0D numerical solution

andreapiccolo89

September 2022

1 Introduction

I take the work done during the coding friday and expanding a bit.

- Setting up the equations starting from the definition of necking;
  1. Rheological consideration;
  2. Introduction of the main Forces (F_B and F_D)
  3. Definition of $\tau_{B,0}$ and τ;
  4. Definition of the dimensional equation to solve, with the characteristic scaling
- Definition of the characteristic dimensions and additional relation
- Derivation of the τ
  1. Rearranging the equation associated with the force balance;
  2. Introduction of the following adimensional number: Λ,Ψ and γ
  3. Definition of τ as a function of $\frac{dD}{dt}$,τ_B and Λ,Ψ and γ.
- Final equation and introduction of the therminology

2 Derivation 0D numerical equations

2.1 Main reference equation and introductory definitions

We have:

$$\dot{\varepsilon} = -\frac{1}{D}\frac{dD}{dt} \tag{1}$$

and

$$\dot{\varepsilon} = B_n\tau^n + B_d\tau \tag{2}$$

this can be expressed as:

$$\dot{\varepsilon} = B_n\tau^n\left[1 + \frac{B_d}{B_n}\tau^{1-n}\right] \tag{3}$$

transition stress:

$$\tau_t^{n-1} = \frac{B_d}{B_n} \tag{4}$$

Put that in the equation above:

$$\dot{\varepsilon} = B_n \tau^n \left[1 + \left(\frac{\tau}{\tau_t} \right)^{1-n} \right] \tag{5}$$

Now, I would like to add a further definition. In my numerical approach I imposed that η_d is $\eta_d = \Xi \eta_n$ at τ_B (the buoyancy stress imposed by the stalled slab). Where Ξ is the viscosity contrast between the two mechanisms at the reference condition (i.e. τ_B).

From this reference condition, I can derived $B_{n|d}$. Therefore, this strategy allows me to do the following:

$$B_d = \frac{1}{2\eta_n \Xi} \tag{6}$$

$$B_n = \frac{\tau_B^{1-n}}{2\eta_n \Xi} \tag{7}$$

then:

$$\tau_t^{n-1} = \frac{\frac{1}{2\eta_n \Xi}}{\frac{\tau_B^{1-n}}{2\eta_n}} \tag{8}$$

doing some arrangments:

$$\sqrt[n-1]{\tau_t^{n-1}} = \sqrt[n-1]{\frac{\tau_B^{n-1}}{\Xi}} \tag{9}$$

and by conveniently introducing $\xi = \Xi^{\frac{1}{1-n}}$

$$\tau_t = \xi \tau_B \tag{10}$$

So, let's come back to the equation 1 and rearrange it to describe the problem that I want to solve:

$$\frac{dD}{dt} = -D \left\{ B_n \tau^n \left[1 + \left(\frac{\tau}{\xi \tau_B} \right)^{1-n} \right] \right\} \tag{11}$$

The stress in the necking region is given by:

$$\tau = \frac{1}{2} \frac{F_B + F_D}{D} \tag{12}$$

where:

$$F_B = \Delta \varrho g L_0 D_0 \tag{13}$$

$$F_D = -2 \frac{dD}{dt} \eta_{eff,0}^{UM} \left(\frac{D_0}{D} \right)^2 \frac{L_0 \alpha}{s} \tag{14}$$

The equation from which I would like to start the derivation is the following

$$0 = -D^* l_c \left\{ (B_n^* \tau_c^{-n} {t_c}^{-1}) (\tau^* \tau_c)^n \left[1 + \left(\frac{(\tau^* \tau_c)}{\xi(\tau_{B,0}^* \tau_c)} \right)^{1-n} \right] \right\} - \left(\frac{dD^*}{dt^*} \frac{l_c}{t_c} \right) \quad (15)$$

Where $\tau^* = \tau/\tau_c$ represents the not dimensional effective stress. While D^* represents the not dimensional thickness.

2.2 Characteristic value and additional relations

Now let's start defining additional and important relation and what I believe are the most important characteristic length:

$$l_c = D_0 \quad (16)$$
$$\tau_c = \tau_{B,0} \quad (17)$$
$$= \frac{F_B}{2D_0} \quad (18)$$
$$= \frac{\Delta \varrho g L_0 D_0}{2D_0} \quad (19)$$
$$= \frac{\Delta \varrho g L_0}{2} \quad (20)$$
$$t_c = \frac{1}{\dot{\varepsilon}_c} \quad (21)$$
$$\dot{\varepsilon}_c = \left\{ B_n \tau_c^n \left[1 + \left(\frac{\tau_c}{\xi \tau_B} \right)^{1-n} \right] \right\} \quad (22)$$
$$\frac{dD}{dt}_c = l_c \dot{\varepsilon}_c \quad (23)$$
$$= \frac{l_c}{t_c} \quad (24)$$

Additionally, it can be possible define an other quantity $\eta_{eff,0}^S$. This quantity is the effective viscosity of the slab at given reference condition (i.e. $\tau = \tau_{B,0} = \tau_c$). This quantity allows to define the $\tau_{B,0}$ in this alternative manner:

$$\tau_c = \tau_{B,0} = 2\eta_{eff,0}^S \dot{\varepsilon}_c \quad (25)$$

Then, we can tackle the problem represent by τ and introducing other useful relation for the derivation:

$$\tau_B = \frac{\tau_{B,0} D_0}{D} [Pa] \quad (26)$$
$$\tau_B^* = \frac{D_0}{D} \quad (27)$$

2.3 Derivation of the effective stress

The basic equation are set up. In the following part I derive a definition for τ, the effective stress (12), using the relations in (14).

$$\tau = \frac{F_B}{2D}\left(1 + \frac{F_D}{F_B}\right) \tag{28}$$

$$= \frac{\tau_{B,0} D_0}{D}\left(1 + \frac{-2\frac{dD}{dt}\eta^{UM}_{eff,0}\left(\frac{D_0}{D}\right)^2 \frac{L_0}{s}}{2\tau_{B,0}D_0}\right) \tag{29}$$

$$\tag{30}$$

The next part of the derivation is exploitivng some relation that I wrote above ($\tau_{B,0} = 2\eta^S_{eff}\dot{\varepsilon}_c$ and $\frac{dD}{dt}_c = \dot{\varepsilon}_c D_0$)

$$\tau = 2\eta^S_{eff,0}\dot{\varepsilon}_c \frac{D_0}{D}\left(1 + \frac{-\frac{dD}{dt}\eta^{UM}_{eff,0}\left(\frac{D_0}{D}\right)^2 \frac{L_0}{s}}{2\eta^S_{eff,0}\dot{\varepsilon}_c D_0}\right) \tag{31}$$

$$\left(1 - \frac{\Psi L_0 \alpha}{2s}\left(\frac{D_0}{D}\right)^2 \frac{dD}{dt}\left(\frac{dD}{dt}_c\right)^{-1}\right) \tag{32}$$

then the equation for the effetive stress becomes

$$\tau = \tau_{B,0}\frac{D_0}{D}\left(1 - \frac{\Psi L_0 \alpha}{2s}\left(\frac{D_0}{D}\right)^2 \frac{dD}{dt}\left(\frac{dD}{dt}_c\right)^{-1}\right) \tag{33}$$

which is properly not dimensionalized with $\tau_c = \tau_{B,0}$, and using additional characteristic value defined above:

$$\frac{\tau}{\tau_c} = \tau^*_B\left(1 - \frac{\Psi L_0 \alpha}{2s}\left(\tau^*_B\right)^2 \frac{dD^*}{dt^*}\right) \tag{34}$$

Additionally i can introduce the following term:

$$\gamma = \frac{L_0 \alpha}{2s} \tag{35}$$

Which represent adimensional group concerning the characteristic wavelegnth of deformation w.r.t the length of the slab and the scale of the convection. Yielding:

$$\frac{\tau}{\tau_c} = \tau^*_B\left(1 - \gamma\Psi{\tau^*_B}^2 \frac{dD^*}{dt^*}\right) \tag{36}$$

or introducing $\Lambda = \gamma\Psi$:

$$\frac{\tau}{\tau_c} = \tau^*_B\left(1 - \Lambda{\tau^*_B}^2 \frac{dD^*}{dt^*}\right) \tag{37}$$

3 Final equation

$$0 = -D^* \left\{ B_n^* (\tau^*)^n \left[1 + \left(\frac{\tau^*}{\xi} \right)^{1-n} \right] \right\} - \left(\frac{dD^*}{dt^*} \right) \tag{38}$$

$$= -D^* \left\{ B_n^* \left(\tau_B^* \left(1 - \Lambda {\tau_B^*}^2 \frac{dD^*}{dt^*} \right) \right)^n \left[1 + \left(\frac{\tau_B^* \left(1 - \Lambda {\tau_B^*}^2 \frac{dD^*}{dt^*} \right)}{\xi} \right)^{1-n} \right] \right\} - dD^* dt^* \tag{39}$$

since we define $\xi = \frac{1}{\sqrt[n-1]{\Xi}}$ which is equivalent to $\xi = \Xi^{\frac{1}{1-n}}$. Then $\frac{1}{\xi} = \frac{1}{\Xi^{\frac{1}{1-n}}}$ which allows to extract $\frac{1}{\Xi}$

$$0 = -D^* \left\{ B_n^* \left(\tau_B^* \left(1 - \Lambda {\tau_B^*}^2 \frac{dD^*}{dt^*} \right) \right)^n \left[1 + \frac{1}{\Xi} \left(\tau_B^* \left(1 - \Lambda {\tau_B^*}^2 \frac{dD^*}{dt^*} \right) \right)^{1-n} \right] \right\} - \frac{dD^*}{dt^*} \tag{40}$$

$$B_n^* = \frac{B_n}{(\tau_{B,0})^{-n} \dot{\varepsilon}_c} \qquad \tau_B^* = \frac{\tau_B}{\tau_{B,0}} = \frac{D_0}{D} \tag{41}$$

$$D^* = \frac{D}{D_0} \qquad \frac{dD^*}{dt} = \frac{\frac{dD}{dt}}{D_0 \dot{\varepsilon}_c} \tag{42}$$

$$\Lambda = \gamma \Psi \qquad \gamma = \frac{L_0 \alpha}{s} \tag{43}$$

$$\Psi = \frac{\eta_{eff,0}^{UM}}{\eta_{eff,0}^{S}} \qquad \Xi = \frac{\eta_{d,0}^{S}}{\eta_{d,0}^{S}} \tag{44}$$