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A lot of papers describes the definition and the properties of moving average and linear filters (see for example (Ladiray 2018)). He we summarize some of the main results.

Let p et f two integers, a moving average M_θ or M is defined by a set of coefficients $\theta = (\theta_{-p},\dots,\theta_f)'$ such as:

$$M(X_t) = \sum_{k=-p}^{+f} \theta_k X_{t+k}$$

- $p+f+1$ is called the *moving average order*.
- When $p=f$ the moving average is said to be *centered*. If we also have $\forall k:\ \theta_{-k} = \theta_k$, the moving average M_θ is said to be *symmetric*. In this case, the quantity $2h+1$ is called the *bandwith*.

2.1 Gain and phase shift functions

Let $X_t = \mathrm{e}^{i\omega t}$, the result of the moving average M_θ in X_t is:

$$Y_t = M_\theta X_t = \sum_{k=-p}^{+f} \theta_k \,\mathrm{e}^{i\omega(t+k)} = \left(\sum_{k=-p}^{+f} \theta_k \,\mathrm{e}^{i\omega k}\right) \cdot X_t.$$

The function $\Gamma_\theta(\omega) = \sum_{k=-p}^{+f} \theta_k e^{i\omega k}$ is called the *transfer function*. It can be rewritten as:

$$\Gamma_\theta(\omega) = G_\theta(\omega)\,\mathrm{e}^{-i\Phi_\theta(\omega)}$$

where $G_\theta(\omega) = |\Gamma_\theta(\omega)|$ is the *gain* or *amplitude* function and $\Phi_\theta(\omega)$ is the *phase shift* or *time shift* function[1]. For all symmetric moving average we have $\Phi_\theta(\omega) = 0$.

To sum up, applying a moving average to an harmonic times series affects in in two different ways :

- by multiplying it by an amplitude coefficient $G_\theta(\omega)$;
- by "shifting" it in time by $\Phi_\theta(\omega)/\omega$, which directly affects the detection of turning points[2].

[1] This function is sometimes represented as $\phi_\theta(\omega) = \frac{\Phi_\theta(\omega)}{\omega}$ to mesure the phase shift in number of periods.

[2] When $\Phi_\theta(\omega)/\omega > 0$ the time shift is positive: a turning point is detected with delay.

2.2 Desirable properties of a moving average

The moving average are often constructed under some specific constraints. In the report we will focus on two constraints:

- the preservation of certain kind of trends;
- the variance reduction.

2.2.1 Trend preservation

Is is often desirable for a moving average to conserve certain kind of trends. A moving average M_θ conserve a function of the time $f(t)$ if $\forall t : M_\theta f(t) = f(t)$.

We have the following properties for the moving average M_θ:

- To conserve a constant series $X_t = a$ we need

$$\forall t : M_\theta(X_t) = \sum_{k=-p}^{+f} \theta_k X_{t+k} = \sum_{k=-p}^{+f} \theta_k a = a \sum_{k=-p}^{+f} \theta_k = a$$

  the sum of the coefficients of the moving average $\sum_{k=-p}^{+f} \theta_k$ must then be equal to 1.

- To conserve a linear trend $X_t = at + b$ we need:

$$\forall t : M_\theta(X_t) = \sum_{k=-p}^{+f} \theta_k X_{t+k} = \sum_{k=-p}^{+f} \theta_k [a(t+k)+b] = at \sum_{k=-p}^{+f} k\theta_k + b \sum_{k=-p}^{+f} \theta_k = at + b$$

  which is equivalent to:

$$\begin{cases} \sum_{k=-p}^{+f} \theta_k & = 1 \\ \sum_{k=-p}^{+f} k\theta_k & = 0 \end{cases}$$

- In general, it can be shown that M_θ conserve a polynomial of degree d if and only if:

$$\sum_{k=-p}^{+f} \theta_k = 1 \text{ and } \forall j \in [\![1, d]\!] : \sum_{k=-p}^{+f} k^j \theta_k = 0$$

2.2.2 Variance reduction

All time series are affected by noise that can blur the signal extraction. Hence, we seek to reduce the variance of the noise. The sum of the sum of the squares of the coefficients $\sum_{k=-p}^{+f} \theta_k^2$ is the *variance reduction* ratio.

Indeed, let $\{\varepsilon_t\}$ a sequence of independent random variables with$\mathbb{E}\left[\varepsilon_t\right]=0\mathbb{E}$, $\mathbb{V}\left[\varepsilon_t\right]=\sigma^2$.

$$\mathbb{V}\left[M_\theta \varepsilon_t\right]=\mathbb{V}\left[\sum_{k=-p}^{+f}\theta_k\varepsilon_{t+k}\right]=\sum_{k=-p}^{+f}\theta_k^2\mathbb{V}\left[\varepsilon_{t+k}\right]=\sigma^2\sum_{k=-p}^{+f}\theta_k^2$$

2.2.3 Quality indicators (?)

To compare the different moving average / filters the following indicators are used:

$$\begin{cases}\text{constant bias} = \text{bias0} & =\sum_{k=-p}^{+f}\theta_k\\ \text{linear bias} = \text{bias1} & =\sum_{k=-p}^{+f}k\theta_k\\ \text{quadratic bias} = \text{bias2} & =\sum_{k=-p}^{+f}k^2\theta_k\\ \text{variance reduction} & =\sum_{k=-p}^{+f}\theta_k^2\end{cases}$$

3 Local polynomial filters

In this section we detail the filters that arised from fitting a local polynomial to our time series as described by (Proietti and Luati 2008).

We assume that our time series y_t can be decomposed as

$$y_t=\mu_t+\varepsilon_t$$

where μ_t is the signal (trend) and $\varepsilon_t\overset{i.i.d}{\sim}\mathcal{N}(0,\sigma^2)$ is the noise. We assume that μ_t can be locally approximated by a polynomial of degree d of the time t between y_t and the neighboring observations $\left(y_{t+j}\right)_{j\in[\![-h,h]\!]}$. Then $\mu_t\simeq m_t$ with:

$$\forall j\in[\![-h,h]\!]: y_{t+j}=m_{t+j}+\varepsilon_{t+j},\quad m_{t+j}=\sum_{i=0}^{d}\beta_i j^i$$

This signal extraction problem is then equivalent to the estimation of $m_t=\beta_0$. In matrix notation we can write:

$$\underbrace{\begin{pmatrix} y_{t-h} \\ y_{t-(h-1)} \\ \vdots \\ y_t \\ \vdots \\ y_{t+(h-1)} \\ y_{t+h} \end{pmatrix}}_{y} = \underbrace{\begin{pmatrix} 1 & -h & h^2 & \cdots & (-h)^d \\ 1 & -(h-1) & (h-1)^2 & \cdots & (-(h-1))^d \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & h-1 & (h-1)^2 & \cdots & (h-1)^d \\ 1 & h & h^2 & \cdots & h^d \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \beta_d \end{pmatrix}}_{\beta} + \underbrace{\begin{pmatrix} \varepsilon_{t-h} \\ \varepsilon_{t-(h-1)} \\ \vdots \\ \varepsilon_t \\ \vdots \\ \varepsilon_{t+(h-1)} \\ \varepsilon_{t+h} \end{pmatrix}}_{\varepsilon}$$

Two parameters are crucial in determining the accuracy of the approximation:

- the degree d of the polynomial;
- the number of neighbored, *bandwith* $H = 2h + 1$.

In order to estimate β we need $H \geq d + 1$ and the estimation is done by the weighted least squares (WLS), which consists of minimizing the following objective function:

$$S(\hat{\beta}_0, \dots, \hat{\beta}_d) = \sum_{j=-h}^{h} \kappa_j (y_{t+j} - \hat{\beta}_0 - \hat{\beta}_1 j - \cdots - \hat{\beta}_d j^d)^2$$

where κ_j is a set of weights called *kernel*. We have $\kappa_j \geq 0$, $\kappa_{-j} = \kappa_j$ and with $K = diag(\kappa_{-h}, \dots, \kappa_h)$, the estimate of β can be written as $\hat{\beta} = (X'KX)^1 X'Ky$. With $e_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}'$, the estimate of the trend is:

$$\hat{m}_t = e_1 \hat{\beta} = w'y = \sum_{j=-h}^{h} w_j y_{t-j} \text{ with } w = KX(X'KX)^{-1} e_1$$

To conclude, the estimate of the trend $\hat{m}_t$ can be obtained applying the symmetric filter w to y_t[3]. Moreover, $X'w = e_1$ so:

$$\sum_{j=-h}^{h} w_j = 1, \quad \forall r \in [\![1, d]\!] : \sum_{j=-h}^{h} j^r w_j = 0$$

Hence, the filter w preserve deterministic polynomial of order d.

[3] Due to the symmetry of the kernel weights κ_j.

3.1 Different kernels

In signal extraction, we generally look for weighting observations according to their distance from time t: this is the role of the kernel function. For that, we introduce a kernel function κ_j, $j=0,\pm1,\dots,\pm h$ with $\kappa_j\geq 0$ and $\kappa_j=\kappa_{-j}$. An important class of kernels is the Beta kernels. In the discrete, up to a proportional factor (so that $\sum_{j=-h}^h\kappa_j=1$):

$$\kappa_j=\left(1-\left|\frac{j}{h+1}\right|^r\right)^s$$

with $r>0$, $s\geq 0$. The following kernels are considered in this report:

- $r=1,s=0$ uniform kernel:
$$\kappa_j^U=1$$
- $r=s=1$ triangle kernel:
$$\kappa_j^T=\left(1-\left|\frac{j}{h+1}\right|\right)$$
- $r=2,s=1$ Epanechnikov (or Parabolic) kernel:
$$\kappa_j^E=\left(1-\left|\frac{j}{h+1}\right|^2\right)$$
- $r=s=2$ biweight kernel:
$$\kappa_j^{BW}=\left(1-\left|\frac{j}{h+1}\right|^2\right)^2$$
- $r=2,s=3$ triweight kernel:
$$\kappa_j^{TW}=\left(1-\left|\frac{j}{h+1}\right|^2\right)^3$$
- $r=s=3$ tricube kernel:
$$\kappa_j^{TC}=\left(1-\left|\frac{j}{h+1}\right|^3\right)^3$$
- Gaussian kernel[4]:
$$\kappa_j^G=\exp\left(-\frac{j^2}{4h}\right)$$
- Henderson kernel (see section 3.1.1 for its construction):
$$\kappa_j=\left[1-\frac{j^2}{(h+1)^2}\right]\left[1-\frac{j^2}{(h+2)^2}\right]\left[1-\frac{j^2}{(h+3)^2}\right]$$
- Trapezoidal kernel:
$$\kappa_j^{TP}=\begin{cases}\frac{1}{3(2h-1)} & \text{if } j=\pm h\\ \frac{2}{3(2h-1)} & \text{if } j=\pm(h-1)\\ \frac{1}{2h-1} & \text{otherwise}\end{cases}$$

The figure 1 summarises the coefficients of the different kernels. Analysing the coefficients we can already anticipate some properties of the associated filters:

[4]The gaussian kernel is generally defined as $\mathrm{e}^{-\frac{x^2}{2\sigma^2}}$. Here we take $\sigma^2=2/h$ with the following idea: the biggest h is, the narrowest the filter is.

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- For h high:
  - the Henderson kernel is equivalent to the triweight kernel (since $h+1 \sim h+2 \sim h+3$, $\kappa_j^H \sim \kappa_j^{TW}$)

  A un horizon plus court, trapezoidal pas très loin triweight. Si on augmente h, Henderson devrait être équivalent que le triweight.

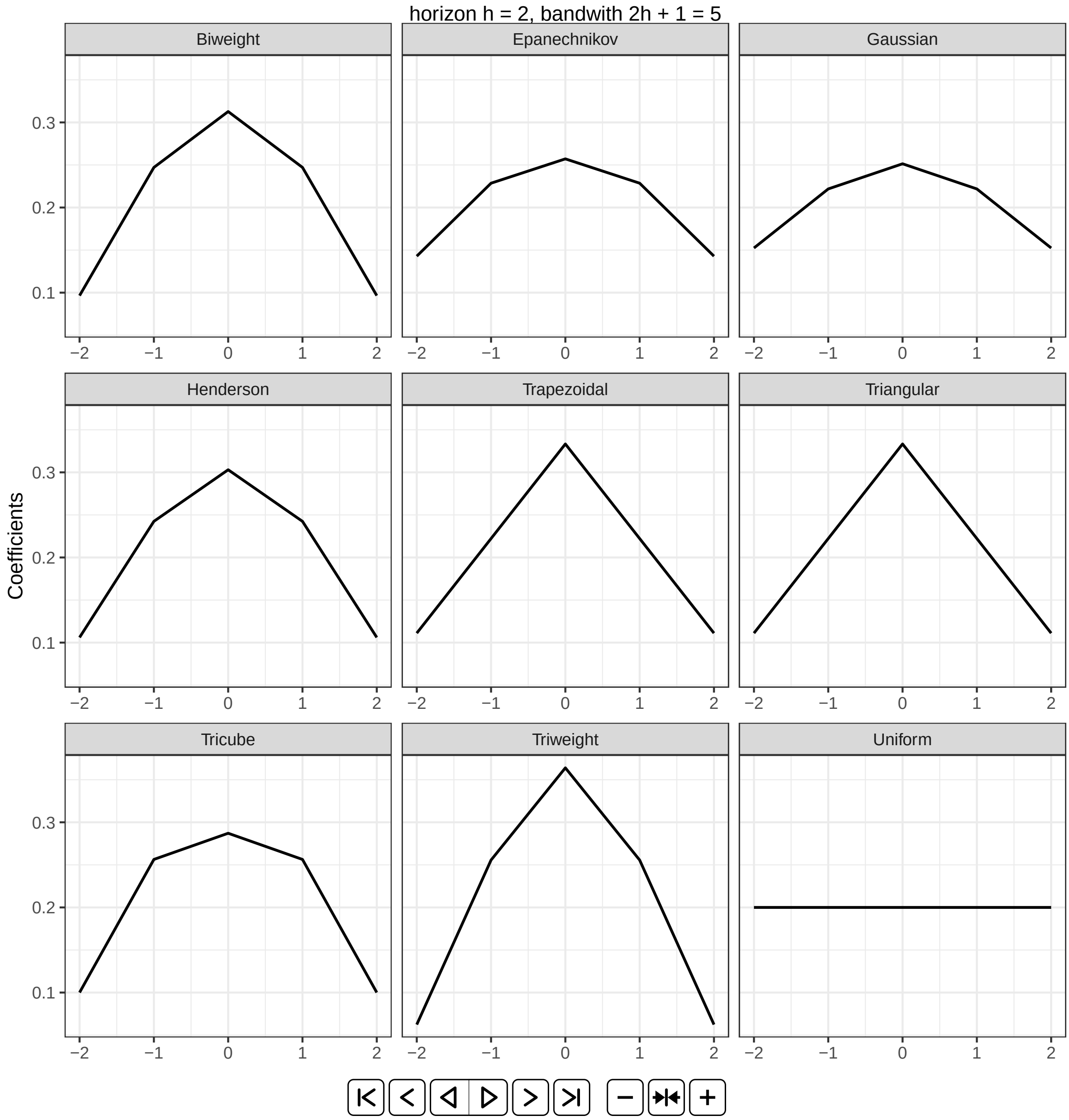

Figure 1: Coefficients of the different kernels for h from 2 to 80.

Note: to see the animation the PDF must be open with Acrobat Reader, KDE Okular, PDF-XChange or Foxit Reader. Otherwise you will only be able to see the results for $h = 2$.

3.1.1 Specific symmetric filters

When $p=0$ (local adjustment by a constant) we obtain the Nadaraya-Watson's estimator.

With the uniform kernel we obtain the Macaulay's filter. When $p=0,1$, this is the arithmetic moving average: $w_j = w = \frac{1}{2h+1}$.

Epanechnikov utilise le filtre suivant :

$$\kappa_j^E = \left(1 - \left|\frac{j}{h+1}\right|^2\right)$$

TCe noyau est souvent recommandé comme le noyau optimal minimisant la moyenne quadratique d'une estimation par polynôme local.

LOESS is a local polynomial regression using a tricube kernel. Sa particularité est qu'il utilise la bande passante du plus proche voisin.

Le filtre de Henderson est un cas particulier de l'ajustement cubique $p=3$. Henderson a cherché la fonction de poids qui donnait l'estimation la plus lisse de la tendance pour h fixé et $p=3$. Le critère de lissage utilisé par Henderson est basé sur la variance des différences d'ordre 3 des estimations de la tendances Le critère de lissage adopté par Henderson est basé sur la variance des troisièmes différences des estimations de la tendance. Plus la variance est petite, plus le degré de lissage est élevé, car l'accélération de la tendance est sujette à moins de variations.

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Proietti, Tommaso, and Alessandra Luati. 2008. "Real Time Estimation in Local Polynomial Regression, with Application to Trend-Cycle Analysis." *Ann. Appl. Stat.* 2 (4): 1523–53. https://doi.org/10.1214/08-AOAS195.

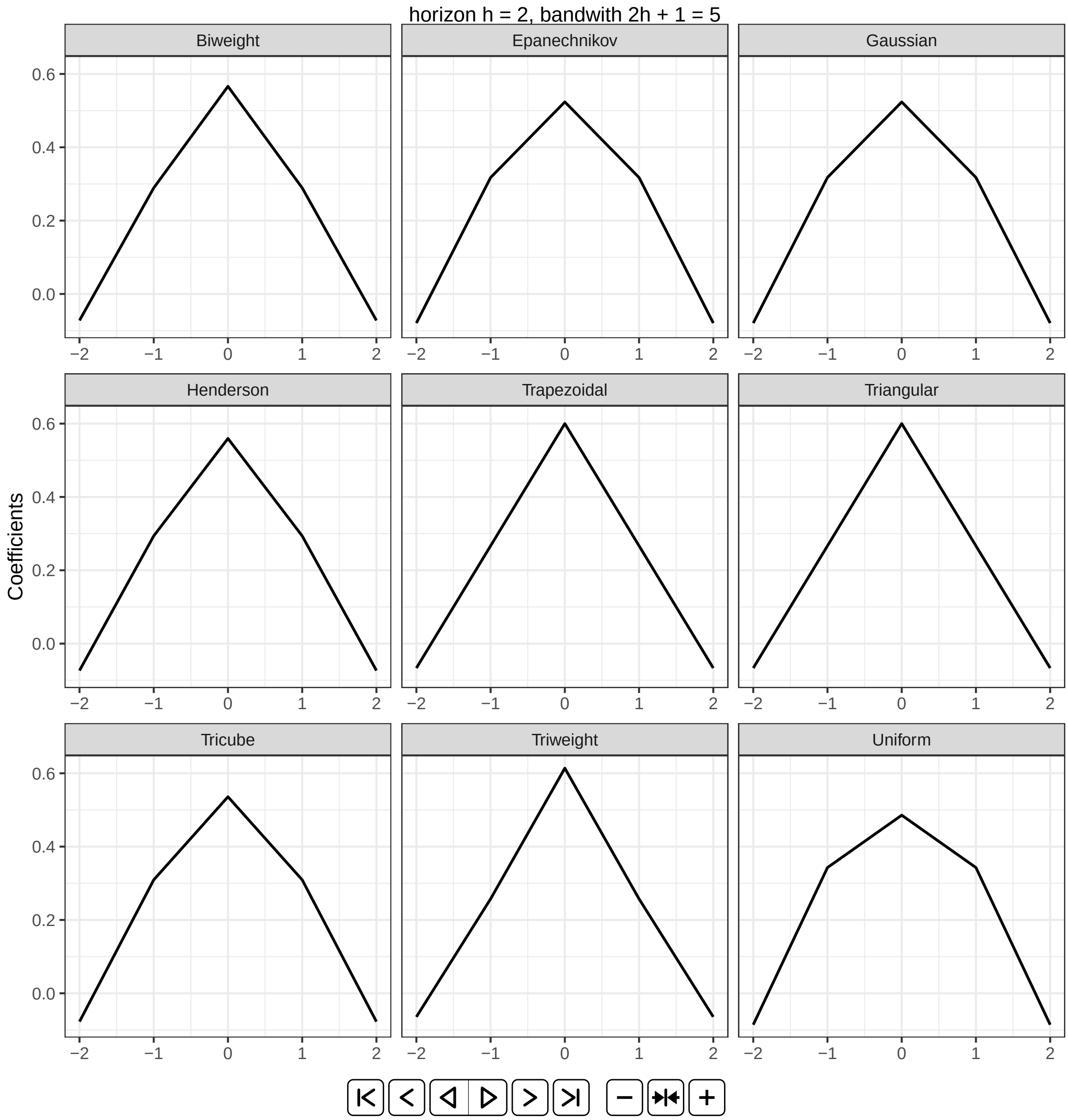

Figure 2: Coefficients of symmetric filters associated to the differents kernels for h from 2 to 80.
Note: to see the animation the PDF must be open with Acrobat Reader, KDE Okular, PDF-XChange or Foxit Reader. Otherwise you will only be able to see the results for $h = 2$.