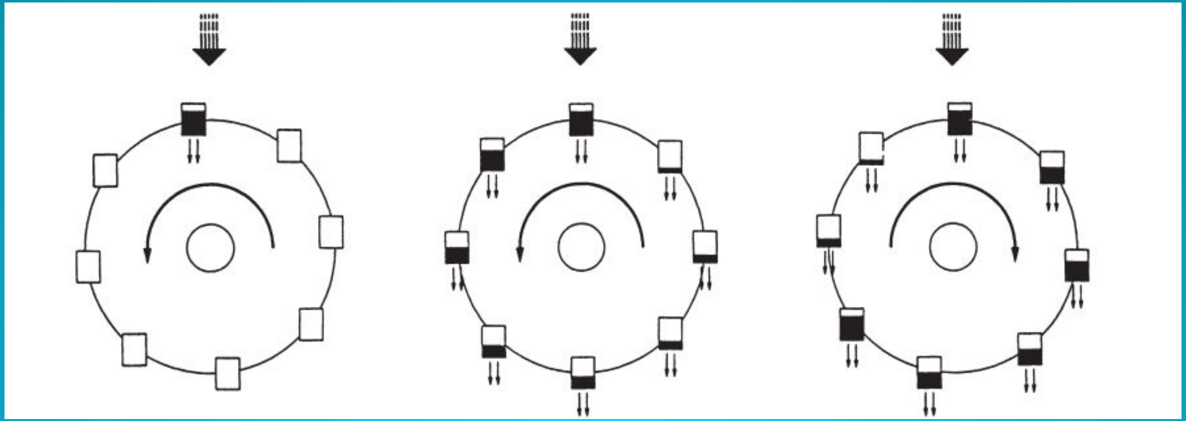
Malkus Waterwheel

What is it?

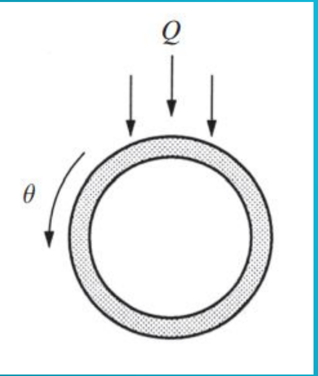
PH 567 : Presentation

Chaotic/Lorenz Waterwheel

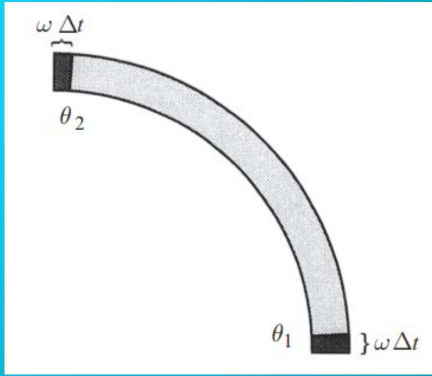
Img by : Harvard Natural Sciences Lecture Demonstration(2020, May 14) Chaotic Waterwheel [Video]. Youtube. URL
https://www.youtube.com/watch?app=desktop&v=Lx8gMBJBlP8



The motion of waterwheel after onset of tap inflow



Side view of the theoretically modelled wheel



Section of the wheel being analysed

Image credits : Strogatz, S. H. Nonlinear Dynamics and Chaos: With Applications To Physics, Biology, Chemistry, And Engineering

MASS CONSERVATION

$$\Delta M = \Delta t\left[\int_{\theta_1}^{\theta_2} Q\,d\theta - \int_{\theta_1}^{\theta_2} Km\,d\theta\right] + m(\theta_1)\,\omega\Delta t - m(\theta_2)\,\omega\Delta t.$$

$$\frac{\partial m}{\partial t} = Q - Km - \omega\frac{\partial m}{\partial \theta}$$

Cause of Nonlinearity

Rate of mass change in a cup

Mass inflow by the tap

Water leaked due to hole at bottom

Water spilled to neighbouring cups

MOMENT BALANCE

Integro-Differential Equation

$$\frac{d[I\omega]}{dt} = -\nu\omega + gR\int_0^{2\pi} m(\theta, t)\sin\theta d\theta$$

Net torque on system

Damping due to axle friction

Torque due to gravitational force

HOW TO SOLVE THESE EQUATIONS?

FOURIER SERIES !

Writing the mass and inflow rate as sum of fourier components a, b and q respectively

$$m(\theta, t) = \sum_{n=0}^{\infty} \left[a_n(t)\sin n\theta + b_n(t)\cos n\theta\right].$$

$$Q(\theta) = \sum_{n=0}^{\infty} q_n \cos n\theta.$$

Why no sin(θ) term ????

The water is being poured symmetrically from top
(Unlike asymmetric flow which drive the rotation in one side only)

Putting the fourier series in the mass conservation equation,

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right] = -\omega\frac{\partial}{\partial \theta}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right]$$
$$-K\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right] + \sum_{n=0}^{\infty} q_n \cos n\theta$$

Equating the coefficients of sine terms,

$$\dot{a}_n = n\omega b_n - K a_n.$$

Equating the coefficients of cosine terms,

$$\dot{b}_n = -n\omega a_n - K b_n + q_n.$$

Why only sine and cosine terms?

Because sine and cosine are orthogonal in a period

Image credits : Steven Strogatz

Putting the fourier series in the moment balance equation,

$$
\begin{aligned}
I\dot{\omega} &= -v\omega + g\mathrm{R}\int_0^{2\pi}\left[\sum_{n=0}^{\infty} a_n(t)\sin n\theta + b_n(t)\cos n\theta\right]\sin\theta\, d\theta \\
&= -v\omega + g\mathrm{R}\int_0^{2\pi} a_1 \sin^2\theta\, d\theta \\
&= -v\omega + \pi g\mathrm{R} a_1
\end{aligned}
$$

$\forall\ n \geq 2$, the three equations are **decoupled.**

Hence, for $n = 1$,

$$
\begin{aligned}
\dot{a}_1 &= \omega b_1 - K a_1 \\
\dot{b}_1 &= -\omega a_1 - K b_1 + q_1 \\
\dot{\omega} &= (-v\omega + \pi g\mathrm{R} a_1)/I\,.
\end{aligned}
$$

FIXED POINTS

Q. What is fixed point of a system?

$$\dot{x} = 0$$

fixed points represent **equilibrium** (steady)
solutions, means if x= x* initially, then x (t) = x* for all time
Can be both **stable** or **unstable**

Q. Why analyse Fixed Points?

We can analyse the stability of the system about the fixed point by linearizing the equation

Fixed points

$$\dot{a}_1 = 0 \implies a_1 = \frac{\omega}{k} b_1$$
$$\dot{b}_1 = 0 \implies \omega a_1 = q_1 - K b_1$$
$$\dot{\omega} = 0 \implies a_1 = \frac{\nu \omega}{\pi g R}$$

$$\omega^2 = \frac{\pi g R q_1}{\nu} - K^2$$

- $(a_1, b_1, \omega) = (0, q_1/k, 0)$ => Wheel is stationary
- Two solutions, $\pm\,\omega$ (steady rotation in either direction), iff $\dfrac{\pi g R q_1}{K^2 \nu} > 1$

Transforming the variables to x, y and z

$$\sigma = \frac{1}{K}\frac{\nu}{I_{total}}$$

$$\rho = \frac{q_1}{K^2}\frac{\pi R g}{\nu}$$

$$x = \frac{\omega}{K},\ y = \frac{\rho K}{q_1}a_1,\ z = \rho - \frac{\rho K}{q_1}b_1$$

The Lorenz equations

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = \rho x - y - xz$$
$$\dot{z} = xy - bz$$

σ : Prandtl number
ρ : Rayleigh number
b : aspect ratio

Transformation credits : Illing et al., 2012

Two types of fixed points in Lorenz system

1. $(x, y, z) = (0, 0, 0) \quad \forall\, \varrho$

2. $x = y = \pm\sqrt{b(\rho - 1)},\ z = \rho - 1,\ \rho > 1$

c^{+}

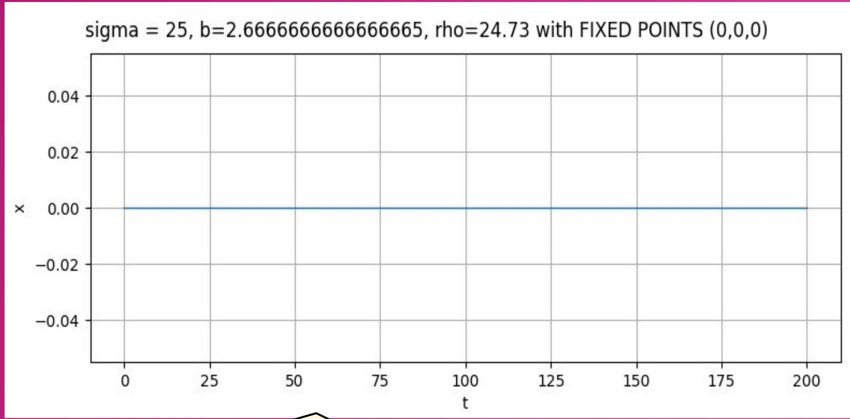
c^{-}

As $\varrho \to 1^{+}$, c^{+} & c^{-} $\to (0,0,0)$

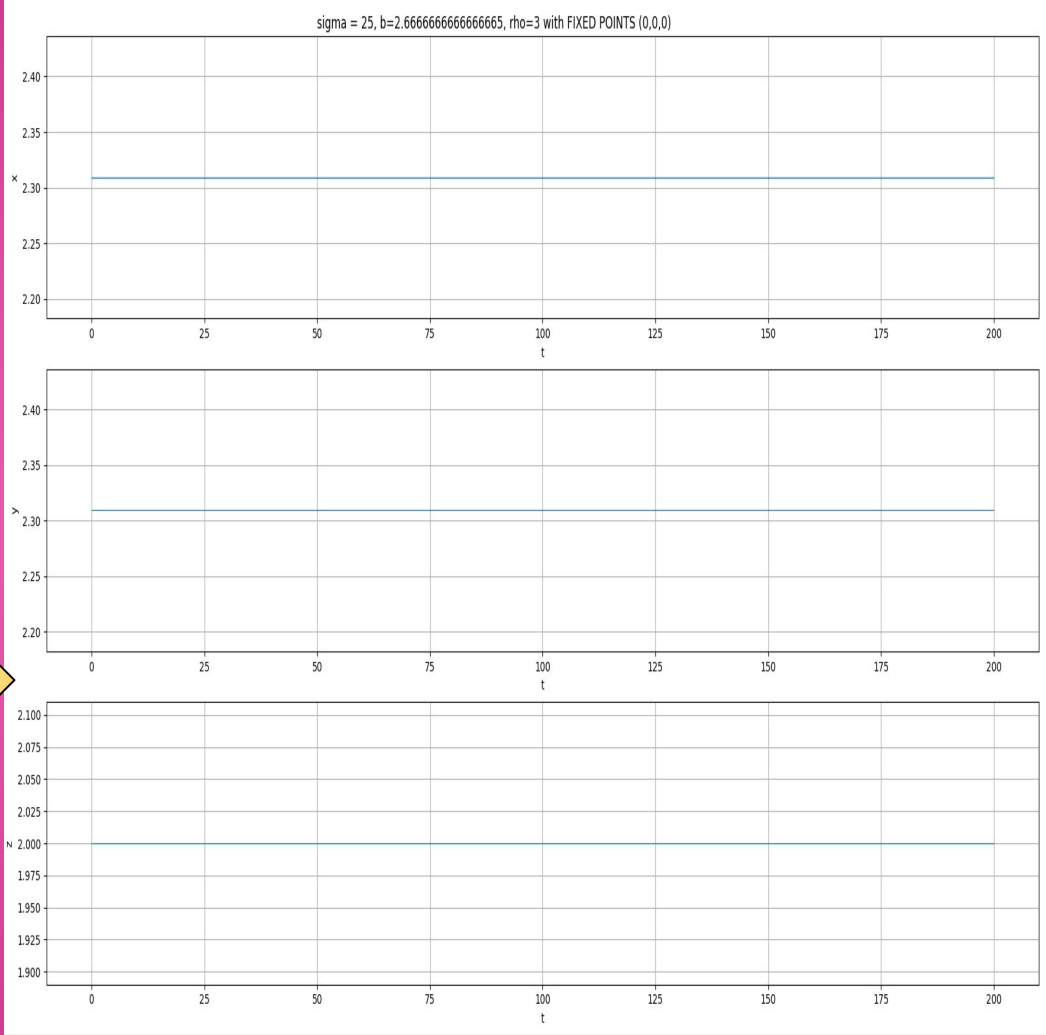
ϱ is ratio of driving force to resistive force

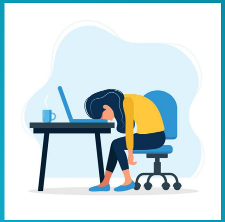
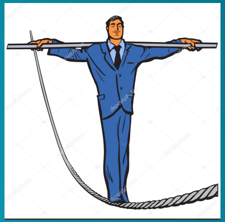
PHYSICAL SANITY CHECK :

- Small ϱ - angular velocity dies to o
- Medium ϱ - angular velocity becomes constant
- Large ϱ - angular velocity (x) shows chaos



Fixed point for different parameters



Let’s check the stability about the fixed points -

Linearise ‘em equations

1. Linearising the system about origin by omitting the ***xy*** and ***xz*** nonlinearities in the Lorentz equations,

$$\begin{aligned}\dot{x} &= \sigma(y - x)\\ \dot{y} &= \rho x - y\\ \dot{z} &= -bz\end{aligned}$$

$$\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix} = \begin{pmatrix}-\sigma & \sigma\\ \text{��} & -1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

Eigenvalues of the system - all negative => Stable fixed point

- all positive => Unstable fixed point
- both positive and negative => Saddle point

Stable in one direction, unstable in other

$$\lambda_1 = -b,$$

$$\lambda_{2,3} = \frac{-\sigma - 1 \pm \sqrt{(\sigma+1)^2 + 4\sigma(\rho - 1)}}{2}$$

For $0<\rho<1$, the system has negative eigenvalues

STABLE F.P.

Credits: Matthew D. Johnston at the University of Wisconsin – Madison : Math 415 (Fall 2014)

FUN FACT : Origin is globally asymptotically stable when **0<ρ<1**

Proved by Lyapunov Function

$$L(x, y, z) = \frac{x^2}{2\sigma} + \frac{y^2}{2} + \frac{z^2}{2}$$

$$\frac{dL}{dt} = \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial y}\frac{dy}{dt} + \frac{\partial L}{\partial z}\frac{dz}{dt}$$

$$= -\left(x - \frac{\rho+1}{2}y\right)^2 - \left(1 - \left(\frac{\rho+1}{2}\right)^2\right)y^2 - bz^2$$

L'(t) < 0 for 0<ρ<1 => Ellipsoid L collapsing to origin eventually

When $\rho>1$: 3 Fixed Points exist, but origin is not stable

For -

$$1 < \rho < \rho_H = \frac{\sigma(\sigma+b+3)}{\sigma-b-1}$$

The 2 fixed points are stable - Proof little complicated :(

We observe a "PITCHFORK" bifurcation at $\rho=1$ with the other two fixed points as c^+ & c^-, and a "SUBCRITICAL HOPF" bifurcation at $\rho=\rho_H$

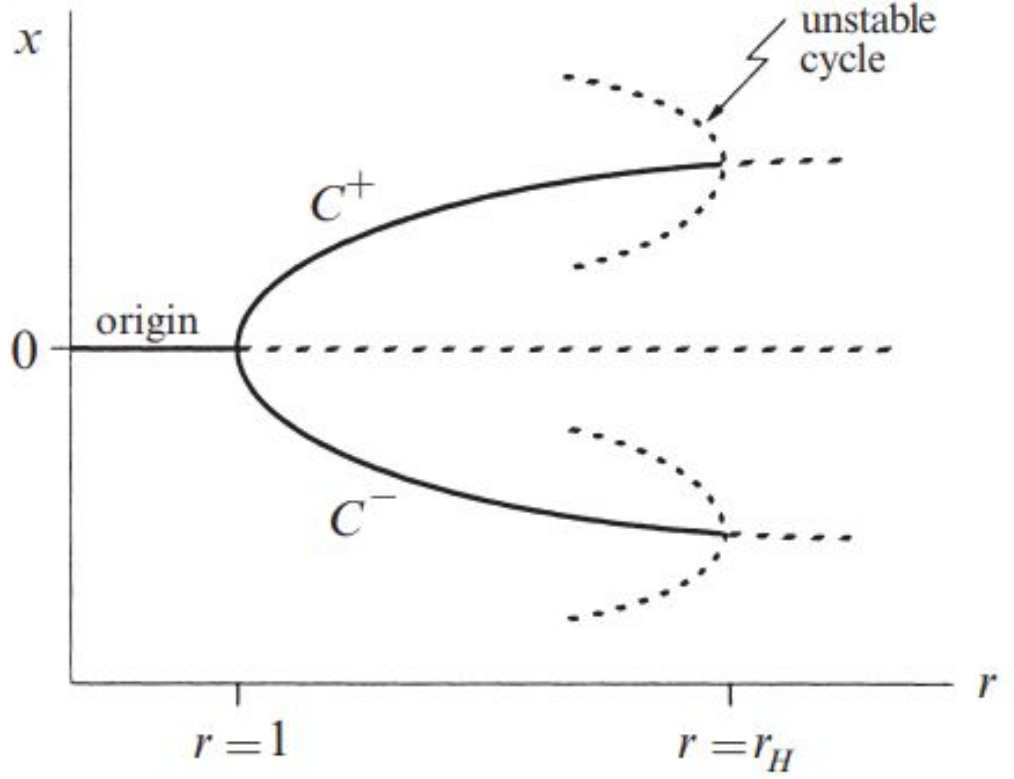
For $\sigma = 10$ and $b=8/3$: standard parameters, $\rho_H = 24.74$

Credits : Hirsch, Morris W.; Smale, Stephen; Devaney, Robert (2003). Differential Equations, Dynamical Systems, & An Introduction to Chaos (Second ed.).

Pitchfork bifurcation vs Hopf bifurcation

- When a stable limit cycle surrounds an unstable equilibrium point: Supercritical Hopf bifurcation ⤀ Periodic behaviour eventually
- Limit cycle is unstable and surrounds a stable equilibrium point : Subcritical Hopf bifurcation
  - Behavior Before Bifurcation - If perturbed, the system might either return to the stable fixed point or diverge away
  - Behavior After Bifurcation - it might exhibit large-amplitude oscillations or be drawn to a distant attractor

- Pitchfork bifurcation - A system's fixed point changes stability, and new fixed points emerge



x
unstable
cycle
C^+
origin
0
C^-
r
$r = 1$
$r = r_H$

When $\rho>\rho_H$: Lose the stability of these 2 points also

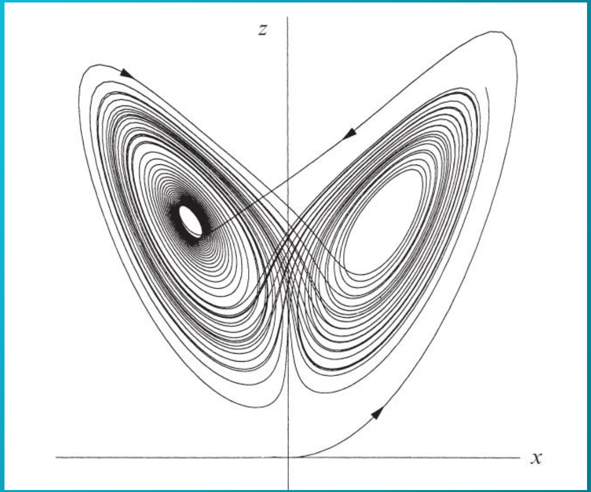
Where does the solution goes now?

- Stable Fixed Point ? NO
- Remain unbounded ? NO
- Move towards a stable limit cycle? NO

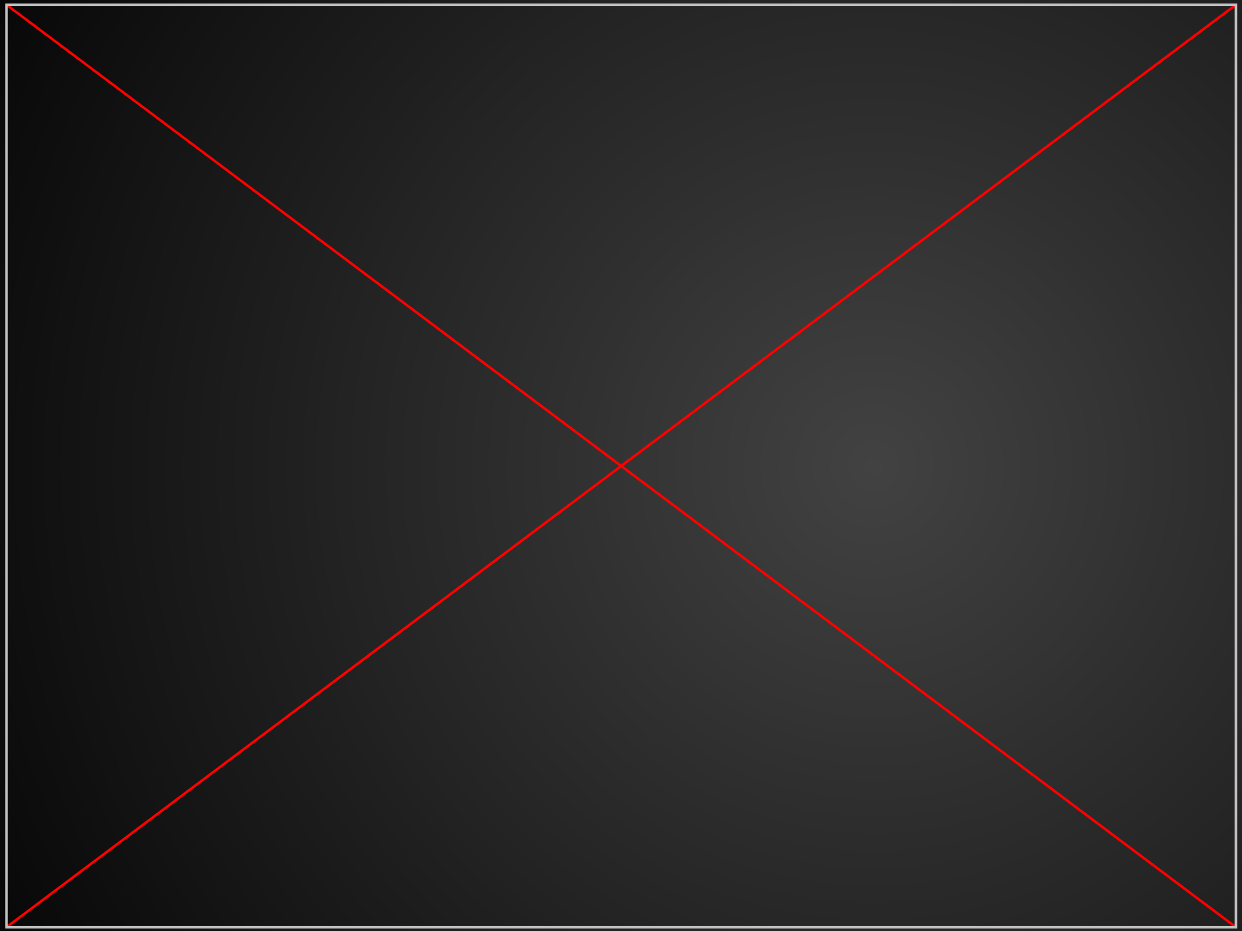
IT GOES TO A **STRANGE DISTANT ATTRACTOR**

STRANGE ATTRACTOR

- After removing initial transients, the solution settles into an irregular/aperiodic oscillation that persists as t ↣ ∞
- A wonderful structure emerges if the solution is visualized as a trajectory in state space
- After the hopf bifurcation, if ρ is increased further, It enters into a strange distant butterfly attractor.
- The number of rotations made on either side is not fixed and may vary on the conditions




IT’S TIME FOR SIMULATION



$\sigma = 10$

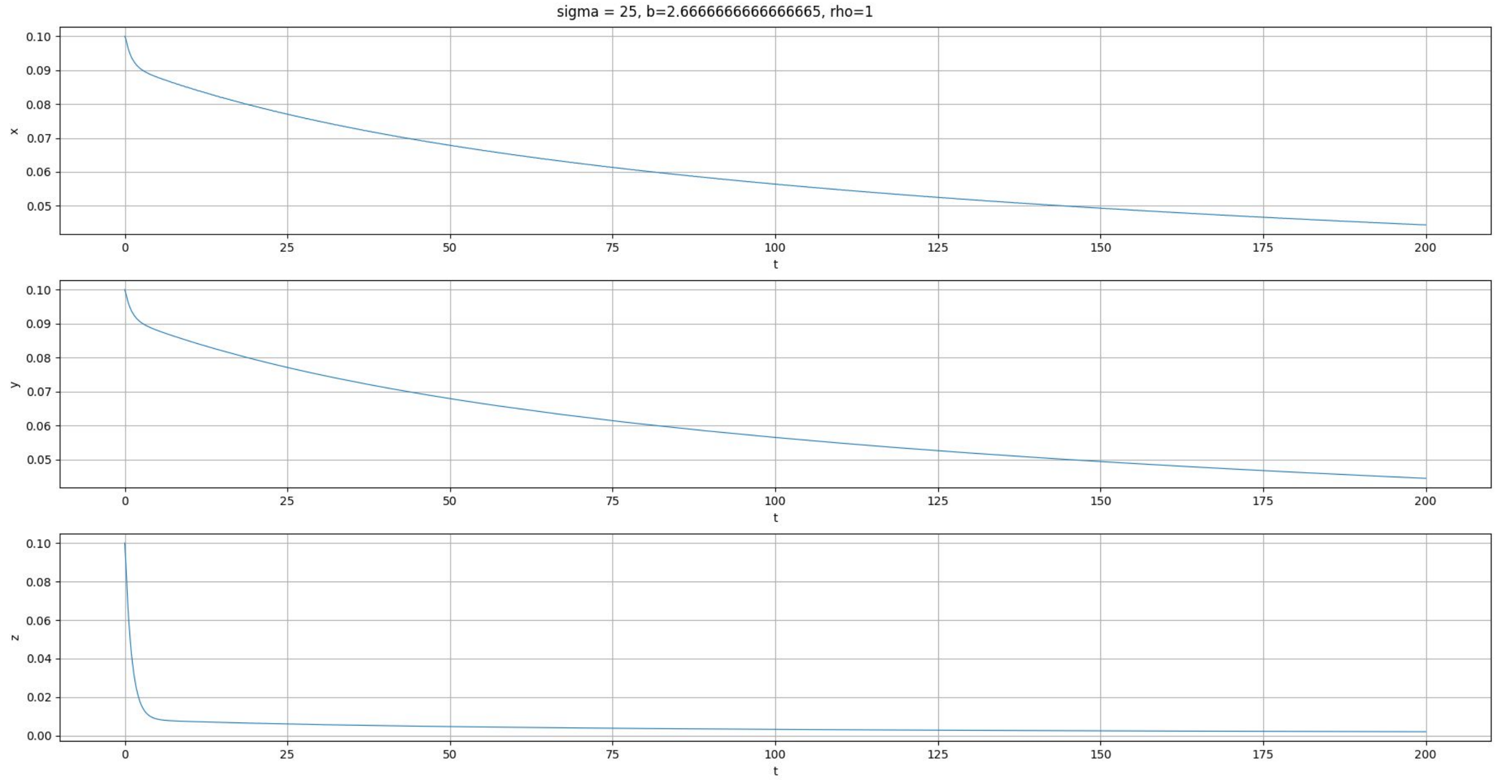
$\rho = 28$

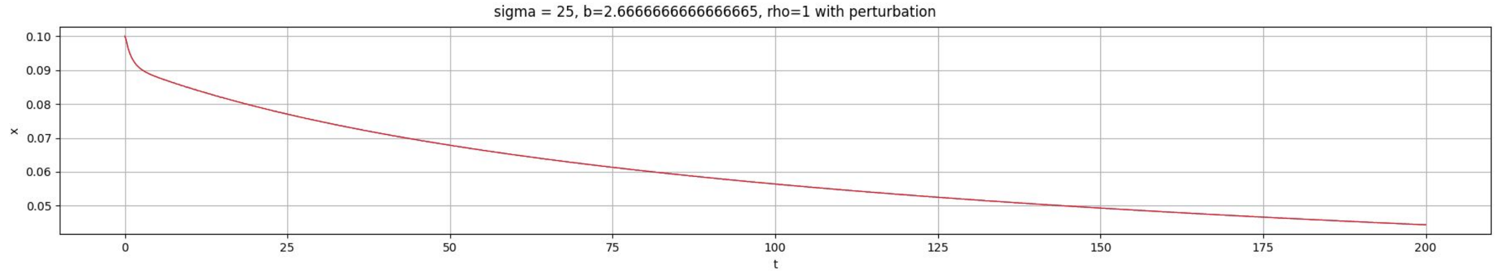
$b = 8/3$

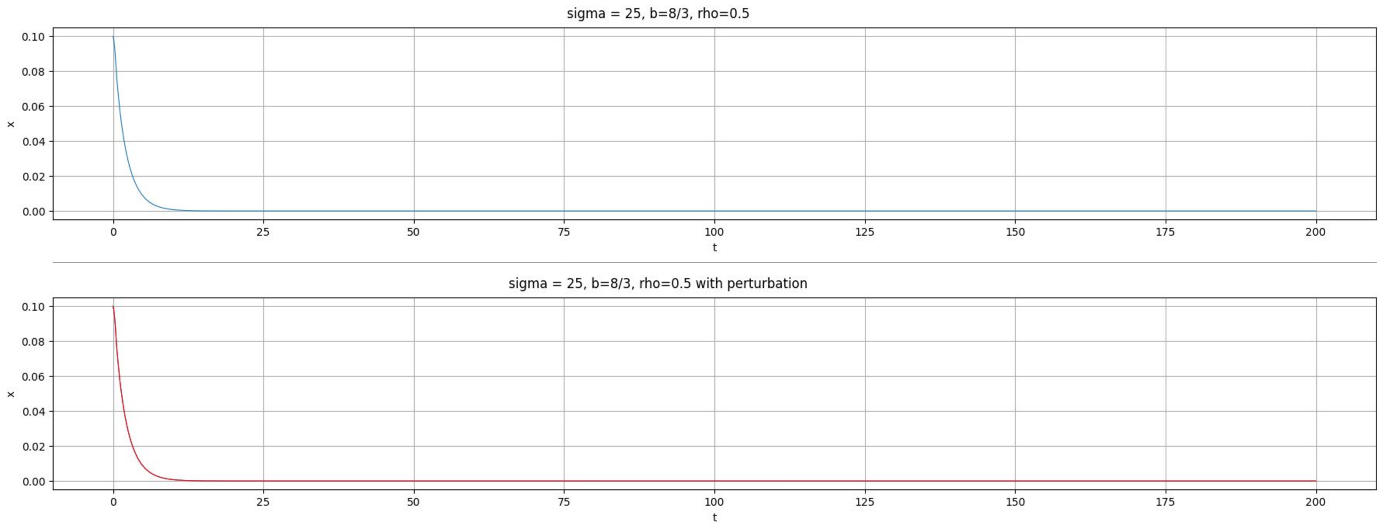
Red - `[1.0,1.0,1.0]`

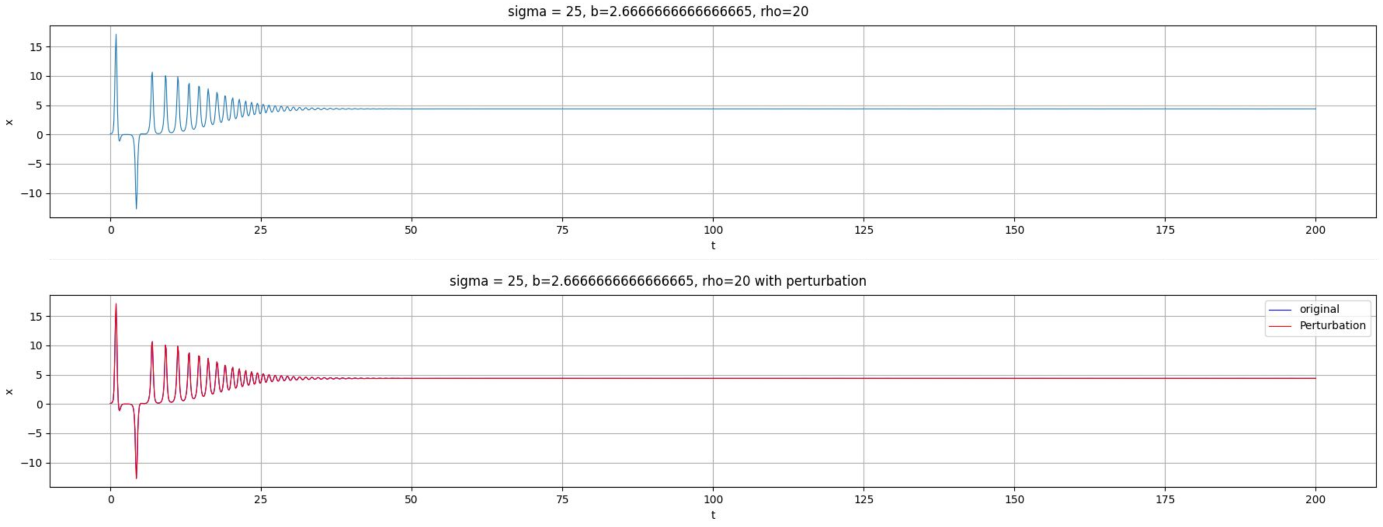
Blue - `[1.1,1.0,1.0]`

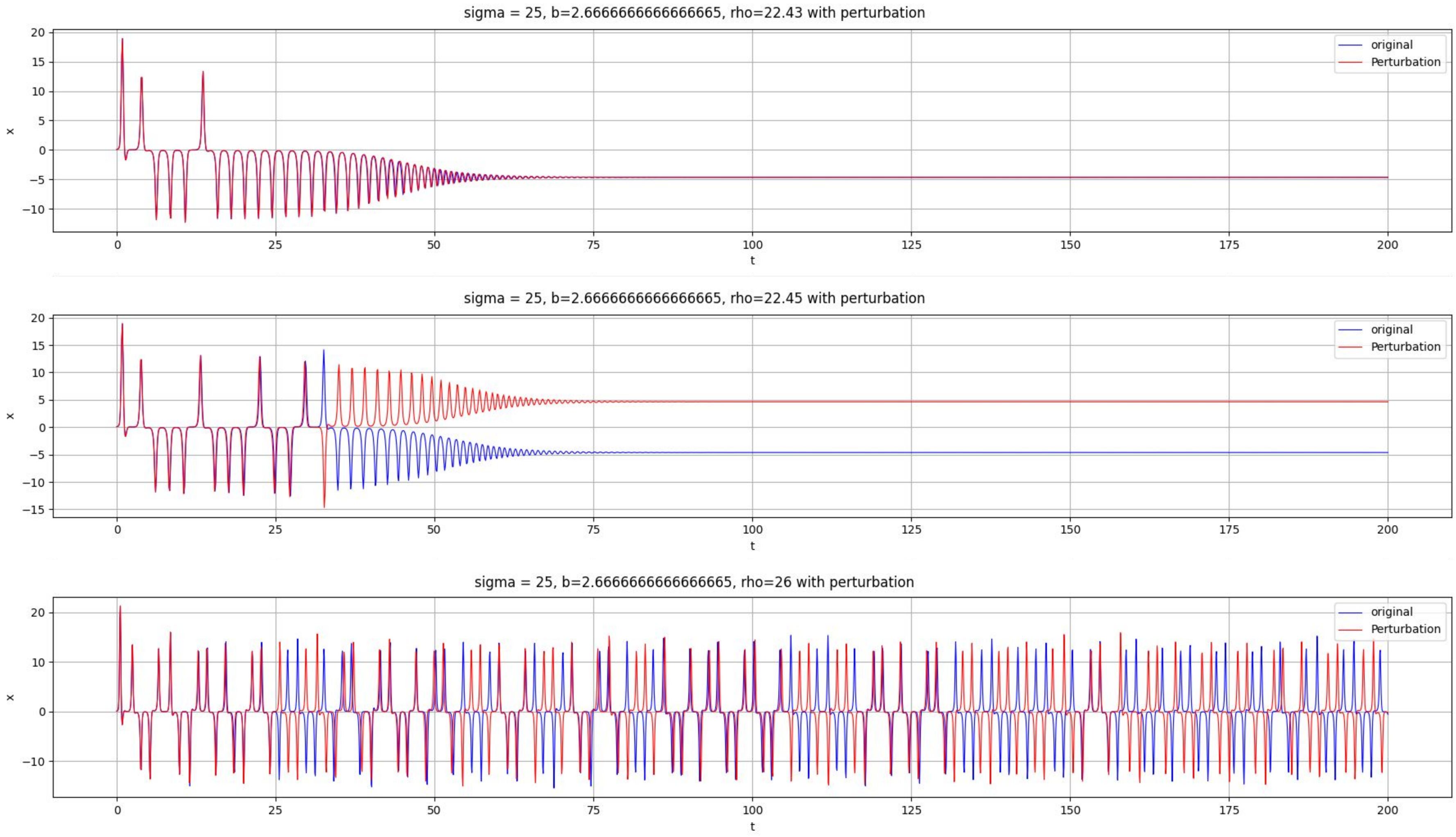
Green - `[1.0,1.1,1.0]`


sigma = 25, b=2.6666666666666665, rho=1
x
y
z
t
0
25
50
75
100
125
150
175
200
0.10
0.09
0.08
0.07
0.06
0.05
0.10
0.08
0.06
0.04
0.02
0.00

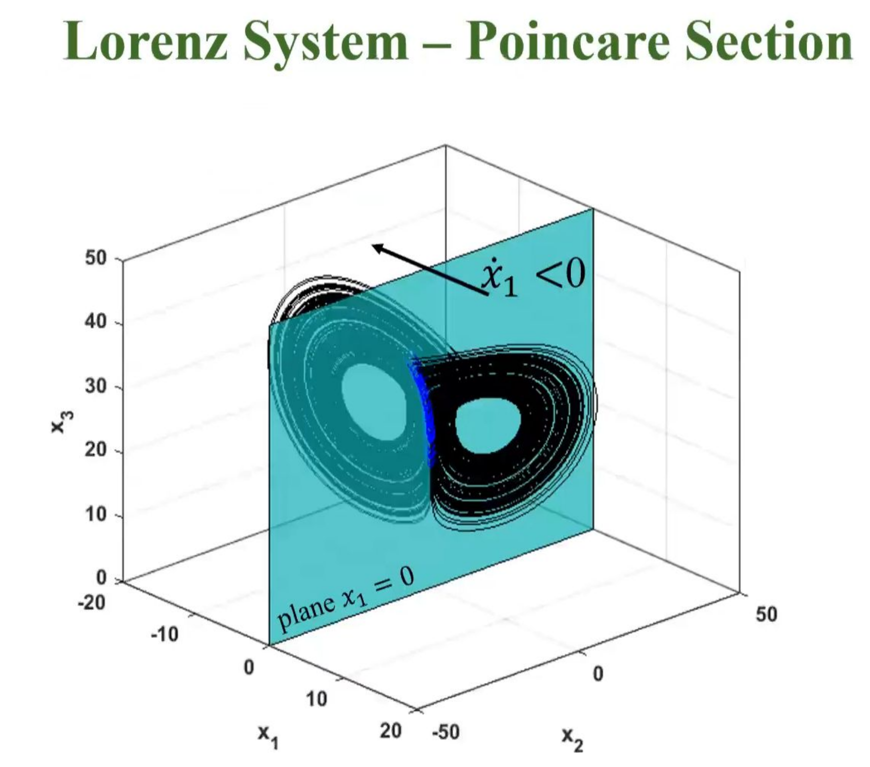

sigma = 25, b=2.6666666666666665, rho=1 with perturbation
x
t
0
25
50
75
100
125
150
175
200
0.10
0.09
0.08
0.07
0.06
0.05

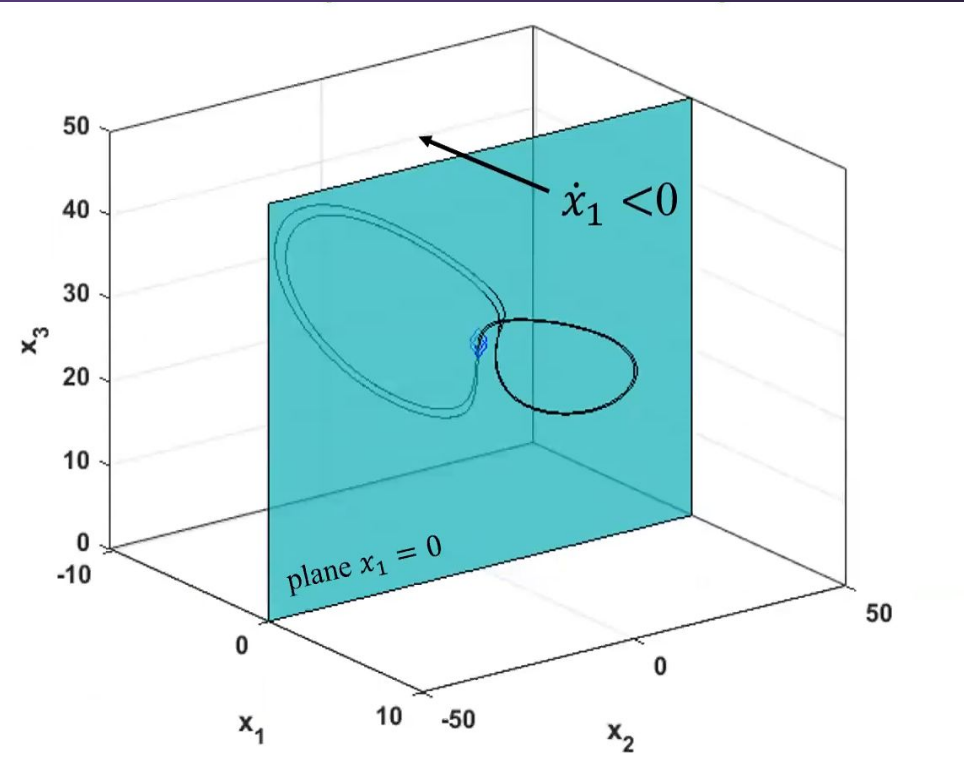

sigma = 25, b=8/3, rho=0.5
x
t
sigma = 25, b=8/3, rho=0.5 with perturbation
x
t

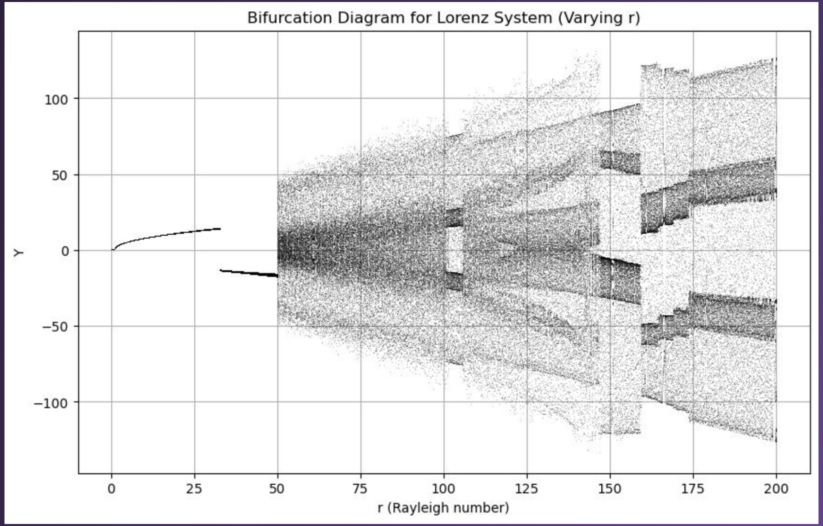

sigma = 25, b=2.6666666666666665, rho=20
x
t
sigma = 25, b=2.6666666666666665, rho=20 with perturbation
original
Perturbation
x
t


sigma = 25, b=2.6666666666666665, rho=22.43 with perturbation
original
Perturbation
x
t
sigma = 25, b=2.6666666666666665, rho=22.45 with perturbation
original
Perturbation
x
t
sigma = 25, b=2.6666666666666665, rho=26 with perturbation
original
Perturbation
x
t

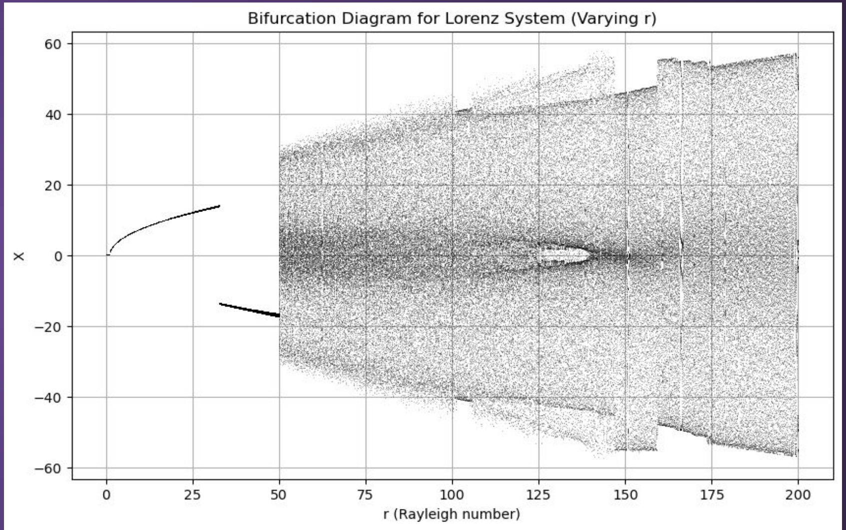
Poincare Cut:

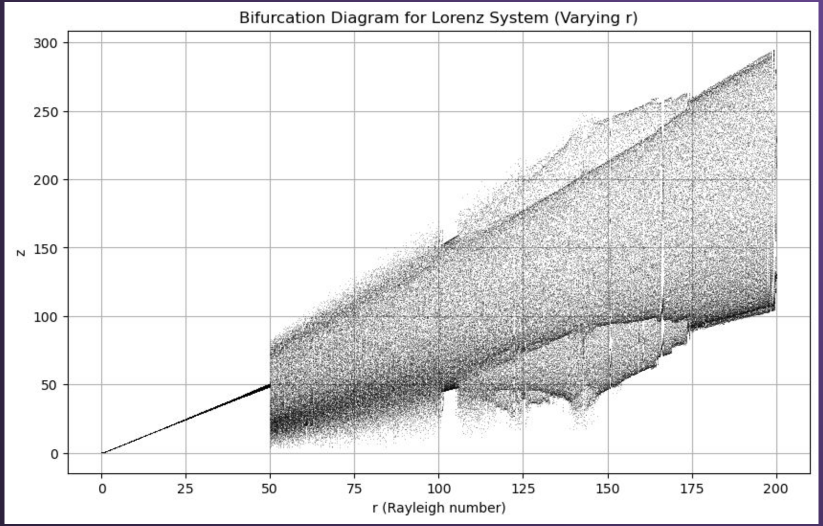






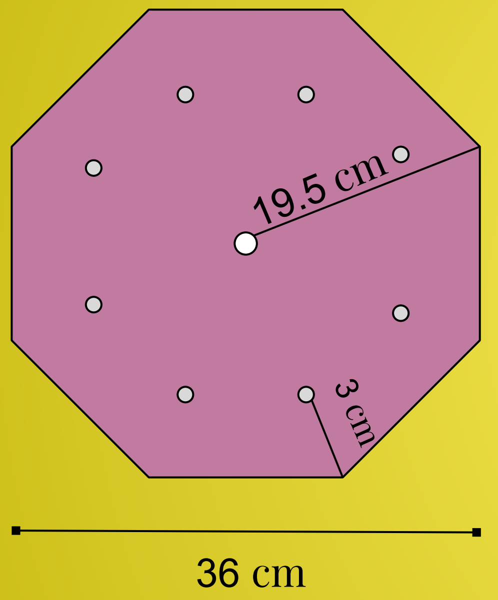
σ=10, b = 8/3, ρ_H = 63.33
Point of chaos infliction



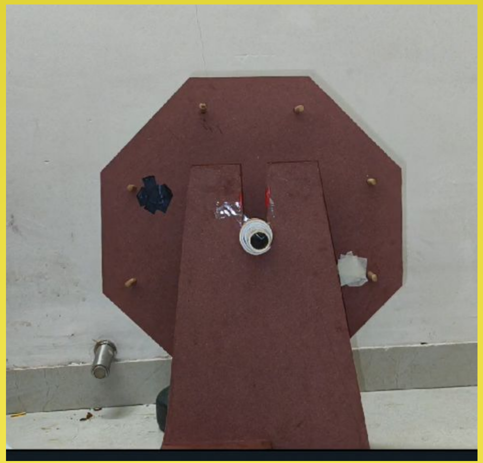



IT'S TIME FOR
EXPERIMENTS

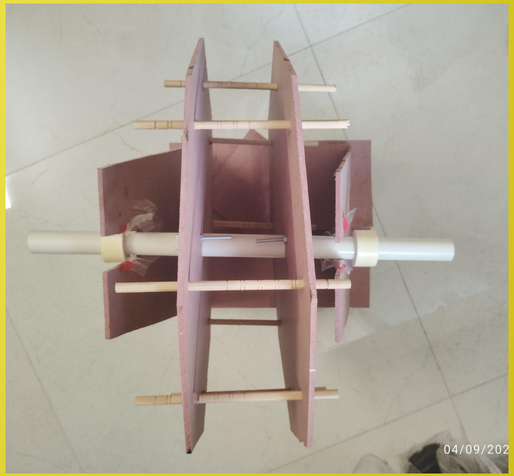
Setup Explanation



Wheel dimensions (not drawn to scale)



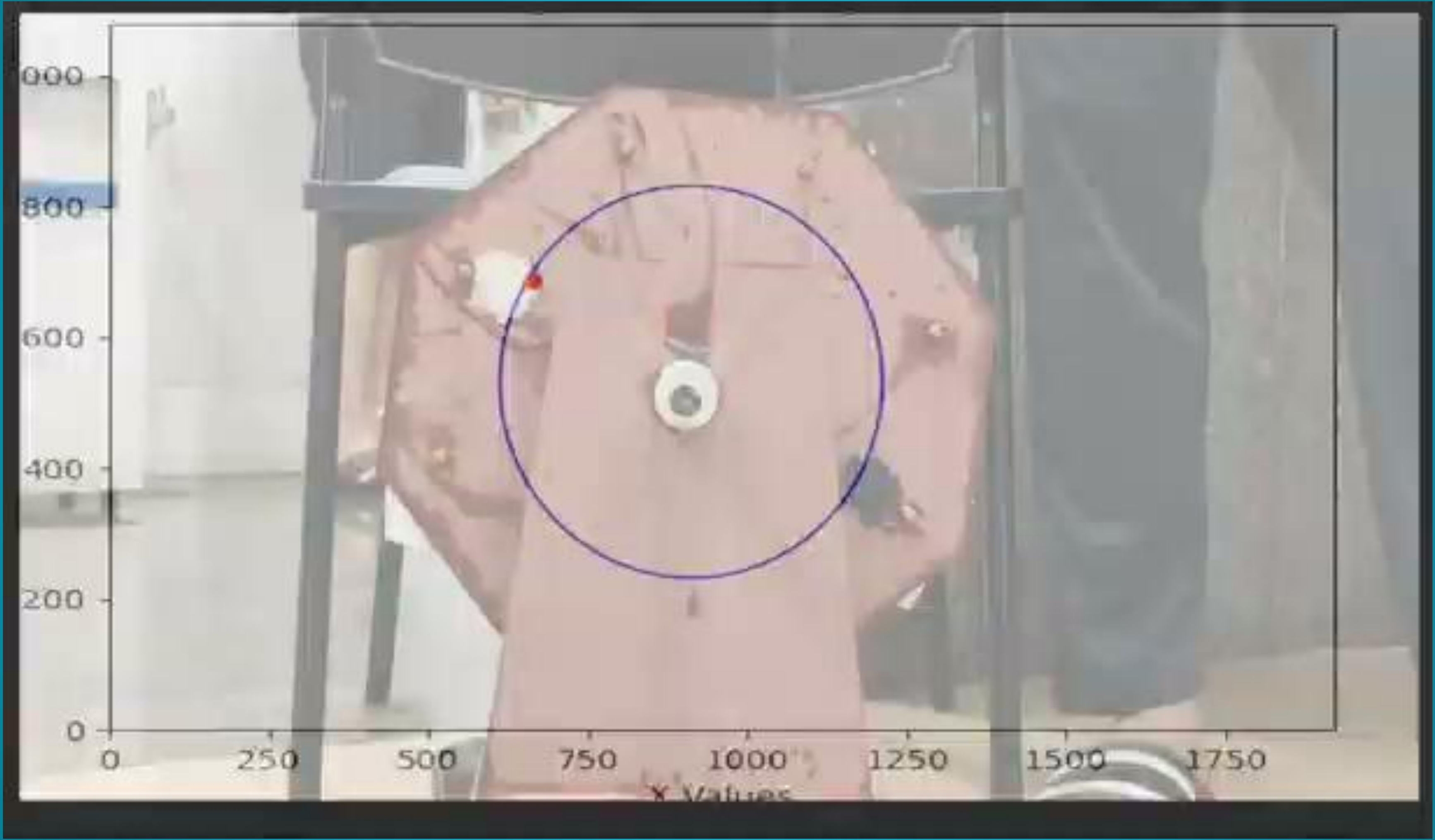
Front view of setup



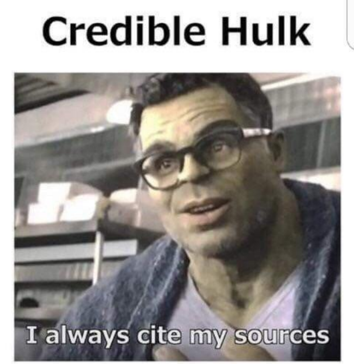
Top view of setup

Properties	Value
Material used for wheel	MDF
Density	600-800 kg/m3
Thickness of wheel	4mm
Moment of Inertia of wheel + water	0.027255 kg-m2
Damping coefficient	0.0092669

Experimental Condition	Value
Inflow rate	0.1754 kg/sec
q1 [4]	1.67×10^{-4} kg/sec
Leakage coefficient	0.008793 1/sec


000
800
600
400
200
0
0
250
500
750
1000
1250
1500
1750
X Values

Angular velocity vs frames:



References :-

1. *Strogatz, Steven, author. (2015). Nonlinear dynamics and chaos : with applications to physics, biology, chemistry, and engineering.*
2. *Lucas Illing, Rachel F. Fordyce, Alison M. Saunders, Robert Ormond; Experiments with a Malkus–Lorenz water wheel: Chaos and Synchronization . Am. J. Phys. 1 March 2012; 80 (3): 192–202.*
3. *Hirsch, Morris W.; Smale, Stephen; Devaney, Robert (2003). Differential Equations, Dynamical Systems, & An Introduction to Chaos (Second ed.). Boston, MA: Academic Press. ISBN 978-0-12-349703-1.*
4. *Lucas Illing, Rachel F. Fordyce, Alison M. Saunders, Robert Ormond; Experiments with a Malkus–Lorenz water wheel: Chaos and Synchronization . Am. J. Phys. 1 March 2012; 80 (3): 192–202.*

Behind the scenes :

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