MATH 4032: HOMEWORK 7 (DUE APRIL 14 AT 11:59PM)

You are strongly encouraged to typeset your homework solutions using LaTeX.
Warm-up and bonus questions are not for credit.

Warm-up. *The space of multivariate polynomials.*

What is the dimension of the space of all polynomials in n variables of degree at most d?

Problem 1. *Oddtown.*

Fix a positive integer n. An **oddtown** is a set system $\mathcal{F} \subseteq \mathcal{P}([n])$ such that:

- for all $A \in \mathcal{F}$, $|A|$ is odd;
- for all distinct $A, B \in \mathcal{F}$, $|A \cap B|$ is even.

(a) Construct an oddtown $\mathcal{F}$ of size n.

(b) Show that $|\mathcal{F}| \leqslant n$ for every oddtown $\mathcal{F}$.

Hint. Work over the field $\mathbb{Z}_2$.

Problem 2. *Eventown.*

Fix a positive integer n. An **eventown** is a set system $\mathcal{F} \subseteq \mathcal{P}([n])$ such that:

- for all $A \in \mathcal{F}$, $|A|$ is even;
- for all distinct $A, B \in \mathcal{F}$, $|A \cap B|$ is even.

(a) Construct an eventown $\mathcal{F}$ of size $2^{\lfloor n/2 \rfloor}$.

Suppose that $\mathcal{F}$ is an eventown and let M be the matrix over $\mathbb{Z}_2$ whose columns are the characteristic vectors of the members of $\mathcal{F}$.

(b) Show that $M^\top M = 0$, where $M^\top$ denotes the transpose of M.

(c) Use part *(b)* to deduce that the rank of M is at most $\lfloor n/2 \rfloor$.

(d) Conclude that $|\mathcal{F}| \leqslant 2^{\lfloor n/2 \rfloor}$.

Problem 3. *Frankl–Wilson mod p.*

Prove the "mod p" version of the Frankl–Wilson theorem:

Let p be a prime number and let $L \subseteq \mathbb{Z}_p$ be a set of size $|L| = \ell$. Suppose that $\mathcal{F} \subseteq \mathcal{P}([n])$ is a set system such that:

- for all $A \in \mathcal{F}$, $|A| \pmod{p} \notin L$;
- for all distinct $A, B \in \mathcal{F}$, $|A \cap B| \pmod{p} \in L$.

Then $|\mathcal{F}| \leqslant \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{\ell}$.

Problem 4. *Zero, one, three!*

Let n be a positive integer and let $\mathcal{F} \subseteq \binom{[n]}{k}$ be a $\{0, 1, 3\}$-intersecting system of sets of size k.

(a) Show that $|\mathcal{F}| = O(n^3)$.

(b) Show that if $k \pmod{6} \notin \{1, 3\}$, then $|\mathcal{F}| = O(n^2)$.

Hint. To deal with even k, bound the number of sets in $\mathcal{F}$ that can contain any one element $x \in [n]$.

Remark. In this problem, we write $|\mathcal{F}| = O(f(n))$ to mean that there is some constant $c > 0$, independent of n and k, such that $|\mathcal{F}| \leqslant c f(n)$.