limits_{x\to\infty} \dfrac{-2e^{-\frac{1}{x^2}}}{x} = 0$

3. OBLICI $(0^0), (1^\infty), (\infty^0)$

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{g(x)\ln f(x)} = e^{\lim_{x\to a} g(x)\ln(f(x))} = e^{(0\cdot\infty)} \ldots$$

Zad. Izračunajte $\lim_{x\to 0^+} \frac{x\cdot\cos x + a\sin x}{x^3}$ u ovisnosti o $a\in\mathbb{R}$

$$\lim_{x\to 0^+} \frac{x\cdot\cos x + a\sin x}{x^3} = \left(\frac{0}{0}\right) \overset{L'H}{=} \lim_{x\to 0^+} \frac{\cos x - x\cdot\sin x + \overbrace{\sin x}^{0} + a\cos x}{3x^2} =$$

$$\lim_{x\to 0^+} \frac{\cos x - x\sin x + a\cos x}{3x^2} = \lim_{x\to 0^+} \frac{(a+1)\cos x - x\sin x}{3x^2} =$$

$$\lim_{x\to 0} \left(\frac{(a+1)\cos x}{3x^2} - \frac{x\cdot\sin x}{3x^2}\right) = \begin{cases} a=-1 & \lim_{x\to 0^+} = \frac{-1}{3} \\ a>-1 & \lim_{x\to 0^+} = +\infty \\ a<-1 & \lim_{x\to 0^+} = -\infty \end{cases}$$

$$\lim_{x\to 0^+} \frac{(a+1)\cos x}{3x^2} = \frac{(a+1)\cdot 1}{0^+}$$