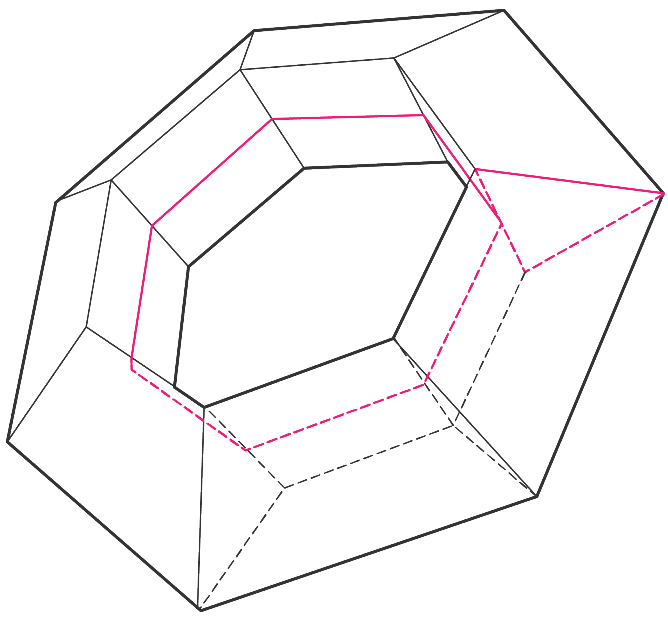
Discrete Mesh Operators



Abstract: All modern computational geometric systems, from FEM simulations, signal processing (image/video) analysis and commercial applications like in visual effects, require the extensive use of the theory of discrete differential operators as the bridge between conventional mathematical modelling of surfaces and PDES in smooth settings to computers. This work is an attempt to coherently translate and survey many of these operators as comprehensively as possible as to use them more efficiently in research and industrial setttings.

Keywords: Discrete Differential Geometry, Finite Element Methods, PDEs, Computational Geometry, Visual Effects, Signal Processing

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Chapter 1 Introduction

The current market. The (digital) world is *covered* in meshes: evaluated at more than USD 238 *billion* [**?**] in 2024, and expected to reach around USD 372 *billion* by 2030, the computer graphics community is in virtually every segment of the industrial and scientific world. From *Finite Element Methods* (FEM) employed for the simulations of fluids, stellar coronae and static deformations, to stunning award-winning visual effects accomplished by *Weta* and *Pixar*, and a variety of robotics, machine learning and medical applications, **meshes** are the most used backbone to all these areas, underlying the majority of the used software and mathematical constructions.

The applications of most interest to this study relate to *2-triangulated* (triangle) meshes used in practical geometrical steps to many algorithms and artistic endeavours, be it texture wrapping, variational cutting, and mesh reconstruction from point primitives, as well to the disciplines of physical simulations with mesh based methods, such as *FEM/BEM/ISPEM*, which involve *3-triangulated* (tetrahedron) as well as standard 2-triangulated meshes. A sample application is the one in producing topologically stable *geodesic tracings* on broken or highly deformed objects, with the intent of later producing accurate transformations or calculating partial differential equations on them by means of a metric compensation, and the calculation of fluid flow on curved surfaces:

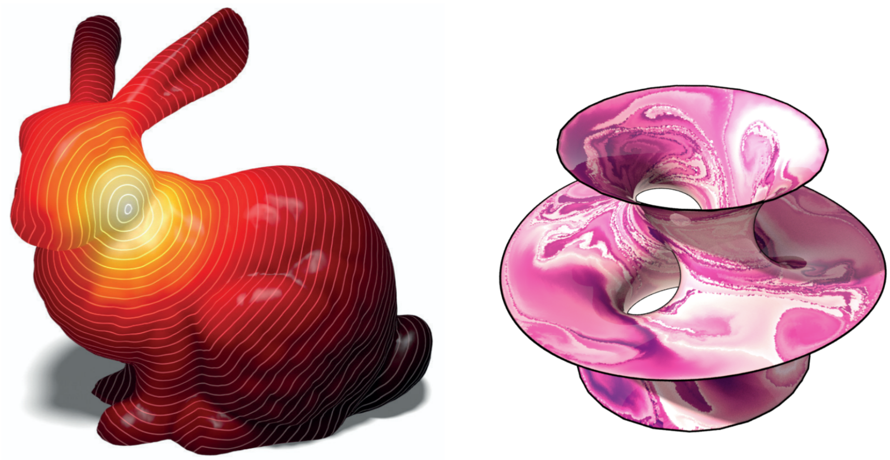


Figure 1.1. Production of geodesics on a bunny mesh by means of the *heat method* [?], and simulation of a soap film on Costa's minimal surface [?]. Source: Crane, Chern.

Given their plethora of applications and usefulness, it makes sense to develop very robust mathematical methods to manipulate meshes in physical and mathematical contexts. However, the difficulties for this rise rapidly when the context for current industrial standards come into play: with over *a 100 different file formats* (`.obj, .wav` ...) [**?**], and more than *30 major* software packages of manipulating said files, we then have 4 different coordinate system conventions, different spline parametrization conventions, and dozens of small variations in geometric primitives and data structures.

Mesh quality. Further, supposing you take all this into account, we still have the actual *quality* of the meshes to take into consideration. The vast majority of geometric data, period, is extremely *irregular*, being mostly harvested from physical sensors and observation equipment, but even purely digitally produced data is full of degenerate artifacts due to poor practices and mesh production algorithms. This means a large part of geometry processing algorithms on meshes *fail* for big datasets, since they can't handle all the possible edge cases unless otherwise explictly told to.

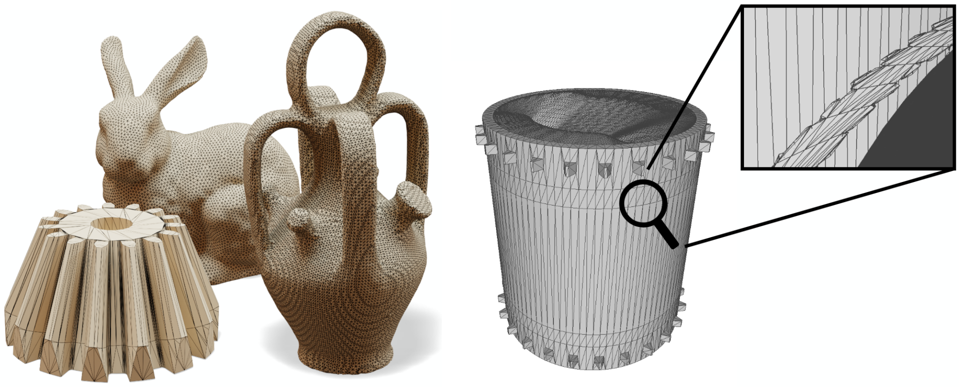


Figure 1.2. Difficulties in mesh quality vary from extremely high polygon density, sheared triangles, unpatched holes, lack of connectivity, highly skewed geometry, etc. Source: Nicholas Sharp.

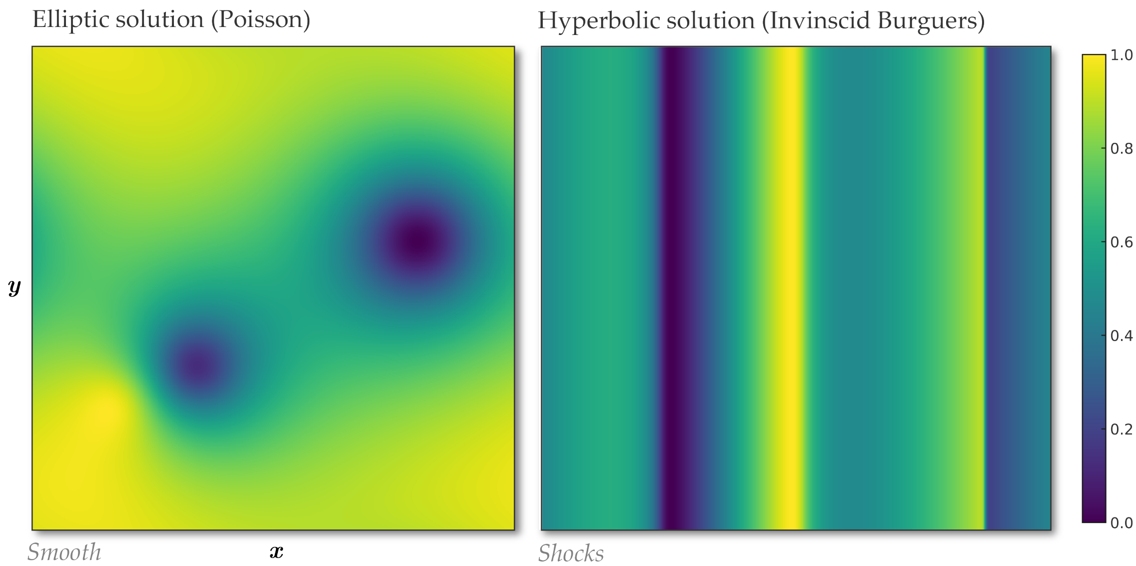
Therefore, either you need to *manually* clean up your meshes, or produce one from scratch given some other geometric data, which is a very laborious, time consuming procedure, requiring artistics and technical expertise. Consider the manual meshing process of a *Formula 1 car* [**?**], which even for simple models can take up to a week, or almost any mesh production for CFD simulation.

A great part of mesh processing algorithms still need to be manually tuned and selected by experts, be it researchers or artists, to be used, and this manual tweaking can often *break* the desired properties of the algorithm in the first place. The greatest offenders are the so-called *variational crimes* [**?**] in many FEM solvers, which break the behavior and accuracy of variational (e.g Garlekin) methods by poor geometric considerations, all of which can be fixed by more geometric-centry algorithms, such as by *Finite Element Exterior Calculus* [**?**]; these errors are *exceptionally* dangerous, because, as reported many times [**?**], they can introduce very dangerous vulnerabilities in airplanes, boats, and other critical equipment and infrastructure, since the related models don't actually converge to the physically real answer, leading to underestimations of stress, and etc. Therefore, even more than just convencience, new geometric methods are needed for safety.

A simpe illustration of this phenomena is by the distinction in the simulation of *hyperbolic* or *elliptic* differential operators. Altough elliptic operators are generally as smooth as the input data and conditions permit them to be, hyperbolic operators usually contain naturally discontinuous and badly-behaved solutions [**?**], such as shocks and solitons, therefore many methods fail to even properly capture the actual node-behvaior, much less convergence, of the solutions. An example is in the solutions of the *Poisson problem*, a smooth "screened" Laplacian problem, and of the *Inviscid Burguers problem*, a fairly simple nonlinear wave equation with dissipation in fluid dynamics. Because of the presence of modes proportional to the gradient, it develops sharp gradients which then become *shocks* [**?**].

1.1 Justification

Mesh agnosticism All of this leads to the question of how to efficiently produce new algorithms that can process *any* mesh, regardless of its quality or origin? It is of note



$$\Delta u = f \quad \text{On} \quad \Omega$$
$$u = g \quad \text{On} \quad \partial\Omega$$

$$\mathcal{B}[u] = \partial_t u + \partial_x \left(\frac{1}{2} u^2 \right) = 0 \quad \text{On } \Omega$$
$$u = g \quad \text{On } \partial\Omega$$

Figure 1.3. Comparison of an elliptic operator, the Poisson problem, and the hyperbolic inviscid Burguers problem, which contains jump discontinuities and shocks Source: Author

how appreciably hard this problem is, given that in such an active landscape it hasn't been satisfactorily solved yet, but it is not *impossible*. In the last 10-15 years, new techniques have been slowly implemented by research groups such as the *Geometry Collective* and *Ulrich Pinkall's group* and many more as to create *mesh agnostic* methodologies.

These are done under the general guise of *discrete analysis* and *discrete differential geometry*, or *DDG*. The *lemma* of DDG is similar to the "quantization" procedure of physics; given the *discretization* of a smooth shape, is it possible to, in the limit of *homogeneity*, that is, of *infinite* mesh resolution, recover all the given results of regular geometry [**?**]? Further, is it possible to make similar analogues to theorems even in low resolution scenarios?

For example, naturally, the *extrinsic curvature* of an object may defined on a parametrized curve $\gamma(s)$ as $\kappa = \langle N, \ddot{\gamma}(s) \rangle$, that is, the normal projected onto the "acceleration". For any C^2-curve, this is of course well-defined and well behaved, but what of a discrete, polygonal curve? Any attempt to take the second derivative of a piece-wise discrete function will lead to the curvature being *infinite*. Nevertheless, we can produce *multiple* "curvature measures of said" curve that, in the infinite polygon limit, mimic the exact smooth curvature [**?**].

Taking this idea to its limit, we then finally arrive at the project's full proposal, which is a subproject of the area of DDG to a full coverage of **discrete mesh differential operators**, which is a map between the often known differential operators in the smooth $C^\infty(\mathcal{M})$ setting to the discrete, piecewise setting. This mapping allows for a full reproduction of the results of the theory of *partial differential equations, algebraic topology* and more onto the computational setting.

Spectral geometry processing. Let's use a simple example of the discretization procedure, and perhaps its most important. Given the regular *Hodge-Laplace* operator ($\boldsymbol{\Delta}$) on a smooth manifold $\mathcal{M}$, we may define the *discrete* equivalent by means of a *intrinsic Delaunay triangulation* procedure [**?**].

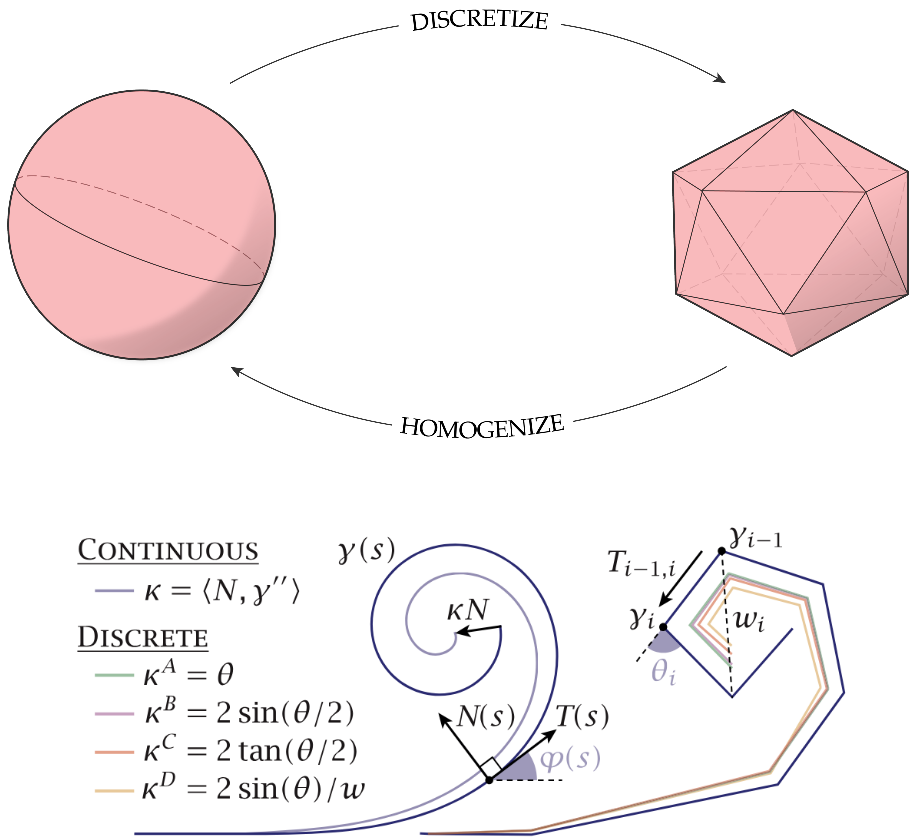


Figure 1.4. The *lemma* of DDG (top), and the variety of possible curvature measures on a discrete polygonal curve (bottom). Source: Author, Crane.

Definition 1 (Discrete Hodge-Laplace operator on 0-forms) Let $(\mathcal{M}, g)$ be a smooth oriented Riemannian surface and $\Delta := \delta d + d\delta$ the Hodge–Laplace operator, where d is the exterior derivative and $\delta = (-1)^{nk+1} \star d \star$ is the codifferential on k-forms in dimension n. On 0-forms $f \in \Omega^0(\mathcal{M})$, this reduces to

$$\Delta f = \delta(df),$$

the usual Laplace–Beltrami operator on functions.

Let $\mathbf{M} = (\mathbf{V}, \mathbf{E}, \mathbf{F})$ be a simplicial surface mesh approximating $\mathcal{M}$, with vertices $\mathbf{V}$, edges $\mathbf{E}$, and triangular faces $\mathbf{E}$. For an interior edge $ij \in \mathbf{E}$ shared by two triangles (i, j, k) and (i, j, ℓ), let α_{ij}, β_{ij} denote the angles opposite ij in those triangles. The *cotangent weight* for edge ij is

$$w_{ij} \;=\; \tfrac{1}{2}\big(\cot \alpha_{ij} + \cot \beta_{ij}\big),$$

while for a boundary edge ij (incident to a single face with opposite angle α_{ij}) one sets $w_{ij} = \frac{1}{2} \cot \alpha_{ij}$.

The **discrete Hodge–Laplace operator** (a.k.a. cotangent Laplacian) acting on 0-

forms $u : V \to \mathbb{R}$ is the matrix $L \in \mathbb{R}^{|V|\times|V|}$ with entries

$$(Lf)_{ij} = \frac{1}{2a_i} \sum_{i,j\ \in\ E} (\cot \alpha_{ij} + \cot \beta_{ij})\ (f_i - f_j)$$

Then for any function $u : V \to \mathbb{R}$, the discrete Laplacian is $(Lu)(i) = \sum_{j:\, ij\in E} w_{ij} \big(u(i) - u(j)\big)$. This operator is symmetric, has rows summing to zero, and converges to the smooth Laplace–Beltrami operator under mesh refinement.

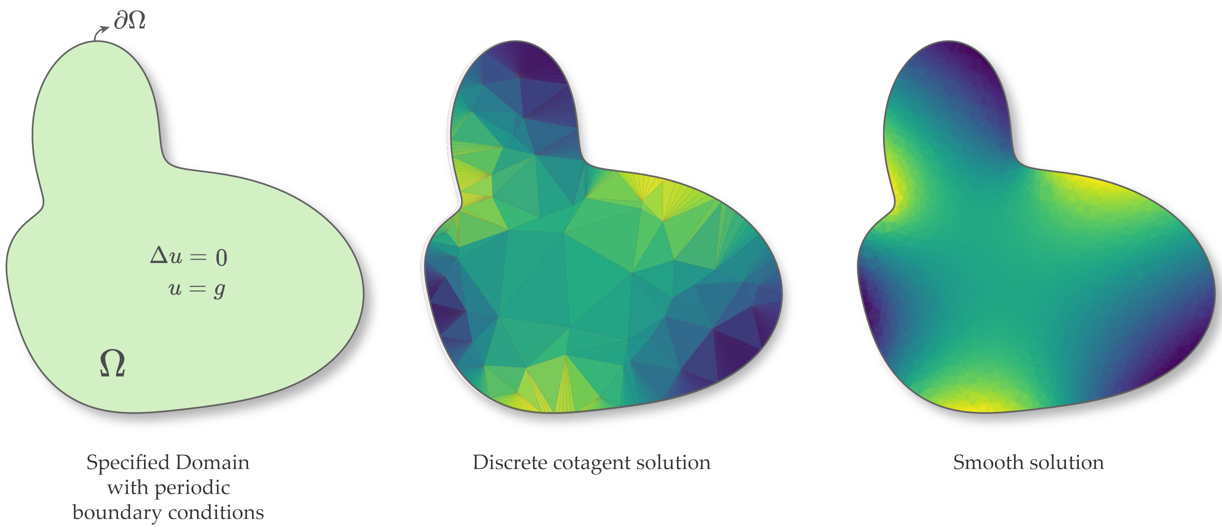


Figure 1.5. Comparison of the solutions of the discrete cotagent laplacian on a discretized mesh and the smooth laplacian. Source: Author

This operator is very much the "swiss-army knife" [?] of geometry processing algorithms; the main motivation comes to that the Laplacian is the lowest order differential operator that is self-adjoint and elliptic, and has the largest significance physically as one can prove that any operator that commutes with rotations and translations is just a linear combination of the laplacian. For the general eigenvalue Helmholtz problem where a solution will always be $u_k(\mathbf{x}, t) = \alpha_k(t)\phi_k(\mathbf{x})$:

$$\boldsymbol{\Delta}\phi_k = \lambda_k\phi_k, \quad \lambda_k \in \mathbb{C}$$

We may have solutions [?]:

$$\alpha_k(t) = \begin{cases} \exp(-\lambda_k t) & \text{Heat equation} \\ \exp\left(i\sqrt{\lambda_k} t\right) & \text{Wave equation} \\ \exp(i\lambda_k t) & \text{Schrödinger equation} \end{cases}$$

This permits a variety of deductions, from estimating the boundary behavior (in his famous essay *Can One Hear the Shape of a Drum?* [?]) as well as estimating the actual deformation properties of the volume and area elements, estimating distances, so on and so forth.

However, in the same vein as the discrete extrinsic curvature, there are *many* Laplacian choices that need to respect the smooth laplacian, all of which with their own pros and cons [?] (The positive-semi definitess was only recently fixed with the intrinsic Delaunay Laplacian, for example). That is, there are many equivalent choices that approach the limit of the smooth one.

This introduces the problem of choice: what are the **error metrics** one needs in order to determine the best discrete operator? More, what is a general procedure of constructing them if they're generally non-unique? This lack of well-posedness forces us to make some kind of general *availability* criteria, as to wheter or not the properties of an operator D in the smooth setting are availble to us in the discrete one.

Beyond $0-$forms. Beyond the simple Laplacian, we also have the famous *Dirac operator*, its famous "square root", and the fundamental equation of relativistic quantum theory.
Folloiw

Definition 2 (Discrete extrinsic Dirac operator)

$$D\phi \tag{1.1}$$

We $dz \wedge d\bar{z} = -2i|df|^2$

1.2 Previous work

Discrete analysis. A large amount of work on discrete operator theory has already been done, even previous to the advent of much of modern computing, by means of *discrete analysis*. Examples include the famous *Regge Calculus* of general relativity [?], or in the study of topology and homotopical algebra by means of simplicial complexes and sheaves [?]. The *expansion* of these discrete theories for computational purposes has also been massively sucessful and applied in a dozen or so different mathematical areas; a lot of these results however are quite *scattered*, so a thorough literature review is necessary.

Specific to our purposes is most of the literature related to the *discrete exterior calculus* (DEC) [?, ?], including extensions for bundle-valued elements [?] and methods therein as to implement all of the basic differential operators and simplicial complex theory in computers, such as *discrete connections* [?], *homology* [?, ?] and *cohomology* measures, and other differential geometric quantities [?]. A couple of works have already implement partly or purely functional libraries of DEC such as in the Rust language [?], or more sophisticated Clifford algebraic libraries such as in Javascript ganja.js.

Differential taxonomy. Somewhat general frameworks for discrete differential operators in meshes are already present [?, ?, ?], including well posed parabolic differential operators [?]. Of particular interest to the the author is in the distinctions and manipulation of *intrinsic, extrinsic, scalar* and *hypercomplex* operators. The first two categories verify how much on the normal vectors of the given mesh the operator depends on; the more it is solely defined by edge/vertex relationns, the more intrinsic it is, and the two other categories distinguish between operators that act on and output $0-$forms $f(\mathbf{x})$, regular functions, and operators that take as either input or output $(p, q)-$valued forms or general hypercomplex numbers, such as *quaternions* [?, ?, ?], spinors [?, ?], and multivectors/Clifford numbers in general [?, ?, ?, ?, ?]. Of future interest is the implementation of techniques of the physical sciences [?] into these methodologies.

Based on previous surveys [?], we have made a small table of the current landscape of operators (not ehxaustive):

	Intrinsic	Extrinsic
Scalar	• Laplace–Beltrami (cotan) • Graph Laplacian • Anisotropic Laplacians • Hessian operators • Schrödinger/Hamiltonian • Connection Laplacian	• Dirichlet-to-Neumann • Volumetric Laplacian • Shape operator • Mean curvature flow • Curvature Laplacian
Hypercomplex	• Intrinsic Dirac • Quaternionic Laplacian • Spinor Dirac • Cauchy–Riemann / Beltrami	• Extrinsic Dirac • Relative Dirac • Conformal deformation ops • Spinorial shape ops

Table 1.1. Dictionary of mesh operators by intrinsic/extrinsic and scalar vs. hypercomplex.

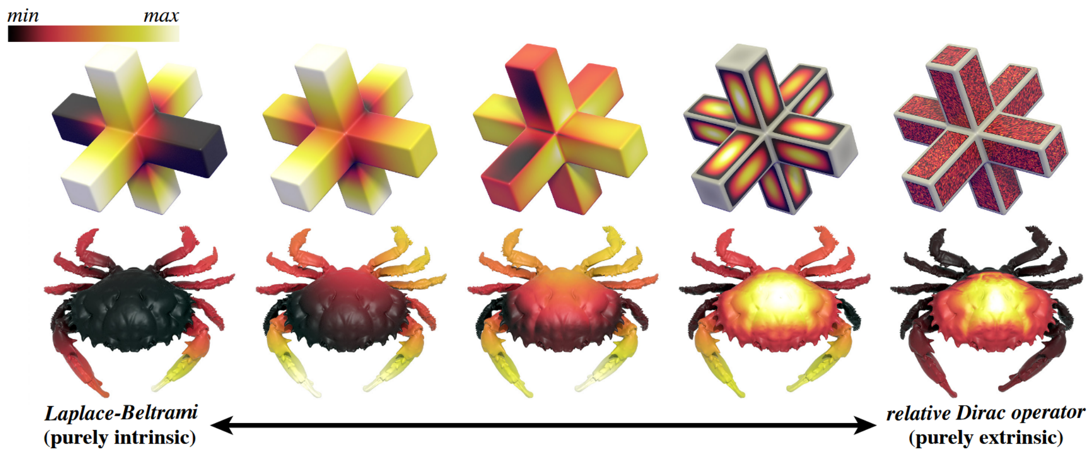


Figure 1.6. Gradient between intrinsic scalar (Laplace) and extrinsic hypercomplex (Dirac) operators. Source: Crane.

Chapter 2 Objectives

2.1 Reproduction of State-of-the-Art

Of our main goals, the first is a complete implementation of the *already* existing literature, in an open-source fashion, such that the operators and algorithms can be easily used by both the *GIVA* Lab as well as other interested researchers and industry professionals.

2.2 Specific goals

A list of resources that will be studied and solved

1. Lawrence C. Evans - Partial Differential Equations [?]
2. Berline, Getzler and Verne - Dirac Operators and Heat Kernels [?]
3. Jayme Vaz Jr - Representation of surfaces using spinor operators [?]
4. Jayme Vaz Jr - The Dirac Operator over Abelian Finite Groups [?]

A list of desired useful algorithms for all mesh processing applications in the GIVA Lab:

1. Trivial connections on discrete surfaces [?]
2. Discrete Torsion of Connection Forms on Simplicial Meshes [?]
3. Spin transformations on discrete surfaces [?]
4. Globally optimal direction fields [?]
5. The Vector Head Method [?]
6. Symmetric Moving Frames [?]
7. Affine Heat Method [?]

A list of operators to be initially implemented in Python/Rust:

1. A Dirac Operator for Extrinsic Shape Analysis
2. A Laplacian for Nonmanifold Triangle Meshes [?]

Chapter 3 Work plan

- A1. Initial literature review and definition of the problem;
- A2. Continuous monitoring of relevant literature;
- A3. Study and research to determine the SSI approach to be used, aiming for a trade-off between geometric accuracy performance — acquiring familiarity with GPU processing.
- A4. Development and implementation of the tools on which the numerical methods adopted by the approach chosen in activity A3 will be based (AA framework, spatial acceleration structures on CPU and GPU, etc.);
- A5. Development and open-source implementation of the selected mesh operators;
- A6. Writing the text for the qualification exam required by the institution;
- A7. Study and research to determine how ;
- A8. Development and implementation of a;
- A9. Publication of the results obtained throughout the project;
- A10. Writing the dissertation and defense.

3.1 Timeline

CHAPTER 4 MATERIALS AND METHODS

Due to the nature of the research being only a software endeavour, no extra materials will be necessary (besides coffee!).

4.1 Methodology

Activities related to literature review (A1, A2) and study/research (A3, A7) will be carried out by the scholarship recipient without a predefined structure; weekly meetings between the scholarship recipient and the advisor will be used to guide the direction of the research in order to maintain the schedule. Activities related to development (A4, A5, A8) will be carried out mainly by the scholarship recipient, with monitoring by the advisor through version control systems (VCS), in addition to the aforementioned weekly meetings. Activities related to scientific writing or publication (A6, A9, A10) will be carried out by the scholarship recipient and reviewed by the advisor.

Chapter 5 Result Analysis

5.1 Expected results

APPENDIX A A

Appendix one text goes here.

Definition 3 (Well-posed problem ("good behavior")) A problem for a PDE is *well-posed* if (i) it has a solution, (ii) the solution is unique, and (iii) the solution depends continuously on the data.

Definition 4 (Classical solution) Let $k \geq 1$ be the order of the PDE. A *classical solution* is a function u that is at least C^k so that all derivatives appearing in the PDE exist and are continuous; we then solve the PDE in the classical sense subject to (well-posedness) items (i)–(iii) when applicable.

Definition 5 (Weak (variational) solution — elliptic case) For a divergence-form elliptic operator

$$Lu \;=\; -\sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u$$

on a domain U, with right-hand side f, define the bilinear form

$$B[u,v] \;:=\; \int_U \Big(\sum_{i,j} a_{ij}\, u_{x_i} v_{x_j} \;+\; \sum_i b_i\, u_{x_i} v \;+\; c\, u\, v \Big)\, dx.$$

We call $u \in H_0^1(U)$ a *weak solution* of $Lu = f$ with $u|_{\partial U} = 0$ if

$$B[u,v] \;=\; (f,v)_{L^2(U)} \quad \text{for all } v \in H_0^1(U).$$

Definition 6 (Weak (variational) solution — parabolic case) For the parabolic problem $u_t + Lu = f$ on $U \times (0,T)$ in divergence form with the time–dependent bilinear form $B[\cdot,\cdot;t]$, a function

$$u \in L^2(0,T; H_0^1(U)), \qquad u' \in L^2(0,T; H^{-1}(U))$$

is a *weak solution* if, for a.e. $t \in (0,T)$ and every $v \in H_0^1(U)$,

$$(u'(t), v)_{H^{-1},H_0^1} \;+\; B[u,v;t] \;=\; (f(t),v)_{L^2(U)}.$$

Definition 7 (Regularity (of weak solutions)) The *regularity problem* asks whether a weak solution is in fact smoother (e.g. belongs to higher Sobolev spaces or is C^∞). For elliptic problems, a typical gain reads: if $-\Delta u = f$ with $f \in H^m$, then (morally) $u \in H^{m+2}$, i.e. u has "two more derivatives in L^2" than f.

Definition 8 (Blow-up) A solution u *blows up in finite time* if there exists $0 < t_* < \infty$ such that some natural norm or quantity associated with $u(\cdot, t)$ becomes unbounded as $t \uparrow t_*$. For instance, in the nonlinear heat example in Evans, one obtains

$$\lim_{t\uparrow t_*} \int_U u(x,t)\, w_1(x)\, dx \;=\; +\infty$$

(for a fixed positive first eigenfunction w_1), and in this situation one says that u blows up at time t_*.

Definition 9 (Uniform ellipticity (second-order operator)) Let

$$Lu = -\sum_{i,j=1}^{n} (a_{ij}(x) u_{x_i})_{x_j} \;+\; \sum_{i=1}^{n} b_i(x)\, u_{x_i} \;+\; c(x)\, u$$

$$Lu = -\sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j} \;+\; \sum_{i=1}^{n} b_i(x)\, u_{x_i} \;+\; c(x)\, u,$$

with $a_{ij} = a_{ji}$. We say L is *(uniformly) elliptic* if there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \;\geq\; \theta\, |\xi|^2 \quad \text{for a.e. } x \in U \text{ and all } \xi \in \mathbb{R}^n.$$

Definition 10 (Uniform parabolicity) For the operator $\partial_t + L$ as above (with $a_{ij} = a_{ji}$ possibly depending on (x,t)), we say $\partial_t + L$ is *(uniformly) parabolic* if there exists $\theta > 0$ such that

$$\sum_{i,j=1}^{n} a_{ij}(x,t)\, \xi_i \xi_j \;\geq\; \theta\, |\xi|^2 \quad \text{for all } (x,t) \in U \times (0,T) \text{ and all } \xi \in \mathbb{R}^n.$$

Definition 11 (Hyperbolic first-order system (linear)) Consider

$$u_t + \sum_{j=1}^{n} B_j(x,t)\, u_{x_j} = f(x,t), \qquad u : \mathbb{R}^n \times [0,\infty) \to \mathbb{R}^m.$$

For $y \in \mathbb{R}^n$ set $B(x,t;y) = \sum_{j=1}^{n} y_j B_j(x,t)$. The system is called *hyperbolic* if, for each (x,t) and each y, the matrix $B(x,t;y)$ is diagonalizable over $\mathbb{R}$ (equivalently: it has m real eigenvalues and a basis of eigenvectors).

Definition 12 (Jump discontinuity of a BV function) Let $u \in BV(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open. A point $x \in \Omega$ belongs to the *jump set* J_u if there exist two distinct values $u^+(x), u^-(x) \in \mathbb{R}$ and a unit vector $\nu(x) \in \mathbb{S}^{n-1}$ such that

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho^+(x,\nu)|} \int_{B_\rho^+(x,\nu)} |u(y) - u^+(x)| \, dy = 0,$$

and

$$\lim_{\rho \to 0^+} \frac{1}{|B_\rho^-(x,\nu)|} \int_{B_\rho^-(x,\nu)} |u(y) - u^-(x)| \, dy = 0,$$

where $B_\rho^\pm(x,\nu) = \{ y \in B_\rho(x) : \pm(y - x) \cdot \nu > 0 \}$ are the half-balls centered at x with radius ρ, oriented by ν.

At such a point $x \in J_u$, the function u has a *jump discontinuity* with one-sided traces $u^+(x)$ and $u^-(x)$ separated by the normal direction $\nu(x)$.

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