Induction

1. Define your logical predicate P(n).
2. Prove that P(0) is true.
3. Prove that P(n) implies P(n+1).
   –often done by proving P(n) assuming all P(1), P(2), ..., P(n-1) are true
   –these are termed weak and strong induction, respectively

Problem 1

Prove that for all n in the nonnegative integers,

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

Solution

For the base case, our $P(0)$ is really $P(1)$. The above formula is somewhat ill-defined, but let's assume it represents the sum from 1 to n. In that case, $P(n = 1)$ can be proved trivially:

$$1 = \frac{1(1+1)}{2} 1 = 1$$

For the case of $P(n)$, we assume $P(n)$ to be true (inductive hypothesis), and directly compute $P(n+1) = P(n) + (n+1)$ as

$$\begin{aligned} \text{P(n+1)} \quad &= \frac{n(n+1)}{2} + (n+1) \\ &= \frac{n^2 + n + 2n + 2}{2} \\ &= \frac{(n+1)(n+2)}{2} \end{aligned}$$

yielding the expression to be proved for $P(n+1)$, and completing the proof by induction.

Problem 2

Prove that every integer greater than 1 has a prime divisor.

Hint: two cases to consider for a given integer n. Use strong induction.

Solution

Case 1: n is prime, and divides itself.

Case 2: n is not prime, so it must be a compound of two non-prime integers. Crucially, these integers are also *smaller* than n.

If we let $n = a\dot{b}$, with $a, b < n$, then by our inductive hypothesis, both a and b have prime divisors.

Then let a/d be an integer, with d any prime divisor of a.

The quantity $(n/a)(a/d)$ is the product of two integers, and thus the simplification n/d must also be an integer.

Thus proving that n has a prime divisor.

Problem 3

Prove that all trees with n vertices contain $n-1$ edges.

Solution

Base case: The smallest tree has $n = 1$ vertex, and $n - 1 = 0$ edges.

Consider an arbitrary tree with n vertices.

Remove an arbitrary edge from the tree. The result will be two disjoint trees (proof by contradiction: otherwise, the original tree would contain a cycle), let these be called tree S with $k \in [1..n]$ vertices, and tree T with $n - k$ vertices.

By our inductive hypothesis, tree S contains $k - 1$ edges and tree T contains $(n - k) - 1$ edges.

Consider merging these trees by adding an edge between any two vertices $s \in S$ and $t \in T$. The result must be a tree, no cycle can be created since there was no path from s to t betwen two disjoint trees.

The resulting tree will contain $(k - 1) + (n - k - 1) + 1 = n - 1$ edges, thus proving the result by induction.

Problem 4

Any convex polygon P with $n \geq 3$ vertices can be decomposed into a set of $n-2$ triangles whose interiors do not overlap.

Solution

Base case: smallest polygon has $n = 3$ vertices, equivalent to a single triangle.

Next, consider an arbitrary convex polygon with n vertices.

By definition of convexity, we can add an edge between any two nonadjacent vertices in P. This edge is guaranteed not to overlap any edges of the polygon.

This added edge decomposes P into two smaller nonoverlapping polygons, Q and R. Q has $k \in [3..(n-3)]$ vertices, and R has $n-k+2$ vertices (the +2 from the two vertices the polygons share).

Thus, by our inductive hypothesis, Q can be decomposed into $(k-3)$ triangles and R can be decomposed into $(n-k)$ triangles.

The sum of the number of triangles in these two smaller polygons is $n-3$ triangles to compose our original polygon P, thus proving the statement by induction.