CS 3430: S26: Scientific Computing Addendum on Chudnovsky

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A Note on Reading the Slides on Chudnovsky

The Chudnovsky method has a lot of fascinating mathematics (number theory and calculus, mostly) behind it. Since we are focusing on using it to approximate π, this math is relevant, but tangential to our purpose.

In the following slides, I put

$$\boxed{\text{boxes around the major steps/formulas}}$$

that help you understand the pseudocode later on. I kept some intermediate derivation steps in this deck for those of you who are mathematically inclined. You can skip them on first reading and come back to them later if you want.

Some references at the end of this slide deck give you more rigorous background on the foundations of this beautiful algorithm.

Chudnovsky Method

The Chudnovsky method, named after the two brothers who invented this algorithm in the late 1980's, is a state of the art algorithm that has broken many world records. It is based on the following formula

$$\boxed{\frac{1}{\pi} = 12\sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(545,140,134k + 13,591,409)}{(3k)!(k!)^3 (640,320)^{3k+3/2}}.}$$

A proof of the formula can be found in reference [5] at the end this slide deck.

Simplification of the Chudnovsky Formula

While the formula stated on the previous slide is the backbone of the Chudnovsky algorithm, it is not used in it as is. So, let us work with it to get it in the form used in the algorithm.

First, we notice the following expression in the denominator.

$$640,320^{3k+3/2}$$

Since

$$640,320^{3k+3/2} = 640,320^{3k} 640,320^{3/2},$$

$640,320^{3/2}$ can be factored out, because it is independent of k. Also, observe that

$$\begin{array}{lcll} 640,320^{3/2} & = & (640,320^{1/2})^3 & = \\ (\sqrt{640,320})^2\sqrt{640,320} & = & 640,320\sqrt{64 \cdot 10,005} & = \\ 640,320 \cdot 8\sqrt{10,005}. & & & \end{array}$$

Simplification of the Chudnovsky Formula

Here is the original formula after the refactoring of the constant.

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (545,140,134k + 13,591,409)}{(3k)!(k!)^3 (640,320)^{3k+3/2}} =$$

$$\frac{12}{640,320 \cdot 8\sqrt{10,005}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (545,140,134k + 13,591,409)}{(3k)!(k!)^3 (640,320)^{3k}} =$$

$$\frac{1}{426,880\sqrt{10,005}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (545,140,134k + 13,591,409)}{(3k)!(k!)^3 (640,320)^{3k}},$$

which looks slightly more manageable but still a bit complicated. So we will work on it some more.

Simplification of the Chudnovsky Formula

We let

$$\boxed{f(k) = \frac{(-1)^k (6k)!}{(3k)!(k!)^3 (640,320)^{3k}}}$$

and rewrite the last formula on the previous slide in terms of $f(k)$ to get

$$\frac{1}{\pi} = \frac{1}{426,880\sqrt{10,005}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(545,140,134k + 13,591,409)}{(3k)!(k!)^3 (640,320)^{3k}} =$$

$$\frac{1}{426,880\sqrt{10,005}} \sum_{k=0}^{\infty} f(k)(545,140,134k + 13,591,409).$$

Now we go to work on $f(k)$ to get a closed form for it.

Simplification of the Chudnovsky Formula

Let us take a look at $\frac{f(k)}{f(k-1)}$, $k > 0$.

$$
\begin{aligned}
\frac{f(k)}{f(k-1)} &= \\
\frac{(-1)^k (6k)!}{(3k)!(k!)^3(640{,}320)^{3k}} \frac{(3(k-1))!((k-1)!)^3 640{,}320^{3(k-1)}}{(-1)^{k-1}(6(k-1))!} &= \\
\frac{(-1)^k}{(-1)^{k-1}} \frac{(6k)!}{(6(k-1))!} \frac{(3(k-1))!}{(3k)!} \frac{((k-1)!)^3}{(k!)^3} \frac{(640{,}820)^{3k-3}}{(640{,}820)^{3k}}.
\end{aligned}
$$

Simplification of the Chudnovsky Formula

We work on the components of the last formula on the previous slide fraction by fraction.

$$\frac{(-1)^k}{(-1)^{k-1}} = -1;$$

$$\frac{(6k)!}{(6(k-1))!} = \frac{6k(6k-1)(6k-2)(6k-3)(6k-4)(6k-5)(6k-6)!}{(6k-6)!} =$$

$$6k(6k-1)(6k-2)(6k-3)(6k-4)(6k-5);$$

$$\frac{(3(k-1))!}{(3k)!} = \frac{(3k-3)!}{(3k)!} = \frac{1}{3k(3k-1)(3k-2)};$$

$$\frac{((k-1)!)^3}{(k!)^3} = \frac{1}{k^3};$$

$$\frac{(640{,}820)^{3k-3}}{(640{,}820)^{3k}} = \frac{1}{640{,}320^3}.$$

Simplification of the Chudnovsky Formula

So, here's what we get

$$\frac{f(k)}{f(k-1)} = \frac{(-1)^k}{(-1)^{k-1}} \frac{(6k)!}{(6(k-1))!} \frac{(3(k-1))!}{(3k)!} \frac{((k-1)!)^3}{(k!)^3} \frac{(640{,}820)^{3k-3}}{(640{,}820)^{3k}} =$$

$$\frac{(-1)6k(6k-1)(6k-2)(6k-3)(6k-4)(6k-5)}{3k(3k-1)(3k-2)(640{,}320)^3 k^3} =$$

$$\frac{-(6k-1)(2k-1)(6k-5)}{1{,}093{,}905{,}886{,}003{,}200k^3} = \frac{-(6k-1)(2k-1)(6k-5)}{Ck^3},$$

where $C = 10,939,058,860,032,000$. Quite a number! We know that $f(0) = 1$ (this can be directly computed). Then

$$f(k) = f(k-1)\frac{-(6k-1)(2k-1)(6k-5)}{Ck^3}$$

Simplification of the Chudnovsky Formula

By iterating the last equation on the previous slide for $f(k-2)$, $f(k-3)$, etc., we get

$$\boxed{f(k) \quad = \quad \prod_{j=1}^{k} \frac{-(6j-1)(2j-1)(6j-5)}{Cj^3}.}$$

This is way better! Now we plug our findings into the equation

$$\frac{1}{\pi} = \frac{1}{426,880\sqrt{10,005}} \sum_{k=0}^{\infty} f(k)(545,140,134k + 13,591,409)$$

to get

$$\frac{1}{\pi} = \frac{1}{426,880\sqrt{10,005}} \Big[f(0)(545,140,134 \cdot 0 + 13,591,409) +$$

$$\sum_{k=1}^{\infty} \Big(\prod_{j=1}^{k} \frac{-(6j-1)(2j-1)(6j-5)}{Cj^3} \Big)(545,140,134k + 13,591,409) \Big].$$

Simplification of the Chudnovsky Formula

The last equation on the previous slide simplifies to

$$\frac{1}{\pi} = \frac{1}{426,880\sqrt{10,005}}\Bigg[13,591,409+$$

$$\sum_{k=1}^{\infty}\left(\prod_{j=1}^{k}\frac{-(6j-1)(2j-1)(6j-5)}{Cj^3}\right)(545,140,134k+13,591,409)\Bigg],$$

which is much closer to the form of the equation that the Chudnovsky algorithm works with.

The Formula

The formula

$$\frac{1}{\pi} = \frac{1}{426,880\sqrt{10,005}}\Bigg[13,591,409+$$

$$\sum_{k=1}^{\infty}\left(\prod_{j=1}^{k}\frac{-(6j-1)(2j-1)(6j-5)}{Cj^3}\right)(545,140,134k+13,591,409)\Bigg].$$

is still an eyeful, for sure. So, let's start simplifying it by introducing the following 4 constants:

$$C_1 = 426,880\sqrt{10,005};$$

$$C_2 = 13,591,409;$$

$$C_3 = C = 10,939,058,860,032,000;$$

$$C_4 = 545,140,134.$$

Introduction of 4 Constants into the Formula

Now I can fit it on a single line with my LaTeX:

$$\frac{1}{\pi} = \frac{1}{C_1}\left[C_2 + \sum_{k=1}^{\infty}\left(\prod_{j=1}^{k}\frac{-(6j-1)(2j-1)(6j-5)}{C_3 j^3}\right)(C_4 k + C_2)\right],$$

where

1. $C_1 = 426,880\sqrt{10,005}$;
2. $C_2 = 13,591,409$;
3. $C_3 = 10,939,058,860,032,000$;
4. $C_4 = 545,140,134$.

This is a smaller eyeful, but an eyeful, nonetheless. So, let's simplify it some more.

Abstraction of the Product in the Formula

The product component of the Chudnovsky formula on the previous slide, inside the sum $\sum_{k=1}^{\infty}$, is as follows:

$$\prod_{j=1}^{k} \frac{-(6j-1)(2j-1)(6j-5)}{C_3 j^3} = \frac{\prod_{j=1}^{k} -(6j-1)(2j-1)(6j-5)}{\prod_{j=1}^{k} C_3 j^3}.$$

We can abstract the numerator and denominator of this ratio into two separate products:

$$P(a,b) = \prod_{j=a}^{b-1} -(6j-1)(2j-1)(6j-5);$$

$$Q(a,b) = \prod_{j=a}^{b-1} C_3 j^3.$$

Abstraction of the Product in the Formula

If we do one more functional abstraction by letting $f(j) = -(6j-1)(2j-1)(6j-5)$ (**NB:** we are reusing the same function symbol f; in other words, the $f(j)$ on this slide is not the same as $f(k)$ on the previous slides; $f(k)$ has served its purpose) and $g(j) = C_3 j^3$, we can write $P(a,b)$ and $Q(a,b)$ as

$$P(a,b) = \prod_{j=a}^{b-1} f(j);$$

$$Q(a,b) = \prod_{j=a}^{b-1} g(j).$$

Discovering Patterns in $P(a, b)$ and $Q(a, b)$

Let us instantiate $P(a, b)$ with some values to establish a pattern that will help us define the base case for the recursive version of the Chudnovsky algorithm later in the lecture.

$$P(1,2) = \prod_{j=1}^{2-1} f(j) = f(1);$$

$$P(2,3) = \prod_{j=2}^{3-1} f(j) = f(2);$$

$$P(1,3) = \prod_{j=1}^{3-1} f(j) = f(1)f(2) = P(1,2)P(2,3).$$

Note that $P(1,3) = P(1,2)P(2,3)$ and $1 < 2 < 3$. Why did I state the obvious $1 < 2 < 3$? To emphasize the fact that to compute $P(1,3)$ on the interval $[1,3]$, we can compute $P(1,2)$ on the subinterval $[1,2]$, and $P(2,3)$ on the subinterval $[2,3]$. Effectively, we split the interval $[1,3]$ into $[1,2]$ and $[2,3]$.

Discovering Patterns in $P(a, b)$ and $Q(a, b)$

One more instantiation:

$$P(1,4) = \prod_{j=1}^{4-1} f(j) = \prod_{j=1}^{3} f(j) = f(1)f(2)f(3) =$$

$$P(1,2)P(2,4) = \Big(f(1)\Big)\Big(f(2)f(3)\Big) =$$

$$P(1,3)P(3,4) = \Big(f(1)f(2)\Big)\Big(f(3)\Big).$$

So, $P(1,4) = P(1,2)P(2,4) = P(1,3)P(3,4)$, $1 < 2, 3 < 4$. In other words, to compute $P(1,4)$, we can compute $P(1,2)P(2,4)$ or $P(1,3)P(3,4)$. I will introduce a little technical jargon here and state that we can compute $P(1,4)$ by *splitting* the interval $[1,4]$ on m such that $1 < m < 4$. Said another way, we can split $[1,4]$ into $[1,2]$ and $[2,4]$ or into $[1,3]$ and $[3,4]$ and obtain the same numerical result in either case.

Discovering Patterns in $P(a, b)$ and $Q(a, b)$

We can generalize the splitting computation on the previous slide as follows. Let m be an integer such that $a < m < b$. Then

$$P(a,b) = \prod_{j=a}^{b-1} f(j) =$$

$$\Bigg(f(1)\cdots f(m-1)\Bigg)\Bigg(f(m)\cdots f(b-1)\Bigg) =$$

$$P(a,m)P(m,b).$$

Or, more succinctly,

$$\boxed{P(a,b) = P(a,m)P(m,b), a < m < b.}$$

Discovering Patterns in $P(a, b)$ and $Q(a, b)$

The same generalization applies to $Q(a, b)$, i.e., if $a < m < b$. Then

$$Q(a, b) = \prod_{j=a}^{b-1} g(j) =$$

$$\left(g(1) \cdots g(m-1)\right)\left(g(m) \cdots g(b-1)\right) =$$

$$Q(a, m)Q(m, b).$$

Or, more succinctly,

$$\boxed{Q(a, b) = Q(a, m)Q(m, b), a < m < b.}$$

A Rewrite of the Formula

We go back to the previous version of the Chudnovsky Formula

$$\frac{1}{\pi} = \frac{1}{C_1}\left[C_2 + \sum_{k=1}^{\infty}\left(\prod_{j=1}^{k}\frac{-(6j-1)(2j-1)(6j-5)}{C_3 j^3}\right)(C_4 k + C_2)\right]$$

and rewrite it as

$$\boxed{\frac{1}{\pi} = \frac{1}{C_1}\left[C_2 + \sum_{k=1}^{\infty}\frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2)\right],}$$

where C_1, C_2, C_3, and C_4 are the previously defined constants.

Abstraction of the Infinite Sum in the Formula

Now we take

$$\frac{1}{\pi} = \frac{1}{C_1}\left[C_2 + \sum_{k=1}^{\infty} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2)\right]$$

and abstract the inifinite sum as

$$S(1,\infty) = \sum_{k=1}^{\infty} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2).$$

With this abstraction in hand, $1/\pi$ rewrites as

$$\frac{1}{\pi} = \frac{C_2 + S(1,\infty)}{C_1},$$

and we have

$$\boxed{\pi = \frac{C_1}{C_2 + S(1,\infty)}.}$$

Beautiful, but Infinite!

Let's take a moment and reflect on our journey so far. We transformed

$$\frac{1}{\pi} = \frac{1}{426,880\sqrt{10,005}} \Bigg[13,591,409 +$$

$$\sum_{k=1}^{\infty} \left(\prod_{j=1}^{k} \frac{-(6j-1)(2j-1)(6j-5)}{1,093,905,886,003,2000j^3} \right) (545,140,134k + 13,591,409) \Bigg]$$

into

$$\pi = \frac{C_1}{C_2 + S(1,\infty)},$$

which is much simpler!

Beautiful, But Infinite!

However,

$$\pi = \frac{C_1}{C_2 + S(1, \infty)}$$

is not computable, because of ∞. So, we can only approximate π by taking the following two steps:

$$S(a, b) = \sum_{k=a}^{b-1} \frac{P(a, k+1)}{Q(a, k+1)}(C_4 k + C_2)$$

and

$$\boxed{\pi \approx \frac{C_1}{C_2 + S(1, n)}, n > 1.}$$

Discovering a Pattern in $S(a, b)$

Let's play with $S(a, b)$ to see if we can discover a splitting pattern in it as we did with $P(a, b)$ and $Q(a, b)$. So, we instantiate simple values and compute.

$$S(1,2) = \sum_{k=1}^{2-1} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2) =$$

$$\sum_{k=1}^{1} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2) = \frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2].$$

$$S(2,3) = \sum_{k=2}^{3-1} \frac{P(2,k+1)}{Q(2,k+1)}(C_4 k + C_2) =$$

$$\sum_{k=2}^{2} \frac{P(2,k+1)}{Q(2,k+1)}(C_4 k + C_2) = \frac{P(2,3)}{Q(2,3)}[C_4 \cdot 2 + C_2].$$

Discovering a Pattern in $S(a, b)$

$$S(1,3) = \sum_{k=1}^{3-1} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2) =$$

$$\sum_{k=1}^{2} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2) =$$

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2].$$

Discovering a Pattern in $S(a, b)$

OK, we have

$$S(1,3) = \frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2].$$

If we recall that $P(a, b) = P(a, m)P(m, b)$ and $Q(a, b) = Q(a, m)Q(m, b)$, $a < m < b$. Then we have

$$S(1,3) = \frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2] =$$

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,2)P(2,3)}{Q(1,2)Q(2,3)}[C_4 \cdot 2 + C_2] =$$

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,2)}{Q(1,2)}\left(\frac{P(2,3)}{Q(2,3)}[C_4 \cdot 2 + C_2]\right),$$

because $P(1,3) = P(1,2)P(2,3)$ and $Q(1,3) = Q(1,2)Q(2,3)$.

Discovering a Pattern in $S(a, b)$

To recap what we computed on the previous slide:

$$S(1,3) = \frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,2)}{Q(1,2)} \frac{P(2,3)}{Q(2,3)}[C_4 \cdot 2 + C_2].$$

We computed a few slides ago that

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] = S(1,2)$$

and

$$\frac{P(2,3)}{Q(2,3)}[C_4 \cdot 2 + C_2] = S(2,3).$$

Thus,

$$S(1,3) = S(1,2) + \frac{P(1,2)}{Q(1,2)} S(2,3), 1 < 2 < 3.$$

Discovering a Pattern in $S(a, b)$

Let us compute $S(1, 5)$.

$$S(1,5) = \sum_{k=1}^{5-1} \frac{P(1,k+1)}{Q(1,k+1)}(C_4 k + C_2) =$$

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2] + \frac{P(1,4)}{Q(1,4)}[C_4 \cdot 3 + C_2] + \frac{P(1,5)}{Q(1,5)}[C_4 \cdot 4 + C_2] =$$

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2] +$$
$$\left(\frac{P(1,4)}{Q(1,4)}[C_4 \cdot 3 + C_2] + \frac{P(1,5)}{Q(1,5)}[C_4 \cdot 4 + C_2] \right) =$$

Discovering a Pattern in $S(a, b)$

$$= \frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2] +$$

$$\left(\frac{P(1,3)P(3,4)}{Q(1,3)Q(3,4)}[C_4 \cdot 3 + C_2] + \frac{P(1,3)P(3,5)}{Q(1,3)Q(3,5)}[C_4 \cdot 4 + C_2] \right) =$$

$$\frac{P(1,2)}{Q(1,2)}[C_4 \cdot 1 + C_2] + \frac{P(1,3)}{Q(1,3)}[C_4 \cdot 2 + C_2] +$$

$$\frac{P(1,3)}{Q(1,3)} \left(\frac{P(3,4)}{Q(3,4)}[C_4 \cdot 3 + C_2] + \frac{P(3,5)}{Q(3,5)}[C_4 \cdot 4 + C_2] \right) =$$

$$S(1,3) + \frac{P(1,3)}{Q(1,3)} S(3,5).$$

Generalization of the Discovered Pattern in $S(a, b)$

So, if we compute $S(1,5)$ and split the interval $[1,5]$ on $m = 3$, we get

$$S(1,5) = S(1,3) + \frac{P(1,3)}{Q(1,3)} S(3,5).$$

If we compute $S(1,5)$ and split the interval $[1,5]$ on $m = 2$, we get

$$S(1,5) = S(1,2) + \frac{P(1,2)}{Q(1,2)} S(2,5).$$

And, in general, if we split the interval $[a, b]$, $a < b$, on m such that $a < m < b$, then

$$\boxed{S(a,b) = S(a,m) + \frac{P(a,m)}{Q(a,m)} S(m,b).}$$

Note that if the interval is $[a, a+1]$, there is no need to split, because we can compute it directly with the $S(a, b)$ formula (cf.,e.g., the previous slides where we computed $S(1,2)$ and $S(2,3)$; the sum in the case of a unit inverval of only 1 element). So the unit interval $[a, a+1]$ is the *base case*.

Auxiliary Functions for the Formula

So, let us restate all the auxiliary equations we have derived for the interval $[a, b]$ and $a < m < b$.

$$P(a, b) = P(a, m)P(m, b);$$

$$Q(a, b) = Q(a, m)Q(m, b);$$

$$S(a, b) = S(a, m) + \frac{P(a, m)}{Q(a, m)} S(m, b).$$

Let us add one more equation, which we can derive in the same fashion as we did $P(a, b)$, $Q(a, b)$ and $S(a, b)$

$$\boxed{R(a, b) = Q(a, b)S(a, b) = Q(m, b)R(a, m) + P(a, m)R(m, b).}$$

Auxiliary Functions for the Formula

From the definition of $R(a, b)$ we have

$$S(a, b) = \frac{R(a, b)}{Q(a, b)}.$$

We use the above equation to write

$$\pi \approx \frac{C_1}{C_2 + S(1, n)}, n > 1,$$

as

$$\boxed{\pi \approx \frac{C_1}{C_2 + \frac{R(1,n)}{Q(1,n)}} = \frac{C_1 Q(1, n)}{C_2 Q(1, n) + R(1, n)}, n > 1,}$$

which is the version of the approximation formula used in the Chudnovsky algorithm, which we can now define.

Binary Splitting: Base Case

The workhorse of the Chudnovsky algorithm is *binary splitting*. Why *binary*? Because the interval $[a, b]$ in all the auxiliary functions, i.e., $P(a, b)$, $Q(a, b)$, and $R(a, b)$, is always split into 2 subintervals in the middle m, i.e., $[a, m]$ and $[m, b]$, where m is the integer value of the floor of $(a + b)/2$.

In the base case, we have the interval $[a, a + 1]$, for which we set the values of the following three variables.

1. $Pab = -(6a - 1)(2a - 1)(6a - 5)$;
2. $Qab = C_3 a^3$;
3. $Rab = Pab(C_4 a + C_2)$.

No need for Sab, by the way, because it is already integrated into Rab.

Binary Splitting: Recursive Case

So, we split the interval $[a, b]$ into to two subintervals in the middle m, and recurse on the subintervals $[a, m]$ and $[m, b]$ until we hit the unit interval, i.e., the base case when the recursion stops, because it can be computed directly. Here it is, in pseudocode.

```
C_1 = 426880 * SQUARE_ROOT(10005)
C_2 = 13591409
C_3 = 10939058860032000
C_4 = 545140134
BINARY_SPLIT(a, b)
   IF b == a + 1 ## this is the base case: unit interval no need to split.
      THEN
         Pab = -(6*a-1)*(2*a-1)(6*a-5)
         Qab = C_3 * (a**3)
         Rab = Pab * (C_4 * a + C_2)
      END THEN
   ELSE ## this is the recursive case
      m = (a + b) // 2 ## we split in the middle and
      Pam, Qam, Ram = BINARY_SPLIT(a, m) ## 1) recurse on [a,m]
      Pmb, Qmb, Rmb = BINARY_SPLIT(m, b) ## 2) recurse on [m,b]
      ## we assemble the results by multiplying P's and Q,s
      ## and computing the R.
      Pab = Pam * Pmb
      Qab = Qam * Qmb
      Rab = Qmb * Ram + Pam * Rmb
   END ELSE
   RETURN Pab, Qab, Rab
```

The Chudnovsky Algorithm

And here (finally!) is the Chudnovsky algorithm to approximate π.

```
CHUDNOVSKY_PI(1, n)
    ASSERT n > 1
    P1n, Q1n, R1n = BINARY_SPLIT(1, n)
    RETURN (C_1 * Q1n) / (C_2 * Q1n + R1n)
```

References

1. Wikipedia, en.wikipedia.org/wiki/Pi.
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