Subspaces and Dimension

Lecture 3

Proposition 1. $\text{Span}(S)$ is a subspace of V

Lecture 4

Lemma 1. $v \in \text{Span}(S) \iff \text{Span}(S) = \text{Span}(S \cup \{v\})$

Theorem 1. The following are equivalent (and the definition of a basis)

1. S is a minimal spanning set
2. S is maximally linearly independent
3. Every $v \in V$ can be written as a unique linear combination of elements in S

Lecture 5

Lemma 2. (Steinitz) Let $A = \{v_1, \dots, v_m\}$, $B = \{w_1, \dots, w_n\}$ be linearly independent sets in V with $m \geq n$. For every $0 \leq j \leq n$, we can renumber elements of B such that the following set is linearly independent: $\{v_1, v_2, \dots, v_j, w_{j+1}, \dots, w_m\}$

Theorem 2. Assume $S = \{s_1, \dots, s_n\}$ is a basis for V, then every other basis V has n elements.

Remark 1.

1. Dimension depends on V, not the choice of basis
2. Suppose V is a vector space with dimension n. If T is a linearly independent set in V, then $\dim(T) \leq n$
3. Any linearly independent set can be completed to a basis
4. $W \subseteq V$ is a subspace, then $\dim(W) \leq \dim(V)$ with equality iff $W = V$

Lecture 6

$B = \{b_1, \dots, b_n\}$ and $C = \{c_1, \dots, c_n\}$ are bases of V. Define the following:

$$b_k = \sum_{i=1}^{n} m_{ik} c_i, \quad {}_C M_B = \begin{bmatrix} m_{11} & m_{12} & \cdots & m_{1n} \\ m_{21} & m_{22} & \cdots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{n1} & m_{n2} & \cdots & m_{nn} \end{bmatrix}$$

Theorem 3. For any $v \in V$, $[v]_C = {}_C M_B [v]_B$

Lecture 7

Remark 2. Let f be a linear map, $f : V \to W$

1. $f(0_V) = 0_w$
2. $\mathrm{Ker}(f) = \{v \in V \mid f(v) = 0\}$ is a subspace of V
3. $\mathrm{Im}(f) = \{f(v) \mid v \in V\}$ is a subspace of W

Theorem 4. Let V and W be vector spaces over $\mathbb{F}$. Let $B = \{b_1, \dots, b_n\}$ be a basis for V and let $\{t_1, \dots, t_n\}$ be any elements of W. Then there exists a unique linear map such that $f(b_i) = t_i$ for all $i = 1, \dots n$ (Note: this does not require W to be finite dimensional)

Theorem 5. Let V be a finite dimensional vector space, then $V \cong \mathbb{F}^n$

Corollary 1. If $T : V \to W$ is an isomorphism and $B = \{b_1, \dots, b_n\}$ be a basis of V. Then $\{T(b_1), \dots, T(b_n)\}$ is a basis for W (in particular, $\dim(V) = \dim(W)$.

Theorem 6.
Lecture 8

Let $T : V \to W$ and V finite dimensional. If $\mathrm{Im}(T)$ is finite dimensional. Then

$$\dim(V) = \dim(\mathrm{Ker}(T)) + \dim(\mathrm{Im}(T))$$

Theorem 7. (First Iso) Let $T : V \to W$ be a linear surjective map. then $V/\mathrm{Ker}(T) \cong W$

Theorem 8. $W_1, W_2 \subseteq V$ are subspaces. Then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

Proposition 2. Let $f : V \to W$ be a linear map and $\dim(V) = \dim(W)$. Then f is an isomorphism if f is surjective or injective

Lecture 9

Theorem 9. (Fitting's Lemma) Let V be finite dimensional and $T : V \to V$ be a linear operator. Then there exists a decomposition $V = U \oplus W$ such that

1. U and W are T-invariant subspaces of V
2. $T|_U$ is nilpotent
3. $T|_W$ is an isomorphism

Theorem 10.

1. Let $U, W \subseteq V$ be subspaces such that $V = U \oplus W$. Then we can define the following map as a projection:
$$P : V \to V, \quad u + w \mapsto u \quad [(u, w) \mapsto (u, 0)]$$
2. If $P : V \to V$ is any projection operator and $U = \mathrm{Im}(P)$ and $W = \mathrm{Ker}(P)$, then $V = U \oplus W$ and $P(v) := P(u + w) = u$

Linear Maps and Matrices

Lecture 10

Theorem 11. Let $T : V \to W$ be a linear map with B being a basis for V (of size n) and C being a basis of W of size m.

1. There exists a unique $m \times n$ matrix ${}_C[T]_B$ such that: $[Tv]_C = {}_C[T]_B \cdot [v]_B \quad$ for all $v \in V$
2. If $S, T \in \mathrm{Hom}(V, W)$, then ${}_C[S+T]_B = {}_C[S]_B + {}_C[T]_B$ and ${}_C[\lambda T]_B = \lambda {}_C[T]_B$
3. For every $M \in M_{m\times n}(\mathbb{F})$, there is a linear map $T : V \to W$ such that ${}_C[T]_B = M$. This implies that
$$\mathrm{Hom}(V, W) \to M_{m\times n}(\mathbb{F}), \quad T \mapsto {}_C[T]_B$$
is an isomorphism
4. If $R : W \to U$, with D being a basis for U, then ${}_D[R \circ T]_B = {}_D[R]_C \cdot {}_C[T]_B$

Determinants

Lecture 11

Theorem 12. Let $n \geq 2$. There exists a unique group homomorphism, "sgn" such that

$$\mathrm{sgn} : S_n \to \{\pm 1\}$$

(with respect to multiplication) such that $\mathrm{sgn}((kl)) = -1$

Lecture 12

Theorem 13. Let $\mathbb{F}$ be a field. There exist a unique map, $\det : M_n(\mathbb{F}) \to \mathbb{F}$ such that

1. $\det(I_n) = 1_{\mathbb{F}}$
2. det is a multilinear map i.e.
$$\det(v_1, \ldots, v_i + \lambda v_i', \ldots, v_n) = \det(v_1, \ldots, v_i, \ldots, v_n) + \lambda \det(v_1, \ldots, v_i', \ldots, v_n)$$
3. $\det(A) = 0$ if A has an identical pair of columns.

Corollary 2. For all $\tau \in S_n$,

$$\det(v_1, \ldots, v_i, \ldots, v_n) = \det(v_{\tau(1)}, \ldots, v_{\tau(i)}, \ldots, v_{\tau(n)}) = \mathrm{sgn}(\tau) \det(v_1, \ldots, v_i, \ldots, v_n)$$

Lecture 13

Remark 3. There is an injective group homomorphism $S_n \to GL_2(\mathbb{F})$ defined by sending $\sigma \mapsto T_\sigma$, where T_σ is defined as: $T_\sigma(e_i) = e_{\sigma(i)}$. (Exercise: check this is a group homomorphism)

Theorem 14. For any $A, B \in M_n(\mathbb{F})$, $\det(AB) = \det(A)\det(B)$

Lecture 14

Theorem 15. $A^{ij} = (-1)^{i+j} A_{ij}$

Corollary 3.

$$\begin{aligned} det(A) &= a_{1j}A^{1j} + \cdots + a_{nj}A^{nj}, forallj \in 1, \ldots, n \\ &= a_{i1}A^{i1} + \cdots + a_{in}A^{in}, foralli \in 1, \ldots, n \end{aligned}$$

Theorem 16. Define $\mathrm{Adj}(A) = (c_{ij})$, where $c_{ij} = A^{ji}$, then

$$\mathrm{Adj}(A)A = A\mathrm{Adj}(A) = \det(A)I_n$$

Systems of Linear Equations

Lecture 15

Corollary 4. Any homogeneous system on m equations with n unknowns acn be reduced to a system of m' equations with n unknowns, with $m' \leq n$.

Lecture 16

Remark 4.

- Every matrix is equivalent by row reduction (*row equivalent*) to a unique matrix in reduced echelon form
- Two matrices have the same row space $\iff$ they are row equivalent

Theorem 17. Let $A \in M_{m \times n}(\mathbb{F})$, then $rk_R(A) = rk_C(A)$ (dimension of span of columns is equal to the dimension of span of rows of a matrix)

Corollary 5. Dimension of space of solutions: $n - rk_R(A)$

Theorem 18. (Cramer's Rule) Let $A \in M_n(\mathbb{F})$ with non-zero determinant. Then there exists a unique solution $x_1, x_2, \ldots, x_n$ to a system $Ax = b$ given by

$$x_i = \frac{\det(A_i)}{\det(A)} =: \frac{\det([v_1|v_2|\ldots|v_{i-1}|b|v_{i+1}|\ldots|v_n])}{\det(A)}$$

Lecture 17

Corollary 6. Let $A \in M_n(\mathbb{F})$ and EA is in reduced echelon form. Then A is invertible iff $EA = I_n$

Dual Spaces

Lecture 18

Remark 5. By Corollary 3.8.2 in Goren's notes,

$$\dim(\mathrm{Hom}(V, W)) = \dim(V)\dim(W).$$

This implies that $V^\star := \mathrm{Hom}(V, \mathbb{F})$ (space of linear maps from V to $\mathbb{F}$) has dimension equal to the dimension of V

Proposition 3. Let V be a finite dimensional space with a basis $B = \{b_1, \ldots, b_n\}$. Then there exists a unique basis $B^\star = \{f_1, \ldots, f_n\}$ of $V^\star$ such that

$$f_i(b_j) = \delta_{ij}$$

Proposition 4. There exists a natural isomorphism such that $V \cong V^\star$

Lecture 19

Proposition 5. Let $U^\perp$ be the annihilator of $U \subseteq V$ (a finite dimension space), then:

- $U^\perp$ is a subspace
- $U \subseteq W \subseteq V \implies W^\perp \subseteq U^\perp$
- $\dim(U^\perp) = \dim(V) - \dim(U)$
- $(U^\perp)^\perp = U$
- $(U_1 + U_2)^\perp = U_1^\perp \cap U_2^\perp$
- $(U_1 \cap U_2)^\perp = U_1^\perp + U_2^\perp$

Proposition 6. Let $U \subset V$ (finite dimensional). There exists a natural isomorphism such that $U^\star \cong V^\star/U^\perp$ (where $U^\perp \cong (V/U)^\star$)

Lecture 20

Proposition 7. Let T be a linear map between finite dimension spaces V and W and let $T^\star$ denote the "dual map". Then

1. $T^{\star\star} = T$
2. If B is a basis for V and C is a basis for W, then (lowercase t denotes transpose):

$${}_{B^\star}[T^\star]_{C^\star} = {}_C[T]_B^t$$

3. $\mathrm{Im}(T^\star) = (\ker(T))^\perp$
4. $\ker(T^\star) = (\mathrm{Im}(T))^\perp$

Inner Product Spaces

Theorem 19. (Cauchy-Schwarz) Let V be an inner product space. For all $u, v \in V$,

$$|\langle u, v\rangle| \leq ||u|| \cdot ||v||$$

Lecture 21

Proposition 8. $v \mapsto ||v||$ is a function such that

- $||v|| \geq 0$ with equality if and only if $v = 0$
- $||\alpha v|| = |\alpha|\ ||v||$
- $||u + v|| \leq ||u|| + ||v||$

Corollary 7. $d(u, v)$ is a function such that

- $d(u, v) \geq 0$ with equality if and only if $v = u$
- $d(u, v) = d(v, u)$
- $d(u, v) \leq d(u, w) + d(w, v)$

Theorem 20. (Gram-Schmidt) Let $\{s_1, \ldots, s_n\}$ be a basis for V. Then there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ such that

$$\text{span}(\{s_1, \ldots, s_n\}) = \text{span}(\{v_1, \ldots, v_n\})$$

Lecture 22

Proposition 9. Let $U \subseteq V$ be finite dimensional vector spaces and let $U^\perp$ denote the orthogonal complement to U. Then:

$$V = U \oplus U^\perp, \quad (U^\perp)^\perp = U$$

Theorem 21. If $\{v_1, \ldots, v_r\}$ be an orthonormal basis for U, then

1. $P(v) = \sum_{i=1}^{r} \langle v, v_i\rangle v_i$
2. $v - P(v) \in U^\perp$
3. $P(v) \in U$ is the closest vector in U to $v \in V$

where $P : V \to U$ denotes the orthogonal projection onto U

Eigenvalues

Lecture 23

Theorem 22. If $A \sim B$, then $\Delta_A(t) = \Delta_B(t)$

Theorem 23. $\Delta_A(\lambda) = 0 \iff \lambda$ is an eigenvalue for T (where A is a matrix representing $T : V \to V$; they have the same characterstic polynomial).

Proposition 10. If $A \in M_n(\mathbb{F})$, then $\Delta_A(t) = t^n - \text{Tr}(A)t^{n-1} + \cdots + (-1)^n \det(A)$

Lecture 24

Remark 6. $\Delta_A(t)$ is independent of basis

Lemma 3. $1 \leq m_g \leq m_a \leq n$ for a given eigenvalue

Theorem 24. *A is diagonalizable iff there exists a basis B such that ${}_B[T]_B$ is diagonal (with just eigenvalues) iff V has a basis of eigenvectors [do!]*

Theorem 25. $\dim(V) < \infty$ with $T : V \to V$. Then T is diagonalizable if and only if $m_g(\lambda) = m_a(\lambda)$ for all λ of T.

Lecture 25

Lemma 4. $A \in M_n(\mathbb{F})$ with $f \in \mathbb{F}[t]$ monic. Then

$$(tI_n - A)(B_a t^a + B_{a-1}t^{a-1} + \cdots + B_0) = f(t)I_n \iff f(A) = 0$$

for some $B_a, B_{a-1}, \ldots, B_0 \in M_n(\mathbb{F})$.

Theorem 26. (Cayley Hamilton) $A \in M_n(\mathbb{F})$, then $\Delta_A(A) = 0$.

Proposition 11. $m_A(t)$ is a monic polynomial of minimal degree such that $m_A(A) = 0$. Then

1. If $f(A) = 0$ for some $f \in \mathbb{F}[t]$, then $m_A(t)|f(t)$
2. $m_A(t) \mid \Delta_A(t) \mid (m_a(t))^n$

Lecture 26

Lemma 5. Suppose $f \in \mathbb{F}[t]$ satisfies $f(T) = 0$ (i.e. $m_T(t)|f(t)$) and factors $f(t) = g(t)h(t)$ with $\gcd(f, h) = 1$. Then

1. $\ker(g(T))$ and $\ker(h(T))$ are T-invariant,
2. $V = \ker(g(T)) \oplus \ker(h(T))$ with $\ker(g(T)) = \mathrm{Im}(h(T))$ and $\ker(h(T)) = \mathrm{Im}(g(T))$.
3. Assume $f(t) = m_T(t)$ then we also get

$$g(t) = \text{min poly } T\big|_{\ker(g(T))}, \quad h(t) = \text{min poly } T\big|_{\ker(h(T))}$$

Theorem 27. Let $\dim(V) < \infty$ and $T : V \to V$ with

$$m_T(t) = f_1(t)^{n_1} \cdots f_r(t)^{n_r}$$

with f_i irreducible and distinct. Set $W_i = \ker(f_i(T)^{n_i})$ then

1. W_i is T-invariant i.e. $T(W_i) \subseteq W_i$
2. $V = W_1 \oplus \cdots \oplus W_r$ i.e. every $v \in V$ can be written uniquely as

$$v = w_1 + \cdots + w_r, \quad w_i \in W_i$$

3. $f_i(t)^{n_i}$ is the minimal polynomial of

$$T_i := T\big|_{W_i} : W_i \to W_i.$$

Lecture 27

Theorem 28. A matrix A is diagonalizable:

1. iff $\Delta_A(t)$ can be split into linear factors with $m_a(\lambda) = m_g(\lambda)$ for all λ,

2. iff $\Delta_A(t)$ plits into distinct linear factors

Remark 7. Method 1 and Method 2 (proof is exercise kms) for computing if a matrix is diagonalizable

Jordan Canonical Form

Lecture 28

Proposition 12. Let $U : V \to V$ be nilpotent (with V finite dimensional). Then there exists a basis B such that ${}_B[U]_B$ is block diagonal with standard nilpotent matrices.

Theorem 29. Let $T : V \to V$ such that

$$\Delta_T(t) = (t - \lambda_1)^{n_1} \cdots (t - \lambda_r)^{n_r}; \quad m_T(t) = (t - \lambda_1)^{m_1} \cdots (t - \lambda_r)^{m_r}.$$

Then there exists a basis B such that ${}_B[T]_B$ is block diagonal with Jordan blocks such that

1. The maximal block size for λ_i is m_i

2. Total size of blocks for λ_i is n_i

3. Number of blocks for λ_i is $m_g(\lambda_i)$

4. Number of blocks for λ_i of size b is given by

$$2\text{null}(U_i^b) - (\text{null}(U_i^{b-1}) + \text{null}(U_i^{b+1}))$$

Lecture 29

Examples

Applications (Chapter 10)

Lecture 30

Let $\dim(V) < \infty$ and $T : V \to V$ is a linear operator. Then there exists a unique operator (*adjoint operator*)

$$T^\star : V \to V, \quad \text{s.t} \quad \langle Tu, v\rangle = \langle u, T^\star v\rangle, \quad \forall u, v \in V.$$

Further, if B is an orthonormal basis, then $[T^\star]_B = [T]_B^\star = \overline{[T]_B^t}$

Lecture 31

Theorem 30. Let $\dim(V) < \infty$ and consider a self-adjoint $T : V \to V$. Then every eigenvalue is *real* and if $\lambda \neq \mu$ (distinct eigenvalues) then $E_\lambda \perp E_\mu$

Lecture 32

Let $\dim(V) < \infty$ and $T : V \to V$ be self-adjoint. Then there exists an orthonormal basis B such that $[T]_B$ is diagonal.

Corollary 8. If A is hermitian, there exists a unitary matrix U such that $U^\star AU$ is diagonal [over $\mathbb{R}$, symmetric matrices are always diagonal]

Lecture 33

Theorem 31. Let $T : V \to V$ be a normal operator. Then there exists an orthonormal basis B such that $[T]_B$ is diagonal.

Remark 8. A normal operator gives the following decomposition:

$$V = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_r}$$

with $E_{\lambda_i} \perp E_{\lambda_j}$ if $\lambda_i \neq \lambda_j$.

Lecture 34