MATH 249 - Honours Complex Variables

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Chapter 1

Functions of a Complex Variable

1.1 Preliminaries

Definition 1.1.1. A **function of a complex variable** is a function f with argument $z = x+iy \in \mathbb{C}$ such that it can be decomposed in the following way:

$$w = f(z) = u(x, y) + iv(x, y)$$

where w is simply the value of f evaluated at z and $u(x, y) := \mathcal{R}(f)$, the *real* part of f and $v(x, y) := \mathcal{I}(f)$, the *imaginary* part

1.1.1 Limits

Notions of limits and continuity follow similarly to functions taking inputs on the reals;

$$\lim_{z \to z_0} f(z) = w_0 \text{ if } \forall \epsilon > 0, \exists \delta \text{ s.t. } |f(z) - w_0| < \epsilon \text{ whenever } |z - z_0| < \delta$$

1.1.2 Derivatives

We define the notion of differentiation in the following way

$$f'(z_0) = \frac{d}{dz} f(z)|_{z_0} = \lim_{\Delta z \to 0} \left[\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \right]$$

where we don't care how $\Delta z \to 0$; it is *path independent.* As a result, we can say that

$$f'(z) = \frac{d}{dx} f = u_x + iv_x$$

which will be particularly important in the following theorem.

1.2 Cauchy Riemann Equations

The Cauchy Riemann (CR) Equations are equations that determine differentiation of a complex function. In particular, the CR equations are both necessary and sufficient for $f(z)$ to be differentiable at a point z_0. Let $f(z) = u(x, y) + iv(x, y)$, then the CR equations are the following:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

abbreviated $u_x = v_y$ and $u_y = -v_x$.

Theorem 1.2.1. $f(z) = u(x,y) + iv(x,y)$ has a derivative at z_0 iff the CR equations hold at z_0

Proof.
"$\Rightarrow$" Suppose $f(z) = u(x,y) + iv(x,y)$ has a derivative at z_0. Since the derivative is independent of path, we look at two paths in particular: $\Delta z = \Delta x$ (thus $\Delta y = 0$) and $\Delta z = \Delta y$ (thus $\Delta x = 0$). It might be helpful to draw these paths but just picture the limit coming horizontally and vertically, respectfully. In the first case:

$$\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = \lim_{\Delta x\to 0}\frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x} + i\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} = u_x + iv_x =: f_x.$$

In the second case, we take the limit vertically, thus:

$$\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = \lim_{\Delta y\to 0}\frac{\Delta u + i\Delta v}{i\Delta y} = \lim_{\Delta y\to 0}\frac{\Delta u}{i\Delta y} + \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} = \frac{1}{i}u_y + v_y = -iu_y + v_y =: f_y.$$

Since the derivative is path independent, we require that $f_x = f_y$, i.e.

$$u_x + iv_x = v_y - iu_y \implies u_x = v_y \text{ and } u_y = -v_x$$

"$\Leftarrow$" We are given a function $f(z) = u(x,y) + iv(x,y)$ such that the CR equations are satisfied with u and v being continuously differentiable functions. To this end, we incorporate the "Fundamental Increment Lemma", which says the following about continuously differentiable functions:

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y) = u_x\Delta x + u_y\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y$$

with $\epsilon_{1,2}\to 0$ as $\Delta x, \Delta y \to 0$. We apply this to both u and v;

$$\begin{aligned}
\Delta u + i\Delta v &= u_x\Delta x + u_y\Delta y + \epsilon_1\Delta x + \epsilon_2\Delta y + i[v_x\Delta x + v_y\Delta y\epsilon_3\Delta x + \epsilon_4\Delta y] \\
&= u_x\Delta x + u_y\Delta y + i[v_x\Delta x + v_y\Delta y]\epsilon_1\Delta x + i\epsilon_3\Delta x + \epsilon_2\Delta y + i\epsilon_4\Delta y \\
&=: u_x\Delta x + u_y\Delta y + i[v_x\Delta x + v_y\Delta y] + \delta_1\Delta x + \delta_2\Delta y \\
&\overset{\text{CR}}{=} u_x\Delta x + -v_x\Delta y + i[v_x\Delta x + u_x\Delta y] + \delta_1\Delta x + \delta_2\Delta y \\
&= u_x(\Delta x + i\Delta y) + iv_x(\Delta x + i\Delta y) + \delta_1\Delta x + \delta_2\Delta y \\
&=: u_x(\Delta z) + iv_x(\Delta z) + \delta_1\Delta x + \delta_2\Delta y
\end{aligned}$$

where, since $\epsilon_{1,2,3,4}\to 0$ as $\Delta x, \Delta y \to 0$, we have that $\delta_{1,2}\to 0$ as well. Dividing through by Δz, we get that:

$$\frac{\Delta f}{\Delta z} := \frac{\Delta u + i\Delta v}{\Delta z} = u_x + iv_x + \delta_1\frac{\Delta x}{\Delta z} + \delta_2\frac{\Delta y}{\Delta z}$$

where we have that $|\Delta x/\Delta z| \le 1$, $|\Delta y/\Delta z| \le 1$ and that $\delta_{1,2}\to 0$ as $\Delta z \to 0$. Taking $\Delta z\to 0$, we get that:

$$\lim_{\Delta z\to 0}\frac{\Delta f}{\Delta z} = u_x + iv_x + \lim_{\Delta z\to 0}\left(\delta_1\frac{\Delta x}{\Delta z} + \delta_2\frac{\Delta y}{\Delta z}\right) = u_x + iv_x$$

□

Proposition 1.2.1. If $f(z)$ is differentiable at $z_0 \implies f$ is continuous at z_0.

Proof.

$$\lim_{z\to z_0}[f(z) - f(z_0)] = \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}(z-z_0)\right] = \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right]\lim_{z\to z_0}(z-z_0) = f'(z_0)\cdot 0 = 0$$

□

Remark 1. However the converse of the above theorem is not *always* true. Take $f(z) = zz^\star = (x+iy)(x-iy) = x^2+y^2$, with $u(x,y) = x^2+y^2$ and $v(x,y) = 0$. The CR equations are only satisfied at $(0,0)$, so the derivative only exists there but nowhere else; *not* analytic

Definition 1.2.1. A function $f(z)$ is called **analytic** at z_0 if f has a derivative in a neighborhood of z_0.

Definition 1.2.2. A function $f(z)$ is called **entire** if it is analytic *everywhere*

1.3 Conjugate Harmonic Functions

Definition 1.3.1. Let $u : \mathbb{R}^n \to \mathbb{R}$, we say that f is **harmonic** if $\nabla^2 u = \sum_{i=1}^n u_{x_i x_i} = 0$

Let $f(z) = u(x,y) + iv(x,y)$ that satisfy the CR equations. Then:

$$\nabla^2 u = u_{xx} + u_{yy} = (u_x)_x + (u_y)_y \overset{\text{CR}}{=} (v_y)_x + (-v_x)_y = v_{xy} - v_{yx} = 0$$

and similarly for v. Thus, as long as f is analytic at z_0, then $\nabla^2 u = \nabla^2 v = 0$.

Example 1.3.1. Let $u = x^3 - 3xy^2$. Finding the *conjugate harmonic function* v amount to finding a v that solves the CR equations. Should get: $v(x,y) = 3x^2y - y^3 + C$, where we can set $C = 0$. Then, to write $u + iv$ as a function of $z = x + iy$, we can simply evaluate $u(x,0) + iv(x,0)$ and use this as our expression for $f(z)$. Should get: $f(z) = z^3$

1.4 Families of Curves

Suppose $u(x,y) = C$, for some constant C and $y = y(x)$. Then we have that:

$$\frac{d}{dx}(u(x,y(x)) = 0 \implies u_x + u_y \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -u_x/u_y =: [m]_{u=C}$$

and similarly denote $[M]_{v=C} := -v_x/v_y$. We get the following relation:

$$[m]_{u=C}[M]_{v=C} = -1 = (u_x v_x)/(u_y v_y)$$

and then say the curves are **mutually orthogonal**. For example, check with $f(z) = x^2 - y^2 + i2xy$

Chapter 2

Elementary (Trancendental) Functions

2.1 Exponential

We define the exponential of a complex number in the following (obvious) way:

$$e^z = e^{x+iy} = e^x e^{iy}.$$

Recall the expansion for exponential functions:

$$e^a = \sum_{k=0}^{\infty} \frac{a^k}{k!}$$

and apply this to e^{iy} in the above term for e^z; we arrive at

$$\begin{aligned} e^{iy} &= 1 + (iy) + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + \cdots \\ &= \left[1 - \frac{y^2}{2!} + \frac{y^4}{4!} + \cdots\right] + i\left[y - \frac{y^3}{3!} + \cdots\right] = \cos(y) + i\sin(y) \end{aligned}$$

thus our final expression for the complex exponential is:

$$e^z = e^x(\cos(y) + i\sin(y) = e^x\cos(y) + ie^x\sin(y) =: u(x,y) + iv(x,y)$$

2.1.1 Properties

1. $e^z = e^x\cos(y) + ie^x\sin(y))$
2. e^z is entire (easy to check that CR equations hold for all x, y) and $\frac{d}{dz}e^z = u_x + iv_x = e^z$
3. $|e^z| = e^x|\sqrt{\cos^2(y) + \sin^2(y)}| = e^x$; unbounded
4. $\exp[z + i2\pi] = \exp[x + i(y + 2\pi)] = e^x[\cos(y + 2\pi) + i\sin(y + 2\pi)] = e^x[\cos(y) + i\sin(y)] = e^z$; periodic function

2.2 Trigonometric

Our favorite trigonometric functions can be written as linear combinations of the exponential function;

$$\cos(y) = \frac{1}{2}\left(e^{iy} + e^{-iy}\right), \sin(y) = \frac{1}{2i}\left(e^{iy} - e^{-iy}\right)$$

and so we can define

$$\begin{aligned}
\cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
&= \frac{1}{2}\left(e^{ix-y} + e^{-ix+y}\right) \\
&= \frac{1}{2}\left(e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(-x) + i\sin(-x)\right) \\
&= \frac{1}{2}\left(e^{-y}(\cos(x) + i\sin(x)) + e^{y}(\cos(x) - i\sin(x)\right) \\
&= \frac{1}{2}\left(\cos(x)(e^{y} + e^{-y}) - i\sin(x)(e^{y} + e^{-y})\right) \\
&=: \cos(x)\cosh(y) - i\sin(x)\cosh(y)
\end{aligned}$$

and similarly, given that $\sinh(y) := \frac{1}{2}(e^{y} - e^{-y})$, we have that

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) = \cdots = \sin(x)\cosh(y) + i\cos(x)\sinh(y).$$

We also define $\tan(z) = \cos(z)/\sin(z), \operatorname{cotan}(z), \sec(z), \csc(z)$ in the obvious ways.

2.2.1 Entire

Easy to verify that $\cos(z)$ and $\sin(z)$ are entire with $\frac{d}{dz}\cos(z) = -\sin(z)$ and $\frac{d}{dz}\sin(z) = \cos(z)$. Furthermore, $\nabla^2 u = 0, \nabla^2 v = 0$ in both cases.

2.2.2 Zeros (Roots)

We say that $\cos(z) = 0$ if both the real and imaginary parts are zero i.e.

$$\cos(z) = 0 \implies \cos(x)\cosh(y) = 0 \text{ and } \sin(x)\sinh(y) = 0.$$

First note that $\cosh(y) \neq 0,\ \forall y$ thus the first term is only zero for all $x = (2n+1)\frac{\pi}{2}$, for all $n \geq 0$. This implies the following for the imaginary term:

$$\sin\left[(2n+1)\frac{\pi}{2}\right]\sinh(y) = 0 \implies (-1)^n\sinh(y) = 0 \implies y = 0$$

and hence, $\cos(z) = 0$ for $z = (2n+1)\frac{\pi}{2} + i0 = (2n+1)\frac{\pi}{2}$

2.2.3 Unbounded

These functions are also unbounded;

$$\begin{aligned}
|\cos(z)|^2 = \cos(z)\cos(z)^\star &= \cos^2(x)\cosh^2(y) + \sin^2(x)\sinh^2(y) \\
&= \cos^2(x)(1 + \sinh^2(y)) + (1 - \cos^2(x)\sinh^2(y) \\
&= \cos^2(x) + \sinh^2(y) \implies \quad \text{unbounded}
\end{aligned}$$

and similarly, $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y)$

2.2.4 Identities

1. $\cos^2(z) + \sin^2(z) = 1$

2. $\cos(z_1 \pm z_2) = \cos(z_1)\cos(z_2) \mp \sin(z_1)\sin(z_2)$

3. $\sin(z_1 \pm z_2) = \sin(z_1)\cos(z_2) \pm \cos(z_1)\sin(z_2)$

4. $\cos(2z) = \cos^2(z) - \sin^2(z)$

5. $\sin(2z) = 2\cos(z)\sin(z)$

6. $\tan(z) = (\sin(2x + i\sinh(y))/(\cos(2x) + \cosh(2y))$

2.3 Hyperbolic

Hyperbolic functions are *also* the same!

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z}) = \cdots = \cosh(x)\cos(y) + i\sinh(x)\sin(y)$$

$$\sinh(z) = \frac{1}{2}(e^z -^{-z}) = \cdots = \sinh(x)\cos(y) + i\cosh(x)\sin(y)$$

where the "$\cdots$" are left to be verified.

2.3.1 Derivative

It is easy to check that these functions are entire (i.e. u_x, u_y, v_x, v_y are continuous and satisfy the CR equations everywhere) and that $\frac{d}{dz}\cosh(z) = \sinh(z)$ and $\frac{d}{dz}\sinh(z) = \cosh(z)$.

2.3.2 Unbounded

Both $\cosh(z)$ and $\sinh(z)$ are unbounded;

$$|\cosh(z)|^2 = \sinh^2(x) + \cos^2(y)$$

$$|\sinh(y)|^2 = \sinh^2(x) + \sin^2(y)$$

2.3.3 Zeros

The zeros are slightly different in this case. Let's first look at $\cosh(z) = 0$. By the relationship found above, we require that $\cosh(x)\cos(y) = 0$ and that $\sinh(x)\sin(y) = 0$. Note that $\sinh(x) = 0$ at $x = 0$ and $\cosh(x) \neq 0\ \forall x$; thus we require that $\cos(y) = 0$. This is true for all $y = \frac{(2n+1)}{2}\pi$, for all $n \in \mathbb{Z}$. Thus, we have that $\cosh(z) = 0$ if $z = 0 + i\frac{(2n+1)}{2}\pi = i\frac{(2n+1)}{2}\pi$.

By using similar arguments, one can show $\sinh(z) = 0$ for all $z = in\pi$, for all $n \in \mathbb{Z}$.

2.3.4 Identities

1. $\cosh^2(z) - \sinh^2(z) = 1$
2. $\sinh(2z) = 2\sinh(z)\cosh(z)$
3. $\cosh(2z) = \cosh^2(z) + \sinh^2(z)$
4. Check addition/subtraction inside the arguments

2.3.5 Relation between trigonometric and hyperbolic functions

The following relationships can be readily verified:

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2} = \frac{e^{-z} + e^{z}}{2} = \cosh(z)$$

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

$$\sin(iz) = i\sinh(z), \quad \sinh(iz) = i\sin(z)$$

2.4 Logarithmic

Definition 2.4.1. Let $z = r(\cos(\theta) + i\sin(\theta)) = re^{i\theta}$, then we define the **(natural) logarithm** of $z \in \mathbb{C}$ as follows:

$$\log(z) = \log(re^{i\theta}) = \ln(r) + \log(e^{i\theta}) = \ln(r) + i\theta$$

We remark that $\log(z)$ is multi-valued, which doesn't suit us well. To make it single-valued, we consider the following definition;

Definition 2.4.2. The **principal value** of $\log(z)$ is $\log(z) = \ln(r) + i\theta$, where $\theta \in (-\pi, \pi]$. This is single valued but discontinuous at $\theta = \pi$.

Example 2.4.1. Check that $\log(-2) = \ln(2) + i\pi$.

Definition 2.4.3. The **general branch** is $\log(z) = \ln(r) + i\theta$, for $\theta \in (\theta_0, \theta_0 + 2\pi)$

Example 2.4.2. Using the Polar-CR equations, derived in HW1: $\frac{d}{dz}\log(z) = z^{-1}$

2.4.1 Products

Let $\log(z_1) = \ln(r_1) + i(\theta_1 + 2n\pi)$ and $\log(z_2) = \ln(r_2) + i(\theta_2 + 2m\pi)$. Then their sum is as follows

$$\begin{aligned}\log(z_1) + \log(z_2) &= \ln(r_1) + i(\theta_1 + 2n\pi) + \ln(r_2) + i(\theta_2 + 2m\pi) \\ &= \ln(r_1 r_2) + i(\theta_1 + \theta_2) + i2\pi(n+m).\end{aligned}$$

Now let $z_1 z_2 = r_1 r_2 \exp(i(\theta_1 + \theta_2))$, then

$$\log(z_1 z_2) = \ln(r_1 r_2) + i(\theta_1 + \theta_2) + i2p\pi$$

and so we get equality for certain values of m, n and p.

2.5 Complex Exponents

Definition 2.5.1. Let $c \in \mathbb{C}$, then $z^c = \exp[c\log(z)]$

Example 2.5.1.

$$i^{-2i} = \exp[-2i\log(i)] = \exp[-2i(\log(1) + i\pi/2 + i2n\pi)] = \exp[-2i(i\pi/2 + i2n\pi)] = \exp(\pi + 4n\pi)$$

with $PV\ e^{\pi}$

2.5.1 Multivalued Exponential

Let $e^z := \exp[z\log(e)]$, but this is multivalued as follows:

$$\log(e) = \ln(e) + in2\pi = 1 + in2\pi \implies e^z = \exp[z(1 + in2\pi)]$$

and thus: $\log(z) = \ln(z) + in2\pi$; for $n \in \mathbb{Z}$. Henceforth, we call $\exp[z]$ the **principal value** of e^z; when $n = 0$.

2.5.2 Derivatives

We compute the derivative of z^c, for some complex c in the following way: pick a branch of $\log(z)$ i.e. $\log(z) = \ln(r) + i\theta$, where $r > 0$ and $\theta \in (\theta_0, \theta_0 + 2\pi)$. Now

$$z^c = \exp[c\ln(z)] =: \exp[w] \implies w = c\log(z).$$

Hence, we use chain rule to compute the derivative

$$\frac{d}{dz}z^c = \frac{d}{dz}\exp[w] = \frac{d}{dw}\exp[w]\frac{dw}{dz} = \exp[w]\frac{c}{z} = z^c\frac{c}{z} = cz^{c-1}.$$

We get similar results looking at the function c^z; for some complex c.

$$c^z = \exp[z\log(c)] =: \exp[w]; w = z\log(c).$$

Again, using chain rule:

$$\frac{d}{dz}c^z = \frac{d}{dz}\exp[w] = \frac{d}{dw}\exp[w]\frac{dw}{dz} = \exp[w]\log(c) = c^z\log(c)$$

. A special case of this is e^z; the derivative is just e^z.

2.5.3 Inverses

An important consequence of the natural logarithm is the ability to derive a closed form representation of an *inverse trigonometric function*. We outline the procedure in the following example.

Proposition 2.5.1.

$$\sin^{-1}(z) = -i\log[iz + \sqrt{1 - z^2}]$$

Proof. Let $w = \sin^{-1}(z)$, then:

$$\begin{aligned}
&\implies z = \sin(w) = \frac{1}{2i}\left(e^{iw} - e^{-iw}\right) \\
&\implies 2iz = e^{-iw}\left(e^{2iw} - 1\right) \\
&\implies \left(e^{iw}\right)^2 - (2iz)e^{iw} - 1 = 0 \\
&\implies e^{iw} = \frac{2iz \pm \sqrt{-4z^2 + 4}}{2} = iz \pm \sqrt{1 - z^2} \\
&\implies iw = \log[iz + \sqrt{1 - z^2}] \\
&\implies w = \sin^{-1}(z) = -i\log[iz + \sqrt{1 - z^2}].
\end{aligned}$$

□

Remark 2. All this to say that most of our previous understanding about derivatives over $\mathbb{R}$ holds true over $\mathbb{C}$. Some notions are new, such as periodicity, and the CR equations, but nothing that breaks your current intuition.

Chapter 3

Integration

3.1 Integration with complex variables

Complex integration can be boiled down to line integrals. Let C be a curve in complex space from α to β. We represent this curve parametrically with $x = \phi(t)$ and $y = \psi(t)$, with $t \in [a, b]$. Thus, $z = x + iy$ and we write

$$\alpha = \phi(a) + i\psi(a), \quad \beta = \phi(b) + i\psi(b).$$

Definition 3.1.1. For a suitable function f, we define the integral along the curve C as follows:

$$\int_C f(z)\ dz = \int_a^b f(\phi(t) + i\psi(t))(\phi'(t) + i\psi'(t))\ dt.$$

From this definition, we have the following "theorem", which allows us to bound the integral.

Theorem 3.1.1. (ML Theorem) Let C be some curve with length L. Then:

$$\left|\int_C f(z)\ dz\right| \leq L \max |f|.$$

Proof.

$$\begin{aligned}
\left|\int_C f(z)\ dz\right| &= \left|\int_a^b f(\phi(t) + i\psi(t))(\phi'(t) + i\psi'(t))\ dt\right| \\
&\leq \int_a^b |f(\phi(t) + i\psi(t)|\ |\phi'(t) + i\psi'(t)|\ dt \\
&\leq \max |f| \int_a^b \sqrt{\phi'(t)^2 + \psi'(t)^2}\ dt = L \max |f|.
\end{aligned}$$

□

Example 3.1.1. Let C be the semi-circle in the upper-half plane with radius R (oriented CCW). Evaluate the following limit:

$$\lim_{R\to\infty} \int_C \frac{\log(z)}{z^2}\ dz.$$

Proof. Using the ML-theorem,

$$\left|\int_C \frac{\log(z)}{z^2}\ dz\right| \leq M\pi R.$$

To find M, we do some bounding:

$$\left|\frac{\log(z)}{z^2}\right| = \frac{|\ln(R) + i\theta|}{R^2} = \frac{\sqrt{\ln(R)^2 + \theta^2}}{R^2} \leq \frac{\ln(R) + \pi}{R^2}.$$

Thus, by l'Hopital's Rule:

$$\lim_{R\to\infty}\left|\int_C \frac{\log(z)}{z^2}\ dz\right| \leq \lim_{R\to\infty}\frac{\pi^2 + \pi\ln(R)}{R} = 0$$

□

3.2 Integration Theorems

Theorem 3.2.1. (Cauchy's Theorem) Let $f(z)$ be analytic within and on a simple, closed curve C (henceforth written as SCC) with $f'(z)$ continuous there as well. Then

$$\oint_C f(z)\ dz = 0.$$

Proof. Let D be the bounded domain within the SCC C. Then:

$$\begin{aligned}\oint_C f(z)\ dz &= \oint_C u(x,y) + iv(x,y)\ (dx + idy)\\ &= \oint_C u\ dx - v\ dy + i\oint_C v\ dx + u\ dy\\ &\overset{\bigstar}{=} \iint_D (-v_x - u_y)\ \ dA + i\iint_D (u_x - v_y)\ dA\\ &\overset{\text{C.R}}{=} 0 + i0 = 0,\end{aligned}$$

where *Green's Theorem* was used at $\bigstar$. □

Remark 3. The above theorem is in fact true even when the derivative of $f(z)$ is not continuous!

Theorem 3.2.2. (Cauchy-Goursat Theorem for MCDs) Let C_1 and C_2 be two SC curves, with C_1 embedded in C_2 (creating a donut of sorts) with domain D between the two curves. Assume $f(z)$ is analytic within/on C_1 and C_2. Then

$$\oint_{C_1} f(z)\ dz = \oint_{C_2} f(z)\ dz.$$

Theorem 3.2.3. Let $f(z)$ be analytic in a domain D. The integral is path independent.

Proof. Take z_1 and z_2 in D. Let C_1 be a curve going from z_1 to z_2 and then C_2 is the curve from z_2 to z_1 (together make a SCC C). Then, by Cauchy's Theorem and some manipulation:

$$\oint_C f(z)\ dz = 0 \implies \int_{C_1} f(z)\ dz = \int_{C_2} f(z)\ dz.$$

□

Theorem 3.2.4. If f is analytic in a domain D with $z_0, z \in D$, then:

$$\int_{z_0}^{z} f(w)\, dw = F(z), \quad \frac{dF}{dz} = f(z).$$

Theorem 3.2.5. (Cauchy's Integral Formula) Take a SCC C and let $g(z)$ be a function with a singularity at z_0 within C. That is, we can write $g(z) = f(z)/(z - z_0)$, with $f(z)$ analytic within and on C. Then

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z_0}\, dw.$$

Proof. We create an ϵ-contour around the point z_0, denoted C_ϵ. That is, let $w - z_0 = \epsilon e^{i\theta}$ for some $\epsilon > 0$. By Cauchy-Goursat Theorem and some algebraic manipulation, we have that

$$\oint_C \frac{f(w)}{w - z_0}\, dw = \oint_{C_\epsilon} \frac{f(w)}{w - z_0}\, dw \implies \oint_C \frac{f(w)}{w - z_0}\, dw = \oint_{C_\epsilon} \frac{f(w) - f(z_0)}{w - z_0}\, dw + f(z_0) \oint_{C_\epsilon} \frac{dw}{w - z_0}.$$

We first evaluate the second term:

$$f(z_0) \oint_{C_\epsilon} \frac{dw}{w - z_0} = f(z_0) \int_0^{2\pi} \frac{i\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\, d\theta = f(z_0) 2\pi i.$$

To complete the proof, we show that

$$\lim_{\epsilon \to 0} \oint_{C_\epsilon} \frac{f(w) - f(z_0)}{w - z_0}\, dw = 0.$$

To this end:

$$\begin{aligned}
\lim_{\epsilon \to 0} \left| \oint_{C_\epsilon} \frac{f(w) - f(z_0)}{w - z_0}\, dw \right| &\leq \lim_{\epsilon \to 0} 2\pi\epsilon \left| \frac{f(w) - f(z_0)}{w - z_0} \right| \\
&< \lim_{\epsilon \to 0} 2\pi\epsilon = 0.
\end{aligned}$$

□

Theorem 3.2.6. (General Cauchy Formula) Let $f(z)$ be analytic within and on a SC C. We have that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w - z_0)^{n+1}}\, dw.$$

3.2.1 Motivating examples

To motivate the importance of the Cauchy Integral Formula, we present some examples. Consider the contour C given by the square with vertices $(\pm 2, \pm 2)$ (with the countour being CCW). We want to compute the contour integral of the following functions:

$$f_1(z) = \frac{e^{-z}}{z-(\pi/2)i}, \; f_2(z) = \frac{\cos(z)}{z(z^2+8)}, \; f_3(z) = \frac{z}{2z+1}, \; f_4(z) = \frac{\tan(z/2)}{(z-x_0)}.$$

Example 3.2.1. We start with f_1 above. Note that there is a singularity, $z_0 = \frac{\pi}{2}$, within the contour. Using the Cauchy Integral Formula, we simply let the numerator be the "analytic" part and the singularity is the denominator:

$$\oint_C \frac{e^{-z}}{z-(\pi/2)i}\; dz = 2\pi i \exp(-(\pi/2)i) = 2\pi i(\cos(\pi/2) - i\sin(\pi/2)) = 2\pi.$$

Example 3.2.2. There are multiple options for singularities: $\pm i2\sqrt{2}$ and 0. However, only $z_0 = 0$ is within the contour. The "analytic" part is then $\cos(z)/(z^2+8)$, and so:

$$\oint_C \frac{\cos(z)}{z(z^2+8)}\; dz = 2\pi i \cos(0)/(0^2+8) = \frac{i\pi}{4}.$$

Example 3.2.3. With some re-writing, we see that there is a singularity at $z_0 = -1/2$:

$$\oint_C \frac{z}{2z+1}\; dz = \frac{1}{2}\oint_C \frac{z}{z+0.5}\; dz = \frac{1}{2}2\pi i(-1/2) = \frac{-i\pi}{2}.$$

Example 3.2.4. Let $x_0 \in \mathbb{R}$. There are three cases to consider: $|x_0| > 2, |x_0| < 2$ and $x_0 = \pm 2$. In the first case, there is no singularity and so the contour integral of f_4 evaluates to 0. The second case is an easy application of the Cauchy Integral Formula:

$$\oint_C \frac{\tan(z/2)}{(z-x-0)} = 2\pi i \tan(x_0/2).$$

The last case is the most intricate and uses what is called a *keyhole* to compute the contour integral. WLOG, let us consider $x_0 = 2$ and take the contour

$$C_\epsilon : z = 2 + \epsilon e^{i\theta}, \quad \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

Visually, this is a small (outward?) bump along the contour of the square at $x_0 = 2$. Taking the limit of $\epsilon \to 0$, we retrieve the original contour integral (as it is the only singularity present. We perform a change of variable with θ,

$$\oint_{C_\epsilon} \frac{\tan(z/2)}{z-2}\; dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\tan\left(\frac{2+\epsilon e^{i\theta}}{2}\right)\epsilon e^{i\theta}}{\epsilon e^{i\theta}}\; d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\left(\frac{2+\epsilon e^{i\theta}}{2}\right)\; d\theta.$$

Moving in the limit and by continuity of $\tan(\cdot)$:

$$\lim_{\epsilon\to 0}\oint_{C_\epsilon} \frac{\tan(z/2)}{z-2}\; dz = \lim_{\epsilon\to 0}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan\left(\frac{2+\epsilon e^{i\theta}}{2}\right)\; d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \tan(1)\; d\theta = \frac{\pi^2}{4}.$$

3.3 More theorems

Theorem 3.3.1. If $f(z)$ is analytic at z_0, then its derivatives are analytic at z_0.

Theorem 3.3.2. (Cauchy Estimates) Let $f(z)$ be analytic within and on a SCC C. Take z in C and let $M := \max|f|$. Then:

$$\left|f^{(n)}(z)\right| \leq \frac{n!M}{r^n}.$$

Proof. Using the General Cauchy Integral formula and the ML theorem, we get the desired bound

$$\left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z_0)^{n+1}}\ dw\right| \leq \frac{n!}{2\pi}(2\pi r)\frac{\max|f|}{r^{n+1}} = \frac{n!\max|f|}{r^n}.$$

$\square$

Corollary 3.3.1. (Liouville's Theorem) If $f(z)$ is analytic everywhere and bounded, then f is constant.

Proof. Since f is bounded, $M := \max|f|$ is well defined. We use the Cauchy Estimate for the first derivative of f and take the limit of $r \to \infty$ (since f is analytic everywhere). However:

$$|f^{(1)}(z_0)| = \frac{M}{r} \longrightarrow 0.$$

Thus the derivative is zero and f must be constant. $\square$

Theorem 3.3.3. (Gauss Mean Value Theorem) The value $f(z_0)$ is determined by the boundary values of (any?) circle with z_0 as the center.

Proof. Let C be a SCC with z_0 at the center and arbitrary radius r i.e. $C : z = z_0 + re^{i\theta}$. Using the Cauchy Integral Formula and a simple change of variable, we get the desired result:

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\ dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f\left(z_0+re^{i\theta}\right)\ ire^{i\theta}}{re^{i\theta}}\ d\theta = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0+re^{i\theta}\right)\ d\theta =: [f_{\text{mean}}]_C$$

$\square$

Example 3.3.1. A cool example: take $f(z) = e^z$; by the GMVT,

$$1 = e^0 = \frac{1}{2\pi}\int_0^{2\pi} f\left(0+e^{i\theta}\right)\ d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{e^{i\theta}}\ d\theta \implies \int_0^{2\pi} e^{e^{i\theta}} = 2\pi.$$

Theorem 3.3.4. (Maximum Principle) A harmonic (analytic) function, f, achieves the maximal value on the boundary of its (compact) domain, not at an interior point.

Proof. Suppose f attains the maximum at an interior point, p. If f is not constant, then by the GMVT, there exists a ball around p where $f(p)$ is determined by the boundary values of that ball. Since f is not constant, it will have to attain higher/lower values around this boundary; $f(p)$ is not maximal. $\square$

Chapter 4

Application: Poisson Problems

This course introduced the applications of complex analysis to solve various problems in partial differential equations. In this chapter, we exploit the symmetry in two PDE problems using not much more than the Cauchy Integral Formula and some algebraic manipulation.

4.1 Upper Half Plane

4.1.1 Problem Setup

Find (a bounded) $\psi(x, y)$ in the upper half plane ($x \in \mathbb{R}, y > 0$) that solves the following PDE:

$$\nabla^2\psi(x, y) = 0 \quad \text{such that} \quad \psi(x, 0) = g(x)$$

for some $g : \mathbb{R} \to \mathbb{R}$, called the *boundary function*. We claim that

$$\psi(x, y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{g(\eta)}{(\eta - x)^2 + y^2}\, d\eta.$$

4.1.2 Poisson Integral Formula for the UHP

In this derivation, we replace ψ with f for typing feasibility.

Let $f(z) = u(x, y) + iv(x, y)$ be analytic in the UHP. We consider the contour $C = C_R + [-R, R]$. Consider a $z = x + iy$ within the contour; the Cauchy Integral formula then says

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw = \frac{1}{2\pi i} \int_{C_R} \frac{f(w)}{w - z}\, dw + \frac{1}{2\pi i} \int_{-R}^{R} \frac{u(\eta, 0) + iv(\eta, 0)}{\eta - (x + iy)}\, d\eta.$$

However, taking $z^\star = x - iy$ (the conjugate of z), which lies outside the contour, we have that

$$f(z^\star) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z^\star}\, dw = 0$$

because there is no singularity. We subtract $f(z) - f(\star)$ and do some algebraic manipulation:

$$\begin{aligned}
f(z) = f(z) - f(z^\star) &= \frac{1}{2\pi i}\oint_C f(w)\left(\frac{1}{(w-z)} - \frac{1}{(w-z^\star)}\right)\, dw \\
&= \frac{1}{2\pi i}\oint_C f(w)\left(\frac{2iy}{(w-z)(w-z^\star)}\right)\, dw \\
&= \frac{2iy}{2\pi i}\oint_C \frac{f(w)}{(w-z)(w-z^\star)}\, dw \\
&= \frac{y}{\pi}\int_{C_R} \frac{f(w)}{(w-z)(w-z^\star)}\, dw + \frac{y}{\pi}\int_{-R}^{R} \frac{u(\eta,0)+iv(\eta,0)}{[\eta-(x+iy)][\eta-(x-iy)]}\, d\eta \\
&= \frac{y}{\pi}\int_{C_R} \frac{f(w)}{(w-z)(w-z^\star)}\, dw + \frac{y}{\pi}\int_{-R}^{R} \frac{u(\eta,0)+iv(\eta,0)}{(\eta-x)^2+y^2}\, d\eta.
\end{aligned}$$

Since we are interested in the entire UHP, we take $R \to \infty$ in the above expression. Note that since f is harmonic, u and v represent potentials and so they are bounded, hence f is bounded. Using the ML theorem, we get the following bound on the first term:

$$\left|\frac{y}{\pi}\int_{C_R} \frac{f(w)}{(w-z)(w-z^\star)}\, dw\right| \le \max|f| \cdot R\pi \cdot O(R^{-2}) = O(R^{-1}).$$

Taking the limit, the first term disappears and we are left with a closed form solution to the UHP. Note that this means:

$$f(z) = \underbrace{\frac{y}{\pi}\int_{\mathbb{R}} \frac{u(\eta,0)}{(\eta-x)^2+y^2}\, d\eta}_{u(x,y)} + i\,\underbrace{\frac{y}{\pi}\int_{\mathbb{R}} \frac{v(\eta,0)}{(\eta-x)^2+y^2}\, d\eta}_{v(x,y)}.$$

4.2 Circle

4.2.1 Problem Setup

Find (a bounded) $\psi(r,\theta)$ in the circle $(0 < r < R, \theta \in (0, 2\pi))$ that solves the following PDE:

$$\nabla^2\psi(r,\theta) = 0 \quad \text{such that} \quad \psi(R,\theta) = g(\theta).$$

We claim that

$$\psi(r,\theta) = \frac{R^2 - r^2}{2\pi}\int_0^{2\pi} \frac{g(\varphi)}{R^2 + r^2 - 2Rr\cos(\varphi-\theta)}\, d\varphi.$$

4.2.2 Poisson Integral Formula for the Circle

Again, we substitute ψ with f for this proof. Take f to be analytic within and on the circle C defined by $|z| \le R$ (for a fixed $R > 0$). Thus, an interior point is given by $re^{i\theta}$, where $0 < r < R$. Define ω to be a value on the boundary, given by $Re^{i\varphi}$ (both θ and ϕ are between 0 and 2π).

This approach is the same as the previous problem; we look for an appropriate point outside the circle that we can exploit. Take $z' := R^2/z^\star = e^{i\theta}(R^2/r)$; this point is outside the circle. By way of the above derivation:

$$f(z) = f(z) - f(z') = \frac{1}{2\pi i}\oint_C \frac{(z-z')f(\omega)}{(\omega-z)(\omega-z')}\, d\omega$$

where $\omega = Re^{i\varphi}$. After performing the necessary change of variables, we just need to do some algebraic manipulation:

$$\begin{aligned}
f\big(re^{i\theta}\big) &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{\big((r-(R^2/r))e^{i\theta}\big)\, f\big(Re^{i\varphi}\big) iRe^{i\varphi}}{\big(Re^{i\varphi}-re^{i\theta}\big)\big(Re^{i\varphi}-(R^2/r)e^{i\theta}\big)}\, d\varphi \\
&= \frac{r^2-R^2}{2\pi}\int_0^{2\pi} \frac{f\big(Re^{i\varphi}\big)e^{i(\theta+\varphi)}}{\big(Re^{i\varphi}-re^{i\theta}\big)\big(re^{i\varphi}-Re^{i\theta}\big)}\, d\varphi \\
&= \frac{r^2-R^2}{2\pi}\int_0^{2\pi} \frac{f\big(Re^{i\varphi}\big)}{e^{-i(\theta+\varphi)}\big(Re^{i\varphi}-re^{i\theta}\big)\big(re^{i\varphi}-Re^{i\theta}\big)}\, d\varphi \\
&= \frac{r^2-R^2}{2\pi}\int_0^{2\pi} \frac{f\big(Re^{i\varphi}\big)}{\big(Re^{-i\theta}-re^{-i\varphi}\big)\big(re^{i\varphi}-Re^{i\theta}\big)}\, d\varphi \\
&= \frac{r^2-R^2}{2\pi}\int_0^{2\pi} \frac{f\big(Re^{i\varphi}\big)}{-R^2-r^2+2Rr\cos(\varphi-\theta)}\, d\varphi \\
&= \frac{R^2-r^2}{2\pi}\int_0^{2\pi} \frac{f\big(Re^{i\varphi}\big)}{R^2+r^2-2Rr\cos(\varphi-\theta)}\, d\varphi.
\end{aligned}$$

The second-to-last equality requires quite a bit of algebraic manipulation but nothing impossible.

Recall that, in the context of the Poisson problem, $f\big(Re^{i\varphi}\big)$ is the boundary value of the PDE.

4.3 Circle: Series Solution

Consider the same Poisson problem as in the above section. We propose another closed form solution to the problem with a very different approach. We first note that, with a bit of algebraic manipulation, the following relation is true (given the work we did above):

$$f(z) = f\big(re^{i\theta}\big) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{1-\frac{r}{R}e^{i(\theta-\varphi)}} - \frac{1}{1-\frac{R}{r}e^{i(\theta-\varphi)}}\right) f\big(Re^{i\varphi}\big)\, d\varphi.$$

We can write the first term in the parenthesis as an infinite (geometric) sum, since

$$\left|\frac{r}{R}e^{i(\theta-\varphi)}\right| = r/R < 1 \implies \big(1-(r/R)e^{i(\theta-\varphi)}\big)^{-1} = \sum_{n=0}^{\infty}\big[(r/R)e^{i(\theta-\varphi)}\big]^n.$$

We also note that the second term can be written as follows:

$$\frac{1}{1-(R/r)e^{i(\theta-\varphi)}} = \frac{(r/R)e^{-i(\theta-\varphi)}}{1-(r/R)e^{-i(\theta-\varphi)}} = \sum_{n=1}^{\infty}\big[(r/R)e^{-i(\theta-\varphi)}\big]^n.$$

Putting this all together:

$$\begin{aligned}
f\big(re^{i\theta}\big) &= \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=0}^{\infty}\big[(r/R)e^{i(\theta-\varphi)}\big]^n + \sum_{n=1}^{\infty}\big[(r/R)e^{-i(\theta-\varphi)}\big]^n\right) f\big(Re^{i\varphi}\big)\ d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1+\sum_{n=1}^{\infty}(r/R)^n\big[e^{in(\theta-\varphi)}+e^{-in(\theta-\varphi)}\big]\right) f\big(Re^{i\varphi}\big)\ d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left(1+2\sum_{n=1}^{\infty}(r/R)^n\cos((\theta-\varphi)n)\right) f\big(Re^{i\varphi}\big)\ d\varphi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f\big(Re^{i\varphi}\big)\left[1+2\sum_{n=1}^{\infty}(r/R)^n\left(\cos(n\phi)\cos(n\theta)+\sin(n\phi)\sin(n\theta)\right)\right]\ d\varphi.
\end{aligned}$$

If $f(z) = u(r,\theta) + iv(r,\theta)$, we get that

$$u(r,\theta) = a_0 + \sum_{n=1}^{\infty}(r/R)^n\Big[a_n\cos(n\theta) + b_n\sin(n\theta)\Big]$$

where $a_0 = \dfrac{1}{2\pi}\int_0^{2\pi} u(R,\varphi)\ d\varphi$ (average of the boundary function) and

$$a_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\varphi)\cos(n\varphi)\ d\varphi, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} u(R,\varphi)\sin(n\varphi)\ d\varphi.$$

Chapter 5

Fourier Series and Fourier Transform

5.1 Fourier Series

Definition 5.1.1. A **Complete Fourier Series** of a function is written as

$$f(\theta) = a_0 + \sum_{n=1}^{\infty} \Big[a_n \cos(n\theta) + b_n \sin(n\theta) \Big]$$

with a_0, a_n and b_n as defined in the previous chapter (there's no "radius" business here though).

Definition 5.1.2. A **Fourier Cosine Series** is a series where the function in question is even, which forces $b_n \equiv 0$. Similarly, a **Fourier Sine Series** is an odd function with $a_0 = a_n \equiv 0$

5.2 Fourier Transform

5.2.1 Definitions

To get a full derivation of the Fourier Transform, I would suggest using pretty much any other reference on complex variables.

Definition 5.2.1. Let $f \in L^1$. The **Fourier Transform** of f is defined as

$$F(\omega) := \mathcal{F}[f] = \int_{\mathbb{R}} e^{-i\omega\xi} f(\xi)\ d\xi.$$

Definition 5.2.2. The **Inverse Fourier Transform** of $F(w)$ is given by the following:

$$f(x) := \mathcal{F}^{-1}[F] = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega x} F(\omega)\ d\omega.$$

Example 5.2.1. Fix $a > 0$. Let $f(t) := \mathbb{1}_{|t|<a}$. The Fourier transform is easy to compute:

$$F(\omega) = \int_{\mathbb{R}} e^{i\omega\tau} f(\tau)\ d\tau = \int_{-a}^{a} e^{i\omega\tau}\ d\tau = \int_{-a}^{a} \cos(\omega\tau)\ d\tau + i \int_{-a}^{a} \sin(\omega\tau)\ d\tau = \frac{2}{\omega}\sin(a\omega).$$

Note that $\sin(\cdot)$ is an odd function so the integral is zero (this is a very common/useful trick).

Definition 5.2.3. Assume $\int_0^\infty f(x)\ dx$ exists. The **Fourier Cosine Transform** (respectively, **Inverse Fourier Cosine Transform**) is defined to be

$$F_c(\omega) := \mathcal{F}_c[f] = \int_0^\infty \cos(\omega\tau) f(\tau)\ d\tau, \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos(\omega x) F_c(\omega)\ d\omega.$$

Definition 5.2.4. Assume $\int_0^\infty f(x)\, dx$ exists. The **Fourier Sine Transform** (respectively, **Inverse Fourier Sine Transform**) is defined to be

$$F_s(\omega) := \mathcal{F}_s[f] = \int_0^\infty \sin(\omega\tau) f(\tau)\ d\tau, \quad f(x) = \frac{2}{\pi}\int_0^\infty \sin(\omega x) F_s(\omega)\ d\omega.$$

Definition 5.2.5. We denote the **convolution** of two functions f and g to be

$$(f \star g)(t) = \int_{\mathbb{R}} f(\tau) g(t-\tau)\ d\tau.$$

The following example builds on the concept of the **Laplace Transform**. An appropriate table will be provided on the midterm/final should these types of problems come up.

Example 5.2.2. Let $f(x) = e^{-ax}$, for some $a > 0$. Compute the Fourier Cosine Transform and re-compute $f(x)$.

$$F_c(\omega) = \int_0^\infty e^{-a\tau}\cos(\omega\tau)\ d\tau = \mathcal{L}\big[\cos(\omega\tau)\big]_{s=a} = \frac{a}{a^2+\omega^2}.$$

$$e^{-ax} = \frac{2}{\pi}\int_0^\infty \cos(\omega x)\frac{a}{a^2+\omega^2}\ d\omega \implies \int_0^\infty \frac{\cos(\omega x)}{a^2+\omega^2}\ d\omega = \frac{\pi}{2a}e^{-a|x|},$$

since $\cos(\cdot)$ is an even function.

5.2.2 Theorems

Theorem 5.2.1. (Parseval's Theorem) Suppose $f \in L^1 \cap L^2$ with $F(\omega)$ denoting its Fourier transform. Then

$$\|F(\omega)\|_{L^2}^2 = 2\pi\|f(t)\|_{L^2}^2.$$

Proof. Consider $f, g \in L^1 \cap L^2$ and denote their Fourier transforms by $F(\omega)$ and $G(\omega)$, respectively. Let the complex conjugate of $G(\omega)$ be written as

$$G^\star(\omega) = \int_{\mathbb{R}} e^{i\omega\tau} g(\tau)\ d\tau.$$

Now we take the inner product of F and G and use Fubini's Theorem (denoted by "(F)"):

$$\begin{aligned}\langle F, G\rangle &= \int_{\mathbb{R}} F(\omega) G^\star(\omega)\ d\omega = \int_{\mathbb{R}} F(\omega)\left[\int_{\mathbb{R}} e^{i\omega\tau} g(\tau)\ d\tau\right] d\omega \\ &\overset{\text{(F)}}{=} \int_{\mathbb{R}} g(\tau)\left[\int_{\mathbb{R}} F(\omega) e^{i\omega\tau}\ d\omega\right] d\tau = 2\pi\int_{\mathbb{R}} g(\tau) f(\tau)\ d\tau = 2\pi\langle f(t), g(t)\rangle.\end{aligned}$$

Taking $f \equiv g$, we get the desired result. $\square$

Theorem 5.2.2. (Convolution Property) Take $f, g \in L^1$, then:

$$\mathcal{F}[(f \star g)] = F(\omega)G(\omega)$$

Proof.

$$\begin{aligned} F(\omega)G(\omega) &= \int_{\mathbb{R}} e^{-i\omega\tau} f(\tau)\ d\tau \int_{\mathbb{R}} e^{-i\omega\xi} g(\xi)\ d\xi = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\omega(\tau+\xi)} f(\tau)g(\xi)\ d\tau d\xi \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-i\omega t} f(\tau)g(t-\tau)\ d\tau dt \qquad (t = \tau + \xi) \\ &= \int_{\mathbb{R}} e^{-i\omega t}(f \star g)(t)\ dt = \mathcal{F}[(f \star g)]. \end{aligned}$$

□

5.3 Applications

5.3.1 Solving "convolution equations"

Let $0 < a < b$ and consider the following equation:

$$\int_{\mathbb{R}} \frac{f(\tau)}{(t-\tau)^2 + a^2}\ d\tau = \frac{1}{t^2 + b^2}$$

and suppose you want to search for what function $f(\cdot)$ is. We start by taking the Fourier transform of both sides, since the LHS is a convolution (i.e. the *convolution property* is very useful here) and we are already familiar with the Fourier transform of the RHS (see Example 5.2.2). For simplicity, we denote

$$g(t) = (t^2 + a^2)^{-1}, \quad h(t) = (t^2 + b^2)^{-1},$$

and so, by the convolution property, our equation to solve is $F(\omega)G(\omega) = H(\omega)$ (denote the Fourier transforms of f, g and h respectively). Note that $G(\omega)$ and $H(\omega)$ are the same up to changing a and b. To this end:

$$\begin{aligned} H(\omega) = \int_{\mathbb{R}} \frac{e^{-i\omega\tau}}{\tau^2 + b^2}\ d\tau &= \int_{\mathbb{R}} \frac{\cos(\omega\tau)}{\tau^2 + b^2}\ d\tau - i\int_{\mathbb{R}} \frac{\sin(\omega\tau)}{\tau^2 + b^2}\ d\tau \\ &= 2\int_0^\infty \frac{\cos(\omega\tau)}{\tau^2 + b^2}\ d\tau + 0 = \frac{\pi}{b}e^{-b|\omega|} \end{aligned}$$

since $\sin(\cdot)$ is odd (here we used Example 5.2.2). This implies that $G(\omega) = (\pi/a)e^{-a|\omega|}$. Diving through, we get that $F(\omega) = (a/b)e^{-(b-a)|\omega|}$. We now just need to compute the inverse Fourier transform but again, this is easy:

$$\begin{aligned} f(t) &= \frac{a}{2\pi b}\int_{\mathbb{R}} e^{i\omega t}e^{-(b-a)|\omega|}\ d\omega = \frac{a}{2\pi b}\int_{\mathbb{R}} e^{-(b-a)|\omega|}\cos(\omega t)\ d\omega + i\frac{a}{2\pi b}\int_{\mathbb{R}} e^{-(b-a)|\omega|}\sin(\omega t)\ d\omega \\ &= \frac{a}{\pi b}\int_0^\infty e^{-(b-a)|\omega|}\cos(\omega t)\ d\omega + 0 = \frac{a}{\pi b}\mathcal{L}[\cos(\omega t)]_{s=b-a} = \frac{a}{\pi b}\frac{(b-a)}{(b-a)^2 + t^2}. \end{aligned}$$

Proposition 5.3.1.
$$\int_0^\infty \cos(ax)e^{-m^2x^2}\ dx = \frac{\sqrt{\pi}}{2m}\exp[-a^2/(4m^2)].$$

5.3.2 Heat Diffusion: Semi-infinite bar

Consider the standard heat equation with *Dirichlet Boundary Conditions*:

$$\frac{1}{\alpha^2}\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u(0,t) = 0,\ u(x,0) = f(x)$$

for all $x, t > 0$. We make the following ansatz: $u(x,t) = X(x)T(t)$ i.e. the solution is separable. Plugging this into the PDE, we get the following *eigenvalue equations*

$$\frac{\partial^2 X}{\partial x^2} = \lambda X, \quad \frac{\partial T}{\partial t} = (\alpha^2\lambda)T$$

where $\lambda \in \mathbb{R}$. It is easy to check that, in the case of $\lambda \geq 0$, $X(x)$ is unbounded (which is not desired). Since $\lambda < 0$, we can re-write $\lambda = -\omega^2$. We now solve the eigenvalue equations:

$$\frac{\partial^2 X}{\partial x^2} = -\omega^2 X \implies X(x) = A\cos(\omega x) + B\sin(\omega x)$$

and since the boundary condition is $X(0) = 0$, we get $X(x) = B\sin(\omega x)$. Similarly:

$$\frac{\partial T}{\partial t} = -\alpha^2\omega^2 T \implies T(t) = C\exp[-\alpha^2\omega^2 t].$$

Putting these together, we can write

$$u(x,t) = \int_0^\infty u_\omega(x,t)\ d\omega, \quad u_\omega(x,t) := D(\omega)\sin(\omega x)\exp[-\alpha^2\omega^2 t].$$

Using the other initial condition, we can get a closed-form expression for $D(\omega)$:

$$u(x,0) = \int_0^\infty D(\omega)\sin(\omega x)\ d\omega = f(x) \implies D(\omega) = \mathcal{F}_s^{-1}[f] = \frac{2}{\pi}\int_0^\infty \sin(\omega\xi)f(\xi)\ d\xi.$$

We plug this into our expression for $u(x,t)$ and use Fubini's Theorem and the above proposition (the proof should be done in one of the assignments):

$$\begin{aligned}
u(x,t) &= \frac{2}{\pi}\int_0^\infty\int_0^\infty \sin(\omega\xi)f(\xi)\sin(\omega x)e^{-\alpha^2\omega^2 t}\ d\xi d\omega \\
&= \frac{2}{\pi}\int_0^\infty f(\xi)\left[\int_0^\infty \sin(\omega\xi)\sin(\omega x)e^{-\alpha^2\omega^2 t}\ d\omega\right] d\xi \\
&= \frac{2}{\pi}\int_0^\infty f(\xi)\left[\frac{1}{2}\int_0^\infty (\cos(\omega(\xi - x)) - \cos(\omega(\xi + x)))e^{-\alpha^2\omega^2 t}\ d\omega\right] d\xi \\
&= \frac{1}{\sqrt{4\pi\alpha^2 t}}\int_0^\infty f(\xi)\left(\exp\left[-\frac{(\xi - x)^2}{4\alpha^2 t}\right] - \exp\left[-\frac{(\xi + x)^2}{4\alpha^2 t}\right]\right) d\xi \\
&=: \int_0^\infty f(\xi)G(x,t,\xi)\ d\xi.
\end{aligned}$$

5.3.3 Poisson's formula for the UHP via Fourier Transforms

Let $u(x, y)$ be the solution to the following PDE:

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = f(x), \; \lim_{y\to\infty} u(x,y) = 0.$$

We again assume that $u(x, y) = X(x)Y(y)$ (the solution is separable). The PDE is the the following eigenvalue equation(s):

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \frac{-1}{X}\frac{\partial^2 X}{\partial x^2} = \lambda.$$

Note that the limit condition is equivalent to $\lim_{y\to\infty} Y(y) = 0$ so, taking $\lambda = \omega^2$, we get that $Y_\omega(y) = A(\omega)e^{-|\omega|y}$. Similarly, soling the ODE for X yields $X_\omega(x) = B(\omega)e^{i\omega x} + C(\omega)e^{-i\omega x}$. Putting these together, we can write

$$u(x,y) = \int_{\mathbb{R}} u_\omega(x,y)\; d\omega = \int_{\mathbb{R}} \left[\widetilde{B}(\omega)e^{i\omega x} + \widetilde{C}(\omega)e^{-i\omega x}\right] e^{-|\omega|y}\; d\omega.$$

By symmetry (imo, it's magic), we can re-write the problem as

$$u(x,y) = \int_{\mathbb{R}} D(\omega)e^{-i\omega x}e^{-|\omega|y}\; d\omega.$$

Implementing the boundary condition, we get a closed-form expression for $D(\omega)$:

$$u(x,0) = f(x) = \int_{\mathbb{R}} D(\omega)e^{-i\omega x}\; d\omega \implies D(\omega) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\omega\xi} f(\xi)\; d\xi.$$

Putting this all together (we drop the $\sin(\cdot)$ term because the integrand is odd):

$$\begin{aligned}
u(x,y) &= \frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\xi)e^{i\omega(\xi - x)}e^{-|\omega|y}\; d\xi d\omega \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)\left[\int_{\mathbb{R}} \big(\cos(\omega(\xi - x) + i\sin(\omega(\xi - x))\big)e^{-|\omega|y}\; d\omega\right] d\xi \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} f(\xi)\left[\int_{\mathbb{R}} \cos(\omega(\xi - x))e^{-|\omega|y}\; d\omega\right] d\xi \\
&= \frac{1}{\pi}\int_{\mathbb{R}} f(\xi)\left[\int_0^\infty \cos(\omega(\xi - x))e^{-\omega y}\; d\omega\right] d\xi \\
&= \frac{1}{\pi}\int_{\mathbb{R}} f(\xi)\mathcal{L}[\cos(\omega(\xi - x))]_{s=y}\; d\xi \\
&= \frac{1}{\pi}\int_{\mathbb{R}} f(\xi)\frac{y}{(\xi - x)^2 + y^2}\; d\xi = \frac{y}{\pi}\int_{\mathbb{R}} \frac{f(\xi)}{(\xi - x)^2 + y^2}\; d\xi.
\end{aligned}$$

Chapter 6

Residues

6.1 Intro to Series

Recall that a power series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

converges provided $|z - z_0| < R$, with $R := \lim_{N\to\infty} |a_N / a_{N+1}|$ (the "radius" of a circle/interval of convergence).

6.1.1 Recovering Taylor Series

Let $f(z)$ be analytic within a circle C_0 but not on the boundary. Let C_1 be a SCC contour completely within C_0; then $f(z)$ is analytic within and on C_1. Take $z \in C_1$; by Cauchy's Integral formula,

$$f(z) = \frac{1}{2\pi i} \oint_{C_1} \frac{f(y)}{y - z}\, dy.$$

Recall the following expression from HW1:

$$\frac{1}{1-\alpha} = 1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} + \frac{\alpha^n}{1-\alpha}.$$

We use this to expand the denominator of the above contour integral as follows:

$$\frac{1}{y - z} =$$

6.2 Laurent Series

6.2.1 Singularities